EEE360C: Algorithms

Proofs

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Definition

A Proof

- a statement is either *true* or *false*.
  - $1 = 0$ is *false*
  - $\exists t : \cos(t) = t$ is *true*
  - $\forall a, b, c, n : (n > 2) \wedge (a^n + b^n = c^n) \Rightarrow a = b = c = 0$ is true (though it's difficult to prove)
- some statements may be true or false depending on the values assigned to variables:
  - $3x = 5$
  - $x^2 + y^2 - 4xy > 0$

Proofs

A mathematical proof is a "convincing" argument expressed in the language of mathematics

- it should contain enough detail to convince someone with reasonable background in the subject

Terminology

Some Terminology

Proof Terminology

- *Definition*: an unambiguous explanation of terms
- *Proposition*: a statement that is claimed to be true
- *Theorem*: a major result
- *Lemma*: a minor result; often used on the way to proving a theorem
- *Corollary*: something that follows from something just proved
- *Axioms*: basic assumptions or truths

Terminology (cont.)

Forms of Theorems

A theorem can be reduced to stating “if A then B.” The following are all equivalent:

- If A is true then B is true
- A implies B
- $A \Rightarrow B$
- A only if B
- A is sufficient for B
- B is true whenever A is true

The Forward-Backward Method

The Forward-Backward Method

A good technique to approaching a proof is to work from both directions. Start by first writing both the statements A and B. In the forward direction: “given A, what else do I know?” In the backward direction: “how would I show B?”

An Example

If a right triangle xyz with sides of length x and y and a hypotenuse of length z has area $z^2/4$, then the triangle xyz is isosceles.

The Forward-Backward Method (cont.)

An Example

If a right triangle xyz with sides of length x and y and a hypotenuse of length z has area $z^2/4$, then the triangle xyz is isosceles.

A right triangle xyz has area $z^2/4$

A1 $xy/2 = z^2/4$ (area = 1/2 base $\times$ height)

A2 $x^2 + y^2 = z^2$ (Pythagorean theorem)

A3 $(x^2 + y^2)/4 = xy/2$ (substituting for z^2)

A4 $(x^2 + y^2) = 2xy$ (multiplying through by 4)

A5 $x^2 - 2xy + y^2 = 0$ (rearranging)

A6 $(x - y)^2 = 0$ (factoring)

B2 $(x - y) = 0$

B1 $x = y$

B triangle xyz is isosceles

The Forward-Backward Method (cont.)

An Example

If a right triangle xyz with sides of length x and y and a hypotenuse of length z has area $z^2/4$, then the triangle xyz is isosceles.

A Condensed Proof

From the hypothesis and the definition of the area of a triangle, $xy/2 = z^2/4$. By Pythagoras, $x^2 + y^2 = z^2$. On substituting $x^2 + y^2$ for z^2, we obtain $(x - y)^2 = 0$. Hence $x = y$ and the triangle is isosceles.

Tools

Proof Tools

- part of our proof is just algebraic manipulation
- other pieces also drew upon external information
  - e.g., the definition of isosceles triangle, the theorem stating the area of a triangle, the Pythagorean theorem
- in general, a proof will draw upon definitions, axioms, and previously proven theorems
- be careful to avoid a circular proof (i.e., where a step in your proof relies on the theorem you're trying to prove).

Truth Tables

Notations

- $A \Rightarrow B$: “implies”
- $\overline{B} \Rightarrow \overline{A}$: “contrapositive”
- $B \Rightarrow A$: “converse”
- $\overline{A} \Rightarrow \overline{B}$: “inverse”
- $A \Leftrightarrow B$: “equivalence” or “if-and-only-if” or “iff”

A	B	$\overline{A}$	$\overline{B}$	$A \Rightarrow B$	$\overline{B} \Rightarrow \overline{A}$	$B \Rightarrow A$	$\overline{A} \Rightarrow \overline{B}$	$A \Leftrightarrow B$
F	F	T	T	T	T	T	T	T
F	T	T	F	T	T	F	F	F
T	F	F	T	F	F	T	T	F
T	T	F	F	T	T	T	T	T

Quantifiers

- $\exists$: there exists an object with a certain property such that something happens
- $\forall$: for all objects with a given property, something happens

Specialization

- x' has a certain property
- $\forall x$ with a certain property, something happens
- the something happens for x'

Choose

- $\forall x$ with a certain property, something happens.
- Let x' be such that the certain property holds
- something happens for x'

Examples

An Example

If s and t are rational and $t \neq 0$, then s/t is rational.

A s and t are rational and $t \neq 0$

A1 $\exists$ integers p, q, $q \neq 0$ such that $s = p/q$

A2 Let a, b be integers such that $b \neq 0$ and $s = a/b$

A3 $\exists$ integers p, q, $q \neq 0$ such that $t = p/q$

A4 Let c, d be integers such that $d \neq 0$ and $t = c/d$

A5 $t \neq 0 \Rightarrow c \neq 0$

A6 $\frac{s}{t} = \frac{a/b}{c/d} = \frac{ad}{bc}$

A7 Let $p = ad$ and $q = bc$

B2 $bc \neq 0$, $\frac{s}{t} = \frac{ad}{bc}$

B1 $\exists$ integers p, q, $q \neq 0$ such that $s/t = p/q$

B s/t is rational

The Example: The EE360C Way

If s and t are rational numbers and $t \neq 0$, then s/t is rational.

