Control Engineering Applications of V. I. Zubov's Construction Procedure for Lyapunov Functions*

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Summary—The determination of a stability domain of a control system, the motion of which is described by nonlinear differential equations, is often the object of intensive experimental and theoretical attack. This paper, partly tutorial and partly a presentation of new results, describes a method for obtaining a solution to this problem proposed recently by the Russian mathematician, V. I. Zubov.

The tutorial part outlines the fundamental principles of V. I. Zubov's procedure for constructing Lyapunov functions for nonlinear systems. If the construction problem can be solved, it leads to a Lyapunov function which uniquely defines the exact boundary of the stability region.

For the application of the method, several simple examples are treated in which the exact stability region is found in analytic closed form. Since the construction procedure requires the solution of a linear partial differential equation, there are many cases for which an exact analytic solution is not possible. In some of these cases, however, it is possible to construct an approximate series solution which is always at least as good as the usual quadratic form Lyapunov function.

The series construction procedure has been programmed (in IBM 7070 FORTRAN language) for a broad class of differential equations of the second order. A simple example solved by the digital computer program is described.

INTRODUCTION

IN RECENT YEARS, a great deal of attention has been devoted to the direct method of Lyapunov as a practical tool for studying the stability of nonlinear control systems. From an engineering standpoint, the systems of greatest interest are those which possess the quality of asymptotic stability in some domain of the space of the initial values of the system state variables. The usual procedure in studying the stability of such systems by means of the direct method is to obtain a Lyapunov function which assures the asymptotic stability of the system in the small and then, by modifying the Lyapunov function, to obtain a domain of finite extent in which asymptotic stability is assured. Because of the nonuniqueness of the Lyapunov function, the guaranteed stability domain may or may not be a good approximation to the actual domain of stability of the system. This, coupled with the fact that there had been no general method for constructing a suitable Lyapunov function, has led to doubts about the practicality of the direct method.

The work of a Russian mathematician, V. I. Zubov, which has not received much attention by engineers in this country, to some extent alleviates a great deal of this difficulty [1]. By using the results of this work, a Lyapunov function can always be found for asymptotically stable systems; and in cases in which the linear partial differential equation involved in the procedure can be solved in closed form, the Lyapunov function obtained uniquely defines the exact stability domain of the system. In the cases in which the partial differential equation cannot be solved in closed form, a series solution is possible which will approximate the exact stability domain of the system.

This paper is devoted to an exposition of the main portions of the work done by Zubov, with some additional results obtained by the authors. The problems considered are as follows:

1) The existence of a Lyapunov function.
2) The equations for the boundary of the domain of asymptotic stability of the perturbed motion.
3) The necessary and sufficient conditions for global asymptotic stability.
4) An approximate method of obtaining the domain of asymptotic stability.

This paper is divided into four main parts.

The first part is a restatement of the theoretical development as given by Zubov. Here, not all of the theorems stated are proved and some of the proofs given are heuristic. Except for some omissions and modifications, this section roughly follows the work by Zubov [1].

The second, very short part of this paper gives some specific examples of systems for which the construction problem can be solved in closed form. Most of these examples originated with Zubov or other prominent researchers in this field.

The third part establishes a relationship between control system performance measures and the Lyapunov functions generated by Zubov's construction procedure.

The fourth part of this paper concerns the numerical computation of the series solution used to approximate the stability boundary. The Van der Pol equation is used as an example. The results of the computer computations are shown in a series of curves which give the stability domain as obtained by analog computation and the various approximations as obtained by the digital computer program.

In the Appendix, merely for convenience, some of the important definitions and theorems of the direct method of Lyapunov are stated.

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Zubov's Theoretical Development

Notation

Consider a system of ordinary differential equations of the type

$$\dot{x}_i = \frac{dx_i}{dt} = f_i(x_1, \cdots, x_n)$$

$$[f_i(0, \cdots, 0) = 0, \ (i = 1, \cdots, n)], \quad (1)$$

where the unperturbed motion $x_1 = \cdots = x_n = 0$ is asymptotically stable. If a point $(x_1^0, \cdots, x_n^0)$ belongs to the domain of asymptotic stability, the integral curve or trajectory given by $x_1 = x_1(t), \cdots, x_n = x_n(t)$ for $t > 0$, which passes through this point at $t = 0$ must have the property,

$$x_1^2 + x_2^2 + \cdots + x_n^2 \to 0 \qquad \text{for } t \to \infty. \quad (2)$$

Likewise, if an integral curve which passes through some point $(x_1^0, \cdots, x_n^0)$ at some time, say $t = 0$, has property (2), then the point $(x_1^0, \cdots, x_n^0)$ belongs to the domain of asymptotic stability of the unperturbed motion. In case every point of the space of the state variables $(x_1, \cdots, x_n)$ belongs to the domain of asymptotic stability, then the unperturbed motion is asymptotically stable in the whole, or globally asymptotically stable. Whether the domain of asymptotic stability be finite or infinite, it is denoted by the letter A. The boundary of the domain of asymptotic stability is itself an integral curve of system (1) [2]. If a point P lies on this boundary of the integral curves, then in any arbitrarily small neighborhood of this point lie points which belong to the domain of asymptotic stability and other points which do not belong to this domain.

The right sides of (1) are assumed to be such that

$$f_i(x_1, \cdots, x_n) = \sum_{k=1}^{n} a_{ik}x_k + \sum_{m_1+\cdots+m_n \geqq 2}^{\infty} P_i(m_1, \cdots, m_n) \cdot x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}, \qquad (i = 1, \cdots, n),$$

where the a_{ik} and the $P_i(m_1, \cdots, m_n)$ are real numbers. It is also assumed that this series always converges. Further, assume that the equations of the linear approximation

$$\dot{x}_i = \sum_{k=1}^{n} a_{ik}x_k \qquad (i = 1, \cdots, n), \quad (3)$$

are globally asymptotically stable, *i.e.*, the roots $\lambda_1, \cdots, \lambda_n$ of the characteristic equation

$$|B - \lambda I| = 0 \quad (4)$$

have negative, nonzero real parts. Here B denotes the matrix of the a_{ik} and I is the unit matrix.

