# The Output Properties of Volterra Systems (Nonlinear Systems with Memory) Driven by Harmonic and Gaussian Inputs

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**Abstract—Troublesome distortions often occur in communication systems. For a wide class of systems such distortions can be computed with the help of Volterra series.**

**Results, both old and new, which will aid the reader in applying Volterra-series-type analyses to systems driven by sine waves or Gaussian noise are presented.**

**The $n$-fold Fourier transform $G_n$ of the $n$th Volterra kernel plays an important role in the analysis. Methods of computing $G_n$ from the system equations are described and several special systems are considered. When the $G_n$ are known, items of interest regarding the output can be obtained by substituting the $G_n$ in general formulas derived from the Volterra series representation. These items include expressions for the output harmonics, when the input is the sum of two or three sine waves, and the power spectrum and various moments, when the input is Gaussian. Special attention is paid to the case in which the Volterra series consists of only the linear and quadratic terms.**

## PART I: STATEMENT OF RESULTS AND EXAMPLES

### I. Introduction

VOLTERRA SERIES were introduced into nonlinear circuit analysis in 1942 by Wiener. Later Wiener [1] extended the theory and applied it in a general way to a number of problems including FM spectra. Since Wiener's early work many reports and papers have dealt with the subject.

Volterra series are particularly useful in calculating small (but troublesome) distortions in communication systems and have been used recently to determine the distortion produced in various types of amplifiers [2]–[6]. The distortion due to filters in an FM system can also be expressed as a Volterra series.

The object of this paper is to present results, both old and new, that will aid the reader in applying Volterra-series-type analyses to systems driven by sine waves or Gaussian noise. The events which led to this paper began when we learned of Mircea's [7] elegant series for the power spectrum of the distortion produced by filters in an FM system and of its extension by Mircea and Sinnreich [5] to systems described by Volterra series. In papers [8]–[10], which we published before we were aware of Mircea's work, we gave the second- and third-order modulation terms in Mircea's FM series. In a subsequent exchange of letters [11] (to the Editor of the PROCEEDINGS OF THE IEEE) with Mircea some new ideas were brought out. The present paper extends these ideas.

In content our paper resembles a 1955 paper by Deutsch [12] and (to a lesser degree) the 1959 report by George [13], in which a number of the topics discussed here are treated somewhat differently.

Volterra series have been described as "power series with memory" which express the output of a nonlinear system in "powers" of the input $x(t)$. A substantial number of the systems encountered in communication problems can be represented as Volterra series. We shall write the series for the typical system as

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!}\int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n g_n(u_1, \cdots, u_n) \prod_{r=1}^{n} x(t-u_r) \quad (1)$$

where $y(t)$ is the output, $x(t)$ the input, and the kernels $g_n(u_1, \cdots, u_n)$ describe the system.[1] It will be noted that the first-order kernel $g_1(u_1)$ is simply the familiar impulse response of a linear network. The higher order kernels can thus be viewed as higher order impulse responses which serve to characterize the various orders of nonlinearity.

The coefficient $1/n!$ in (1) is not used by most writers. We insert it because it simplifies many of our equations. Some authors allow the kernels to be unsymmetric functions of the $u$'s; however, symmetry is necessary for the results presented here. If the response of a system is obtained as a series of the form (1) containing an unsymmetric kernel say $\gamma_n$, in place of $g_n$, a symmetric kernel can be obtained by "symmetrization." This process consists of permuting the subscripts on the $u_i$ in all $n!$ ways, and taking $g_n$ to be $1/n!$ times the sum of the resulting $\gamma_n$.

The $n$-fold Fourier transform

$$G_n(f_1, \cdots, f_n) = \int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n g_n(u_1, \cdots, u_n) \cdot \exp\left[-j(\omega_1 u_1 + \cdots + \omega_n u_n)\right] \quad (2)$$

where $\omega_i = 2\pi f_i$ plays an important role in the analysis. $G_0$ is identically zero because our Volterra series starts with $n=1$ (instead of $n=0$, which would imply an active system, i.e., an output without an input). Also, $G_1(f_1)$ will be recognized as the familiar transfer function of a linear network. Thus the transform of the $n$th-order Volterra kernel is seen to be analogous to an $n$th-order transfer function. We shall refer to $G_n(f_1, \cdots, f_n)$ as the "$n$th-order Volterra transfer function." Since $g_n(u_1, \cdots, u_n)$ is a symmetric function of $u_1, \cdots, u_n$, it follows that $G_n(f_1, \cdots, f_n)$ is a symmetric function of $f_1, \cdots, f_n$. As discussed in Section III, in many cases $G_n$ can be obtained without first computing $g_n$.

Suppose that the $G_n$, $n=1, 2, \cdots$, for a particular system are known. Suppose further that the input $x(t)$ to the system in (1) consists of 1) one or more sine waves, 2) Gaussian noise, 3) a sine wave plus Gaussian noise, or 4) a random pulse train. Then we can obtain expressions for a number of items of interest regarding the output $y(t)$ by substituting the $G_n$ in formulas derived from the Volterra series for $y(t)$. The leading terms in some of these formulas are listed in Section II. This list is intended to be a guide to the complete formulas derived in the later sections.

The complete formulas are infinite series in which the labor of computing the $n$th term increases rapidly as $n$ increases. Fortunately, in the study of communication systems it is often possible to neglect

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[1] The arguments of $g_n(u_1, \cdots, u_n)$ and its Fourier transform $G_n(f_1, \cdots, f_n)$ in (2) will occasionally be omitted for brevity when the meaning is clear.

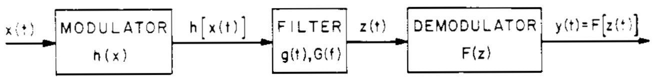


Fig. 1. Modulator–filter–demodulator system.

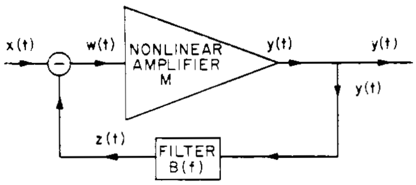


Fig. 2. Feedback system with nonlinear amplifier.

modulation terms (i.e., terms in the Volterra series) of order higher than the second or third.

In practice it appears that Volterra series do not enable us to do anything that cannot be done otherwise. However, a direct attack on modulation problems often leads to a morass of algebra. The Volterra series approach has the virtue that many such problems can be treated in an orderly way by first computing the $G_n$ and then substituting them in the appropriate general formulas.

For convenience, the paper is separated into two major parts: I, Statement of Results and Examples and II, Derivation of Formulas. A summary form of the leading terms of the principal results follows these introductory remarks. Then methods are presented for computing the Volterra transfer functions $G_n(f_1, \cdots, f_n)$ for several types of systems. Finally, a number of examples are worked out to illustrate the use of the formulas. Among these are 1), the general modulator–filter–demodulator system, shown in Fig. 1, which is used to obtain the leading terms in Mircea's series for FM distortion and 2) the nonlinear system with feedback, shown in Fig. 2, recently treated by Narayanan [6].

Part II consists largely of derivations of the various results. These include expressions for the output and its spectrum for a number of inputs. In addition, moments of higher order are considered for the case of a Gaussian input in order to obtain expressions for the cumulants of the output distribution. Formulas are listed which give information about the probability density of $y(t)$ when the cumulants are known. Known results concerning the distribution of quadratic forms are applied to obtain an expression for the probability density of $y(t)$ when $x(t)$ is Gaussian and terms beyond the second are omitted in the Volterra series for $y(t)$.

## II. Leading Terms in Formulas

This section lists the leading terms of formulas which give information about the output $y(t)$ for a number of inputs $x(t)$ when the Volterra transfer functions $G_n$ are known. Most of the leading terms listed do not go beyond $G_3(f_1, f_2, f_3)$. Note that $G_n$ is symmetric.

Although the list is intended as a guide to the complete formulas given in the later sections, the leading terms often suffice. In fact, the reader should not expect too much practical help from the complete formulas. Usually only two or three terms beyond those listed in this section can be used with present-day computers because of the rapid increase in complexity.

### A. Sinusoidal Inputs

When $x(t) = P \cos pt$, (137) gives the complete series

$$y(t) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} \left(\frac{P}{2}\right)^n \frac{\exp[j(2k-n)pt]}{k!(n-k)!} G_{k,n-k}(f_p) \tag{3}$$

where $p = 2\pi f_p$ and $G_{k,n-k}(f_p)$ denote $G_n(f_1, \cdots, f_n)$ with the first $k$ of the $f_i$ equal to $f_p$ and the remaining $n-k$ equal to $-f_p$. For the example of the memoryless nonlinearity in the next section, (23) shows that $G_n(f_1, \cdots, f_n)$ reduces to a constant $a_n$, and the series (3) may either converge or diverge depending upon $P$ and the $a_n$. The leading terms in (3) give

$$\begin{aligned} y(t) = &\left[\frac{P^2}{4} G_2(f_p, -f_p) + \cdots\right] \\ &+ e^{jpt}\left[\frac{P}{2} G_1(f_p) + \frac{P^3}{16} G_3(f_p, f_p, -f_p) + \cdots\right] \\ &+ e^{j2pt}\left[\frac{P^2}{8} G_2(f_p, f_p) + \cdots\right] \\ &+ e^{j3pt}\left[\frac{P^3}{48} G_3(f_p, f_p, f_p) + \cdots\right] \\ &+ e^{-jpt}\left[\frac{P}{2} G_1(-f_p) + \frac{P^3}{16} G_3(-f_p, -f_p, f_p) + \cdots\right] \\ &+ e^{-j2pt}\left[\frac{P^2}{8} G_2(-f_p, -f_p) + \cdots\right] \\ &+ e^{-j3pt}\left[\frac{P^2}{48} G_3(-f_p, -f_p, -f_p) + \cdots\right] + \cdots. \end{aligned} \tag{4}$$

Replacing $pt$ by $(pt+\varphi)$ in the exponents in (3) and (4) gives $y(t)$ when $x(t) = P \cos (pt+\varphi)$.

When $x = P \cos pt + Q \cos qt$ where $p$ and $q$ are incommensurable, the complete series for the $\exp[j(Np+Mq)t]$ component of $y(t)$ is given by (139). The leading terms in the dc and some of the lower order components of $y(t)$ are

$$\begin{gathered} \left[\frac{P^2}{4} G_2(f_p, -f_p) + \frac{Q^2}{4} G_2(f_q, -f_q)\right] \\ e^{jpt}\left[\frac{P}{2} G_1(f_p) + \frac{P^3}{16} G_3(f_p, f_p, -f_p) + \frac{PQ^2}{8} G_3(f_p, f_q, -f_q)\right] \\ e^{j2pt} \frac{P^2}{8} G_2(f_p, f_p) \qquad e^{j(p+q)t} \frac{PQ}{4} G_2(f_p, f_q) \\ e^{j(2p+q)t} \frac{P^2 Q}{16} G_3(f_p, f_p, f_q). \end{gathered} \tag{5}$$

Changing the signs of $q$ and $f_q$ in the $\exp[j(p+q)t]$ component gives the $\exp[j(p-q)t]$ component, and so on. When $p$ and $q$ are not incommensurable, some of the components coalesce. For example, if $q = 2p$ and $x(t) = P \cos pt + Q \cos 2pt$, the $2p-q$ and $-2p+q$ terms combine with the dc terms in (5) to give for the leading terms in the new dc component

$$\begin{aligned} &\left[\frac{P^2}{4} G_2(f_p, -f_p) + \frac{Q^2}{4} G_2(2f_p, -2f_p) + \frac{P^2 Q}{16} G_3(f_p, f_p, -2f_p)\right. \\ &\qquad \left. + \frac{P^2 Q}{16} G_3(-f_p, -f_p, 2f_p)\right]. \end{aligned} \tag{6}$$

Similarly, the leading terms in the new $\exp(jpt)$ component are given by the sum of the $\exp(jpt)$ component in (5) with $f_q = 2f_p$ plus

$$e^{jpt} \frac{PQ}{4} G_2(-f_p, 2f_p) \tag{7}$$

which is the contribution of the $(-p+q)$ term when $q = 2p$. This term is obtained from the $(p+q)$ term in (5) by changing the signs of $p$ and $f_p$ and then setting $q = 2p$ and $f_q = 2f_p$.

When $x(t) = P \cos pt + Q \cos qt + R \cos rt$, the complete series for the $\exp[j(Np+Mq+Lr)t]$ component of $y(t)$ is given by (140).

For example, the leading term in the exp $[j(p+q+r)t]$ component of $y(t)$ is

$$e^{j(p+q+r)t}\frac{PQR}{8}G_3(f_p, f_q, f_r). \tag{8}$$

Changing the signs of $r$ and $f_r$ gives the leading term in the exp $[j(p+q-r)t]$ component, and so on.

When $x(t)$ is equal to the sum of an infinite number of sinusoidal components in the sense that it possesses the Fourier transform $X(f)$, i.e.,

$$x(t) = \int_{-\infty}^{\infty} e^{j\omega t}X(f)df, \qquad \omega = 2\pi f \tag{9}$$

then substitution in (1) shows that $y(t)$ and its Fourier transform $Y(f)$ are given by

$$\begin{aligned} y(t) &= \sum_{n=1}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty} df_1\cdots\int_{-\infty}^{\infty} df_n G_n(f_1,\cdots,f_n) \\ &\quad\cdot e^{j(\omega_1+\cdots+\omega_n)t}\prod_{r=1}^{n} X(f_r) \\ Y(f) &= \frac{1}{1!}G_1(f)X(f) + \frac{1}{2!}\int_{-\infty}^{\infty} df_1 G_2(f_1, f-f_1)X(f_1)X(f-f_1) \\ &\quad + \frac{1}{3!}\int_{-\infty}^{\infty} df_1\int_{-\infty}^{\infty} df_2 G_3(f_1, f_2, f-f_1-f_2)X(f_1)X(f_2) \\ &\quad \cdot X(f-f_1-f_2) + \cdots. \end{aligned} \tag{10}$$

### B. Gaussian Noise Input

In the following, the input $x(t)$ is a zero-mean stationary Gaussian process with the two-sided power spectrum $W_x(f)$. The output $y(t)$ is a stationary process, and the ensemble averages $\langle [y(t)]^l\rangle$ and associated cumulants $\kappa_l$ do not change with $t$.

The leading terms in the complete series (147) for $\langle y(t)\rangle$ are

$$\begin{aligned} \langle y(t)\rangle &= \frac{1}{1!2}\int_{-\infty}^{\infty} df_1 W_x(f_1)G_2(f_1,-f_1) + \frac{1}{2!2^2}\int_{-\infty}^{\infty} df_1\int_{-\infty}^{\infty} df_2 \\ &\quad W_x(f_1)W_x(f_2)G_4(f_1,-f_1,f_2,-f_2) + \cdots. \end{aligned} \tag{11}$$

The leading terms in the complete series (177) for $\langle y^2(t)\rangle$ are

$$\begin{aligned} \langle y^2(t)\rangle &= \langle y(t)\rangle^2 + \int_{-\infty}^{\infty} df_1 W_x(f_1)G_1(f_1)G_1(-f_1) \\ &\quad + \int_{-\infty}^{\infty} df_1\int_{-\infty}^{\infty} df_2 W_x(f_1)W_x(f_2)[G_1(f_1)G_3(-f_1,f_2,-f_2) \\ &\quad + \tfrac{1}{2}G_2(f_1,f_2)G_2(-f_1,-f_2)] + \cdots. \end{aligned} \tag{12}$$

The triple integral comprising the third-order term not shown in (12) is given in (180) for the second cumulant $\kappa_2 = \langle y^2(t)\rangle - \langle y(t)\rangle^2$. The complete series for $\langle y^2(t)\rangle$ is a special case of the complete series for $\langle y(t+\tau)z(t)\rangle$ given by (152), (156)–(158). Here $z(t)$ is defined by a Volterra series obtained from the series (1) for $y(t)$ by replacing $g_n(u_1,\cdots,u_n)$ by a different kernel $g_n'(u_1,\cdots,u_n)$. Both $y(t)$ and $z(t)$ have the same Gaussian input $x(t)$. The $G_n'(f_1,\cdots,f_n)$ which appears in (157) is the Fourier transform of $g_n'(u_1,\cdots,u_n)$.

The first cumulant for the probability density of $y(t)$ is $\kappa_1 = \langle y(t)\rangle$, the second is $\kappa_2 = \langle y^2(t)\rangle - \langle y(t)\rangle^2$, and from (180) the leading terms in $\kappa_3$ and $\kappa_4$ are

$$\kappa_3 = 3\int_{-\infty}^{\infty} df_1\int_{-\infty}^{\infty} df_2 W_x(f_1)W_x(f_2)G_1(f_1)G_1(f_2)G_2(-f_1,-f_2) + \cdots$$

$$\begin{aligned} \kappa_4 &= 4\int_{-\infty}^{\infty} df_1\int_{-\infty}^{\infty} df_2\int_{-\infty}^{\infty} df_3 W_x(f_1)W_x(f_2)W_x(f_3)G_1(f_1)G_1(f_2) \\ &\quad\cdot[G_1(f_3)G_3(-f_1,-f_2,-f_3) + 3G_2(-f_1,f_3)G_2(-f_2,-f_3)] \\ &\quad + \cdots. \end{aligned} \tag{13}$$

The next terms beyond those shown in (13) are given in (180), but the general forms of the series for $\kappa_3$ and $\kappa_4$ are not known. The use of the first four cumulants to obtain information about the probability density of $y(t)$ is discussed in Part II and an example is furnished by (84).

The leading terms in the Mircea–Sinnreich [5] series (160) for the two-sided power spectrum $W_y(f)$ of $y(t)$ are shown in

$$\begin{aligned} W_y(f) &= \langle y(t)\rangle^2\delta(f) \\ &\quad + W_x(f)\left|G_1(f) + \frac{1}{2}\int_{-\infty}^{\infty} df_1 W_x(f_1)G_3(f,f_1,-f_1) + \cdots\right|^2 \\ &\quad + \frac{1}{2!}\int_{-\infty}^{\infty} df_1 W_x(f_1)W_x(f-f_1)|G_2(f_1,f-f_1) + \cdots|^2 \\ &\quad + \frac{1}{3!}\int_{-\infty}^{\infty} df_1\int_{-\infty}^{\infty} df_2 W_x(f_1)W_x(f_2)W_x(f-f_1-f_2) \\ &\quad \cdot|G_3(f_1,f_2,f-f_1-f_2) + \cdots|^2 + \cdots \end{aligned} \tag{14}$$

where $\langle y(t)\rangle$ is the dc component of $y(t)$ given by (11) and $\delta(f)$ is the unit impulse function. The right side of (14) shows all of the second-order terms (those which, when written out, contain the product of exactly two $W_x$) but only some of the third-order terms. All of the third-order terms, and some of the fourth- and fifth-order, would be shown if the double integral containing $G_5$ and the single integral containing $G_4$ were added inside the absolute value signs on the second and third lines, respectively, of (14).

When the number of terms in the Volterra series (1) is finite, the series (14) terminates and gives an exact expression for $W_y(f)$. When (14) does not terminate, its application to FM suggests that it may be an asymptotic series [8], [10], [14].

### C. Sine Wave Plus Noise Input

In the following, $x(t) = P\cos pt + I_N(t)$, where $I_N(t)$ is a zero-mean stationary Gaussian process with two-sided power spectrum $W_I(f)$.