The Proof

Let a, b be integers such that $s = a/b$ ($b \neq 0$). Such integers must exist because s is rational. Similarly, let c, d be integers such that $t = c/d$ ($d \neq 0$). Since $t \neq 0$, it must be true that $c \neq 0$. Then, substituting, $s/t = (a/d)/(c/d) = ad/bc$. $bc \neq 0$ (since both b and c are nonzero). Therefore, s/t is rational because there exist integers p, q such that s/t is p/q.

Another Example

- *Def:* $f : S \to T$ is onto iff $\forall t \in T, \exists s \in S : f(s) = t$
- *Def:* Let $f : X \to Y$ and $g : Y \to Z$ be functions, then $g \bullet f : X \to Z$ is the function such that $(g \bullet f)(x) = g(f(x))$

Proposition: **if** $f : X \to Y$ **is onto and** $g : Y \to Z$ **is onto, then** $g \bullet f : X \to Z$ **is onto.**

A	$f : X \to Y, g : Y \to Z$ are onto
A1	Let $c \in Z$
A2	$\forall z \in Z, \exists y \in Y$ such that $g(y) = z$
A3	$\exists y \in Y$ such that $g(y) = c$
A4	Let b be such a y: $b \in Y, g(b) = c$
A5	$\forall y \in Y, \exists x \in X$ such that $f(x) = y$
A6	$\exists x \in X$ such that $f(x) = b$
A7	Let a be such an x: $a \in X, f(a) = b$
A8	Let x of **[B2]** be a
A9	$(g \bullet f)(a) = g(f(a)) = g(b) = c$
B3	$(g \bullet f)(a) = c$
B2	$\exists x \in X$ such that $(g \bullet f)(x) = c$
B1	$\forall z \in Z, \exists x \in X$ such that $(g \bullet f)(x) = z$
B	$g \bullet f : X \to Z$ is onto
QED	(quod erat demonstrandum)

And in EE360C Style

If $f : X \to Y$ is onto and $g : Y \to Z$ is onto, then $g \bullet f : X \to Z$ is onto.

The Proof

For any $c \in Z$, we can find a $b \in Y$ such that $g(b) = c$. (Such a b must exist because g is onto.) Similarly, let $a \in X$ be such that $f(a) = b$ (again, a must exist because f is onto). Then, given any selected $c \in Z$, $(g \bullet f)(a) = c$, i.e., some $a \in X$ can be found to make the claim true. Therefore $g \bullet f : X \to Z$ is onto.

Methodologies

Proof by Contradiction

Proof By Contradiction

We assume that the negation of our proposition is true and show that it leads to a contradictory statement.

An Example

Theorem: There are infinitely many prime numbers.

Proof: Suppose there is a finite number of prime numbers. Then you can list them in order: $p_1, p_2, \ldots, p_n$. Consider the number $q = p_1 p_2 \ldots p_n + 1$. The number q is either prime or composite. If we divide any of the listed primes p_i into q, there would be a remainder of 1. Thus q cannot be composite. Therefore q is a prime number that is not listed among the primes listed above, contradicting the assumption that our list $p_1, p_2, \ldots, p_n$ lists all of the prime numbers.

Proof by Induction

Three Steps to an Inductive Proof

- Start with verifying the *base case*.
- Then assume the n^{th} case.
- And use that to prove the $(n+1)^{st}$ case.

An Example

Prove that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

- **Base case**: show it's true for $n = 0$: $0 = \frac{0(0+1)}{2}$
- **Inductive step**: show that if it holds for n then it holds for $n+1$. That is, use:
  $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ to show that:
  $0 + 1 + 2 + \cdots + (n+1) = \frac{(n+1)((n+1)+1)}{2}$
- Substituting in the right hand side of the equation for the sum to n to most of the left hand side of the equation for the sum to $n+1$ gives us:

$$\frac{n(n+1)}{2} + (n+1) = \frac{(n+1)((n+1)+1)}{2}$$

  which is true.

Another Induction Example

Prove that the sum of the first n odd positive integers is n^2.

The Proof

- **Base case**: the sum of the first one odd positive integers is 1^2. This is true since the sum of the first odd positive integer is 1.
- **Inductive step**: show that if it holds for n, then it holds for $n+1$. If the proposition is true for n, then $1+3+5+\cdots+(2n-1)=n^2$. Then we must show that $1+3+5+\cdots+(2n-1)+(2n+1)=(n+1)^2$. We can prove this algebraically.

One More Induction Example

Prove that if S is a finite set with n elements, then S has 2^n subsets.

The Proof

- **Base case**: a set S of size 0 has one subset (the empty set); $2^0 = 1$.
- **Inductive step**: assume that every set with n elements has 2^n subsets. Prove that by adding one element to the set S, we increase the number of subsets to 2^{n+1}. Let T be a set with $n + 1$ elements. Then it is possible to express $T = S \cup \{a\}$ where a is one of the elements of T and $S = T - \{a\}$. The subsets of T can be obtained by the following. For each subset X of S, there are exactly two subsets of T, namely X and $X \cup \{a\}$. Since there are 2^n subsets of S, there are 2×2^n subsets of T, which is 2^{n+1}.

Questions?