MATH 174E: Mathematical Finance
Lecture Note

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CHAPTER 1

Introduction to mathematical finance

1. Derivatives

The *Market* is an economic system where participants exchange various assets in order to maximize their profit. An *Asset* is an entity that has a trade value in the market, whose value may change (randomly) over time. Whereas assets could be as real as rice grains, cloths, and semiconductors, they could also be as abstract as the right to purchase 100 shares of certain company's stock for \$5000 in three months. The latter is an example of options, which is a popular form of 'derivatives'.

Definition 1.1 (Derivative)**.** A *derivative* is a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables[1].

In most cases the variables underlying derivatives are the prices of traded assets. Derivatives can be dependent on almost any variable. Diversity of the form of assets in the market means the form of possible trades can be more flexible. For instance, typical cycle of town home in LA could be as long as five years, but large banks could trade their mortgages in a much shorter time period.

Three basic form of derivatives are *forward contract*, *futures contract*, and *options*.

Definition 1.2 (Forward contract)**.** A *forward contract* is a direct agreement between two parties to buy or sell an asset at a certain future time for a certain price. A *spot contract* is an agreement to buy or sell an asset almost immediately.

One of the parties in a forward contract assumes a *long position* and agrees to buy the underlying asset on a certain specified future date for a certain specified price, which is called the *delivery price*. The other party assumes a *short position* and agrees to sell the asset on the same date for the same price.

Proposition 1.3. *Consider a forward contract on an asset, whose price is given by $(S_t)_{t\geq 0}$. Let T denote the expiration date and K be the delivery price. Then*

$$\textit{payoff from a long position per unit asset} = S_T - K \tag{1}$$

$$\textit{payoff from a short position per unit asset} = K - S_T. \tag{2}$$

PROOF. If in long position, then at the day of maturity T, one should purchase one unit of the asset for the delivery price K. Since the actual price of one unit of at time T is S_T, the profit is $S_T - K$. Since this is a contract between two parties, the profit from short position should be $K - S_T$ so that the sum of the total profit from both parties is zero. □

Definition 1.4 (Futures contract)**.** A *futures contract* is an indirect agreement between two parties to buy or sell an asset at a certain time in the future for a certain price, called the *futures prices*, through a third party.

In contrast to the forward contracts, futures contracts are normally traded on an exchange, which specifies standardized features of the contract. As the two parties to the contract do not necessarily know each other, the exchange also provides a mechanism that gives the two parties a guarantee that the contract will be honored.

[1] Mathematically speaking, functions of the underlying (random) variables.

Example 1.5. The futures price in a futures contract, like any other price, is determined by the laws of supply and demand. Consider in a corn futures contract market in July, the current futures prices for each bushel is 500 cents. If, at a particular time, more traders wish to sell rather than buy October corn futures contract, the futures price will go down. New buyers then enter the market so that a balance between buyers and sellers is retained. If more traders wish to buy rather than sell October corn, the price goes up. New sellers then enter the market and abalance between buyers and sellers is retained.

Definition 1.6 (Options)**.** A *call option* gives the holder (long) the right to buy the underlying asset by a certain date for a certain price. A *put option* gives the holder (long) the right to sell the underlying asset by a certain date for a certain price. The price in the contract is known as the *exercise price* or *strike price*; the date in the contract is known as the *expiration date* or *maturity*. *American options* can be exercised at any time up to the expiration date. *European options* can be exercised only on the expiration date itself.

Buying or holding a call or put option is a *long position* since the investor has the right to buy or sell the security to the writing investor at a specified price. Selling or writing a call or put option is the opposite and is a *short position*, since the writer is obligated to sell the shares to or buy the shares from the long position holder, or buyer of the option.

Example 1.7 (Long and short positions in options)**.** Say an individual buys (goes long) one Apple (AAPL) American call option from a call writer for \$28 (the writer is short the call). Say the strike price on the option is \$280 and AAPL currently trades for \$310 on the market. The writer takes the premium payment of \$28 but is obligated to sell AAPL at \$280 if the buyer decides to exercise the contract at any time before it expires. The call buyer who is long has the right to buy the shares at \$280 before expiration from the writer.

One other hand, say an individual buys (goes long) one Apple (AAPL) European put option from a put writer for \$28 (the writer is short the call). Say the strike price on the option is \$290 and AAPL currently trades for \$310 on the market. The writer takes the premium payment of \$28 but is obligated to buy AAPL at \$290 if the buyer decides to exercise the contract at any time before it expires. ▲

For each real number $a \in \mathbb{R}$, we denote $a^+ = \max(a, 0)$ and $a^- = -\min(0, a)$.

Proposition 1.8. *Consider a European call/put option on an asset, whose price is given by* $(S_t)_{t\geq 0}$*. Let* T *denote the expiration date and* K *be the strike price. Then ignoring the cost of the option,*

$$\text{payoff from a long position in call option per unit asset} = (S_T - K)^+ \tag{3}$$

$$\text{payoff from a long position in put option per unit asset} = (K - S_T)^+. \tag{4}$$

PROOF. If in long position, then one has the right to buy one unit of the asset for strike price K at time T. If $S_T \geq K$, then using the option gives profit of $S_T - K \geq 0$; otherwise, one should not exercise the option and get profit 0. Combining these two cases, the profit is $(S_T - K)^+$. On the other hand, if in short position, then one has the right to sell one unit of the asset for the strike price K. If $S_T < K$, then using the option results in profit $K - S_T \geq 0$; otherwise, one should not exercise the option and get profit 0. Hence in this case the profit of the option is $(K - S_T)^+$. □

2. Types of traders

Three broad categories of traders can be identified: hedgers, speculators, and arbitrageurs. *Hedgers* use derivatives to reduce the risk that they face from potential future movements in a market variable. *Speculators* use them to bet on the future direction of a market variable. *Arbitrageurs* take offsetting positions in two or more instruments to lock in a profit.

2.1. Hedgers. The purpose of hedging is to reduce risk. However, there is no guarantee that the outcome with hedging will be better than the outcome without hedging.

Example 2.1 (Hedging using options, Excerpted from [Hul03])**.** Consider an investor who in May of a particular year owns 1,000 shares of a particular company. The share price is \$28 per share. The investor is concerned about a possible share price decline in the next 2 months andwants protection. The investor could buy ten July put option contracts on the company's stock with a strike price of \$27.50. Each contract is on 100 shares, so this would give the investor the right to sell a total of 1,000 shares for a price of \$27.50. If the quoted option price is \$1, then each option contract would cost $100 \times \$1 = \$100$ and the total cost of the hedging strategy would be $10 \times \$100 = \$1000$.

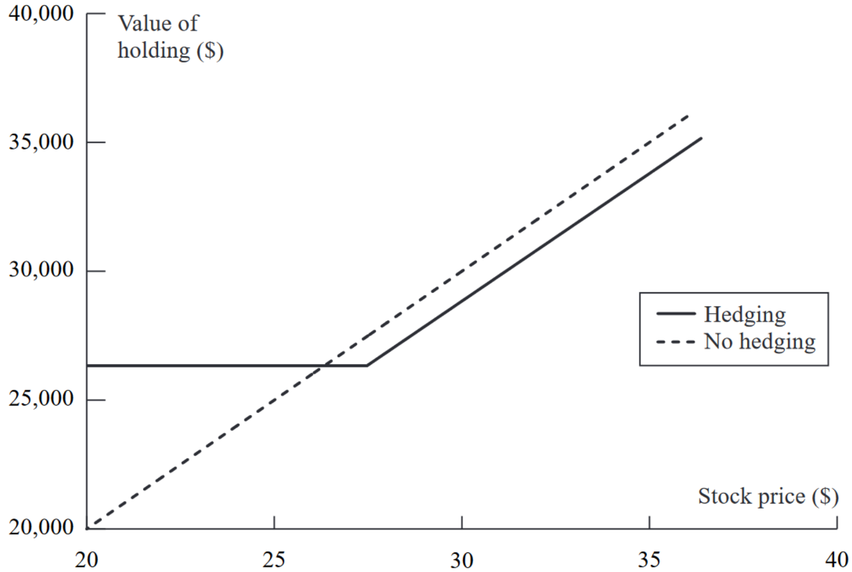

FIGURE 1. Value of the stock holding in 2 months with and without hedging

The strategy costs \$1,000 but guarantees that the shares can be sold for at least \$27.50 per share during the life of the option. If the market price of the stock falls below \$27.50,the options will be exercised, so that \$27,500 is realized for the entire holding. When the cost of the options is taken into account, the amount realized is \$26,500. If the market price stays above \$27.50, the options are not exercised and expire worthless. However, in this case the value of the holding is always above \$27,500 (or above \$26,500 when the cost of the options is taken into account). Figure1.4 shows the net value of the portfolio (after taking the cost of the options into account) as a function of the stock price in 2 months.The dotted line shows the value of the portfolio assuming no hedging. ▲

2.2. Speculators. While hedgers want to avoid exposure to adverse movements in the price of an asset, speculators are willing to take a position in the market. Either they are betting that the price of the asset will go up or they are betting that it will go down.

Example 2.2 (Speculating using options, Excerpted from [Hul03])**.** Suppose that it is October and a speculator considers that a stock is likely to increase in value over the next 2 months. The stock price is currently \$20, and a 2-month call option with a \$22.50 strike price is currently selling for \$1. There are two possible alternatives, assuming that the speculator is willing to invest \$2,000. One alternative is to purchase 100 shares; the other involves the purchase of 2,000 call options (i.e., 20 call option contracts). Suppose that the speculator's hunch is correct and the price of the stock rises to \$27 by December.The first alternative of buying the stock yields a profit of

$$100 \times (\$27 - \$20) = \$700. \tag{5}$$

However, the second alternative is far more profitable. A call option on the stock with astrike price of \$22.50 gives a payoff of \$4.50, because it enables something worth \$27 tobe bought for \$22.50. The total payoff from the 2,000 options that are purchased under the second alternative is

$$2000 \times \$4.50 = \$9000. \tag{6}$$

Subtracting the original cost of the options yields a net profit of

$$\$9000 - \$2000 = \$7000. \tag{7}$$

The options strategy is, therefore, 10 times more profitable than directly buying the stock.

Options also give rise to a greater potential loss. Suppose the stock price falls to \$15 by December. The first alternative of buying stock yields a loss of

$$100 \times (\$20 - \$15) = \$500. \tag{8}$$

On the other hand, because the call options expire without being exercised, the options strategy would lead to a loss of \$2,000, which is the original amount paid for the options. ▲

2.3. Arbitrageurs. Arbitrageurs are a third important group of participants in futures, forward, and options markets. Arbitrage involves locking in a riskless profit by simultaneously entering into transactions in two or more markets.

Example 2.3 (Kimchi premium)**.** In late 2017, the price of most cryptocurrency (e.g., Bitcoin, ripple, and Etherium) were much higher in Korean cryptocurrency exchange market (e.g., bithum) than in U.S. market (e.g., coinbase). Sometimes the price gap was more than 40%. This means that one can purchase 1 BTC for \$10,000 from the U.S. market and sell it in the Korean market for \$14,000 equivalent of Korean Won. Of course, this absurd arbitrage opportunity did not last long. ▲

The market is an extremely complicated dynamical system, so in order to mathematically analyze some of its properties, one needs to simplify the situation by assuming some hypothesis. One of the classical assumption in economics is that each participants are infinitely reasonable in maximizing their profit, so if somewhere in the market there exists any opportunity to obtain some profit with no risk, the participants take advantage of that 'arbitrage opportunity' until it is no longer available. Hence, if we want to analyze the 'typical behavior' of the market, we may simply assume that at any time and anywhere in the market, there exists no arbitrage opportunity. This is called the *principle of no arbitrage*.

Hypothesis 2.4 (Principle of no arbitrage)**.** *The market instantaneously reaches equilibrium so that there is no arbitrage opportunity of earn sure profit.*

Exercise 2.5 (Pricing a forward contract)**.** Suppose one share of the stock of a company A is now worth \$60. Suppose an investor could lend money for one year at 5% interest rate. Under the assumption of no arbitrage, what is the right price for the 1-year forward contract of one share of this stock? (See also, Proposition 5.2).

Example 2.6 (Convergence of futures price to spot price)**.** One of the many consequence of the no arbitrage principle is that, as the delivery period for a futures contract is approached, the futures price converges to the spot price of the underlying asset.

To illustrate this, we first suppose that the futures price is above the spot price during the delivery period. Traders then have a clear arbitrage opportunity:

(1) Sell (i.e., short) a futures contract.
(2) Buy the asset.
(3) Make delivery.

These steps are certain to lead to a profit equal to the amount by which the futures price exceeds the spot price. As traders exploit this arbitrage opportunity, the futures price will fall.

Suppose next that the futures price is below the spot price during the delivery period. Companies interested in acquiring the asset will find it attractive to enter into a long futures contract and then wait for delivery to be made. As they do so, the futures price will tend to rise. The result is that the futures price is very close to the spot price during the delivery period. ▲

3. Mathematical formulation of no-arbitrage principle

In this section we formalize the principle of no-arbitrage, which is the fundamental axiom in mathematical finance. We also derive the principle of replication, which states that two portfolios that have the same value in the future must have the same value at the current time. We will use this frequently in pricing derivatives such as forward and future contracts.

We model the market as a probability space $(\Omega, \mathbb{P})$, where Ω consists of sample paths ω of the market, which describes a particular time evolution scenario. For each event $E \subseteq \Omega$, $\mathbb{P}(E)$ gives the probability that the event E occurs. A *portfolio* is a collection of assets that one has at a particular time. The value of a portfolio A at time t is denoted by $V^A(t)$. If t denotes the current time, then $V^A(t)$ is a known quantity. However, at a future time $T \geq t$, $V^A(T)$ depends on how the market evolves during $[t, T]$, so it is a random variable.

Definition 3.1. A portfolio A at current time t is said to be an *arbitrage portfolio* if its value V^A satisfies the following property:

$$\text{There exists a future time } T \geq t \text{ such that } \mathbb{P}(V^A(T) \geq V^A(t)) = 1 \text{ and } \mathbb{P}(V^A(T) > V^A(t)) > 0. \tag{9}$$

Hypothesis 3.2 (Principle of no arbitrage)**.** *An arbitrage portfolio does not exist.*

Theorem 3.3 (Monotonicty Theorem)**.** *Consider two portfolios A and B at current time t. Under the assumption of no-arbitrage, the followings hold:*

(i) *If $T \geq t$ and $\mathbb{P}(V^A(T) \geq V^B(T)) = 1$, then $V^A(t) \geq V^B(t)$.*

(ii) *If $T \geq t$, $\mathbb{P}(V^A(T) \geq V^B(T)) = 1$, and $\mathbb{P}(V^A(T) > V^B(T)) > 0$, then $V^A(t) > V^B(t)$.*

PROOF. Consider the difference portfolio $C = A - B$, where $-B$ means we go short B (This assumes that we can go short and hold negative amounts of an asset at will). Fix $T \geq t$ and suppose $\mathbb{P}(V^A(T) \geq V^B(T)) = 1$. Since $V^C = V^A - V^B$, this implies $\mathbb{P}(V^C(T) \geq 0) = 1$. If $V^A(t) - V^A(B) = V^C(t) < 0$,

$$\mathbb{P}\left(V^C(T) > V^C(t)\right) \geq \mathbb{P}\left(V^C(T) \geq 0\right) = 1. \tag{10}$$

This shows C is a arbitrage portfolio, which contradicts the no-arbitrage principle. Hence we must have $V^A(t) - V^A(B) = V^C(t) \geq 0$, so $V^A(t) \geq V^B(t)$, as desired. This shows (i). A similar argument also shows (ii). □

Corollary 3.4 (Principle of replication)**.** *Consider two portfolios A and B at current time t. Under the assumption of no-arbitrage, we have the following implication: For any $T \geq t$,*

$$\mathbb{P}(V^A(T) = V^B(T)) = 1 \quad \Longrightarrow \quad V^A(t) = V^B(t). \tag{11}$$

PROOF. By the monotonicity theorem above, we have $V^A(t) \geq V^B(t)$ and $V^A(t) \leq V^B(t)$, and hence $V^A(t) = V^B(t)$. □

Example 3.5. The converse of the monotonicity theorem is not necessarily true. For example, consider two portfolios at time 0: (A) depositing 1 at constant rate r; (B) investing 1 at a stock in the spot market. The value of portfolio A at time t is e^{rt}, whereas the value of portfolio B at time t depends on the stock market. No-arbitrage principle implies that one cannot know for sure that whether the stock value will likely to go up or down in the future. However, the market could evolve in any possible way to yield different values for different portfolios. After all, there are more successful investors and under-performing funds. ▲

4. Interest rates

If we deposit N at interest rate r per annum, compounded annually, then after T years we get the amount $N(1+r)^T$. Here N is called the *notional* or *principal*. Suppose $N = 1$ for simplicity. If we invest 1 at rate r compounded semi-annually, then we have $(1+r/2)$ after six months and $(1+r/2)^{2T}$ after T years. In general, if we invest 1 at rate r compounded m times per annum, after T years we get $(1+r/m)^{mT}$. What happens if we let $m \to \infty$? That is, if the interested rate r is compounded continuously in time, what is the total amount after T years? The answer is given by the following simple fact

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{mT} = e^{rT}. \tag{12}$$

In words, a unit amount compounded continuously at rate r becomes e^{rT} after T years.

Exercise 4.1. If $c_m \to 0$, $a_m \to \infty$, and $c_m a_m \to \lambda$ as $m \to \infty$, then show that

$$\lim_{m\to\infty}(1+c_m)^{a_m} = e^{\lambda}. \tag{13}$$

(Hint: Take log and use L'Hospitals.)

Exercise 4.2 (Continuous to discrete compounding)**.** Suppose the continuously compounded rate for period T is r. Let r_m be the equivalent rate with compounding frequency m.

(i) Argue that r_m has to satisfy

$$e^{rT} = \left(1+\frac{r_m}{m}\right)^{mT}. \tag{14}$$

(ii) Show that r_m is given by

$$r_m = m\left(e^{r/m}-1\right). \tag{15}$$

Now we consider a converse question. If we have \$1 at time T and the continuously compounded interest rate r is constant, then how much does it worth at a past time $t \le T$? If it worth x at time t, then we can simply solve

$$xe^{r(T-t)} = 1, \tag{16}$$

so we find $x = e^{-r(T-t)}$. In general, this is the value of any asset that pays 1 at time T.

Definition 4.3. A *zero coupon bound* (ZCB) with maturity T is an asset that pays 1 at time T (and nothing else).

Let $Z(t,T)$ denote the price at time $t \le T$ of a ZCB with maturity T. By definition, $Z(T,T) = 1$.

Proposition 4.4. *Suppose the continuously compounded interest rate during $[t,T]$ is a constant r. Then*

$$Z(t,T) = e^{-r(T-t)}. \tag{17}$$

PROOF BY REPLICATION. Consider the following two portfolios at time $t \le T$:

Portfolio A: One ZCB with maturity T.

Portfolio B: $e^{-r(T-t)}$ of cash deposited at rate r.

Both portfolios worth 1 at time T, so they must worth the same at a prior time t. □

Exercise 4.5. Give a proof of Proposition 4.4 using no arbitrage principle.

Exercise 4.6. Let $Z(t,T)$ denote the price at time $t \le T$ of a ZCB with maturity T. Suppose the annually compounded rate during $[t,T]$ is a constant r_A. Show that

$$Z(t,T) = (1+r_A)^{-(T-t)}. \tag{18}$$

Exercise 4.7 (Time-dependent interest rate)**.** Suppose the continuously compounded interest rate during $[t,T]$ is a time-dependent variable $r(s)$, $t \le s \le T$.

(i) For each $s \in [s, T]$, let $V(s)$ denote the value of \$1 deposited at time $t \leq T$. Show that

$$V(T) = \exp\left(\int_t^T r(u)\,du\right). \tag{19}$$

(ii) If $r(t) \equiv r$ a constant, then deduce that

$$V(T) = e^{r(T-t)}. \tag{20}$$

(iii) Let $Z(t,T)$ denote the price at time $t \leq T$ of a ZCB with maturity T. Show that

$$Z(t,T) = \exp\left(-\int_t^T r(u)\,du\right). \tag{21}$$

Exercise 4.8 (Annuities)**.** An *annuity* is a series of fixed cashflows C at specified times $T_1 < T_2 < \cdots < T_n$. (e.g., rent, lease, and other monthly payments)

(i) Let V denote the value of an annuity at current time $t \leq T_1$. Show that

$$V = C\sum_{i=1}^{n} Z(t, T_i), \tag{22}$$

where $Z(t, T_i)$ denote the price at time $t \leq T_1$ of a ZCB with maturity T_i, for each $1 \leq i \leq n$.

(ii) Suppose an annuity pays $C = 1$ for M years and that the annually compounded zero rates are r during that period. Then show that

$$V = \sum_{i=1}^{M} (1+r)^{-i} = \frac{1}{r}\left(1-(1+r)^{-M}\right). \tag{23}$$

5. Determination of Forward prices

5.1. Forward value and forward price. Consider a forward contract on an asset with price $(S_t)_{t\geq 0}$, delivery price K, and maturity T. If an investor wants to take a long position in this forward contract, what would be the right delivery price that he/she should agree? If one already has a forward contract with maturity T and delivery price K, what is the actual value of this contract at time $t \leq T$ prior to the expiration? In this section, we study these questions using two strategies: 1) *Replication* and 2) *No arbitrage principle*.

To begin, suppose an investor A wants to go long forward contract one share of a stock at time t, maturity T, delivery price $K = \$50$. This contract will only be established if there is another investor B willing to take a short position and sell the stock at delivery price \$50 at maturity T. However, if B believes that the delivery price \$50 is \$10 less than what it should be, then taking a short position in this forward contract would cost B a loss of \$10. In this case either A and B agree to the 'fair' delivery price \$60, or A gives extra \$10 to B in order to establish the forward contract. In this case, the 'value' of this forward contract at time t is \$10, and the 'forward price' is \$60.

Definition 5.1. Suppose a forward contract with maturity T and delivery price K is given. For each time $t \leq T$, let $V_K(t,T)$ denote the *value* $V_K(t,T)$ of the foward contract (in long position). The *forward price*, denoted by $F(t,T)$, is the special value of K such that $V_K(t,T) = 0$.

Since $V_{F(t,T)}(t,T) = 0$, one can always enter a forward contract in either position at time t with delivery price $F(t,T)$: Somewhere in the market there is someone willing to take the opposite position with no upfront fee. Somewhat surprisingly, the forward price $F(t,T)$ depends only on the current stock price S_t, interest rate r, and duration of the contract $T-t$ (not on the actual stock price S_T at maturity).

Proposition 5.2 (Forward price)**.** *For an asset paying no income with price $(S_t)_{t\geq 0}$ (e.g., stock without dividends), we have*

$$F(t,T) = S_t e^{r(T-t)}. \tag{24}$$

PROOF BY REPLICATION. Consider the following two portfolios at time t:

Portfolio A: One unit of stock,

Portfolio B: Long one forward contract with delivery price K and maturity T, plus $Ke^{-r(T-t)}$ of cash deposit.

The value of portfolio A at time T is S_T. On the other hand, recall that $V_K(T,T) = S_T - K$, so the value of portfolio B at time T is $(S_T - K) + K = S_T$. Hence the two portfolios have the same value S_T at time T. So they must have the same value also at prior time $t \leq T$. This gives

$$S_t = V_K(t,T) + Ke^{r(T-t)} \tag{25}$$

By definition of the forward price $F(t,T)$, letting $K = F(t,T)$ gives

$$S_t = V_{F(t,T)}(t,T) + F(t,T)e^{-r(T-t)} = F(t,T)e^{-r(T-t)}. \tag{26}$$

This shows the assertion. □

Proof by no arbitrage principle is similar to proof by contradiction in mathematics. Assume the assertion is not true, and construct an arbitrage portfolio that produces sure profit; This will contradict the no arbitrage principle.

PROOF BY NO ARBITRAGE PRINCIPLE. Suppose $F(t,T) > S_t e^{r(T-t)}$, which means the 'balanced delivery price' $F(t,T)$ at time t is more than the true price. In this case we perform the following transactions:

1. Borrow S_t cash from bank at interest rate r at time t until T.
2. With the cash S_t, buy the stock at its current market price at time t.
3. Go short one forward contract at time t (i.e., 'sell the stock forward') at its forward price $F(t,T)$ (with no cost).

Now at maturity T, we must sell the stock for price $F(t,T)$ under the terms of the forward contract (being in the short position). We need to pay the loan amount $S_t e^{r(T-t)}$. But then we get a positive profit

$$F(t,T) - S_t e^{r(T-t)} > 0, \tag{27}$$

which contradicts no arbitrage principle.

On the other hand, suppose $F(t,T) < S_t e^{r(T-t)}$. Then we take the following steps:

1. Go long one forward contract at time t (i.e., 'buy the stock forward') at its forward price $F(t,T)$ (with no cost).
2. Sell the stock for the current market price S_t at time t.
3. Deposit S_t cash at interest rate r at time t until T.

At maturity T, we get cash $S_t e^{r(T-t)}$. According to the forward contract, we have to buy back the stock for the delivery price $F(t,T)$ (being in the long position). But then we have a positive profit

$$S_t e^{r(T-t)} - F(t,T) > 0, \tag{28}$$

which contradicts no arbitrage principle. □

Exercise 5.3 (Forward on asset paying known income)**.** Suppose an asset with price $(S_t)_{\geq t}$ pays income during the lift time of the forward contract, which is of value $I > 0$ at present time t (e.g., dividends, coupons, rent). We will show that

$$F(t,T) = (S_t - I)e^{r(T-t)}. \tag{29}$$

(i) Give a replication proof of the above statement.
(ii) Give a no-arbitrage proof of the above statement.