By a Lyapunov function is meant a function $v(x_1, \cdots, x_n)$ which is positive definite, *i.e.*, $v(x_1, \cdots, x_n) > 0$ if $(x_1, \cdots, x_n) \neq (0, \cdots, 0)$ and $v(0, \cdots, 0) = 0$ and which has a time derivative which, when taken along the integral curves of system (1), is a negative definite function,

$$w(x_1, \cdots, x_n) = \left.\frac{dv}{dt}\right|_{\text{taken along the integral curves}} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i(x_1, \cdots, x_n), \quad (5)$$

i.e., $w(x_1, \cdots, x_n) < 0$ when $(x_1, \cdots, x_n) \neq (0, \cdots, 0)$, $w(0, \cdots, 0) = 0$. This corresponds to the usual notation but is opposite to that used in the paper by Zubov.

Systems which satisfy all the conditions imposed on them by the above are not wholly general; but, because engineers, when speaking of stable systems, are usually speaking of asymptotically stable systems, they do have wide applicability. From this standpoint, the most serious restriction is the one which requires that the linear approximation be asymptotically stable. This eliminates from any consideration those systems that have zero real part roots of the characteristic equation (4) and hence all systems which are critical cases, although it is possible for critical cases to be asymptotically stable. In effect, this means that before the theory proposed by Zubov is applied, the differential equations of the linear approximation should first be examined to ascertain that these equations indicate asymptotic stability.

The description and limitations above apply to n-dimensional systems. Following Zubov, henceforth only two-dimensional systems are discussed, but with the understanding that there is no theoretical difficulty in extending the various theorems and concepts to n-dimensional systems. Additional discussion on this point is given in the work by Zubov. Note, however, that even though there is no theoretical difficulty, there could be very great computational difficulty for systems of higher dimension.

Several Partial Differential Equations

In two dimensions, system (1) can be written as

$$\dot{x} = f_1(x, y), \qquad \dot{y} = f_2(x, y) \quad (6)$$

where $f_1(x, y)$ and $f_2(x, y)$ are functions which satisfy all the requirements previously discussed.

Zubov's construction procedure concerns the solution to a partial differential equation. This equation is discussed in the present section and its application is shown in succeeding sections. The basic partial differential equation considered is

$$\frac{\partial u}{\partial x} f_1(x, y) + \frac{\partial u}{\partial y} f_2(x, y) = -\theta(x, y)[1 + f_1^2 + f_2^2][1 - u(x, y)] \quad (7)$$

where $\theta(x, y)$ is a positive definite quadratic form, or $\theta(x, y)$ can be assumed to be a positive definite form of power $2m$, $m \geqq 1$. For such choices of $\theta(x, y)$ according to a theorem of Lyapunov, the function $u(x, y)$ can be uniquely determined in the form of a convergent power series going to zero for $x=y=0$,

$$u(x, y) = u_2(x, y) + u_3(x, y) + \cdots \tag{8}$$

where $u_m(x, y)$ is a homogeneous form relative to x and y of the mth power [3].

Instead of (7) and (8), consider a simplified set of equations

$$\frac{\partial v}{\partial x} f_1(x, y) + \frac{\partial v}{\partial y} f_2(x, y) = -\phi(x, y)[1 - v(x, y)]. \tag{9}$$

Under the proper conditions which involve the continuation of solutions to system (6) for $t \in (-\infty, +\infty)$, the function v can be uniquely determined when $\phi(x, y)$ is a quadratic form. However, if these conditions are not satisfied, let $\phi(x, y) = \theta(x, y)[1 + f_1^2 + f_2^2]$; later when the quadratic form $\phi(x, y)$ is manipulated, the quadratic form part of $\phi(x, y)$ is meant.

Similar to the above, $v(x, y)$ can be found in a series form

$$v(x, y) = v_2(x, y) + v_3(x, y) + \cdots \tag{10}$$

where $v_m(x, y)$ is a homogeneous form relative to x and y of the mth power. To determine the $v_m(x, y)$, (10) is substituted into (9), yielding the system of recurrence relations

$$\frac{\partial v_2}{\partial x} f_{11} + \frac{\partial v_2}{\partial y} f_{21} = -\phi_2(x, y)$$

$$f_{11}(x, y) = a_{11}x + a_{12}y, \qquad f_{21}(x, y) = a_{21}x + a_{22}y$$

$$\frac{\partial v_m}{\partial x} f_{11} + \frac{\partial v_m}{\partial y} f_{21} = R_m(x, y) \quad (m = 3, 4, \cdots) \tag{11}$$

where $\phi_2(x, y)$ is a quadratic form of the variables x and y and $R_m(x, y)$ is a function of the mth degree in x, y which is known if each of the $v_2(x, y), v_3(x, y), \cdots, v_{m-1}(x, y)$ have already been determined. Thus, from system (11), the sequence $v_2(x, y), \cdots, v_n(x, y)$ is easily found. From a theorem of Lyapunov, the function $v_2(x, y)$ is a positive definite quadratic form [3].

If the solutions of system (6) are considered, then $x = x(t, x^0, y^0)$, $y = y(t, x^0, y^0)$. Substituting these values of x and y into the function $v(x, y)$, and applying (9) the derivative of v along the integral curves is obtained

$$\frac{dv}{dt} = \dot{v}(x, y) = -\phi(x, y)[1 - v(x, y)]. \tag{12}$$

By separation of variables and integration

$$v(x, y) = 1 - [1 - v(x^0, y^0)] \exp [J(t)] \tag{13}$$

$$J(t) = \int_0^t \phi(x, y) dt. \tag{14}$$

Expressions (13) and (14) will be of some use later.

Without going into the proof, the $v(x, y)$ obtained as a solution to (9) is valid at any point (x, y) of the domain of asymptotic stability of the unperturbed motion of system (6).

