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Iterative design of time-varying stabilizers for multi-input systems in chained form[1]

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Abstract

This paper proposes an alternative solution to the global stabilization of nonholonomic multi-input chained form systems investigated in recent contributions [13, 18]. A systematic design, which is reminiscent of integrator backstepping methods, is presented to generate a new class of smooth time-varying dynamic stabilizers. The proof of stability is straightforward and the algorithm finds its application in adaptive control of nonholonomic systems and tracking control of mobile robot.

Keywords: Chained nonholonomic systems; Integrator backstepping; Time-varying controller; Global stabilization

1. Introduction

In this paper, the problem of smooth feedback stabilization is addressed for a control system described by

$$\begin{aligned}
\dot{x}_0 &= u_0,\\
\dot{x}_{i1} &= x_{i2}u_0,\\
&\vdots\\
\dot{x}_{i(n_i-1)} &= x_{in_i}u_0,\\
\dot{x}_{in_i} &= u_i, \quad 1\leqslant i\leqslant m
\end{aligned} \tag{1}$$

where $x = (x_0, x_1, \ldots, x_m) \in \mathbb{R}^N$, with $N = 1 + \sum_{i=1}^{m} n_i$ and $x_i = (x_{i1}, \ldots, x_{in_i})$ for $1\leqslant i\leqslant m$, and $u = (u_0, u_1, \ldots, u_m) \in \mathbb{R}^{m+1}$ represent the state and the input, respectively.

System (1) is an equivalent version of the so-called chained form system with $(m+1)$ inputs, m chains and single-generator introduced by Murray and Sastry in [12]. By abuse of terminology, a system with the form of (1) is referred to as a *multi-input chained form system*. This class of systems has received renewed attention (see, e.g., [12, 14–16, 18]). As shown in the recent study of mechanical nonholonomic systems, many physical systems with nonholonomic constraints can be converted into a chained form control system (1) by using a change of coordinates and state feedback. Systems of this kind include a car with multiple trailers, the knife-edge, a rigid spacecraft with two torque actuators and a fire truck [12, 14, 9, 2].

Necessary and sufficient conditions which bring a driftless affine system $\dot{\xi} = \sum_{i=0}^{m} g_i(\xi)v_i$ into a multi-input chained form system (1) are given in [18, 2]. The nonexistence of a time-invariant smooth feedback stabilizer for system (1) is confirmed by the violation of Brockett's necessary condition for feedback stabilization [1]. It was proved in [4] that a *time-varying* feedback law exists for feedback stabilization of a general class of controllable systems without drift including

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(1) in particular. Constructive approaches have been derived in [13, 16, 18, 14].

In [13], Pomet employs the Jurjevic–Quinn method to design time-periodic stabilizing control laws for a class of controllable driftless systems to which systems (1) belong. In [16], Center Manifold techniques along with an "averaging" transformation and certain saturation functions are used to achieve global stabilization of a two-input nonholonomic system in power form, which is equivalent to a chained form system (1) with two-inputs and a single-chain. The method proposed in [16] was recently extended to the general $(m+1)$-input, m-chain and single-generator case in [18]. In [14], Samson considers the feedback stabilization of system (1) with two-inputs and a single-chain. His method consists of first transforming the system (1) into a skew-symmetric chain form and then applying Lyapunov-like functions for control design and Barbalat's lemma for stability analysis.

In this paper, a new systematic design scheme is built for systems of the form (1) with $(m+1)$-inputs and m-chains. The proposed design procedure is stepwise and relies upon the canonical structure of the system (1). It is in the spirit of the integrator backstepping ideas (see, e.g., [3, 17, 8]). The smooth control laws derived in the sequel are dynamic and time-varying (not necessarily time-periodic), as opposed to the static and time-periodic controllers in [13, 16, 18]. In contrast to [13, 16, 18, 14], the stability analysis for the closed-loop system in question is very simple and comes from a direct application of LaSalle's invariance principle [10].