The ensemble average of $x(t)$ at time $t$ is $\langle x(t)\rangle = P\cos pt$. Similarly, the ensemble average of $y(t)$ consists of a sum of sinusoidal harmonics of cos $pt$. The complete expression for $\langle y(t)\rangle$ at time $t$ is given by (164) and the leading terms are shown in

$$\begin{aligned} \langle y(t)\rangle &= \left[\frac{P^2}{4}G_2(f_p,-f_p) + \frac{1}{2}\int_{-\infty}^{\infty} df_1 W_I(f_1)G_2(f_1,-f_1) + \cdots\right] \\ &\quad + e^{jpt}\left[\frac{P}{2}G_1(f_p) + \frac{P^3}{16}G_3(f_p,f_p,-f_p)\right. \\ &\quad \left.+ \frac{P}{4}\int_{-\infty}^{\infty} df_1 W_I(f_1)G_3(f_1,-f_1,f_p) + \cdots\right] \\ &\quad + e^{j2pt}\left[\frac{P^2}{8}G_2(f_p,f_p) + \frac{P^2}{16}\int_{-\infty}^{\infty} df_1\right. \\ &\quad \left. W_I(f_1)G_4(f_1,-f_1,f_p,f_p) + \cdots\right] \\ &\quad + e^{j3pt}\left[\frac{P^3}{48}G_3(f_p,f_p,f_p) + \cdots\right] + \cdots \\ &\quad + \{\text{terms with } -p, -f_p \text{ for } p, f_p \text{ in } e^{jkpt}[\cdots], \\ &\qquad k = 1, 2, \cdots\}. \end{aligned} \tag{15}$$

where $f_p = p/2\pi$. When $I_N(t)$ is identically zero, $W_I(f)$ is zero, and (15) reduces to (4) for $y(t)$ when $x = P\cos pt$. When $P$ is zero, (15) reduces to (11) for $\langle y(t)\rangle$ when $x(t)$ is Gaussian.

The complete expression for the power spectrum of $y(t)$ is given by (175). The leading terms are shown in

$$\begin{aligned}
W_y(f) = {} & \{\text{spikes due to dc and sine waves in } \langle y(t)\rangle\} \\
& + W_I(f)\Big|G_1(f) + \frac{P^2}{4}G_3(f_p, -f_p, f) \\
& + \frac{1}{2}\int_{-\infty}^{\infty} df_1 W_I(f_1) G_3(f_1, -f_1, f) + \cdots\Big|^2 \\
& + W_I(f - f_p)\Big|\frac{P}{2}G_2(f_p, f - f_p) + \cdots\Big|^2 \\
& + W_I(f - 2f_p)\Big|\frac{P^2}{8}G_3(f_p, f_p, f - 2f_p) + \cdots\Big|^2 + \cdots \\
& + \{\text{terms with } -f_p \text{ for } f_p \text{ in } W_I(f - kf_p)|\cdots|^2, k = 1, 2, \cdots\} \\
& + \frac{1}{2!}\int_{-\infty}^{\infty} df_1 W_I(f_1) W_I(f - f_1)|G_2(f_1, f - f_1) + \cdots|^2 \\
& + \frac{1}{2!}\int_{-\infty}^{\infty} df_1 W_I(f_1) W_I(f - f_1 - f_p) \\
& \cdot\Big|\frac{P}{2}G_3(f_1, f_p, f - f_1 - f_p) + \cdots\Big|^2 \\
& + \frac{1}{2!}\int_{-\infty}^{\infty} df_1 W_I(f_1) W_I(f - f_1 + f_p) \\
& \cdot\Big|\frac{P}{2}G_3(f_1, -f_p, f - f_1 + f_p) + \cdots\Big|^2 \\
& + \frac{1}{3!}\int_{-\infty}^{\infty} df_1 \int_{-\infty}^{\infty} df_2 W_I(f_1) W_I(f_2) W_I(f - f_1 - f_2) \\
& \cdot|G_3(f_1, f_2, f - f_1 - f_2) + \cdots|^2 + \cdots. \qquad (16)
\end{aligned}$$

The spikes in $W_y(f)$ due to the dc and sine waves in $\langle y(t)\rangle$ can be computed from (15) for $\langle y(t)\rangle$. The spike due to the component $A_k(f_p, P)\exp(jkpt)$, $k = \cdots -1, 0, \cdots$, is $\delta(f - kf_p)|A_k(f_p, P)|^2$. When $P$ is zero, (16) reduces to (14) for $W_y(f)$ when $x(t)$ is Gaussian. When $I_N(t)$ is identically zero, $W_y(f)$ consists only of the spikes due to the sinusoidal components of $y(t)$.

*D. Random Pulse Train Input*

Finally we state a result of some interest which has not been thoroughly studied. The input is the pulse train

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \delta(t - nT). \qquad (17)$$

When the $a_n$ are identically distributed independent random variables whose probability density is even about $a_n = 0$, and the Volterra series for $y(t)$ stops with the quadratic term, it can be shown that the ensemble average of $y(t)$ consists of the periodic part (of period $T$)

$$\langle y(t)\rangle = \frac{\langle a^2\rangle}{2T}\sum_{m=-\infty}^{\infty} e^{j2\pi mt/T}\int_{-\infty}^{\infty} df_1 G_2\left(f_1, \frac{m}{T} - f_1\right) \qquad (18)$$

and that the power spectrum of $y(t)$ is

$$\begin{aligned}
W_y(f) = {} & \{\text{spikes due to } \langle y(t)\rangle\} + \frac{\langle a^2\rangle}{T}|G_1(f)|^2 \\
& + \frac{[\langle a^4\rangle - 3\langle a^2\rangle^2]}{4T}\left|\int_{-\infty}^{\infty} df_1 G_2(f_1, f - f_1)\right|^2 \\
& + \frac{\langle a^2\rangle^2}{2T^2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} df_1 G_2(f_1, f - f_1) \\
& \cdot G_2^*\left(f_1 - \frac{m}{T}, f - f_1 + \frac{m}{T}\right). \qquad (19)
\end{aligned}$$

Here $\langle a^m\rangle$ is the $m$th moment of $a_n$, and $G_2(f_1, f_2)$ is assumed to be such that the integrals and the sum converge. The asterisk denotes conjugate complex. Equations (18) and (19) can be proved by using the first two terms in (10) for $y(t)$ and the results

$$\sum_{n=-\infty}^{\infty} e^{-j\omega nT} = T^{-1}\sum_{n=-\infty}^{\infty}\delta(f - nT^{-1}), \qquad \omega = 2\pi f$$

$$W_y(f) = \int_{-\infty}^{\infty} d\tau \int_0^T dt\, T^{-1}\langle y(t+\tau) y^*(t)\rangle e^{-j\omega\tau}.$$

If the pulse shape is $F(t)$ instead of $\delta(t)$, the input is

$$x(t) = \sum_{n=-\infty}^{\infty} a_n F(t - nT) \qquad (20)$$

instead of (17), and the corresponding $\langle y(t)\rangle$ and $W_y(f)$ can be obtained by replacing $G_1(f_1)$, $G_2(f_1, f_2)$ in (18) and (19) by $S(f_1)G_1(f_1)$, $S(f_1)S(f_2)G_2(f_1, f_2)$ where

$$S(f) = \int_{-\infty}^{\infty} e^{-j\omega t} F(t)dt. \qquad (21)$$

## III. Computation of Volterra Transfer Functions

Considerable work has been done on the determination of the kernels $g_n(u_1, \cdots, u_n)$ and their Fourier transforms, the Volterra transfer functions $G_n(f_1, \cdots, f_n)$, by measurements made on the system [13], [15], [16]. Here we are principally interested in the calculation of the Volterra transfer functions when the system equations are known and the system can be represented by a Volterra series (which is not always the case).

One of the simplest of these calculations is furnished by the memoryless case

$$y(t) = \sum_{n=1}^{\infty} a_n[x(t)]^n/n! \qquad (22)$$

to which the Volterra series reduces when $g_n(u_1, \cdots, u_n)$ is equal to $a_n\,\delta(u_1)\cdots\delta(u_n)$ where $\delta(u)$ is the unit impulse function. Here $a_n$ is a constant and, from definition (2) of the Volterra transfer function,

$$G_n(f_1, \cdots, f_n) = a_n. \qquad (23)$$

This case is useful for checking the more complicated results.

This section is divided into three parts. The first describes a summation notation which is convenient in dealing with expressions which hold for general values of $n$. The second and third parts are concerned with two methods of computing $G_n$, namely, the "harmonic input" method and the "direct expansion" method.

The harmonic input method is useful in computing $G_n(f_1, \cdots, f_n)$ for the first few values of $n$ and can be used to obtain the expressions for $G_1(f_1)$, $G_2(f_1, f_2)$, and $G_3(f_1, f_2, f_3)$ listed for the examples in this section. The derivations of the expressions listed for $G_n(f_1, \cdots, f_n)$ for arbitrary $n$ are sketched in Sections V-A and V-B and make use of the direct expansion method.

*A. Summation Notation*

In order to explain the summation notation used to deal with $G_n$ for general values of $n$, we take for illustration the formula, derived in Section V-A,

$$G_n^{(l)}(f_1, \cdots, f_n) = l! \sum_{(v;l,n)} \sum_N' G_{v_1}(f_1, \cdots, f_{v_1}) G_{v_2}(f_{v_1+1}, \cdots, f_{v_1+v_2}) \cdots G_{v_l}(f_\mu, \cdots, f_n) \quad (24)$$

for the $n$-fold Fourier transform of the $n$th kernel in the Volterra series for $[y(t)]^l$, $l$ being a positive integer, and $1 \le l \le n$. $G_n^{(l)}(f_1, \cdots, f_n)$ is zero for $l > n$ and $G_n^{(n)}(f_1, \cdots, f_n)$ is equal to $n! G_1(f_1) G_1(f_2) \cdots G_1(f_n)$.

In (24) $\mu = v_1 + v_2 + \cdots + v_{l-1} + 1 = n - v_l + 1$ and $(v;\ l,\ n)$ beneath the leftmost $\sum$ denotes summation over sets of integers $v_i$ such that

$$v_1 + v_2 + \cdots + v_l = n, \qquad 1 \le v_1 \le v_2 \le \cdots \le v_l. \quad (25)$$

In other words, the summation is taken over those partitions of $n$ which have $l$ parts. The second summation $\sum_N'$ in (24) extends over the $N$ "nonidentical" products that can be obtained by permuting the subscripts on the $f$'s. $G_n(f_1, \cdots, f_n)$ is symmetric, and "identical" is used in the sense that $G_2(f_2, f_1)$ is identical with $G_2(f_1, f_2)$, and $G_1(f_2)G_1(f_1)$ is identical with $G_1(f_1)G_1(f_2)$. The number of terms in $\sum_N'$ is

$$N = n!/v_1! v_2! \cdots v_l! r_1! r_2! \cdots r_k! \quad (26)$$

where $r_1$ is the number of equal $v$'s in the first run of equalities in the arrangement $v_1 \le v_2 \le \cdots \le v_l$, $r_2$ the number in the second run, and so on. When the $v$'s are unequal, the $r$'s do not appear.

Sometimes, as in the case of Table I, given later, a table of partitions is of help in writing out the terms in the summation.

When $n=2$ and $l=2$, we have $v_1 = v_2 = 1$, $r_1 = 2$, $N = 2!/(1!1!2!) = 1$ and (24) becomes

$$\frac{1}{2!} G_2^{(2)}(f_1, f_2) = G_1(f_1) G_1(f_2). \quad (27)$$

When $n=3$ and $l=2$, we have $v_1 = 1$, $v_2 = 2$, $N = 3$, and

$$\frac{1}{2!} G_3^{(2)}(f_1, f_2, f_3) = \sum_3' G_1(f_1) G_2(f_2, f_3) = (1)(23) + (2)(13) + (3)(12) \quad (28)$$

in an abbreviated notation. When $n=4$ and $l=2$, there are two 2-part ($l$-part) partitions of $n$, $1+3=4$ and $2+2=4$, which give $v_1 = 1$, $v_2 = 3$, $N = 4$ and $v_1 = 2$, $v_2 = 2$, $N = 3$, respectively. Hence

$$\frac{1}{2} G_4^{(2)}(f_1, f_2, f_3, f_4) = (1)(234) + (2)(134) + (3)(124) + (4)(123) + (12)(34) + (13)(24) + (14)(23). \quad (29)$$

The number of products of $G$'s in the expressions for $G_2^{(2)}/2!$, $G_3^{(2)}/2!$, $G_4^{(2)}/2!$ are (by counting them in (27)–(29)) 1, 3, 7, respectively. These are the Stirling numbers of the second kind $S(n, 2)$ for $n = 2, 3, 4$. In general, the number of products in the sum (24) for $G_n^{(l)}(f_1, \cdots, f_n)/l!$ is $S(n, l)$. This can be shown by taking all of the $a_n$ to be unity in the memoryless case (22) (so that all of the $G_n$ are equal to unity) and using the fact that $l!\,S(n, l)$ is the coefficient of $x^n/n!$ in the expansion of $(e^x - 1)^l$.

### B. Harmonic Input Method

This method relies on the fact that a harmonic input must result in a harmonic output when (1) holds. Thus, when $x(t)$ is the sum

$$x(t) = \exp(j\omega_1 t) + \exp(j\omega_2 t) + \cdots + \exp(j\omega_n t) \quad (30)$$

where $\omega_i = 2\pi f_i$, $i = 1, 2, \cdots, n$, and the $\omega_i$ are incommensurable, $G_n$ is given by

$$G_n(f_1, \cdots, f_n) = \{\text{coefficient of the } \exp[j(\omega_1 + \cdots + \omega_n)t] \text{ term in the expansion of } y(t)\}. \quad (31)$$

This result follows from (1). It enables us to compute $G_1(f_1)$, $G_2(f_1, f_2), \cdots$ in succession. Thus, when we replace $x(t)$ by $\exp(j\omega_1 t)$ in the system equations and assume

$$y(t) = \sum_{k=1}^{\infty} c_k \exp(jk\omega_1 t). \quad (32)$$

$G_1(f_1)$ is equal to $c_1$. Similarly, $G_2(f_1, f_2)$ is equal to $c_{11}$ where

$$x(t) = \exp(j\omega_1 t) + \exp(j\omega_2 t)$$

$$y(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{kl} \exp[j(k\omega_1 + l\omega_2)t] \quad (33)$$

$$c_{00} = 0 \qquad c_{10} = G_1(f_1) \qquad c_{01} = G_1(f_2)$$

and $G_3(f_1, f_2, f_3)$ is equal to $c_{111}$ in an analogous triple sum where

$$c_{000} = 0 \qquad c_{110} = G_2(f_1, f_2)$$
$$c_{100} = G_1(f_1) \qquad c_{010} = G_1(f_2), \cdots. \quad (34)$$

To illustrate the use of this method, consider a system described by the equation

$$y(t) = x(t) + \varepsilon [x'(t)]^2 x''(t) \quad (35)$$

which arises in some forms of the quasi-static approximation to filtered FM. Here $\varepsilon$ is a constant and the prime denotes differentiation with respect to $t$. Setting $x(t)$ equal to $\exp(j\omega_1 t)$ gives $G_1(f_1) = 1$. Setting $x(t)$ equal to $\exp(j\omega_1 t) + \exp(j\omega_2 t)$ shows that $G_2(f_1, f_2)$ is zero. Setting $x(t)$ equal to the sum of three exponentials and picking out the coefficient of $\exp[j(\omega_1 + \omega_2 + \omega_3)t]$ in (35) gives $G_3$. Therefore,

$$G_1(f_1) = 1 \qquad G_2(f_1, f_2) = 0$$
$$G_3(f_1, f_2, f_3) = 2\varepsilon \omega_1 \omega_2 \omega_3 (\omega_1 + \omega_2 + \omega_3) \quad (36)$$

and $G_n$ is zero for $n > 3$. In the Volterra series associated with (35), $g_1(u_1)$ is $\delta(u_1)$ and $g_3$ is the sum of products of derivatives of impulse functions.

A broader application of the method can be illustrated by considering the system specified by the nonlinear differential equation

$$F(d/dt)y + \sum_{l=2}^{\infty} a_l y^l = x(t) \quad (37)$$

with the condition that $y(t)$ vanish identically when $x(t)$ does. It is assumed that one and only one such solution exists and that the system is stable. $F(d/dt)$ is a polynomial in $d/dt$, and the coefficients in $F(d/dt)$ and the coefficients $a_l$ are independent of $t$, $x$, and $y$. The first three $G_n$, derived from (30) through (34), and the recurrence relation derived in Section V-B are

$$G_1(f_1) = 1/F(j\omega_1)$$

$$G_2(f_1, f_2) = -2a_2 G_1(f_1) G_1(f_2)/F(j\omega_1 + j\omega_2)$$

$$G_3(f_1, f_2, f_3) = -\frac{2a_2 \sum_3' G_1(f_1) G_2(f_2, f_3) + 6a_3 G_1(f_1) G_1(f_2) G_1(f_3)}{F(j\omega_1 + j\omega_2 + j\omega_3)}$$

$$G_n(f_1, \cdots, f_n) = -\frac{\sum_{l=2}^{n} a_l G_n^{(l)}(f_1, \cdots, f_n)}{F(j\omega_1 + \cdots + j\omega_n)}. \quad (38)$$

The last equation is a recurrence relation because $G_n^{(l)}$ is given by (24) and (for $2 \le l \le n$) the right side of (24) is a combination of some or all (depending on $l$) of

$$G_1(f_1),\ G_2(f_1, f_2), \cdots, G_{n-1}(f_1, \cdots f_{n-1}).$$

As a specific example of the use of these results, let the input $x(t)$ be the voltage applied to a unit inductance connected in series with a slightly nonlinear resistance. The output $y(t)$ is the current through the circuit and is that solution of the Riccati equation

$$\frac{dy}{dt} + \alpha y + \varepsilon y^2 = x(t) \tag{39}$$

which tends to zero when $x(t)$ does. The existence and stability of such a solution is to be expected on physical grounds, provided $\alpha$ and $\varepsilon$ are such that the resistance $\alpha + \varepsilon y$ is almost never negative during the operation of the circuit. We regard $\varepsilon$ to be so small that $\varepsilon y$ is almost always small compared to $\alpha$.

In applying (37) to (39), we take $F(d/dt)$ to be $(d/dt)+\alpha$, $a_2 = \varepsilon$, and obtain from (38), with $\omega_i = 2\pi f_i$,

$$\begin{aligned}
G_1(f_1) &= (\alpha + j\omega_1)^{-1} \\
G_2(f_1, f_2) &= \frac{(-2\varepsilon)[\alpha + j(\omega_1 + \omega_2)]^{-1}}{(\alpha + j\omega_1)(\alpha + j\omega_2)} \\
G_3(f_1, f_2, f_3) &= \frac{(-2\varepsilon)^2[\alpha + j(\omega_1 + \omega_2 + \omega_3)]^{-1}}{(\alpha + j\omega_1)(\alpha + j\omega_2)(\alpha + j\omega_3)} \\
&\quad \cdot \sum\nolimits_3' \frac{1}{\alpha + j\omega_2 + j\omega_3} \\
G_n(f_1, \cdots, f_n) &= -\varepsilon[\alpha + j(\omega_1 + \cdots + \omega_n)]^{-1} G_n^{(2)}(f_1, \cdots, f_n)
\end{aligned} \tag{40}$$

where $G_n^{(2)}$ is given by (24) and in $G_3$

$$\sum\nolimits_3' \frac{1}{\alpha + j\omega_2 + j\omega_3} = \frac{1}{\alpha + j\omega_2 + j\omega_3} + \frac{1}{\alpha + j\omega_1 + j\omega_3} + \frac{1}{\alpha + j\omega_1 + j\omega_2}. \tag{41}$$

Another application of the harmonic input method, which is somewhat similar to the nonlinear differential equation system in (37), is concerned with the $G_n$ for the voltage $y(t)$ across the nonlinear device shown in Fig. 3. The voltage $x(t)$ is applied to the series combination of the nonlinear device, defined by

$$I(t) = \sum_{l=1}^{\infty} a_l[y(t)]^l \tag{42}$$

and the linear admittance $H(f)$. As illustrated by the work of Deutsch [12], [17], a knowledge of the series for $y(t)$ is the key to the solution of a number of nonlinear problems. The Volterra transfer functions for $y(t)$ are

$$\begin{aligned}
G_1(f_1) &= H(f_1)/[a_1 + H(f_1)] \\
G_2(f_1, f_2) &= -2a_2 G_1(f_1)G_1(f_2)/[a_1 + H(f_1 + f_2)] \\
&\;\;\vdots \\
G_n(f_1, \cdots, f_n) &= -\frac{\sum_{l=2}^{n} a_l G_n^{(l)}(f_1, \cdots, f_n)}{a_1 + H(f_1 + \cdots + f_n)}
\end{aligned} \tag{43}$$

where the last equation is derived in Section V-B. For $n > 1$ these equations differ from (38) only in having the denominator $a_1 + H(f_1 + \cdots + f_n)$ instead of $F(j\omega_1 + \cdots + j\omega_n)$. They furnish a set of recurrence relations for $G_n(f_1, \cdots, f_n)$ which are essentially due to Deutsch [12], [17]. Our $G_n(f_1, \cdots, f_n)$ is $n!$ times the symmetrized version of Deutsch's functions $Q_n(\omega_1, \cdots, \omega_n)$ [with $\omega_i = 2\pi f_i$]. For the circuit of Fig. 3, the multiplier $K(\omega_1)$ appearing in Deutsch's $Q_1(\omega_1)$ is unity.