If a substitution is made where $0 \leqq v < 1$

$$\mathbf{v} = -\ln (1 - v), \tag{15}$$

another useful partial differential equation is obtained

$$\frac{\partial \mathbf{v}}{\partial x} f_1(x, y) + \frac{\partial \mathbf{v}}{\partial y} f_2(x, y) = -\phi(x, y). \tag{16}$$

Statements and equations similar to those for v can be made also for $\mathbf{v}$. The function $\mathbf{v}$ plays some importance in a connection between Zubov's construction procedure and the calculation of performance indexes for control systems. This will be discussed further in a later section.

Zubov's Principal Theorems

Theorem 1: The function $v(x, y)$, the solution to (9), is a Lyapunov function establishing the asymptotic stability of the unperturbed motion $x=y=0$ of system (6).

Proof: By the discussion in the preceding section, $v(x, y)$ can be found in series form. Because the leading term in this series is a positive definite quadratic form (the proof of this requires only that the roots of the characteristic equation (4) have negative real parts), $v(x, y)$ is positive definite in some neighborhood of the origin. Eq. (9) gives the derivative of $v(x, y)$ along the integral curves of system (6) and since near the origin (9) reduces to $dv/dt \approx -\phi$ (where ϕ is positive definite) obviously dv/dt is negative in a neighborhood of the origin, in fact, dv/dt is negative whenever $0 \leqq v < 1$. Moreover, by Theorem 15 of the Appendix, v is possessed of an infinitely small upper bound. Thus, the function $v(x, y)$ satisfies the three requirements of Theorem 17 of the Appendix and is, therefore, a Lyapunov function establishing the asymptotic stability of the unperturbed motion. QED

Definition 1: Let A designate the set of the initial values (x^0, y^0) which make up the domain of asymptotic stability of the unperturbed motion $x=y=0$. Thus, A is the domain of asymptotic stability.

Theorem 2: If $(x, y) \in A$, then

$$0 \leqq v(x, y) < 1. \tag{17}$$

Proof: Since A is the domain of asymptotic stability, then, for any $(x^0, y^0) \in A$, (2) must hold or, in other words, considering $v(x, y)$ as it varies along a trajectory,

$$v[x(t, x^0, y^0), y(t, x^0, y^0)] = v(t). \tag{18}$$

For $(x^0, y^0) \in A$, $v(t) \rightarrow 0$ as $t \rightarrow \infty$. From (13)

$$v(x^0, y^0) = 1 - [1 - v(x, y)] \exp [-J(t)]. \tag{19}$$

Allowing $t \rightarrow \infty$,

$$v(x^0, y^0) = 1 - \exp [-J(\infty)]. \tag{20}$$

Since the integrand of (14) is greater than zero for all $(x, y) \neq (0, 0)$, then necessarily $J(\infty) \geqq 0$, and the inequality is strict if $(x^0, y^0) \neq (0, 0)$. Therefore, from

(20), $0 \leqq v(x^0, y^0) < 1$, where again the left-hand inequality is strict if $(x^0, y^0) \neq 0$. QED

Remarks: By simple calculations it can easily be seen that Theorem 1 applies to v. The analog of Theorem 2 is stated as a corollary; its proof is simple:

Corollary 2.1: If $(x, y) \in A$, then

$$0 \leqq v(x, y) < \infty. \tag{21}$$

Definition 2: Let $\lambda \in (0, 1)$. Consider the set of points containing (0, 0) which is determined by the condition $0 \leqq v(x, y) < \lambda$. Define $G(\lambda)$ to be this set.

Now, without proof, several important theorems are stated.

Theorem 3: The limiting value of the function $v(x, y)$ as $(x, y) \to (\xi, \eta)$ from the inside of domain A is equal to one whatever the point (ξ, η) lying on the boundary of domain A.

Theorem 4: Whatever the value $\lambda \in (0, 1)$, $G(\lambda)$ is a bounded domain inside domain A.

Theorem 5: The curve $v(x, y) = 1$, if it exists, is an integral curve of system (6).

Theorem 6: For a fixed $\phi(x, y)$, the solution to (9) is uniquely determined inside domain A.

Theorem 7: The boundary of domain A is a family of curves $v(x, y) = 1$.

Theorem 8: In order for the unperturbed motion of system (6) to be asymptotically stable in the whole, it is necessary and sufficient that $v(x, y) < 1$ for all (x, y).

For each of the above theorems there is an analogous theorem which can be stated for v. Now consider another system of differential equations,

$$\frac{dx}{dt} = \frac{\partial v(x, y)}{\partial y}, \qquad \frac{dy}{dt} = -\frac{\partial v(x, y)}{\partial x} \tag{22}$$

where $v(x, y)$ is a solution to (9). Obviously, solutions to (22) are of the form, $\dot{v} = 0$

$$v(x, y) = \lambda, \qquad \lambda = \text{constant}. \tag{23}$$

Hence, $x = y = 0$ is a center for this system of differential equations, and the curve $v(x, y) = \lambda_1$ is enclosed completely within the curve $v(x, y) = \lambda_2$ with no common points if $\lambda_1 < \lambda_2$. This leads to Theorem 9.

Theorem 9: To every system of differential equations (6) can be related an entire class of systems of equations (22), depending on the choice of the function $\phi(x, y)$, such that each closed curve given by (23) has no points in common with another closed curve given by (23) with a different λ, and the boundary of domain A will be the only common integral curve of systems (6) and (22) for $0 < \lambda \leqq 1$.

In particular, this means that if the domain A is bounded by a limit cycle, (23) defines closed curves also for $\lambda > 1$.

As mentioned previously, the theorems above can be generalized to an n-dimensional system of the type in (1). Zubov actually does this in his paper in a separate section, where most of the theorems above are restated. Zubov also considers the problem of the existence of a function $\phi(x_1, \cdots, x_n)$ which corresponds to $\phi(x, y)$ here, and states conditions for which such functions exist in extremely simple form.

In another section, Zubov discusses the possibility of constructing a set of differential equations which has a previously given stability boundary. The possibilities for the application of the information contained in this section are obvious and may eventually turn out to be much more important than the content of some of his other sections.

Thus far the theory has been discussed from the standpoint that $v(x, y)$ could be exactly determined, but in many practical cases this is simply not true. In the paper, Zubov gives practical methods for approximating the $v(x, y)$ required so that the domain of guaranteed stability actually approaches the complete stability domain. However, before beginning this discussion some examples are considered for which the appropriate $v(x, y)$ can be obtained exactly and then a relation between control systems and the Zubov construction procedure is stated.