Proposition 5.4 (Value of forward contract)**.** *Consider a forward contract with maturity T and delivery price K. Let $V_K(t,T)$ and $F(t,T)$ denote its value and forward price at time $t \leq T$. Under constant continuously compounded rate r,*

$$V_K(t,T) = (F(t,T) - K)e^{-r(T-t)}. \tag{30}$$

More generally, if $Z(t,T)$ denotes the value at time t of a ZCB maturing at T, then

$$V_K(t,T) = (F(t,T) - K)Z(t,T). \tag{31}$$

PROOF. We only show the first assertion, as the second can be shown by a similar argument. Let $(S_t)_{\geq t}$ denote the price of the stock. Suppose $V_K(t,T) > (F(t,T) - K)e^{-r(T-t)}$. First assume $F(t,T) \geq K$. Consider taking two forward contracts at time t: Long one forward contract with delivery price $F(t,T)$ (at no cost), and short one forward contract with delivery price K. Then $V_K(t,T) > 0$, so when going short one forward contract, one gets $V_K(t,T)$ in cash. Deposit this into bank at time t. The value of this portfolio at time T is

$$(S_T - F(t,T)) + (K - S_T) + V_K(t,T)e^{r(T-t)} = -(F(t,T) - K) + V_K(t,T)e^{r(T-t)} > 0. \tag{32}$$

Second, assume $F(t,T) < K$. We then take two forward contracts at time t: Short one forward contract with delivery price $F(t,T)$ (at no cost), and long one forward contract with delivery price K. Then $V_K(t,T) < 0$, so when going long one forward contract, one gets $-V_K(t,T) > 0$ in cash. Deposite this at bank at time t. The value of this portfolio at time T is

$$(F(t,T) - S_T) + (S_T - K) - V_K(t,T)e^{r(T-t)} = (F(t,T) - K) - V_K(t,T)e^{r(T-t)} > 0. \tag{33}$$

So in both cases, we get arbitrage opportunity. Argue the other case $V_K(t,T) < (F(t,T) - K)e^{r(T-t)}$ similarly. □

Exercise 5.5. Construct an arbitrage portfolio assuming $V_K(t,T) < (F(t,T) - K)e^{-r(T-t)}$ in the proof of Proposition 5.4.

Exercise 5.6 (Forward on stock paying dividends)**.** Suppose a stock with price $(S_t)_{t\geq 0}$ pays dividends at a known dividend yield q, expressed as a percentage of the stock price on a continually compounded per annum basis. We will show that the forward price of this stock at time t with maturity T is given by

$$F(t,T) = S_t e^{(r-q)(T-t)}. \tag{34}$$

(i) Consider a portfolio A at time t consisting of $e^{-q(T-t)}$ units of stock, with dividends all reinvested in the stock. Show that at the value of this portfolio at time T equals S_T.
(ii) Consider another portfolio B at time t consisting of one long forward contract with delivery price K plus $Ke^{-r(T-t)}$ cash. Show that the value of this portfolio at time T equals S_T.
(iii) Conclude that

$$S_t e^{-q(T-t)} = V_K(t,T) + Ke^{-r(T-t)}. \tag{35}$$

Let $K = F(t,T)$ and deduce (34).

Exercise 5.7 (Forward on currency)**.** Suppose X_t is the price at time t in dollars of one unit of other currency. (e.g., at the time of writing, £1 = \$1.22). Let $r_\$$ and r_f be the zero rate for dollar and the foreign currency, both constant and continuously compounded.

(i) Use Exercise 5.6 to deduce

$$F(t,T) = X_t e^{(r_\$ - r_f)(T-t)}. \tag{36}$$

(ii) Show the conclusion of (i) directly using replication or no-arbitrage argument.

5.2. Forward rates and LIBOR. In this subsection, we consider forward contract on zero coupon bonds. This will naturally lead to the concept of forward rates.

Fix $T_1 \leq T_2$, and consider a forward contract with maturity T_1 on a ZCB with maturity T_2. At the time of contract $t \leq T_1$, the underlying ZCB with maturity T_2 has value $Z(t,T_2)$. Let $F(t,T_1,T_2)$ denote the forward price of this forward contract, which makes the value of the contract zero at time t. In other words, with this as the delivery price, one can establish this forward contract at long position with no upfront cost to the counter party.

Proposition 5.8 (Forward on zero coupon bonds)**.** *Fix $t \le T_1 \le T_2$, and let $F(t, T_1, T_2)$ denote the forward price of the forward contract with maturity T_1 on a ZCB with maturity T_2. Then*

$$F(t, T_1, T_2) = \frac{Z(t, T_2)}{Z(t, T_1)}. \tag{37}$$

PROOF. We give a replication argument. Consider the following two portfolios at time $t \le T_1$:

Portfolio A: One ZCB with maturity T_2

Portfolio B: [One long forward contract on one ZCB maturing at T_2 with delivery price K] + [K ZCBs with maturity T_1].

The value of portfolio A is 1 at time T_2 by definition. For portfolio B, at time T_1, one has to buy one ZCB with maturity T_2 at delivery price K, according to the forward contract. One can do this by selling the K ZCBs with maturity T_1 for K. Now that we have one ZCB with maturity T_2 at time T_1, at time T_2, the value of portfolio B is also 1. Hence the two portfolios have the same value at time T_2, so they must also have the same value at time t. This gives

$$Z(t, T_2) = V_K(t, T_1) + KZ(t, T_1). \tag{38}$$

Now letting $K = F(t, T_1, T_2)$, we obtain

$$Z(t, T_2) = F(t, T_1, T_2)Z(t, T_1). \tag{39}$$

This shows the claim. (Can one give a no-arbitrage argument?) □

Exercise 5.9 (Forward interest rates)**.** A *forward rate* at current time t for period $[T_1, T_2]$, $t \le T_1$, is the rate agreed at t at which one can lend money during $[T_1, T_2]$. Denote this forward rate by f_{12}. Suppose that the current zero rates (continuously compounded) during $[t, T_1]$ and $[t, T_2]$ are r_1 and r_2, respectively.

(i) Consider two portfolios: (A) Deposit x during $[t, T_2]$ at rate r_2; (B) Deposit y during $[t, T_1]$ at rate r_1 and go short one forward contract to lend $ye^{r_1(T_1-t)}$ at rate f_{12} during $[T_1, T_2]$. Show that at time T_2, their values are $xe^{r_2(T_2-t)}$ and $ye^{r_1(T_1-t)}e^{f_{12}(T_2-T_1)}$, respectively.

(ii) For the choice

$$x = e^{r_2(T_2-t)}, \qquad y = e^{-r_1(T_1-t)-f_{12}(T_2-T_1)}, \tag{40}$$

show that the two portfolios in (i) have the same value 1 at time T_2. Conclude that $x = y$, and deduce

$$f_{12} = \frac{r_2(T_2 - t) - r_1(T_1 - t)}{T_2 - T_1}. \tag{41}$$

(iii) Show the conclusion of (ii) using a no-arbitrage argument. (Hint: Compare the two portfolios in (i) with $x = y = 1$. If one always has higher value than the other at time T_2, one can create an arbitrage opportunity.)

Next, we consider derivatives built on top of interest rates. In order to do so, we first introduce the concept of 'LIBOR', which is the rate at which banks borrow or lend to each other[2] On the current day t, LIBOR rates for periods $\alpha = 1$ (twelve-month LIBOR, often abbreviated as 12mL), $\alpha = 0.5$ (six-month LIBOR, 6mL), $\alpha = 0.25$ (three-month LIBOR, 3mL) etc are published. For example, say a bank A lends (resp., deposits) N from (resp., into) another bank at time t for a short period $[t, t+\alpha]$ for some investment purpose. At maturity $t+\alpha$, bank A needs to pay back (resp., receive) $N(1+r)$ for some $r > 0$. All interest is paied at the maturity (resp., term of the deposit), and there is no interim compounding. The *LIBOR rate* (or LIBOR fix) $L_t[t, t+\alpha]$ is defined so that $r = \alpha L_t[t, t+\alpha]$.

[2]LIBOR is an acronym for the London InterBank Offered Rate.

Definition 5.10. A *forward rate agreement* (FRA) is a forward contract to exchange two cashflows. Namely, the buyer of the FRA with maturity T, duration α, and delivery price (or fixed rate) K agrees at current time $t \le T$ to do the following:

$$\text{At time } T+\alpha, \text{ pay } \alpha K \text{ and recieve } \alpha L_T[T, T+\alpha]. \tag{42}$$

Let $V_K(t, T, T+\alpha)$ denote the value of this FRA at time t. The *forward LIBOR rate*, denoted by $L_t[T, T+\alpha]$, is the value of K such that $V_K(t, T, T+\alpha)$ equals 0 at time $t \le T$.

Unlike the LIBOR fix $L_t(t, t+\alpha)$, the LIBOR rate $L_t(T, T+\alpha)$ is a random variable since it depends on the market at future time T.

Proposition 5.11. *Consider a FRA with maturity T, duration α, and fixed rate K.*

(i) *The forward LIBOR rate is given by*

$$L_t[T, T+\alpha] = \frac{Z(t,T) - Z(t, T+\alpha)}{\alpha Z(t, T+\alpha)}. \tag{43}$$

(ii) *Let $V^{FL}(t)$ denote the value of the LIBOR payment $\alpha L_t(T, T+\alpha)$ at time t. Then*

$$V^{FL}(t) = Z(t,T) - Z(t, T+\alpha). \tag{44}$$

PROOF. Consider the following two portfolios at time t:

Portfolio A: [One long FRA with maturity T, duration α, and delivery price K] + [$(1+\alpha K)$ ZCBs maturing at time $T+\alpha$]

Portfolio B: One ZCB maturing at time T.

According to (42) for FRA, the value of the portfolio A at time $T+\alpha$ equals

$$\alpha(L_T[T, T+\alpha] - K) + (1+\alpha K) = \alpha L_T[T, T+\alpha] + 1. \tag{45}$$

On the other hand, for portfolio B, when we get 1 for the payoff of ZCB at its maturity T, we can deposit it during $[T, T+\alpha]$ with interest rate $L_T[T, T+\alpha]$ at no cost, by definition of $L_T[T, T+\alpha]$. Hence the value of portfolio B at time $T+\alpha$ equals $1+\alpha L_T[T, T+\alpha]$, which is the same as that of portfolio A at time $T+\alpha$. Hence they have the same value at time t. This gives

$$V_K(t, T, T+\alpha) + (1+\alpha K) Z(t, T+\alpha) = Z(t,T). \tag{46}$$

Now plugging in $K = L_t(T, T+\alpha)$ gives

$$(1+\alpha L_t[T, T+\alpha]) Z(t, T+\alpha) = Z(t,T). \tag{47}$$

Solving this for $L_t[T, T+\alpha]$ then gives (i).

For the second assertion, note that the value at t of the fixed payment αK made at time $T+\alpha$ is $\alpha K Z(t, T+\alpha)$. Hence from (46) we have

$$V^{FL}(t) = V_K(t, T, T+\alpha) + \alpha K Z(t, T+\alpha) = Z(t,T) - Z(t, T+\alpha). \tag{48}$$

This shows (ii). □

6. Interest rate swaps

A *swap* is an agreement between two counterparties to exchange a series of cashflow at agreed dates. Cashflows are calculated on a notional amount, which we may assume to be 1. A swap has a start date T_0, maturity T_n, and payment dates $T_1, T_2, \cdots, T_n$. In a standard or *vanila* swap, we assume that the gap between consecutive payment dates are a constant α.

One counterparty (the *payer*) pays a fixed amount αK at each payment date $T_i + \alpha$ accumulated during $[T_i, T_i+\alpha]$, which is called the *fixed leg* of the swap. Here K is called the *fixed rate*. On the other hand, the other counterparty (the *reviever*) receives variable amount $\alpha L_{T_i}[T_i, T_i+\alpha]$ at time $T_i+\alpha$ (i.e., LIBOR fixing at T_i for the priod $[T_i, T_i+\alpha]$ payed at $T_i+\alpha$).

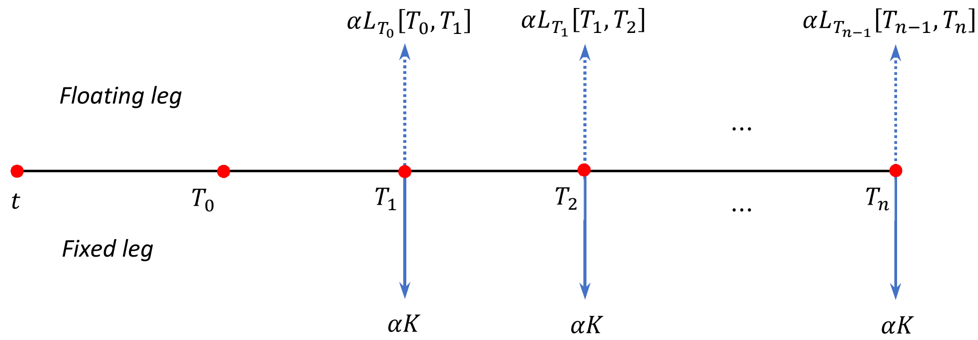

FIGURE 2. Illustration of swap. On pay dates $T_1, \cdots, T_n$, payer pays regular amount (leg) αK to, and receives floating leg $\alpha L_{T_i}[T_i, T_{i+1}]$ from, the receiver.

Example 6.1 (Swap between IBM and the World Bank in 1981, [Hul03])**.** The birth of the over-the-counter swap market can be traced to a currency swap negotiated between IBM and the World Bank in 1981. The World Bank had borrowings denominated in U.S. dollars while IBM had borrowings denominated in German deutsche marks and Swiss francs. The World Bank (which was restricted in the deutsche mark and Swiss franc) agreed to make interest payments on IBM's borrowings while IBM in return agreed to make interest payments on theWorld Bank's borrowings. Since that first transaction in 1981, the swap market has seen phenomenal growth. ▲

Next, we compute the value of the fixed leg and the floating leg in a swap. Denote

$$P_t[T_0, T_n] = \sum_{i=1}^{n} \alpha Z(t, T_i), \tag{49}$$

which is called the *pv01* of the swap, the present value of receiving 1 times α at each payment date.

Proposition 6.2. *Consider a vanila swap with start date T_0, pay dates $T_1, \cdots, T_0$, and fixed rate K. Let $V_K^{FXD}(t)$ and $V^{FL}(t)$ denote the values of its fixed leg and floating leg, respectively. Then*

$$V_K^{FXD}(t) = KP_t[T_0, T_n] \tag{50}$$

$$V^{FL}(t) = Z(t, T_0) - Z(t, T_n). \tag{51}$$

PROOF. Note that the fixed leg is equivalent to an annuity paying K times the accrual factor α at each payment date. So we have

$$V_K^{FXD}(t) = K \sum_{i=1}^{n} \alpha Z(t, T_i) = KP_t[T_0, T_n]. \tag{52}$$

On the other hand, floating leg is a sequence of regular LIBOR payments, and measure the value of each LIBOR payments at time t by that of its forward contract. Namely, using Proposition 5.11,

$$V^{FL}(t) = \sum_{i=1}^{n} \alpha L_t[T_{i-1}, T_i] Z(t, T_i) \tag{53}$$

$$= \sum_{i=1}^{n} [Z(t, T_{i-1}) - Z(t, T_i)] = Z(t, T_0) - Z(t, T_n). \tag{54}$$

This shows the assertion. □

Exercise 6.3. Consider a vanila swap with start date T_0, pay dates $T_1, \cdots, T_n$, and fixed rate K.

(i) Consider two portfolios at time t: (A) [Receiving regular LIBOR payments during $[T_1, T_n]$] + [one ZCB maturing at T_n], and (B) [One ZCB maturing at T_0]. Show that one can replicate (A) by (B) by repeatedly reinvesting 1 at LIBOR deposit during each period $[T_{i-1}, T_i]$, $i = 1, \cdots, n$.

(ii) Conclude that $V^{FL}(t) = Z(t,T_0) - Z(t,T_n)$.

Analogously as for the forward price, the *forward swap rate* at time t for a swap from T_0 to T_n is defined to be the special value $y_t[T_0,T_n]$ of the fixed rate K for which the value of the swap at t is zero.

Corollary 6.4. *Consider a vanila swap with start date T_0, pay dates $T_1,\cdots,T_0$, and fixed rate K. Then the forward swap rate $y_t[T_0,T_n]$ is given by*

$$y_t[T_0,T_n] = \frac{Z(t,T_0) - Z(t,T_n)}{P_t[T_0,T_n]}. \tag{55}$$

PROOF. According to Proposition 6.2, by letting $K = y_t[T_0,T_n]$, we have

$$y_t[T_0,T_n]P_t[T_0,T_n] = V^{FXD}_{y_t[T_0,T_n]}(t) = V^{FL}(t) = Z(t,T_0) - Z(t,T_n). \tag{56}$$

This shows the assertion. □

Corollary 6.5. *Consider a vanila swap with start date T_0, pay dates $T_1,\cdots,T_0$, and fixed rate K. Then its value $V^{SW}_K(t)$ at current time $t \le T_0$ is given by*

$$V^{SW}_K(t) = (y_t[T_0,T_n] - K)P_t[T_0,T_n]. \tag{57}$$

PROOF. Using Proposition 6.2 and Corollary, we have

$$\begin{aligned} V^{SW}_K(t) &= V^{FL}(t) - V^{FXD}_K(t) && (58)\\ &= [Z(t,T_0) - Z(t,T_n)] - KP_t[T_0,T_n] && (59)\\ &= y_t[T_0,T_n]P_t[T_0,T_n] - KP_t[T_0,T_n] && (60)\\ &= (y_t[T_0,T_n] - K)P_t[T_0,T_n]. && (61) \end{aligned}$$

This shows the assertion. □

Compare the conclusion of the above corollary with the value of a forward contract

$$V_K(t,T) = (F(t,T) - K)e^{-r(T-t)}. \tag{62}$$

given in Proposition 5.4.

Example 6.6 (A numerical example)**.** OIS rates are the risk-free rates used by traders to value derivatives. An *overnight indexed swap* (OIS) involves exchanging a fixed OIS rate for a floating rate. The floating rate is calculated by assuming that someone invests at the (very low risk) overnight rate, reinvesting the proceeds each day. Suppose that at time $t = 0$, certain fixed OIS rates can be exchanged for floating rates in the market and gives the following OIS zero rates:

$$\text{6 months: 3.7\%,}\quad \text{12 months: 4.2\%,}\quad \text{18 months: 4.4\%,}\quad \text{24 months: 4.9\%.} \tag{63}$$

In other words, this means that the values of the zero coupon bonds with maturity 6, 12, 18, and 24 months (0.5, 1, 1.5, and 2 years) are given by

$$Z(0,0.5) = e^{-0.037\times 0.5},\quad Z(0,1) = e^{-0.042\times 1},\quad Z(0,1.5) = e^{-0.044\times 1.5},\quad Z(0,2) = e^{-0.049\times 2}. \tag{64}$$

Denote $T_0 = 0$, $T_1 = 0.5$, $T_2 = 1$, $T_3 = 1.5$, and $T_4 = 2$.

(i) *Forward LIBOR rates.* According to Proposition 5.11, we can compute the forward LIBOR rates $L_0[T_i,T_{i+1}]$ as below:

$$L_0[0,0.5] = \frac{Z(0,0) - Z(0,0.5)}{0.5\times Z(0,0.5)} = \frac{1 - e^{-0.037\times 0.5}}{0.5\times e^{-0.037\times 0.5}} = 3.73\% \tag{65}$$

$$L_0[0.5,1] = \frac{Z(0,0.5) - Z(0,1)}{0.5\times Z(0,1)} = \frac{e^{-0.037\times 0.5} - e^{-0.042\times 1}}{0.5\times e^{-0.042\times 1}} = 4.75\% \tag{66}$$

$$L_0[1,1.5] = \frac{Z(0,1) - Z(0,1.5)}{0.5\times Z(0,1.5)} = \frac{e^{-0.042\times 1} - e^{-0.044\times 1.5}}{0.5\times e^{-0.044\times 1.5}} = 4.85\% \tag{67}$$

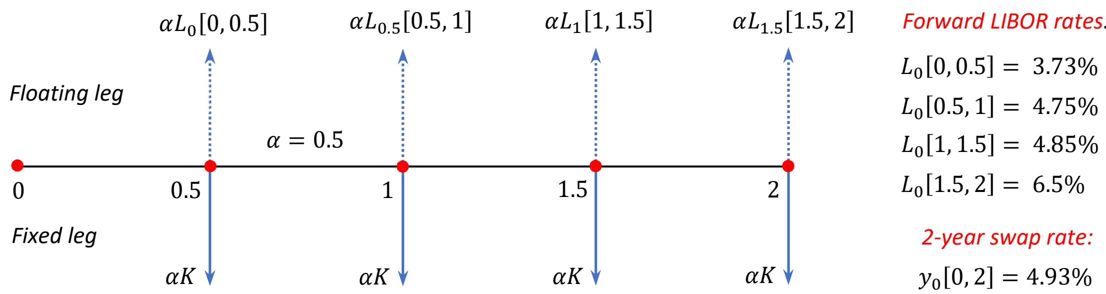

FIGURE 3. Illustration of swap example.

$$L_0[1.5.2] = \frac{Z(0,1.5) - Z(0,2)}{0.5 \times Z(0,2)} = \frac{e^{-0.044\times 1.5} - e^{-0.049\times 2}}{0.5 \times e^{-0.049\times 2}} = 6.50\% \tag{68}$$

(ii) *Forward swap rates.* Consider a two-year swap with start date $T_0 = 0$ and semi-annual payment dates $T_1 = 0.5$, $T_2 = 1$, $T_3 = 1.5$, $T_4 = 2$ with $\alpha = 0.5$ (6 months) with fixed rate K. According to Corollary 6.4, we can compute the forward two-years swap rate $y_0[0,2]$ as below:

$$y_0[0,2] = \frac{Z(0,0) - Z(0,2)}{P_0[0,2]} = \frac{1 - Z(0,2)}{0.5(Z(0,0.5) + Z(0,1) + Z(0,1.5) + Z(0,2))} \tag{69}$$

$$= \frac{1 - e^{-0.049\times 2}}{0.5(e^{-0.037\times 0.5} + e^{-0.042\times 1} + e^{-0.044\times 1.5} + e^{-0.049\times 2})} = 4.93\%. \tag{70}$$

(iii) *Value of the swap.* The value of the fixed leg $V_K^{FXD}(0)$ at current time 0 is

$$V_K^{FXD}(0) = 0.5Ke^{-0.037\times 0.5} + 0.5Ke^{-0.042\times 1} + 0.5Ke^{-0.044\times 1.5} + 0.5Ke^{-0.049\times 2} \tag{71}$$

$$= K \times 1.8916. \tag{72}$$

According to Proposition 6.2, the value $V^{FL}(0)$ of the floating leg at current time $t = 0$ is

$$V^{FL}(0) = Z(0,0) - Z(0,2) = 1 - e^{-0.049\times 2} = 0.0933 \tag{73}$$

Hence the value $V_K^{SW}(0)$ of the two-year swap at the current time $t = 0$ is

$$V_K^{SW}(0) = 0.0933 - K \times 1.8916 = (0.0493 - K)1.8916. \tag{74}$$

▲

Exercise 6.7. Rework Example 6.6 with following OIS zero rates:

$$\text{6 months: } 3.5\%, \quad \text{12 months: } 3.7\%, \quad \text{18 months: } 4.15\%, \quad \text{24 months: } 4.3\%. \tag{75}$$

Example 6.8 (Swaps as difference between bonds)**.** A *fixed rate bond* with notional N and coupon K pays αKN at fixed dates $T_1, \cdots, T_n$, and N at T_n, where we have $T_{i+1} = T_i + \alpha$. A *floating rate bond* with notional N pays LIBOR coupons $\alpha N L_{T_{i-1}}[T_{i-1}, T_i]$ at T_i for $i = 1, \cdots, n$ and N at T_n. Let $N = 1$, and denote by $B_K^{FXD}(t)$ and $B^{FL}(t)$ the price of the fixed and floating rate bonds, respectively.

Consider a swap with notional $N = 1$ where we pay a fixed rate K and receive LIBOR instead. Let $V_K^{SW}(t)$ denote the value of this swap at the current time t. Note that these are the values of the fixed and floating leg plus the value $Z(t,T_n)$ of the ZCB. Hence

$$V_K^{SW}(t) = B^{FL}(t) - B_K^{FXD}(t) = (V^{FL}(t) + Z(t,T_n)) - (V_K^{FL}(t) + Z(t,T_n)) = V^{FL}(t) - V_K^{FL}(t). \tag{76}$$

Hence the value of this swap is the same as that of the interest rate swap in Corollary 6.5.

Furthermore, according to Proposition 6.2, note that

$$B^{FL}(t) = V^{FL}(t) + Z(t,T_n) = Z(t,T_0). \tag{77}$$

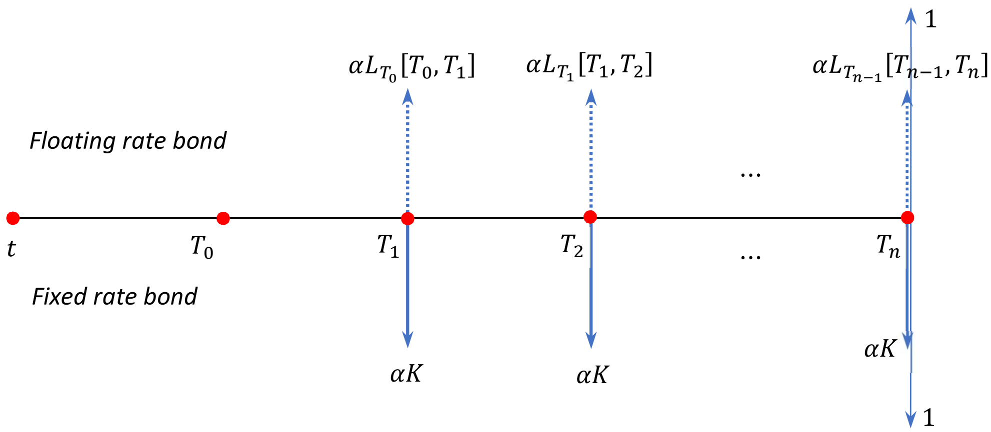

FIGURE 4. Exchanging fixed rate and floating rate bonds.

Hence the value of the floating rate bond at time t equals that of a ZCB with maturity T_0. In particular, if $t = T_0$, then $B^{FL}(T_0) = 1$, regardless of the actual interest rates. In other words, a floating rate bond has no interest rate exposure. In contrary, the fixed rate bond has known cashflows but its value changes. ▲

7. Futures contracts

A futures contract is a derivative that is essentially a contract to trade an underlying asset at a fixed time in the future, just like the forward contract. Similarly to a forward contract, a *futures contract* (or future) has a specified maturity T, an underlying asset (whose price at time t is denoted by S_t), and a futures price $\Phi(t, T)$ at which one can go long or short the contract at no cost at time t. The futures price at maturity T is defined to be $\Phi(T, T) = S_T$.

The key distinction between the futures and forwards lies in the cashflows during the lifetime of a contract. In particular, the holder of a futures contract receives (or pays) changes in the futures price over the life of the contract, and not just at maturity.

***Forward*:** At time t, we can go long one forward contract with maturity T with delivery price $F(t, T)$ at no cost. There is no cashflow up to time T, and at maturity, we receive (pay if negative) $S_T - F(t, T) = F(T, T) - F(t, T)$.

