A computational method for the determination of attraction regions

W. Fermín Guerrero-Sánchez[1], J.F. Guerrero-Castellanos[2],Vladimir V. Alexandrov[1]
[1]Facultad de Ciencias Físico-Matemáticas,[2]Facultad de Ciencias de la Electrónica,
Benemérita Universidad Autónoma de Puebla (BUAP)
Ciudad universitaria, 72570, Puebla, México
willi@fcfm.buap.mx,fguerrero@ece.buap.mx,valex@fcfm.buap.mx

Abstract— **The region of attraction of nonlinear dynamical system can be considered using an analytical R-function that can be written like an infinite series where each term of the series has the homogeneous form of degree $n \geq 2$ this function allows to determine and to come near to the region of attraction of a nonlinear system around the point of equilibrium located in the origin. The analytical function and the sequence of this Taylor polynomials are constructed by a recurrence formula using the coefficients of the power series expansion of f at 0. [10]. This paper describes a novel computational method using the Software MATHEMATICA for obtaining a solution to this problem, which was proposed by the Russian mathematician, V. I. Zubov. In order to evaluate the method, two examples are treated in which the exact attraction region is found in analytic closed form. Since the construction procedure requires the solution of a linear partial differential equation, there are many cases for which an exact analytic solution is not possible. In some of these cases, however, it is possible to construct an approximate series solution which is always at least as good approximation of the usual quadratic form of Lyapunov functions. The "trajectory reversing method"is presented as a powerful numerical technique for low order systems. Then an analytical procedure based on the same topological approach is developed, and a comparison is made with the classical Zubov method.**

Key words : Zubov method, Trajectory reversing method, Lyapunov function, Domain of attraction.

I. Introduction

Let be the following system of differential equations:

$$\dot{x} = f(x) \tag{1}$$

where $f : R^n \longrightarrow R^n$ is a function of class C^1 on R^n with $f(0) = 0$. (i.e $x = 0$ is a steady state of (1). If the steady state $x = 0$ is asymptotically stable [6], then the set $D_a(0)$ of all initial states x^0 for which the solution $x(t; 0, x^0)$ of the initial value problem

$$\dot{x} = f(x), \quad x(0) = x^0 \tag{2}$$

tends to 0 as $t \longrightarrow \infty$, is open and connected and it is called the domain of attraction(domain of asymptotic stability)of 0. The results of Barbashin [1], Barbashin-Krasovski [3] and of Zubov [14]- [15], have probably been the first results concerning the exact determination of $D_a(0)$. The problem of determining stability regions (region of attraction) of nonlinear dynamical systems is of fundamental importance for many disciplines in ingineering and the sciences

The numerous methods proposed in the literature for estimating the stability region can be roughly divided into two classes [5] : those using Lyapunov functions, and all others.
The estimated of the region of attraction based on methods that use functions of Lyapunov usually it determines a subset of the region of true attraction. Recently, methods using computer generated Lyapunov functions [9], [11] have been proposed. Another method, belonging to the Lyapunov function approach, is the Zubov method [4] Theoretically, this method provides the true stability region via the solution of a partial differential equation. Recent advance includes the maximal Lyapunov function [13].
Zubov´s method offers a technique for computing the entire stability region via the optimal Lyapunov function. However, constructing this optimal Lyapunov function entails solving a set of nonlinear partial differential equations which are difficult, if not imposible, to solve. Because of this dificulty, several techniques have been proposed for approximating the solution of the p.d.e. but they have not proven very successful. Another method, called the trajectory-reversing method, was proposed [11],[7] in which the estimation of the stability region is synthesized from a number of system trayectories obtained by integrating the system equation. In this article examples will appear in which methods were applied both, the algebraic and numerical developments were realised with Mathematica 6.0.

II. Preliminaries

In this section some theorems and definitions appear [8]-[12]-[2]-[5] that they helped us to consider regions of attraction of nonlinear a dynamic system

Theorem 1: **Attractor Theorem**: Let $V(x)$ be a continuously differentiable function and let D_a denote the region where $V(x) < c$. Assume that within D_a

$$V(x) > 0 \quad for \quad x \neq \hat{x} \tag{3}$$

$$\dot{V}(x) \leq 0 \tag{4}$$

Corollary 1: If 4 is replaced by

$$\dot{V}(x) < 0 \quad for \quad all \quad x \neq \hat{x} \in \quad D_a \tag{5}$$

then the equilibrium at $\hat{x}$ is asymptotically stable and every solution $x(t)$ in D_a approaches $\hat{x}$ as $t \rightarrow \infty$.

