The Closed-Form Solution of True Proportional Navigation

M. GUELMAN
Ministry of Defence
Tel Aviv, Israel

Abstract

The closed-form solution of the equations of motion of an ideal missile pursuing a nonmaneuvering target according to the true proportional navigation law is obtained. An analysis of the solution is performed, and the conditions necessary for the missile to reach the target are determined.

Manuscript received December 5, 1974.

This work was supported by the U.S. Air Force Avionics Laboratory, Wright Patterson Air Force Base, under AFOSR Grant 74-2879.

I. Introduction

Most modern air-to-air and surface-to-air missile systems use a form of proportional navigation in the homing phase of flight.

In proportional navigation, control accelerations are generated proportional to the measured rate of rotation of the interceptor-target line of sight.

In the literature two basic forms of proportional navigation have been considered. These two forms are generally labeled pure proportional navigation (PPN) [2] and true proportional navigation (TPN) [1]. In PPN, commanded accelerations are applied normal to the missile velocity. In TPN, commanded accelerations are applied in a direction normal to the interceptor-target line of sight. In both cases no closed-form solution is available, and linearized analysis was applied to study these two forms of proportional navigation.

Applying qualitative methods for the analysis of PPN [2] it was demonstrated that, provided a set of conditions relating the ratio of velocities and the constant of navigation are fulfilled, capture can be assured for any initial conditions expect for a precisely defined particular case.

In this study we determine the closed-form solution of the differential equations describing the trajectories of a missile pursuing a nonmaneuvering target according to the true proportional navigation law. The solution was analyzed and it is demonstrated that capture is restricted for the cases where the initial conditions belong to a determined circle, defined as the circle of capture. In particular, it can be shown that even if the missile is initially approaching the target there exists an entire region of initial conditions where capture cannot be assured. This strongly differentiates the two forms of proportional navigation as opposed to previous linearized analysis where equivalent results were obtained for both cases [1], [3].

An analysis is also made of the behavior of the rate of rotation of the line of sight for the case where capture is assured. New results relating to the boundedness of the line of sight rate are demonstrated.

II. Problem Statement

Consider a target T and missile M as points in a plane moving with velocities V_T and V_M, respectively, as shown in Fig. 1. The system can be described in a relative system of coordinates with its center at T and axis T_x along the straight line trajectory of the target.

In TPN [1], the missile acceleration commanded a_M is applied *normal to the line of sight,* as opposed to PPN [2] where the missile acceleration commanded is applied *normal to the missile velocity,* as depicted in Fig. 2.

The equations of motion of the missile are derived in the following form. Letting a dot over a symbol denote differentiation with respect to time, the components of the relative velocity from missile to target, in polar coordinates, are

$$V_r = \dot{r} = V_M \cos \alpha - V_T \cos \theta \tag{1}$$

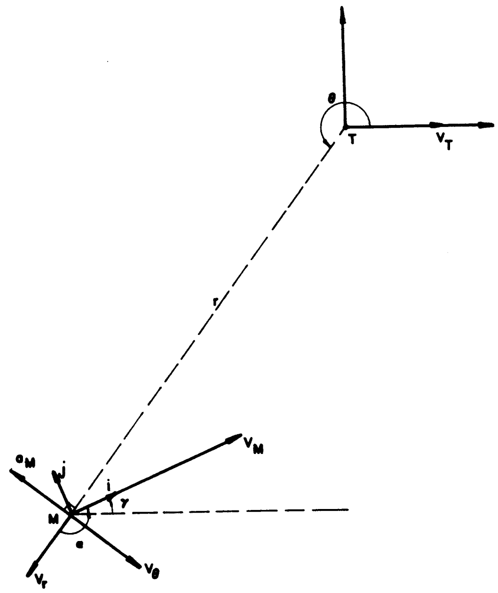


Fig. 1. Planar pursuit, true proportional navigation.