It is worth noting that smooth stabilizing feedbacks are well suitable for nonlinear and adaptive stabilization of controlled systems with a triangular structure (see, e.g., [3, 17, 8]). In fact, as shown in [5, 6], the stabilization scheme presented in this paper may be easily adapted to nonholonomic control systems with parametric uncertainty, whereas it is not the case for the static control laws proposed in [13, 16, 18, 14]. More recently in [7], we showed that the integrator backstepping methodology as used in this paper can be exploited successfully for global tracking control of mobile robots.

It should be mentioned that smooth time-varying controllers presented in this paper do not achieve global asymptotic stability with exponential convergence for systems (1) (see [14]). Nonsmooth feedback laws have been sought to gain the exponential rate of convergence (see, for instance, [15, 11]).

This paper is organised as follows. Section 2 describes the control design algorithm which is used to solve the problem. Section 3 formulates the main stabilization result. Along the way, a problem of partial-state regulation is solved Section 4 presents simulation results based on the fire truck example. Finally, Section 5 draws some conclusions.

2. Controller design

The main purpose of this section is to develop a systematic control design procedure for system (1). The stepwise design procedure is in principle an iterative use of the now popular integrator backstepping idea (see, e.g., [3, 17, 8]). The backstepping-based stabilization scheme presented in the sequel should not be confused with the application of the integrator backstepping idea considered in [9]. Assuming that a driftless nonholonomic control system is stabilizable, the authors of [9] proved that its dynamic extension (i.e., the original system augmented with integrators) is stabilizable.

First of all, define a finite number of functions ϕ_k by

$$\begin{aligned}
&\phi_1(u_0) = u_0^{2l_1+1},\\
&\phi_k(u_0, \dot{u}_0, \ldots, u_0^{(k-1)}) = \dot{\phi}_{k-1}/u_0,\\
&\quad 2 \leqslant k \leqslant \max\{n_1 - 1, \ldots, n_m - 1\},
\end{aligned} \tag{2}$$

where l_1 is a nonnegative integer greater than $\max\{n_1 - 3, \ldots, n_m - 3\}$.

As it can be directly verified, the ϕ_k's are smooth (i.e., C^∞) functions.

Step 1: Consider the x_1-subsystem of (1), i.e.,

$$\begin{aligned}
\dot{x}_{11} &= x_{12}\, u_0,\\
&\vdots\\
\dot{x}_{1(n_1-1)} &= x_{1n_1} u_0,\\
\dot{x}_{1n_1} &= u_1,
\end{aligned} \tag{3}$$

where u_1 is considered as the control input, with u_0 understood to be a smooth function of time.

Step 1.1: Denote $z_{11} = x_{11}$ and decompose x_{12} as

$$x_{12} = \alpha_{11} + z_{12}, \tag{4}$$

where α_{11} is an intermediate stabilizing function of t and x_{11}.

Differentiating the function $V_{11} = \frac{1}{2} z_{11}^2$ along the solutions of (3) yields

$$\dot{V}_{11} = z_{11}\alpha_{11} u_0 + z_{11} z_{12} u_0. \tag{5}$$

Setting

$$\alpha_{11}(x_{11},\phi_1) = -c_{11}z_{11}\phi_1(u_0) \tag{6}$$

with $c_{11} > 0$, (5) gives

$$\dot{V}_{11} = -c_{11}z_{11}^2 u_0^{2l_1+2} + z_{11}z_{12}u_0. \tag{7}$$

By virtue of (4) and (6), one obtains the following z_{11}-equation:

$$\dot{z}_{11} = -c_{11}z_{11}u_0^{2l_1+2} + z_{12}u_0. \tag{8}$$

Notice that for arbitrary $u_0(t)$, $\alpha_{11} = 0$ whenever $x_{11} = 0$.

Step 1.2: Recall $z_{12} = x_{12} - \alpha_{11}$ and decompose x_{13} as

$$x_{13} = \alpha_{12} + z_{13}, \tag{9}$$

where α_{12} is an intermediate stabilizing function of t and (x_{11}, x_{12}).