Theorem 2: **Zubov´s Theorem:** For a region D_a with $x = 0$ interior to D_a to be the domain of attraction to the origin, it is necessary and sufficient that there exist two functions $V(x)$ and $\psi(x)$ such that:

(i) $\psi(x)$ is continuous for all x and $V(x)$ is continuous on D_a;
(ii) $V(0) = \psi(0) = 0$;
(iii) $\psi(x) > 0$ for all nonzero x
(iv)$0 < V(x) < 1$ for all nonzero $x \in D_a$
(v) If $x^* \in \partial D$ (boundary of D_a), then $lim_{x \longrightarrow x^*} V(x) = 1$
If $\|x\| \longrightarrow \infty, x \in D_a$, then $lim_{\|x\| \longrightarrow \infty} V(x) = 1$;
(vi) $V(x)$ satisfies the partial differential equation

$$\dot{V} = [\nabla V(x)]^T f(x) = -\psi(x)(1 - V)\sqrt{1 + f^T f} \tag{6}$$

Unless otherwise specified, we assume that the system $\dot{x} = f(x)$, $f(0) = 0$ does not have a finite escape time [12]. For this case (6) can be replaced with

$$[\nabla V(x)]^T f(x) = -\psi(x)(1 - V) \tag{7}$$

to determine the domain of attraction for the origin, we choose a positive definitive continuous function $\psi(x)$ and solve the partial differential equation (7) for $V(x)$, subject to the boundary condition $V(0) = 0$. The domain of attraction is the region defined by $0 \leq V(x) < 1$.
The function $\psi(x)$ can be chosen as needed, to facilite solving (7). However, it is sufficient to choose $\psi(x)$ as quadratic form. The requirement that $\Psi(x)$ be positive definite can be related to $\psi(x) \geq 0$, provided $\psi[x(t)]$is not identically zero along a trajectory in the region $0 \leq V(x) < 1$

Theorem 3: The function $V(x, y)$, solution to $\frac{\partial V}{\partial x} f_1(x, y) + \frac{\partial V}{\partial y} f_2(x, y) = -\psi(x, y)(1 - V)$ is a Lyapunov function establishing the asymptotic stability of the unperturbed motion $x = y = 0$ of system $\dot{x} = f_1(x, y)$, $\dot{y} = f_2(x, y)$
If a substitution is made where $0 \leq V < 1$

$$V^* = -ln(1 - V) \tag{8}$$

another useful partial differential equation is a obtained

$$\frac{\partial V^*}{\partial x} f_1(x, y) + \frac{\partial V^*}{\partial y} f_2(x, y) = -\psi(x, y) \tag{9}$$

Statements and equations similar to those for V can be made also for V^*

Definition 1: Let D_a designate the set of the initial values (x^0, y^0) stability of the unperturbed motion $x = y = 0$. Thus D_a is the domain of asymptotic stability.

Theorem 4: If $(x, y) \in D_a$, then $0 \leq V(x, y) < 1$.

Definition 2: Let $\lambda \in (0, 1)$ Consider the set of points containing $(0, 0)$ which is determined by the condition $0 \leq V(x, y) < \lambda$. Define $G(\lambda)$ to be this set.

Theorem 5: The limiting value of the function $V(x, y)$ as $(x, y) \longrightarrow (\xi, \eta)$ from the inside of domain D_a is equal to one whatever the point (ξ, η) lying on the boundary of domain D_a.

Theorem 6: The curve $V(x, y) = 1$, if it exists, is an integral curve of system $\dot{x} = f_1(x, y)$, $\dot{y} = f_2(x, y)$

Theorem 7: For a fixed $\psi(x, y)$, the solution to $\dot{x} = f_1(x, y)$, $\dot{y} = f_2(x, y)$ is uniquely determined inside D_a.

Theorem 8: The boundary of domain D_a is a family of curves $V(x, y) = 1$.

Theorem 9: In order for for the unperturbed motion of $\dot{x} = f_1(x, y)$, $\dot{y} = f_2(x, y)$ to be asymptotically stable in the whole, it is necessary and sufficient that $V(x, y) < 1$ for all (x, y).
the demonstrations of the theorems can be seen in [8]

II-A. Zubov´s Recursive Procedure [12]

If is often not possible to find a closed-form solution to Zubov´s partial differential equation. However, for a very large class of systems, there is a recursive procedure for constructing the solution. Consider the case where the right-hand sides of the differential equations

$$\dot{x} = f(x), \quad f(0) = 0 \tag{10}$$

possess continuous partial derivatives of all orders. Then $f(x)$ can be expanded in a Taylor series about $x = 0$, yielding

$$\dot{x} = Ax + g(x) \tag{11}$$

where

$$A = \frac{\partial f(0)}{\partial x} \tag{12}$$

is the matrix for the linearized equations of motion and $g(x)$ is a power series

$$g(x) = g_2(x) + g_3(x) + \cdots, \tag{13}$$

where g_m consists of all terms that are homogeneous of degree $m \geq 2$, that is,

$$g_m = \sum_{\sum n_j = m} c_{n_1 \ldots n_{n_x}} x_1^{n_1} x_2^{n_2} \cdots x_{n_x}^{n_{n_x}} \tag{14}$$