Examples of Solutions in Closed Form

Example 1: (from Zubov, [1])

$$\begin{aligned} \dot{x} &= -x + y + x(x^2 + y^2), \\ \dot{y} &= -x - y + y(x^2 + y^2). \end{aligned} \tag{24}$$

Assume a function $\phi(x, y) = 2(x^2 + y^2)$,

$$\frac{\partial v}{\partial x}(-x + y + x^3 + xy^2) + \frac{\partial v}{\partial y}(-x + y^3 - y + yx^2) = -2(x^2 + y^2)[1 - v]. \tag{25}$$

One way to solve this partial differential equation is by using the recurrence relations of (11). Assuming $v_2(x, y) = ax^2 + bxy + cy^2$ and substituting into (11), it turns out that $a = c = 1$, $b = 0$. As a check, $v_2(x, y) = x^2 + y^2$ is substituted into (25) to see how close $v_2(x, y)$ is to $v(x, y)$. It turns out that $v_2(x, y) = (x^2 + y^2)$ satisfies (25). Therefore, the required $v(x, y) = (x^2 + y^2)$ and the stability boundary is given by the equation

$$x^2 + y^2 = 1 \tag{26}$$

which is more or less obvious from the original (24).

Example 2: (from Hahn, [4])

$$\dot{x} = -x + 2x^2 y, \qquad \dot{y} = -y. \tag{27}$$

Assume a function

$$\phi(x, y) = 2(x^2 + y^2)$$

and the partial differential equation for v becomes

$$\frac{\partial v}{\partial x}(-x + 2x^2 y) + \frac{\partial v}{\partial y}(-y) = -2(x^2 + y^2). \tag{28}$$

Eq. (28) can be solved by elementary integration techniques to yield

$$\mathrm{v} = + y^2 + \frac{x^2}{(1 - xy)} \cdot$$

By a theorem analogous to Theorem 7 for v, the boundary of the stability domain is given by $\mathrm{v} = \infty$; therefore the stability domain of (27) is given by the equation, $xy = 1$.

Example 3:

$$\dot{x} = -2x + 2y^4, \qquad \dot{y} = -y. \tag{29}$$

Choose $\phi(x, y) = 24(x^2 + y^2)$ and consider the equation

$$\frac{\partial \mathrm{v}}{\partial x}(-2x + 2y^4) + \frac{\partial \mathrm{v}}{\partial y}(-y) = -24(x^2 + y^2). \tag{30}$$

The solution to (30), obtained by elementary integration procedures, is

$$\mathrm{v} = 6x^2 + 12y^2 + 4xy^4 + y^8. \tag{31}$$

Since the stability boundary is at points where $\mathrm{v} = \infty$, the system is globally asymptotically stable, which is obvious from a direct examination of (29).

Relation to Control System Performance Measures

The stability of any control system which can be described by equations of type (1) can be studied by means of the Zubov construction procedure, as well as with many other procedures for studying stability of control systems. However, the relation between the functions v or v obtained in the Zubov construction procedure and the physically significant measures of control system performance are much deeper lying than the construction of the asymptotic stability domain.

It is possible to establish a simple relation between the Zubov functions v and v and the integral measures of error (IME), such as those recently reviewed by Schultz and Rideout [5]. For this discussion the requirements on $\phi(x, y)$, except that it be non-negative, are disregarded. From (20), (15) and (14),

$$\mathrm{v}(x^0, y^0) = \int_0^\infty \phi(x(t), y(t))dt. \tag{32}$$

Since $\phi(x, y)$ is required only to be non-negative and to vanish at the origin, by identifying one of the coordinates, say x, with the error e, ϕ can be chosen as any of the usual positive measures of error, such as e^2, $|e|$, etc. Specifically, if $\phi = e^2$, then

$$\mathrm{v} = \int_0^\infty e^2 dt,$$

which is the integral-squared error performance measure. Of course, this assumes that the system has a region of asymptotic stability. The curves of constant v within the stability domain then give the contours on which the initial positions of the system state variables must lie in order for the system to accumulate a specified error according to some performance index.

This idea can be extended. Since the generalized performance measures discussed, for example in Bellman *et al.* [6], are integrals of non-negative functions of the coordinates, any of these non-negative functions can be considered as corresponding to the $\phi(x, y)$, and from this a Lyapunov function constructed according to the principles roughly outlined here. The choice of ϕ will affect the shape of the constant v or v curves within the region of asymptotic stability, but all admissible functions ϕ will give the same stability domain, since, as stated in Theorem 9, the $v = 1$ contours are independent of the choice of ϕ.

The details of this general connection between performance indexes and the v and v functions need more attention and research. In a recent paper by Kalman and Bertram [7] a Lyapunov function is defined as

$$V(x_1, \cdots, x_n) = \int_0^\infty \rho(x_1(t), \cdots, x_n(t))dt \tag{33}$$

where $\rho(x_1, \cdots, x_n)$ is the error criterion. This equation is of exactly the same form as could be obtained from (13) and (15) of this paper. In other words, the function V defined by Kalman and Bertram corresponds to the function v of this paper. However, v is a Lyapunov function which, if it exists, gives the exact stability boundary of the region of asymptotic stability of the unperturbed motion which is a slight extension of the results in Kalman and Bertram [7]. In addition, the approximations to v permit the evaluation of regions of guaranteed stability.

Approximate Solutions

General Principles

In many cases, (9) and (16) cannot be solved for v and v exactly. The next best thing is to be able to approximate the exact solutions. But since it is the domain of asymptotic stability which is usually of interest, the approximate solution should at the same time insure that the domain indicated is actually a part of the domain of asymptotic stability, and further it should permit the possibility of approaching as closely as desired to the exact stability domain.

In order to approach this problem the solution to (9) is first approximated by a quadratic form denoted by v_2. It is necessary to find a curve $v_2 = c_1$ which is entirely contained in the domain of asymptotic stability of the origin. First a lemma is needed.