***Futures*:** At time t, we can go long a futures contract with price $\Phi(t, T)$ at no cost. Let $t = t_0 < t_1 < \cdots < t_n = T$, where t_i is the ith day from the contract made at time t. Each day up to the maturity, we receive (pay if negative) the *mark-to-market* change (or *variation margin*) $\Phi(t_i, T) - \Phi(t_{i-1}, T)$.

Note that the total amount of the mark-to-market we receive over the lifetime of a futures contract equals

$$\sum_{i=1}^{n} [\Phi(t_i, T) - \Phi(t_{i-1}, T)] = \Phi(T, T) - \Phi(t, T) = S_T - \Phi(t, T). \tag{78}$$

However, since the constituent payment is made each day, the overall value of the payment at maturity T may not equal to $S_T - \Phi(t, T)$.

Remark 7.1. Whereas forward contracts are made over-the-counter, Futures contracts are traded on electronic exchanges such as CME (formerly the Chicago Mercantile Exchange), CBOT (Chicago Board of Trade), NYMEX (New York Mercantile Exchange) and LIFFE (London International Financial Futures and Options Exchange, pronounced ‘life’). Each futures market participant deposits initial margin at the exchange, and receives (or posts) additional variation margin as prices move up (or down). Initial margin is usually established to be a size that can cover 99% of five-day moves. Note that margin—and in particular the variation margin—are exposed to interest.

Proposition 7.2. *Consider a forward and futures contract with maturity T on an asset with price $(S_t)_{t\geq 0}$. Assume constant continuously compounded interest rate r. Then for any $t \leq T$,*

$$\Phi(t,T) = F(t,T). \tag{79}$$

In words, the futures and forward prices are the same.

PROOF. Denote $t = t_0 < t_1 < \cdots < t_n = T$, where t_i is the ith day from the contract made at time t. Denote $\Delta = t_i - t_{i-1}$ (e.g., $\Delta = 1/365$). We first consider the following futures trading strategy.

(1) At time $t = t_0$, go long $e^{-r(n-1)\Delta}$ futures contract with futures price $\Phi(t_0,T)$ (with no cost).

(2) At time t_1, we receive variation margin of

$$[\Phi(t_1,T) - \Phi(t_0,T)]e^{-r(n-1)\Delta}. \tag{80}$$

Invest this amount at rate r. (If negative, borrow this amount from bank at rate r to pay.) Also increase the position to $e^{-r(n-2)\Delta}$ futures contracts at futures price $\Phi(t_1,T)$.

(3) In general, at time t_i, we receive variation margin of

$$[\Phi(t_i,T) - \Phi(t_{i-1},T)]e^{-r(n-i)\Delta}. \tag{81}$$

Invest this amount at rate r. (If negative, borrow this amount from bank at rate r to pay.) Also increase the position to $e^{-r(n-i-1)\Delta}$ futures contracts at futures price $\Phi(t_i,T)$.

At time $T = t_n$, the total value of this strategy is

$$\sum_{i=1}^{n} [\Phi(t_i,T) - \Phi(t_{i-1},T)]e^{-r(n-i)\Delta}e^{r(n-i)\Delta} = \Phi(T,T) - \Phi(t,T) = S_T - \Phi(t,T). \tag{82}$$

Now we compare the following two portfolios at time t:

Portfolio A: $[e^{-rn\Delta}\Phi(t,T)$ of cash$]+[e^{-r(n-1)\Delta}$ futures contract maturing at T with futures price $\Phi(t,T)]$

Portfolio B: $[e^{-rn\Delta}F(t,T)$ of cash$]+[$One long forward contract maturing at T with delivery price $F(t,T)]$

According to the previous discussion, the value of portfolio A at time T is

$$\Phi(t,T) + (S_T - \Phi(t,T)) = S_T. \tag{83}$$

On the other hand, the value of portfolio B at time T is

$$F(t,T) + (S_T - F(t,T)) = S_T. \tag{84}$$

It follows that they have the same value at time t. This gives

$$e^{-rn\Delta}\Phi(t,T) = e^{-rn\Delta}F(t,T). \tag{85}$$

Canceling out $e^{rn\Delta}$ then shows the assertion. □

The difference $\Phi(t,T) - F(t,T)$ in future and forward prices is called *futures convexity correction.* Proposition 7.2 shows that the futures convexity correction is zero under constant interest rate. In general, this is nonzero when the value S_T of the asset at maturity of the contract is correlated with the interest rate, which is expressed in terms of the money market account. Recall that the *money market account* M_t is the value at time t of 1 invested at time 0. For instance, when the interest is continuously compounded at constant rate r, then $M_t = e^{rt}$. (See also Exercise 4.7 for time-dependent rates.)

The following is a general result concerning the futures convexity correction, whose proof goes beyond the scope of this course.

Theorem 7.3. *Consider a forward and futures contract with maturity T on an asset with price $(S_t)_{t\geq 0}$. Let M_t denote the money market account at time t. For each $t \leq T$, we have*

$$\Phi(t,T) - F(t,T) \propto \mathrm{Cov}(S_T, M_T). \tag{86}$$

PROOF. Omitted. See [HK04]. □

Remark 7.4. For two random variables X and Y, their *covariance* is defined by

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]. \tag{87}$$

It is elementary to show that $\mathrm{Cov}(X,Y) = 0$ if X and Y are independent. Thus, Theorem 7.3 implies that $\Phi(t,T) = F(t,T)$ whenever S_T and M_T are independent. In particular, if the interest rate is constant, then S_T and M_T are independent, so Proposition 7.2 follows from Theorem 7.3.

8. Basic Properties of options

8.1. Recap of options. We have briefly introduced options in Section 1. Fix an asset A with price $(S_t)_{t\geq 0}$. Recall that a *European option* with strike (or *exercise price*) K and *maturity* T on asset A is the right (but not the obligation) to buy or sell the asset for K at time T. The execution of this right is called the *exercise* of the option. (see Def 1.6). Below are different types of options:

European: Exercise only at T.
American: Exercise at any time $t \leq T$.
Bermudan: Exercise at finite set of times $T_0, \cdots, T_n \leq T$.

European options are common on FRAs, and known as caps and floors (See [Hul03, Sec. 29]); American options are common on stocks; Bermudan options are common on swaps, where they are based on mortgages and call able bonds (see [Hul03, Sec. 26.3]). In this note, we will only focus on European and American options.

Recall that there are two sides to every option contract: On one side is the investor who has taken the long position (i.e., has bought the option); On the other side is the investor who has taken a short position (i.e., has sold or *written* the option). The writer of an option receives cash up front, but has potential liabilities later. For instance, if X goes short a European option on one share of a stock with strike price \$10 for option price \$1, then X receives up-front cash \$1, and at maturity, he/she must sell one share of the stock for price \$10, if the one on the long position decides to exercise the option.

In Section 1 and especially in Proposition 1.8, we have seen that the value of a (long) European option with strike K and maturity T at time T (i.e., *payoff*) is given as below:

$$\text{call:} \quad (S_T - K)^+ = \max(S_T - K, 0), \tag{88}$$

$$\text{put:} \quad (K - S_T)^+ = \max(K - S_T, 0). \tag{89}$$

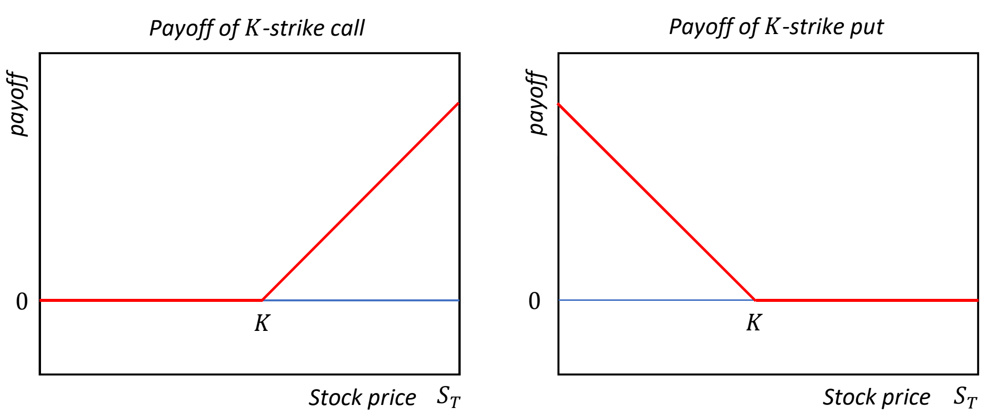

FIGURE 5. Payoff of European call and put options with strike K.

The payoff of short European option is the negative of that of the corresponding long options.

Let $F(t,T)$ denote the forward price of the asset A at time t with maturity T. At time $t \le T$, we say a call option with strike K maturing at T is:

$$\begin{array}{rlrl} \textit{at-the-money} & \text{if } S_t = K & \textit{at-the-money-forward} & \text{if } S_t = F(t,T) \\ \textit{in-the-money} & \text{if } S_t > K & \textit{in-the-money-forward} & \text{if } S_t > F(t,T) \\ \textit{out-the-money} & \text{if } S_t < K & \textit{out-the-money-forward} & \text{if } S_t < F(t,T) \end{array} \tag{90}$$

8.2. Put-call parity. Let $C_K(t,T)$ (resp., $P_K(t,T)$) denote the value of a European call (resp., put) option on an asset of price $(S_t)_{t\ge 0}$ with stike K and maturity T.

Proposition 8.1. *For all $t \le T$, $C_K(t,T) \ge 0$ and $P_K(t,T) \ge 0$.*

PROOF. Note that

$$C_K(T,T) = (S_T - K)^+ \ge 0, \qquad P_K(T,T) = (K - S_T)^+ \ge 0. \tag{91}$$

By the monotonicity theorem (Thm 3.3), it follows that

$$C_K(t,T) \ge 0, \qquad P_K(t,T) \ge 0. \tag{92}$$

□

Put-call parity states that long one call plus short one put equals long one forward contract. Similarly, long one call equals long one forward plus long one put. Hence we can always convert from a call to a put by trading the forward.

Proposition 8.2 (Put-call parity)**.** *Let $C_K(t,T)$, $P_K(t,T)$ be as before. Let $V_K(t,T)$ denote the value of a forward option on the same asset with delivery price K and maturity T. Let $F(t,T)$ denote the forward price.*

$$C_K(t,T) - P_K(t,T) = V_K(t,T). \tag{93}$$

In particular, for at-the-money-forward options,

$$C_{F(t,T)}(t,T) - P_{F(t,T)}(t,T) = 0. \tag{94}$$

PROOF. We compare the following two portfolios at time $t \le T$:

Portfolio A: One long European call and one short European put on A, both with strike K and maturity T.

Portfolio B: One long forward contract on A with delivery price K and maturity T.

At time T, the value of Portfolio A is

$$(S_T - K)^+ - (K - S_T)^+ = \begin{cases} (S_T - K) - 0 & \text{if } S_T \ge K \\ 0 - (K - S_T) & \text{if } S_T \ge K \end{cases} \tag{95}$$

$$= S_T - K = V_K(T,T), \tag{96}$$

where the last expression is the value of Portfolio B at time T. Hence by replication, they must have the same value at time $t \le T$. This gives the first. The second assertion follows from the first by noting that $V_{F(t,T)}(t,T) = 0$. □

Remark 8.3. Note that a forward can be replicated by a holding of stock and cash, as we saw in the proof of Proposition 5.2. Hence combined with put-call parity, we can in principle convert a call to a put (and vice versa) directly by a holding of stock and cash.

8.3. Bounds on call prices. In later sections, we will develop a systematic way to price options. In this subsection, we first derive some preliminary upper and lower bounds on option prices.

Proposition 8.4. *Fix a stock A with price $(S_t)_{t\geq 0}$. Consider European call on A with strike K and maturity T and let its value be denoted by $C_K(t,T)$. Let $Z(t,T)$ be the value at time t of a ZCB maturing at time T. Then*

$$\max(0, S_t - KZ(t,T)) \leq C_K(t,T) \leq S_t. \tag{97}$$

PROOF. Since $C_K(T,T) = (S_T - K)^+ \leq S_T$, by monotonicity thoerem (Thm 3.3), we have $C_K(t,T) \leq S_t$. For the lower bound, we compare the following two portfolios at time $t \leq T$:

Portfolio A: [One long European call] + [K ZCBs maturing at T].
Portfolio B: [One share of stock A]

At time T, we have

$$(S_T - K)^+ + K = \begin{cases} (S_T - K) + K & \text{if } S_T \geq K \\ 0 + K & \text{if } S_T < K \end{cases} \tag{98}$$

$$\leq S_T. \tag{99}$$

This shows that the value of Portfolio A is at most that of Portfolio B at time T. By the monotonicty theorem, we conclude that this hold at time $t \leq T$ as well. Hence

$$C_K(t,T) + KZ(t,T) \leq S_t. \tag{100}$$

Since $C_K(t,T) \geq 0$ by Proposition 8.1, this shows the lower bound on $C_K(t,T)$ in the assertion. □

Exercise 8.5. In this exercise, we will show that the value of an American call and European call on a stock without dividend are the same.

Let $\widetilde{C}_K(t,T)$ and $C_K(t,T)$ denote the value at time t of the American and European call with strike K and maturity T on a stock with price $(S_t)_{t\geq 0}$. Assume that the stock does not pay dividends.

(i) Argue that for any $t \leq T$, $\widetilde{C}_K(t,T) \geq C_K(t,T)$.
(ii) Let $\tau \in [t,T]$ denote the (random)
Show that if the American option is not exercised before time T, then $\widetilde{C}_K(t,T) = C_K(t,T)$.
(iii) Suppose that the American option is exercised at some time $s \in [t,T)$. Show that

$$\widetilde{C}_K(s,T) = S_s - K \leq S_s - KZ(s,T) \leq C_K(s,T), \tag{101}$$

where $Z(s,T)$ denotes the value at s of ZCB maturing at T.
(iv) Show that if $\widetilde{C}_K(t,T) > C_K(t,T)$, then one can create an arbitrage opportunity. (Consider two cases when the American option is exercised before T or not.) Conclude that $\widetilde{C}_K(t,T) = C_K(t,T)$. Why does this result makes sense?
(v) Suppose the stock pays dividends. Is it still true that the American and European call options have the same price?

8.4. Bull and bear spreads. A *spread* is a portfolio consisting of multiple European call options on the same asset with the same maturity but possibly different strike prices. By suitably combining multiple options, one can reduce the risk of the portfolio at the expense of giving up the possibility of yielding higher profit.

A *bull spread* of strike $K_1 < K_2$ and maturity T can be constructed using two call options. Namely, consider long one call option with strike K_1 and short one call option with strike K_2, both with maturity T. Note that this spread has value $C_{K_1}(t,T) - C_{K_2}(t,T)$ at time t. At maturity T, this equals

$$(S_T - K_1)^+ - (S_T - K_2)^+ = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 \leq S_T \leq K_2 \\ K_2 - K_1 & \text{if } K_2 \leq S_T. \end{cases} \tag{102}$$

On the other hand, a *bear spread* can strike $K_1 < K_2$ and maturity T can be constructed by combining one long put option with strike K_2 and one short put option with strike K_1, both with maturity T. This spread has value $P_{K_2}(t,T) - P_{K_1}(t,T)$ at time t. At maturity T, this equals

$$(K_1 - S_T)^+ - (K_2 - S_T)^+ = \begin{cases} K_2 - K_1 & \text{if } S_T \le K_1 \\ K_2 - S_T & \text{if } K_1 \le S_T \le K_2 \\ 0 & \text{if } K_2 \le S_T. \end{cases} \tag{103}$$

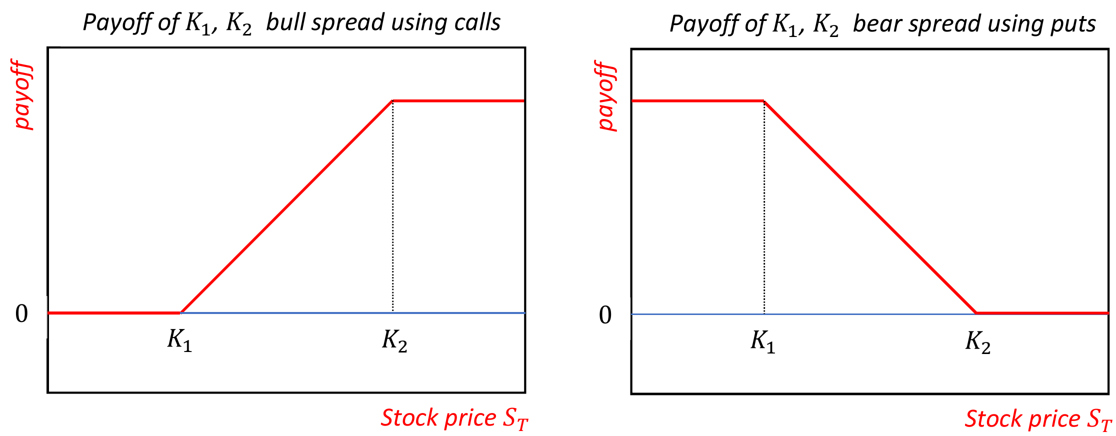

FIGURE 6. Payoff a call spread with strike K_1 and K_2.

Proposition 8.6. *For each $K_1 < K_2$ and $t \le T$,*

$$C_{K_1}(t,T) \ge C_{K_2}(t,T), \qquad P_{K_1}(t,T) \le F_{K_2}(t,T). \tag{104}$$

In particular, the bear spread using calls and put spread using puts have non-negative value at $t \le T$.

PROOF. Follows easily by the monotonicity theorem, □

Proposition 8.7. *For each $K_1 < K_2$ and $t \le T$, the followings hold:*

$$0 \le C_{K_1}(t,T) - C_{K_2}(t,T) \le Z(t,T)(K_2 - K_1), \tag{105}$$
$$0 \le P_{K_2}(t,T) - P_{K_1}(t,T) \le Z(t,T)(K_2 - K_1). \tag{106}$$

PROOF. The leftmost inequalities follow from Proposition 8.6. Recall that according to Proposition 5.4, the value $V_K(t,T)$ of a forward option with delivery price K and maturity T is given by

$$V_K(t,T) = (F(t,T) - K)Z(t,T). \tag{107}$$

Hence by the put-call parity (Prop. 8.2),

$$C_{K_1}(t,T) - F_{K_1}(t,T) = (F(t,T) - K_1)Z(t,T), \tag{108}$$
$$C_{K_2}(t,T) - F_{K_2}(t,T) = (F(t,T) - K_2)Z(t,T). \tag{109}$$

Subtracting these two equations,

$$\big(C_{K_1}(t,T) - C_{K_2}(t,T)\big) + \big(F_{K_2}(t,T) - F_{K_1}(t,T)\big) = (K_2 - K_1)Z(t,T). \tag{110}$$

Since each terms in the left hand side is non-negative by Proposition 8.6, each of them must be at most the right hand side. □

Exercise 8.8. Give an alternative proof of Proposition 8.7 by comparing the following two portfolios at time $t \le T$:

Portfolio A: [One long European call with strike K_2 maturing at T] + [$(K_2 - K_1)$ ZCBs maturing at T].

Portfolio B: [One long European call with strike K_1 maturing at T].

Example 8.9 (Profit of bull and bear spreads)**.** Recall that we have discussed a bull spread using call options and a bear spread using put options and their values at maturity. Since we might be using non-equilibrium strike prices for the options, there might be costs in going long or short for different options.

For instance, consider a bull spread consisting of a long call with strike 10 and short call with strike 30. The value at time t of this bull spread is $C_{10}(t,T) - C_{30}(t,T)$, which is nonnegative by Proposition 8.6. If this equals zero, then this bull spread gives an arbitrage opportunity, so it has to be strictly positive. This means that there is a cost, say $c_{10,30} > 0$, in involving into this spread. Then the *profit* at maturity of this bull spread is given by

$$C_{10}(T,T) - C_{30}(T,T) - c_{10,30} = (S_T - 10)^+ - (S_T - 30)^+ - c_{10,30}, \tag{111}$$

which is a random variable that takes both positive and negative values with positive probability. The profit of this bull spread as well as that of its constituent call options are shown in Figure 7 (left). Similarly,

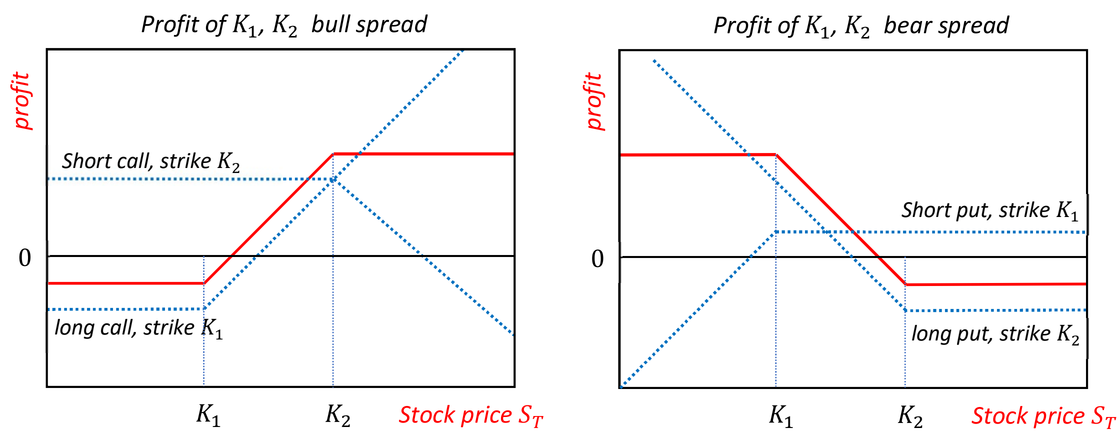

FIGURE 7. Profit of bull (using call) and bear (using put) spreads with strike K_1 and K_2.

one can think of the profit of the bear spread consisting of a long put with strike K_2 and a short put with stike K_1, where $K_1 < K_2$. One can similarly argue that the initial value of this bear spread is strictly positive, so there is a cost in getting into this portfolio. The corresponding profit is depicted in Figure 7 (right). ▲

Exercise 8.10 (Bull and bear spreads with negative value)**.** Consider an asset with price $(S_t)_{t\geq 0}$. We are going to construct bull and bear spreads on this asset with strictly negative initial value. Fix $K_1 < K_2$ and $T > 0$.

(i) Consider a bull spread consisting of a long put with strike K_1 and a short put with strike K_2. Compute its payoff and draw its graph. Show that it is non-positive for all values of S_T.

(ii) Use a no-arbitrage argument to show that the bull spread in (i) has strictly negative initial value. Hence one receives an up-front payment, say c_{K_1,K_2}, when entering into this spread. Let $K_1 = 10$, $K_2 = 30$, and $c_{K_1,K_2} = 15$. Compute the profit of this spread at maturity and draw its graph.

(iii) Consider a bull spread consisting of a long call with strike K_2 and a short call with strike K_1. Compute its payoff and draw its graph. Show that it is non-positive for all values of S_T.

(iv) Do (ii) for the bull spread in (iii).

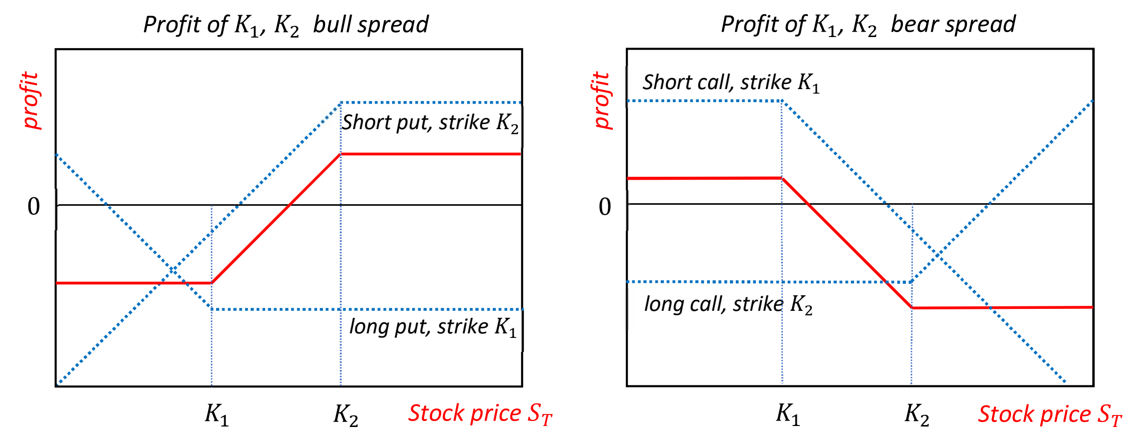

FIGURE 8. Profit of bull (using put) and bear (using call) spreads with strike K_1 and K_2.

CHAPTER 2

Essentials in option pricing

1. Hedging and replication in the two-state world

In the previous note, we have determined the price and value of the forward contracts by using no-arbitrage principle (or replication principle), without assuming how the price of the underlying asset evolves in time. On the contrary, in case of the options, we do need to consider the evolution of market to determine their price and value. In this section, we start with the most basic example – 1-step 2-state (binomial) model. Even though this model is very simple, it contains many of the essential ideas in option pricing.

Recall that we model the market as a probability space $(\Omega, \mathbb{P})$, where Ω consists of sample paths ω of the market, which describes a particular time evolution scenario. For each event $E \subseteq \Omega$, $\mathbb{P}(E)$ gives the probability that the event E occurs. A *portfolio* is a collection of assets that one has at a particular time. The value of a portfolio A at time t is denoted by V_t^A. If t denotes the current time, then V_t^A is a known quantity. However, at a future time $T \geq t$, V_T^A depends on how the market evolves during $[t, T]$, so it is a random variable. Also recall the definition of an arbitrage portfolio:

Definition 1.1. A portfolio A at current time t is said to be an *arbitrage portfolio* if its value V^A satisfies the followings:
(i) $V^A(t) \leq 0$.
(ii) There exists a future time $T \geq t$ such that $\mathbb{P}(V^A(T) \geq 0) = 1$ and $\mathbb{P}(V^A(T) > 0) > 0$.

Example 1.2 (A 1-step binomial model)**.** Suppose we have an asset with price $(S_t)_{t\geq 0}$. Consider a European call option at time $t = 0$ with strike $K = 110$ and maturity $t = 1$ (year). Suppose that $S_0 = 100$ and at time 1, S_1 takes one of the two values 120 and 90 according to a certain distribution. One can imagine flipping a coin with unknown probability, and according to whether it lands heads (H) or tail (T), the stock value S_1 takes values $S_1(H) = 120$ and $S_1(T) = 90$. Assume annually compounded interested rate $r = 4\%$. Can we determine its current value $c = C_{110}(0, 1)$? We will show $c = 14/3 \approx 4.48$ by using two arguments – hedging and replication.