As a specific example of this application we consider a linear

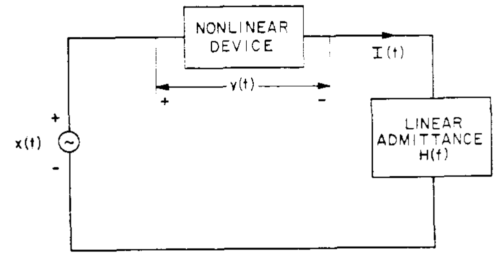


Fig. 3. Nonlinear device in series with linear admittance.

inductance $L$ in series with the square-law device $I(t) = a_2[y(t)]^2$. The circuit equation is

$$\beta \frac{d}{dt} y^2 + y = x(t), \qquad \beta = La_2. \tag{44}$$

This differs somewhat from the Riccati equation (39) for an inductance in series with a nonlinear resistance. The admittance function for the inductance is $H(f) = 1/j\omega L$ and the only coefficient $a_l$ which is not zero is $a_2$. Substitution in (43) gives, upon omitting the arguments,

$$\begin{aligned}
G_1 &= 1 \\
G_2 &= -2a_2 j(\omega_1 + \omega_2)L = -2j\beta(\omega_1 + \omega_2) \\
G_3 &= 2(-2j\beta)^2(\omega_1 + \omega_2 + \omega_3)^2
\end{aligned} \tag{45}$$

where the expression for $G_3$ is obtained from the recurrence relation

$$G_n = -j\beta(\omega_1 + \cdots + \omega_n)G_n^{(2)}(f_1, \cdots, f_n)$$

and (28) for $G_3^{(2)}$. Although $G_4$ is a symmetric polynomial of the third degree in $\omega_1, \cdots, \omega_4$, use of (29) for $G_4^{(2)}$ shows that $G_4$ is not as simple as might be expected.

### C. Direct Expansion Method

An alternative to "probing" the system, as in the harmonic input method, is to manipulate the defining system equations until they are brought into the form (1) of a Volterra series expansion. The Volterra transfer functions can then be found by taking the $n$-fold Fourier transform of the Volterra kernels using (2). This was precisely the technique used in the trivial memoryless case (22). The chief value of the direct expansion method seems to be in the derivation of expressions which hold for general values of $n$, as in Section V-B. When $n$ is small, it appears simpler to use harmonic inputs.

An example of a nonlinear system that can be analyzed by the direct expansion method is the modulator–filter–demodulator system shown in Fig. 1.

The system output is given by

$$y(t) = F\left\{\int_{-\infty}^{\infty} g(u)h[x(t-u)]du\right\}. \tag{46}$$

It is assumed that the modulator and demodulator functions can be expanded in the power series

$$\begin{aligned}
h(x) &= \sum_{\nu=0}^{\infty} h_\nu x^\nu/\nu! \qquad & F(z) &= \sum_{l=1}^{\infty} F_l(z - z_0)^l/l! \\
z_0 &= h_0 \int_{-\infty}^{\infty} g(u)du \qquad & F(z_0) &= 0
\end{aligned} \tag{47}$$

and the impulse response $g(t)$ of the filter is related to the filter transfer function $G(f)$ by

$$G(f) = \int_{-\infty}^{\infty} e^{-j\omega t} g(t)dt. \tag{48}$$

TABLE I

| Partition | $l$ | $N$ | Terms in $G_4(f_1, f_2, f_3, f_4)$ |
|---|---|---|---|
| 4 | 1 | 1 | $F_1 h_4 G(f_1 + f_2 + f_3 + f_4)$ |
| 1 + 3 | 2 | 4 | $F_2 h_1 h_3 \Sigma'_4 G(f_1) G(f_2 + f_3 + f_4)$ |
| 2 + 2 | 2 | 3 | $F_2 h_2^2 \Sigma'_3 G(f_1 + f_2) G(f_3 + f_4)$ |
| 1 + 1 + 2 | 3 | 6 | $F_3 h_1^2 h_2 \Sigma'_6 G(f_1) G(f_2) G(f_3 + f_4)$ |
| 1 + 1 + 1 + 1 | 4 | 1 | $F_4 h_1^4 G(f_1) G(f_2) G(f_3) G(f_4)$ |

The Volterra transfer functions, i.e., the Fourier transforms of the kernels in the Volterra series for $y(t)$ obtained from (46), are

$$G_1(f_1) = F_1 h_1 G(f_1)$$

$$G_2(f_1, f_2) = F_1 h_2 G(f_1 + f_2) + F_2 h_1^2 G(f_1) G(f_2)$$

$$G_3(f_1, f_2, f_3) = F_1 h_3 G(f_1 + f_2 + f_3) + F_2 h_1 h_2 \textstyle\sum'_3 G(f_1) G(f_2 + f_3) + F_3 h_1^3 G(f_1) G(f_2) G(f_3)$$

$$G_n(f_1, \cdots, f_n) = \sum_{l=1}^{n} F_l \sum_{(\nu; l, n)} h_{\nu_1} \cdots h_{\nu_l} \textstyle\sum'_N G(f_1 + f_2 + \cdots + f_{\nu_1}) G(f_{\nu_1 + 1} + \cdots + f_{\nu_1 + \nu_2}) \cdots G(f_\mu + \cdots + f_n). \quad (49)$$

The last equation is derived in Section V-B. Here, $\sum'_3$ is a sum of the type shown in (28), and the sums denoted by $\sum_{(\nu; l, n)}$ and $\sum'_N$ are defined in connection with (24)–(29).

We have

$$\sum_{l=1}^{n} \sum_{(\nu; l, n)} = \sum_{\pi(n)} \quad (50)$$

where $\pi(n)$ beneath the $\sum$ denotes summation over all partitions of $n$. The number of parts in the partition is $l$ and the $\nu$'s are the parts. The parts are related by (25). The general form (49) for $G_n$ can be written almost immediately from either 1) the table of partitions given in [18, pp. 831–832], where the values of $N$ are listed in the column labeled $M_3$, or from 2) the table of Bell polynomials given in [19, p. 125]. Table I illustrates the procedure for $n = 4$. The value of $G_4$ is given by the sum of the terms in the last column.

Let $B_n$ be the number of different products in $G_n(f_1, \cdots, f_n)$ when they are counted in a manner which gives [see (49)] $B_1 = 1$ for $G_1(f_1)$, $B_2 = 2$ for $G_2(f_1, f_2)$, $B_3 = 1 + 3 + 1 = 5$ for $G_3(f_1, f_2, f_3)$, and (see Table I) $B_4 = 1 + 4 + 3 + 6 + 1 = 15$ for $G_4$. When $G(f) \equiv 1$, $g(u)$ is $\delta(u)$, $y(t)$ is $F\{h[x(t)]\}$, the system is memoryless, and $G_n$ is the coefficient of $[x(t)]^n/n!$ in the expansion of $y(t)$. Setting $h_\nu = 1$ and $F_l = 1$ gives $y = \exp[e^x - 1] - 1$ from (47) and $G_n = B_n$ from (49). Hence $B_n$ is the coefficient of $x^n/n!$ in the expansion of $\exp[e^x - 1]$, and from this a recurrence relation for $B_n$ can be obtained. $B_n$ increases rapidly with $n$. For example, $B_5 = 52$ and $B_6 = 203$. The $B_n$ are the Bell numbers [19, p. 192].

A particular case of the modulator–filter–demodulator system is furnished by the phase-modulation system shown in Fig. 4 and discussed further in the next section. Briefly, the input $x(t)$ is used to phase modulate a carrier wave that passes through a filter having a transfer function $K(f)$. The output $\theta(t)$ is taken as the variable portion of the phase angle of the filter output. This corresponds to the system of Fig. 1 with $h(x) = \exp(jx)$, $F(z) = ln\, z$, and $z_0 = 1$. Then

$$h_\nu = j^\nu \qquad F_l = (-1)^{l-1}(l-1)! \quad (51)$$

and substitution in (49) leads to, as shown in Section IV-C, the Volterra transfer functions for $\theta(t)$. For $n = 1$ and $n = 2$ these are

$$G_{\theta 1}(f_1) = \frac{1}{2}[\Gamma(f_1) + \Gamma^*(-f_1)]$$


Fig. 4. Phase-modulation system.

$$G_{\theta 2}(f_1, f_2) = \frac{j}{2}[\Gamma(f_1 + f_2) - \Gamma(f_1)\Gamma(f_2) - \Gamma^*(-f_1 - f_2) + \Gamma^*(-f_1)\Gamma^*(-f_2)] \quad (52)$$

where the asterisk denotes conjugate complex and $\Gamma(f) = K(f_0 + f)/K(f_0)$, $f_0$ being the carrier frequency $\omega_0/(2\pi)$. The expression (52) for $G_{\theta 2}(f_1, f_2)$ with $\Gamma(f) = \exp(-jbf^2)$ has been used to study the distortion which occurs when an FM wave travels through the ionosphere [9]. The structure of the general $G_{\theta n}(f_1, \cdots, f_n)$ term given here is much the same as the structure of the intermodulation functions introduced by Mircea [7] in his studies of FM distortion.

The other general example of the use of the direct expansion method for determining the Volterra transfer functions is the feedback system shown in Fig. 2.

The system input is $x(t)$ and the output is $y(t)$. The system equations relating $x(t)$ and $y(t)$ are

$$y(t) = \sum_{l=1}^{\infty} \frac{1}{l!} \int_{-\infty}^{\infty} du'_1 \cdots \int_{-\infty}^{\infty} du'_l m_l(u'_1, \cdots, u'_l) \prod_{q=1}^{l} w(t - u'_q) \quad (53)$$

$$w(t) = x(t) - z(t) \quad (54)$$

$$z(t) = \int_{-\infty}^{\infty} b(v) y(t - v) dv \quad (55)$$

where the filter transfer function $B(f)$ [the Fourier transform of $b(v)$] and the symmetric $n$-fold Fourier transforms $M_n(f_1, \cdots, f_n)$ of the Volterra kernels $m_n(u_1, \cdots, u_n)$ are assumed to be known.

The problem is to determine the Volterra transfer function $G_n(f_1, \cdots, f_n)$, i.e., the $n$-fold Fourier transform of the $n$th kernel in the series (1) for $y(t)$. For $n = 1$, 2, and 3 the answer can be obtained by the harmonic input method and is

$$G_1(f_1) = [1 + M_1(f_1)B(f_1)]^{-1} M_1(f_1)$$

$$G_2(f_1, f_2) = [1 + M_1(f_1 + f_2)B(f_1 + f_2)]^{-1} K_1(f_1) K_1(f_2) M_2(f_1, f_2)$$

$$\begin{aligned} G_3(f_1, f_2, f_3) = {} & [1 + M_1(f_1 + f_2 + f_3)B(f_1 + f_2 + f_3)]^{-1} \\ & \cdot \{[K_1(f_1)K_2(f_2, f_3)M_2(f_1, f_2 + f_3) \\ & + K_1(f_2)K_2(f_3, f_1)M_2(f_2, f_3 + f_1) \\ & + K_1(f_3)K_2(f_1, f_2)M_2(f_3, f_1 + f_2)] \\ & + K_1(f_1)K_1(f_2)K_1(f_3)M_3(f_1, f_2, f_3)\} \end{aligned} \quad (56)$$

where

$$K_1(f_1) = 1 - B(f_1)G_1(f_1)$$

$$K_n(f_1, \cdots, f_n) = -B(f_1 + \cdots + f_n)G_n(f_1, \cdots, f_n), \qquad n > 1. \quad (57)$$

The expression for $G_2(f_1, f_2)$ depends on $G_1(f_1)$ through $K_1(f_1)$; and the expression for $G_3(f_1, f_2, f_3)$ depends on $G_1(f_1)$ and $G_2(f_2, f_3)$ through $K_1(f_1)$ and $K_2(f_1, f_2)$. Therefore the second and third of (56) are recurrence relations.

The three expressions (56) are equivalent to three given by Narayanan [6]. The $\mu_n(f_1, \cdots, f_n)$, $G_n(f_1, \cdots, f_n)$, and $\beta(f)$ used by Narayanan are equal to, in our notation, $M_n(f, \cdots, f_n)/n!$, $G_n(f_1, \cdots, f_n)/n!$, and $B(f)$, respectively.

The $n$th recurrence relation for $n > 1$, obtained in Section V-B by the direct expansion method, is

$$G_n(f_1,\cdots,f_n) = \left[1 + M_1(f_1+\cdots+f_n)B(f_1+\cdots+f_n)\right]^{-1} \cdot \sum_{l=2}^{n} \sum_{(v;l,n)} \sum\nolimits_N' K_{v_1}(f_1,\cdots,f_{v_1})K_{v_2}(f_{v_1+1},\cdots,f_{v_1+v_2}) \cdots K_{v_l}(f_\mu,\cdots,f_n)M_l(f_1+\cdots+f_{v_1}, f_{v_1+1}+\cdots+f_{v_1+v_2},\cdots,f_\mu+\cdots+f_n). \quad (58)$$

Here the integer $N$ and the summation over the sets $(v;l,n)$ of integers $v_i$ are the same as in (24).

The $K_n$ are the $n$-fold Fourier transforms of the kernels in

$$w(t) = \sum_{i=1}^{\infty} \frac{1}{i!}\int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_i k_i(u_1,\cdots,u_i)\prod_{r=1}^{i} x(t-u_r) \quad (59)$$

which can be obtained by substituting the series (1) for $y(t)$ in (55) to get a series for $z(t)$, and then substituting this series for $z(t)$ in (54), $w(t)=x(t)-z(t)$.

## IV. Illustrative Examples

The diversity of the list of formulas in Section II giving properties of the output of nonlinear devices for a variety of inputs makes it desirable to illustrate their use by applying them to practical problems of interest. In this section some of the examples used in Section III to illustrate computation of the Volterra transfer functions will be treated further to obtain output properties of interest for specific input signals.

### A. Quasi-Static Filtered FM

The system equation and Volterra transfer functions for this case are, from Section III-B,

$$y(t) = x(t) + \varepsilon[x'(t)]^2x''(t)$$

$$G_1(f_1)=1, \qquad G_3(f_1,f_2,f_3) = 2\varepsilon\omega_1\omega_2\omega_3(\omega_1+\omega_2+\omega_3) \quad (60)$$

where $\omega=2\pi f$ and where all of the remaining $G_n$ are zero. When $x(t)$ is Gaussian, (11) shows that $\langle y(t)\rangle$ is zero. From (14) the leading terms in the power spectrum for $y(t)$ are

$$W_y(f) = W_x(f)\left|1 - \varepsilon(2\pi f)^2\int_{-\infty}^{\infty} df_1 W_x(f_1)(2\pi f_1)^2\right|^2 + \frac{1}{3!}\int_{-\infty}^{\infty} df_1 \int_{-\infty}^{\infty} df_2 W_x(f_1)W_x(f_2)W_x(f-f_1-f_2) \cdot |2\varepsilon f_1 f_2(f-f_1-f_2)f(2\pi)^4|^2. \quad (61)$$

This is also the exact expression for $W_y(f)$ because an examination of the complete expression (160) for $W_y(f)$ shows that all of the remaining terms are zero for this case. When we use the relation $W_{x'}(f) = (2\pi f)^2W_x(f) = \omega^2W_x(f)$ between the power spectrum of $x(t)$ and its time derivative $x'(t)$, (61) goes into

$$W_y(f) = W_x(f)\left|1 - \varepsilon\omega^2\int_{-\infty}^{\infty} df_1 W_{x'}(f_1)\right|^2 + \frac{4\varepsilon^2\omega^2}{3!}\int_{-\infty}^{\infty} df_1\int_{-\infty}^{\infty} df_2 W_{x'}(f_1)W_{x'}(f_2)W_{x'}(f-f_1-f_2). \quad (62)$$

When $x(t)$ in (60) is equal to $P\cos pt + I_n(t)$, where $I_n(t)$ is Gaussian, (15) and (16) show that the periodic part of $y(t)$ is the ensemble average

$$\langle y(t)\rangle = P\left\{1 - \tfrac{1}{4}P^2\varepsilon p^4 - \varepsilon p^2\int_{-\infty}^{\infty} df_1 W_{I'}(f_1)\right\}\cos pt + \tfrac{1}{4}P^3\varepsilon p^4\cos 3pt \quad (63)$$

and that $y(t)$ has the power spectrum

$$W_y(f) = \{\text{Four spikes due to the } \exp(\pm jpt) \text{ and } \exp(\pm j3pt) \text{ components of } \langle y(t)\rangle\} + W_I(f)\left|1 - \tfrac{1}{2}P^2\varepsilon p^2\omega^2 - \varepsilon\omega^2\int_{-\infty}^{\infty} df_1 W_{I'}(f_1)\right|^2 + \tfrac{1}{16}P^4\varepsilon^2p^4\omega^2[W_{I'}(f-2f_p) + W_{I'}(f+2f_p)] + \tfrac{1}{2}P^2\varepsilon^2p^2\omega^2\int_{-\infty}^{\infty} df_1 W_{I'}(f_1)[W_{I'}(f-f_1-f_p) + W_{I'}(f-f_1+f_p)] + \frac{4\varepsilon^2\omega^2}{3!}\int_{-\infty}^{\infty} df_1\int_{-\infty}^{\infty} df_2 W_{I'}(f_1)W_{I'}(f_2)W_{I'}(f-f_1-f_2) \quad (64)$$

where $2\pi f_p = p$. When $P$ is zero, (64) reduces to the expression (62) for $W_y(f)$ when $x(t)$ consists of Gaussian noise alone.

### B. Series Inductance and Nonlinear Resistance

Suppose that the voltage $x(t) = P\cos pt + Q\cos qt$ is applied to the series combination of the unit inductance and nonlinear resistance $(\alpha+\varepsilon y)$ described in Section III-B [(39)]. What is the leading term in the $(p-q)$ component of the current $y(t)$ when $\varepsilon$ is small? Changing the signs of $p$ and $q$ in (5) for the $(p+q)$ term to obtain the $\pm(p-q)$ terms and substituting $G_2(f_1, f_2)$ from (40) shows that (assuming $\alpha$ and $\varepsilon$ real) the desired leading term is

$$2\,\mathrm{Re}\left[e^{j(p-q)t}\frac{PQ}{4}\,\frac{(-2\varepsilon)[\alpha+j(p-q)]^{-1}}{(\alpha+jp)(\alpha-jq)}\right]. \quad (65)$$

When the voltage $x(t)$ applied to the series combination is Gaussian, the leading terms (out to order $\varepsilon^2$) in the power spectrum of the current $y(t)$ are, from (14),

$$W_y(f) = \langle y(t)\rangle^2\delta(f) + W_x(f)\left|G_1(f) + \frac{1}{2}\int_{-\infty}^{\infty} df_1 W_x(f_1)G_3(f,f_1,-f_1)\right|^2 + \frac{1}{2}\int_{-\infty}^{\infty} df_1 W_x(f_1)W_x(f-f_1)|G_2(f_1,f-f_1)|^2 + O(\varepsilon^4) \quad (66)$$

where, from (40) and (11),

$$G_1(f) = (\alpha+j\omega)^{-1}, \qquad \omega=2\pi f$$

$$|G_2(f_1,f-f_1)|^2 = \frac{4\varepsilon^2(\alpha^2+\omega^2)^{-1}}{(\alpha^2+\omega_1^2)[\alpha^2+(\omega-\omega_1)^2]}$$

$$\langle y(t)\rangle = -\varepsilon\alpha^{-1}\int_{-\infty}^{\infty} df_1 W_x(f_1)/(\alpha^2+\omega_1^2) \quad (67)$$

and from (40) $G_3(f, f_1, -f_1)$ is $O(\varepsilon^2)$. When $\varepsilon$ is small and $x(t)$ is Gaussian, the probability density $p(y)$ of $y(t)$ is almost, but not quite, normal. As discussed in Section VIII-B, the departure of $p(y)$ from normality can be estimated from the values of the cumulants $\kappa_1$, $\kappa_2$, $\kappa_3$, $\kappa_4$. However, when the $G_n$ from (40) are used, the integrals (13) for $\kappa_3$, $\kappa_4$ are quite complicated and we shall not stop to evaluate them here. A less complicated example is given later in Section IV-D in connection with noise through a square-law device followed by a filter.