Lemma 1: Whatever the integral curve L, lying on the boundary of domain A, it is possible to find a value c_4 such that the $v_2 = c_4$ curve is tangent to the integral curve L at some point (x^0, y^0).

If there is an integral curve lying on the boundary of the stability domain, then part of the boundary must be a finite distance away from the origin. Since the distance between the origin and the curve $v_2 = c$ can be

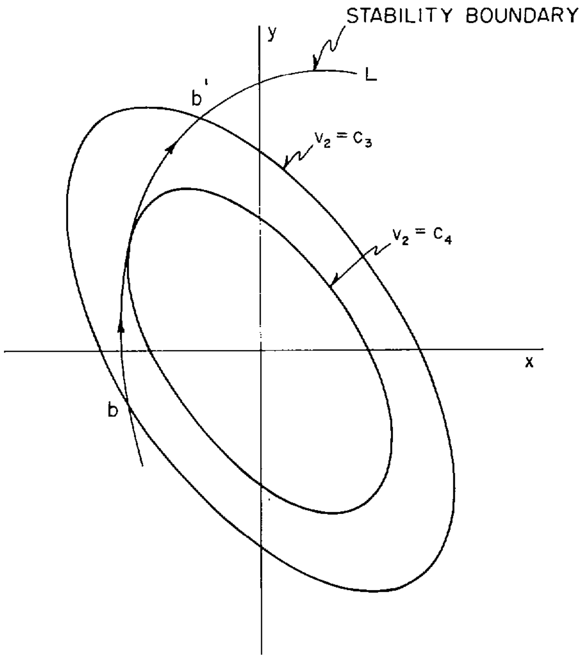

Fig. 1—One of the $v_2=c$ curves is tangent to the stability boundary.

made as large as desired by making c as large as necessary, there must be a value c_3 for which the curve $v_2=c_3$ intersects the stability boundary in at least two points. Denote these points by b and b' and the times of intersection starting from b and traveling to b' along the integral curve L, by $t(b)$ and $t(b')$, respectively. (See Fig. 1.) Then, along this integral curve, the general shape of the variation of $v_2(t)$ must be as in Fig. 2, since $v_2(t)$ is a continuous function of time which assumes the same value, *i.e.*, c_3, at the times $t(b)$ and $t(b')$. Evidently, $\dot{v}_2(t)=0$ at some time, say t_4, when $t(b)<t_4<t(b')$. Designate by c_4 the value of $v_2(t_4)$. Clearly, since $\dot{v}_2=0$ at t_4 along this integral curve, $v_2=c_4$ is a curve which is tangent to L.

Now some of Zubov's important theorems on the approximation of the stability domain can be stated and proved.

Definition 3: Consider the set of all points (x, y) other than $(x, y)=(0, 0)$ for which $\dot{v}_2(x, y)=0$ and designate this set by W_2.

Find the largest and smallest values of v_2 on W_2 and designate these by c_2 and c_1 respectively.

Theorem 10: The curve $v_2(x, y)=c_1$ is wholly contained in A.

Proof: It is necessary only to show that $\dot{v}_2$ is negative definite inside $v_2=c_1$. Certainly $\dot{v}_2$ is negative throughout a sufficiently small neighborhood of the origin, since, near the origin, $\dot{v}_2$ is approximately equal to $-\phi$. If, inside $v_2=c_1$, $\dot{v}_2$ becomes positive, then, of necessity, $\dot{v}_2$ would have to pass through zero somewhere other than at (0, 0) at a point where $v_2<c_1$. But this is not possible since by definition, c_1 is the smallest value of v_2 for which $\dot{v}_2=0$. Hence, inside $v_2=c_1$, v_2 is positive definite and $\dot{v}_2$ is negative definite and further, v_2 is possessed of an infinitely small upper bound which

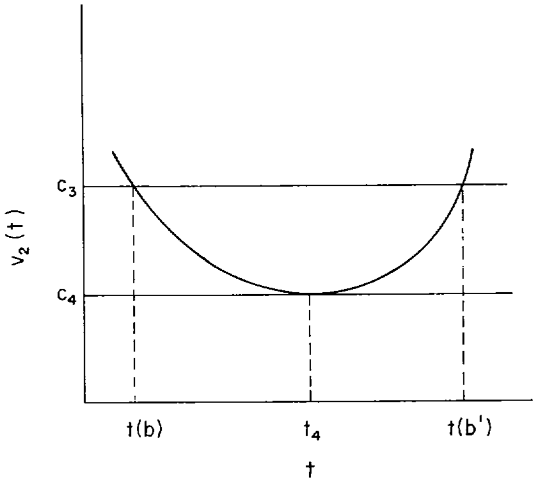

Fig. 2—Variation of $v_2(t)$ along the trajectory which bounds the region of stability.

indicates that the domain inside $v_2=c_1$ is a domain of asymptotic stability. QED

Definition 4: Domain A is bounded if it can be enclosed in a circle $x^2+y^2=R^2$ where R is some sufficiently large but finite number. A is unbounded if no such finite R exists.

Theorem 11: If the stability domain A is bounded, all curves $v_2=c_1$ are bounded regardless of the exact form of the admissible $\phi(x, y)$. If any $v_2=c_1$ curve is unbounded for some admissible $\phi(x, y)$, the stability domain is unbounded.

This theorem follows directly from the fact that each of the $v_2=c_1$ curves is entirely contained in domain A.

