Extended Lyapunov stability theorem and its applications in control system with constrained input

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Abstract—**The Lyapunov stability theorem has been proposed for more than 100 years, and it is still one of the most important theories in control science and other fields. In this paper, a new stability theorem (Extended Lyapunov stability theorem) is proposed and proved to be different from Lyapunov stability theorem. The Lyapunov stability theorem demands that the time derivative of Lyapunov function is negative. But according to Extended Lyapunov stability theorem, the system can still keep stable when the time derivative of Lyapunov function is positive in some horizon and even in infinite horizon . So the conditions of Extended Lyapunov stability theorem is widely relaxed compared with the Lyapunov stability theorem. Inputs of actual systems are always limited by energy, under this background, a control law is designed to make system stable according to Extended Lyapunov stability theorem. And it can be proved that no Lyapunov function can be found to make the system stable. So with the help of Extended Lyapunov stability theorem better results can be obtained than those by Lyapunov stability theorem. At last, the numerical simulation result shows that Extended Lyapunov stability theorem is greatly different from Lyapunov stability theorem, such as the time derivative of energy function can be positive in infinite horizon. So it can be concluded that Extended Lyapunov stability theorem contains Lyapunov stability theorem and also it can be used more widely in many fields.**

Keywords- *Constrained input; Global terminal; Intelligent backstepping; Differential bomb*

I. INTRODUCTION

The Lyapunov stability theorem has been proposed for more than 100 years (1-2). It is not only one of the most important theories in control science but also one of the most important tools to solve nonlinear control problems. It is also considered to be very important by many researchers in other fields that is associated with nonlinear problems, such as in mathematical speciality, Lyapunov functions are viewed as the basic tools to analyse the stabilities of nonlinear systems. The Lyapunov theory is very important because it can analyse the stabilities of the nonlinear systems by constructing Lyapunov function in the viewpoint of energy, and there is no other method that can be more efficient than Lyapunov function method in analysing the stability of nonlinear systems.

But Lyapunov stable condition is not the necessary condition of judging the stability of a system(3-5). The requirement that the time derivative of Lyapunov function needs to be negative is relaxed by Extended Lyapunov stability theorem. According to Extended Lyapunov stability theorem, the system can be proved to be stable even the derivative of the Lyapunov function is positive definite. In this paper , we can see that no Lyapunov function can be found to prove the system is stable , but a control law can be designed to make the system stable with the help of Extended Lyapunov stability theorem. In fact , , this is always the situation that no Lyapunov function can be found to prove the system stable due to the limits of input in many actual nonlinear systems. Then this control law can be designed based on Extended Lyapunov stability theorem to make the system stable.

Almost all inputs of actual systems are limited by energy, so our research work under this background is very meaningful. A simple tracking system with constrained input is investigated and it can be proved that no Lyapunov function can be found to prove the system is stable. With the help of Extended Lyapunov stability theorem, a stable control law is designed. And at last, the simulation results show that the Extended Lyapunov stability theorem is greatly different form Lyapunov function method.

II. EXTENDED LYAPUNOV STABILITY THEOREM AND ITS ANALYSIS

Consider the general system $\dot{x} = f(x,t)$, and it satisfies: $f(0,t)=0$, and we assume $\dot{V}(x)$ is meaningful near the field of 0, where $\dot{V}(x)$ is the one order continuous time derivative of $V(x)$.

Lemma 1: (Lyapunov stability theorem) The system is asymptotically stable nearing 0 if there exists a positive definite function $V(x)>0$, which satisfy $\dot{V}(x)<0$ and $\dot{V}(x)$ is negative definite.

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Lemma 2: The system is unstable nearing 0 if there exists a positive definite function $V(x) > 0$, which satisfy $\dot{V}(x) > 0$ and $\dot{V}(x)$ is positive definite.

Theory 1: (Extended Lyapunov stability theorem) The system is stable near 0 if there is a positive definite function $V(x)>0$ and a positive constant $a \geq 0$ such that $\lim_{t\to\infty}\int_0^t \dot{V}(x)g(\dot{V})dt \leq a$ where $g(\dot{V})$ is defined as $g(\dot{V}) = sign(\dot{V})+1 \geq 0$ and $sign(x)$ is the sign function(it is defined as :

$$sign(x) = \left\{\begin{matrix} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{matrix}\right\}).$$

Assume for all t, we have : $\dot{V}(x) > 0$, then $\lim_{t\to\infty}\int_0^t \dot{V}(x)g(\dot{V})dt = \lim_{t\to\infty}\int_0^t \dot{V}(x)dt \leq a$.