### C. Filtered Phase Modulation

Here we illustrate the use of (49) for $G_n(f_1,\cdots,f_n)$ by considering a special case of the modulator–filter–demodulator system of Fig. 1, namely, the phase-modulation system shown in Fig. 4. In this system the filter produces undesirable distortion. The input is $x(t)$, the output is $\theta(t)$, and $K(f)$ is the filter transfer function. When

$x(t)$ is zero, the filter output is $a \cos(\omega_0 t + b)$ where [with $\omega_0 = 2\pi f_0$], $K(f_0) = K_0^*(-f_0) = a \exp(jb)$, the envelope factor $R(t)$ in the filter output is unity, and $\theta(t)$ is zero. From phase-modulation theory [8]–[10]

$$\theta(t) = \operatorname{Im} y(t)$$
$$y(t) = \ln\left[\int_{-\infty}^{\infty} du\, \gamma(u) e^{jx(t-u)}\right]$$
$$\gamma(u) = \int_{-\infty}^{\infty} df\, e^{j\omega u}\Gamma(f), \qquad \omega = 2\pi f$$
$$\Gamma(f) = K(f_0 + f)/K(f_0). \tag{68}$$

In this example the system output is $\theta(t)$ instead of $y(t)$. Here $y(t)$ is equal to $\ln R(t) + j\theta(t)$. When the filter is absent, $K(f)$ is independent of the frequency $f$, $\Gamma(f) = 1$, $\gamma(u) = \delta(u)$, $y(t) = jx(t)$, and $\theta(t)$ is equal to $x(t)$. We are interested in analyzing the difference, usually small, between $\theta(t)$ and $x(t)$ when the filter is present.

In order to apply the formulas listed in Section II to $\theta(t)$, we need the Fourier transforms $G_{\theta n}(f_1, \cdots, f_n)$ of the kernels in the Volterra series for $\theta(t)$. By splitting the Volterra series for $y(t)$ into its real and imaginary parts and using $\theta(t) = \operatorname{Im} y(t)$, it can be shown that

$$G_{\theta n}(f_1, \cdots, f_n) = [G_n(f_1, \cdots, f_n) - G_n^*(-f_1, \cdots, -f_n)]/(2j) \tag{69}$$

where $G_n(f_1, \cdots, f_n)$ is the Fourier transform of the $n$th kernel for $y(t)$. Indeed we have the general result that, when $x(t)$ is real, the Fourier transforms of the kernels in the series for the real and imaginary parts, $y_R(t)$ and $y_I(t)$, of $y(t)$ are

$$\begin{aligned}
&[G_n(f_1, \cdots, f_n) + G_n^*(-f_1, \cdots, -f_n)]/2, && \text{for } y_R(t)\\
&[G_n(f_1, \cdots, f_n) - G_n^*(-f_1, \cdots, -f_n)]/(2j), && \text{for } y_I(t).
\end{aligned} \tag{70}$$

Comparison of (68) and (46) for $y(t)$ shows that $g(u) = \gamma(u)$, $h(x) = \exp(jx)$, and $F(z) = \ln z$. Hence $G(f)$ goes into $\Gamma(f)$, the coefficients in the expansion of $h(x)$ are $h_\nu = j^\nu$, and the equation to determine $z_0$ becomes $z_0 = h_0\Gamma(0) = 1$. Expanding $F(z)$ about $z = z_0 = 1$ gives $F_l = (-1)^{l-1}(l-1)!$ Then the general equations (49) show that the Fourier transforms of the kernels in the Volterra series for $y(t)$ defined by the phase-modulation equations (68) are

$$\begin{aligned}
G_1(f_1) &= j\Gamma(f_1),\\
G_2(f_1, f_2) &= j^2[\Gamma(f_1 + f_2) - \Gamma(f_1)\Gamma(f_2)],\\
G_3(f_1, f_2, f_3) &= j^3[\Gamma(f_1 + f_2 + f_3) - \Gamma(f_1)\Gamma(f_2 + f_3)\\
&\quad - \Gamma(f_2)\Gamma(f_1 + f_3) - \Gamma(f_3)\Gamma(f_1 + f_2)\\
&\quad + 2\Gamma(f_1)\Gamma(f_2)\Gamma(f_3)],\\
&\vdots\\
G_n(f_1, \cdots, f_n) &= j^n \sum_{l=1}^{n} (-1)^{l-1}(l-1)! \sum_{(\nu;l,n)} \sum{}_N' \Gamma(f_1 + \cdots + f_{\nu_1})\\
&\quad \cdot \Gamma(f_{\nu_1+1} + \cdots + f_{\nu_1+\nu_2}) \cdots \Gamma(f_\mu + \cdots + f_n).
\end{aligned} \tag{71}$$

The expressions for $G_{\theta 1}$ and $G_{\theta 2}$ obtained by substituting (71) in the general relation (69) for $G_{\theta n}$ are

$$\begin{aligned}
G_{\theta 1}(f_1) &= \frac{j}{2j}[\Gamma(f_1) + \Gamma^*(-f_1)]\\
G_{\theta 2}(f_1, f_2) &= \frac{j^2}{2j}[\Gamma(f_1 + f_2) - \Gamma(f_1)\Gamma(f_2) - \Gamma^*(-f_1 - f_2)\\
&\quad + \Gamma^*(-f_1)\Gamma^*(-f_2)]
\end{aligned} \tag{72}$$

as also shown in (52). When $\Gamma^*(-f)$ is equal to $\Gamma(f)$, as it is when the filter transfer function $K(f)$ is "symmetrical" about the carrier frequency $f_0$, $G_{\theta n}$ is zero if $n$ is even and is equal to $-j\, G_n$ if $n$ is odd.

Now that the $G_{\theta n}$ are known, information regarding $\theta(t)$ for various inputs $x(t)$ can be obtained by replacing $y(t)$ by $\theta(t)$ and $G_n(f_1, \cdots, f_n)$ by $G_{\theta n}(f_1, \cdots, f_n)$ in the formulas listed in Section II. For example, when $x(t) = P\cos pt + Q\cos qt$, the $\exp[j(p-q)t]$ component in $\theta(t)$ has the leading term [upon changing the signs of $q$ and $f_q$ in the $\exp[j(p+q)t]$ term in (5)]

$$e^{j(p-q)t}\frac{PQ}{4}G_{\theta 2}(f_p, -f_q) = e^{j(p-q)t}\frac{PQ}{8}j$$
$$\cdot[\Gamma(f_p - f_q) - \Gamma(f_p)\Gamma(-f_q) - \Gamma^*(-f_p + f_q) + \Gamma^*(-f_p)\Gamma^*(f_q)]. \tag{73}$$

Mircea [20] has discussed the case when $x(t) = P\cos pt$, and has given [7] the structure of the general term in the series for the power spectrum $W_\theta(f)$ when $x(t)$ is Gaussian (see also [8]–[11]). The leading terms in the sum of the linear and second-order modulation terms in $W_\theta(f)$ are, upon substitution in the Mircea–Sinnreich series (14),

$$W_x(f)|G_{\theta 1}(f)|^2 + \frac{1}{2}\int_{-\infty}^{\infty} df_1\, W_x(f_1)W_x(f - f_1)|G_{\theta 2}(f_1, f - f_1)|^2 \tag{74}$$

where $G_{\theta 1}$ and $G_{\theta 2}$ are given by (72).

Some insight into the region where the Volterra series approach is useful can be obtained by considering the case $x(t) = P\cos pt$. The usual expression for $\theta(t)$ for this case is, in our notation,

$$\theta(t) = \arctan\left[\frac{\operatorname{Im} S}{\operatorname{Re} S}\right]$$
$$S = \sum_{n=-\infty}^{\infty} j^n J_n(P)\Gamma(nf_p)e^{jnpt} \tag{75}$$

where $J_n(P)$ is the Bessel function of order $n$. When $\Gamma(f) \equiv 1$, $S$ is equal to $\exp(j\,P\cos pt)$, and $\theta(t)$ is equal to the input $x(t)$. When the filter bandwidth is large, $S$ remains close to $\exp(j\,P\cos pt)$. When $P$ is small enough to make $|S-1| < 1$ for all values of $t$, $\theta(t)$ can be expanded in a convergent power series in $P$ by using

$$\theta(t) = \operatorname{Im}\ln[1 + (S - 1)]. \tag{76}$$

On the other hand, the complete series (3) shows that

$$\theta(t) = \sum_{n=1}^{\infty}\left(\frac{P}{2}\right)^n \sum_{k=0}^{n} \frac{\exp[j(2k-n)pt]}{k!(n-k)!} G_{\theta(k,n-k)}(f_p) \tag{77}$$

where the subscript $\theta(k, n-k)$ on $G$ denotes $G_{\theta n}(f_1, \cdots, f_n)$ with the first $k$ of the $f_i$ equal to $f_p$ and the remaining $n-k$ equal to $-f_p$. Therefore, (77) gives the power series expansion of $\arctan[\operatorname{Im} S/\operatorname{Re} S]$ in powers of $P$. In an FM system $P$ is the "deviation ratio." This indicates that when the Volterra series analysis is applied to FM systems, it is most useful for systems employing a small deviation ratio—as is the case in many microwave radio relay systems.

### D. Filtered Square-Law Detector

The system equation for a square-law device followed by a filter is

$$y(t) = \int_{-\infty}^{\infty} b(u)x^2(t - u)du \tag{78}$$

when $x(t)$ is the input to the square-law device and $y(t)$ is the filter output. We are interested in the probability density $p(y)$ of $y(t)$ when $x(t)$ is a Gaussian noise with power spectrum $W_x(f) = (2\pi)^{-1/2} \exp(-f^2/2)$, and the filter impulse response $b(u)$ and its Fourier transform $B(f)$ are

$$b(u) = \beta(2\pi)^{1/2} \exp(-2\pi^2\beta^2u^2)$$
$$B(f) = \exp[-f^2/(2\beta^2)]. \tag{79}$$

The effective passband of the filter extends from $-\beta$ to $+\beta$ and the effective band of the noise from $-1$ to $+1$. This is a simpler case than one considered by Slepian [21] in which $x(t)$ is $RLC$ Gaussian noise, $b(u)=1$ for $|u|<T/2$, and $b(u)=0$ for $|u|>T/2$.

First some general remarks. Averaging both sides of (78) and using $\langle x^2(t)\rangle=1$ shows that $\langle y(t)\rangle=B(0)=1$, irrespective of the bandwidth $\beta$. Since $b(u)\geq 0$, $p(y)$ is zero when $y<0$. When $\beta=\infty$, the filter has no effect, $y(t)=x^2(t)$, and for $y>0$

$$p(y) = (2\pi y)^{-1/2} \exp(-y/2). \tag{80}$$

When $\beta\to 0$, the filter bandwidth tends to zero and we expect [22] $p(y)$ to tend to a normal law of vanishing width centered on the mean value $y=1$:

$$p(y) \to (2\pi\beta)^{-1/2} \exp[-(y-1)^2/(2\beta)]. \tag{81}$$

As $\beta$ increases from 0 to $\infty$, $p(y)$ changes from (81) to (80).

Now we apply some of the results mentioned in Section II for Volterra series. The system equation (78) corresponds to a Volterra series with all of its kernels zero except for $n=2$:

$$g_2(u_1, u_2) = 2b(u_1)\delta(u_2 - u_1)$$
$$G_2(f_1, f_2) = 2B(f_1 + f_2). \tag{82}$$

From the more complete series (180) corresponding to (11) and (13) we see that the cumulants $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ for $p(y)$ are, respectively, the integrals of $(W)(1,-1)/2$, $(WW)(1,2)(-1,-2)/2$, $(WWW)(1,2)\cdot(-1,3)(-2,-3)$, $3(WWWW)(1,2)(-1,3)(-2,4)(-3,-4)$. Here $(W)$, $(WW)$, $(1,2),\cdots$ denote $W_x(f_1)$, $W_x(f_1)W_x(f_2)$, $G_2(f_1,f_2),\cdots$ much as in (180). The Gaussian form of the integrands permits the integrations to be performed. It is found that

$$\kappa_1 = 1 \qquad \kappa_2 = 2/c, \qquad c = (4\beta^{-2}+1)^{1/2}$$
$$\kappa_3 = 32/(3c^2+1) \qquad \kappa_4 = 96/[c(c^2+1)]. \tag{83}$$

These values of $\kappa_1$ and $\kappa_2$ (the mean and variance) agree with those obtained from the limiting forms (80) and (81) of $p(y)$. When $\beta$ becomes small, $c\to 2/\beta$ and the values of $\kappa_2, \kappa_3, \kappa_4$ approach $\beta$, $8\beta^2/3$, $12\beta^3$, respectively. Thus, the variance tends to $\beta$, and (181) shows that the skewness $\gamma_1$ tends to $(8/3)\beta^{1/2}$ and the excess $\gamma_2$ to $12\beta$. The Edgeworth series (182) shows how the normal law (81) is approached:

$$p(y)=\beta^{-1/2}\{Z(u)-\tfrac{4}{9}\beta^{1/2}Z^{(3)}(u)+\beta[\tfrac{1}{2}Z^{(4)}(u)+\tfrac{8}{81}Z^{(6)}(u)]-\cdots\} \tag{84}$$

where $u=(y-1)/\beta^{1/2}$ and $Z(u)$ is equal to $(2\pi)^{-1/2}\exp(-u^2/2)$. From (184) the peak of $p(y)$ occurs at $y_0\approx 1-(4\beta/3)$ where $p(y_0)$ is about $[1+(49\beta/54)]$ times higher than the peak value $(2\pi\beta)^{-1/2}$ of a normal law with the variance $\beta$.

At this stage we go to the special results given in Section VIII-C for the two-term Volterra series. For the Gaussian forms of $W_x(f)$ and $G_2(f_1,f_2)$ in our example, it is convenient to work with row 4 of Table II. For our example the integral equation is

$$\lambda\Psi(f) = (2\pi)^{-1/2}\exp(-f^2/2)\int_{-\infty}^{\infty} df_1 2\Psi(f_1) \cdot\exp[-(f-f_1)^2/(2\beta^2)]. \tag{85}$$

The $k$th eigenvalue and eigenfunction are found to be

$$\lambda_k = \frac{4}{c+1}\left(\frac{c-1}{c+1}\right)^k$$

$$\Psi_k(f) = A_k\exp[-\tfrac{1}{4}(c+1)f^2]H_k\left[f\left(\frac{c}{2}\right)^{1/2}\right] \tag{86}$$

where $A_k$ depends only on $k$ and the Hermite polynomial is given by

$$H_k(x) = e^{x^2}\left(-\frac{d}{dx}\right)^k e^{-x^2} \quad H_0(x)=1 \qquad H_1(x)=2x.$$

That (86) is a solution of (85) can be verified with the help of

$$\int_{-\infty}^{\infty} e^{-px^2+qx}H_k(rx)dx = \left(\frac{\pi}{p}\right)^{1/2}\left(1-\frac{r^2}{p}\right)^{k/2}\exp\left[\frac{q^2}{4p}\right]H_k\left[\frac{qr}{2p}\left(1-\frac{r^2}{p}\right)^{-1/2}\right]. \tag{87}$$

In our example the parameter $\xi_k$ is zero because the first term of the two-term Volterra series is missing, and we need only $\lambda_k$ in order to calculate $p(y)$ and the cumulants $\kappa_m$. The integral (186) for $p(y)$ becomes

$$p(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz e^{-jyz}/\prod_{k=0}^{\infty}[1-4jz\rho^k(c+1)^{-1}]^{1/2} \tag{88}$$

where $\rho=(c-1)/(c+1)$. The series in (188) for the $m$th cumulant gives

$$\kappa_m = \frac{(m-1)!}{2}\sum_{k=0}^{\infty}\lambda_k^m = \frac{(m-1)!2^{2m-1}}{(c+1)^m-(c-1)^m}. \tag{89}$$

For $m=1,2,3,4$ this agrees with (83). Computing $\kappa_m$ from the integrals in (188) is essentially the same as evaluating the multiple integrals used to obtain (83).

To illustrate the procedure when the first term of the two-term Volterra series is present, we consider

$$y(t) = \int_{-\infty}^{\infty} du\, g_1(u)x(t-u) + \int_{-\infty}^{\infty} du\, b(u)x^2(t-u) \tag{90}$$

where

$$g_1(u)=\alpha\beta_1(2\pi)^{1/2}\exp(-2\pi^2\beta_1^2u^2) \quad G_1(f)=\alpha\exp[-f^2/(2\beta_1^2)]. \tag{91}$$

To calculate $p(y)$ and $\kappa_m$, we now need the parameter $\xi_k$ in addition to $\lambda_k$ [which is still given by (86)]. The orthonormalization relation for $\Psi_k(f)$ given in row 4 of Table II turns out to be that for Hermite polynomials and gives $j^k c^{1/4}(2\pi k!2^k)^{-1/2}$ for the normalization constant $A_k$ in (86). When the normalized $\Psi_k(-f)$ is substituted in the integral for $\xi_k$ given in the last column of Table II and the result evaluated with the help of (87), it is found that when $k$ is odd, $\xi_k$ is zero; and when $k$ is even,

$$\xi_{2n} = \alpha\frac{c^{-1/4}}{n!2^n}\left[\frac{(2n)!2}{a'+1}\right]^{1/2}\left[\frac{a'-1}{a'+1}\right]^n, \qquad a'=(2\beta_1^{-2}+1)/c. \tag{92}$$

We still have $\kappa_1=1$, but from (188) $\kappa_m$ for $m\geq 2$ is the sum of (89) plus

$$\begin{aligned}\frac{m!}{2}\sum_{n=0}^{\infty}\xi_{2n}^2\lambda_{2n}^{m-2} &= \alpha^2c^{-1/2}\frac{m!}{a'+1}\left(\frac{4}{c+1}\right)^{m-2}\sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}\cdot\left[\left(\frac{a'-1}{a'+1}\right)^2\left(\frac{c-1}{c+1}\right)^{2m-4}\right]^n\\ &= \alpha^2c^{-1/2}m!4^{m-2}[(a'+1)^2(c+1)^{2m-4} - (a'-1)^2(c-1)^{2m-4}]^{-1/2}\end{aligned} \tag{93}$$

where $(\alpha)_0=1$ and $(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$ when $n>0$. The series always converges when $m\geq 2$, and for $m=2,3,4$ gives values in agreement with those obtained from the general multiple integrals (180).