Choosing $\psi(x)$ as a quadratic form, the solution to Zubov´s partial differential equation

$$\frac{\partial V}{\partial x}[Ax + g(x)] = -\psi(x)[1 - V] \tag{15}$$

with boundary condition $V(0) = 0$ can then be written as an infinite series

$$V(x) = V_2(x) + V_3(x) + \cdots, \tag{16}$$

where each function $V_m(x)$ in the series is a homogeneous form of degree m

$$V_m = \sum_{\sum n_j = m} b_{n_1 \ldots n_{n_x}} x_1^{n_1} x_2^{n_2} \cdots x_{n_x}^{n_{n_x}} \tag{17}$$

The coefficients of each $V_m(x)$ it can be calculated replacing (13), (16) in the equation(15) obtaining:

$$[\sum_{n=2} \frac{\partial V_n}{\partial x}] = [Ax + \sum_{n=2} g_n(x) + \cdots] = -\psi(x)[1 - (\sum_{n=2} V_n)] \tag{18}$$

For a quadratic $\psi(x)$, expanding both sides (18) and equating terms of the same degree produces the recursive relations in order to determine $V(x)$maximal function of Lyapunov

$$\frac{\partial V_2}{\partial x}Ax = -\psi \quad (19)$$
$$\frac{\partial V_3}{\partial x}Ax = \frac{\partial V_2}{\partial x}g_2 \quad (20)$$
$$\frac{\partial V_4}{\partial x}Ax = \psi V_2 g_3 - \frac{\partial V_3}{\partial x}g_2 \quad (21)$$
$$\vdots \quad (22)$$
$$\frac{\partial V_m}{\partial x}Ax = \psi V_{m-2} - \sum_{j=2}^{m-1}\frac{\partial V_{2j}}{\partial x}g_{m+1-j} \quad (23)$$

Replacing V_2 in the equation (19) and equaling coefficients of the corresponding terms with the left and right side a system of equations takes place in V_2. After determining $V_2(x)$ the second equation (20) it can to be used to determine the coefficients of V_3. This process it is repeated until determining each element of the series of $V(x)$, to determine the coefficients of V_m a is reduced to solve a system of $m+1$ linear equations east process it easily solves with the use of a computer and an appropriate software. If the series that represents a $V(x)$ it can be represented in closed form, then the attraction domain corresponds to the region defined by $0 \leq V(x) \leq 1$. Of another way it represents the domain of great attraction but is obtained taking several terms from the series of Lyapunov $V(x)$.

II-B. *Optimization Procedure*

The optimization process [12] is the following. For an equilibrium at the origin, choose a positive definitive Lyapunov function $V(x)$ for which $\dot{V}(x)$ is negative semidefinite(preferably negative definite) at least in some neighborhood of the origin (excluding origin). Then solve the following optimization problem:

$$min \quad V(x) \quad subjectto \quad a \quad \dot{V}(x) \leq 0 \quad (24)$$

If this problem has a solution $x^* \neq 0$, and if the region $V(x) < V^* \equiv V(x^*)$ is bounded, with $\dot{V}(x) \leq 0$ in the region and not identically zero except at the origin, then the region $V(x) < V^*$ is the largest estimate for the domain of attraction, based on the specified Lyapunov function $V(x)$.

Candidate solutions for the optimization problem (24) can be found by using the first-order necessary conditions for optimality. Assuming that $\frac{\partial V(x^*)}{\partial x} \neq 0$ the conditions reduce to

$$\dot{V}(x^*) = 0 \quad (25)$$
$$\frac{\partial V(x^*)}{\partial x} = \gamma \frac{\partial \dot{V}(x^*)}{\partial x} \quad (26)$$

where $\gamma > 0$ These results imply that the surfaces $V(x) = V^*$ and $\dot{V}(x) = 0$ are tangent at x^*

