Shadow Vector Field Construction for small h

Symbolic Code

In[1729]:=
```
Clear["Global`*"];
Remove["Global`*"];
(*Helper Functions *)
dot[f_, y_, k_] := Module[{i, fdot},
  fdot = D[f, y] * f;
  For[i = 2, i ≤ k, i++,
   fdot = D[fdot, y] * f];
  fdot]
computeYTilde[fi_, y_, h_, n_] := Module[{i, yTilde},
  yTilde = y + h * fi;
  i = 2;
  While[i ≤ n, yTilde = yTilde + h^i / i! * dot[fi, y, i - 1]; i++];
  yTilde
 ]

computeShadowVF[fOriginal_, ϕ_, h_, y_, n_, k_] := Module[
  { fShadow = Table[fShadow_i, {i, 1, k}], deltaf = Table[deltaf_i, {i, 1, k}], yTilde, i},
  fShadow_1 = fOriginal; deltaf_1 = fOriginal;
  For[i = 1, i < k, i++, (*k = no of terms in shadow vf *)
   yTilde = computeYTilde[fShadow_i, y, h, n]; (* n = No of terms in expansion *)
   deltaf_i+1 = D[ϕ - yTilde, {h, i + 1}] / (i + 1) !;
   fShadow_i+1 = fShadow_i + h^(i) * deltaf_i+1
  ];
  fShadow_k]
```

In[1991]:=
```
(*Symbolic Testing and usage of above function *)
f = y^2;
ϕ = y + h * f; n = 2; k = 3;
fShadow = computeShadowVF[f, ϕ, h, y, n, k];
Simplify[fShadow /. {h → 0.01}]
```

Out[1994]= $y^2\left(1. - 0.01\,y + 0.00025\,y^2 - 6.\times 10^{-6}\,y^3\right)$

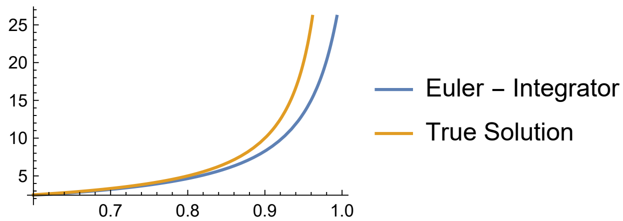
Test Numerically

In[2209]:=
```
(* Plot True output and integrated output together*)
Clear["Global`*"];
Remove["Global`*"];
```