When $m=1$, the left side of (93) is of the form $\sum \xi_k^2/(2\lambda_k)$. In Section VIII-C it is pointed out that if this series converges to a value $S$, and if all of the $\lambda_k$ are positive, then $y(t)$ is never less than $-S$. Putting $m=1$ in (93) and replacing $c$ and $a'$ by their expressions in terms of the bandwidths $\beta$ and $\beta_1$ shows that the series converges when $\beta^2 < 2\beta_1^2$ and gives

$$S = \frac{\alpha^2 \beta_1^2}{4\beta}(2\beta_1^2 - \beta^2)^{-1/2}. \tag{94}$$

The inequality $y(t) \geq -S$ is a special case of a more general result which we owe to Pollak. Thus, assuming that $b(u)$ is never negative in (90),

$$g_1(u)x(t-u) + b(u)x^2(t-u) \geq -g_1^2(u)/(4b(u))$$

$$y(t) \geq -\int_{-\infty}^{\infty} g_1^2(u)du/(4b(u)). \tag{95}$$

For the $g_1(u)$ and $b(u)$ of our example, the integral converges when $\beta^2 < 2\beta_1^2$ and again gives $y(t) \geq -S$.

Finally, when $\beta \to \infty$, the second integral in (90) for $y(t)$ becomes $x^2(t)$, and $c \to 1$, $\lambda_0 \to 2$, $\lambda_k \to 0$ for $k>0$. However, the $\Psi_k(f)$ and the $\xi_k$ computed from them do not change markedly. The exponent in the factor $Q(z)$ in the integral (186) for $p(y)$ now contains the sum $\sum \xi_k^2$ which, from (188), is equal to $\kappa_2 - \lambda_0^2/2$.

In general, when the second term in a two-term Volterra series is $a_2 x(t)^2/2$, analogy with the above example and row 4 of Table II suggests that $\lambda_0 = a_2\sigma^2$, $\Psi_0(f) = W_x(f)/\sigma$, $\lambda_k = 0$ for $k>0$, and

$$Q(z) = (1 - j\lambda_0 z)^{-1/2} \exp\left\{-\frac{z^2}{2}\left[\kappa_2 + j\lambda_0 z \xi_0^2 (1 - j\lambda_0 z)^{-1}\right]\right\}$$

$$\xi_0 = \int_{-\infty}^{\infty} df\, G_1(f) W_x(f)/\sigma$$

$$\kappa_2 = \int_{-\infty}^{\infty} df\, W_x(f) G_1(f) G_1(-f) \tag{96}$$

where $\sigma^2 = \langle x^2(t) \rangle$.

# PART II: DERIVATION OF FORMULAS

## V. Formulas Associated with the Direct Expansion Method

The direct expansion method is useful in dealing with Volterra transfer functions of arbitrary order. The expansion is usually accomplished with the help of Maclaurin's series and di Bruno's formula for the $n$th derivative of a function of a function. Frequently, the resulting expansion is a Volterra series with unsymmetrical kernels which must be symmetrized.

Proofs of the general expressions for the Volterra transfer functions listed in Section III are sketched here.

### A. Volterra Series for $[y(t)]^l$

This section is devoted to the derivation of an expression for the $n$th kernel $g_n^{(l)}(u_1, \cdots, u_n)$ in the Volterra series for $[y(t)]^l$, where $l$ is a positive integer.

We introduce the function $H(\zeta)$ which is obtained from the Volterra series (1) for $y(t)$ by replacing the $x(t-u_i)$ by $\zeta x(t-u_i)$. The time $t$ enters $H(\zeta)$ as a parameter, and $H(1)$ is equal to $y(t)$. Let $F(z)$ be $z^l$. Then

$$[H(\zeta)]^l = F[H(\zeta)] = \sum_{n=1}^{\infty} \frac{\zeta^n}{n!}\left[\frac{d^n}{d\zeta^n} F[H(\zeta)]\right]_{\zeta=0}. \tag{97}$$

The $n$th derivative may be evaluated by di Bruno's formula for the derivative of a function of a function:

$$\frac{d^n}{d\zeta^n} F[H(\zeta)] = \sum_{k=1}^{n} F^{(k)}[H(\zeta)] \sum_{(v;k,n)} N(v_1, v_2, \cdots, v_k) \cdot H^{(v_1)}(\zeta) H^{(v_2)}(\zeta) \cdots H^{(v_k)}(\zeta) \tag{98}$$

where, with $k$ replacing $l$, the summation notation is the same as in (24) and $N(v_1, \cdots, v_k)$ is the $N$ given by (26).

The $k$th derivative $F^{(k)}(z)$ is $l(l-1)\cdots(l-k+1)z^{l-k}$. Since $H(\zeta)$ is 0 for $\zeta=0$, the value of $F^{(k)}[H(0)]$ is 0 for $k \neq l$ and is $l!$ for $k=l$. Differentiating the series for $H(\zeta)$ obtained by inserting $\zeta$ in the Volterra series (1) for $y(t)$ shows that

$$H^{(v)} = (0)\left[\frac{d^v}{d\zeta^v} H(\zeta)\right]_{\zeta=0} = \int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_v g_v(u_1, \cdots, u_v) \cdot \prod_{r=1}^{v} x(t-u_r). \tag{99}$$

When these values are substituted in (98), the result is, for $l \leq n$,

$$\left[\frac{d^n}{d\zeta^n} F[H(\zeta)]\right]_{\zeta=0} = l! \sum_{(v;l,n)} N(v_1, \cdots, v_l) \int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n \cdot g_{v_1}(u_1, \cdots, u_{v_1}) \cdots g_{v_l}(u_\mu, \cdots, u_n) \prod_{r=1}^{n} x(t-u_r). \tag{100}$$

For $l>n$ the right side of (100) is zero.

Substituting (100) in (97), setting $\zeta=1$, and taking the $(v; l, n)$ summation inside the integrations gives a series for $[y(t)]^l$. This series can be converted into a Volterra series by symmetrizing the products

$$g_{v_1}(u_1, \cdots, u_{v_1}) \cdots g_{v_l}(u_\mu, \cdots, u_n) \equiv P(u_1, u_2, \cdots, u_n) \tag{101}$$

where, of course, the $g_v$ are symmetric.

The symmetric function formed by permuting the $n$ subscripts in $P(u_1, \cdots, u_n)$ and adding can be reduced to the right side of [see (120)]

$$\frac{1}{n!}\sum_{n!} P(u_1, \cdots, u_n) = \frac{1}{N}\sum\nolimits_N' P(u_1, \cdots, u_n) \tag{102}$$

where $N$ is given by (26) and is the same as the $N(v_1, \cdots, v_l)$ in (100). As in (24) $\sum_N'$ denotes summation over $N$ nonidentical products. Let $u_1$ be assigned an arbitrary numerical value, $u_2$ a different but otherwise arbitrary value, and so on. Then $P(u_1, \cdots, u_n)$ will have a definite numerical value. In Section V-C (102) will be obtained by counting the permutations which leave this value of $P(u_1, \cdots, u_n)$ unchanged.

Thus, the effect of symmetrizing the products (101) is to replace $N(v_1, \cdots, v_l)$ in (100) by the sum $\sum_N'$. This leads to

$$g_n^{(l)}(u_1, \cdots, u_n) = l! \sum_{(v;l,n)} \sum\nolimits_N' g_{v_1}(u_1, \cdots, u_{v_1}) \cdots g_{v_l}(u_\mu, \cdots, u_n). \tag{103}$$

Taking the $n$-fold Fourier transform of both sides of (103) gives (24) for $G_n^{(l)}(f_1, \cdots, f_n)$:

$$G_n^{(l)}(f_1, \cdots, f_n) = l! \sum_{(v;l,n)} \sum\nolimits_N' G_{v_1}(f_1, \cdots, f_{v_1}) \cdots G_{v_l}(f_\mu, \cdots, f_n). \tag{104}$$

### B. Volterra Transfer Functions of Arbitrary Order

The results regarding $[y(t)]^l$ and symmetrization are used in this section to derive the expressions stated in Section III for $G_n(f_1, \cdots, f_n)$ when $n$ is arbitrary.

First consider the differential equation (37). When $x(t)$ is taken to be $\exp(j\omega t)$ and $y(t)$ is assumed to be the series of exponentials (32), substitution in (37) and equating coefficients of $\exp(j\omega_1 t)$ gives $c_1 = G_1(f_1) = 1/F(j\omega_1)$. For $n>1$, take $x(t)$ to be the sum of $n$ exponentials and assume that $y(t)$ can be expanded in an $n$-fold series, analogous to (33) for the case $n=2$, in which the coefficient of $\exp[j(\omega_1+\cdots+\omega_n)t]$ is $G_n(f_1,\cdots,f_n)$. Then $[y(t)]^l$ can be expanded in a similar series in which the coefficient of $\exp[j(\omega_1+\cdots+\omega_n)t]$ is $G_n^{(l)}(f_1,\cdots,f_n)$. Substituting these series in the differential equation (37), equating coefficients of $\exp[j(\omega_1+\cdots+\omega_n)t]$, and noting that $G_n^{(l)}$ is zero when $l>n$ gives

$$F(j\omega_1+\cdots+j\omega_n)G_n(f_1,\cdots,f_n)+\sum_{l=2}^{n} a_l G_n^{(l)}(f_1,\cdots,f_n)=0 \tag{105}$$

from which the desired recurrence relation (38) follows.

The system equation for the system shown in Fig. 3, is obtained by equating the series (42) for $I(t)$ to the convolution integral for the current through the admittance $H(f)$:

$$\sum_{l=1}^{\infty} a_l[y(t)]^l = \int_{-\infty}^{\infty} h(\tau)[x(t-\tau)-y(t-\tau)]d\tau \tag{106}$$

where $h(t)$ is the Fourier transform of $H(f)$. Taking $x(t)$ to be $\exp(j\omega_1 t)$, assuming the series (32) for $y(t)$, and equating coefficients of $\exp(j\omega_1 t)$ in (106) gives

$$a_1 G_1(f_1) = H(f_1) - G_1(f_1)H(f_1). \tag{107}$$

For $n>1$, taking $x(t)$ to be the sum of $n$ exponentials and equating coefficients of $\exp[j(\omega_1+\cdots+\omega_n)t]$ gives

$$a_1 G_n(f_1,\cdots,f_n)+\sum_{l=2}^{n} a_l G_n^{(l)}(f_1,\cdots,f_n) = -G_n(f_1,\cdots,f_n)H(f_1+\cdots+f_n). \tag{108}$$

The recurrence relations (43) for the circuit of Fig. 3 follow from (107) and (108).

We turn now to Fig. 1. The system equations for the modulator–filter–demodulator system are given by (46) and (47). To start the derivation of the expression (49) for $G_n(f_1,\cdots,f_n)$, we define $H(\zeta)$ by

$$H(\zeta) = \int_{-\infty}^{\infty} g(u)h[\zeta x(t-u)]du \tag{109}$$

where $t$ is regarded as a parameter in $H(\zeta)$. The function $F[H(\zeta)]$ is equal to $y(t)$ when $\zeta=1$. Also, by direct expansion,

$$F[H(\zeta)] = \sum_{n=0}^{\infty} \frac{\zeta^n}{n!}\left[\frac{d^n}{d\zeta^n}F[H(\zeta)]\right]_{\zeta=0} \tag{110}$$

where we shall evaluate the $n$th derivative by di Bruno's formula (98). For this we need

$$\begin{aligned} H^{(\nu)}(0) &= h_\nu \int_{-\infty}^{\infty} g(u)[x(t-u)]^\nu du \\ H(0) &= z_0 \\ F[H(0)] &= F(z_0) = 0 \\ F^{(l)}[H(0)] &= \left[\frac{d^l}{dz^l}F(z)\right]_{z=H(0)} = F_l \end{aligned} \tag{111}$$

which follow from (47) and (109). Using di Bruno's formula in (110) and setting $\zeta=1$ gives

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!}\sum_{l=1}^{n} F_l \sum_{(\nu;l,n)} N(\nu_1,\cdots,\nu_l)H^{(\nu_1)}(0)\cdots H^{(\nu_l)}(0). \tag{112}$$

We rewrite the integral for $H^{(\nu)}(0)$ as

$$\begin{aligned} H^{(\nu)}(0) &= h_\nu \int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_\nu \varphi_\nu(u_1,\cdots,u_\nu)x(t-u_1)\cdots x(t-u_\nu) \\ \varphi_1(u_1) &= g(u_1) \\ \varphi_\nu(u_1,\cdots,u_\nu) &= g(u_1)\delta(u_2-u_1)\cdots\delta(u_\nu-u_1), \qquad \nu>1 \end{aligned} \tag{113}$$

where $\varphi_\nu(u_1,\cdots,u_\nu)$ can be regarded as a symmetric function possessing the $\nu$-fold Fourier transform $G(f_1+\cdots+f_\nu)$, $G(f)$ being the Fourier transform (48) of $g(t)$. Then (112) becomes

$$\begin{aligned} y(t) &= \sum_{n=1}^{\infty} \frac{1}{n!}\int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n \gamma_n(u_1,\cdots,u_n)\prod_{r=1}^{n} x(t-u_r) \\ \gamma_n(u_1,\cdots,u_n) &= \sum_{l=1}^{n} F_l \sum_{(\nu;l,n)} h_{\nu_1}\cdots h_{\nu_l} N(\nu_1,\cdots,\nu_l)\varphi_{\nu_1}(u_1,\cdots,u_{\nu_1})\cdots\varphi_{\nu_l}(u_\mu,\cdots,u_n) \end{aligned} \tag{114}$$

where $\gamma_n(u_1,\cdots,u_n)$ is usually not symmetric when $n>2$. To convert (114) into a Volterra series, we symmetrize $\gamma_n(u_1,\cdots,u_n)$ with the help of (102). The effect of the symmetrization is to replace $N(\nu_1,\cdots,\nu_l)$ by the sum $\sum_N'$ taken over $N$ nonidentical products. Thus, (114) goes into the Volterra series (1) with the symmetric kernel

$$g_n(u_1,\cdots,u_n) = \sum_{l=1}^{n} F_l \sum_{(\nu;l,n)} h_{\nu_1}\cdots h_{\nu_l} \cdot \sum_N' \varphi_{\nu_1}(u_1,\cdots,u_{\nu_1})\cdots\varphi_{\nu_l}(u_\mu,\cdots,u_n). \tag{115}$$

This kernel has the $n$-fold Fourier transform $G_n(f_1,\cdots,f_n)$ stated in (49) as we wished to show.

Finally we sketch the derivation of the general recurrence relation (58) which gives $G_n(f_1,\cdots,f_n)$ for the feedback system of Fig. 2. Replacing $t$ by $t-u_q'$ in the Volterra series (59) for $w(t)$ in "powers" of $x(t)$ gives a series for $w(t-u_q')$. The product $\prod_{q=1}^{l} w(t-u_q')$ in the system equation (53) for $y(t)$ can then be written as an $l$-fold sum of an $i_1+\cdots+i_q=n$-fold integral. Changing the order of summation and using

$$\sum_{l=1}^{\infty}\sum_{i_1=1}^{\infty}\cdots\sum_{i_l=1}^{\infty} = \sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{i_1+i_2+\cdots+i_l=n}$$

where the summand is of the form $A_{i_1 i_2\cdots i_l}$, leads to a series for $y(t)$:

$$y(t) = \sum_{n=1}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty} du_1\cdots\int_{-\infty}^{\infty} du_n \varphi_n(u_1,\cdots,u_n)\prod_{r=1}^{n} x(t-u_r) \tag{116}$$

where

$$\varphi_n(u_1,\cdots,u_n) = \sum_{l=1}^{n}\frac{1}{l!}\sum_{i_1+\cdots+i_l=n}\frac{n!}{i_1!\cdots i_l!}\int_{-\infty}^{\infty} du_1'\cdots\int_{-\infty}^{\infty} du_l' \cdot m_l(u_1',\cdots,u_l')k_{i_1}(u_1-u_1',\cdots,u_{i_1}-u_1')\cdots k_{i_l}(\cdots,u_n-u_l'). \tag{117}$$

Comparison of (116) with the Volterra series (1) for $y(t)$ shows that $g_n(u_1,\cdots,u_n)$ is the symmetrized version of $\varphi_n(u_1,\cdots,u_n)$; i.e., $G_n(f_1,\cdots,f_n)$ is the symmetrized version of $n$-fold Fourier transform $\Phi_n(f_1,\cdots,f_n)$ of $\varphi_n$. It turns out that the expression for $\Phi_n$ can be obtained from the right side of (117) by replacing the $l$-fold integral by a product $K_{i_1}K_{i_2}\cdots K_{i_l}M_l$ which has the same form (with $\nu$'s replaced by $i$'s) as the product $K_{\nu_1}K_{\nu_2}\cdots K_{\nu_l}M_l$ appearing in the general expression (58) for $G_n(f_1,\cdots,f_n)$. This product (and also the function $M_l$ with arguments $f_1+\cdots+f_{i_1}$, etc.) has the same type of symmetry as the product $P(f_1,\cdots,f_n;i_1,\cdots,i_l)$ discussed in Section V-C. Symmetrizing $\Phi_n$ with the help of (123), setting the result equal to $G_n$, noting that the term for $l=1$ in $\Phi_n$ is the only one containing $G_n$, and solving for $G_n$ completes the derivation of (58) for $G_n$.

### C. Symmetrization of Products of Symmetric Functions

Let "SV" stand for "symmetrized version of" and denote by $\{\mathrm{SV}\, F(f_1, f_2, \cdots, f_n)\}$ the function obtained by symmetrizing the arbitrary function $F(f_1, f_2, \cdots, f_n)$:

$$\{\mathrm{SV}\, F(f_1, \cdots, f_n)\} = \frac{1}{n!} \sum_{n!} F(f_1, \cdots, f_n). \tag{118}$$

Here the subscript $n!$ on $\sum$ denotes that the summation extends over all $n!$ permutations of the subscripts on the $f$'s. Define the product $P(\cdots)$ by

$$P(f_1, \cdots, f_n; i_1, \cdots, i_l) \equiv s_{i_1}(f_1, \cdots, f_{i_1}) s_{i_2}(f_{i_1+1}, \cdots, f_{i_1+i_2}) \cdots s_{i_l}(f_{n-i_l+1}, \cdots, f_n)$$

$$i_1 + i_2 + \cdots + i_l = n, \qquad 1 \le i_q \le n, \qquad q = 1, 2, \cdots, l \tag{119}$$

where the functions $s_i(\cdots)$ are symmetric functions of their arguments.

We first prove that

$$\{\mathrm{SV}\, P(f_1, \cdots, f_n; v_1, \cdots, v_l)\} = \frac{1}{N} \sum_N' P(f_1, \cdots, f_n; v_1, \cdots, v_l) \tag{120}$$

where $n$, $l$, and the set of integers $v_i$ are given. The $v$'s are integers such that, as in (25)

$$v_1 + v_2 + \cdots + v_l = n, \qquad 1 \le v_1 \le v_2 \cdots \le v_l. \tag{121}$$

The sum on the right side of (120) is over all nonidentical products where, as in connection with (26), the number of such products in the sum is

$$N = n!/(v_1! \cdots v_l! r_1! \cdots r_p!). \tag{122}$$

Here $r_1$ is the number of equal $v$'s in the first run of equal $v$'s in the arrangement $v_1 \le v_2 \le \cdots \le v_l$, $r_2$ the number in the second run, and so on.

To prove (120) note that for a given set of values of $v_1, v_2, \cdots, v_l$ the value of $P(\cdots)$ is not changed when the $f_k$ in the argument of $s_v(\cdots)$ are permuted. There are $v_1!v_2! \cdots v_l!$ such permutations. Furthermore, the value of $P(\cdots)$ is not changed by permuting the $s_v$ with equal subscripts (equal $v$'s). Permutations of this sort do not violate the condition $v_1 \le v_2 \le \cdots \le v_l$. There are $r_1!r_2! \cdots r_p!$ such permutations. The $n!$ quantities $P(\cdots)$ given by the $n!$ permutations of $f_1, \cdots, f_n$ can be sorted into sets according to the value of $P(\cdots)$. The number of members in each set is the same, namely, $v_1!v_2! \cdots v_l! r_1! \cdots r_p! = M$ and the number of sets is $N = n!/M$. Changing the summation over the $n!$ permutations [see (118)] to a summation over the $N$ sets then gives the desired relation (120).