Theorem 12: If, for any admissible $\phi(x, y)$, the value of c_2 is finite, the domain of stability A is bounded and its boundary lies in the region

$$c_1 \leqq v_2(x, y) \leqq c_2. \tag{34}$$

Proof: To prove this it is expedient to show that the boundary of the stability domain cannot pass outside of the region given by (34). The proof of Theorem 10 shows that the stability boundary does not pass inside domain $v_2(x, y)<c_1$. An analogous proof will show that no integral curve can pass outside the region $v_2(x, y)\leqq c_2$. From the definition of c_4 in the proof of Lemma 1, $c_1\leqq c_4\leqq c_2$. Thus, it follows that at least one point on the stability boundary, the point where $v_2=c_4$ and $\dot{v}_2=0$, is a point of the region $v_2\leqq c_2$. Designate this point (x^0, y^0). It must now be shown that the stability boundary does not go outside the region $v_2\leqq c_2$. Assume to the contrary that the stability boundary does pass outside this region. Then, as shown in Fig. 3, it must touch the curve $v_2=c_2$. In an ϵ-neighborhood of (x^0, y^0), there are integral curves which are part of the unstable domain and integral curves which are part of the stable domain. The theorem on integral continuity guarantees that, if ϵ is chosen small enough, all the integral curves originating in this neighborhood will remain near the

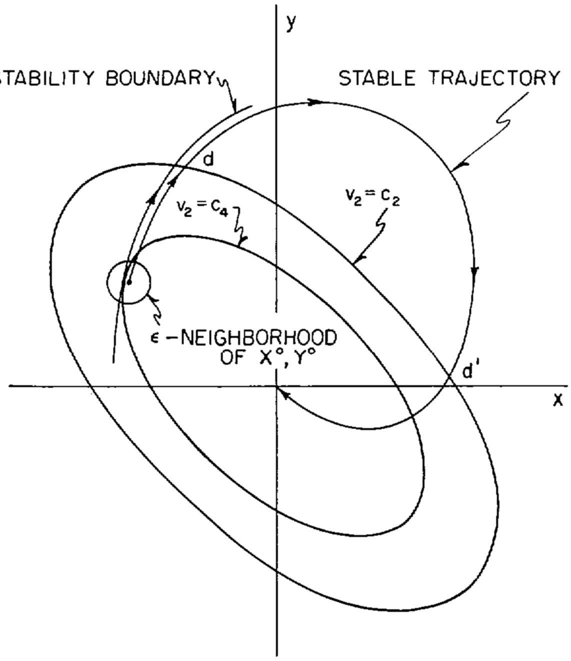

Fig. 3—Assumed behavior of the stability boundary leads to a contradiction.

stability boundary long enough for some of the stable ones to pass from the inside to the outside of the curve $v_2 = c_2$. Since these are stable integral curves, eventually they must pass back into the domain $v_2 < c_2$. Moving along one such integral curve, designate the time at which the curve goes out of domain $v_2 < c_2$ by $t = t(d)$, and the time at which it re-enters this domain by $t = t(d')$. In the interval $t(d) < t < t(d')$, $\dot{v}_2(x, y)$ must go from positive values, through zero, to negative values. Hence, at some point outside $v_2 = c_2$, there is a point for which $\dot{v}_2 = 0$. But this clearly contradicts the conditions by which region (34) was constructed. Therefore, no stable integral curve passes outside of the curve $v_2(x, y) = c_2$. This implies that the boundary of the stability region can at most be tangent to this curve from the inside. Therefore the theorem is proved. QED

The theorems above concerned only a quadratic form $v_2(x, y)$, but it is clear some of them can be extended to cover a series form of the solutions to (9). For this, make the following designations:

$$v^{(n)}(x, y) = v_2(x, y) + v_3(x, y) + \cdots + v_n(x, y) \tag{35}$$

$$w^{(n)}(x, y) = \text{all points } (x, y) \text{ on which } \dot{v}^{(n)}(x, y) = 0 \tag{36}$$

$$c_1^{(n)} = \min \left[v^{(n)}(x, y) \text{ on } w^{(n)}(x, y) \right] \tag{37}$$

$$A^{(n)} = \text{all points } (x, y) \text{ for which } v^{(n)}(x, y) \leq c_1^{(n)}. \tag{38}$$

These yield the following theorems:

Theorem 13: The curve $v^{(n)}(x, y) = c_1^{(n)}$ is wholly contained in A.

Theorem 14: If A is bounded, all curves $v^{(n)}(x, y) = c_1^{(n)}$ are bounded irrespective of the form of admissible $\phi(x, y)$. If any $v^{(n)}(x, y) = c_1^{(n)}$ curve is unbounded for some admissible $\phi(x, y)$, the stability domain is unbounded.

As mentioned previously, one object of the series solution for (9) is to be able to come as close to the stability boundary as desired. One might think that the more terms used in the expansion of v, the closer one could come to the correct expression for the stability boundary, but this does not necessarily follow; that is, it is not necessarily true that

$$A^{(n)} \subset A^{(n+1)}. \tag{39}$$

The numerical computation which follows later is one case in which the relation (39) is not true (for some values of n) as is made clear from the figures. This difficulty is not discussed in the paper by Zubov [1]. One solution which has worked in practice is to use the function $\boldsymbol{v}$ in computations, rather than v. This would appear to be a fruitful area for further research, however.

A Computational Example

In this section, numerical solutions of Zubov's equation are considered. The well-known Van der Pol equation

$$\ddot{x}_1 + \epsilon(1 - x_1^2)\dot{x}_1 + x_1 = 0 \tag{40}$$

is taken as an example. To reduce this to a pair of coupled first-order equations, introduce the variable x_2 defined by

$$x_2 = \dot{x}_1 + \epsilon\left(x_1 - \frac{x_1^3}{3}\right) \tag{41}$$

and thus obtain

$$\dot{x}_1 = x_2 - \epsilon\left(x_1 - \frac{x_1^3}{3}\right) \tag{42}$$

$$\dot{x}_2 = -x_1 \tag{43}$$

which has the form of (6).

As can be seen from (43), the trajectories plot x_1 against its (negative) time-integral. This contrasts to the usual phase plane which plots x_1 vs its time derivative. The principal advantage of this procedure is that the stability boundary is better approximated by a polynomial. The results can be transferred to the phase plane by solving (41) for $\dot{x}_1$, if desired.

Assume a formal solution to (9) in the form

$$v^{(n)} = \sum_{j=2}^{n} \sum_{k=0}^{j} d_{jk} x_1^{j-k} x_2^{k}. \tag{44}$$

The series solution is computed recursively, beginning with the quadratic terms, using recursion relations of the form (11). Then approximate the stability boundary by using Theorem 10. Reviewing this procedure, the series is truncated by including terms up to nth order. The set of points other than $(x_1, x_2) = (0, 0)$ is found for which $\dot{v}^{(n)} = 0$, and this set is designated by $w^{(n)}$. The smallest value of $v^{(n)}$ on $w^{(n)}$ is designated $c_1^{(n)}$ and $v^{(n)} = c_1^{(n)}$ is an approximation to the stability boundary.