II-C. *The trayectory Reversing Method*

Let the evolution of an autonomous nonlinear system be described by the equation

$$\dot{x} = f(x) \quad (27)$$

where $x(t) \in R^n$, $f : R^n \longrightarrow R^n$ and satisfies the well-known sufficient conditions for the existence and the uniquences of each solution $x(t, x^0)$ from initial condition $x(0) = x^0$. It is also assumed that the origin es an equilibrium point

$$f(0) = 0 \quad (28)$$

and that it is isolated and asymptotically stable. System may admit other equilibrium points satisfying

$$f(x) = 0 \quad (29)$$

The region of asymptotic stability of the origin is defined as the set of all points x_0 such that

$$lim_{t \longrightarrow \infty} x(t, x^0) = 0 \quad (30)$$

and will be denoted by Ω (simply connected), with boundary $\partial\Omega$, It is known, that Ω is an open invariant set and that $\partial\Omega$ is formed by whole trajectories of system (27).
Time reversing in (27) (backward integration of (27)) is equivalent to considering the system

$$\dot{x} = -f(x) \quad (31)$$

which is characterized by the same trayectories in state space as (27) but with reversed arrows on them. So, beyond other modifications, the origin becomes unstable and it seems evident that the asymptotic behavior of trayectories starting in the RAS Ω is related to its boundary $\partial\Omega$ and always provides information about it. The following theorem provides sufficient conditions to such an enlargement.

Theorem 10: Given the system (27) with continuous right member, if the origin is asymptotically stable i.e. there exists a positive definite Ltapunov function $V(x)$ such that:
a) $\Omega_0 = \{x : V(x) < k_0\}$ is simply connected with boundary $\partial\Omega$
b) $\dot{V} < 0 \ \forall x \in \{x : V(x) \leq k_0\}, \quad x \neq 0$ Then the RAS may be approximated arbitrarily well by means of a convergent sequence of simply connected domains generated by the backward integration technique, starting from the initial RAS estimate Ω_0,[8]

III. Numerical examples

Example One

In this first example the trajectory reversing method would be used, later the method of Zubov would be used, in this example it would observe that an analytical solution for the equation of Zubov can be found giving like result the region of exact stability around the equilibrium point.

$$\dot{x} = -x + y + x(x^2 + y^2) \quad (32)$$
$$\dot{y} = -x - y + y(x^2 + y^2) \quad (33)$$

The origin is the equilibrium point, the boundary of the region of attraction is determined solving the system of differential equations with a negative time

```
nsol[tval_,a_,b_,t1_] := ({x[t],y[t]}
/.First[NDSolve[{x'[t] ==
x[t] - y[t] - x[t] (x[t]^2 + y[t]^2),
 y'[t] == x[t] + y[t] -
y[t] (x[t]^2 + y[t]^2),x[0]==a,y[0]==b},
{x[t],y[t]},{t,0,-t1}]])
/.t -> tval
```

```
nphase[t1_]:=ParametricPlot[
 Evaluate[Flatten[Table[
 nsol[t,a,b,t1], {a,-.7,.7,.1},
 {b,-.7,.7,.1}],1]],
 {t,0,-t1}, AspectRatio -> Automatic]
 nphase[-1.4]
```

The attraction region is in the Figure 1 :

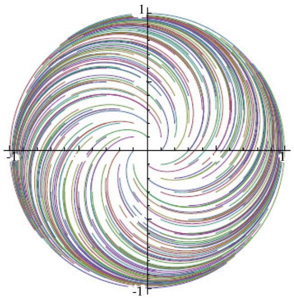

Fig. 1. Region of attraction for the example 1

Now we will apply the method of Zubov, proposing a function of Lyapunov in the form of an infinite series of the form (55) and a function $\psi(x) = 2(x^2 + y^2)$ using the recurrence relation (13)-(17) the first differential equation is solved that has the form

$$\frac{\partial v^{(2)}}{\partial x}(-x + y + x^3 + xy^2) + \frac{\partial v^{(2)}}{\partial y}(-x + y^3 - y + yx^2) = -2(x^2 + y^2)(1 - v^{(2)})$$

with

$$v^{(2)} = d_{20}x^2 + d_{11}xy + d_{02}y^2 \tag{34}$$

the linear system of equations is solved d_{20}, d_{11}, d_{02} are the following values $d_{11} = 0, d_{20} = 1, d_{02} = 1$, then the quadratic term has the form

$$v^{(2)} = x^2 + y^2 \tag{35}$$

that it satisfies the differential equation with Zubov, the function of Lyapunov is $V = v^{(2)} = (x^2 + y^2)$ and therefore boundary of the attraction region this given by the equation

$$x^2 + y^2 = 1 \tag{36}$$

in this case a form closed for the solution of the equation of Zubov´s was obtained and the boundary is the same that the found one by the trayectory reversing method