Differentiating the function $V_{12} = V_{11} + \frac{1}{2}z_{12}^2$ along the solutions of (3) yields

$$\dot{V}_{12} = -c_{11}z_{11}^2 u_0^{2l_1+2} + z_{12}\left(z_{13} + \alpha_{12} + z_{11} - \frac{\partial\alpha_{11}}{\partial x_{11}}x_{12} - \frac{\partial\alpha_{11}}{\partial\mu_{11}}\frac{\dot{\phi}_1}{u_0}\right)u_0. \tag{10}$$

By (2), $\dot{\phi}_1/u_0 = \phi_2(u_0,\dot{u}_0) = (2l_1+1)u_0^{2l_1-1}\dot{u}_0$. Then, setting

$$\alpha_{12}(x_{11},x_{12},\phi_1,\phi_2) = -c_{12}z_{12}\phi_1(u_0) - z_{11} + \frac{\partial\alpha_{11}}{\partial x_{11}}x_{12} + \frac{\partial\alpha_{11}}{\partial\mu_{11}}\phi_2(u_0,\dot{u}_0) \tag{11}$$

with $c_{12} > 0$, (10) implies

$$\dot{V}_{12} = -(c_{11}z_{11}^2 + c_{12}z_{12}^2)u_0^{2l_1+2} + z_{12}z_{13}u_0. \tag{12}$$

With the help of (9) and (11), one obtains the following z_{12}-equation:

$$\dot{z}_{12} = -z_{11}u_0 - c_{12}z_{12}u_0^{2l_1+2} + z_{13}u_0. \tag{13}$$

Notice that for arbitrary $u_0(t)$, $\alpha_{12} = 0$ whenever $x_{11} = x_{12} = 0$.

Step 1.j ($3 \leqslant j \leqslant n_1 - 1$): Assume that smooth functions α_{1k} ($1 \leqslant k \leqslant j-1$) have been designed such that, with

$$z_{1(k+1)} = x_{1(k+1)} - \alpha_{1k}(x_{11},\ldots,x_{1k},\phi_1,\ldots,\phi_k), \tag{14}$$

the time derivative of $V_{1(j-1)} = \frac{1}{2}z_{11}^2 + \cdots + \frac{1}{2}z_{1(j-1)}^2$ satisfies

$$\dot{V}_{1(j-1)} = -\sum_{k=1}^{j-1} c_{1k}z_{1k}^2 u_0^{2l_1+2} + z_{1(j-1)}z_{1j}u_0. \tag{15}$$

Decompose $x_{1(j+1)}$ as

$$x_{1(j+1)} = \alpha_{1j} + z_{1(j+1)}, \tag{16}$$

where α_{1j} is an intermediate stabilizing function of t and $(x_{11},\ldots,x_{1j})$.

Differentiating the function $V_{1j} = V_{1(j-1)} + \frac{1}{2}z_{1j}^2$ along the solutions of (3) yields

$$\dot{V}_{1j} = -\sum_{k=1}^{j-1} c_{1k}z_{1k}^2 u_0^{2l_1+2} + z_{1j}\left(z_{1(j+1)} + \alpha_{1j} + z_{1(j-1)} - \sum_{k=1}^{j-1}\frac{\partial\alpha_{1(j-1)}}{\partial x_{1k}}x_{1(k+1)} - \sum_{k=1}^{j-1}\frac{\partial\alpha_{1(j-1)}}{\partial\mu_{1k}}\frac{\dot{\phi}_k}{u_0}\right)u_0. \tag{17}$$

Taking note of (2) and setting

$$\alpha_{1j}(x_{11},\ldots,x_{1j},\phi_1,\ldots,\phi_j) = -c_{1j}z_{1j}\phi_1(u_0) - z_{1(j-1)} + \sum_{k=1}^{j-1}\frac{\partial\alpha_{1(j-1)}}{\partial x_{1k}}x_{1(k+1)} + \sum_{k=1}^{j-1}\frac{\partial\alpha_{1(j-1)}}{\partial\mu_{1k}}\phi_{(k+1)} \tag{18}$$

with $c_{1j} > 0$, (17) implies

$$\dot{V}_{1j} = -\sum_{k=1}^{j} c_{1k}z_{1k}^2 u_0^{2l_1+2} + z_{1j}z_{1(j+1)}u_0. \tag{19}$$

By virtue of (16) and (18), the following z_{1j}-equation is established:

$$\dot{z}_{1j} = -z_{1(j-1)}u_0 - c_{1j}z_{1j}u_0^{2l_1+2} + z_{1(j+1)}u_0. \tag{20}$$

Notice that for arbitrary $u_0(t)$, $\alpha_{1j} = 0$ whenever $x_{11} = \cdots = x_{1j} = 0$.