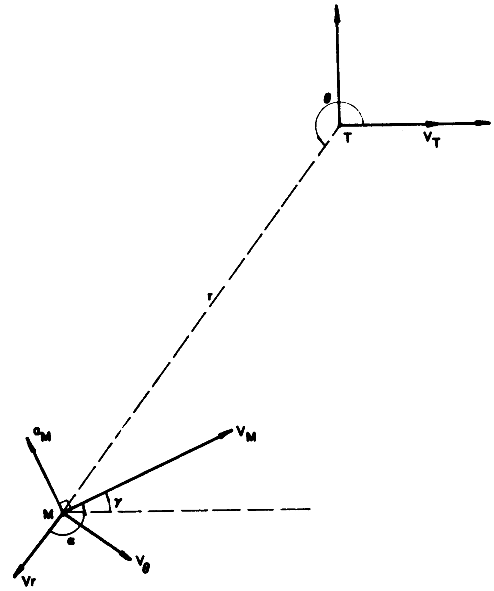


Fig. 2. Planar pursuit, pure proportional navigation.

$$V_\theta = r\dot{\theta} = V_M \sin\alpha + V_T \sin\theta. \tag{2}$$

In proportional navigation the interceptor acceleration is proportional to the line of sight angular rate

$$a_M = c\dot{\theta} \tag{3}$$

where c is generally defined in TPN as [1]

$$c = -\lambda V_{r_0} \tag{4}$$

with λ the navigation constant.

From the kinematics of a point,

$$a_M = \dot{V}_M + \omega \times V_M \tag{5}$$

where

$$\omega = \dot{\gamma} k \tag{6}$$

is the angular velocity of the missile system of coordinates (i, j, k) with respect to an inertial reference.

Developing (5) for the TPN case,

$$\dot{\gamma} = -(a_M \cos\alpha)/V_M \tag{7}$$

$$\dot{V}_M = -a_M \sin\alpha. \tag{8}$$

From Fig. 1,

$$\gamma = \alpha + \theta - 2\pi. \tag{9}$$

Differentiating (9) with respect to time and rearranging,

$$\dot{\alpha} = \dot{\gamma} - \dot{\theta}. \tag{10}$$

Replacing $\dot{\gamma}$ from (7) into (10),

$$\dot{\alpha} = [-(a_M \cos\alpha)/V_M] - \dot{\theta}. \tag{11}$$

Differentiating (1) and (2) with respect to time,

$$\dot{V}_r = \dot{V}_M \cos\alpha - V_M \sin\alpha\dot{\alpha} + V_T \sin\theta\dot{\theta} \tag{12}$$

$$\dot{V}_\theta = \dot{V}_M \sin\alpha + V_M \cos\alpha\dot{\alpha} + V_T \cos\theta\dot{\theta}. \tag{13}$$

Replacing $\dot{V}_M$ and $\dot{\alpha}$ from (8) and (11), respectively, into (12) and (13),

$$\dot{V}_r = (V_M \sin\alpha + V_T \sin\theta)\,\dot{\theta} \tag{14}$$

$$\dot{V}_\theta = -a_M - (V_M \cos\alpha - V_T \cos\theta)\,\dot{\theta}. \tag{15}$$

Introducing in the right hand terms of (14) and (15) V_r and V_θ instead of their values as defined in (1) and (2) yields

$$\dot{V}_r = V_\theta \dot{\theta} \tag{16}$$

$$\dot{V}_\theta = -a_M - V_r \dot{\theta}. \tag{17}$$

Finally, replacing V_r by $\dot{r}$ and V_θ by $r\dot{\theta}$ in (16) and (17) and rearranging,

$$\ddot{r} - r\dot{\theta}^2 = 0 \tag{18}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -a_M. \tag{19}$$

Replacing a_M from (3) into (19) and rewriting (18) gives

$$\ddot{r} - r\dot{\theta}^2 = 0 \tag{20}$$

$$r\ddot{\theta} + (2\dot{r} + c)\dot{\theta} = 0. \quad (21)$$

Equations (20) and (21) are the two well known equations of TPN [1]. The solution of this nonlinear system of differential equations will provide the trajectories of the missile in the relative system of coordinates previously defined.