Example two The system

$$\dot{x_1} = x_2 \tag{37}$$
$$\dot{x_2} = -(1 - x_1^2)x_1 - x_2 \tag{38}$$

The system has three equilibrium points $\hat{x} = 0$ y $\hat{x} = [\pm 1 \quad 0]^T$ The linearized equations of motion indicate that the origin is at least locally asymptotically stable. We use Zubov´s recursive procedure to estimate the domain of attraction for the origin, we apply the trayectory reversing method a (37)

$$\dot{x_1} = -x_2 \tag{39}$$
$$\dot{x_2} = (1 - x_1^2)x_1 + x_2 \tag{40}$$

```
wsol[tval_,a_,b_,t1_]:=
 ({x[t], y[t]} /.
First[NDSolve[{x'[t] == -y[t],
y'[t] == y[t] + (1 - x[t]*x[t])
x[t], x[0] == a,
y[0] == b}, {x[t], y[t]},
{t, 0, -t1}]]) /. t ->
tval
```

```
wphase[t1_] :=
 ParametricPlot[
  Evaluate[ Flatten[Table[
     wsol[t, a, b, t1], {a, -.8, .8, .1},
 {b, -.8, .8, .1}], 1]], {t,0, -t1}
  , AxesLabel -> {"x", "y"}]
  reg1 = wphase[-1.4]
```

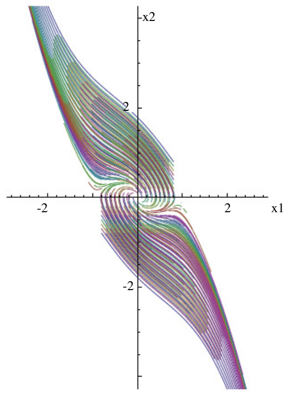

Fig. 2. Domain of attraction by the trajectory reversing method.

The attraction region is in the Figure 2. We apply the method of Zubov to approximate us to the boundary of the stability region. A function of Lyapunov sets out as a series of powers, the first term of the series calculates using the equation (55) it is replaced in recurrence relation (13)-(17) begins with the first differential equation

$$y\frac{\partial v^{(2)}}{\partial x} + (-x - y)\frac{\partial v^{(2)}}{\partial y} = -x^2 \tag{41}$$

three linear algebraic equations are obtained

$$-d_{11} = -1$$
$$-2d_{02} - d_{11} + 2d_{20} = 0$$
$$-2d_{02} + d_{11} = 0 \quad (42)$$

the solution of the system of linear equations is: $d_{11} = 1, d_{20} = 1, d_{02} = 1/2$ the first term of the series takes the form

$$v^{(2)} = x^2 + xy + \frac{y^2}{2} \quad (43)$$

the following step is to apply the optimization criterion

$$\dot{v} = \frac{\partial v^{(2)}}{\partial t} = 0 \quad (44)$$
$$\frac{\partial v^{(2)}}{\partial x} - \gamma\frac{\partial \dot{v}}{\partial y} = 0 \quad (45)$$
$$\frac{\partial v^{(2)}}{\partial y} - \gamma\frac{\partial \dot{v}}{\partial y} = 0 \quad (46)$$

the method of Newton-Raphson is applied and are the following roots $x = 1, y = 0$, the function is evaluated $\hat{V} = v^{(2)}$ y the following equation is obtained $x^2 + xy + \frac{y^2}{2} = 1$ graphic and the first region of attraction is obtained Figure 3,

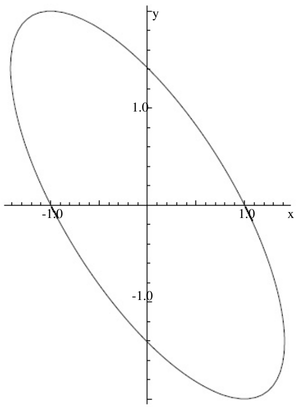

Fig. 3. First estimated of the attraction region .

the following step is to determine the second term of the series $v^{(3)} = d_{30}x^3 + d_{21}x^2y + d_{12}xy^2 + d_{03}y^3$, of recurrence relation (13)-(17) the second differential equation is taken

$$y\frac{v^{(3)}}{\partial x} + (-x - y)\frac{\partial v^{(3)}}{\partial y} = 0 \quad (47)$$

when solving the system of linear algebraic equations is that: $d_{30} = d_{21} = +d_{12} = +d_{03} = 0$ therefore $v^{(3)} = 0$, it can to verify that $v^n = 0$ for n odd. In order to calculate the third term $v^{(4)} = d_{40}x^4 + d_{31}x^3y + d_{22}x^2y^2 + d_{13}x^1y_3 + d_{04}y^4$ and the order terms superior is used the following relation of recurrence

$$\frac{\partial v^{^{(m)}}}{\partial x} + \frac{\partial v^{^{(m)}}}{\partial y}(x - y) = \psi v^{(m-2)} - x^3\frac{\partial v^{m-2}}{\partial y}; \quad m \geq 4 \quad (48)$$

when replacing $v^{(4)}$ they are obtained 5 equations linear they are solved and the values are obtained to determine $v^{(4)}$ the function of Lyapunov with two terms is obtained $V = v^{(2)} + v^{(4)}$ the optimization criterion is applied, finally the second region of attraction is obtained Figure 4, which corresponds to the intersection of two closed curves

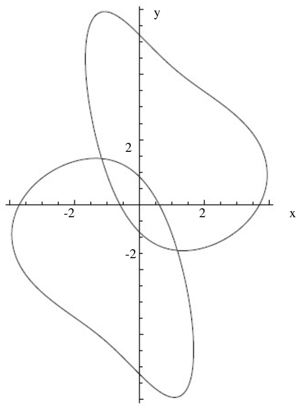

Fig. 4. The intersection is the estimate region of attraction.