Step 1.n_1: At this step, $u_1 = x_{1(n_1+1)} = \alpha_{1n_1}(x_1, u_0, \dot{u}_0, \ldots, u_0^{(n_1-1)})$. By choosing

$$u_1 = -c_{1n_1}z_{1n_1} - z_{1(n_1-1)}u_0 + \sum_{k=1}^{n_1-1}\frac{\partial\alpha_{1(n_1-1)}}{\partial x_{1k}}x_{1(k+1)}u_0 + \sum_{k=1}^{n_1-1}\frac{\partial\alpha_{1(n_1-1)}}{\partial\mu_{1k}}\dot{\phi}_k \tag{21}$$

the time derivative of the function

$$V_{1n_1} = V_{1(n_1-1)} + \tfrac{1}{2}z_{1n_1}^2 = \tfrac{1}{2}z_{11}^2 + \cdots + \tfrac{1}{2}z_{1n_1}^2 \tag{22}$$

along the solutions of (3) satisfies

$$\dot V_{1n_1} = -\sum_{k=1}^{n_1-1} c_{1k} z_{1k}^2 u_0^{2l_1+2} - c_{1n_1} z_{1n_1}^2. \tag{23}$$

Under the new coordinates $z_1 = (z_{11}, \ldots, z_{1n_1})$, the x_1-system (3) in closed loop with (21) is transformed into

$$\begin{aligned}
\dot z_{11} &= -c_{11} z_{11} u_0^{2l_1+2} + z_{12} u_0,\\
&\vdots\\
\dot z_{1(n_1-1)} &= -z_{1(n_1-2)} u_0 - c_{1(n_1-1)} z_{1(n_1-1)} u_0^{2l_1+2} \\
&\quad + z_{1n_1} u_0,\\
\dot z_{1n_1} &= -z_{1(n_1-1)} u_0 - c_{1n_1} z_{1n_1}.
\end{aligned} \tag{24}$$

Step i ($2 \leqslant i \leqslant m$): Repeating the same procedure in Step 1 for the x_i-subsystem of (1),

$$\begin{aligned}
\dot x_{i1} &= x_{i2} u_0,\\
&\vdots\\
\dot x_{i(n_i-1)} &= x_{in_i} u_0,\\
\dot x_{in_i} &= u_i
\end{aligned} \tag{25}$$

with u_i viewed as the control input, there exist intermediate stabilizing control functions α_{ik} ($1 \leqslant k \leqslant n_i$) such that

$$\alpha_{ik}(0, \ldots, 0, \mu_1, \ldots, \mu_k) = 0 \quad \forall (\mu_1, \ldots, \mu_k) \in \mathbb{R}^k. \tag{26}$$

Further, under the new coordinates $z_i = (z_{i1}, \ldots, z_{in_i})$ defined by

$$\begin{aligned}
z_{i1} &= x_{i1},\\
z_{ik} &= x_{ik} - \alpha_{i(k-1)}(x_{i1}, \ldots, x_{i(k-1)}, \phi_1, \ldots, \phi_{k-1}),\\
&\qquad 2 \leqslant k \leqslant n_i
\end{aligned} \tag{27}$$

and in closed loop with

$$u_i = \alpha_{in_i}(x_i, u_0, \ldots, u_0^{(n_i-1)}), \tag{28}$$

the system (25) is put into

$$\begin{aligned}
\dot z_{i1} &= -c_{i1} z_{i1} u_0^{2l_1+2} + z_{i2} u_0,\\
&\vdots\\
\dot z_{i(n_i-1)} &= -z_{i(n_i-2)} u_0 - c_{i(n_i-1)} z_{i(n_i-1)} u_0^{2l_1+2} \\
&\quad + z_{in_i} u_0,\\
\dot z_{in_i} &= -z_{i(n_i-1)} u_0 - c_{in_i} z_{in_i},
\end{aligned} \tag{29}$$