We next prove that, given $n$ and $l$,

$$\left\{\mathrm{SV} \frac{1}{l!} \sum_{i_1+\cdots+i_l=n} \frac{n!}{i_1! \cdots i_l!} P(f_1, \cdots, f_n; i_1, \cdots, i_l)\right\} = \sum_{(v;l,n)} \sum_N' P(f_1, \cdots, f_n; v_1, \cdots, v_l) \tag{123}$$

where the summation on the left is taken over the integers $i_1, \cdots, i_l$ such that $i_1 + \cdots + i_l = n$ and $1 \le i_q \le n$ [see (119)]. The $(v; l, n)$ beneath the $\sum$ on the right denotes summation over all sets of integers $v$ which satisfy (121). To prove (123), note that corresponding to each set $v_1, v_2, \cdots, v_l$ in $(v; l, n)$ there are $l!/(r_1! \cdots r_p!)$ sets of $i_1, i_2, \cdots, i_l$ in which the values of the $i$'s are scrambled values of the $v$'s. For any one of these sets of $i$'s we have 1) $i_1! \cdots i_l! = v_1! \cdots v_l!$ and 2)

$$\{\mathrm{SV}\, P(f_1, \cdots, f_n; i_1, \cdots, i_l)\} = \{\mathrm{SV}\, P(f_1, \cdots, f_n; v_1, \cdots, v_l)\}. \tag{124}$$

Therefore, the left side of (123) can be written as

$$\begin{aligned} &\frac{1}{l!} \sum_{i_1+\cdots+i_l=n} \frac{n!}{i_1! \cdots i_l!} \{\mathrm{SV}\, P(f_1, \cdots, f_n; i_1, \cdots, i_l)\} \\ &= \frac{1}{l!} \sum_{v_1+\cdots+v_l=n} \frac{n!}{v_1! \cdots v_l!} \frac{l!}{r_1! \cdots r_p!} \{\mathrm{SV}\, P(f_1, \cdots, f_n; v_1, \cdots, v_l)\} \\ &= \sum_{(v;l,n)} N\{\mathrm{SV}\, P(f_1, \cdots, f_n; v_1, \cdots, v_l)\} \\ &= \sum_{(v;l,n)} \sum_N' P(f_1, \cdots, f_n; v_1, \cdots, v_l) \end{aligned} \tag{125}$$

which gives the desired relation (123).

## VI. Simple Properties of the Output

Here the Volterra series (1) and the expression (2) for the Volterra transfer function are recast in forms suited to deal with harmonic and Gaussian inputs. The new forms are illustrated by applying them to obtain 1) the general form of the expressions listed in Section II-A for $y(t)$ when $x(t)$ is a sine wave or the sum of two or more sine waves, and 2) the expression for the dc value of $y(t)$ when $x(t)$ is Gaussian. When $x(t)$ is Gaussian, the dc value of $y(t)$ is equal to the expected value, or ensemble average, $\langle y(t) \rangle$.

### A. General Relations

The derivation of the formulas whose leading terms are listed in Section II-A is simplified by writing the product $x(t-u_1) \cdots x(t-u_n)$ in (1) as the coefficient of $\alpha_1 \alpha_2 \cdots \alpha_n$ in the expansion of $\exp[\alpha_1 x(t-u_1) + \cdots + \alpha_n x(t-u_n)]$; i.e., as the result of operating on the exponential function by

$$D_\alpha^n \equiv \left. \frac{\partial^n}{\partial \alpha_1 \cdots \partial \alpha_n} \right|_{\alpha_1 = \cdots = \alpha_n = 0}. \tag{126}$$

Thus

$$x(t-u_1) \cdots x(t-u_n) = D_\alpha^n \exp\left[\sum_{s=1}^n \alpha_s x(t-u_s)\right] \tag{127}$$

$$= D_\alpha^n \left[\sum_{s=1}^n \alpha_n x(t-u_s)\right]^n \Big/ n! \tag{128}$$

and the Volterra series (1) for $y(t)$ can be rewritten in the following two ways:

$$y(t) = \sum_{n=1}^\infty \frac{1}{n!} \int_{-\infty}^\infty du_1 \cdots \int_{-\infty}^\infty du_n g_n(u_1, \cdots, u_n) \cdot D_\alpha^n \exp\left[\sum_{s=1}^n \alpha_s x(t-u_s)\right] \tag{129}$$

$$y(t) = \sum_{n=1}^\infty \frac{1}{n!} \int_{-\infty}^\infty du_1 \cdots \int_{-\infty}^\infty du_n g_n(u_1, \cdots, u_n) \cdot D_\alpha^n \left[\sum_{s=1}^n \alpha_s x(t-u_s)\right]^n \Big/ n!. \tag{130}$$

The series (130) for $y(t)$ and the method of calculating $G_n(f_1, \cdots, f_n)$ by taking $x(t)$ to be the sum of $n$ exponential terms leads to the useful expression

$$G_n(f_1, \cdots, f_n) = \frac{1}{n!} \int_{-\infty}^\infty du_1 \cdots \int_{-\infty}^\infty du_n g_n(u_1, \cdots, u_n) D_\alpha^n \prod_{r=1}^n A_n(f_r) \tag{131}$$

where

$$A_n(f) = \sum_{s=1}^n \alpha_s e^{-j\omega u_s}, \qquad \omega = 2\pi f. \tag{132}$$

The following result [8] will be used in conjunction with the rewritten Volterra series (129) for $y(t)$ when $x(t)$ is Gaussian with the two-sided power spectrum $W_x(f)$. Let $L$ be a linear operator (operating on functions of $t$) such that

$$L[e^{j\omega t}] = H(f)e^{j\omega t}, \qquad \omega = 2\pi f. \tag{133}$$

Then

$$\langle \exp\{L[x(t)]\}\rangle = \exp\left[\frac{1}{2}\int_{-\infty}^{\infty} df W_x(f)H(f)H(-f)\right]. \tag{134}$$

*B. Harmonic Input*

When $x(t)=P\cos pt$, $p=2\pi f_p$, the rightmost sum in (130) is

$$\sum_{s=1}^{n} \alpha_s x(t-u_s) = \frac{1}{2}P\sum_{s=1}^{n} \alpha_s(e^{jpt-jpu_s}+e^{-jpt+jpu_s}) = \frac{1}{2}P[e^{jpt}A_n(f_p)+e^{-jpt}A_n(-f_p)] \tag{135}$$

where $A_n$ is given by (132). From the binomial theorem

$$\left[\sum_{s=1}^{n}\alpha_s x(t-u_s)\right]^n \Big/ n! = \left(\frac{P}{2}\right)^n \sum_{k=0}^{n} \frac{\exp[j(2k-n)pt]}{k!(n-k)!} A_n^k(f_p)A_n^{n-k}(-f_p). \tag{136}$$

Substuting this in (130) for $y(t)$ and using (131) for $G_n$ gives

$$y(t) = \sum_{n=1}^{\infty}\left(\frac{P}{2}\right)^n \sum_{k=0}^{n} \frac{\exp[j(2k-n)pt]}{k!(n-k)!} G_{k,n-k}(f_p). \tag{137}$$

Here $G_{k,n-k}(f_p)$ denotes $G_n(f_1,\cdots,f_n)$, with the first $k$ of the $f_i$ equal to $+f_p$ and the remaining $n-k$ equal to $-f_p$.

Selecting the terms in (137) for which $2k-n=N\geq 0$ shows that the $\exp(jNt)$ component of $y(t)$ is

$$e^{jNpt}\sum_{l=0}^{\infty}\frac{(P/2)^{2l+N}}{(N+l)!\,l!}G_{N+l,l}(f_p). \tag{138}$$

The value of $G_{0,0}(f_p)$, which occurs when $N=0$, is zero because $G_0\equiv 0$. Changing the signs of $p$ and $f_p$ in (138) gives the $\exp(-jNpt)$ component of $y(t)$. For $x(t)=P\cos(pt+\varphi)$, $y(t)$ is given by (137) and (138) with $pt$ replaced by $pt+\varphi$. These expressions for the output components are similar to formulas given by Mircea [7].

The same type of argument shows that when $x(t)=P\cos pt + Q\cos qt$, the $\exp[j(Np+Mq)t]$ component in $y(t)$ is, for $M\geq 0$ and $N\geq 0$,

$$e^{j(Np+Mq)t}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(P/2)^{2l+N}(Q/2)^{2k+M}}{(N+l)!\,l!\,(M+k)!\,k!}G_{N+l,l;M+k,k}(f_p,f_q) \tag{139}$$

where $2\pi f_p=p$, $2\pi f_q=q$, and the four subscripts on $G$ mean that it is equal to $G_n(f_1,\cdots,f_n)$ with $n=N+2l+M+2k$ and the first $N+l$ of the $f_i$ equal to $f_p$, the next $l$ equal to $-f_p$, the next $M+k$ equal to $f_q$, and the last $k$ equal to $-f_q$. Changing the signs of $p$ and $f_p$ in (139) gives the $\exp[j(-Np+Mq)t]$ component of $y(t)$, and so on.

Similarly, when $x(t)=P\cos pt+Q\cos qt+R\cos rt$, the $\exp[j(Np+Mq+Lr)t]$ component in $y(t)$ is, for $M\geq 0$, $N\geq 0$, and $L\geq 0$,

$$e^{j(Np+Mq+Lr)t}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(P/2)^{2l+N}(Q/2)^{2k+M}(R/2)^{2i+L}}{(N+l)!\,l!\,(M+k)!\,k!\,(L+i)!\,i!} \cdot G_{N+l,l;M+k,k;L+i,i}(f_p,f_q,f_r) \tag{140}$$

where the order of $G_n(f_1,\cdots,f_n)$ is $n=N+2l+M+2k+L+2i$.

When phase angles appear in the cosine terms in $x(t)$, we replace $Npt$ by $Npt+N\varphi_p$, $Mqt$ by $Mqt+M\varphi_q$, etc., in the exponential terms in (139) and (140).

*C. Expected Value of $y(t)$ for Gaussian Input*

When $x(t)$ is a zero-mean stationary Gaussian process, the expected value of $y(t)$ is the ensemble average $\langle y(t)\rangle$ obtained by averaging both sides of (129) and using (134) to show that

$$\left\langle \exp\sum_{s=1}^{n}\alpha_s x(t-u_s)\right\rangle = \exp J_{nAA} \tag{141}$$

$$J_{nAA} = \frac{1}{2}\int_{-\infty}^{\infty} df\, W_x(f)A_n(f)A_n(-f). \tag{142}$$

Here the subscript $nAA$ is suggested by (142) and (150), $W_x(f)$ is the two-sided power spectrum of $x(t)$, and $A_n(f)$ is defined by (132). The integral (142) for $J_{nAA}$ is obtained by identifying (141) with (134) and using

$$L[e^{j\omega t}] = e^{j\omega t}\sum_{s=1}^{n}\alpha_s e^{-j\omega u_s} = e^{j\omega t}A_n(f) \tag{143}$$

to show that $H(f)$ goes into $A_n(f)$. Thus the expression (129) for $y(t)$ gives

$$\langle y(t)\rangle = \sum_{n=1}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty}du_1\cdots\int_{-\infty}^{\infty}du_n g_n(u_1,\cdots,u_n)D_\alpha^n \exp J_{nAA}. \tag{144}$$

The next step is to expand $\exp J_{nAA}$. It is convenient to introduce the operator $Q_k[h(x)]$ which denotes a $k$-fold integration with respect to $x_1, x_2, \cdots, x_k$, with limits $\pm\infty$. The integrand is $h(x_1)\cdots h(x_k)$ times the function of $x_1,\cdots,x_k$ represented by all of the terms lying to the right of $Q_k[h(x)]$. $Q_0[h(x)]$ denotes the identity operator. When we substitute

$$\exp J_{nAA} = \sum_{\mu=0}^{\infty}\frac{1}{\mu!}J_{nAA}^{\mu} = 1+\sum_{\mu=1}^{\infty}\frac{1}{\mu!\,2^\mu}Q_\mu[W_x(f)]\prod_{r=1}^{\mu}A_n(f_r)A_n(-f_r) \tag{145}$$

in the series (144) for $\langle y(t)\rangle$, the 1 in (145) corresponding to $\mu=0$ contributes nothing because for $n\geq 1$ the value of $D_\alpha^n$ operating on 1 is 0. Interchanging the order of the $n$ and $\mu$ summations gives

$$\langle y(t)\rangle = \sum_{\mu=1}^{\infty}\frac{1}{\mu!\,2^\mu}Q_\mu[W_x(f)]\sum_{n=1}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty}du_1\cdots \cdot\int_{-\infty}^{\infty}du_n g_n(u_1,\cdots,u_n)D_\alpha^n\prod_{r=1}^{\mu}A_n(f_r)A_n(-f_r). \tag{146}$$

Since $A_n(f_r)$ is a homogeneous linear function of the $\alpha$'s, the product in (146) is of degree $2\mu$ in the $\alpha$'s; and all of the terms in the $n$-sum are 0 except the one for $n=2\mu$. Setting $n=2\mu$ and using (131) for $G_n$ leads to

$$\langle y(t)\rangle = \sum_{\mu=1}^{\infty}\frac{1}{\mu!\,2^\mu}Q_\mu[W_x(f)]G_{2\mu}(f_1,-f_1,f_2,-f_2,\cdots,f_\mu,-f_\mu) \tag{147}$$

the first two terms of which have been given in (11).

A somewhat similar expression for $\langle y(t)\rangle$ has been given by Deutsch [12].

## VII. Power Spectra

In this section the two-sided power spectrum $W_y(f)$ of $y(t)$ is computed for two cases. In the first, the input $x(t)$ is zero-mean stationary Gaussian noise with power spectrum $W_x(f)$ (the Mircea–Sinnreich case). In the second, the input $x(t)$ is a sine wave plus zero-mean stationary Gaussian noise, $P \cos pt + I_N(t)$. In both cases the ensemble average $\langle y(t+\tau)y^*(t)\rangle$ is computed and then its Fourier transform taken to get $W_y(f)$.

### A. The Expected Value of $y(t+\tau)z(t)$ for Gaussian Input

Let $y(t)$ be given by the Volterra series (1) and $z(t)$ by a similar series with $g'_n(u_1, \cdots, u_n)$ in place of $g_n(u_1, \cdots, u_n)$. Both have the same Gaussian noise input $x(t)$. The steps in calculating the ensemble average $\langle y(t+\tau)z(t)\rangle$ are similar to, but more complicated than, those used to calculate $\langle y(t)\rangle$.

From the rewritten Volterra series (129)

$$\langle y(t+\tau)z(t)\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n!m!}\int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n \int_{-\infty}^{\infty} dv_1 \cdots \int_{-\infty}^{\infty} dv_m g_n(u_1, \cdots, u_n) g'_m(v_1, \cdots, v_m) D_\alpha^n D_\beta^m \left\langle \exp\left[\sum_{s=1}^{n} \alpha_s x(t+\tau-u_s) + \sum_{s=1}^{m} \beta_s x(t-v_s)\right]\right\rangle. \quad (148)$$

The ensemble average of the exponential function is again given by (134), but now $H(f)$ is determined by

$$L[e^{j\omega t}] = e^{j\omega t}[e^{j\omega\tau}A_n(f) + B_m(f)] = e^{j\omega t}H(f) \quad (149)$$

where $A_n(f)$ is still given by (132) and $B_m(f)$ by (132) with $n$, $\alpha$, $u$ replaced by $m$, $\beta$, $v$. Therefore, the ensemble average of the exponential function in (148) can be written as exp $K$, where

$$K = \frac{1}{2}\int_{-\infty}^{\infty} df\, W_x(f)[A_n(f)A_n(-f) + B_m(f)B_m(-f) + 2e^{j\omega\tau}A_n(f)B_m(-f)] = J_{nAA} + J_{mBB} + J_{nmAB}. \quad (150)$$

The evenness of $W_x(f)$ has been used to obtain the term containing $\exp(j\omega\tau)$; $J_{nAA}$ is the integral containing $A_n(f)A_n(-f)$; and so on.

Expanding $\exp J_{nmAB}$ in the same way as was $\exp J_{nAA}$ in (145) gives

$$e^K = \exp[J_{nAA} + J_{mBB}] \cdot \left[1 + \sum_{k=1}^{\infty} \frac{Q_k[W_x(f)]}{k!} \prod_{r=1}^{k} e^{j\omega_r\tau} A_n(f_r)B_m(-f_r)\right]. \quad (151)$$

When this is substituted for the ensemble average in (148), the contribution of $\exp[J_{nAA} + J_{mBB}]$ times 1, the 1 being the term in (151) corresponding to $k=0$, is $\langle y(t)\rangle\langle z(t)\rangle$. This follows from the series (144) for $\langle y(t)\rangle$. The contribution of the remaining portion, i.e., the portion arising from the $k \geq 1$ terms in (151), can be obtained by changing the order of the summations and integrations. After the change the double $m$, $n$-sum can be written as the product of the $m$-sum and the $n$-sum. Therefore, the substitution of (151) in (148) yields

$$\langle y(t+\tau)z(t)\rangle = a_0b_0 + \sum_{k=1}^{\infty} \frac{Q_k[W_x(f)]}{k!} e^{j(\omega_1+\cdots+\omega_k)\tau} a_k(f_1, \cdots, f_k) b_k(f_1, \cdots, f_k) \quad (152)$$

where $a_0 = \langle y(t)\rangle$, $b_0 = \langle z(t)\rangle$, and for $k > 0$,

$$a_k(f_1, \cdots, f_k) = \sum_{n=1}^{\infty} \frac{1}{n!}\int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n g_n(u_1, \cdots, u_n) D_\alpha^n e^{J_{nAA}} \prod_{r=1}^{k} A_n(f_r). \quad (153)$$

The function $b_k(f_1, \cdots, f_k)$ is given by an expression obtained by replacing $n$, $u$, $g_n$, $\alpha$, $J_{nAA}$, $A_n(f_r)$ in (153) by $m$, $v$, $g'_m$, $\beta$, $J_{mBB}$, $B_m(-f_r)$. For example, if $z(t) \equiv y(t)$, then $b_k(f_1, \cdots, f_k) = a_k(-f_1, \cdots, -f_k)$.