The process is repeated until obtaining but the great region of attraction.
The second estimate region of attraction, this given by the function of Lyapunov

$$x^2 - \frac{x^4}{14} + xy + \frac{y^2}{2} - \frac{1}{7}x^2y^2 - \frac{1}{14}xy^3 - \frac{y^4}{56} = ,3542 \quad (49)$$

The third region of attraction to see Figure 5 this given by the function of following Lyapunov:

$$x^2 - \frac{x^4}{14} + \frac{31x^6}{378} + xy + \frac{1}{4}x^5y + \frac{y^2}{2} - \frac{1}{7}x^2y^2 + \frac{17}{252}x^4y^2 - \frac{1}{4}xy^3 + \frac{19}{378}x^3y^3 - \frac{y^4}{56} + \frac{5}{168}x^2y^4 + \frac{5}{504}xy^5 + \frac{5}{3024}y^6 = ,9849 \quad (50)$$

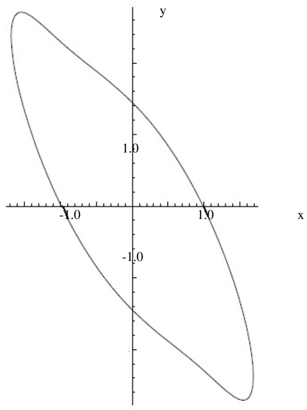

Fig. 5. Third estimate region of attraction.

As it can be observed of this process of iteration between but terms have the series of the function of Lyapunov but close we will be of the exact region, in Figure 6 are the three calculated regions of attraction, if we continued with the process the series of regions it converges to the region of attraction of nonlinear the dynamic system.

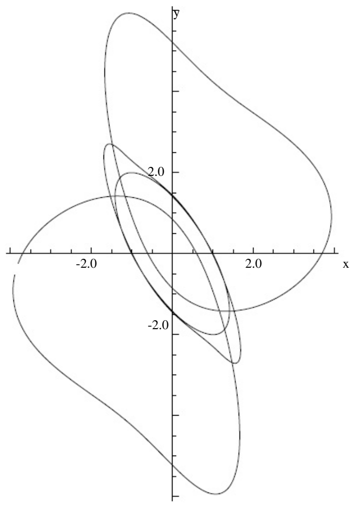

Fig. 6. Superposition of the three estimate regions of attraction.

IV. Conclusions

In this work one is like applying the method of Zubov´s in order to estimate the region of exact attraction, this is obtained in those nonlinear systems to where the function of Lyapunov is written like an infinite series, the calculation of the elements of series and graphic the corresponding ones to the considered regions can be realised of a simple way with a computer and an appropriate software, in this article mathematica 6.0 use, it allows to realise symbolic and numerical calculation.
The method of the reversible trajectories is implemented of an easy way in this software in order to show to the region of attraction without the aid of a function of Lyapunov and power to compare with the method proposed by Zubov.

V. ACKNOWLEDGMENT

This work was financed by the projet of Red temática de colaboración académica 2009 “Problemas directos e inversos en biología e ingeniería” of the academic corps, BUAP-CA-33. The authors want to address a special thanks to the academic corps “Sistemas de Potencia para Tracción, Calidad y Generación de Energía” of the Facultad de Ciencias de la Electrónica, BUAP, for its support and for the useful discussions.

VI. Appendix

In this section one is as to in series write of equivalent form the development of Taylor of an analytical function based on homogeneous polynomials

Theorem 11: **Taylor´s theorem** Let B be a ball in R^n centered at a point x_0, and f be a real-valued function defined on the closure $\bar{B}$ having $n+1$ continuous partial derivatives at every point. Taylor's theorem asserts that for any $x \in B$.