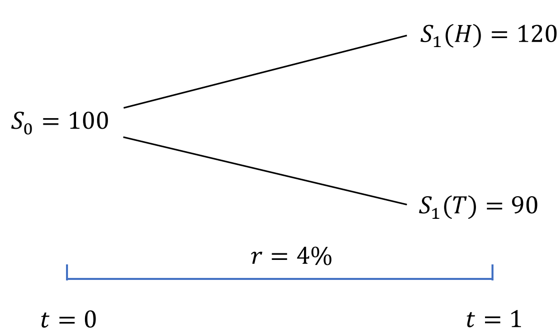

FIGURE 1. 1-step binomial model

First we give a 'hedging argument' for option pricing. Consider the following portfolio at time $t = 0$:
Portfolio A: [x shares of the stock] + [y European call options with strike 110 and maturity 1].

The cost of entering into this portfolio (at time $t=0$) is $100x+cy$. Hence the profit of this portfolio takes the following two values

$$\begin{cases} V_1^A(H)-(100x+cy)(1.04)=[120x+y(120-110)^+]-[104x+(1.04)cy]=16x+(10-(1.04)c)y \\ V_1^A(T)-(100x+cy)(1.04)=[90x+y(90-110)^+]-[104x+(1.04)cy]=-14x-(1.04)cy. \end{cases} \tag{112}$$

In order for a perfect hedging, consider choosing the values of x and y such that the profit of this portfolio at maturity is the same for the two outcomes of the stock. Hence we must have

$$16x+(10-(1.04)c)y=-14x-(1.04)cy. \tag{113}$$

Solving this, we find

$$3x+y=0. \tag{114}$$

Hence if the above equation is satisfied, the profit of portfolio A is

$$V_1^A-(104x+(1.04)cy)=-14x-(1.04)c(-3x)=((3.12)c-14)\,x. \tag{115}$$

If $(3.12)c>14$, then portfolio A is an arbitrage portfolio; If $(3.12)c<14$, then the 'dual' of portfolio A, which consists of $-x$ shares of the stock and $-y$ European call options, is an arbitrage portfolio. Hence assuming no-arbitrage, the only possible value of c is $c=14/(3.12)$.

Second, consider the following portfolio:

Portfolio B: [λ shares of the stock] + [μ ZCBs maturing at 1].

Here we want to choose λ and μ so that portfolio B replicates the European call option in this example. By matching out the payoff in the two cases, this gives

$$\begin{cases} 120\lambda+\mu=(120-110)^+=10 \\ 90\lambda+\mu=(90-110)^+=0. \end{cases} \tag{116}$$

Solving this, we find $\lambda=1/3$ and $\mu=-30$. Hence for such choices, portfolio B and long one European call has the same value at maturity. By the monotonicity theorem, their values at current time $t=0$ must also be the same. Hence we get

$$C_{110}(0,1)=100\lambda+\mu Z(0,1) \tag{117}$$

$$=100(1/3)+(-30)(1.04)^{-1}=\frac{14}{3.12}. \tag{118}$$

This is called a replication argument for option pricing. ▲

2. The fundamental theorem of asset pricing

The observation we made in Example 1.2 can be generalized into the so-called 'fundamental theorem of asset pricing'. For its setup, consider a market where there are n different time evolution $\omega_1,\cdots,\omega_n$ between time $t=0$ and $t=1$, each occurs with a positive probability. Suppose there are assets $A^{(1)},\cdots,A^{(m)}$, whose price at time t is given by $S_t^{(i)}$ for $i=1,2,\cdots,m$ (see Figure 2). For each $1\le i\le m$ and $1\le j\le n$, define

$$\alpha_{i,j}=\begin{pmatrix}\text{profit at time } t=1 \text{ of buying one share of asset } A^{(i)} \\ \text{at time } t=0 \text{ when the market evolves via } \omega_j.\end{pmatrix} \tag{119}$$

Let $\mathbf{A}=(\alpha_{i,j})$ denote the $(m\times n)$ matrix of profits:

$$\mathbf{A}:=\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{m,1} & \alpha_{m,2} & \cdots & \alpha_{m,n} \end{bmatrix}. \tag{120}$$

Consider the following portfolio at time $t=0$:

Portfolio A: [x_1 shares of asset $A^{(1)}$] $+\cdots+$ [x_m shares of asset $A^{(m)}$].

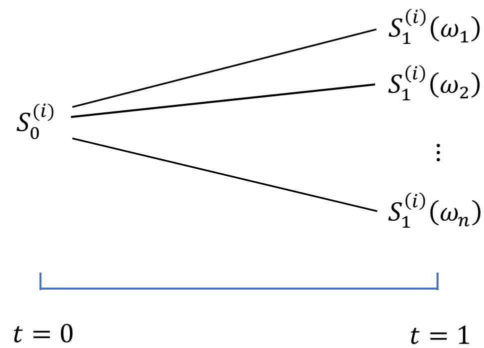

FIGURE 2. 1-step n-state model for asset $A^{(i)}$.

Theorem 2.1 (The fundamental theorem of asset pricing)**.** *Consider portfolio A and the profit matrix* $\mathbf{A}=(\alpha_{i,j})$ *as above. Then exactly one of the followings hold:*

(i) *There exists an investment allocation* $(x_1,\cdots,x_m)$ *such that portfolio A is an arbitrage portfolio, that is, the n-dimensional row vector*

$$[x_1,x_2,\cdots,x_m]\begin{bmatrix}\alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n}\\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n}\\ \vdots & \vdots & \cdots & \vdots\\ \alpha_{m,1} & \alpha_{m,2} & \cdots & \alpha_{m,n}\end{bmatrix} \tag{121}$$

has nonzero coordinates and at least one strictly positive coordinate.

(ii) *There exists a strictly positive probability distribution* $\mathbf{p}^*=(p_1^*,\cdots,p_n^*)$ *under which the expected profit of each asset is zero:*

$$\begin{bmatrix}\alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n}\\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n}\\ \vdots & \vdots & \cdots & \vdots\\ \alpha_{m,1} & \alpha_{m,2} & \cdots & \alpha_{m,n}\end{bmatrix}\begin{bmatrix}p_1^*\\ p_2^*\\ \vdots\\ p_n^*\end{bmatrix}=\begin{bmatrix}0\\0\\ \vdots\\0\end{bmatrix}. \tag{122}$$

Remark 2.2. Theorem 2.1 (i) states that the portfolio A is an arbitrage portfolio for some $(x_1,\cdots,x_m)$. The probability distribution $\mathbf{p}^*$ in the above theorem is called the *risk-neutral probability distribution.* Hence Theorem 2.1 states that there is no way to make A into an arbitrage portfolio if and only if there exists a risk-neutral probability distribution under which the expected profit of each asset $A^{(i)}$ is zero.

Example 2.3 (Example 1.2 revisited)**.** Consider the situation described in Example 1.2. Let $A^{(1)}$ be the asset of price $(S_t)_{t\geq 0}$ and $A^{(2)}$ denote the European call with strike $K=110$ on this asset with maturity T. Then the matrix $\mathbf{A}$ of profits is given by

$$\mathbf{A}=\begin{bmatrix}16 & -14\\ 10-(1.04)c & -(1.04)c\end{bmatrix}, \tag{123}$$

where $c=C_{110}(0,1)$ denotes the price of this European option. Assuming no-arbitrage, the fundamental theorem implies that there exists risk-neutral probability distribution $\mathbf{p}^*=(p_1^*,p_2^*)$ such that $\mathbf{A}(\mathbf{p}^*)^T=\mathbf{0}^T$. Namely,

$$\begin{cases}16p_1^*-14p_2^*=0\\ (10-(1.04)c)p_1^*-(1.04)cp_2^*=0\end{cases} \tag{124}$$

Since $p_1^* + p_2^* = 1$, the first equation implies $p_1^* = 7/15$ and $p_2^* = 8/15$. Then from the second equation, we get

$$(1.04)c = 10p_1^* = \frac{14}{3}. \tag{125}$$

This gives $c = 14/(3.12)$. ▲

Exercise 2.4. Rework Examples 1.2 and 2.3 with following parameters:

$$S_0 = 100, \quad S_1(H) = 130, \quad S_1(T) = 80, \quad r = 5\%, \quad K = 110. \tag{126}$$

PROOF OF THEOREM 2.1. Suppose (i) holds with $\mathbf{x} = (x_1, \cdots, x_n)$. We want to show that (ii) cannot hold. Fix a strictly positive probability distribution $\mathbf{p} = (p_1, \cdots, p_n)'$, where $'$ denotes the transpose so that $\mathbf{p}$ is an n-dimensional column vector. By (i), we have

$$\mathbf{x}(\mathbf{Ap}) = (\mathbf{xA})\mathbf{p} > 0. \tag{127}$$

It follows that $\mathbf{Ap}$ cannot be the zero vector in $\mathbb{R}^n$. Hence (122) cannot hold for $\mathbf{p}^* = \mathbf{p}$, as desired.

Next, suppose that (ii) holds for some strictly positive probability distribution $\mathbf{p}^* = (p_1^*, \cdots, p_n^*)'$. We use a linear algebra argument to show that (i) does not hold. For each m-dimensional row vector $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$, one can correspond a n-dimensional column vector $\mathbf{xA} \in \mathbb{R}^n$. The condition (122) says that $\mathbf{Ap}^* = \mathbf{0}$, where T denotes the transpose. Hence for each $\mathbf{x} \in \mathbb{R}^m$, by using associativity of matrix multiplication,

$$(\mathbf{xA})\mathbf{p}^* = \mathbf{x}\left(\mathbf{Ap}^*\right) = \mathbf{x0} = 0. \tag{128}$$

This shows that the image of the linear map $\mathbf{x} \mapsto \mathbf{xA}$, which is a linear subspace of $\mathbb{R}^n$, is orthogonal to the strictly positive vector $\mathbf{p}^*$. Hence this linear subspace intersects with the positive orthant $\{(y_1, \cdots, y_n) \,|\, y_1, \cdots, y_n \geq 0\}$ only at the origin. This shows that (i) does not hold, as desired. □

Exercise 2.5 (A 1-step 3-state model)**.** Consider a 1-step 3-state model, where there are three possible time evolution $\omega_1, \omega_2, \omega_3$ and a stock with current price $S_0 = 100$ and time-1 price $S_1(\omega_1) = 120$, $S_1(\omega_1) = 110$, $S_1(\omega_1) = 90$. Assume one-time interest rate $r = 4\%$ during $[0,1]$. Now consider a European call option on this stock with strike price $K = 110$.

(i) Consider the following portfolio

$$[\Delta_0 \text{ shares of the stock}] + [\text{Short one call option with strike 110 and maturity 1}]. \tag{129}$$

Can one find Δ_0 such that the above portfolio is perfectly hedged?

(ii) Consider the following portfolio

$$[\Delta_0 \text{ shares of the stock}] + [x \text{ cash}]. \tag{130}$$

Can one find Δ_0 and x such that the above portfolio replicates one long European call option?

(iii) Write down the (2×3) profit matrix whose first and second row corresponds to one share of the stock and one long European call option, respectively. Show that there exists a risk-neutral probability distribution $\mathbf{p}^* = (p_1^*, p_2^*, p_3^*)$, but it is not necessarily unique.

(iv) Based on (i)-(iii), how should one price the European call option at time $t = 0$? Is it possible at all? Give your reasoning.

3. Binomial tree

In this section, we consider the general binomial model, where each time the stock price can go up or down by multiplicative factors u and d, respectively.

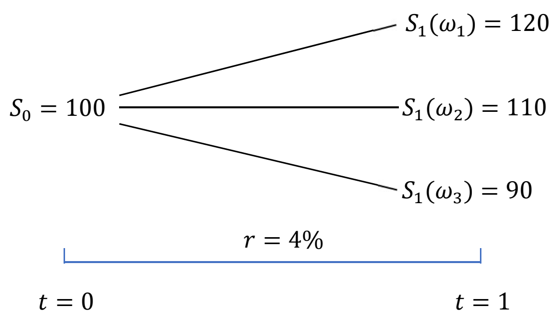

FIGURE 3. A 1-step 3-state model

3.1. 1-step binomial model. Suppose we have an asset with price $(S_t)_{t\geq 0}$. Assume that a future time $t = 1$, the stock price S_1 can take two values $S_1(H) = S_0 u$ and $S_1(T) = S_0 d$ according to an invisible coin flip, where $u, d > 0$ are multiplicative factors for upward and downward moves for the stock price during the period $[0,1]$. Assume the aggregated interested rate during $[0,1]$ is $r > 0$, so that the value of ZCB maturing at 1 is given by $Z(0,1) = 1/(1+r)$.

Consider a general European option on this stock, whose value at time $t = 0,1$ is denoted V_t. We would like to determine its initial value (price at $t = 0$) V_0 in terms of its payoff V_1. In the previous section, we have seen that there are three ways to proceed: 1) hedging argument, 2) replication argument, and 3) risk-neutral probability distribution.

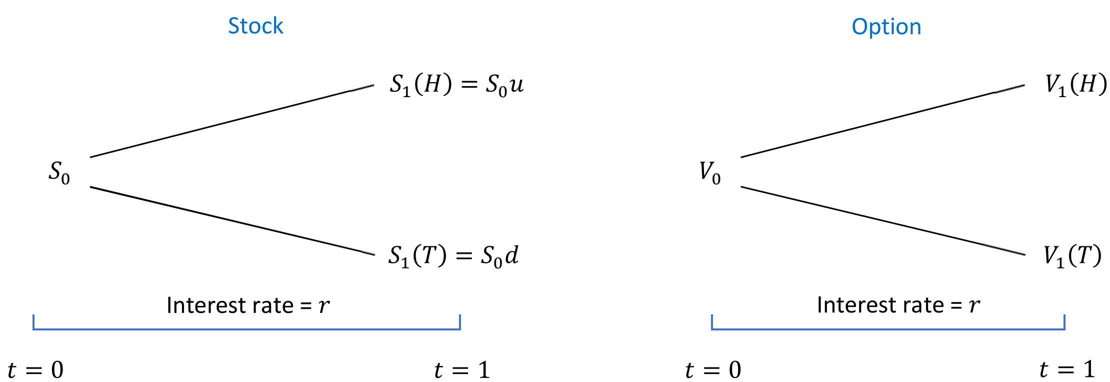

FIGURE 4. 1-step binomial model with general European option

Proposition 3.1. *In the above binomial model, the followings hold.*

(i) *There exists a risk-neutral probability distribution* $\mathbf{p}^* = (p_1^*, p_2^*)$ *if and only if*

$$0 < d < 1 + r < u. \tag{131}$$

Furthermore, if the above condition holds, $\mathbf{p}^*$ *is uniquely given by*

$$p_1^* = \frac{(1+r)-d}{u-d}, \qquad p_2^* = \frac{u-(1+r)}{u-d}. \tag{132}$$

(ii) *Suppose (131) holds. Then the initial value V_0 of the European option is given by*

$$V_0 = \frac{1}{1+r}\mathbb{E}_{\mathbf{p}^*}[V_1] = \frac{1}{1+r}\left(\frac{(1+r)-d}{u-d}V_1(H) + \frac{u-(1+r)}{u-d}V_1(T)\right). \tag{133}$$

PROOF. To begin, we first need to compute the (2×2) profit matrix $\mathbf{A} = (\alpha_{i,j})$, whose rows and columns correspond to the two kinds of assets (stock and European option) and outcomes (coin flips), respectively. We find

$$\mathbf{A} = \begin{bmatrix} S_1(H) - S_0\cdot(1+r) & S_1(T) - S_0\cdot(1+r) \\ V_1(H) - V_0\cdot(1+r) & V_1(T) - V_0\cdot(1+r) \end{bmatrix}. \tag{134}$$

The risk-neutral probability $\mathbf{p}^* = (p_1^*, p_2^*)'$ satisfies $\mathbf{A}\mathbf{p}^* = \mathbf{0}$, so

$$\begin{cases} [S_1(H) - S_0\cdot(1+r)]p_1^* + [S_1(T) - S_0\cdot(1+r)]p_2^* = 0 \\ [V_1(H) - V_0\cdot(1+r)]p_1^* + [V_1(T) - V_0\cdot(1+r)]p_2^* = 0. \end{cases} \tag{135}$$

Using the fact that $p_1^* + p_2^* = 1$, the first equation gives

$$p_1^* = \frac{S_0(1+r) - S_1(T)}{S_1(H) - S_1(T)} = \frac{(1+r)-d}{u-d}, \qquad p_2^* = \frac{S_1(H) - S_0(1+r)}{S_1(H) - S_1(T)} = \frac{u-(1+r)}{u-d}. \tag{136}$$

Hence the desired expression for the risk-neutral probabilities p_1^* and p_2^* holds. Note that this gives a strictly positive probability distribution if and only if (131) holds. This shows (i). (Why does this condition make sense?)

Assuming (131), the second equation in (135) then gives

$$V_0\cdot(1+r) = V_1(H)p_1^* + V_1(T)p_2^*. \tag{137}$$

The right hand side can be regarded as the expectation $\mathbb{E}_{\mathbf{p}^*}[V_1]$ of the value V_1 of the European option at time $t=1$ under the risk-neutral probability distribution $\mathbf{p}^*$. Then (ii) follows from (i). □

Proposition 3.2. *In the 1-step binomial model as before, consider the following portfolios:*

Portfolio A: [Δ_0 shares of the stock] + [Short one European option].

Portfolio B: [Δ_0 shares of the stock] + [x cash],

Then the followings hold:

(i) *Portfolio A is perfectly hedged (i.e., constant payoff at time $t=1$) if and only if*

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}. \tag{138}$$

(ii) *Portfolio B replicates long one European option if and only if we have (138) and*

$$x = \frac{1}{1+r}\frac{S_1(H)V_1(T) - S_1(T)V_1(H)}{S_1(H) - S_1(T)}. \tag{139}$$

Furthermore,

$$V_0 = x + \Delta_0 S_0 = \frac{1}{1+r}\left(V_1(H)\frac{(1+r)-u}{u-d} + V_1(T)\frac{u-(1+r)}{u-d}\right). \tag{140}$$

PROOF. To show (i), we equate the two payoffs of portfolio A at time $t=1$ and obtain

$$\Delta_0 S_1(H) - V_1(H) = \Delta_0 S_1(T) - V_1(T). \tag{141}$$

Solving this for Δ_0 shows the assertion.

To show (ii), note that portfolio A replicates long one European option if and only if

$$\begin{cases} \Delta_0 S_1(H) + x\cdot(1+r) = V_1(H) \\ \Delta_0 S_1(T) + x\cdot(1+r) = V_1(T), \end{cases} \tag{142}$$

or in matrix form,

$$\begin{bmatrix} S_1(H) & 1+r \\ S_1(T) & 1+r \end{bmatrix}\begin{bmatrix} \Delta_0 \\ x \end{bmatrix} = \begin{bmatrix} V_1(H) \\ V_1(T) \end{bmatrix}. \tag{143}$$

This is equivalent to

$$\begin{bmatrix} \Delta_0 \\ x \end{bmatrix} = \frac{1}{(1+r)(S_1(H) - S_1(T))}\begin{bmatrix} 1+r & -1-r \\ -S_1(T) & S_1(H) \end{bmatrix}\begin{bmatrix} V_1(H) \\ V_1(T) \end{bmatrix}, \tag{144}$$

which is also equivalent to (138) and (139), as desired.

Lastly, suppose portfolio B replicates long one European option. Then by the monotinicty theorem, intitial value V_0 of the European option should equal to the value of portfolio A at time $t=0$. This shows

$$V_0 = x + \Delta_0 S_0. \tag{145}$$

Using (139), we also have

$$x+\Delta_0 S_0 = \frac{1}{1+r}\frac{S_1(H)V_1(T)-S_1(T)V_1(H)}{S_1(H)-S_1(T)} + \frac{V_1(H)S_0 - V_1(T)S_0}{S_1(H)-S_1(T)} \tag{146}$$

$$= \frac{1}{1+r}\left(V_1(H)\frac{S_0\cdot(1+r)-S_1(H)}{S_1(H)-S_1(T)} + V_1(T)\frac{S_1(H)-S_0\cdot(1+r)}{S_1(H)-S_1(T)}\right) \tag{147}$$

$$= \frac{1}{1+r}\left(V_1(H)\frac{(1+r)-u}{u-d} + V_1(T)\frac{u-(1+r)}{u-d}\right). \tag{148}$$

This shows (ii). (Remark: By using Proposition 3.1, one can avoid using the monotonicity theorem here.) □

Example 3.3 (Excerpted from [Dur99])**.** Suppose a stock is selling for \$60 today. A month from now it will be either \$80 or \$50, i.e., $u = 4/3$ and $d = 5/6$. Assume the interest rate $r = 1/18$ for this period. Then according to Proposition 3.2, the risk-neutral probability distribution $\mathbf{p}^* = (p_1^*, p_2^*)$ is given by

$$p_1^* = \frac{(1+\frac{1}{18})-\frac{5}{6}}{\frac{4}{3}-\frac{5}{6}} = \frac{4}{9}, \qquad p_2^* = \frac{5}{9}. \tag{149}$$

Now consider a European call option on this stock with strike $K = 65$ maturing in a month. Then $V_1(H) = (80-65)^+ = 15$ and $V_1(T) = (50-65)^+ = 0$. By Proposition 3.2, the initial value V_0 of of this European option is

$$V_0 = \frac{1}{1+(1/18)}\mathbb{E}_{\mathbf{p}^*}[V_1] = \frac{18}{19}\cdot 15\cdot\frac{4}{9} = \frac{120}{19} = 6.3158. \tag{150}$$

Working in an investment bank, you were able to sell 10,000 calls to a customer for a slightly higher price each at \$6.5, receiving up-front payment of \$65,000. At maturity, the overall profit is given by

$$\begin{cases} (19/18)\cdot \$65,000 - 10,000\cdot(80-65)^+ = -\$81,389 & \text{if stock goes up} \\ (19/18)\cdot \$65,000 - 10,000\cdot(50-65)^+ = \$68,611 & \text{if stock goes down.} \end{cases} \tag{151}$$

Being worried about losing a huge amount if the stock goes up, you decided to hedge and lock the profit. According to Proposition 3.2, the hedge ratio Δ_0 is given by

$$\Delta_0 = \frac{(80-65)^+ - (50-65)^+}{80-50} = \frac{15}{30} = \frac{1}{2}. \tag{152}$$

Since you have shorted 10,000 calls, this means you need to buy 5,000 shares of the stock owing $5,000\cdot 60 - \$65,000 = \$235,000$ to the bank. This forms a portfolio of

$$[5,000 \text{ shares of stock}] + [10,000 \text{ short calls}]. \tag{153}$$

The overall profit at maturity is then

$$\begin{cases} 5,000\cdot \$80 - 10,000\cdot(80-65)^+ - (19/18)\cdot \$235,000 = \$1,944 & \text{if stock goes up} \\ 5,000\cdot \$50 - 10,000\cdot(50-65)^+ - (19/18)\cdot \$235,000 = \$1,944 & \text{if stock goes down.} \end{cases} \tag{154}$$

▲

Exercise 3.4. Rework Example 3.3 for the following parameters:

$$S_0 = 50, \quad S_1(T) = 70, \quad S_1(H) = 40, \quad r = 4\%, \quad K = 60. \tag{155}$$

3.2. The 2-step binomial model. In this subsection, we consider the 2-step binomial model. Namely, staring at the current time $t=0$, we flip two coins at time $t=1$ and $t=2$ to determine the market evolution. More precisely, now the sample space of the outcomes is $\Omega=\{HH,HT,TH,TT\}$, and the stock price S_t for $t=0,1,2$ are determined by the 2-step time evolution. We also consider a general European stock option with value V_t for $t=0,1,2$, and we also assume the interest rate for each period $[0,1]$ and $[1,2]$ are some constant $r>0$,

As a developer of the European stock option, we determine the payoff of the option $V(\omega)$ at time $t=2$ depending on the 2-step market evolution $\omega\in\Omega$. We would like to determine the right price of this option at time $t=0$. This can be done by using the same argument as in the 1-period case backward in time.

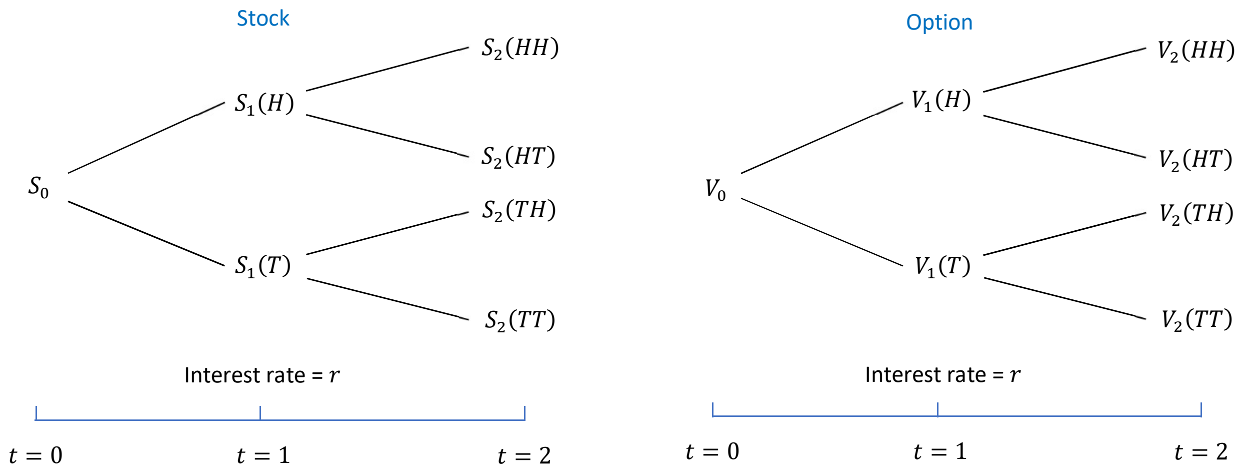

FIGURE 5. 2-step binomial model with general European option.