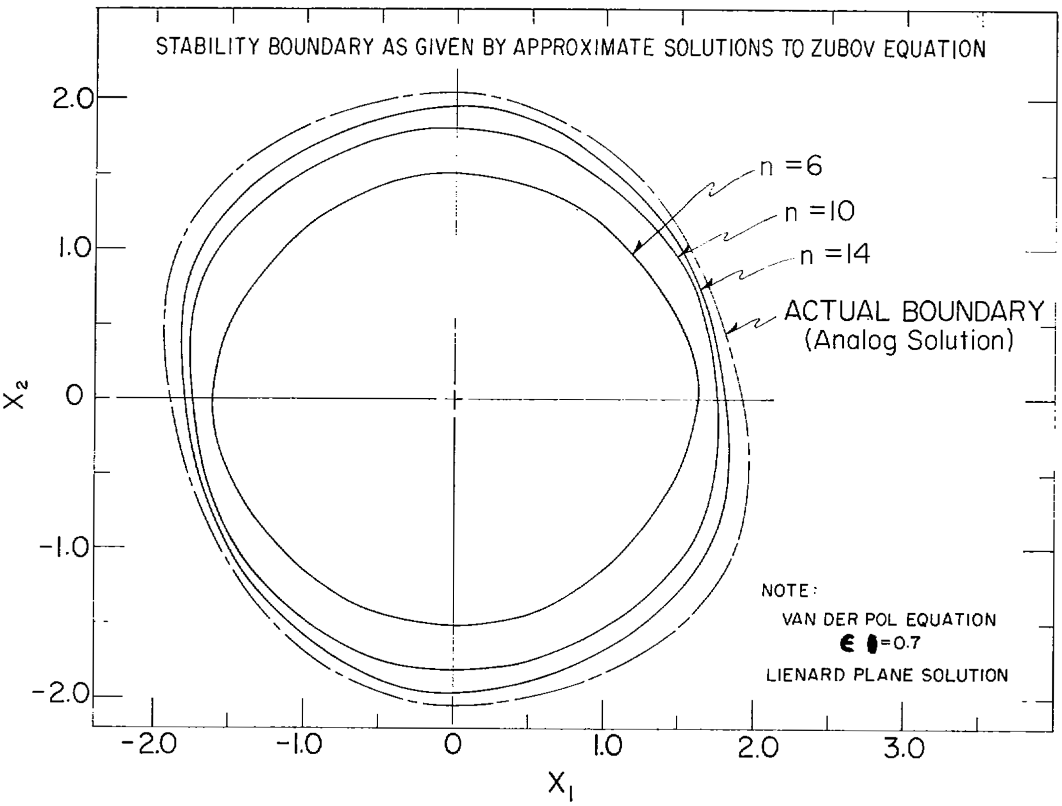

Fig. 4—Stability boundary as given by approximate solutions to Zubov equation.

Of course, a different value of c_1 is found for each value of n.

Some results are shown in Fig. 4, which compares the results obtained with different values of n to the exact stability boundary, as obtained by analog computer. The constant ϵ is taken as 0.7. The series computations were all performed using the semidefinite function $\phi = x_1^2$. This function was chosen because, for this particular quadratic form of ϕ, the corresponding $n=2$ approximation to the stability region includes the largest area or is the best quadratic approximation to v. Here, the quadratic approximation is a circle of radius $\sqrt{3}$ (not shown on Fig. 4).

It is of interest to determine whether higher order approximations will improve the $n=2$ estimate; but there is no assurance, in general, that higher approximations will give better results. Here, for example, the $n=6$ approximation is completely enclosed by the $n=2$ approximation. On the other hand, the $n=10$ approximation completely encloses both the $n=2$ and $n=6$ approximations and is, in turn, completely enclosed by the $n=14$ approximation.

Thus, a surprisingly high order of approximation must be used in order to improve upon the quadratic results. Part of the reason for this can be seen from Fig. 5, in which various approximations to the Zubov function v are plotted against x_1 for constant x_2. The "exact" solution used for the comparison was computed from (13) using the analog computer to evaluate the integral J of (14). Note that the curves are very flat in

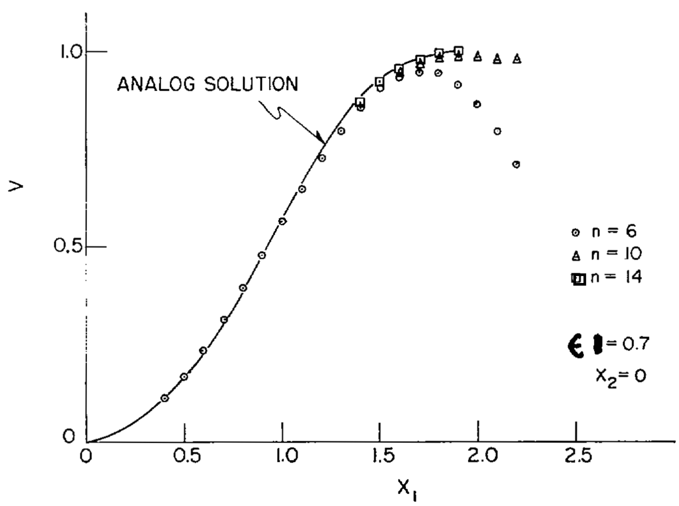

Fig. 5—Comparison of digital and analog solutions of the Zubov equation—Van der Pol example—Liénard plane.

the vicinity of the stability boundary, so that a small error in v results in a relatively large error in the location of the boundary. At the same time, although both the $n=6$ and the $n=10$ curves closely approximate the exact solution, the minimum of $v^{(n)}$ when $\dot{v}^{(n)}=0$ is 0.937 for $n=6$ and 0.984 for $n=10$ compared to the exact solution, $v=1$. Because of the flatness of the curves of v against x, these errors of only a few per cent in v produce a significant difference in the estimate of the stability boundary. In all cases, however, the computation guarantees that the estimated stability region is entirely contained in the actual stability region.

