EE360C: Algorithms
University of Texas at Austin | Homework #2
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Homework #2

Problem 1: Graphs

Show that in any undirected graph, there is a path from any vertex with odd degree to some other vertex of odd degree.

Solution

For some graph G, pick an arbitrary vertex v of odd degree. Take a subset of the graph that is connected to v via some path (i.e., the connected component of G that contains v), call this G_v. Since G_v is itself a graph, it must have total degree even (because every edge connects two vertices). Because v is of odd degree, there must be another vertex of odd degree in G_v. Since G_v is connected, there must exist a path between v and this other vertex. (There is also a totally legitimate, though longer, proof by induction.)

Problem 2: First preferences in Stable Marriage

Decide whether the following statement is true or false. If it is true, give a short proof. If it is false, give a counter example.

In every instance of the Stable Marriage problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

Solution

False. Counter-example. Consider the case with two women (w_1 and w_2) and two men (m_1 and m_2). w_1 prefers m_2 over m_1; w_2 prefers m_1 over m_2. m_1 prefers w_1 over w_2; m_2 prefers w_2 over w_1. There are two possible stable matches (there are two perfect matchings, and theyre both stable): $\{(m_1, w_1), (m_2, w_2)\}$ and $\{(m_1, w_2), (m_2, w_1)\}$. By inspection, in all four pairings in these two matchings, one of the participants did get their top priority, while the other one did not.

Problem 3: Truthfulness in Stable Marriage

For this problem, we will explore the issue of *truthfulness* in the Stable Matching Problem and specifically in the Gale-Shapley algorithm. The basic question is: can a man or a woman end up better off by lying about his or her preferences? More concretely we suppose each participant has a true preference order. Now consider a woman w. Suppose w prefers man m to m', but both m and m' are low on her list of preferences. Can it be the case that by switching the order of m and m' on her list of preferences (i.e., by falsely claiming that she prefers m' to m) and running the algorithm with this false preference list, w will end up with a man m'' that she truly prefers to both m and m'? (We can ask the same question for men but will focus on the case of women for the purposes of this question.)

Resolve this question by doing one of the following two things:

(a) Give a proof that, for any set of preference lists, switching the order of a pair on the list cannot improve a woman's partner in the Gale-Shapley algorithm; or

(b) Give an example of a set of preference list for which there is a switch that would improve the partner of a woman who switched preferences.

Solution

It is possible for a woman to "game" the system.
Assume we have three men, m_1 to m_3 and three women w_1 to w_3 with the following (true) preferences:

- m_1: w_3, w_1, w_2
- m_2: w_1, w_3, w_2
- m_3: w_3, w_1, w_2
- w_1: m_1, m_2, m_3
- w_2: m_1, m_2, m_3
- w_3: m_2, m_1, m_3

First consider a possible execution of Gale-Shapley with these true preference lists. First m_1 proposes to w_3 then m_2 proposes to w_1. Then m_3 proposes to w_3 and w_1 and gets rejected, finally proposes to w_2 and is accepted. This execution forms pairs (m_1, w_3), (m_2, w_1), and (m_3, w_2), thus pairing w_3 with m_1, who is her second choice.
Now consider what happens when w_3 pretends that her preferences are m_2, m_3, m_1 instead. Then the execution might unfold as follows. Man m_1 proposes to w_3, m_2 to w_1, then m_3 to w_3. She accepts the proposal, leaving m_1 alone. m_1 proposes to w_1, which causes her to leave m_2, who consequently proposes to w_3 (which is exactly what w_3 wants).

Problem 4: Strong and weak instability in Stable Marriage

The stable matching problem, as described in the text, assumes that all men and women have a fully ordered list of preferences. In this problem, we will consider a version of the problem in which men and women can be *indifferent* between certain options. As before, we have a set M of n men and a set W of n women. Assume each man and each woman ranks the members of the opposite gender, but now we allow ties in the ranking. For example (with $n = 4$), a woman could say that m_1 is ranked in first place; second place is a tie between m_2 and m_3 (she has no preference between them); and m_4 is in last place. We will say that *w prefers m to m′* if m is ranked higher than m' on her preference list (they are not tied).

With indifferences in the rankings, there could be two natural notions of stability. And for each, we can ask about the existence of stable matchings.

(a) A *strong instability* in a perfect matching S consists of a man m and a woman w, such that both m and w prefer each other over their current partners in S. Does there always exist a perfect matching with no strong instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a strong instability or give an algorithm that is guaranteed to find a perfect matching with no strong instability.

Solution

The answer is Yes. A simple way to think about it is to break the ties in some fashion and then run the stable matching algorithm on the resulting preference lists. We can, for example, break the ties lexicographically—that is, if a man m is indifferent between two women w_i and w_j, then w_i appears on m's preference list before w_j if $i < j$ and if $j < i$, w_j appears before w_i. Similarly, if w is indifferent between two men m_i and m_j, then m_i appears on w's preference list before m_j if $i < j$ and if $j < i$, m_j appears before m_i.
Now that we have concrete preference lists, we run the stable matching algorithm. We claim that the matching produced would have no strong instability. But this claim is true because any strong instability would be an instability for the match produced by the original algorithm in the original situations, yet we know that this is not the case.

(b) A *weak instability* in a perfect matching S consists of a man m and a woman w, such that their partners in S are w' and m', respectively, and one of the following holds:

- m prefers w to w', and w either prefers m to m' or is indifferent between these two choices; or
- w prefers m to m', and m either prefers w to w' or is indifferent between these two choices.

In other words, the pairing between m and w is either preferred by both, or preferred by one while the other is indifferent. Does there always exist a perfect matching with no weak instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a weak instability; or give an algorithm that is guaranteed to find a perfect matching with no weak instability.

Solution

The answer is No. The following is a simple counterexample. Let $n = 2$ and m_1, m_2 be the two men and w_1, w_2 the two women. Let m_1 be indifferent between w_1 and w_2 and let both women prefer m_1 to m_2. The choices of m_2 are insignificant. There is no mathcing without weak stability in this example, since regardless of who was matched with m_1, the other woman together with m_2 would form a weak instability.