where the c_{ik}'s are arbitrary positive constants. More importantly, the time derivative of the function

$$V_{in_i} = \tfrac{1}{2}z_{i1}^2 + \cdots + \tfrac{1}{2}z_{in_i}^2 \tag{30}$$

along the solutions of (29) satisfies

$$\dot V_{in_i} = -\sum_{k=1}^{n_i-1} c_{ik} z_{ik}^2 u_0^{2l_1+2} - c_{in_i} z_{in_i}^2. \tag{31}$$

Remark 1. It is of interest to note that Eqs. (23) and (31) hold as long as u_0 is a smooth function of time. This important observation will be used in the stability analysis in Section 3.

Step m + 1: As already mentioned in Remark 1, the previous steps do not impose any condition on u_0 except the smoothness.

Denote $p = \max\{n_1 - 1, \ldots, n_m - 1\}$ and let u_0 be an output of the following nonlinear time-varying system:

$$\begin{aligned}
\dot y_1 &= y_2,\\
&\vdots\\
\dot y_{p-1} &= y_p,\\
\dot y_p &= a_0 x_0 + a_1 y_1 + \cdots \\
&\quad + a_p y_p + \kappa(z_1, \ldots, z_m) \sin(t),\\
u_0 &= y_1,
\end{aligned} \tag{32}$$

where the a_j's are real numbers such that $s^{p+1} - a_p s^p - \cdots - a_1 s - a_0$ is a Hurwitz polynomial and κ is a smooth function such that

$$\kappa(z_1, \ldots, z_m) = 0 \Leftrightarrow z_i = 0, \quad 1 \leqslant i \leqslant m. \tag{33}$$

Note that, with the choice (32), $u_0^{(k)} = y_{k+1}$ for all $0 \leqslant k \leqslant p-1$. As a result, the stabilizing control functions α_{ik} defined above are implementable and do not require any differentiator.

3. Main results

Motivated by Remark 1 and a remark made by Samson in [14], we first give a result on the regulation of partial-state $(x_1,\dots,x_m)$ of system (1).

To this end, in place of (32), one implements the following *static* control law for u_0:

$$u_0 = -c_0x_0 + \kappa_0(t), \tag{34}$$

where $c_0 > 0$ and κ_0 is a smooth function of positive time such that

$$-c_0\int_0^t e^{-c_0(t-\tau)}\kappa_0(\tau)\,d\tau + \kappa_0(t) \not\to 0, \quad \text{as } t\to\infty \tag{35}$$

and $\kappa_0(t)$ as well as its time derivatives of order up to p are bounded. Examples of such time-varying functions $\kappa_0(t)$ include $\sin(t)$, $\cos(2t)$ and $(t/t+1)\sin(2t)$.

The following proposition shows that the time-varying control laws defined in (34), (21) and (28) achieve the desired partial-state regulation.

Proposition 1 (Partial-state regulation). *All solutions of the closed-loop system comprised of* (1), (34), (21) *and* (28) *are well defined on* $[0,\infty)$ *and bounded. Furthermore,*

$$\lim_{t\to\infty}(|x_1(t)|+\cdots+|x_m(t)|) = 0. \tag{36}$$

Proof. By construction of u_0 in (34),

$$x_0(t) = e^{-c_0t}x_0(0) + \int_0^t e^{-c_0(t-\tau)}\kappa_0(\tau)\,d\tau. \tag{37}$$

Thus, $x_0(t)$ and $u_0(t)$ are bounded on $[0,\infty)$. It can immediately be verified that the time derivatives of u_0 of order up to p are bounded on $[0,\infty)$.

From (23) and (31), it follows that $(z_1(t),\dots,z_m(t))$ are bounded and therefore, with (27), $(x_1(t),\dots,x_m(t))$ are bounded. The first statement of Proposition 1 is then proved.