The approximate series solution with $n=14$ is practically identical to the analog computer solution. This close agreement indicates the practicality of using the series solution to Zubov's equation in the numerical computation of integral measures of error for nonlinear systems. Note that this information can be obtained from a plot of $v=-\ln(1-v)$ without computing the trajectories of the nonlinear system.

The results presented in Figs. 4 and 5 were computed by a code written in FORTRAN language and run on an IBM 7070 computer. The same basic program has been incorporated in a code which computes the series solution for any system described by the equation

$$\ddot{x} + c(x)\dot{x} + r(x)x = 0,$$

where $c(x)$ and $r(x)$ are either convergent power series or polynomials, each of which must have a nonzero constant term.

Fortunately, since it is possible in this case to use a very fast method for finding the coefficients d_{jk} the computing time does not increase very much as the order of approximation is increased. This makes it quite practical to use a tenth order, or even twentieth order, approximation if required.

The code, in its present form, cannot handle systems of higher than second order or systems with nonanalytic nonlinearities. These are areas for further research.

Conclusions

Zubov's construction procedure gives, in principle, a method for constructing a Lyapunov function for asymptotically stable linear or nonlinear systems of differential equations. Thus, the first problem formulated, that of the existence of a Lyapunov function, is solved by Theorem 1.

Where the Zubov construction can be carried out in closed form, it gives the exact stability boundary of the domain of asymptotic stability. Thus, the second problem formulated, that of determining the exact stability domain of the system, is solved by Theorem 7.

By examining the v function obtained in closed form as a result of this construction procedure, it can be immediately determined whether a system is globally asymptotically stable. Thus, the third problem formulated is solved by Theorem 8.

In cases in which the construction cannot be carried out in closed form, the method permits an approximation of the exact stability domain in terms of a domain in which the asymptotic stability is guaranteed. However, this approximation is not entirely satisfactory because convergence of the guaranteed stability domain to the exact stability domain does not appear to be rapid. Thus, even though the fourth problem formulated is at least partially solved by Theorems 13 and 14, there is opportunity for much more research into this problem.

In the example considered, the results of the application of the construction procedure indicate that if enough terms are included in the approximate solution, much useful information is obtained about both the system stability and system performance indexes.

The procedure proposed by Zubov by no means solves all the problems regarding the stability studies of systems of differential equations. There remain computational difficulties for systems of higher dimension, convergence difficulties in the approximate solutions, systems which cannot be represented by equations of the form of (1), etc. On the other hand, the method is a significant step forward which affords an insight into the meaning of Lyapunov functions as related to control system theory and does solve some of the problems which appear to have been obstacles to a definitive study of nonlinear systems.

Appendix

Here some of the major definitions and theorems of Lyapunov's Second Method are given along with a significant proof for one of the theorems.

Definition 4: The unperturbed motion, $x_1 = \cdots = x_n = 0$ of system (1) is stable if for every real number $\epsilon > 0$, there exists a real number $\delta(\epsilon) > 0$ such that $x_{10}^2 + \cdots + x_{n0}^2 < \delta$ implies

$$x_1^2(t) + \cdots + x_n^2(t) < \epsilon \qquad \text{for all } t > 0.$$

Definition 5: The unperturbed motion $x_1 = \cdots = x_n = 0$ of system (1) is asymptotically stable if it is stable and if, in addition

$$x_1^2(t) + \cdots + x_n^2(t) \to 0 \qquad \text{as } t \to +\infty.$$

Definition 6: A function $v(x_1, \cdots, x_n)$ is said to possess an infinitely small upper bound if

$$\lim v(x_1, \cdots, x_n) = 0 \qquad \text{for } (x_1, \cdots, x_n) \to (0, \cdots, 0)$$

in any manner whatever.

Theorem 15: A function $v(x_1, \cdots, x_n)$, which in some neighborhood of the origin has a convergent power series development in the x_i without a constant term is possessed of an infinitely small upper bound.[1]

Theorem 16: If in some region, S, containing the origin, there exists a positive definite function $v(x_1, \cdots, x_n)$ such that its derivative $\dot{v}$, taken by virtue of the differential equations of system (1) is not positive, then the unperturbed motion, $x_1 = \cdots = x_n = 0$ is stable.

Theorem 17: If, in some region, S, containing the origin, there exists a positive definite function $v(x_1, \cdots, x_n)$, possessed of an infinitely small upper bound, such that its derivative $\dot{v}$ by virtue of (1) is negative definite, then the unperturbed motion, $x_1 = \cdots = x_n = 0$ is asymptotically stable.

Proof of Theorem 17: Since the hypotheses include all those of Theorem 16, the unperturbed motion is stable.

[1] See especially Hahn [4], p. 5.

Since $\dot{v}$ is negative definite, v must decrease continuously along the integral curves of system (1) as t increases indefinitely. Since $v \geqq 0$, there must exist a limiting value, say v_0 such that

$$\lim v(t) = v_0 \geqq 0.$$

If $v_0 = 0$ for all initial values in S, then the theorem is proved. But suppose there is at least one initial value such that $v_0 > 0$. Then there is a set of nonzero values of the $(x_1, \cdots, x_n)$ for which $v(x_1, \cdots, x_n) \geqq v_0$. Over this set of nonzero values, since $\dot{v}$ is negative definite, $\dot{v} < 0$. Let $-a$ be the least upper bound of $\dot{v}$ over this set of values. From the relation, $\dot{v} < -a$

$$v(t) = v(0) + \int_0^t \dot{v} dt \leqq v(x_1^0, \cdots, x_n^0) - a(t - 0),$$

we see that at some time, say t_1,

$$v(t_1) \leqq v_0.$$

At any later time, the last inequality will be strict which contradicts our statement that $v(t) \geqq v_0$. Since the assumption that $v_0 > 0$ led to this contradiction, then it must be true that $v_0 = 0$, which, because $v = 0$ only at $(0, \cdots, 0)$, proves the theorem. QED

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