$$\begin{aligned} f(x) = f(x_0) + \frac{1}{1!}\sum_{j_1}^{n} \frac{\partial f}{\partial x_{j_1}}(x_0)h_{j_1} + \\ \frac{1}{2!}\sum_{j_1,j_2}^{n} \frac{\partial^2 f}{\partial x_{j_1}\partial x_{j_2}}(x_0)h_{j_1}h_{j_2} + \cdots \\ + \frac{1}{k!}\sum_{j_1,j_2\ldots h_k}^{n} \frac{\partial^k f}{\partial x_{j_1}\partial x_{j_2}}(x_0)h_{j_1}h_{j_2} + \ldots h_{j_k} \\ + \frac{1}{(k+1)!}\sum_{j_1,j_2\ldots h_{k+1}}^{n} \frac{\partial^{k+1} f}{\partial x_{j_1}\partial x_{j_2}}(x_0 + \\ \theta h)h_{j_1}h_{j_2} + \ldots h_{j_{k+1}} \end{aligned} \quad (51)$$

where $h = x - x_0 = (h_2, h_1, \cdots, h_n)$ and $\theta \in (0,1)$ I number that it depends on h.
Taylor's theorem for 2-dimension or 3-dimension
It is f a class function C^4 in the open disc $B(\mathbf{a}, r) \subset R^2$. For each $h \in R^2$ with $\|h\| < r$ where $\theta \in (0,1)$ such that

$$\begin{aligned} f(a+h) = f(a) + \frac{\partial f}{\partial x}(a)h_1 + \frac{\partial f}{\partial y}(a)h_2 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(a)h_1^2 + \\ \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(a)h_1h_2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(a)h_2^2 + \\ \frac{1}{6}\frac{\partial^3 f}{\partial x^3}(a)h_1^3 + \frac{1}{2}\frac{\partial^3 f}{\partial x^2 \partial y}(a)h_1^2h_2 + \\ \frac{1}{2}\frac{\partial^3 f}{\partial \partial y^2}(a)h_1h_2^2 + \\ \frac{1}{6}\frac{\partial^3 f}{\partial \partial y^3}(a)h_2^3 + \frac{1}{4!}f^{(4)}(x_0 + \theta h)(h) \end{aligned} \quad (52)$$

This expression can be written of a compact way as it follows:

$$\begin{aligned} f(a+h) = f(a) + d_{10}h_1^1h_2^0 + d_{01}h_1^0h_2^1 + \\ d_{20}h_1^2h_2^0 + d_{11}h_1^1h_2^1 + d_{02}h_1^0h_2^2 \\ + d_{30}h_1^3h_2^0 + d_{21}h_1^2h_2^1 + d_{12}h_1^3h_2^1 + \\ d_{03}h_1^0h_2^3 + \frac{1}{4!}f^{(4)}(x_0 + \theta h)(h) \end{aligned} \quad (53)$$

The first terms are homogeneous terms of first degree, the three following they are of degrees 2 and next the 4 terms are of homogeneous degree of degree 3. We can return to write development of Taylor in another simplified form but.

$$\begin{aligned} f(a+h) = f(a) + d_{10}h_1^1h_2^0 + d_{01}h_1^0h_2^1 + \\ v^{(2)} + v^{(3)} + \frac{1}{4!}f^{(4)}(x_0 + \theta h)(h) \end{aligned} \quad (54)$$

where the homogeneous terms can be written like:

$$v^n = \sum_{j=2}^{n}\sum_{k=0}^{j} d_{(j-k)k}x_1^{j-k}x_2^k \quad (55)$$

this expression can be developed in Mathematica like:

```
Do[Print[v_{j}=
Sum[d_{j-k}_{k}h_{1}^{j-k}h_{2}^{k},
{k,0,j}]],{j,2,6}]
```

Taylor series for the case in that the dimension is 3. It is f a class function C^4 in open disc $B(\mathbf{a}, r) \subset R^3$. For each $h \in R^3$ with $\|h\| < r$, $\theta \in (0,1)$ such that

$$\begin{aligned} f(a+h) = f(a) + \frac{\partial f}{\partial x}(a)h_1 + \frac{\partial f}{\partial y}(a)h_2 + \frac{\partial f}{\partial z}(a)h_3 + \\ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(a)h_1^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(a)h_2^2 + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}(a)h_2^2 + \\ \frac{\partial^2 f}{\partial x \partial y}(a)h_1h_2 + \frac{\partial^2 f}{\partial x \partial z}(a)h_1h_2 + \frac{\partial^2 f}{\partial y \partial z}(a)h_2h_3 + \\ \frac{1}{6}\frac{\partial^3 f}{\partial x^3}(a)h_1^3 + \frac{1}{6}\frac{\partial^3 f}{\partial y^3}(a)h_2^3 + \frac{1}{6}\frac{\partial^3 f}{\partial z^3}(a)h_2^3 + \\ \frac{1}{2}\frac{\partial^3 f}{\partial x^2 \partial y}(a)h_1^2h_2 + \frac{1}{2}\frac{\partial^3 f}{\partial x^2 \partial z}(a)h_1^2h_3 + \frac{1}{2}\frac{\partial^3 f}{\partial x \partial y^2}(a)h_1h_2^2 + \\ \frac{1}{2}\frac{\partial^3 f}{\partial x \partial z^2}(a)h_1h_3^2 + \frac{1}{2}\frac{\partial^3 f}{\partial y^2 \partial z}(a)h_2^2h_3 + \frac{1}{2}\frac{\partial^3 f}{\partial y \partial z^2}(a)h_2h_3^2 \\ + \frac{\partial^3 f}{\partial x \partial y \partial z}(a)h_1h_2h_3 \\ \frac{1}{4!}f^{(4)}(x_0+\theta h)(h) \end{aligned} \tag{56}$$