To prove the second statement, introduce

$$\psi(t) = \sum_{i=1}^{m}\sum_{k=1}^{n_i-1} c_{ik}z_{ik}(t)^2\,u_0(t)^{2l_i+2} + \sum_{i=1}^{m} c_{in_i}z_{in_i}(t)^2. \tag{38}$$

This function ψ is uniformly continuous since its derivative is bounded. Further, in view of (23) and (31), it is L^1. Hence, a direct application of Barbalat's lemma concludes that $\psi(t)$ converges to zero as t goes to ∞.

Thanks to (35), $u_0(t)$ does not converge to zero as t goes to ∞. Since the nonnegative function $\sum_{i=1}^m V_{in_i}$ is decreasing and tends to a finite number, it follows by a contradiction argument that $z_{ik}(t)$ goes to 0 for all i and k. Finally, with (26) and (27), property (36) follows readily. □

Next, one turns to the case of global stabilization with the control laws defined in (32), (21) and (28).

Proposition 2 (Global stabilization). *The equilibrium point* $x=0$, $y_1=\cdots=y_p=0$ *of the resulting time-periodic system composed of* (1), (21), (28) *and* (32) *is globally uniformly asymptotically stable.*

Proof. In view of (23) and (31),

$$\sum_{i=1}^{m}|z_i(t)|^2 \leqslant \sum_{i=1}^{m}|z_i(0)|^2. \tag{39}$$

Denoting $y=(x_0,y_1,\dots,y_p)$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1\\ a_0 & a_1 & a_2 & a_3 & \cdots & a_p \end{pmatrix} \in \mathbb{R}^{(p+1)\times(p+1)},$$

$$b = (0,\dots,0,1)^{\mathrm{T}} \in \mathbb{R}^{p+1}. \tag{40}$$

The cascaded system composed of $\dot{x}_0 = u_0$ and (32) is rewritten as

$$\dot{y} = Ay + b\kappa(z_1,\dots,z_m)\sin(t). \tag{41}$$

By construction, A is an asymptotically stable matrix. Therefore, from (39), it follows that $y(t)$ is globally uniformly bounded. Noticing that $u_0^{(k)} = y_{k+1}$ for all $0\leqslant k\leqslant p-1$, one establishes the global boundedness property for the closed-loop solutions $x(t)$, $y_j(t)$ for all $1\leqslant j\leqslant p$. In fact, by making use of the recurrent structure in the definitions (27) and the smoothness of the stabilizing functions α_{ik}'s, the uniform stability of the equilibrium $x=0$, $y_1=\cdots=y_p=0$ can be proved.

To establish the attractiveness, one can invoke LaSalle's invariance principle [10]. Towards this purpose, observe that the closed-loop system is rendered autonomous by introducing the additional system

$$\begin{pmatrix} \dot{\chi}_1 \\ \dot{\chi}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \begin{pmatrix} \chi_1(0) \\ \chi_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{42}$$

Indeed, $\sin(t) = \chi_1(t)$ for all $t \geqslant 0$.

Consider now the extended autonomous system composed of (24), (29), (41) and (42). By applying LaSalle's invariance principle, any trajectory $(z_1(t), \dots, z_m(t), y(t), \chi(t))$ (which has been proved to be bounded) tends to the largest invariant subset contained in the locus E of points defined by $\dot{V}_{in_i} = 0$ for all $1 \leqslant i \leqslant m$. Using (23) and (31), E is given by

$$E = E_1 \cup E_2, \tag{43}$$

where

$$E_1 = \{(z_1, \dots, z_m, y, \chi) \in \mathbb{R}^{N+p+2} \mid y_1 = z_{in_i} = 0, \; 1 \leqslant i \leqslant m\}, \tag{44}$$

$$E_2 = \{(z_1, \dots, z_m, y, \chi) \in \mathbb{R}^{N+p+2} \mid z_{ik} = 0, \; 1 \leqslant k \leqslant n_i, \; 1 \leqslant i \leqslant m\}. \tag{45}$$

It is shown that the largest invariant subset is E_2. Then, all $z_i(t)$'s converge to 0 as t goes to ∞. Property (33) together with (41) implies that $y(t)$ goes to 0 as t goes to ∞. Finally, $x(t)$ tends to 0.