Proposition 3.5. *Consider the 2-step binomial model as before. Define the following quantities*

$$p_1^*(\emptyset)=\frac{(1+r)S_0-S_1(T)}{S_1(H)-S_1(T)},\qquad p_2^*(\emptyset)=\frac{S_1(H)-(1+r)S_0}{S_1(H)-S_1(T)},\tag{156}$$

$$p_1^*(H)=\frac{(1+r)S_1(H)-S_2(HT)}{S_2(HH)-S_2(HT)},\qquad p_2^*(H)=\frac{S_2(HH)-(1+r)S_1(H)}{S_2(HH)-S_2(HT)},\tag{157}$$

$$p_1^*(T)=\frac{(1+r)S_1(T)-S_2(TT)}{S_2(TH)-S_2(TT)},\qquad p_2^*(T)=\frac{S_2(TH)-(1+r)S_1(T)}{S_2(TH)-S_2(TT)}.\tag{158}$$

Suppose $0<p_1^*,p_1^*(H),p_1^*(T)<1$. *Then the followings hold:*

$$V_1(H)=\frac{1}{1+r}\left(V_2(HH)p_1^*(H)+V_2(HT)p_2^*(H)\right)\tag{159}$$

$$V_1(T)=\frac{1}{1+r}\left(V_2(TH)p_1^*(T)+V_2(TT)p_2^*(T)\right),\tag{160}$$

$$V_0=\frac{1}{1+r}\left(V_1(H)p_1^*(\emptyset)+V_1(T)p_2^*(\emptyset)\right)\tag{161}$$

$$=\frac{1}{(1+r)^2}\begin{pmatrix}V_2(HH)p_1^*(\emptyset)p_1^*(H)+V_2(HT)p_1^*(\emptyset)p_2^*(H)\\+V_2(TH)p_2^*(\emptyset)p_1^*(T)+V_2(TT)p_2^*(\emptyset)p_2^*(T)\end{pmatrix}.\tag{162}$$

Remark 3.6. We can give a probabilistic interpretation of the expression for V_0 in the 2-step case as stated in Proposition 3.5. Recall that we do not know the probabilities that the Market goes up or down at each step. Suppose that these probabilities are given as the risk-neutral probabilities in Proposition 3.5; the first coin lands heads with probability p_1^*; depending on whether it lands on heads or tails, the second coin lands on heads with probability $p_1^*(H)$ or $p_1^*(H)$, respectively. This yields a risk-neutral probability distribution $\mathbb{P}^*$ on the space of 2-step market evolution $\Omega = \{HH, HT, TH, TT\}$ by

$$\mathbb{P}^*(\{HH\}) = p_1^* p_1^*(H), \quad \mathbb{P}^*(\{HT\}) = p_1^* p_2^*(H), \quad \mathbb{P}^*(\{TH\}) = p_2^* p_1^*(T), \quad \mathbb{P}^*(\{TT\}) = p_2^* p_2^*(T). \tag{163}$$

Then the result in Proposition 3.5 can be rewritten as

$$V_0 = \frac{1}{(1+r)^2}\mathbb{E}_{\mathbb{P}^*}[V_2]. \tag{164}$$

That is, the price of the European option V_0 is the discounted expectation of its payoff V_2 under the 'risk-neutral probability measure' $\mathbb{P}^*$.

Example 3.7. Consider the following 2-step binomial model, were the stock price S_t for $t = 0,1,2$ are given by

$$S_0 = 10, \quad S_1(H) = 15, \quad S_1(T) = 6, \quad S_2(HH) = 22, \quad S_2(HT) = 12, \quad S_2(TH) = 9, \quad S_2(TT) = 4. \tag{165}$$

Assume constant interest rate of 4% during each step. Consider a European call option on this stock with strike $K = 11$ and maturity $t = 2$. Let V_t denote the value of this European option at time t. Its payoff at maturity, $V_2 = (S_2 - 11)^+$, is given by

$$V_2(HH) = 11, \quad V_2(HT) = 1, \quad V_2(TH) = 0, \quad V_2(TT) = 0. \tag{166}$$

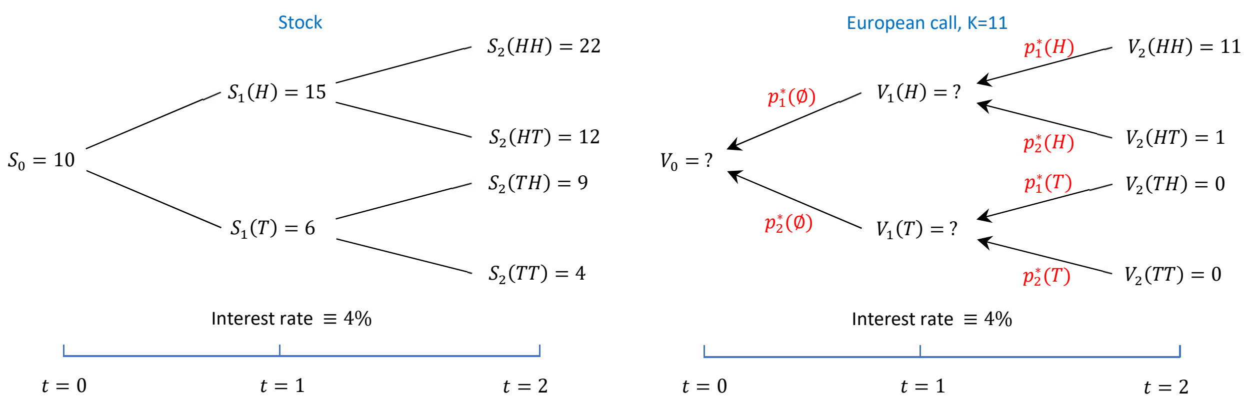

FIGURE 6. 2-step binomial model with a European call option with strike $K = 11$.

To determine V_1 and V_0, we compute the risk-neutral probabilities:

$$p_1^*(\emptyset) = \frac{(1.04)10 - 6}{15 - 6} = \frac{4.4}{9}, \tag{167}$$

$$p_1^*(H) = \frac{(1.04)15 - 12}{22 - 12} = \frac{3.6}{10}, \tag{168}$$

$$p_1^*(T) = \frac{(1.04)6 - 4}{9 - 4} = \frac{2.24}{5}. \tag{169}$$

Then according to Proposition 3.5, we can compute

$$V_1(H) = \frac{1}{1.04}\left(11 \cdot \frac{3.6}{10} + 1 \cdot \frac{6.4}{10}\right) = \frac{39.6 + 6.4}{10.4} = \frac{46}{10.4}, \tag{170}$$

$$V_1(T) = \frac{1}{1.04}\left(0 \cdot \frac{2.24}{5} + 0 \cdot \frac{2.76}{5}\right) = 0, \tag{171}$$

$$V_0 = \frac{1}{1.04}\left(V_1(H)\frac{4.4}{9} + V_1(T)\frac{4.6}{9}\right) = \frac{1}{1.04}\left(\frac{46}{10.4} \cdot \frac{4.4}{9} + 0 \cdot \frac{4.6}{9}\right) = \frac{6325}{3042} = 2.0792 \tag{172}$$

We can also directly compute V_0 as the discounted risk-neutral expectation of the payoff:

$$V_0 = \frac{1}{(1.04)^2}\left(11\cdot\frac{4.4}{9}\cdot\frac{3.6}{10}+1\cdot\frac{4.4}{9}\cdot\frac{6.4}{10}+0\cdot\frac{4.6}{9}\cdot\frac{2.24}{5}+0\cdot\frac{4.6}{9}\cdot\frac{2.76}{5}\right)=2.0792 \tag{173}$$

▲

In the rest of this subsection, we justify Proposition 3.5. To begin, let us first determine $V_1(H)$.

Proposition 3.8. *Consider the 2-step binomial model as before, conditional on the first coin flip being H. Then the followings hold.*

(i) *Define $\mathbf{p}^*(H) = (p_1^*(H), p_2^*(H))$ by*

$$p_1^*(H) = \frac{(1+r)S_1(H)-S_2(HT)}{S_2(HH)-S_2(HT)}, \qquad p_2^*(H) = \frac{S_2(HH)-(1+r)S_1(H)}{S_2(HH)-S_2(HT)}. \tag{174}$$

This defines the risk-neutral probability distribution during the period $[1,2]$ if and only of $0 < p_1^*(H) < 1$.

(ii) *Let $\mathbf{p}_1^*(H)$ be as defined in (i) and suppose $0 < p_1^*(H) < 1$. Then*

$$V_1(H) = \frac{1}{1+r}\mathbb{E}_{\mathbf{p}^*(H)}[V_2\,|\,\text{first coin flip} = H] \tag{175}$$

$$= \frac{1}{1+r}\left(V_2(HH)p_1^*(H)+V_2(HT)p_2^*(H)\right). \tag{176}$$

PROOF. The argument is exactly the same as in the 1-period case. Namely, we first compute the (2×2) profit matrix $\mathbf{A}(H) = (\alpha_{i,j})$ conditional on the first coin flip being H, whose rows and columns correspond to the two kinds of assets (stock and European option) and outcomes (coin flips), respectively. Letting the first and second rows to be for the stock and European option, we find

$$\mathbf{A}(H) = \begin{bmatrix} S_2(HH)-S_1(H)\cdot(1+r) & S_2(HT)-S_1(H)\cdot(1+r) \\ V_2(HH)-V_1(H)\cdot(1+r) & V_2(HT)-V_1(H)\cdot(1+r)\end{bmatrix}. \tag{177}$$

The risk-neutral probability $\mathbf{p}^*(H) = (p_1^*(H), p_2^*(H))$ satisfies $\mathbf{A}(H)\mathbf{p}^*(H)' = \mathbf{0}$, so

$$\begin{cases} [S_2(HH)-S_1(H)\cdot(1+r)]p_1^*(H)+[S_2(HT)-S_1(H)\cdot(1+r)]p_2^*(H)=0 \\ [V_2(HH)-V_1(H)\cdot(1+r)]p_1^*(H)+[V_2(HT)-V_1(H)\cdot(1+r)]p_2^*(H)=0. \end{cases} \tag{178}$$

Using the fact that $p_1^*(H)+p_2^*(H) = 1$, the first equation gives (174). Hence this $\mathbf{p}^*(H)$ defines a valid probability distribution if and only if $0 < p_1^*(H) < 1$. This shows (i).

On the other hand, the second equation in (178) then gives

$$V_0(H)\cdot(1+r) = V_1(HH)p_1^*(H)+V_1(HT)p_2^*(H) = \mathbb{E}_{\mathbf{p}^*(H)}[V_2\,|\,\text{first coin flip}= H]. \tag{179}$$

Then (ii) follows from (i). □

An entirely similar argument shows the following, when the first coin flip is T.

Proposition 3.9. *Consider the 2-step binomial model as before, conditional on the first coin flip being T. Then the followings hold.*

(i) *Define $\mathbf{p}^*(T) = (p_1^*(T), p_2^*(T))$ by*

$$p_1^*(T) = \frac{(1+r)S_1(T)-S_2(TT)}{S_2(TH)-S_2(TT)}, \qquad p_2^*(T) = \frac{S_2(TH)-(1+r)S_1(T)}{S_2(TH)-S_2(TT)}. \tag{180}$$

This defines the risk-neutral probability distribution during the period $[1,2]$ if and only of $0 < p_1^*(T) < 1$.

(ii) *Let* $\mathbf{p}_1^*(T)$ *be as defined in (i) and suppose* $0 < p_1^*(T) < 1$*. Then*

$$V_1(T) = \frac{1}{1+r}\mathbb{E}_{\mathbf{p}^*(T)}[V_2 \,|\, \text{first coin flip} = T] \tag{181}$$

$$= \frac{1}{1+r}\left(V_2(TH)p_1^*(T) + V_2(TT)p_2^*(T)\right). \tag{182}$$

PROOF. Omitted. □

Now that we have an expression for both $V_1(H)$ and $V_1(H)$, we can apply the 1-period case for the interval $[0,1]$ to deduce Proposition 3.5.

PROOF OF PROPOSITION 3.5. The first equation for V_0 follows from the 1-step binomial model. Then the second equation follows by substituting $V_1(H)$ and $V_1(T)$ for the expressions given in Propositions 3.8 and 3.9, respectively. □

3.3. The N-step binomial model. In this subsection, we consider the general N-step binomial model. Namely, staring at the current time $t = 0$, we flip N coins at times $t = 1, 2, \cdots, N$ to determine the market evolution. More precisely, now the sample space of the outcomes is $\Omega = \{H, T\}^N$, which consists of sequences of length N strings of H's or T's. We assume constant interest rate for each periods $[k, k+1]$ for $k = 0, 1, \cdots, N-1$.

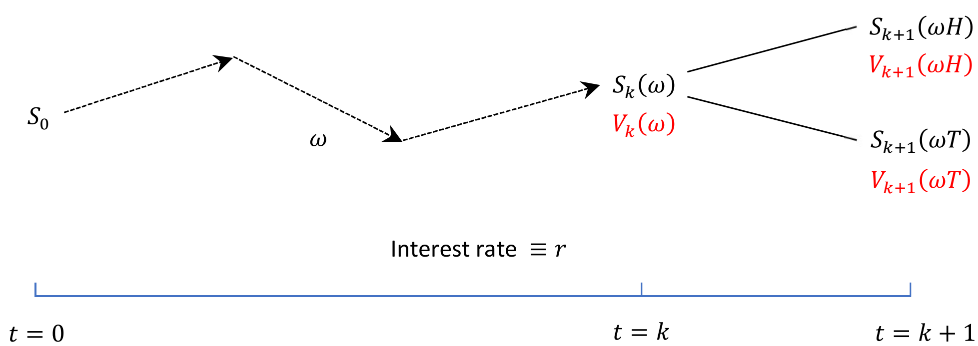

FIGURE 7. Illustration of the N-step binomial model. ω is a sample path of the market evolution in the first k steps. In the next period $[k, k+1]$, either up or down evolution occurs and the same path is extended to ωH or ωT accordingly. S_t and V_t denote the stock price and option payoff.

The following result for the general N-step binomial model is a direct analog of the 2-step case we have discussed in the previous subsection. To better understand its second part, recall the remark on a probabilistic interpretation on the 2-step value formula as in Proposition 3.5.

Proposition 3.10. *Consider the N-step binomial model as above. Consider a European option on this stock wit value* $(V_t)_{0\le t\le N}$.

(i) *For each integer* $0 \le k < N$ *and a sample path* $\omega \in \{H, T\}^k$ *for the first k steps, define the risk-neutral probability distribution* $\mathbf{p}^*(\omega) = (p_1^*(\omega), p_2^*(\omega))$ *by*

$$p_1^*(\omega) = \frac{(1+r)S_k(\omega) - S_{k+1}(\omega T)}{S_{k+1}(\omega H) - S_{k+1}(\omega T)}, \qquad p_2^*(\omega) = \frac{S_{k+1}(\omega H) - (1+r)S_k(\omega)}{S_{k+1}(\omega H) - S_{k+1}(\omega T)}. \tag{183}$$

If $0 < p_1^*(\omega) < 1$*, then*

$$V_k(\omega) = \frac{1}{(1+r)}\mathbb{E}_{\mathbf{p}^*(\omega)}[V_{k+1} \,|\, \text{first } k \text{ coin flips} = \omega] \tag{184}$$

$$= \frac{1}{(1+r)}\left(V_{k+1}(\omega H)p_1^*(\omega) + V_{k+1}(\omega T)p_2^*(\omega)\right). \tag{185}$$

(ii) *Consider N consecutive coin flips such that given any sequence $x_1x_2\cdots x_k$ of the first k flips, the $(k+1)$st coin lands on heads with probability $p_1^*(x_1x_2\cdots x_k)$. Let $\mathbb{P}^*$ denote the induced probability measure (risk-neutral probability measure) on the sample space $\Omega = \{H, T\}^N$. Then*

$$V_0 = \frac{1}{(1+r)^N}\mathbb{E}_{\mathbb{P}^*}[V_N]. \tag{186}$$

PROOF. The argument for (i) is exactly the same as in the proof of Proposition 3.8. For (ii) we use an induction on the number of steps. The base case is verified by (i). For the induction step, we first use (i) to write

$$V_0 = \frac{1}{1+r}(V_1(H)p_1^*(\varnothing) + V_1(T)p_2^*(\varnothing)). \tag{187}$$

Let $X_1, X_2, \cdots, X_N$ be a sequence of N (random) coin flips given by the risk-neutral probability measure $\mathbb{P}^*$. Denote the expectation under $\mathbb{P}^*$ by $\mathbb{E}^*$. By the induction hypothesis, we have

$$V_1(H) = \frac{1}{(1+r)^{N-1}}\mathbb{E}^*[V_N \,|\, X_1 = H], \qquad V_1(T) = \frac{1}{(1+r)^{N-1}}\mathbb{E}^*[V_N \,|\, X_1 = T]. \tag{188}$$

Hence we have

$$V_0 = \frac{1}{(1+r)^N}\left[\mathbb{E}^*[V_N \,|\, X_1 = H]p_1^*(\varnothing) + \mathbb{E}^*[V_N \,|\, X_1 = T]p_2^*(\varnothing)\right] \tag{189}$$

$$= \frac{1}{(1+r)^N}\mathbb{E}^*[\mathbb{E}^*[V_N \,|\, X_1]] = \frac{1}{(1+r)^N}\mathbb{E}^*[V_N], \tag{190}$$

where we have used iterated expectation for the last equality. This shows the assertion. □

Example 3.11 (Callback option)**.** Consider a European option on a stock with price S_t which allows one to buy one share of the stock at time $t = 3$ at its current price S_3, and to sell it at its highest price seen in the past. In other words, the payoff of this option V_3 is given by

$$V_3 = \max_{1\le k\le 3} S_k - S_3. \tag{191}$$

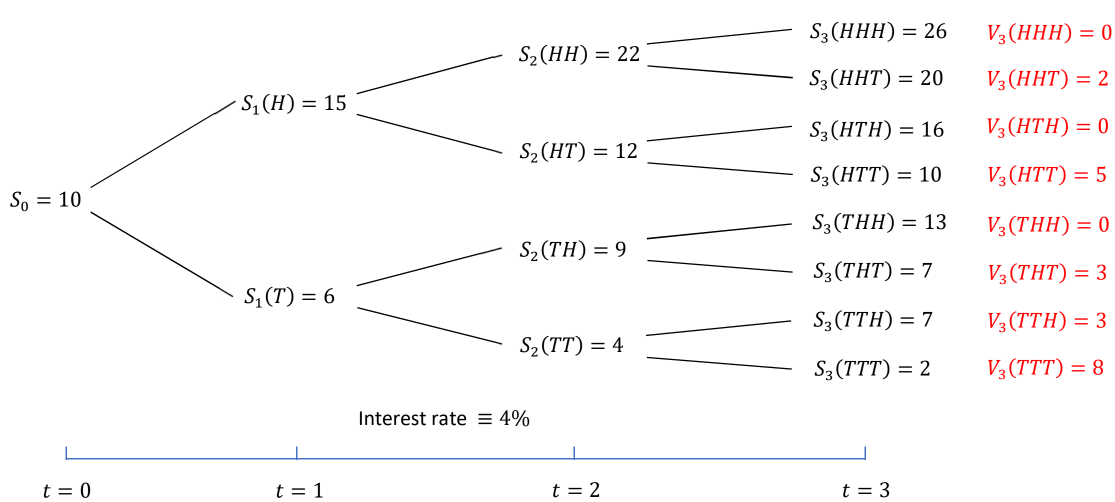

FIGURE 8. Illustration of a 3-step binomial model.

Consider the stock price $(S_t)_{1\le t\le 3}$ follows the following binomial model in Figure 8. Assume constant interest rate $r = \%4$ for each step. The payoff V_3 of the European option at time $t = 3$ is given in red.

In order to compute its price V_0 at time $t = 0$, we first compute the risk-neutral probabilities:

$$p_1^*(\varnothing) = \frac{(1.04)10 - 6}{15 - 6} = \frac{4.4}{9}, \tag{192}$$

$$p_1^*(H) = \frac{(1.04)15-12}{22-12} = \frac{3.6}{10}, \qquad p_1^*(T) = \frac{(1.04)6-4}{9-4} = \frac{2.24}{5}, \tag{193}$$

$$p_1^*(HH) = \frac{(1.04)22-20}{26-20} = \frac{2.88}{6}, \qquad p_1^*(HT) = \frac{(1.04)12-10}{16-10} = \frac{2.48}{6}, \tag{194}$$

$$p_1^*(TH) = \frac{(1.04)9-7}{13-7} = \frac{2.36}{6}, \qquad p_1^*(TT) = \frac{(1.04)4-2}{7-2} = \frac{2.16}{5}. \tag{195}$$

From these data, we can compute the risk-neutral probability distribution $\mathbb{P}^*$ on the sample space $\Omega = \{H, T\}^3$ as

$$\mathbb{P}^*(\{HHH\}) = p_1^*(\emptyset)p_1^*(H)p_1^*(HH) = \frac{4.4}{9}\cdot\frac{3.6}{10}\cdot\frac{2.88}{6} = 0.0845 \tag{196}$$

$$\mathbb{P}^*(\{HHT\}) = p_1^*(\emptyset)p_1^*(H)p_2^*(HH) = \frac{4.4}{9}\cdot\frac{3.6}{10}\cdot\frac{3.12}{6} = 0.0915 \tag{197}$$

$$\mathbb{P}^*(\{HTH\}) = p_1^*(\emptyset)p_2^*(H)p_1^*(HT) = \frac{4.4}{9}\cdot\frac{6.4}{10}\cdot\frac{2.48}{6} = 0.1293 \tag{198}$$

$$\mathbb{P}^*(\{HTT\}) = p_1^*(\emptyset)p_2^*(H)p_2^*(HT) = \frac{4.4}{9}\cdot\frac{6.4}{10}\cdot\frac{3.52}{6} = 0.1836 \tag{199}$$

$$\mathbb{P}^*(\{THH\}) = p_2^*(\emptyset)p_1^*(T)p_1^*(TH) = \frac{4.6}{9}\cdot\frac{2.24}{5}\cdot\frac{2.36}{6} = 0.0901 \tag{200}$$

$$\mathbb{P}^*(\{THT\}) = p_2^*(\emptyset)p_1^*(T)p_2^*(TH) = \frac{4.6}{9}\cdot\frac{2.24}{5}\cdot\frac{3.64}{6} = 0.1389 \tag{201}$$

$$\mathbb{P}^*(\{TTH\}) = p_2^*(\emptyset)p_2^*(T)p_1^*(TT) = \frac{4.6}{9}\cdot\frac{2.76}{5}\cdot\frac{2.16}{5} = 0.1219 \tag{202}$$

$$\mathbb{P}^*(\{TTT\}) = p_2^*(\emptyset)p_2^*(T)p_2^*(TT) = \frac{4.6}{9}\cdot\frac{2.76}{5}\cdot\frac{2.84}{5} = 0.1603. \tag{203}$$

According to Proposition 3.10, we deduce that

$$V_0 = \frac{1}{(1.04)^3}\mathbb{E}_{\mathbb{P}^*}[V_3] = \frac{1}{(1.04)^3}\begin{pmatrix} 0\cdot 0.0845 + 2\cdot 0.0915 + 0\cdot 0.1293 + 5\cdot 0.1836 \\ +0\cdot 0.0901 + 3\cdot 0.1389 + 3\cdot 0.1219 + 8\cdot 0.1603 \end{pmatrix} = 2.8144. \tag{204}$$

We can also compute all the intermediate values of V_t using the backward recursion in Proposition 3.10 (i). ▲

Exercise 3.12 (Put option)**.** Consider the 3-step binomial model with stock price $(S_t)_{0\le t\le 3}$ as given in Figure 8. Assume constant interest rate $r = \%4$ for each step. Consider a European put option with strike $K = 13$ maturing at $t = 3$, so that is payoff V_3 is given by $V_3 = (13 - S_3)^+$. If we denote its value at time t by V_t, compute $(V_t)_{0\le t\le 3}$ for all corresponding sample paths.

Exercise 3.13 (Bull spread)**.** Consider the 3-step binomial model with stock price $(S_t)_{0\le t\le 3}$ as given in Figure 8. Assume constant interest rate $r = \%4$ for each step. Consider a bull spread consisting of European call options with stike $K_1 = 10$ and $K_2 = 30$, both maturing at $t = 3$. If we denote its value at time t by V_t, compute $(V_t)_{0\le t\le 3}$ for all corresponding sample paths.

Exercise 3.14 (Knockout options)**.** Consider the 3-step binomial model with stock price $(S_t)_{0\le t\le 3}$ as given in Figure 8. Assume constant interest rate $r = \%4$ for each step. Consider a European call with strike $K = 11$ and *knockout barrier* at 5, meaning that the option becomes worthless whenever the stock price drops below 5. If we denote its value at time t by V_t, compute $(V_t)_{0\le t\le 3}$ for all corresponding sample paths.

Next, we consider hedging and replication for the N-step binomial model.

Proposition 3.15. *Consider the N-step binomial model with given stock price $S_0, S_1, \cdots, S_N$ and constant 1-step interest rate r. Consider a European option on this stock wit value $(V_t)_{0\le t\le N}$. Consider the following (dynamic) portfolios:*

portfolio A: Short one European option at time $t = 0$ with maturity N, and for each $k = 0, 1, \cdots, N-1$, hold Δ_k shares of stock during $[k, k+1]$.

portfolio B: Starting from initial wealth W_0, for each $k = 0, 1, \cdots, N-1$,

$$[\Delta_k \text{ shares of stock during } [k, k+1]] + [\text{rest of cash deposited at bank}] \tag{205}$$

Then the following holds.

(i) *Then portfolio A is perfectly hedged at each time $t = 0, 1, \cdots, N$ if and only if for each $k = 0, 1, \cdots, N-1$ and $\omega \in \{H, T\}^k$,*

$$\Delta_k(\omega) = \frac{V_{k+1}(\omega H) - V_{k+1}(\omega T)}{S_{k+1}(\omega H) - S_{k+1}(\omega T)}. \tag{206}$$

(ii) *Let $W_0 = V_0$ and choose $(\Delta_k)_{0\le k<N}$ by (206). Then portfolio B replicates long one European call option.*

PROOF. Fix $0 \le k < N$ and a sample path $\omega \in \{H, T\}^k$ of the first k coin flips. Then portfolio A at time $k+1$ given the market evolution ω up to time k if and only if

$$\Delta_k S_{k+1}(\omega H) - V_{k+1}(\omega H) = \Delta_k S_{k+1}(\omega T) - V_{k+1}(\omega T). \tag{207}$$

This is equivalent to (206). This shows (i).