Substituting the series (145) for $\exp J_{nAA}$ in (153) brings in the quantity

$$D_\alpha^n \left[\prod_{q=1}^{\mu} A_n(f'_q)A_n(-f'_q)\right]\left[\prod_{r=1}^{k} A_n(f_r)\right] \quad (154)$$

where $r$, $f$, $f_r$ in (145) have been replaced by $q$, $f'$, $f'_q$. Since $A_n(f)$ is homogeneous and linear in the $\alpha$'s, (154) is zero unless $n = 2\mu + k$. The double sum for $a_k$ taken over $\mu$ and $n$ reduces to a single sum over $\mu$ in which the $\mu$th term is, with $n = 2\mu + k$,

$$\frac{1}{\mu!2^\mu} Q_\mu[W_x(f')] \frac{1}{n!}\int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n \cdot g_n(u_1, \cdots, u_n) \quad \{\text{expression (154)}\}. \quad (155)$$

The expression (131) for $G_n(f_1, \cdots, f_n)$ shows that the multiple integral goes into a $G_n$ function. Consequently, the series (153) for $a_k$, $k > 0$, becomes

$$a_k(f_1, \cdots, f_k) = \sum_{\mu=0}^{\infty} \frac{Q_\mu[W_x(f')]}{\mu!2^\mu} G_{2\mu+k}(f_1, f_2, \cdots, f_k, f'_1, -f'_1, \cdots, f'_\mu, -f'_\mu)$$
$$= G_k(f_1, \cdots, f_k) + \frac{1}{1!2}\int_{-\infty}^{\infty} df'_1 W_x(f'_1) G_{2+k}(f_1, \cdots, f_k, f'_1, -f'_1)$$
$$+ \frac{1}{2!2^2}\int_{-\infty}^{\infty} df'_1 \int_{-\infty}^{\infty} df'_2 W_x(f'_1)W_x(f'_2) G_{4+k}(f_1, \cdots, f_k, f'_1, -f'_1, f'_2, -f'_2) + \cdots. \quad (156)$$

The corresponding expression for $b_k(f_1, \cdots, f_k)$ is obtained by replacing the $G_{2\mu+k}$ functions in (156) by $G'_{2\mu+k}(-f_1, \cdots, -f_k, f'_1, -f'_1, \cdots, f'_\mu, -f'_\mu)$ where $G'_m$ is the Fourier transform of the kernel $g'_m$ in the Volterra series for $z(t)$. The signs of $f_1, \cdots, f_k$ are reversed because $B_m(-f_r)$ replaces $A_n(f_r)$ in the analysis. Since the $G_m$ are symmetric, so are $a_k(f_1, \cdots, f_k)$ and $b_k(f_1, \cdots, f_k)$. The series for $b_k(f_1, \cdots, f_k)$ is

$$b_k(f_1, \cdots, f_k) = G'_k(-f_1, \cdots, -f_k) + \frac{1}{1!2}\int_{-\infty}^{\infty} df'_1 W_x(f'_1) G'_{2+k}(-f_1, \cdots, -f_k, f'_1, -f'_1) + \cdots. \quad (157)$$

Writing out the integrals denoted by $Q_\mu[W_x(f)]$ in (152) gives the required expression

$$\langle y(t+\tau)z(t)\rangle = \langle y(t)\rangle\langle z(t)\rangle + \frac{1}{1!}\int_{-\infty}^{\infty} df_1 e^{j\omega_1\tau} W_x(f_1) a_1(f_1) b_1(f_1)$$
$$+ \frac{1}{2!}\int_{-\infty}^{\infty} df_1 \int_{-\infty}^{\infty} df_2 e^{j(\omega_1+\omega_2)\tau} W_x(f_1)W_x(f_2) a_2(f_1, f_2) b_2(f_1, f_2) + \cdots \quad (158)$$

where the $a_k$ and $b_k$ are given by the series (156) and (157).

The special case $z(t) \equiv y(t)$ has been considered by Deutsch [12] who outlined a procedure for calculating $\langle y(t+\tau)y(t)\rangle$.

### B. Power Spectrum for Gaussian Input

The two-sided power spectrum $W_y(f)$ of $y(t)$, for complex $y(t)$ and Gaussian $x(t)$, is the Fourier transform of the function $\langle y(t+\tau)y^*(t)\rangle$ of $\tau$ obtained by setting $z(t) = y^*(t)$ in (152). Then 1) $g'_m = g_m^*$, 2)

$$G'_{2\mu+k}(-f_1, \cdots, -f_k, f'_1, -f'_1, \cdots, f'_\mu, -f'_\mu) = G^*_{2\mu+k}(f_1, \cdots, f_k, f'_1, -f'_1, \cdots, f'_\mu, -f'_\mu) \quad (159)$$

and 3) (156) and (157) show that $b_k(f_1, \cdots, f_k)$ is equal to $a_k^*(f_1, \cdots, f_k)$. Consequently, $\langle y(t+\tau)y^*(t)\rangle$ is given by the series (152) with $a_k(f_1, \cdots, f_k)b_k(f_1, \cdots, f_k)$ replaced by $|a_k(f_1, \cdots, f_k)|^2$. Multiplying by $\exp(-j\omega\tau)$ and integrating $\tau$ from $-\infty$ to $\infty$ gives, in our notation, the Mircea–Sinnreich [5] series for the power spectrum of $y(t)$:

$$\begin{aligned} W_y(f) &= |a_0|^2\delta(f) \\ &\quad + \sum_{k=1}^{\infty} \frac{Q_k[W_x(f)]}{k!}\delta(f - f_1 - \cdots - f_k)|a_k(f_1, \cdots, f_k)|^2 \\ &= |a_0|^2\delta(f) + W_x(f)|a_1(f)|^2 \\ &\quad + \frac{1}{2!}\int_{-\infty}^{\infty} df_1 W_x(f_1)W_x(f - f_1)|a_2(f_1, f - f_1)|^2 \\ &\quad + \frac{1}{3!}\int_{-\infty}^{\infty} df_1 \int_{-\infty}^{\infty} df_2 W_x(f_1)W_x(f_2)W_x(f - f_1 - f_2) \\ &\quad \cdot |a_3(f_1, f_2, f - f_1 - f_2)|^2 \\ &\quad + \cdots. \end{aligned} \tag{160}$$

Here $a_0 = \langle y(t)\rangle$ and $a_k(f_1, \cdots, f_k)$ for $k>0$ is given by the series (156). Note that the $f$ in the operator $Q_k[W_x(f)]$ takes on the values $f_1, f_2, \cdots, f_k$. It is not related to the $f$ in $\delta(f-f_1-\cdots-f_k)$. The series for $a_0$ is given by the series (147) for $\langle y(t)\rangle$. For $k=1, 2, 3$, the first few terms in (156) are

$$\begin{aligned} a_1(f) &= G_1(f) + \frac{1}{1!2}\int_{-\infty}^{\infty} df_1' W_x(f_1')G_3(f, f_1', -f_1') + \cdots \\ a_2(\rho, \sigma) &= G_2(\rho, \sigma) + \frac{1}{1!2}\int_{-\infty}^{\infty} df_1' W_x(f_1')G_4(\rho, \sigma, f_1', -f_1') + \cdots \\ a_3(\rho, \sigma, \lambda) &= G_3(\rho, \sigma, \lambda) + \cdots. \end{aligned} \tag{161}$$

The leading terms in (160) for $W_y(f)$ have been given in (14).

When $y(t)$ is complex, it follows from (70) that the power spectrum of the real part of $y(t)$ can be obtained by replacing $G_n(f_1, \cdots, f_n)$ by $[G_n(f_1, \cdots, f_n) + G_n^*(-f_1, \cdots, -f_n)]/2$ in the analysis leading to $W_y(f)$. This is equivalent to replacing $|a_k(f_1, \cdots, f_k)|^2$ by $|a_k(f_1, \cdots, f_k) + a_k^*(-f_1, \cdots, -f_k)|^2/4$ in the series (160) for $W_y(f)$ to get the power spectrum of the real part of $y(t)$. Likewise, the power spectrum of the imaginary part of $y(t)$ is obtained by replacing $G_n(f_1, \cdots, f_n)$ by $[G_n(f_1, \cdots, f_n) - G_n^*(-f_1, \cdots, -f_n)]/(2j)$ in the analysis leading to $W_y(f)$, and $|a_k(f_1, \cdots, f_k)|^2$ by $|a_k(f_1, \cdots, f_k) - a_k^*(-f_1, \cdots, -f_k)|^2/4$ in (160).

### C. Power Spectrum for Sine Wave Plus Noise Input

When $x(t) = P\cos pt + I_N(t)$ where $I_N(t)$ is a Gaussian noise having the two-sided power spectrum $W_I(f)$, the ensemble average $\langle y(t)\rangle$ is periodic with period $1/f_p = 2\pi/p$. To obtain $\langle y(t)\rangle$, we proceed as in Section VI-C. Combining (135) and (141) gives

$$\left\langle \exp \sum_{s=1}^{n} \alpha_n x(t - u_s) \right\rangle = \exp\left\{\frac{P}{2}\left[e^{jpt}A_n(f_p) + e^{-jpt}A_n(-f_p)\right]\right\} \exp J_{nAA} \tag{162}$$

where $J_{nAA}$ is given by (142) with $W_x(f)$ replaced by $W_I(f)$. If the right side of (162) is expanded with the help of (145), it becomes

$$\sum_{N=0}^{\infty} \frac{1}{N!}\left(\frac{P}{2}\right)^N \sum_{l=0}^{N} \frac{N!e^{j(2l-N)pt}}{l!(N-l)!}[A_n(f_p)]^l[A_n(-f_p)]^{N-l} \cdot \left[1 + \sum_{\mu=1}^{\infty} \frac{1}{\mu!2^\mu} Q_\mu[W_I(f)] \prod_{r=1}^{\mu} A_n(f_r)A_n(-f_r)\right]. \tag{163}$$

When this is substituted in the series of integrals obtained by taking the ensemble average of (129) and changing the order of summation, the operator $D_\alpha^n$ makes all terms zero except those for which $l + (N-l) + 2\mu = N + 2\mu = n$. The result is

$$\langle y(t)\rangle = \sum_{N=0}^{\infty}\left(\frac{P}{2}\right)^N \sum_{l=0}^{N} \frac{e^{j(2l-N)pt}}{l!(N-l)!} \sum_{\mu=0}^{\infty} \frac{1}{\mu!2^\mu} Q_\mu[W_I(f)] \cdot G_{N+2\mu}(f_1, -f_1, \cdots, f_\mu, -f_\mu, (f_p)_l, (-f_p)_{N-l}) \tag{164}$$

where $G_0 \equiv 0$ and $(f_p)_l$ denotes the string of $l$ arguments $f_p, f_p, \cdots, f_p$. If $\mu$ or $l$ or $N-l$ are zero, the corresponding arguments in $G_{N+2\mu}$ do not appear. The $G_{N+2\mu}$ term corresponding to $\mu=0$ in (164), namely, $G_N((f_p)_l, (-f_p)_{N-l})$, is the same as $G_{l,N-l}(f_p)$ in the notation of (137). The $f$ in $Q_\mu[W_I(f)]$ refers only to $f_1, \cdots, f_\mu$, never to $f_p$. The first few terms of (164) have been given in (15).

The exp $(jnpt)$ component in $\langle y(t)\rangle$ is

$$e^{jnpt}\sum_{\sigma=0}^{\infty}\left(\frac{P}{2}\right)^{2\sigma+|n|} \frac{1}{\sigma!(\sigma+|n|)!} \sum_{\mu=0}^{\infty} \frac{1}{\mu!2^\mu} Q_\mu[W_I(f)] \cdot G_{2\sigma+|n|+2\mu}(f_1, -f_1, \cdots, f_\mu, -f_\mu, (f_p s_n)_{\sigma+|n|}, (-f_p s_n)_\sigma) \tag{165}$$

where $s_n = 1$ for $n \geq 0$ and $s_n = -1$ for $n<0$. When $P$ is zero, (165) becomes (147), and when $W_I(f)$ is zero, (165) becomes (138) with $N = |n|$.

We now obtain $\langle y(t+\tau)z(t)\rangle$ where $z(t)$ is the same as in Section VII-A. For $x(t) = P\cos pt + I_N(t)$,

$$\begin{aligned} &\left\langle \exp\left[\sum_{s=1}^{n} \alpha_s x(t + \tau - u_s) + \sum_{s=1}^{m} \beta_s x(t - v_s)\right]\right\rangle \\ &= \exp\left[\frac{P}{2}\{e^{jpt}[e^{jp\tau}A_n(f_p) + B_m(f_p)] + e^{-jpt}[e^{-jp\tau}A_n(-f_p) + B_m(-f_p)]\}\right] \exp(J_{nAA} + J_{mBB} + J_{nmAB}) \end{aligned} \tag{166}$$

where $J$'s are defined by the integral (150) with $W_x(f)$ replaced by $W_I(f)$, and $A_n(f)$ and $B_m(f)$ are the same as in (149). The contribution of the first term, i.e., unity, in the expansion of $\exp J_{nmAB}$ [see (151)] to the value of the double series for $\langle y(t+\tau)z(t)\rangle$ [obtained by averaging the product of the series (129) for $y(t+\tau)$ and $z(t)$ and using (166)] can be split into the product of the $n$-sum and $m$-sum. As in Section VII-A, this contribution is $\langle y(t+\tau)\rangle\langle z(t)\rangle$ in which the averages can be obtained from (164).

To get the contribution of the remaining terms in the expansion of $\exp(J_{nmAB})$, we expand the first exponential on the right in (166) as

$$\sum_{N=0}^{\infty} \frac{1}{N!}\left(\frac{P}{2}\right)^N \sum_{l=0}^{N} \frac{N!e^{j(2l-N)pt}}{l!(N-l)!} \sum_{\lambda=0}^{l} \sum_{\sigma=0}^{N-l} \frac{l!(N-l)!}{\lambda!(l-\lambda)!\sigma!(N-l-\sigma)!} e^{j(\lambda-\sigma)p\tau} A_n^\lambda(f_p)B_m^{l-\lambda}(f_p)A_n^\sigma(-f_p)B_m^{N-l-\sigma}(-f_p). \tag{167}$$

This leads to

$$\begin{aligned} \langle y(t+\tau)z(t)\rangle &= \langle y(t+\tau)\rangle\langle z(t)\rangle + \sum_{N=0}^{\infty}\left(\frac{P}{2}\right)^N \sum_{l=0}^{N} e^{j(2l-N)pt} \sum_{\lambda=0}^{l}\sum_{\sigma=0}^{N-l} \frac{e^{j(\lambda-\sigma)p\tau}}{\lambda!(l-\lambda)!\sigma!(N-l-\sigma)!} \\ &\quad \cdot \sum_{k=1}^{\infty} \frac{Q_k[W_I(f)]e^{j(\omega_1+\cdots+\omega_k)\tau}}{k!} a_{\lambda,\sigma,k}(f_1, f_2, \cdots, f_k; f_p) b_{l-\lambda, N-l-\sigma, k}(f_1, f_2, \cdots, f_k; f_p) \end{aligned} \tag{168}$$

where

$$a_{\lambda,\sigma,k}(f_1,\cdots,f_k;f_p)=\sum_{\nu=0}^{\infty}\frac{1}{\nu!2^\nu}Q_\nu[W_I(f')]G_{\lambda+\sigma+k+2\nu}(f_1',-f_1',\cdots,f_\nu',-f_\nu',f_1,\cdots,f_k,(f_p)_\lambda,(-f_p)_\sigma) \tag{169}$$

and $b_{\lambda,\sigma,k}(f_1,\cdots,f_k;f_p)$ is obtained from the right side of (169) by replacing $G$ by $G'$ and $f_1,\cdots,f_k$ by $-f_1,\cdots,-f_k$.

The power spectrum of $y(t)$ is given by

$$W_y(f)=\frac{1}{2\pi}\int_0^{2\pi}d(pt)\int_{-\infty}^{\infty}d\tau\, e^{-j\omega\tau}\langle y(t+\tau)y^*(t)\rangle \tag{170}$$

where the ensemble average is given by (168) with $z(t)=y^*(t)$. The $a$'s are defined by (169), and since $G_n'(f_1,\cdots,f_n)$ is equal to $G_n^*(-f_1,\cdots,-f_n)$,

$$\begin{aligned}b_{\lambda,\sigma,k}(f_1,\cdots,f_k;f_p)&=a^*_{\lambda,\sigma,k}(f_1,\cdots,f_k;-f_p)\\&=a^*_{\sigma,\lambda,k}(f_1,\cdots,f_k;f_p).\end{aligned} \tag{171}$$

When the modified ensemble average (168) is substituted in (170), the integration with respect to $(pt)$ eliminates all terms in the summation with respect to $N$ and $l$ except those for which $2l=N$, and the integration with respect to $\tau$ brings in $\delta(f-f_1-f_2\cdots-f_k-\lambda f_p+\sigma f_p)$. Furthermore, if the expression (164) for $\langle y(t)\rangle$ is written as

$$\langle y(t)\rangle=\sum_{n=-\infty}^{\infty}c_n\exp(jnpt) \tag{172}$$

then the contribution of the product $\langle y(t+\tau)\rangle\langle y^*(t)\rangle$ to the right side of (170) is the series of infinite spikes

$$\sum_{n=-\infty}^{\infty}|c_n|^2\delta(f-nf_p). \tag{173}$$

Combining these results gives

$$\begin{aligned}W_y(f)=&\sum_{n=-\infty}^{\infty}|c_n|^2\delta(f-nf_p)\\&+\sum_{l=0}^{\infty}\left(\frac{P}{2}\right)^{2l}\sum_{\lambda=0}^{l}\sum_{\sigma=0}^{l}\frac{1}{\lambda!(l-\lambda)!\sigma!(l-\sigma)!}\sum_{k=1}^{\infty}\frac{1}{k!}Q_k[W_I(f)]\\&\cdot\delta(f-f_1-\cdots-f_k-\lambda f_p+\sigma f_p)a_{\lambda,\sigma,k}(f_1,\cdots,f_k;f_p)\\&\cdot a^*_{l-\sigma,l-\lambda,k}(f_1,\cdots,f_k;f_p).\end{aligned} \tag{174}$$

Changing the order of summation in the four-fold sum so that the $k$-summation is the leftmost, and considering the terms in the $l,\lambda,\sigma$-sum for which $\lambda-\sigma$ is equal to a fixed number $n$, leads to the desired expression

$$\begin{aligned}W_y(f)=&\sum_{n=-\infty}^{\infty}|c_n|^2\delta(f-nf_p)\\&+\sum_{k=1}^{\infty}\frac{1}{k!}Q_k[W_I(f)]\sum_{n=-\infty}^{\infty}\delta(f-f_1-f_2-\cdots-f_k-nf_p)\\&\cdot\left|\sum_{\sigma=0}^{\infty}\left(\frac{P}{2}\right)^{2\sigma+|n|}\frac{1}{\sigma!(\sigma+|n|)!}a_{\sigma+|n|,\sigma,k}(f_1,\cdots,f_k;f_ps_n)\right|^2\end{aligned} \tag{175}$$

where $c_n$ is given by (172) and (165), $a_{\lambda,\sigma,k}(\cdots)$ by (169), the product $f_ps_n$ is equal to $f_p$ when $n\geq 0$, and to $-f_p$ when $n<0$. When $P$ is zero, (175) reduces to (160) for Gaussian $x(t)$. As in (160) the $f$ in $Q_k[W_I(f)]$ takes on only the values $f_1,f_2,\cdots,f_k$, and is not related to the $f$ or the $f_p$ appearing in $\delta(f-f_1-\cdots-f_k-nf_p)$. Also as in (160) the effect of the delta function is to "use up" one of the $k$ integrations denoted by the operator $Q_k[W_I(f)]$ when $k>0$. By noting that (165) is equal to $c_n\exp(jnpt)$ and that $Q_0[W_I(f)]\equiv 1$, we see that the sum of $|c_n|^2\delta(f-nf_p)$ may be regarded as a $k=0$ term, and that (175) can be written as a sum from $k=0$ to $k=\infty$. The first few terms of (175) have been given in (16).

## VIII. Higher Moments and Probability Density

The leading terms in the first four cumulants for $y(t)$ when $x(t)$ is Gaussian noise are derived in Section VIII-A. In Section VIII-B formulas are given to show how these cumulants can be used to obtain information about the probability density of $y(t)$. When the terms beyond the second in a Volterra series vanish, the probability density of $y(t)$ can be expressed as an integral containing certain parameters. The values of the parameters can be obtained by solving an integral equation. In Section VIII-C various forms of the integral equation are listed and methods of computing the cumulants are discussed.

### A. Cumulants

In this section $x(t)$ is taken to be a real zero-mean stationary Gaussian process with two-sided power spectrum $W_x(f)$. The kernels $g_n$ are assumed to be real so that $y(t)$ is real and $G_n(-f_1,\cdots,-f_n)$ is equal to $G_n^*(f_1,\cdots,f_n)$. Since $x(t)$ is stationary, the ensemble averages giving the moments of $y(t)$ do not depend on $t$.