We get $\dot{V}(x(t)) \to 0$.

For example,

$$\lim_{t\to\infty}\int_0^t \dot{V}(x)g(\dot{V})dt \leq a, \dot{V}(x) = \frac{1}{t+2},$$

$$\lim_{t\to\infty}\int_0^t \dot{V}(x)g(\dot{V})dt = \ln(t+2) = \ln\infty - \ln 2$$

$$\dot{V}(x) = \frac{1}{(t+2)^2},$$

$$\lim_{t\to\infty}\int_0^t \dot{V}(x)g(\dot{V})dt = -\frac{1}{t+2} = 0 - -\frac{1}{2} = \frac{1}{2}$$

Proof : we assume there is t_i $\dot{V}(x(t_i)) = 0$

Firstly, assume that there isn't time sequence $\{t_1, t_2, \cdots, t_{2n}\}$ such that $\dot{V}(x) \geq 0$ on time horizon $[t_1,t_2],[t_3,t_4],\cdots,[t_{2n-1},t_{2n}]$, then it means that for all time horizon we get $\dot{V}(x) \leq 0$, and the system can proved to be stable by Lyapunov stability theorem.

Secondly, assume that there exist a time sequence $\{t_1, t_2, \cdots, t_{2n}\}$ such that $\dot{V}(x) \geq 0$ when $t \in [t_1,t_2]\bigcup[t_3,t_4],\cdots,[t_{2n-1},t_{2n}]$. Then we construct sequence as following:

$$S_n = \sum_i^n a_i \text{ , ανδ } a_i = \int_{t_{2i-1}}^{t_{2i}} 2\dot{V}(x)dt > 0 \tag{1}$$

If the sequence $\{t_1, t_2, \cdots, t_{2n}\}$ is finite , then the system can be proved to be stable when $t > t_{2n}$ according to the Lyapunov stability theorem.

If the sequence $\{t_1, t_2, \cdots, t_{2n}\}$ is infinite, then we can get :

$$\lim_{t\to\infty}\int_0^t \dot{V}(x)g(\dot{V}) = \lim_{n\to\infty} S_n \leq a \tag{2}$$

And S_n is increased and convergent so $a_i \to 0$, and we get $\dot{V}(x) \to 0$ when t belongs to the infinite time horizon $[t_1,t_2]\bigcup[t_3,t_4],\cdots,[t_{2n-1},t_{2n}]$. And it means that :

$$\lim_{t\to\infty}\dot{V}(x) \leq 0 \tag{3}.$$

Then it is easy to prove that the system is stable with the help of Lyapunov stability theorem.

III. RESEARCH ON A SAMPLE SYSTEM WITH CONSTRAINED INPUT BY EXTENDED LYAPUNOV STABILITY THEOREM

Take a one order simple system for example:

$$\dot{x}_1 = -0.5x_1 + u \tag{4}$$

Where u is limited by energy so it needs to satisfy $|u| < 20$ and x_1 is the state of the system and its initial value is zero. The goal of the control is to make the system state track to the command x_1^d, where it is defined as

$$x_1^d = -15 + \sum_{n=1}^{\infty}\frac{1}{t-n+1/(10n)}f(n) \tag{5}$$

It derivative exists almost everywhere, where

$$f(n) = \left\{\begin{matrix} 0 & t<n \\ \frac{1}{n^2} & n \leq t < n+(\frac{1}{2})^n \\ 0 & n+(\frac{1}{2})^n \leq t \end{matrix}\right\} \text{ and}$$

$$\dot{x}_1^d = \left\{\begin{matrix} 0 & t<n \\ -\frac{1}{n^2}[\frac{1}{t-n+1/(10n)}]^2 & n \leq t < n+(\frac{1}{2})^n \\ 0 & n+(\frac{1}{2})^n \leq t \end{matrix}\right\}.$$

Obviously, it is easy to prove that:

$$\dot{x}_1^d(n) = -100$$

$$\int_0^{+\infty}\left|\dot{x}_1^d(t)\right|dt \leq \sum_{n=1}^{\infty}100*(\frac{1}{2})^n \leq 100 \tag{6}.$$

Define a new error variable as: $z_1 = x_1 - x_1^d$, Then:

$\dot{z}_1 = \dot{x}_1 - \dot{x}_1^d = -0.5x_1 + u - \dot{x}_1^d$

Remind that: $z_1(1) = x_1(1) - x_1^d(1) = 0 - (-5) = 5 > 0$, to make the system stable, we should chose u to make $\dot{z}_1(1)<0$. And because $-\dot{x}_1^d=-(-100)=100$ and $-20<u<20$, there always exists a small positive constant ε that the input can't provide enough energy to make the system stable when $t\in[1,1+\varepsilon]$ due to the input is limited by energy. To make the system stable, the control law is designed as $u_d=0.5x_1-k_1z_1$. Obviously, the input can provide enough energy to make the system stable when $x_1\le 40$. And then: $\dot{V}=z_1\dot{z}_1=-k_1z_1^2-z_1\dot{x}_1^d$.

It is easy to prove that $\dot{V}\le 0$ when $z_1\ge\frac{|\dot{x}_1^d|}{k_1}=\frac{100}{k_1}$, and so we discuss the situation when $z_1\le\frac{|\dot{x}_1^d|}{k_1}=\frac{100}{k_1}$ and $\dot{V}\ge 0$. We can get:

$$\lim_{t\to\infty}\int_0^t\dot{V}(x)g(\dot{V})\le\lim_{t\to\infty}\int_0^t 2\left|-z_1\dot{x}_1^d\right|dt$$
$$\le\max_{t\in[0,\infty]}|z_1|\left|\int_0^t 2\left|\dot{x}_1^d\right|dt\le 200\max_{t\in[0,\infty]}|z_1|\le 200*\frac{100}{k_1}$$

Then it is easy to prove that the system is stable according to Extended Lyapunov stability theorem.

Consider that the input is limited by energy, we can change the control law as:

$$u=\begin{cases}20 & 20<u_d\\ u_d & -20\le u_d\le 20\\ -20 & u_d<-20\end{cases}\tag{7}$$

Also choose the Lyapunov function as $V=\frac{1}{2}z^2$ and the time derivative of V satisfies:

$$\begin{aligned}\dot{V}&=z_1\dot{z}_1=z_1(-0.5x_1+u-u_d+u_d-\dot{x}_1^d)\\&=z_1(u-u_d)+z_1(-0.5x_1+u_d-\dot{x}_1^d)\end{aligned}.$$

Firstly, consider the situation when u_d satisfies $20<u_d$ and $z_1>0$, this yields $u-u_d<0$ and then we obtain $\dot{V}\le z_1(-0.5x_1+u_d-\dot{x}_1^d)\le -z_1\dot{x}_1^d$

Secondly, consider the situation when u_d satisfies $20<u_d$ and $z_1<0$, it is easy to get $u-u_d<0$ and $\dot{V}=z_1(-0.5x_1+20-\dot{x}_1^d)$. Take the condition $|x_1|\le 40$ in mind, $\dot{V}$ also satisfies $\dot{V}\le z_1(-\dot{x}_1^d)$.

Above all, when $20<u_d$, $\dot{V}$ can be proved to satisfies $\dot{V}\le z_1(-\dot{x}_1^d)$.

In the same way, $\dot{V}$ can be proved to satisfies $\dot{V}\le z_1(-\dot{x}_1^d)$ when $u_d<-20$.

And when $-20\le u_d\le 20$, we can obtain $\dot{V}\le -k_1z_1^2$ obviously.

Above all, we can obtain such inequality as:

$$\lim_{t\to\infty}\int_0^t\dot{V}(x)g(\dot{V})\le\lim_{t\to\infty}\int_0^t 2\left|-z_1\dot{x}_1^d\right|dt\le$$
$$\max_{t\in[0,\infty]}|z_1|\left|\int_0^t 2\left|\dot{x}_1^d\right|dt\le 200\max_{t\in[0,\infty]}|z_1|\le 200*\frac{100}{k_1},$$

So the system is stable according Extended Lyapunov stability theorem if we choose control law as $u=\begin{cases}20 & 20<u_d\\ u_d & -20\le u_d\le 20\\ -20 & u_d<-20\end{cases}$ with the assumptions $|x_1|<40$ and $|u|<20$.

IV. NUMERICAL SIMULATION

To verify the effectiveness of the proposed control law, numerical simulation were done for the result we obtained. Numerical simulation results are shown in Figs 1 to Figs 6. And the control parameters are chosen as $k_1=5$.