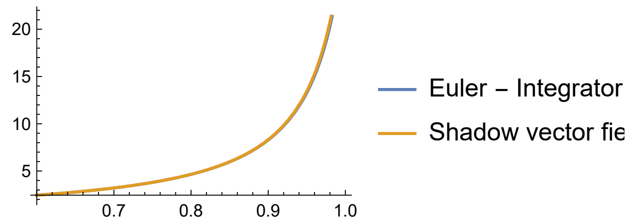
```
(*Helper Functions *)
dot[f_, y_, k_] := Module[{i, fdot},
  fdot = D[f, y] * f;
  For[i = 2, i ≤ k, i++,
   fdot = D[fdot, y] * f];
  fdot]
computeYTilde[fi_, y_, h_, n_] := Module[{i, yTilde},
  yTilde = y + h * fi;
  i = 2;
  While[i ≤ n, yTilde = yTilde + h^i / i! * dot[fi, y, i - 1]; i++];
  yTilde
 ]

computeShadowVF[fOriginal_, ϕ_, h_, y_, n_, k_] := Module[
  { fShadow = Table[fShadow_i, {i, 1, k}], deltaf = Table[deltaf_i, {i, 1, k}], yTilde, i},
  fShadow_1 = fOriginal; deltaf_1 = fOriginal;
  For[i = 1, i < k, i++, (*k = no of terms in shadow vf *)
   yTilde = computeYTilde[fShadow_i, y, h, n]; (* n = No of terms in expansion *)
   deltaf_(i+1) = D[ϕ - yTilde, {h, i + 1}] / (i + 1) !;
   fShadow_(i+1) = fShadow_i + h^(i) * deltaf_(i+1)
  ];
  fShadow_k]

hDiscrete = 0.01;
tFinal = 1; tInitial = 0.6;
nSteps = IntegerPart[tFinal / hDiscrete];
xSol =
  RecurrenceTable[{a[n + 1] == a[n] + hDiscrete * a[n]^2, a[1] == 1}, a, {n, 1, nSteps}];
fEuler = Interpolation[xSol];
p1 = Labeled[Plot[{fEuler[1 + nSteps * t], 1 / (1 - t)}, {t, tInitial, tFinal},
    PlotLegends → {"Euler - Integrator", "True Solution"}], "h = 0.01"];
f = y^2;
ϕ = y + h * f; n = 2; k = 3;
fShadow = computeShadowVF[f, ϕ, h, y, n, k];
fplot[t_] := fShadow /. {y → y[t], h → hDiscrete};
ySol = NDSolveValue[{y'[t] == fplot[t], y[0] == 1},
   y, {t, 0, 1}, Method → {"DiscontinuityProcessing" → None}];
p2 = Labeled[Plot[{fEuler[1 + nSteps * t], ySol[t]}, {t, tInitial, tFinal},
    PlotLegends → {"Euler - Integrator", "Shadow vector field"}], "h = 0.01"];
Grid[{{p1, p2}}]

Clear["Global`*"];
Remove["Global`*"];
(*Helper Functions *)
dot[f_, y_, k_] := Module[{i, fdot},
  fdot = D[f, y] * f;
  For[i = 2, i ≤ k, i++,
```

```
   fdot = D[fdot, y] * f];
  fdot]
computeYTilde[fi_, y_, h_, n_] := Module[{i, yTilde},
  yTilde = y + h * fi;
  i = 2;
  While[i ≤ n, yTilde = yTilde + h^i / i! * dot[fi, y, i - 1]; i++];
  yTilde
 ]

computeShadowVF[fOriginal_, ϕ_, h_, y_, n_, k_] := Module[
  { fShadow = Table[fShadow_i, {i, 1, k}], deltaf = Table[deltaf_i, {i, 1, k}], yTilde, i},
  fShadow_1 = fOriginal; deltaf_1 = fOriginal;
  For[i = 1, i < k, i++, (*k = no of terms in shadow vf *)
   yTilde = computeYTilde[fShadow_i, y, h, n]; (* n = No of terms in expansion *)
   deltaf_(i+1) = D[ϕ - yTilde, {h, i + 1}] / (i + 1) !;
   fShadow_(i+1) = fShadow_i + h^(i) * deltaf_(i+1)
  ];
  fShadow_k]

hDiscrete = 0.1;
tFinal = 1; tInitial = 0.6;
nSteps = IntegerPart[tFinal / hDiscrete];
xSol =
  RecurrenceTable[{a[n + 1] == a[n] + hDiscrete * a[n]^2, a[1] == 1}, a, {n, 1, nSteps}];
fEuler = Interpolation[xSol];
p1 = Labeled[Plot[{fEuler[1 + nSteps * t], 1 / (1 - t)}, {t, tInitial, tFinal},
    PlotLegends → {"Euler - Integrator", "True Solution"}], "h = 0.1"];
f = y^2;
ϕ = y + h * f; n = 2; k = 3;
fShadow = computeShadowVF[f, ϕ, h, y, n, k];
fplot[t_] := fShadow /. {y → y[t], h → hDiscrete};
ySol = NDSolveValue[{y'[t] == fplot[t], y[0] == 1},
   y, {t, 0, 1}, Method → {"DiscontinuityProcessing" → None}];
p2 = Labeled[Plot[{fEuler[1 + nSteps * t], ySol[t]}, {t, tInitial, tFinal},
    PlotLegends → {"Euler - Integrator", "Shadow vector field"}], "h = 0.1"];
Grid[{{p1, p2}}]
```