Clearly, E_2 is an invariant set. If E_2 were not the largest invariant subset, there would exist a trajectory

$$(z_1(t), \dots, z_m(t), y(t), \chi(t)) \in E \quad \forall t \geqslant 0 \tag{46}$$

and a nonempty open interval, say I_0, such that

$$(z_1(t), \dots, z_m(t)) \neq (0, \dots, 0) \quad \forall t \in I_0. \tag{47}$$

This implies that $u_0(t) = 0$ for all $t \in I_0$. Back to the systems (24) and (29), one sees that the z_{ik}'s are constant on I_0 since the derivatives are identically zero. In addition, referring back to the system (41) or (32), the following holds:

$$\left.\begin{aligned} \dot{x}_0 &= 0 \\ 0 &= a_0 x_0 + \kappa(z_1(t), \dots, z_m(t)) \sin(t) \end{aligned}\right\} \quad \forall t \in I_0 \tag{48}$$

which leads to a contradiction with (47) and (33). □

4. Simulation results

The fire truck studied in previous work [2, 18] is a simple example of three-input nonholonomic mechanical system which can be converted into multi-input chained form (1). The dynamics of a fire truck (see Fig. 1) are described by

$$\begin{aligned} \dot{x} &= v_0, \\ \dot{y} &= v_0 \tan\theta_0, \\ \dot{\phi}_0 &= v_1, \\ \dot{\theta}_0 &= \frac{v_0}{L_0} \tan\phi_0 \sec\theta_0, \\ \dot{\phi}_1 &= v_2, \\ \dot{\theta}_1 &= -\frac{v_0}{L_1} \sin(\phi_1 - \theta_0 + \theta_1) \sec\phi_1 \sec\theta_0. \end{aligned} \tag{49}$$

The system (49) is put into a system of the form (1) under the following change of coordinates:

$$\begin{aligned} x_0 &= x, \quad x_{11} = y, \quad x_{12} = \tan\theta_0, \\ x_{13} &= \frac{1}{L_0} \sec^3\theta_0 \tan\phi_0, \quad x_{21} = \theta_1, \\ x_{22} &= -\frac{1}{L_1} \sin(\phi_1 - \theta_0 + \theta_1) \sec\phi_1 \sec\theta_0 \end{aligned}$$

and the following change of feedback:

$$\begin{aligned} u_0 &= v_0, \\ u_1 &= \frac{3v_0}{L_0^2} \sec^5\theta_0 \sin\theta_0 \tan^2\phi_0 + \frac{v_1}{L_0} \sec^3\theta_0 \sec^2\phi_0, \end{aligned}$$

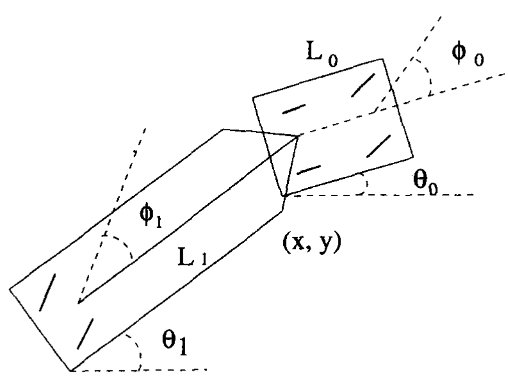

Fig. 1. Kinematic model of the fire truck.