In the same way we can simplify this expression

$$\begin{aligned} f(a+h) = f(a) + d_{100}h_1^1h_2^0h_3^0 + d_{010}h_1^0h_2^1h_3^0 + d_{001}h_1^0h_0^0h_3^1 \\ + d_{200}h_1^2h_2^0h_3^0 + d_{020}h_1^0h_2^2h_3^0 + d_{002}h_1^0h_2^0h_3^2 \\ + d_{110}h_1^1h_1^1h_3^0 + d_{101}h_1^1h_2^0h_3^1 + d_{011}h_1^0h_2^1h_3^1 \\ + d_{300}h_1^3h_2^0h_3^0 + d_{030}h_1^0h_2^3h_3^0 + d_{003}h_1^0h_2^0h_3^3 \\ + d_{210}h_1^2h_2^1h_3^0 + d_{201}h_1^2h_2^0h_3^1 + d_{120}h_1^1h_2^2h_3^0 \\ + d_{102}h_1^1h_2^0h_3^2 + d_{021}h_1^0h_2^2h_3^1 + d_{012}h_1^0h_2^1h_3^2 \\ + d_{111}h_1^1h_2^1h_3^1 + \\ \frac{1}{4!}f^{(4)}(x_0+\theta h)(h) \end{aligned} \tag{57}$$

$f(x)$ it can be expanded in a Taylor series for n variables around $x = 0$, it is possible to be written like:

$$f(x) = Ax + g(x) \tag{58}$$

where $A = \frac{\partial f(0)}{\partial x}$ is the matrix for the linearized equations of motion and $g(x)$ is a power series

$$g(x) = g_2(x) + g_3(x) + \cdots \tag{59}$$

where g_m consists of all terms that are homogeneous of degree $m \geq 2$, that is

$$g_m = \sum_{\sum n_j = m} c_{n_1 n_2 \ldots n_{n_x}} x_1^{n_1} x_2^{n_2} \ldots x_{n_x}^{n_x} \tag{60}$$

References

[1] E. Barbashin. The method of sections in the theory of dynamical systems. *Matem*, (Sb. 29).

[2] Hsiao-Dong Chiang and James S. Trop. Stability region of nonlinear dynamical systems a computer methodology. *IEEE Transaction on Automatic Control*, 24(12):1229–1241, December 1989.

[3] Krasovskii E. Barbashin. On the existence of lyapunov functions in the case of asymptotic stability in the whole. *Prikle, Kat. Mekh. XVII*, pages 345–350, 1954.

[4] W. Hahn. Stability of motion. *New York: Springer-Verlag*.

[5] Morris W. Hirsch Hsiao-Don Chiang and Felix F. Wu. Stability regions of nonlinear autonomous dynamical systems. *IEEE Transaction on Automatic Control*, 33(1), January 1988.

[6] A. M. Lyapunov. Problem general de la stabilite du mouvement (annuals of mathematics studies). *Princeton University Press, Princeton, N.J*, (19), 1949.

[7] R. Genesio M. and A. Vicino. New techniques for constructing asymptotic stability regions for nonlinear systems. *IEEE Trans. Circuits and Syst.*, CAS-31-30(6), June 1984.

[8] S.G. Margolis and W.G. Control engineering applications of v.i. zubov´s construction procedure for lyapunov functions. *IEEE Trans. Circuits and Syst.*, pages 104–113, april 1963.

[9] A.Ñ. Michel N. K. Miller and B. H. Nam. Stability analysis of interconnected systems using computer generated lyapunov function. *IEEE Trans. Circuits and Syst.*, CAS-29:431–440, July 1982.

[10] V. Hahn P. C. Parks. *Stability Theory*.

[11] Michele Tartaglia R. Genesio M. Tartaglia and A. Vicino. On the estimation of asymptotic stability regions: State of art and new proposals. *IEEE Transaction on Automatic Control*, AC-30(8), August 1985.

[12] Walter J. Grantham Thomas L. Vincent. *Nonlinear and Optimal Control Systems*.

[13] A. Vannelli and M. Vidyasagar. Maximal lyapunov functions and domain of attraction for autonomous nonlinear systems. *Automatica*, 21:69–80, Jan. 1985.

[14] V. Zubov. *Stability Theory*.

[15] V. Zubov. *Théorie de la commande*.