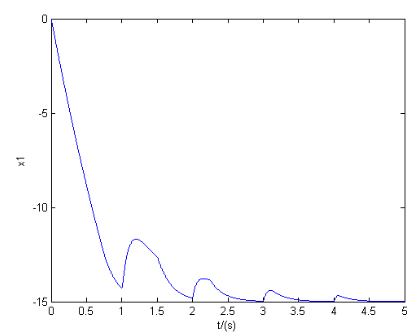

Fig.1. The state x_1

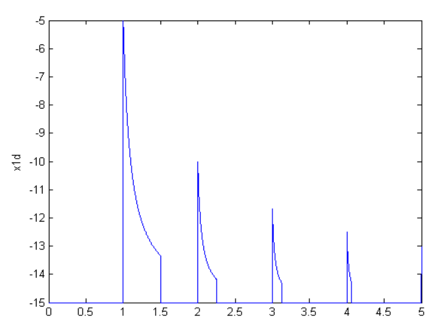

Fig.2. The variable x_1^d

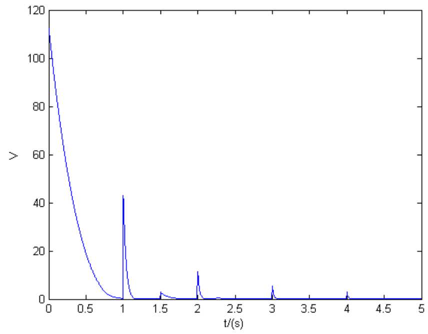

Fig.3. The energy function V

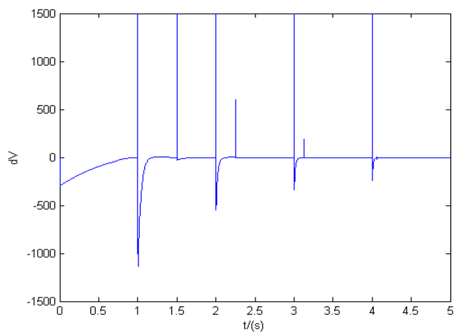

Fig.4. The time derivative of V ($\dot{V}$)

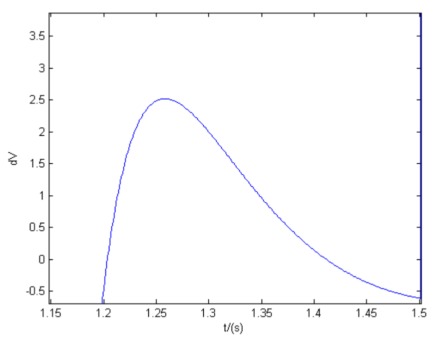

Fig.5. $\dot{V}$ when $t \in [1.1, 1.5]$

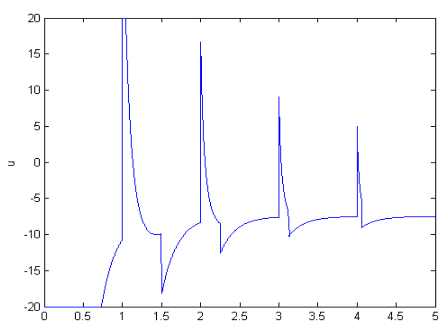

Fig.6. The control input u

Form fig.5 we can found that $\dot{V} \geq 0$ during a long time horizon. Fig.6 shows that the control law can work effectively although it is limited by energy. Fig. 5 shows that the Extended Lyapunov stability theorem is greatly different from Lyapunov stability theorem.

V. CONCLUSIONS

A new stability theorem (Extended Lyapunov stability theorem) is proposed which is considerably different from Lyapunov stability theorem. The Lyapunov stability theorem demands the time derivative of energy function to be negative definite or half-negative definite. This condition is very difficult to be satisfied in many actual nonlinear systems such as systems with constrained input . Then Extended Lyapunov stability theorem is proposed in this paper which relaxes the condition. And according to the Extended Lyapunov stability theorem the system can be proved to be stable when the time derivative of energy function is positive definite in some time horizons, and even when the time horizons are infinite the same result can be obtained. We took a sample system with constrained input as an example to illustrate the situation that the time derivative of energy function is positive in infinite time horizons. The numerical simulation clearly shows that the Extended Lyapunov stable is greatly different from Lyapunov stability theorem because it doesn't need to satisfy the condition that the time derivative of energy function should be negative. So the conclusion is obtained that Extended Lyapunov stability theorem contains Lyapunov stability theorem and it can be used more widely to solve many actual control problems.

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