Let W_k denote the value of portfolio B at time $t = k$. The stochastic process $(W_t)_{0\le t\le N}$ is called the *wealth process*. We will show that if we use the hedging strategy (206) for each $0 \le k < N$, then we have $V_N = W_N$. This is shown by an induction. The base step holds since we take $W_0 = V_0$. For the induction step, suppose we have $W_k = V_k$ for some $0 \le k < N$. Fix any k-step sample path $\omega \in \{H, T\}^k$. Let $x_k(\omega) = W_k(\omega) - \Delta_k S_k(\omega)$ denote the amount of cash holding at time k assuming sample path ω (why?). Then $W_{k+1} = V_{k+1}$ if and only if

$$\begin{cases} \Delta_k S_k(\omega H) + x_k(\omega) \cdot (1+r) = V_{k+1}(\omega H) \\ \Delta_k S_1(\omega T) + x_k(\omega) \cdot (1+r) = V_{k+1}(\omega T). \end{cases} \tag{208}$$

Subtracting these equations gives (207), so $\Delta_k(\omega)$ is given by (206). This completes the induction. $\square$

Exercise 3.16. In Example 3.11, compute the hedge ratio $\Delta_k(\omega)$ for each $0 \le k < N$ and $\omega \in \{H, T\}^k$. Verify that the portfolio A in Proposition 3.15 is indeed perfectly hedged all the time.

4. Conditional expectation and Martingales

4.1. Conditional expectation. Let X, Y be discrete RVs. Recall that the expectation $\mathbb{E}(X)$ is the 'best guess' on the value of X when we do not have any prior knowledge on X. But suppose we have observed that some possibly related RV Y takes value y. What should be our best guess on X, leveraging this added information? This is called the *conditional expectation of X given $Y = y$*, which is defined by

$$\mathbb{E}[X|Y = y] = \sum_x x\mathbb{P}(X = x|Y = y). \tag{209}$$

This best guess on X given $Y = y$, of course, depends on y. So it is a function in y. Now if we do not know what value Y might take, then we omit y and $\mathbb{E}[X|Y]$ becomes a RV, which is called the *conditional expectation of X given Y*.

Exercise 4.1. Let X, Y be discrete RVs. Show that for any function $g : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}_X[Xg(Y)\,|\,Y] = g(Y)E_X[X\,|\,Y]. \tag{210}$$

Exercise 4.2 (Iterated expectation)**.** Let X, Y be discrete RVs. Use Fubini's theorem to show that

$$\mathbb{E}[X] = \mathbb{E}_Y[\mathbb{E}_X[X|Y]]. \tag{211}$$

In order to properly develop our discussion on martingales in the following sections, we need to generalize the notion of conditional expectation $\mathbb{E}[X\,|\,Y]$ of a RV X given another RV Y. Recall that this was the a collection of 'best guesses' of X given $Y=y$ for all y. But what if we only know, say, $Y\ge 1$? Can we condition on this event as well?

More concretely, suppose Y takes values from $\{1,2,3\}$. Regarding Y, the following outcomes are possible:

$$\mathscr{E}_Y := \{\{Y=1\},\{Y=2\},\{Y=3\},\{Y=1,2\},\{Y=2,3\},\{Y=1,3\},\{Y=1,2,3\}\}. \tag{212}$$

For instance, the information $\{Y=1,2\}$ could yield some nontrival implication on the value of X, so our best guess in this scenario should be

$$\mathbb{E}[X\,|\,\{Y=1,2\}] = \sum_x x\mathbb{P}(X=x|\{Y=1,2\}). \tag{213}$$

More generally, for each $A\in\mathscr{E}_Y$, the best guess of X given $A\in\mathscr{E}_Y$ is the following conditional expectation

$$\mathbb{E}[X\,|\,A] = \sum_x x\mathbb{P}(X=x|A). \tag{214}$$

Now, what if we don't know which event in the collection $\mathscr{E}_Y$ to occur? As we did before to define $\mathbb{E}[X\,|\,Y]$ from $\mathbb{E}[X\,|\,Y=y]$ by simply not specifying what value y that Y takes, we simply do not specify which event $A\in\mathscr{E}_A$ to occur. Namely,

$$\mathbb{E}[X\,|\,\mathscr{E}_Y] = \text{best guess on } X \text{ given the information in } \mathscr{E}_Y. \tag{215}$$

In general, this could be defined for any collection of events $\mathscr{E}$ in place of $\mathscr{E}_Y$. Mathematically, we understand $\mathbb{E}[X\,|\,\mathscr{E}]$ as[1]

$$\mathbb{E}[X\,|\,\mathscr{E}] = \text{the collection of } \mathbb{E}[X\,|\,A] \text{ for all } A\in\mathscr{E}. \tag{216}$$

Exercise 4.3 (Jensen's inequality)**.** Let X be any RV with $\mathbb{E}[X]<\infty$. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be any convex function, that is,

$$\varphi(\lambda x+(1-\lambda)y) \le \lambda\varphi(x)+(1-\lambda)(\varphi(y)), \qquad \forall\ \lambda\in[0,1] \text{ and } x,y\in\mathbb{R}. \tag{217}$$

Jensen's inequality states that

$$\varphi(\mathbb{E}[X]) \le \mathbb{E}[\varphi(X)]. \tag{218}$$

(i) Let $c:=\mathbb{E}[X]<\infty$. Show that there exists a line $f(x)=ax+b$ such that $f(c)=\varphi(c)$ and $\varphi(x)\ge f(x)$ for all $x\in\mathbb{R}$.

(ii) Verify the following and prove Jensen's inequality:

$$\mathbb{E}[\varphi(X)] \ge \mathbb{E}[f(X)] = a\mathbb{E}[X]+b = f(c) = \varphi(c) = \varphi(\mathbb{E}[X]). \tag{219}$$

(iii) Let X be RV, A an event, φ be the convex function as before. Show the Jensen's inequality for the conditional expectation:

$$\varphi(\mathbb{E}[X|A]) \le \mathbb{E}[\varphi(X)\,|\,A]. \tag{220}$$

4.2. Definition and examples of martingales. Let $(X_t)_{t\ge0}$ be the sequence of observations of the price of a particular stock over time. Suppose that an investor has a strategy to adjust his portfolio $(M_t)_{t\ge0}$ according to the observation $(X_t)_{t\ge0}$. Namely,

$$M_t = \text{Net value of portfolio after observing } (X_k)_{0\le k\le t}. \tag{221}$$

We are interested in the long-term behavior of the 'portfolio process' $(M_t)_{t\ge0}$. Martingales provide a very nice framework for this purpose.

Martingale is a class of stochastic processes, whose expected increment conditioned on the past is always zero. Recall that the simple symmetric random walk has this property, since each increment is

[1] For more details, see [Dur19].

i.i.d. and has mean zero. Martingales do not assume any kind of independence between increments, but it turns out that we can proceed quite far with just the unbiased conditional increment property.

In order to define martingales properly, we need to introduce the notion of 'information up to time t'. Imagine we are observing the stock market starting from time t. We define

$$\mathscr{E}_t := \text{collection of all possible events we can observe at time } t \tag{222}$$

$$\mathscr{F}_t := \bigcup_{k=1}^{t} \mathscr{E}_k = \text{collection of all possible events we can observe up to time } t. \tag{223}$$

In words, $\mathscr{E}_t$ is the information available at time t and $\mathscr{F}_t$ contains all possible information that we can obtain by observing the market up to time t. We call $\mathscr{F}_t$ the *information* up to time t. As a collection of events, $\mathscr{F}_t$ needs to satisfy the following properties[2]:

(i) (*closed under complementation*) $A \in \mathscr{F}_t \Longrightarrow A^c \in \mathscr{F}_t$;
(ii) (*closed under countable union*) $A_1, A_2, A_3, \cdots \in \mathscr{F}_t \Longrightarrow \bigcup_{k=1}^{\infty} A_k \in \mathscr{F}_t$.

Note that as we gain more and more information, we have

$$\mathscr{F}_s \subseteq \mathscr{F}_t \qquad \forall t \geq s \geq 0. \tag{224}$$

In other words, $(\mathscr{F}_t)_{t\geq 0}$ is an increasing set of information, which we call a *filtration*. The roll of a filtration is to specify what kind of information is observable or not, as time passes by.

Example 4.4. Suppose $(\mathscr{F}_t)_{t\geq 0}$ is a filtration generated by observing the stock price $(X_t)_{t\geq 0}$ of company A in New York. Namely, $\mathscr{E}_t$ consists of the information on the values of the stock price X_t at day t. Given $\mathscr{F}_{10}$, we know the actual values of $X_0, X_1, \cdots, X_{10}$. For instance, X_8 is not random given $\mathscr{F}_{10}$, but X_{11} could still be random. On the other hand, if $(Y_t)_{t\geq 0}$ is the stock price of company B in Hong Kong, then we may have only partial information for $Y_0, \cdots, Y_{10}$ given $\mathscr{F}_t$. ▲

Now we define martingales.

Definition 4.5. Let $(\mathscr{F}_t)_{t\geq 0}$ be a filtration and $(M_t)_{t\geq 0}$ be discrete-time stochastic processes. We call $(M_t)_{t\geq 0}$ a *martingale* with respect to $(\mathscr{F}_t)_{t\geq 0}$ if the following conditions are satisfied: For all $t \geq 0$,

(i) (*finite expectation*) $\mathbb{E}[|M_t|] < \infty$.
(ii) (*measurability*[3]) $\{M_t = m\} \in \mathscr{F}_t$ for all $m \in \mathbb{R}$.
(iii) (*conditional increments*) $\mathbb{E}[M_{t+1} - M_t \,|\, A] = 0$ for all $A \in \mathscr{F}_t$.

When appropriate, we will abbreviate the condition (iii) as

$$\mathbb{E}[M_{t+1} - M_t \,|\, \mathscr{F}_t] = 0. \tag{225}$$

In order to get familiar with martingales, it is helpful to envision them as a kind of simple symmetric random walk. In general, one can subtract off the mean of a given random walk to make it a martingale.

Example 4.6 (Random walks)**.** Let $(X_t)_{t\geq 1}$ be a sequence of i.i.d. increments with $\mathbb{E}[X_i] = \mu < \infty$. Let $S_t = S_0 + X_1 + \cdots + X_t$. Then $(S_t)_{t\geq 0}$ is called a *random walk*. (Think of S_t as the stock price at time t and X_i as the increment of stock price during $[i-1, i]$.) Define a stochastic process $(M_t)_{t\geq 0}$ by

$$M_t = S_t - \mu t. \tag{226}$$

For each $t \geq 0$, let $\mathscr{F}_t$ be the information obtained by observing $S_0, S_1, \cdots, S_t$. Then $(M_t)_{t\geq 0}$ is a martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$. Indeed, we have

$$\mathbb{E}[|M_t|] = \mathbb{E}[|S_t - \mu t|] \leq \mathbb{E}[|S_t| + |\mu t|] = \mathbb{E}[|S_t|] + \mu t < \infty, \tag{227}$$

and for any $m \in \mathbb{R}$,

$$\{M_t = m\} = \{S_t - \mu t = m\} = \{S_t = m + \mu t\} \in \mathscr{F}_t. \tag{228}$$

[2]We are requiring $\mathscr{F}_t$ to be a σ-*algebra*, but we avoid using this terminology.

[3]In this case, we say "M_t is measurable w.r.t. $\mathscr{F}_t$", but we avoid using this terminology.

Furthermore, Since X_{t+1} is independent from $S_0, \cdots, S_t$, it is also independent from any $A \in \mathcal{F}_t$. Hence

$$\mathbb{E}[M_{t+1} - M_t \,|\, A] = \mathbb{E}[X_{t+1} - \mu \,|\, A] \tag{229}$$
$$= \mathbb{E}[X_{t+1} - \mu] = \mathbb{E}[X_{t+1}] - \mu = 0. \tag{230}$$

▲

Example 4.7 (Products of indep. RVs)**.** Let $(X_t)_{t\geq 0}$ be a sequence of independent RVs such that $X_t \geq 0$ and $\mathbb{E}[X_t] = 1$ for all $t \geq 0$. For each $t \geq 0$, let $\mathcal{F}_t$ be the information obtained by observing $M_0, X_0, \cdots, X_t$. Define

$$M_t = M_0 X_1 X_2 \cdots X_t, \tag{231}$$

where M_0 is a constant. Then $(M_t)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$. Indeed, the assumption implies $\mathbb{E}[|M_t|] < \infty$ and that $\{M_t = m\} \in \mathcal{F}_t$ for all $m \in \mathbb{R}$ since M_t is determined by $M_0, X_1 \cdots, X_t$. Furthermore, since X_{t+1} is independent from $X_1, \cdots, X_t$, for each $A \in \mathcal{F}_t$,

$$\mathbb{E}[M_{t+1} - M_t \,|\, A] = \mathbb{E}[M_t X_{t+1} - M_t \,|\, A] \tag{232}$$
$$= \mathbb{E}[(X_{t+1} - 1)(M_0 X_1 \cdots X_t) \,|\, A] \tag{233}$$
$$= \mathbb{E}[X_{t+1} - 1 \,|\, A]\, \mathbb{E}[(M_0 X_1 \cdots X_t) \,|\, A] \tag{234}$$
$$= \mathbb{E}[X_{t+1} - 1]\, \mathbb{E}[(M_0 X_1 \cdots X_t) \,|\, A] = 0. \tag{235}$$

This multiplicative model a reasonable for the stock market since the changes in stock prices are believed to be proportional to the current stock price. Moreover, it also guarantees that the price will stay positive, in comparison to additive models. ▲

Exercise 4.8 (Long range martingale condition)**.** Let $(M_t)_{t\geq 0}$ be a martingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$. For any $0 \leq k < n$, we will show that

$$\mathbb{E}[(M_n - M_k) \,|\, \mathcal{F}_k] = 0. \tag{236}$$

(i) Suppose (236) holds for fixed $0 \leq k < n$. For each $A \in \mathcal{F}_k$, to show that

$$\mathbb{E}[M_{n+1} - M_k \,|\, A] = \mathbb{E}[M_{n+1} - M_n \,|\, A] + \mathbb{E}[M_n - M_k \,|\, A] \tag{237}$$
$$= \mathbb{E}[M_{n+1} - M_n \,|\, A] = 0. \tag{238}$$

(ii) Conclude (236) for all $0 \leq k < n$ by induction.

4.3. The N-step binomial model revisited. In this subsection, we revisit the N-step binomial model in the framework of martingales. First recall the model. Staring at the current time $t = 0$, we flip N coins at times $t = 1, 2, \cdots, N$ to determine the market evolution. The sample space of the outcomes is $\Omega = \{H, T\}^N$, which consists of sequences of length N strings of H's or T's. We assume constant interest rate for each periods $[k, k+1]$ for $k = 0, 1, \cdots, N-1$.

Let $\mathcal{F}_t$ denote the information that we can obtain by observing the market up to time t. For instance, $\mathcal{F}_t$ contains the information of the first t coin flips, stock prices $S_0, S_1, \cdots, S_t$, European option values $V_0, V_1, \cdots, V_t$, and so on. Then $(\mathcal{F}_t)_{t\geq 0}$ defines a natural filtration for the N-step binomial model. Below we reformulate Proposition 3.10.

Proposition 4.9. *Consider the N-step binomial model as above. Let $\mathbb{P}^*$ denote the risk-neutral probability measure defined in Proposition 3.10. Consider a European option on this stock wit value $(V_t)_{0\leq t\leq N}$.*

(i) *The process $\left((1+r)^{-t} V_t\right)_{0\leq t\leq N}$ forms a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ under the risk-neutral probability measure $\mathbb{P}^*$. That is,*

$$\mathbb{E}_{\mathbb{P}^*}\left[(1+r)^{-(t+1)} V_{t+1} - (1+r)^{-t} V_t \,\middle|\, \mathcal{F}_t\right] = 0, \tag{239}$$

which is also equivalent to

$$V_t = \frac{1}{1+r} \mathbb{E}_{\mathbb{P}^*}\left[V_{t+1} \,\middle|\, \mathcal{F}_t\right] = 0. \tag{240}$$

(ii) *We have*

$$V_0 = \mathbb{E}_{\mathbb{P}^*}\left[(1+r)^{-N} V_N\right]. \tag{241}$$

PROOF. To show (i), we first note that, conditioning on the information $\mathscr{F}_t$ up to time t, we know all the coin flips, stock prices, and European option values up to time t. Hence we have

$$\mathbb{E}_{\mathbb{P}^*}\left[(1+r)^{-(t+1)} V_{t+1} \,\middle|\, \mathscr{F}_t\right] = (1+r)^{-(t+1)} \mathbb{E}_{\mathbb{P}^*}\left[V_{t+1} \,\middle|\, \mathscr{F}_t\right] \tag{242}$$

$$= (1+r)^{-(t+1)} \cdot (1+r) V_t = (1+r)^{-t} V_t. \tag{243}$$

This shows that $(1+r)^{-t} V_t$ is a martingale with respect to the natural filtration $(\mathscr{F}_t)_{t\geq 0}$ under the risk-neutral probability measure $\mathbb{P}^*$.

Now (ii) follows from (i) and Exercise 4.8. Namely, for each $A \in \mathscr{F}_0$, by Exercise 4.8,

$$\mathbb{E}_{\mathbb{P}^*}[(1+r)^{-N} V_N - (1+r)^0 V_0 \,|\, \mathscr{F}_0] = 0. \tag{244}$$

This yields

$$V_0 = \mathbb{E}_{\mathbb{P}^*}[V_0 \,|\, \mathscr{F}_0] = \mathbb{E}_{\mathbb{P}^*}[(1+r)^{-N} V_N \,|\, \mathscr{F}_0], \tag{245}$$

as desired. □

Exercise 4.10 (Risk-neutral probabilities make the stock price a martingale)**.** Consider the N-step binomial model with stock price $(S_t)_{0\leq t\leq N}$. Let $\mathbb{P}^*$ denote the risk-neutral probability measure defined in Proposition 3.10 in Lecture note 2. 3.10.

(i) Show that the discounted stock price $(1+r)^{-t} S_t$ forms a martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ under the risk-neutral probability measure $\mathbb{P}^*$. That is,

$$\mathbb{E}_{\mathbb{P}^*}\left[(1+r)^{-(t+1)} S_{t+1} - (1+r)^{-t} S_t \,\middle|\, \mathscr{F}_t\right] = 0, \tag{246}$$

which is also equivalent to

$$S_t = \frac{1}{1+r} \mathbb{E}_{\mathbb{P}^*}\left[S_{t+1} \,\middle|\, \mathscr{F}_t\right] = 0. \tag{247}$$

(Hint: Use the fundamental thm of asset pricing.)

(ii) Show that

$$S_0 = \mathbb{E}_{\mathbb{P}^*}\left[(1+r)^{-N} S_N\right]. \tag{248}$$

5. Pricing American options

In this section, we consider pricing general path-dependent American options under the N-step binomial model.

5.1. Definition and examples of American options. Denote the stock price by $(S_t)_{0\leq t\leq N}$ and assume constant interest rate r. We will be considering a general American option on this stock such that the payoff at time k if exercised is a function on both the stock price S_k and the sample path $\omega \in \{H, T\}^k$ that the market takes up to time k. Namely, we fix a nonnegative function g so that

$$g_k(\omega) = \text{payoff of the American option at time } k \text{ assuming time evolution } \omega. \tag{249}$$

Proposition 5.1. *Consider the path-dependent American option as above. For each sample path ω of length $< N$, let $(p_1^*(\omega), p_2^*(\omega))$ denote the risk neutral probabilities at node ω in the binomial tree. Then the value $(V_k(\omega))_{k,\omega}$ of this American option satisfies the following recursion*

$$V_k(\omega) = \max\left\{ g_k(\omega), \frac{V_{k+1}(\omega H) p_1^*(\omega) + V_{k+1}(\omega T) p_2^*(\omega)}{1+r} \right\}. \tag{250}$$

PROOF. Given a sample path ω of length k, the payoff of the American option at time k, if exercised, is $g_k(\omega)$ by definition. On the other hand, if not exercised, then we can regard the American option as a 1-step European option during $[k, k+1]$ with payoff $V_{k+1}(\omega H)$ and $V_{k+1}(\omega T)$, depending on whether the market goes up or down from ω. Hence the value at time k in this case equals the discounted risk-neutral expectation of $V_{k+1}(\omega *)$, which is the second term in the maximum in the assertion. Since one can decide to exercise or not at time k by comparing these two values, the stated recursion follows. □

Example 5.2 (American put option)**.** Consider an American put option on a stock with price S_t, which allows one to buy one share of the stock at times $t = 0, 1, 2, 3$ for a fixed strike price $K = 11$. Hence the payoff is given by

$$g_k(\omega) = (11 - S_k(\omega))^+. \tag{251}$$

Consider the stock price $(S_t)_{1\le t\le 3}$ follows the following binomial model in Figure 11. Assume constant interest rate $r = \%4$ for each step.

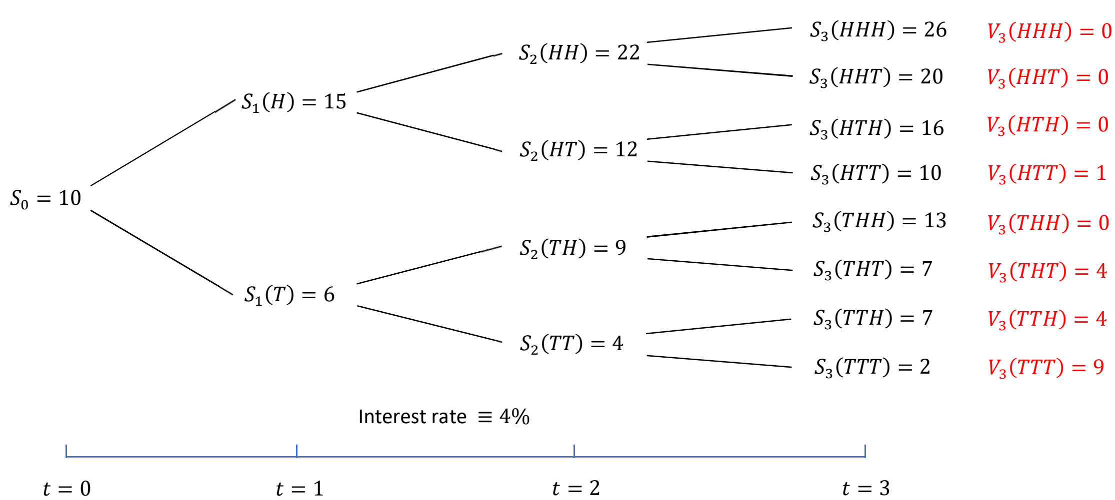

FIGURE 9. Illustration of a 3-step binomial model and an American put option with strike $K = 11$.

First we have $V_3(\omega) = (11 - S_3(\omega))^+$, so

$$V_3(HHH) = 0, \quad V_3(HHT) = 0, \quad V_3(HTH) = 0, \quad V_3(HTT) = 1, \tag{252}$$

$$V_3(THH) = 0, \quad V_3(THT) = 4 \quad V_3(TTH) = 4 \quad V_3(TTT) = 9. \tag{253}$$

Recall the risk-neutral probabilities that makes the discounted stock price $(1+r)^{-t}S_t$ a martingale:

$$p_1^*(\emptyset) = \frac{(1.04)10 - 6}{15 - 6} = \frac{4.4}{9}, \tag{254}$$

$$p_1^*(H) = \frac{(1.04)15 - 12}{22 - 12} = \frac{3.6}{10}, \qquad p_1^*(T) = \frac{(1.04)6 - 4}{9 - 4} = \frac{2.24}{5}, \tag{255}$$

$$p_1^*(HH) = \frac{(1.04)22 - 20}{26 - 20} = \frac{2.88}{6}, \qquad p_1^*(HT) = \frac{(1.04)12 - 10}{16 - 10} = \frac{2.48}{4}, \tag{256}$$

$$p_1^*(TH) = \frac{(1.04)9 - 7}{13 - 7} = \frac{2.36}{6}, \qquad p_1^*(TT) = \frac{(1.04)4 - 2}{7 - 2} = \frac{2.16}{5}. \tag{257}$$

From these data, we can compute V_2 as

$$V_2(HH) = \max\left\{(11 - 22)^+, \frac{0 \cdot (2.88/6) + 0 \cdot (3.12/6)}{1.04}\right\} = \max\{0, 0\} = 0 \tag{258}$$

$$V_2(HT) = \max\left\{(11 - 12)^+, \frac{0 \cdot (2.48/4) + 1 \cdot (1.52/4)}{1.04}\right\} = \max\{0,\ 0.3653\} = 0.3653 \tag{259}$$

$$V_2(TH) = \max\left\{(11-9)^+, \frac{0\cdot(2.36/6)+4\cdot(3.64/6)}{1.04}\right\} = \max\{2, 2.3333\} = 2.3333 \tag{260}$$
$$V_2(TT) = \max\left\{(11-4)^+, \frac{4\cdot(2.16/5)+9\cdot(2.84/5)}{1.04}\right\} = \max\{7, 6.5769\} = 7. \tag{261}$$

For V_1, we have

$$V_1(H) = \max\left\{(11-15)^+, \frac{0\cdot(3.6/10)+0.3653\cdot(6.4/10)}{1.04}\right\} = \max\{0, 0.2248\} = 0.2248 \tag{262}$$
$$V_1(T) = \max\left\{(11-6)^+, \frac{2.3333\cdot(2.24/5)+7\cdot(2.76/5)}{1.04}\right\} = \max\{5, 4.7204\} = 5. \tag{263}$$

Finally, we have the initial value V_0 as

$$V_0 = \max\left\{(11-10)^+, \frac{0.2248\cdot(4.4/9)+5\cdot(4.6/9)}{1.04}\right\} = \max\{1, 2.5629\} = 2.5629. \tag{264}$$

This computes the value of the European put option as well as and the optimal strategy: Stop or continue at each node depending on which value is larger. Observe that when we compute $V_2(TT)$ and $V_1(T)$, it is beneficial to exercise at that point rather than continuing. Hence in this case this American put option has strictly larger value than that of the corresponding European version. ▲

Exercise 5.3. In Example 5.2, compute the price V_0 of the European put option with the same payoff. Verify that the American version of the put option indeed has strictly larger value than the European version.

Example 5.4 (American call option)**.** Consider an American call option on a stock with price S_t, which allows one to buy one share of the stock at times $t = 0,1,2,3$ for a fixed strike price $K = 11$. Hence the payoff is given by

$$g_k(\omega) = (S_k(\omega) - 11)^+. \tag{265}$$

Consider the stock price $(S_t)_{1\le t\le 3}$ follows the following binomial model in Figure 11. Assume constant interest rate $r = \%4$ for each step.