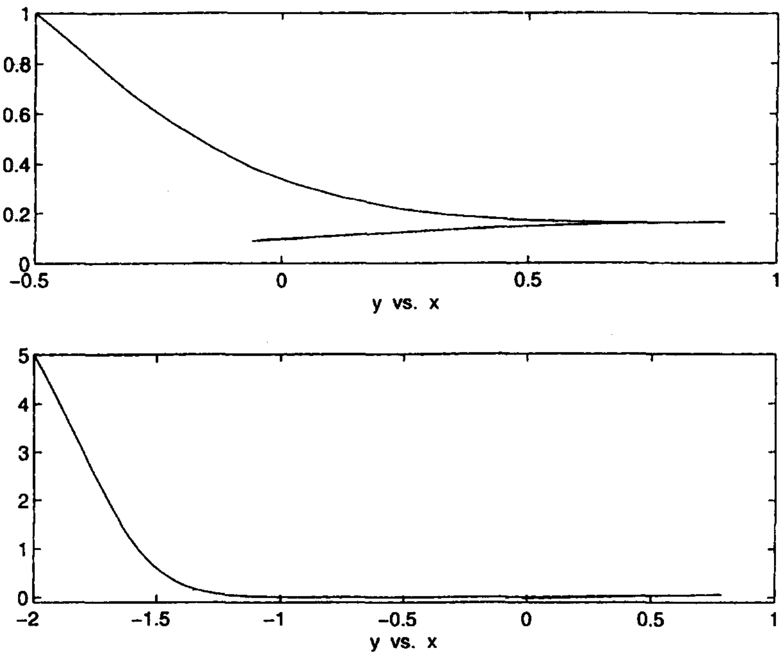

Fig. 2. Parallel parking trajectory with different initial conditions.

$$u_2 = \frac{v_0}{L_0 L_1}\big[\cos(\phi_1 - \theta_0 + \theta_1)\sec\phi_1 \sec\theta_0 - \sin(\phi_1 - \theta_0 + \theta_1)\sec\phi_1 \sin\theta_0 \sec^2\theta_0\big] \times \tan\phi_0 \sec\theta_0 + \frac{v_0}{2L_1^2}\sin(2\phi_1 - 2\theta_0 + 2\theta_1)\sec^2\phi_1 \sec^2\theta_0 - \frac{v_2}{L_1}\big(\cos(\phi_1 - \theta_0 + \theta_1)\sec\phi_1 \sec\theta_0 + \sin(\phi_1 - \theta_0 + \theta_1)\sec\theta_0 \sin\phi_1 \sec^2\phi_1\big).$$

Applying the design procedure in Section 2 gives the following time-varying dynamic stabilizers:

$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = -x_0 - 3y_1 - 3y_2 + 5(|z_1|^2 + |z_2|^2)\sin(t),$$
$$u_0 = y_1$$

and

$$u_1 = -z_{13} - z_{12}u_0 - 3c_1\dot{u}^2 x_{11} - 3c_1 u_0 \ddot{u}_0 x_{11} - (3u_0 + 2)c_1 u_0 \dot{u}_0 x_{12} - c_1 u_0^3 x_{13} - 3u_0^2 \dot{u}_0 z_{12} - u_0^3(z_{13}u_0 - u_0^4 z_{12} - z_{11}u_0) - z_{12}u_0 + c_1 u_0^4 z_{11},$$
$$u_2 = -z_{22} - z_{21}u_0 - 3c_2 u_0^2 \dot{u}_0 z_{21} - c_2 u_0^4 x_{22},$$

where $z_1 = (z_{11}, z_{12}, z_{13})$ and $z_2 = (z_{21}, z_{22})$ are defined by

$$z_1 = (x_{11},\ x_{12} + c_1 u_0^3 x_{11},\ x_{13} + 3c_1 u_0 \dot{u}_0 x_{11} + c_1 u_0^2 x_{12} + u_0^3 z_{12} + x_{11}),$$
$$z_2 = (x_{21},\ x_{22} + c_2 u_0^3 x_{21})$$

with $c_1, c_2 > 0$.

The simulations were performed using MATLAB for a parallel parking maneuver (see Fig. 2).

5. Concluding remarks

A new systematic design procedure is developed in this paper for a class of multi-input nonholonomic systems in chained form. The global stabilization of chained systems is solved by means of a new family of smooth time-varying dynamic feedback laws. The idea lying behind the methodology is a combined application of integrator backstepping and time-varying techniques. It is important to note that the algorithm can be extended to any dynamic model, that is, a system comprised of the kinematic model (1) appended with a chain of integrators. In contrast with previous work [13, 16, 18, 14] leading to static time-varying controllers, the main advantages of the proposed stepwise control design procedure are its simplicity and its easy adaptation to nonholonomic control systems with parametric uncertainty [5, 6]. More recently, the backstepping-based stabilization method presented in this paper is applied to design global tracking controllers for a wheeled mobile robot [7].

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