Out[2226]=

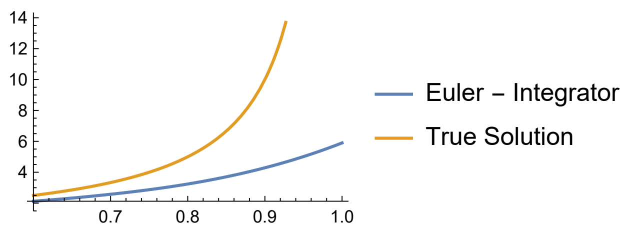


h = 0.01

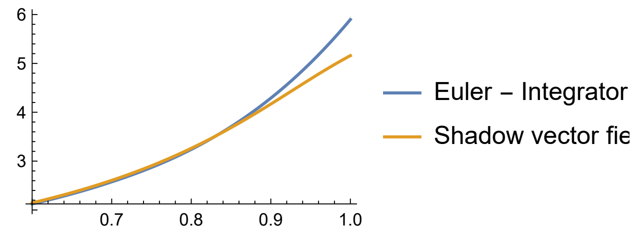


h = 0.01

Out[2244]=



h = 0.1



h = 0.1

Recursive Function for not small h (h = 0.1)

In[2245]:=
```
(* Apply the above functions recursively*)
Clear["Global`*"];
Remove["Global`*"];
dot[f_, y_, k_] := Module[{i, fdot},
  fdot = D[f, y] * f;
  For[i = 2, i ≤ k, i++,
   fdot = D[fdot, y] * f];
  fdot]
computeYTilde[fi_, y_, h_, n_] := Module[{i, yTilde},
  yTilde = y + h * fi;
  i = 2;
  While[i ≤ n, yTilde = yTilde + h^i / i! * dot[fi, y, i - 1]; i++];
  yTilde
 ]

computeShadowVF[fOriginal_, ϕ_, h_, y_, n_, k_] := Module[
  { fShadow = Table[fShadow_i, {i, 1, k}], deltaf = Table[deltaf_i, {i, 1, k}], yTilde, i},
  fShadow_1 = fOriginal; deltaf_1 = fOriginal;
  For[i = 1, i < k, i++, (*k = no of terms in shadow vf *)
   yTilde = computeYTilde[fShadow_i, y, h, n]; (* n = No of terms in expansion *)
   deltaf_i+1 = D[ϕ - yTilde, {h, i + 1}] / (i + 1) !;
   fShadow_i+1 = fShadow_i + h^(i) * deltaf_i+1
  ];
  fShadow_k]
fCleanUp[f_, y_, order_] := Module[{},
  fClean = f /. {y → 0}; i = 1;
  While[i ≤ order,
   fClean = fClean + (D[f, {y, i}] /. {y → 0}) * 1 / i! * y^i; (* Coeff of y^i *)
   i++];
  fClean]
```

In[1879]:=
```
(*Only consider terms upto y^5 *)
f = y^2;
ϕ = y + h * f; n = 2; k = 3;
fNew = f; hStep = 0; deltah = 0.001; hFinal = 0.1;
While[hStep ≤ hFinal,
 fNew = computeShadowVF[fNew, ϕ /. {h → (hStep + δh)}, δh, y, n, k];
 fNew = fNew /. {δh → deltah};
 fNew = fCleanUp[fNew, y, 10];
 hStep = hStep + deltah]
```

In[1883]:=

```
fNew
```

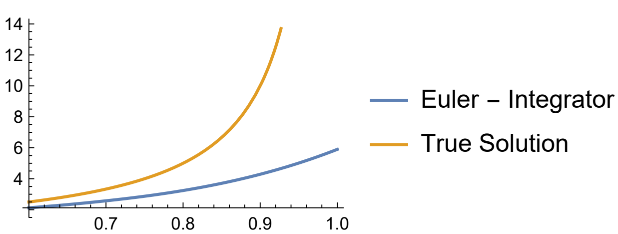
Out[1883]=

$$0. + 1.\, y^2 - 0.101\, y^3 + 0.0128775\, y^4 - 0.00184891\, y^5 + 0.000285504\, y^6 - 0.0000463226\, y^7 + 7.7907 \times 10^{-6}\, y^8 - 1.3466 \times 10^{-6}\, y^9 + 2.37829 \times 10^{-7}\, y^{10}$$