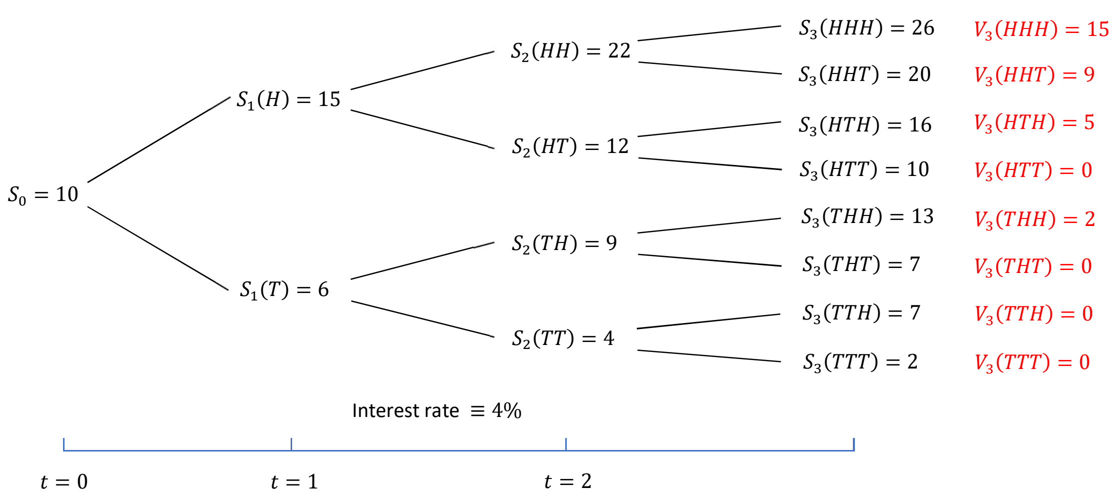

FIGURE 10. Illustration of a 3-step binomial model and an American call option with strike $K = 11$.

First we have $V_3(\omega) = (S_3(\omega) - 11)^+$, so

$$V_3(HHH) = 15, \quad V_3(HHT) = 9, \quad V_3(HTH) = 5, \quad V_3(HTT) = 0, \tag{266}$$
$$V_3(THH) = 2, \quad V_3(THT) = V_3(TTH) = V_3(TTT) = 0. \tag{267}$$

We use the same risk-neutral probabilities we computed in Example 5.2. From these data, we can compute V_2 as

$$V_2(HH) = \max\left\{(22-11)^+, \frac{15\cdot(2.88/6)+9\cdot(3.12/6)}{1.04}\right\} = 11.4230 \tag{268}$$

$$V_2(HT) = \max\left\{(12-11)^+, \frac{5\cdot(2.48/4)+0\cdot(1.52/4)}{1.04}\right\} = 2.9808 \tag{269}$$

$$V_2(TH) = \max\left\{(9-11)^+, \frac{2\cdot(2.36/6)+0\cdot(3.64/6)}{1.04}\right\} = 0.7564 \tag{270}$$

$$V_2(TT) = \max\left\{(4-11)^+, \frac{0\cdot(2.16/5)+0\cdot(2.84/5)}{1.04}\right\} = 0 \tag{271}$$

For V_1, we have

$$V_1(H) = \max\left\{(15-11)^+, \frac{11.4230\cdot(3.6/10)+2.9808\cdot(6.4/10)}{1.04}\right\} = 5.7884 \tag{272}$$

$$V_1(T) = \max\left\{(6-11)^+, \frac{0.7564\cdot(2.24/5)+0\cdot(2.76/5)}{1.04}\right\} = 0.3258. \tag{273}$$

Finally, we have the initial value V_0 as

$$V_0 = \max\left\{(10-11)^+, \frac{5.7884\cdot(4.4/9)+0.3258\cdot(4.6/9)}{1.04}\right\} = 2.8816. \tag{274}$$

Observe that when we compute V_0, V_1, V_2, not exercising and the option always yield a better value. Hence the optimal strategy of using this American call is to wait until the end. Consequently, the price V_0 we have obtained from this American option is the same as that for the corresponding European option. In Exercise 8.5 in Lecture note 1, we have seen that the American call and European call on a stock without dividend have the same value. This was essentially since it is optimal to not exercise the American option until the end, as we can see from this example. ▲

5.2. Optimal strategies and stopping times for American options. In Examples 5.2 and 5.4, we have seen optimal strategies to exercise the given American option. Note that the decision for stopping or exercising an American option at time $t = n$ is based on the information available by that time. To capture this concept more properly, we introduce stopping times.

Definition 5.5 (Stopping time)**.** A random variable τ taking values from $\{1,2,\cdots\}\cup\{\infty\}$ is a *stopping time* with respect to a given filtration $(\mathcal{F}_t)_{t\geq 0}$ if

$$\{\tau = n\} \in \mathcal{F}_n \qquad \forall\, n \in \{1,2,\cdots\}. \tag{275}$$

In other words, think of τ as an algorithm (or decision tree) that tells us to hold or exercise a given American option at each node in the binomial tree.

In the example below, we will see that once we fix a stopping time τ for an American option, then we can compute its value under τ just like in the European case using reverse recursion.

Example 5.6 (American put under a fixed stopping time)**.** Consider the same American put option on a stock with price S_t, which allows one to buy one share of the stock at times $t = 0,1,2,3$ for a fixed strike price $K = 11$. The underlying stock price $(S_t)_{1\leq t\leq 3}$ follows the same 3-step binomial model. Assume constant interest rate $r = \%4$ for each step.

Now suppose we use the following algorithm for this American put:

$$\begin{cases} \text{Exercise at time } t=3 & \text{if the first coin is } H \\ \text{Exercise at time } t=1 & \text{if the first coin is } T. \end{cases} \tag{276}$$

This algorithm can be represented as a stopping time τ where

$$\tau(HHH) = \tau(HHT) = \tau(HTH) = \tau(HTT) = 3 \tag{277}$$

$$\tau(THH) = \tau(THT) = \tau(TTH) = \tau(TTT) = 1. \tag{278}$$

This is indeed a stopping time since

$$\text{At time 0:}\{\tau \neq 1\} \in \mathscr{F}_0 \tag{279}$$
$$\text{At time 1:}\{\tau = 1\} = \{H\}, \{\tau \neq 1\} = \{T\} \in \mathscr{F}_1 \tag{280}$$
$$\text{At time 2:}\{\tau \neq 2\} = \{TT, HT, TH, TT\} \in \mathscr{F}_2 \tag{281}$$
$$\text{At time 3:}\{\tau = 3\} = \{HHH, HHT, HTH, HTT\}, \{\tau = \infty\} = \{THH, THT, TTH, TTT\} \in \mathscr{F}_2 \tag{282}$$

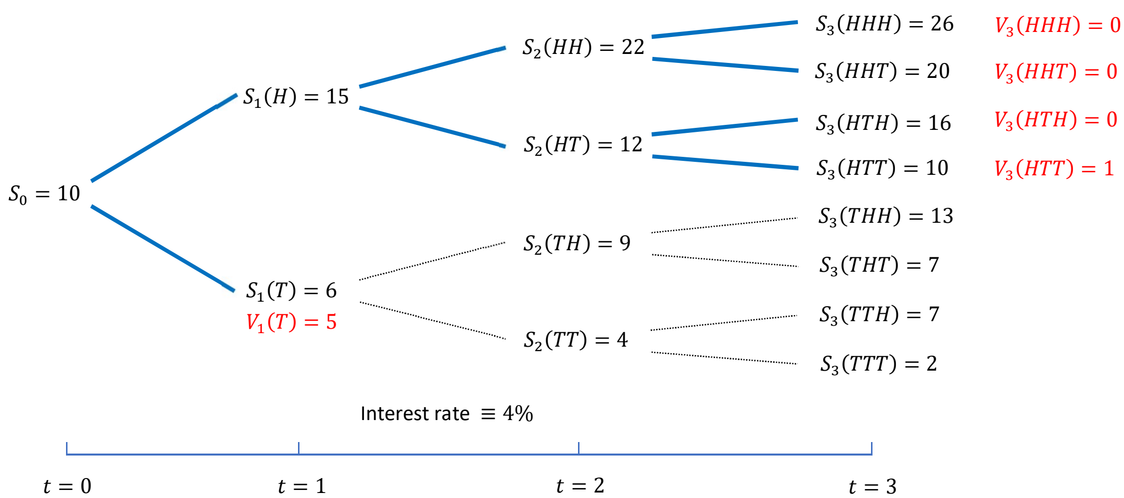

FIGURE 11. Illustration of a 3-step binomial model and an American put option with strike $K = 11$. Bold blue lines represent the decision tree (stopping time) corresponding to the algorithm (276).

Under this stopping time τ, the time that we exercise this American put is given, so we can use the same recursion for the European options to compute V_0. Namely, for each $0 \le k \le 3$ and a sample path $\omega \in \{H, T\}^k$, denote the value of this American put under the stopping time τ as $V_k(\omega \,|\, \tau)$. Then the value at exercise is given by

$$V_3(HHH \,|\, \tau) = (11 - 26)^+ = 0 \tag{283}$$
$$V_3(HHT \,|\, \tau) = (11 - 20)^+ = 0 \tag{284}$$
$$V_3(HTH \,|\, \tau) = (11 - 16)^+ = 0 \tag{285}$$
$$V_3(HTT \,|\, \tau) = (11 - 10)^+ = 1 \tag{286}$$
$$V_1(T \,|\, \tau) = (11 - 6)^+ = 5 \tag{287}$$

From these values at the end of the decision tree given by τ, we can use the recursively compute the values under τ at prior times:

$$V_2(HH \,|\, \tau) = \frac{1}{1+r}\mathbb{E}_{\mathbb{P}^*}[V_3 \,|\, HH] = \frac{0 \cdot (2.88/6) + 0 \cdot (3.12/6)}{1.04} = 0 \tag{288}$$
$$V_2(HT \,|\, \tau) = \frac{1}{1+r}\mathbb{E}_{\mathbb{P}^*}[V_3 \,|\, HT] = \frac{0 \cdot (2.48/4) + 1 \cdot (1.52/4)}{1.04} = 0.3653 \tag{289}$$
$$V_1(H \,|\, \tau) = \frac{1}{1+r}\mathbb{E}_{\mathbb{P}^*}[V_1 \,|\, H] = \frac{0 \cdot (3.6/10) + 0.3653 \cdot (6.4/10)}{1.04} = 0.2248. \tag{290}$$

Finally, we have the initial value $V_0(\emptyset \,|\, \tau)$ as

$$V_0(\emptyset \,|\, \tau) = \frac{1}{1+r}\mathbb{E}_{\mathbb{P}^*}[V_1 \,|\, \emptyset] = \frac{0.2248 \cdot (4.4/9) + 5 \cdot (4.6/9)}{1.04} = 2.5629. \tag{291}$$

Notice that we can also compute $V_0(\varnothing \,|\, \tau)$ as

$$V_0(\varnothing \,|\, \tau) = \mathbb{E}_{\mathbb{P}^*}\left[\frac{(11-S_\tau)^+}{(1+r)^\tau}\right] \tag{292}$$

$$= \frac{5}{(1+r)^1}\mathbb{P}^*(\{T\}) + \frac{0}{(1+r)^3}\mathbb{P}^*(\{HHH\}) + \frac{0}{(1+r)^3}\mathbb{P}^*(\{HHT\}) \tag{293}$$

$$+ \frac{0}{(1+r)^3}\mathbb{P}^*(\{HTH\}) + \frac{1}{(1+r)^3}\mathbb{P}^*(\{HTT\}) \tag{294}$$

$$= 5 \cdot p_2^*(\varnothing) + 1 \cdot p_1^*(\varnothing)\, p_2^*(H)\, p_2^*(HTT) \tag{295}$$

$$= 5 \cdot \frac{4.6}{9} + 1 \cdot \frac{4.4}{9} \cdot \frac{6.4}{10} \cdot \frac{1.52}{4} = 2.5629. \tag{296}$$

Also note that $V_0(\varnothing \,|\, \tau)$ equals the (unconditional) value V_0 of the same American put that we computed in Example 5.2. This is since the stopping time τ we used here agrees with the optimal strategy. ▲

The observation in Example 5.6 leads us to the following general result.

Theorem 5.7. *Consider the N-step binomial model as before and an American option with payoff $g_k(\omega)$ for each sample path ω of length k. Then*

$$V_0 = \max_\tau \mathbb{E}_{\mathbb{P}^*}\left[\frac{g_\tau}{(1+r)^\tau}\right], \tag{297}$$

where the maximum is over all stopping times τ with respect to the filtration $(\mathcal{F}_t)_{0\le t\ge N}$ of the N-step binomial model and $\mathbb{P}^*$ is the risk-neutral probability measure.

PROOF. As before, for each $0 \le k \le N$ and a sample path $\omega \in \{H,T\}^k$, denote the value of this American option under the stopping time τ as $V_k(\omega \,|\, \tau)$. Then the value V_0 of the American option is the maximum of the values $V_0(\varnothing \,|\, \tau)$ we can obtain from all possible stopping times. Hence

$$V_0 = \max_\tau V_0(\varnothing \,|\, \tau). \tag{298}$$

On the other hand, since the exercise time is determined under τ, we can write

$$V_0(\varnothing \,|\, \tau) = \mathbb{E}_{\mathbb{P}^*}\left[\frac{g_\tau}{(1+r)^\tau}\right]. \tag{299}$$

Combining these two equations gives the assertion. □

Exercise 5.8 (A 2-step American option)**.** Consider an American option on a stock with price S_t, which evolves according to the 2-step binomial model with constant interest rate $r = 5\%$ given in Figure 12 (left). The American option is defined by the payoff function $g_k(\omega)$ given in Figure 12 (right).

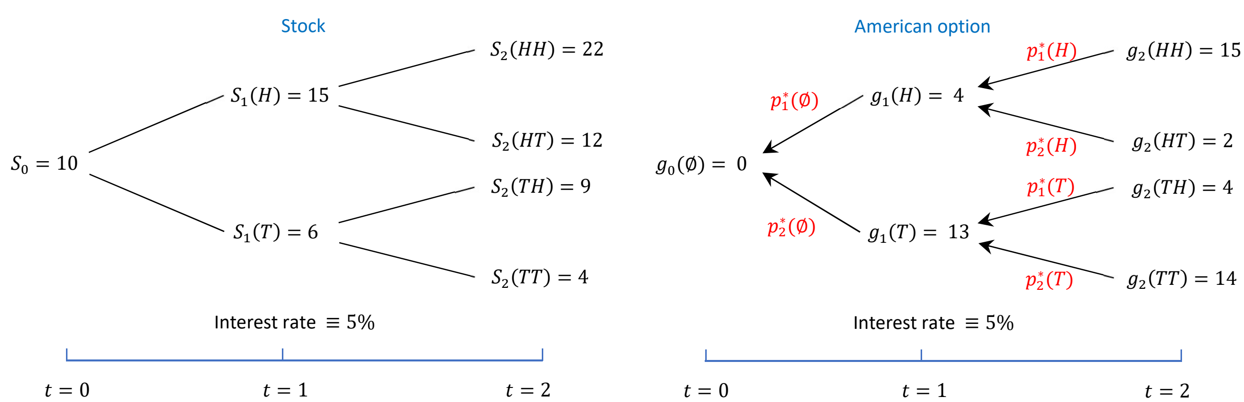

FIGURE 12. A 2-step American option with payoff function $g_k(\omega)$ as given in the right.

(i) Compute the value V_0 of this American option using the recursive formula in Proposition 5.1.
(ii) Write down all possible stopping time τ for this model (there are 5 of them).
(iii) Compute the value $V_0(|\tau)$ of this American option under each stopping time τ. Verify that the maximum value among all τ agrees the result of (i).

Recall that when stock pays no dividend, an American call with payoff $(S_t - K)^+$ is optimal to wait until the end, so it has the same value as its European counterpart. The following theorem generalizes this into more general payoff functions.

Theorem 5.9. *Consider the N-step binomial model as before and an American option with payoff $g(S_k(\omega))$ for each sample path ω of length k. Suppose g is a nonnegative convex function with $g(0) = 0$. Then it is optimal to wait until the end to exercise.*

PROOF. Fix $0 \le k < N$ and a sample path $\omega \in \{H, T\}^k$. Recall that since g is convex, for each $\lambda \in [0,1]$ and $x, y \in \mathbb{R}$, we have

$$g(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda) g(y). \tag{300}$$

By setting $y = 0$ and using the fact that $g(0) = 0$, we have

$$g(\lambda x) \le \lambda g(x) + (1-\lambda) g(0) = \lambda g(x). \tag{301}$$

Also recall that the discounted stock price $(1+r)^{-t} S_t$ is a martingale under $\mathbb{P}^*$ (Exc. 4.10), so by Jensen's inequality (see Exc. 4.3) and the above observation,

$$g\left(S_k(\omega)\right) = g\left(\mathbb{E}_{\mathbb{P}^*}\left[\frac{S_k}{1+r}\,\middle|\,\omega\right]\right) \le \mathbb{E}_{\mathbb{P}^*}\left[g\left(\frac{S_k}{1+r}\right)\,\middle|\,\omega\right] \le \mathbb{E}_{\mathbb{P}^*}\left[\frac{g\left(S_k\right)}{1+r}\,\middle|\,\omega\right]. \tag{302}$$

Note that, if we do not exercise at node ω, then at time $k+1$, we can get at least the values $g(S_k(\omega H))$ or $g(S_k(\omega T))$ (depending on the $k+1$st coin flip) by exercising at time $k+1$. Hence

$$V_{k+1}(\omega H) \ge g(S_k(\omega H)), \qquad V_{k+1}(\omega T) \ge g(S_k(\omega T)). \tag{303}$$

It follows that

$$V_k(\omega\,|\,\text{hold at time } k) = \mathbb{E}_{\mathbb{P}^*}\left[\frac{V_{k+1}}{1+r}\,\middle|\,\omega\right] \ge \mathbb{E}_{\mathbb{P}^*}\left[\frac{g\left(S_k\right)}{1+r}\,\middle|\,\omega\right] \tag{304}$$

$$\ge g(S_k(\omega)) \tag{305}$$

$$= V_k(\omega\,|\,\text{exercise at time } k). \tag{306}$$

Thus at time t given the sample path ω, it is better to continue than to exercise. This shows the assertion. □

Remark 5.10. For an American put with strike $K > 0$, the payoff function g is $g(S_k) = (K - S_k)^+$. This is a convex function, but $g(0) = (K-0)^+ = K \ne 0$. Hence American put does not satisfy the hypothesis of Theorem 5.9. Indeed, in Examples 5.2 and 5.6, we have seen that one can get strictly higher value from an American put by exercising early.

6. Continuous-time limit and Black-Scholes equation

6.1. Continuous-time limit of the binomial model with i.i.d. coin flips. Consider modeling the evolution of stock price during an year. If one measures the stock price every month, then maybe one can try to use the N-step binomial model with $N = 12$; If one uses daily measurements, then one should use $N = 365$; One can use hourly data to go with $N = 8760$, and so on. The question is, what happens if we keep subdividing a given time interval and use finer binomial model? Can we describe the 'limiting model' in some sense?

In this subsection, we consider the binomial model as a discrete model for the continuously evolving stock price, and obtain a continuous-time model by take a limit binomial model as the physical duration of each step goes to zero. This will be the basis of Black-Scholes formula that we will derive in the next subsection.

Suppose there is a stock with price S_T at maturity T. Let S_0 denote the initial stock price, and introduce μ and $\sigma > 0$ by

$$\mathbb{E}\left[\log\left(\frac{S_T}{S_0}\right)\right] = \mu T, \qquad \mathrm{Var}\left(\log\left(\frac{S_T}{S_0}\right)\right) = \sigma^2 T, \tag{307}$$

where the expectation and variance are taken under the physical probability measure on the coin flips. We would like to model the time evolution of this stock price using an N-step binomial model where each step has duration $h = T/N$.

Recall that complete market evolution of the N-step binomial model is determined by the N coin flips, that is, a length N string of H's and T's. We denote the stock price at step k with sample path $\omega \in \{H, T\}^k$ by $S_{kh}(\omega)$. In order to take a continuous-time limit of the binomial model, we make the following assumptions on the general model:

(a) (*constant up- and down-factors*) There exists constants $u, d > 0$ such that

$$S_{(k+1)h}(\omega H) = S_{kh}(\omega)\, u, \qquad S_{(k+1)h}(\omega T) = S_{kh}(\omega)\, d \tag{308}$$

for all $0 \le k < N$ and $\omega \in \{H, T\}^k$.

(b) (*i.i.d. coin flips*) The underlying (physical) coin flips are independent and identically distributed.

Both (a) and (b) are are not realistic assumptions, but they allow us to take a continuous-time limit of the model and obtain stronger results.

To begin, we introduce a the *logarithmic return*

$$X_n = \log\left(\frac{S_{nh}}{S_{(n-1)h}}\right) = \begin{cases} \log u & \text{if the } n\text{th coin lands heads} \\ \log d & \text{if the } n\text{th coin lands tails,} \end{cases} \tag{309}$$

where the second equality follows from the assumption (b). Moreover, by (c), we see that $(X_n)_{n\ge 1}$ is a sequence of i.i.d. RVs. Then observe that

$$\log\left(\frac{S_{nh}}{S_0}\right) = \log\left(\frac{S_h}{S_0} \cdot \frac{S_{2h}}{S_h} \cdots \frac{S_{(n-1)h}}{S_{(n-2)h}} \cdot \frac{S_{nh}}{S_{(n-1)h}}\right) \tag{310}$$

$$= \log\left(\frac{S_h}{S_0}\right) + \log\left(\frac{S_{2h}}{S_h}\right) + \cdots + \log\left(\frac{S_{nh}}{S_{(n-1)h}}\right) \tag{311}$$

$$= X_1 + X_2 + \cdots + X_n. \tag{312}$$

Note that the last expression is a sum of i.i.d. RVs, so we may apply the standard limit theorems such as the law of large numbers (LLN) and the central limit theorem (CLT). Essentially, the passage to continuous-time limit is provided by applying the CLT.

Theorem 6.1 (CLT)**.** *Let $(X_k)_{k\ge 1}$ be i.i.d. RVs and let $S_n = \sum_{k=1}^n X_i$, $n \ge 1$ be a random walk. Suppose $\mathbb{E}[X_1] = \mu < \infty$ and $\mathbb{E}[X_1^2] = \sigma^2 < \infty$. Let $Z \sim N(0,1)$ be a standard normal RV and define*

$$Z_n = \frac{S_n - \mu n}{\sigma\sqrt{n}} = \frac{S_n/n - \mu}{\sigma/\sqrt{n}}. \tag{313}$$

Then $Z_n \Rightarrow Z \sim N(0,1)$ as $n \to \infty$. That is, for all $z \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}(Z_n \le z) = \mathbb{P}(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx. \tag{314}$$

Proposition 6.2. *Consider the N-step binomial model for the period $[0, N]$ under the assumptions (a) and (b). Then as $N \to \infty$,*

$$S_T \Longrightarrow S_0 \exp\left(\mu T + \sigma\sqrt{T} Z\right), \tag{315}$$

where $\Longrightarrow$ denotes convergence in distribution.

PROOF. First note that

$$\log\left(\frac{S_T}{S_0}\right) = \log\left(\frac{S_{Nh}}{S_0}\right) = X_1 + \cdots + X_N. \tag{316}$$

Using (307) and since X_i's are i.i.d., we must have

$$\mu T = \mathbb{E}[X_1 + \cdots + X_N] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_N] = N\mathbb{E}[X_1] \tag{317}$$

$$\sigma^2 T = \mathrm{Var}(X_1 + \cdots + X_N) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_N) = N\mathrm{Var}(X_1). \tag{318}$$

Note that the second equality in the first line follows from the linearity of expectation, whereas the second equality in the second line uses the independence of X_i's (otherwise there will be covariance terms). This shows that we can apply CLT to the sum $X_1 + \cdots + X_N$. Namely, it gives

$$\frac{(\sum_{i=1}^N X_i) - \mu T}{\sigma\sqrt{T}} \Longrightarrow Z \sim N(0,1), \tag{319}$$

where $\Longrightarrow$ denotes convergence in distribution. In other words, as $N \to \infty$,

$$\left(\sum_{i=1}^N X_i\right) \Longrightarrow \mu T + \sigma\sqrt{T} Z. \tag{320}$$

Again using the relation $\log(S_N/S_0) = X_1 + \cdots + X_N$, $S_T = S_{Nh}$, this shows

$$\frac{S_T}{S_0} \Longrightarrow \exp\left(\mu T + \sigma\sqrt{T} Z\right) \tag{321}$$

as $N \to \infty$. This shows the assertion. □

Note that the N-step binomial model was simply a mathematical framework to model the stock price $(S_t)_{0\le t\le T}$. Hence Proposition 6.2 suggests that we may take the limit $N \to \infty$ and simply write

$$S_T = S_0 \exp\left(\mu T + \sigma\sqrt{T} Z\right). \tag{322}$$

Of course, the validity of the above stock pricing model would depend on that of the assumptions (a) and (b) we made.

In order to extend the continuous-time stock pricing formula (322) at maturity T to all times $0 \le t \le T$, we introduce the Brownian motion (a.k.a. Wiener process).

Definition 6.3 (Brownian motion)**.** A continuous-time stochastic process $(B(t))_{0\le t\le T}$ is called a *Brownian motion* if $B(0) = 0$ and it satisfies the following conditions:

(i) (*independent increments*) Whenever $0 = t_0 < t_1 < \cdots < t_k \le T$,

$$B(t_1) - B(t_0),\ B(t_2) - B(t_1),\ \cdots,\ B(t_k) - B(t_{k-1}) \quad \text{are independent.} \tag{323}$$

(ii) (*stationary increment*) The distribution of $B_t - B_s$ for each $s < t$ is $N(0, t-s)$.
(iii) (*continuity*) The map $t \mapsto B(t)$ is continuous.

Proposition 6.4. *Consider the N-step binomial model for the period $[0, N]$ under the assumptions (a) and (b). Linearly interpolate the stock prices $S_0, S_h, \cdots, S_{hN}$ and denote the resulting continuous-time stock price as $(S_{t;N})_{0\le t\le T}$. Then there is a Brownian motion $(B(t))_{0\le t\le T}$ such that for each $0 \le t \le T$,*

$$S_{t;N} \Longrightarrow S_0 \exp\left(\mu t + \sigma B_t\right) \tag{324}$$

as $N \to \infty$.

PROOF. Proof uses Donsker's theorem, which is a 'process-level' CLT for the random walk $\sum_{i=1}^n X_i$. In contrast that CLT shows the last random walk location $\sum_{i=1}^n X_i$ is asymptotically distributioned as a standard normal RV after standardization, Donsker's theorem states that the entire sample path of the random walk from step 0 to step N, after linear interpolation, 'converges' to the sample path of a Brownian motion. Details are beyond the scope of this course and is omitted. □

Definition 6.5. The parameters μ and $\sigma > 0$ in Proposition 6.4 are called the *exponential growth rate* and the *volatility* of the stock.

