Homework #1

You should try to solve these problems by yourself. I recommend that you start early and get help in office hours if needed. If you find it helpful to discuss problems with other students, go for it. **You do not need to turn in these problems. The goal is to be ready for the in class quiz that will cover the same or similar problems.**

Problem 1: Relations

If a relation R is symmetric and transitive, then it is also reflexive. This can be proved in the following way. By symmetry, $a\,R\,b$ implies $b\,R\,a$. Transitivity therefore implies $a\,R\,a$. Is this proof correct? If not, give a counter-example.

Solution

The proof is incorrect. For a relation to be reflexive, for all $a \in A$,$a\ R\ a$. However, for a relation to be symmetric or transitive, not all possible pairs need to exist.
Counterexample: Consider the relation $R = \{(a,a),(a,c),(c,a),(c,c)\}$ from A to A where $A = \{a,b,c\}$. R is both symmetric (i.e., if $a\,R\,b$ then $b\,R\,a$) and transitive (i.e., if $a\,R\,b$ and $b\,R\,c$ then $a\,R\,c$) but it is not reflexive (e.g., $b\,R\,b$ is missing from the relation).

Problem 2: Sets and Counterexamples

Show that for arbitrary sets A, B, and C, taken from the universe $\{1,2,3,4,5\}$ that the following two claims are not aways true by using a simple counter example for each:

(a) if $A \cap B \subseteq C$, then $C \subseteq A \cup B$

Solution

Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$ $A \cap B = \emptyset$ so $A \cap B \subseteq C$ is automatically true for this example, but clearly $3 \in C$ while $3 \notin A \cup B$. Thus $C \subseteq A \cup B$ is false for this example.

(b) if $C \subseteq A \cup B$, then $A \cap B \subseteq C$

Solution

Let $A = \{1,2\}$, $B = \{1,2\}$, and $C = \{1\}$. In this example, $C \subseteq A \cup B$ is true since $1 \in A$. But $A \cap B = \{1,2\}$ in this example and since $2 \in A \cap B$ but $2 \notin C$, it follows that $A \cap B \subseteq C$ is false for this example.

Problem 3: Proofs by Contradiction

Prove each of the following.

(a) $\sqrt{2}$ is irrational.

Solution

Suppose that $\sqrt{2}$ was rational. Then $\sqrt{2} = a/b$ where a and b are positive integers. Also assume that a/b is simplified to its lowest terms. It follows that $2 = a^2/b^2$, or $a^2 = 2b^2$. So the square of a is an even number. Then a is also even (prove, or see **(c)** below). So $a = 2k$ for some k. Then

$$\begin{aligned}
2 &= (2k)^2/b^2 \\
2 &= 4k^2/b^2 \\
2b^2 &= 4k^2 \\
b^2 &= 2k^2
\end{aligned}$$

So b^2 is even, and it follows that b is even. But if both a and b are even, then a/b is not simplified to lowest terms, which is a contradiction.

(b) The sum of an irrational number and a rational number is irrational.

Solution

Suppose that r is rational and i is irrational and $s = i + r$ is rational. Then r can be expressed as $r = p/q$ and s can be expressed as $s = t/u$. Then

$$\begin{aligned}
r + i &= s \\
p/q + i &= t/u \\
i &= t/u - p/q \\
i &= tq/qu - pu/qu \\
i &= (tq - pu)/qu
\end{aligned}$$

But $(tq - pu)$ and qu are rational by definition, which is a contradiction.

(c) If n^2 is even, n is even.

Solution

Assume this is not the case, i.e., that n^2 is even, but n is odd. Because n is odd, you can write it as $2k + 1$. Then, squared, n is $(2k + 1)^2 = 4k^2 + 4k + 1$, which one can rewrite as $2j + 1$, which is odd, not even, a contradiction.

Problem 4: Graphs

Show that any connected, undirected graph $G = (V, E)$ satisfies $|E| \geq |V| - 1$.

Solution

Prove by induction

Base case: Take the simplest connected undirected graph, one with 1 node and 0 edges. Then $|V| = 1$ and $|E| = 0$; clearly $|E| \geq |V| - 1$.

Inductive step: Assume an undirected graph $G = (V, E)$ with n nodes is connected and the property $|E| \geq |V| - 1$ holds. Create a new graph $G' = (V', E')$ that has one additional node (i.e., $|V'| = n + 1$. To maintain the connectivity property, this new node must be connected to at least one other node in the previously connected graph (otherwise, any one of the original n nodes would not be able to reach the $n+1^{st}$ node, and the new graph would not be connected). Therefore $|E'| > |E|$, without loss of generality, let's say $|E'| = |E| + 1$. From the inductive step, $|E| \geq |V| - 1$. Substituting, we have $|E'| - 1 \geq |V'| - 1 - 1$, which reduces to $|E'| \geq |V'| - 1$.

Problem 5: Trees

Show by induction that the number of degree-2 nodes in any non-empty binary tree is 1 fewer than the number of leaves.

Solution

Base Case: Consider a binary tree with one node. There is 1 leaf node, and 0 nodes with two children; 0 is exactly one less than 1.

Inductive Step: Assume the proposition is true for a binary tree of size n. Show that it remains true for a binary tree of size $n+1$. Start with a binary tree with n nodes. To make it a binary tree with $n+1$ nodes: 1) we can add our new node as the first child of a leaf node, 2) as a second child of a non-leaf node that does not already have two children, 3) create a new root, or 4) insert in the middle of the tree. In the first case, we removed one leaf node, added a leaf node, and didn't change the number of nodes with two children. Therefore, if the proposition holds for the tree of size n, it still holds for the tree of size $n + 1$. In the second case, we added a leaf node, but we also "completed" a node, increasing the number of nodes with two children by one, thereby maintaining the proposition. In the third case, the number of leaves stay the same, as does the number of 2-degree nodes. Finally, insertion in-between two pre-existing nodes keeps the number of leaves and 2-degree nodes constant as well.