Substituting $[y(t)]^l$ and $G_{2\mu}^{(l)}$ for $y(t)$ and $G_{2\mu}$ in the series (147) for $\langle y(t)\rangle$ gives a series for the $l$th moment of $y(t)$,

$$\langle[y(t)]^l\rangle=\sum_{\mu=1}^{\infty}\frac{1}{\mu!2^\mu}Q_\mu[W_x(f)]G_{2\mu}^{(l)}(f_1,-f_1,\cdots,f_\mu,-f_\mu) \tag{176}$$

where $G_{2\mu}^{(l)}$ is given in terms of the $G_n$ by (24). The series obtained by substituting (24) in (176) is not the most desirable one because it can be simplified by making use of the symmetry of the $G_n$ together with $W_x(-f)=W_x(f)$ and appropriate changes of sign in the variables of integration. Unfortunately, a general procedure for simplification is not known. However, a simplified form for $l=2$ (for $l=1$ (176) itself is the simplified form) can be obtained by setting $\tau=0$ and $z(t)=y(t)$ in the series (158) for $\langle y(t+\tau)z(t)\rangle$:

$$\langle y^2(t)\rangle=\langle y(t)\rangle^2+\sum_{k=1}^{\infty}\frac{Q_k[W_x(f)]}{k!}a_k(f_1,\cdots,f_k)a_k(-f_1,\cdots,-f_k). \tag{177}$$

The leading terms in (177) are shown in (180) for $\kappa_2=\langle y^2(t)\rangle-\langle y(t)\rangle^2$.

In the following work we shall be more concerned with the cumulants $\kappa_n$ of $y(t)$ than with its moments $\alpha_n$. The cumulants are simpler and appear directly in the discussion of the distribution of $y(t)$ given in Section VIII-B. The first four cumulants are related to the moments by [23, p. 186]:

$$\begin{aligned}\kappa_1&=\alpha_1\\\kappa_2&=\alpha_2-\alpha_1^2\\\kappa_3&=\alpha_3-3\alpha_1\alpha_2+2\alpha_1^3\\\kappa_4&=\alpha_4-3\alpha_2^2-4\alpha_1\alpha_3+12\alpha_1^2\alpha_2-6\alpha_1^4.\end{aligned} \tag{178}$$

The memoryless case (22) in which $y=\sum_1^\infty a_nx^n/n!$ is a useful guide to the general case [the coefficient $a_n$ is unrelated to the $a_k(f_1,\cdots,f_k)$ in (177)]. Here $x$ is a normal random variable with mean 0 and variance $\sigma^2$. The $n$th moment of $x$ is 0 when $n$ is odd and is $1\cdot 3\cdots(n-1)\sigma^n$ when $n$ is even. By first working out the moments for $y$ and then substituting them in (178), we get

$$\kappa_1=\frac{1}{2}a_2\sigma^2+\frac{1}{2!2^2}a_4\sigma^4+\frac{1}{3!2^3}a_6\sigma^6+\cdots$$

$$\kappa_2 = a_1^2\sigma^2 + \left(a_1a_3 + \frac{1}{2}a_2^2\right)\sigma^4 + \left(\frac{1}{4}a_1a_5 + \frac{1}{2}a_2a_4 + \frac{5}{12}a_3^2\right)\sigma^6 + \cdots$$

$$\kappa_3 = 3a_1^2a_2\sigma^4 + \left(\frac{3}{2}a_1^2a_4 + 6a_1a_2a_3 + a_2^3\right)\sigma^6 + \cdots$$

$$\kappa_4 = (4a_1^3a_3 + 12a_1^2a_2^2)\sigma^6 + (2a_1^3a_5 + 18a_1^2a_2a_4 + 36a_1a_2^2a_3 + 12a_1^2a_3^2 + 3a_2^4)\sigma^8 + \cdots. \quad (179)$$

When $y(t)$ is given by the general Volterra series (1), instead of the memoryless power series, the leading terms in the equations corresponding to (179) can be obtained by a similar procedure. The result for $\kappa_1$ is given by (176) with $l=1$, i.e., by (147). The result for $\kappa_2 = \langle y^2(t)\rangle - \langle y(t)\rangle^2$ is given by (177). The results for $\kappa_3$ and $\kappa_4$ require much more work and the use of (178). In order to save space in the following list, $W_x(f)df$ and $G_3(f_1, f_2, f_3)$ are written as simply $(W)$ and (1, 2, 3), etc.

$$\kappa_1 = \frac{1}{2}\int_{-\infty}^{\infty} df_1 W_x(f_1)G_2(f_1, -f_1) + \frac{1}{8}\int_{-\infty}^{\infty} df_1 \int_{-\infty}^{\infty} df_2 W_x(f_1)W_x(f_2)G_4(f_1, f_2, -f_1, -f_2) + \cdots$$

$$= \frac{1}{2}\int (W)(1, -1) + \frac{1}{8}\iint (WW)(1, 2, -1, -2) + \cdots$$

$$\kappa_2 = \int (W)(1)(-1) + \iint (WW)\left[(1)(-1, 2, -2) + \frac{1}{2}(1, 2)(-1, -2)\right] + \iiint (WWW)\left[\frac{1}{4}(1)(-1, 2, -2, 3, -3) + \frac{1}{2}(1,2)(-1, -2, 3, -3) + \frac{1}{4}(1, 2, -2)(-1, 3, -3) + \frac{1}{6}(1, 2, 3)(-1, -2, -3)\right] + \cdots$$

$$\kappa_3 = \iint (WW)3(1)(2)(-1, -2) + \iiint (WWW)\left[\frac{3}{2}(1)(2)(-1, -2, 3, -3) + 3(1)(-1, 2)(-2, 3, -3) + 3(1)(2, 3)(-1, -2, -3) + (1, 2)(-1, 3)(-2, -3)\right] + \cdots$$

$$\begin{aligned}\kappa_4 = &\iiint (WWW)\Big[4(1)(2)(3)(-1, -2, -3) + 12(1)(2)(-1, 3)(-2, -3)\Big] \\ &+ \iiiint (WWWW)\Big[2(1)(2)(3)(-1, -2, -3, 4, -4) \\ &+ \{6(1)(2)(3, 4)(-1, -2, -3, -4) + 12(1)(2)(-1, 3)(-2, -3, 4, -4)\} \\ &+ \{12(1)(-1, 2)(-2, 3)(-3, 4, -4) + 12(1)(2, -3)(3, 4)(-1, -2, -4) + 12(1)(-1, 2)(3, 4)(-2, -3, -4)\} \\ &+ \{6(1)(2)(-1, -2, 3)(-3, 4, -4) + 3(1)(2)(-1, 3, 4)(-2, -3, -4) + 3(1)(2)(-1, 3, -3)(-2, 4, -4)\} \\ &+ 3(1, 2)(-1, 3)(-2, 4)(-3, -4)\Big] + \cdots. \end{aligned} \quad (180)$$

The straightforward derivation of the four-fold integral in $\kappa_4$ could not be carried through because of its complexity. The expression given in (180) is based upon the conjecture that the only terms occurring in $\kappa_l$ are those in $\langle y^l(t)\rangle$ which do not separate into products of integrals. The conjecture is supported by the fact that it agrees with the memoryless case results (179) and with the terms in (180) obtained by the earlier method.

If the conjecture is true, it follows from (176) and (24) that the products of $G$'s in the $\mu$-fold integral in the series for $\kappa_l$ correspond to the $l$-part partitions of $2\mu$. For example, consider the product $(1)(2)(-1, -2)$ in the 2-fold integral in $\kappa_3$. Here $\mu = 2$, $l=3$, and the product corresponds to the 3-part partition $1+1+2$ of $2\mu = 4$. This product appears only in the 2-fold integral in $\kappa_3$. It does not appear in the 2-fold integrals in $\kappa_1$ and $\kappa_2$.

## B. Approximate Probability Density

In this section we review methods of getting information about the probability density $p(y)$ of $y(t)$ from the cumulants $\kappa_n$.

The mean and variance of $y(t)$ are $\kappa_1$ and $\kappa_2$, respectively. The coefficients $\gamma_1$ and $\gamma_2$ of "skewness" and "excess" used by statisticians to compare the skewness and peakedness of $p(y)$ with a normal curve having the same mean and variance are

$$\gamma_1 = \kappa_3/\kappa_2^{3/2} \qquad \gamma_2 = \kappa_4/\kappa_2^2. \quad (181)$$

When the shape of the central portion of $p(y)$ is known approximately from theoretical considerations, it may be possible to use the first four moments (obtained from the first four cumulants) to fit some appropriate curve, for example a Pearson-type curve.

When the central portion of $p(y)$ is known to be almost normal, the deviation from normality is shown by the Edgeworth-type series [23, pp. 221–232]

$$p(y) = \kappa_2^{-\frac{1}{2}}\left\{Z(u) - \left[\frac{1}{6}\gamma_1 Z^{(3)}(u)\right] + \left[\frac{1}{24}\gamma_2 Z^{(4)}(u) + \frac{1}{72}\gamma_1^2 Z^{(6)}(u)\right] + \cdots\right\}. \quad (182)$$

Here $u$ is equal to $(y(t)-\kappa_1)/\kappa_2^{1/2}$ and

$$Z(u) = (2\pi)^{-\frac{1}{2}} \exp(-u^2/2), \qquad Z^{(k)}(u) = (d/du)^k Z(u). \quad (183)$$

The functions $Z^{(k)}(u)$ are tabulated in [18, Table 26.1, pp. 966–973].

When (182) holds, $p(y)$ has its peak at $y = y_0$ where

$$y_0 \approx \kappa_1 - \tfrac{1}{2}\gamma_1\kappa_2^{1/2}$$

$$p(y_0) \approx (2\pi\kappa_2)^{-\frac{1}{2}}\left(1 + \frac{1}{8}\gamma_2 - \frac{1}{12}\gamma_1^2\right). \quad (184)$$

## C. Probability Density of a Two-Term Volterra Series

An expression for the probability density $p(y)$ of the two-term series

$$y(t) = \frac{1}{1!}\int_{-\infty}^{\infty} du_1 g_1(u_1)x(t - u_1) + \frac{1}{2!}\int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2 g_2(u_1, u_2)x(t - u_1)x(t - u_2) \quad (185)$$

when $x(t)$ is Gaussian [with power spectrum $W_x(f)$ and autocorrelation function $R_x(\tau)$], can be obtained by a method which goes back to [24]. The problem is closely related to that of obtaining the distribution of the second term alone, and hence of quadratic forms of normal variates. Problems of this type have been studied by several authors [17], [21], [24]–[26]. Here we state a method for computing

TABLE II

INTEGRAL EQUATIONS TO DETERMINE $\lambda_n$ AND $\xi_n$

$$\lambda F(x) = \int k(x, y) F(y) dy$$

| No. | $F(t)$ or $F(f)$ | $k(t, u)$ or $k(f, f_1)$ | Orthonormalization | $\xi_n$ |
|---|---|---|---|---|
| 1 | $\varphi(t)$ | $\iint dv_1 dv_2 g_2(v_1, v_2) a(t-v_1) a(u-v_2)$ | $\delta_{mn} = \int dt\, \varphi_m(t) \varphi_n(t)$ | $\iint du\, dv\, g_1(u) a(v-u) \varphi_n(v)$ |
| 2 | $\Phi(f)$ | $A(f) A(-f_1) G_2(f, -f_1)$ | $\delta_{mn} = \int df\, \Phi_m(f) \Phi_n(-f)$ | $\int df\, G_1(f) A(f) \Phi_n(-f)$ |
| 3 | $\psi(t)$ | $\int dv\, R_x(t-v) g_2(v, u)$ | $\lambda_n \delta_{mn} = \iint du\, dv\, g_2(u, v) \psi_m(u) \psi_n(v)$ | $\int du\, g_1(u) \psi_n(u)$ |
| 4 | $\Psi(f)$ | $W_x(f) G_2(f, -f_1)$ | $\delta_{mn} = \int df\, \Psi_m(f) \Psi_n(-f) / W_x(f)$ | $\int df\, G_1(f) \Psi_n(-f)$ |
| 5 | $\chi(t)$ | $\int dv\, g_2(t, v) R_x(v-u)$ | $\lambda_n^2 \delta_{mn} = \iint du\, dv\, R_x(u-v) \chi_m(u) \chi_n(v)$ | $\iint du\, dv\, g_1(u) R_x(u-v) \chi_n(v) / \lambda_n$ |
| 6 | $X(f)$ | $W_x(f_1) G_2(f, -f_1)$ | $\lambda_n^2 \delta_{mn} = \int df\, W_x(f) X_m(f) X_n(-f)$ | $\int df\, G_1(f) W_x(f) X_n(-f) / \lambda_n$ |
| | $\psi_n(t) = \int du\, a(u-t) \varphi_n(u)$ | $\Psi_n(f) = A(-f) \Phi_n(f)$ | $R_x(t-u) = \int dv\, a(t-v) a(u-v)$ | $W_x(f) = \lvert A(f) \rvert^2$ |
| | $\chi_n(t) = \int du\, g_2(t, u) \psi_n(u)$ | $X_n(f) = \int df_1\, G_2(f, -f_1) \Psi_n(f_1)$ | $R_x(t-u) = \Sigma_n \psi_n(t) \psi_n(u)$ | $W_x(f) \delta(f-f_1) = \Sigma_n \Psi_n(f) \Psi_n(-f_1)$ |
| | $\lambda_n \psi_n(t) = \int du\, R_x(t-u) \chi_n(u)$ | $\lambda_n \Psi_n(f) = W_x(f) X_n(f)$ | $g_2(u, v) = \Sigma_n \lambda_n^{-1} \chi_n(u) \chi_n(v)$ | $G_2(f_1, f_2) = \Sigma_n \lambda_n^{-1} X_n(f_1) X_n(f_2)$ |
| | $\lambda_n \delta_{mn} = \int dt\, \psi_m(t) \chi_n(t)$ | $\lambda_n \delta_{mn} = \int df\, \Psi_m(f) X_n(-f)$ | $\iint dv\, dw\, g_2(v, w) a(t-v) a(u-w) = \Sigma_n \lambda_n \varphi_n(t) \varphi_n(u)$ | $A(f_1) A(f_2) G_2(f_1, f_2) = \Sigma_n \lambda_n \Phi_n(f_1) \Phi_n(f_2)$ |

*Note:* $A(f)$, $\Phi(f)$, $\Psi(f)$, $X(f)$ are Fourier transforms of $a(t)$, $\varphi(t)$, $\psi(t)$, $\chi(t)$; $|A(f)|^2 = W_x(f)$, $A(-f) = A^*(f)$, the phase of $A(f)$ is otherwise arbitrary; $\delta(t-u) = \Sigma_n \varphi_n(t) \varphi_n(u) = \Sigma_n \lambda_n^{-1} \chi_n(t) \psi_n(u)$.

$p(y)$ based upon these studies. All of the integrations extend from $-\infty$ to $\infty$ unless explicitly written otherwise, and $\sum_n$ denotes summation over all eigenstates of an integral equation.

The first problem is to compute a set of eigenvalues $\lambda_n$ and quantities $\xi_n$. When these are known, $p(y)$ is given by

$$p(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-jyz} Q(z) dz$$

$$Q(z) = \frac{1}{\prod_n (1 - j\lambda_n z)^{\frac{1}{2}}} \exp\left[ -\sum_n \frac{\xi_n^2 z^2}{2(1 - j\lambda_n z)} \right] \tag{186}$$

where $\lambda_n$ and $\xi_n$ are real, and $\arg (1 - j\lambda_n z)^{\frac{1}{2}} = 0$ at $z = 0$. Usually the only practical way to evaluate the integral for $p(y)$ is by numerical integration. When all of the $\lambda_n$ are positive, moving the path of integration upward to $\operatorname{Im} z = \infty$ shows that $p(y)$ is 0 for $y$ less than $-\sum_n \xi_n^2/(2\lambda_n)$, provided the series converges.

The parameters $\lambda_n$ and $\xi_n$ are obtained by solving the most convenient [depending on $g_2(u_1, u_2)$, $R_x(\tau)$ and their Fourier transforms $G_2(f_1, f_2)$, $W_x(f)$] one of the six integral equations listed in Table II. These equations are of the form

$$\lambda F(x) = \int k(x, y) F(y) dy. \tag{187}$$

The kernels are listed in the column labeled "$k(t, u)$ or $k(f, f_1)$" and the eigenfunctions in the column labeled "$F(t)$ or $F(f)$." Solving the integral equation gives the $\lambda_n$ and the eigenfunctions. The $\xi_n$ are then obtained by evaluating the corresponding integral listed in the column labeled "$\xi_n$." If integral equations 1 or 2 in Table II are selected for solution, some freedom of choice remains in selecting $\arg A(f)$ since $A(f)$ is restricted only by $|A(f)|^2 = W(f)$ and $\arg A(-f) = -\arg A(f)$, ($a(t)$ is the Fourier transform of $A(f)$).

All of the eigenvalues $\lambda_n$ are real because the kernel shown in row 1 of the table is a symmetric function of $t$ and $u$.

Table II can be constructed by first taking $x(t)$ to be white noise with $\langle x(t+\tau) x(t) \rangle = \delta(\tau)$ and expanding a typical member $x(t-u)$ of the ensemble as $\sum_n c_n(t) \varphi_n(u)$ where $\varphi_n(u)$ are an as-yet-unspecified orthonormal set. As we go from member to member of the ensemble with $t$ fixed, the $c_n(t)$ behave like independent normal random variables with zero mean and unit variance. At this stage $\varphi_n(u)$ is chosen to be the $n$th eigenfunction of an integral equation having the kernel $g_2(t, u)$. Converting the integral equation into one for which $x(t)$ has the general power spectrum $W_x(f)$ brings in $A(f)$ and leads to 1 in Table II. The arbitrariness associated with $\arg A(f)$ can be removed by introducing $\psi_n(t)$. The eigenfunctions $\psi_n(t)$ and $\chi_m(t)$ are related in essentially the same way, as are the $n$th modal column $\psi_n$ and $m$th modal row $\chi_m$ of a matrix product $Rg$ where $R$ and $g$ are square symmetric matrices

$$[(I\lambda_n - Rg)\psi_n = 0,\ \chi_m(I\lambda_m - Rg) = 0,\ I = \text{unit matrix}].$$

Corresponding to the three integral equations for $\varphi(t)$, $\psi(t)$, $\chi(t)$ are three more corresponding to their Fourier transforms $\Phi(f)$, $\Psi(f)$, $X(f)$.

The cumulants for the probability density $p(y)$ are proportional to the coefficients in the power series expansion of the characteristic function $Q(z)$. From (186) for $Q(z)$ and Table II it is found that

$$\kappa_1 = \frac{1}{2} \sum_n \lambda_n = \frac{1}{2} \int df\, W_x(f) G_2(f, -f)$$

$$\kappa_2 = \sum_n \left( \frac{1}{2} \lambda_n^2 + \xi_n^2 \right) = \frac{1}{2} \int df\, W_x(f) G_2'^{(2)}(f, -f) + \int df\, W_x(f) G_1(f) G_1(-f)$$

$$\kappa_m = \sum_n \left[ \frac{(m-1)!}{2} \lambda_n^m + \frac{m!}{2} \xi_n^2 \lambda_n^{m-2} \right] = \frac{(m-1)!}{2} \int df\, W_x(f) G_2'^{(m)}(f, -f) + \frac{m!}{2} \iint df_1 df_2\, W_x(f_1) \cdot W_x(f_2) G_1(f_1) G_1(f_2) G_2'^{(m-2)}(-f_1, -f_2). \tag{188}$$

The function $G_2'^{(m)}(f_1, f_2)$ is defined by

$$G_2'^{(1)}(f_1, f_2) = G_2(f_1, f_2)$$

$$G_2'^{(k)}(f_1, f_2) = \int df\, W_x(f) G_2'^{(k-1)}(f_1, f) G_2(-f, f_2), \qquad k > 1. \tag{189}$$

Step by step application of orthonormal relation 6 in Table II to the series for $G_2(f_1, f_2)$ in the rightmost column gives

$$G'^{(k)}(f_1, f_2) = \sum_n \lambda_n^{(k-2)} X_n(f_1) X_n(f_2) \tag{190}$$

from which the sum of $\lambda_n^k$ used in (188) can be obtained.

The values of $\kappa_1$, $\kappa_2$, $\kappa_3$, $\kappa_4$ given by (188) agree with those obtained from (180) when $G_n(f_1, \cdots, f_n)$ is zero for $n > 2$.

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