Example 6.6 (MGF of standard normal RV)**.** Given a random variable X, its *moment generating function* (MGF) is defined by the function $t \mapsto \mathbb{E}[e^{tX}]$. A RV X is said to follow the normal distribution $N(\mu, \sigma^2)$ if it has the following probability distribution function (PDF)

$$f_X(x) = \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \tag{325}$$

Furthermore we say X is a standard normal RV if $X \sim N(0,1)$. Below we will show that the MGF of a standard normal RV $Z \sim N(0,1)$ is given by

$$\mathbb{E}[e^{tZ}] = e^{t^2/2}. \tag{326}$$

To see this, we first note

$$\mathbb{E}[e^{tZ}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2+tx}\, dx. \tag{327}$$

By completing square, we can write

$$-\frac{x^2}{2} + tx = -\frac{1}{2}(x^2 - 2tx) = \frac{1}{2}(x-t)^2 + \frac{t^2}{2}. \tag{328}$$

So we get

$$\mathbb{E}[e^{tZ}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2} e^{t^2/2}\, dx = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-t)^2/2}\, dx. \tag{329}$$

Notice that the integrand in the last expression is the PDF of a normal RV with distribution $N(t,1)$. Hence the last integral equals 1, so obtain (326). ▲

Exercise 6.7 (MGF of normal RVs)**.** Show the followings.

(i) Let X be a RV and a, b be constants. Let $M_X(t)$ be the MGF of X. Then show that

$$\mathbb{E}[e^{t(aX+b)}] = e^{bt} M_X(at). \tag{330}$$

(ii) Let $X \sim N(\mu, \sigma^2)$ and $Z \sim N(0,1)$. Using the fact that $\mathbb{E}[e^{tZ}] = e^{t^2/2}$ (Example 6.6) and part (i), show that

$$\mathbb{E}[e^{tY}] = e^{\sigma^2 t^2/2 + t\mu}. \tag{331}$$

Exercise 6.8. Suppose the stock price $(S_t)_{0\le t\le 1}$ during a period of 1 month is given by

$$S_t = S_0 \exp\left(\mu t + \sigma B_t\right), \tag{332}$$

where $(B_t)_{0\le t\le 1}$ is the standard Brownian motion, and the exponential growth rate $\mu = 1\%$ per month and volatility $\sigma = 1$.

(i) What is the expectation of the stock prices S_0, $S_{1/2}$, and S_1? (Hint: Use MGF of normal RVs in the previous exercise.)

(ii) What is the probability that the stock price increase by more than 2% in a month?

Exercise 6.9. Suppose the stock price $(S_t)_{0\le t\le T}$ during a period $[0,T]$ is given by

$$S_t = S_0 \exp\left(\mu t + \sigma B_t\right), \tag{333}$$

where $(B_t)_{0\le t\le T}$ is a Brownian motion.

(i) Using the fact that $B_t \sim N(0,t)$ and the MGF of normal RV, show that

$$\mathbb{E}[\exp\left(\mu t + \sigma B_t\right)] = \exp((\mu + \sigma^2/2)t). \tag{334}$$

(ii) Using (i), show that the discounted stock price $e^{-rt}S_t$ forms a martingale if and only if

$$r = \mu + \frac{\sigma^2}{2}. \tag{335}$$

6.2. Continuous-time limit of the binomial model under the risk-neutral measure. In this subsection, we establish a similar continuous-time limit of the binomial model under the risk-neutral probability measure, without assuming the condition (b) about the underlying coin flips being i.i.d.

Suppose there is a stock with price S_T at maturity T. Let S_0 denote the initial stock price. As before, we would like to model the time evolution of this stock price using an N-step binomial model where each step has duration $h = T/N$. instead of the assumptions (a) and (b) in the previous subsection, here we make the following assumption:

(c) Fix constants μ and $\sigma > 0$. For each $N \geq 1$, the up- and down-factors of the N-step binomial model for the period $[0, T]$ are given by

$$u = \exp(\mu h + \sigma\sqrt{h}), \qquad d = \exp(\mu h - \sigma\sqrt{h}), \tag{336}$$

where $h = T/N$.

Also we will assume that interest is continuously compounded at a constant rate r. So after each step in the binomial model, which has duration $h = T/N$, \$1 deposited at bank becomes \$$e^{rh}$.

Notice that in this setup, we do not need to assume anything about the physical coin flips. However, it is beneficial to consider i.i.d. coin flips to understand the choice of the up- and down-factors in (336). For this see the following exercise.

Exercise 6.10. Suppose that in the N-step binomial model for the interval $[0, T]$, the physical coin flips are i.i.d. with equal probabilities for H and T. Let μ and $\sigma > 0$ be constants satisfying

$$\mathbb{E}\left[\log\left(\frac{S_T}{S_0}\right)\right] = \mu T, \qquad \mathrm{Var}\left(\log\left(\frac{S_T}{S_0}\right)\right) = \sigma^2 T, \tag{337}$$

where $(S_t)_{0\leq t\leq T}$ denotes the stock price during $[0, T]$. Show that the up- and down-factors of the N-step binomial model are given by (336).

Since the up- and down-factors do not depend on the location in the binomial tree, the risk-neutral probabilities also do not depend on the location. More precisely, the risk-neutral probability of going up is given by

$$p^*_{1;h} = \frac{e^{rh} - d}{u - d}. \tag{338}$$

In order to construct the risk-neutral probability measure $\mathbb{P}^*_h$ for the binomial model, consider a sequence of i.i.d. RVs $X_{1;h}, X_{2;h}, \cdots, X_{N;h}$ where

$$\mathbb{P}(X_{i;h} = \log u) = p^*_{1;h}, \qquad \mathbb{P}(X_{i;h} = \log d) = 1 - p^*_{1;h}. \tag{339}$$

Think of these as the N risk-neutral coin flips for the N-step binomial model. Now let $\mathbb{P}^*_h$ denote the risk-neutral probability measure on the space $\{H, T\}^N$ of N risk-neutral coin flips, where for each sample path $\omega \in \{H, T\}^N$ of risk-neutral coin flips,

$$\mathbb{P}^*_h(\{\omega\}) = (p^*_{1;h})^{\#H\text{'s in }\omega}(1 - p^*_{1;h})^{\#T\text{'s in }\omega}. \tag{340}$$

Remark 6.11. In the general binomial model, the risk-neutral coin flips can be made independent but not necessarily identically distributed, as the risk-neutral probability $p^*_1(\omega)$ may depend on the market evolution ω leading to the point of the coin flip. However, when the up- and down-factors are constant as in our case in this section, the risk-neutral probabilities are constant so all risk-neutral coins have the same distribution.

Proposition 6.12. *We have the following asymptotic expressions*

$$p^*_{1;h} = \frac{1}{2} + \frac{r-\mu-\sigma^2/2}{2\sigma}\sqrt{h} + O(h) \tag{341}$$
$$\mathbb{E}[X_{i;h}] = (r-\sigma^2/2)h + O(h\sqrt{h}) \tag{342}$$
$$\mathrm{Var}(X_{i,h}) = \sigma^2 h + O(h\sqrt{h}). \tag{343}$$

PROOF. By using the power series expansion $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$, we can write

$$e^{rh} = 1 + rh + \frac{(rh)^2}{2} + \cdots \tag{344}$$
$$u = 1 + \mu h + \sigma\sqrt{h} + \frac{(\mu h + \sigma\sqrt{h})^2}{2} + \cdots \tag{345}$$
$$d = 1 + \mu h - \sigma\sqrt{h} + \frac{(\mu h - \sigma\sqrt{h})^2}{2} + \cdots. \tag{346}$$

This gives the following asymptotic expression

$$p^*_{1;h} = \frac{\sigma\sqrt{h} + (r-\mu-\sigma^2/2)h + O(h\sqrt{h})}{2\sigma\sqrt{h}} = \frac{1}{2} + \frac{r-\mu-\sigma^2/2}{2\sigma}\sqrt{h} + O(h). \tag{347}$$

For the expectation of $X_{i;h}$, observe that

$$\begin{aligned}
\mathbb{E}[X_{i;h}] &= (\log u)p^*_{1;h} + (\log d)(1-p^*_1) && (348)\\
&= \log(u/d)p^*_{1;h} + \log d && (349)\\
&= 2\sigma\sqrt{h}p^*_{1;h} + (\mu h - \sigma\sqrt{h}) && (350)\\
&= \sigma\sqrt{h} + (r-\mu-\sigma^2/2)h + (\mu h - \sigma\sqrt{h}) + O(h\sqrt{h}) && (351)\\
&= (r-\sigma^2/2)h + O(h\sqrt{h}). && (352)
\end{aligned}$$

For the variance, first note that

$$\begin{aligned}
\mathbb{E}[X_{i;h}^2] &= (\log u)^2 p^*_{1;h} + (\log d)^2(1-p^*_{1;h}) && (353)\\
&= p^*_{1;h}[(\log u - \log d)(\log u + \log d)] + (\log d)^2 && (354)\\
&= p^*_{1;h}(2\sigma\sqrt{h})(2\mu h) + (\mu h - \sigma\sqrt{h})^2 && (355)\\
&= \sigma^2 h + O(h\sqrt{h}). && (356)
\end{aligned}$$

This gives $\mathrm{Var}(X_{i,h}) = \sigma^2 h + O(h\sqrt{h})$, as desired. □

Proposition 6.13. *Consider the N-step binomial model for the interval $[0,T]$ with the assumption (c). Suppose the interest is continuously compounded at a constant rate r. Let $(S_{hk})_{0\le k\le N}$ denote the stock price in the N-step binomial model and let $Z \sim N(0,1)$ denote the standard normal RV. Then under the risk-neutral probability measure $\mathbb{P}^*_h$, as $N\to\infty$,*

$$S_T \Longrightarrow S_0 \exp\big((r-\sigma^2/2)T + \sigma\sqrt{T}Z\big). \tag{357}$$

PROOF. Define the logarithmic return $X_{i;h} = \log(S_{hi}/S_{h(i-1)})$ as in (309). Then under the risk-neutral probability measure $\mathbb{P}^*_h$, we can write

$$\log\left(\frac{S_{hN}}{S_0}\right) = X_{1;h} + X_{2;h} + \cdots + X_{N;h}. \tag{358}$$

From this the argument is essentially the same as in the i.i.d. coin flip case (Prop. 6.2), but here we need to appeal to a stronger type of central limit theorem since the risk-neutral probability $p^*_{1;h}$ depends on

N. In other words, we have a different set of RVs for each N and we want to take the limit $N \to \infty$. This type of situation is captured as the following 'triangular array'

$$N = 1 \qquad X_{1;T} \tag{359}$$
$$N = 2 \qquad X_{1;T/2}, X_{2;T/2} \tag{360}$$
$$N = 3 \qquad X_{1;T/3}, X_{2,T/3}, X_{3,T/3} \tag{361}$$
$$\vdots \tag{362}$$

A more general CLT for the partial sums from the triangular array of RVs (like the one above) is known as the Lindeberg-Feller CLT. The precise statement of this result and its proof is out of the scope of this note.

To apply the Lindeberg-Feller CLT, we first need to compute the mean and variance of the partial sum $X_{1;h} + \cdots + X_{N;h}$. As $X_{1;h}, \cdots, X_{N;h}$ are i.i.d. and $h = T/N$, Proposition 6.12 yields

$$\mathbb{E}[(X_{1;h} + \cdots + X_{N;h})] = N\mathbb{E}[X_{1;h}] = (r - \sigma^2/2)T + O(N^{-3/2}) \tag{363}$$
$$\mathrm{Var}(X_{1;h} + \cdots + X_{N;h}) = N\mathrm{Var}(X_{1;h}) = \sigma^2 T + O(N^{-3/2}). \tag{364}$$

From this it is not hard to check if the Lindeberg condition is satisfied. Therefore by the Lindeberg-Feller CLT,

$$\frac{(X_{1;h} + \cdots + X_{N;h}) - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \Longrightarrow Z \sim N(0,1) \tag{365}$$

as $N \to \infty$. This shows the assertion. $\square$

Theorem 6.14. *Consider the N-step binomial model for the interval $[0,T]$ with the assumption (c). Suppose the interest is continuously compounded at a constant rate r. Linearly interpolate the stock prices $S_0, S_h, \cdots, S_{hN}$ and denote the resulting continuous-time stock price as $(S_{t;N})_{0\le t\le T}$. Then there is a Brownian motion $(B_t)_{0\le t\le T}$ such that under the risk-neutral probability measure $\mathbb{P}^*$, as $N \to \infty$,*

$$S_{t;N} \Longrightarrow S_0 \exp\big((r - \sigma^2/2)t + \sigma B_t\big). \tag{366}$$

PROOF. Argument is the same as before, but apply the process-level Lindeberg-Feller CLT (e.g., [Bro71, Thm. 3]) to show convergence to the Brownian motion. $\square$

The result in Theorem 6.14 suggests the following *continuous-time risk-neutral stock pricing*

$$S_t \overset{d}{=} S_0 \exp\big((r - \sigma^2/2)t + \sigma B_t\big). \tag{367}$$

Proposition 6.15. *The discounted stock price $(e^{-rt}S_t)_{0\le t\le T}$ forms a martingale under the risk-neutral probability measure $\mathbb{P}^*$. That is, for any $0 \le s \le t \le T$, conditional on the information $\mathscr{F}_s$ up to time $0 \le s \le T$,*

$$\mathbb{E}_{\mathbb{P}^*}[e^{-rt}S_t \,|\, \mathscr{F}_s] = e^{-rs}S_s. \tag{368}$$

PROOF. We will sketch the key part of the argument, assuming $s = 0$. Rewriting (367), we have

$$e^{-rt}S_t = S_0 \exp(-\sigma^2 t/2 + \sigma B_t). \tag{369}$$

Taking risk-neutral conditional expectation on both sides, and using the fact that the MGF of a normal RV with distribution $N(0,t)$ is given by $e^{-x^2t/2}$,

$$\mathbb{E}_{\mathbb{P}^*}[e^{-rt}S_t \,|\, \mathscr{F}_0] = S_0 e^{-\sigma^2 t/2}\mathbb{E}_{\mathbb{P}^*}[\exp(\sigma B_t) \,|\, \mathscr{F}_0] = S_0 e^{-\sigma^2 t/2} e^{\sigma^2 t/2} = S_0. \tag{370}$$

Hence the discounted future stock price is a martingale under the risk-neutral probability measure. $\square$

Proposition 6.16. *Consider a European option with payoff $g(S_T)$ on stock with price $(S_t)_{0\le t\le T}$ in the continuous-time model. Then the value V_0 of this European option is given by*

$$V_0 = \mathbb{E}_{\mathbb{P}^*}[e^{-rT}g(S_T)]. \tag{371}$$

PROOF. The same formula holds for the discrete N-step binomial model. The key part of the argument is to show that one can interchange the order of continuous-time limit $N \to \infty$ and the risk-neutral expectation. This is omitted. □

6.3. The Black-Scholes partial differential equation. In this subsection, we derive the Black-Scholes partial differential equation for the value of European options, and obtain a simple formula for the value of the European call option.

The rigorous derivation of the Black-Scholes equation requires a careful development of the theory of Itô integrals and stochastic calculus, which is not going to be covered in this note. Below we present a less rigorous but more transparent derivation of the equation, following the approach in [Dur99]. The starting point is the now-familiar backward recursion for the European option values in the binomial model:

$$V_k(\omega) = \frac{1}{1+r}[V_{k+1}(\omega H)p_1^*(\omega) + V_{k+1}(\omega T)p_2^*(\omega)]. \tag{372}$$

If we use the N-step approximation of the continuous time model for the period $[0,T]$ using the binomial model, then the above recursion describes an instantaneous change (for duration $h = T/N$) of the option values. By taking the limit $N \to \infty$ and using our risk-neutral stock pricing formula (Thm. 6.14), we will obtain a partial differential equation for the European option value in the continuous model, which is the Black-Scholes equation stated below.

Theorem 6.17 (The Black-Scholes equation)**.** *Consider a European option with payoff $g(S_T)$ on stock with price $(S_t)_{0\le t\le T}$ in the continuous-time model. Let $V(t,s)$ denote the value of the option at time $t<T$ when the stock price is s. Then for $0 \le t \le T$,*

$$\frac{\partial V}{\partial t}(t,s) - rV(t,s) + \left(\mu + \frac{\sigma^2}{2}\right)s\frac{\partial V}{\partial s}(t,s) + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2}(t,s) = 0 \tag{373}$$

with the boundary condition $V(T,s) = g(s)$, where r is the continuously compounded interest rate and σ denotes the volatility of the stock.

Remark 6.18. In the rigorous derivation of the Black-Scholes equation using Itô integral and stochastic calculus, one assumes that the stock price is given by a 'geometric Brownian motion'

$$S_t = S_0 \exp(\mu t + \sigma B_t) \tag{374}$$

and derives the following PDE

$$\frac{\partial V}{\partial t}(t,s) - rV(t,s) + rs\frac{\partial V}{\partial s}(t,s) + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2}(t,s) = 0 \tag{375}$$

for the European option value $V(t,s)$ with payoff $g(S_T)$. This equation is equivalent to the one in (6.17) up to the relation $r = \mu + \sigma^2/2$, which guarantees the discounted stock price $e^{-rt}S_t$ to be a martingale. (See Exercise 6.9.)

Before we give the proof, note that since the option payoff $g(S_T)$ depends only on the stock price S_T at maturity and not on the sample path, the value of the option at time $t \in [0,T]$ depends only on the time t and the stock price s at that time t. Hence the value function $V(t,s)$ in the assertion is indeed well-defined[4].

[4] A more rigorous coupling argument (optional): Under the risk-neutral probability meausure $\mathbb{P}^*$, the future stock evolution is independent from the past, so one can couple two stock price evolution so that they evolve the same way once the two path intersects (at the same time and the same price). Then they both end up at the same price S_T at maturity, so the option value corresponding to the two sample paths are the same.

Proposition 6.19. *We have the following asymptotic expressions*

$$\frac{(1-u)p_1^* + (1-d)(1-p_1^*)}{h} = -\left(\mu + \frac{\sigma^2}{2}\right) + O(\sqrt{h}), \tag{376}$$

$$\frac{(1-u)^2 p_1^* + (1-d)^2(1-p_1^*)}{h} = \sigma^2 + O(\sqrt{h}). \tag{377}$$

PROOF. Follows from the power series expansions for p_1^* (341), u (345), and d (346). Details are omitted. □

PROOF OF THEOREM 6.17. Consider the approximation of the continuous-time model for the period $[0,T]$ via the N-step binomial model. Let $h = T/N$ denote the duration of each step in the binomial model with the following constant up- and down-factors

$$u = \exp(\mu h + \sigma\sqrt{h}), \qquad d = \exp(\mu h - \sigma\sqrt{h}), \tag{378}$$

Let $p_1^* = p_{1;h}^* = (e^{rh} - d)/(u-d)$ denote the risk-neutral up-probability. Then[5]

$$e^{rh} V(t-h,s) = [V(t,su)p_1^* + V(t,sd)(1-p_1^*)]. \tag{379}$$

Taking difference with the equation $V(t,s) = V(t,s)p_1^* + V(t,s)(1-p^*)$, we get

$$V(t,s) - e^{rh}V(t-h,s) = p_1^*[V(t,s) - V(t,su)] + (1-p_1^*)[V(t,s) - V(t,sd)]. \tag{380}$$

Dividing by h, this gives

$$\frac{V(t,s)-V(t-h,s)}{h} - \frac{1-e^{rh}}{h}V(t-h,s) = p_1^*\left[\frac{V(t,s)-V(t,su)}{h}\right] + (1-p_1^*)\left[\frac{V(t,s)-V(t,sd)}{h}\right]. \tag{381}$$

Now, we are going to take the limit $h \to 0$, which corresponds to the limit $N \to \infty$. Using definition of derivatives, the left hand side becomes

$$\frac{\partial V}{\partial t}(t,s) - rV(t,s). \tag{382}$$

For the right hand side of (381), we use the second order Taylor expansion of V in the second variable:

$$V(t,s) - V(t,s') = \frac{\partial V}{\partial s}(t,s)\,(s-s') - \frac{\partial^2 V}{\partial s^2}(t,s)\,\frac{(s-s')^2}{2} + O(|s-s'|^3). \tag{383}$$

Using $s' = su$ and $s' = sd$, the right hand side of (381) becomes

$$s\frac{\partial V}{\partial s}(t,s)\left[\frac{(1-u)p_1^* + (1-d)(1-p_1^*)}{h}\right] - \frac{s^2}{2}\frac{\partial^2 V}{\partial s^2}(t,s)\left[\frac{(1-u)^2 p_1^* + (1-d)^2(1-p_1^*)}{h}\right] \tag{384}$$

$$+ h^{-1}O((1-u)^3 + (1-d)^3) \tag{385}$$

Since $1-u = O(h)$ and $1-d = O(h)$ due to (345) and (346), the last term above goes to zero as $h \to 0$. Hence by Proposition 6.19, the above equation converges as $h \to 0$ to

$$-s\left(\mu + \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial s}(t,s) - \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2}(t,s). \tag{386}$$

Thus equating (382) and (386), we obtain the desired partial differential equation. □

In the particular case of European call, we can explicitly compute the solution of the Black-Scholes equation.

[5]Here the value $V(t,s)$ has implicit dependence on h (or on N), but we ignore this for simplicity.

Theorem 6.20. *In the continuous model, the price $C_K(0,T)$ of the European call with payoff $g(S_T) = (S_T - K)^+$ is given by*

$$C_K(0,T) = S_0\Phi(d_1) - e^{-rT}K\Phi(d_2), \tag{387}$$

where $\Phi(x) = \mathbb{P}(N(0,1) \le x)$ and

$$d_1 = \frac{\log(S_0/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}. \tag{388}$$

PROOF. According to Proposition 6.16, we have

$$C_K(0,T) = \mathbb{E}_{\mathbb{P}^*}[e^{-rT}(S_T-K)^+], \tag{389}$$

where $\mathbb{P}^*$ denotes the risk-neutral probability measure. By the risk-neutral stock pricing (Theorem 6.14), we can write

$$S_T \overset{d}{=} S_0 \exp\big((r-\sigma^2/2)T + \sigma B_T\big). \tag{390}$$

Note that under $\mathbb{P}^*$ we have

$$\begin{aligned} S_T \ge K \quad &\Longleftrightarrow \quad S_0 \exp\big((r-\sigma^2/2)T + \sigma B_T\big) \ge K & (391)\\ &\Longleftrightarrow \quad (r-\sigma^2/2)T + \sigma B_T \ge \log(K/S_0) & (392)\\ &\Longleftrightarrow \quad B_T \ge \frac{((\sigma^2/2)-r)T + \log(K/S_0)}{\sigma} =: \alpha. & (393)\end{aligned}$$

Since $B_T \sim N(0,T)$, thus we have

$$\begin{aligned} V_0 &= \int_{S_T \ge K} [S_0 e^{-\sigma^2 T/2 + \sigma y} - e^{-rT}K] \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}}\, dy & (394)\\ &= S_0 e^{-\sigma^2 T/2} \int_{y\ge\alpha} \frac{1}{\sqrt{2\pi T}} e^{\sigma y - \frac{y^2}{2T}}\, dy - e^{-rT}K \int_{y \ge \alpha} \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}}\, dy & (395)\end{aligned}$$

By completing the square in the exponent, we can rewrite the first integral above as

$$\begin{aligned} &S_0 e^{-\sigma^2 T/2} \int_{y\ge\alpha} \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{1}{2T}(y-\sigma T)^2 + \frac{\sigma^2 T}{2}\right) dy & (396)\\ &= S_0 \int_{y\ge\alpha} \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{1}{2T}(y-\sigma T)^2\right) dy & (397)\\ &= S_0 \mathbb{P}(N(\sigma T, T) \ge \alpha) & (398)\\ &= S_0 \mathbb{P}\left(N(0,1) \ge \frac{\alpha - \sigma T}{\sqrt{T}}\right) = S_0 \mathbb{P}\left(N(0,1) \le \frac{\sigma T - \alpha}{\sqrt{T}}\right), & (399)\end{aligned}$$

where the last equality uses the symmetry of the standard normal distribution. Note that $(\sigma T - \alpha)/\sqrt{T} = d_1$ as defined in the assertion. On the other hand,

$$\begin{aligned} e^{-rT}K \int_{y\ge\alpha} \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}}\, dy = e^{-rT}K\mathbb{P}(N(0,T) \ge \alpha) &= e^{-rT}K\mathbb{P}(N(0,1) \ge \alpha/\sqrt{T}) & (400)\\ &= e^{-rT}K\mathbb{P}(N(0,1) \le -\alpha/\sqrt{T}). & (401)\end{aligned}$$

Since $(\sigma T - \alpha)/\sqrt{T} = d_1$, we also have $-\alpha/\sqrt{T} = d_1 - \sigma\sqrt{T} = d_2$. This shows the assertion. □

Remark 6.21. Recall the put-call parity (Prop. 8.2 in Lecture note 1), which states that

$$C_K(0,T) - P_K(0,T) = V_K(0,T), \tag{402}$$

where $V_K(0,T)$ is the value at time $t=0$ of the forward contract on the stock with maturity T and delivery price K. Recall that

$$V_K(0,T) = (F(0,T) - K)e^{-rT} = (S_0 e^{rT} - K)e^{-rT} = S_0 - Ke^{-rT}. \tag{403}$$

Hence (6.20) also gives the following price formula for the European put with payoff $(K-S_T)^+$:

$$
\begin{aligned}
P_K(0,T) &= C_K(0,T) - V_K(0,T) && (404)\\
&= S_0(\Phi(d_1)-1) - e^{-rT}K(\Phi(d_2)-1) && (405)\\
&= e^{-rT}K(1-\Phi(d_2)) - S_0(1-\Phi(d_1)) && (406)\\
&= e^{-rT}K\mathbb{P}(N(0,1)\geq d_2) - S_0\mathbb{P}(N(0,1)\geq d_1) && (407)\\
&= e^{-rT}K\Phi(-d_2) - S_0\Phi(-d_1). && (408)
\end{aligned}
$$

Bibliography

[Bly14] Joseph Blyth, *An introduction to quantitative finance*, Oxford, 2014.

[Dur99] Rick Durrett, *Essentials of stochastic processes*, Springer, 1999.

[Dur19] Rick Durrett, *Probability: Theory and examples*. Vol. 49. Cambridge university press, 2019.

[HK04] Philip Hunt and Joanne Kennedy, *Financial derivatives in theory and practice*, John Wiley & Sons, 2004.

[Hul03] John C Hull, *Options futures and other derivatives*, Pearson Education India, 2003.

[Bro71] B. M. Brown, *Martingale central limit theorems*, The Annals of Mathematical Statistics, 42.1, (1971):59-66.