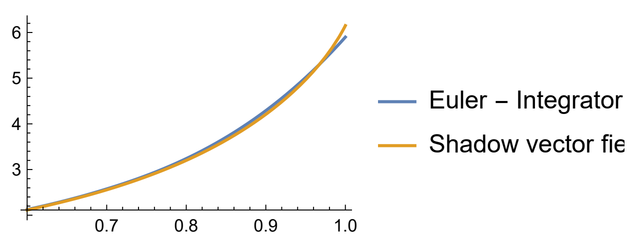
In[2251]:=

```
hDiscrete = 0.1;
tFinal = 1; tInitial = 0.6;
nSteps = IntegerPart[tFinal / hDiscrete];
xSol =
  RecurrenceTable[{a[n+1] == a[n] + hDiscrete * a[n]^2, a[1] == 1}, a, {n, 1, nSteps}];
fEuler = Interpolation[xSol];
p1 = Labeled[Plot[{fEuler[1 + nSteps * t], 1 / (1 - t)}, {t, tInitial, tFinal},
    PlotLegends → {"Euler - Integrator", "True Solution"}], "h = 0.1"];

fNew = 0.` + 1.` y^2 - 0.10100000000000005` y^3 + 0.012877499999999986` y^4 -
   0.001848905999999999` y^5 + 0.00028550427500000016` y^6 - 0.00004632256172500001` y^7 +
   7.790698251123754`*^-6 y^8 - 1.3466010149608202`*^-6 y^9 + 2.3782936655733824`*^-7 y^10;
(* Specific to step size of h = 0.1 *)

fplot[t_] := fNew /. {y → y[t]};
ySol = NDSolveValue[{y'[t] == fplot[t], y[0] == 1},
   y, {t, 0, 1}, Method → {"DiscontinuityProcessing" → None}];
p2 = Labeled[Plot[{fEuler[1 + nSteps * t], ySol[t]}, {t, tInitial, tFinal},
    PlotLegends → {"Euler - Integrator", "Shadow vector field"}], "h = 0.1"];
Grid[{{p1, p2}}]
tInitial = 0.9;
p1 = Labeled[Plot[{fEuler[1 + nSteps * t], 1 / (1 - t)}, {t, tInitial, tFinal},
    PlotLegends → {"Euler - Integrator", "True Solution"}], "h = 0.1"];
p2 = Labeled[Plot[{fEuler[1 + nSteps * t], ySol[t]}, {t, tInitial, tFinal},
    PlotLegends → {"Euler - Integrator", "Shadow vector field"}], "h = 0.1"];
Grid[{{p1, p2}}]
```

Out[2261]=

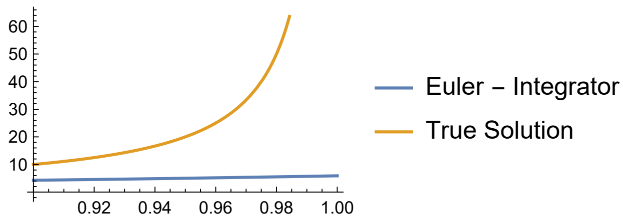


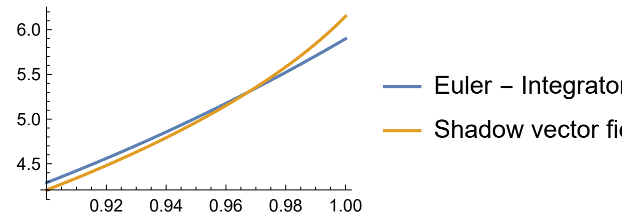
h = 0.1



h = 0.1

Out[2265]=






```
h = 0.1
```

```
h = 0.1
```