III. Closed-Form Solution

In the classical theory of TPN, it is tacitly assumed that the system of differential equations (20) and (21) is not solvable in closed form. Moreover, it is admitted, without proof, that the missile follows a straight line trajectory towards the pursuit end. The analysis that followed only considered small perturbations with respect to this final straight line trajectory.

In this study it is shown that in fact there exists a closed-form solution for system (20), (21) and that this closed-form solution will provide us with the conditions under which the missile captures the target.

Replacing a_M from (3) into (17),

$$\dot{V}_\theta = -(c + V_r)\dot{\theta} \quad (22)$$

$$\dot{V}_r = V_\theta \dot{\theta}. \quad (23)$$

Multiplying (22) by V_θ and (23) by $(c + V_r)$, respectively, and adding them gives

$$\dot{V}_\theta V_\theta + \dot{V}_r(c + V_r) = 0. \quad (24)$$

Rearranging,

$$\tfrac{1}{2}\, d/dt\,(V_r^2 + V_\theta^2) + c\,(dV_r/dt) = 0. \quad (25)$$

Integrating,

$$V_\theta^2 + V_r^2 + 2cV_r = a \quad (26)$$

where

$$a = V_{\theta_0}^2 + V_{r_0}^2 + 2cV_{r_0}. \quad (27)$$

Multiplying (23) by r,

$$r\dot{V}_r = V_\theta^2. \quad (28)$$

Substituting V_θ^2 from (28) into (26),

$$r\dot{V}_r + V_r^2 + 2cV_r = a. \quad (29)$$

Substituting V_r by $\dot{r}$ into (29),

$$r\ddot{r} + \dot{r}^2 + 2c\dot{r} = a. \quad (30)$$

Equation (30) is an equation in r only. At this stage r and θ are separated.

Let us differentiate $r\dot{r}$ with respect to time:

$$(d/dt)\,(r\dot{r}) = r\ddot{r} + \dot{r}^2. \quad (31)$$

Replacing (31) into (30),

$$(d/dt)\,(r\dot{r}) + 2c(dr/dt) = a. \quad (32)$$

Integrating,

$$r\dot{r} + 2cr = at + b \quad (33)$$

where

$$b = r_0\dot{r}_0 + 2cr_0. \quad (34)$$

Let now

$$r = y + mt + n \quad (35)$$

where y is the new independent variable and m and n are two real constants.

Substituting (35) into (33),

$$(y + mt + n)\,\dot{y} + (2c + m)y + (m + 2c)mt + (m + 2c)n = at + b. \quad (36)$$

Let m and n be such that

$$(m + 2c)\,m = a \quad (37)$$

and

$$n = mb/a. \quad (38)$$

With these values of m and n, (37) reduces to

$$(y + mt + n)\,(dy/dt) + ky = 0 \quad (39)$$

where

$$k = m + 2c. \quad (40)$$

Before we proceed further it is important to remark that (37) has two solutions for m:

$$m_1 = -c + \sqrt{c^2 + a} \quad (41)$$

and

$$m_2 = -c - \sqrt{c^2 + a}. \quad (42)$$

Given the fact that (33) fulfills the Cauchy-Lipschitz condition for any real t and $r \neq 0$, it is sufficient to consider the solution for only one of the values of m. The other value will provide the same solution for r. In consequence, let $m = m_1$.

Equation (39) is a homogeneous equation [4] and the variables can be separated. This equation is solved in Appendix I, and the solution for y is

$$(y/y_0)^{m/k}\{[y+(m+k)x]/[y_0+(m+k)x_0]\}=1 \quad (43)$$

where

$$x=t+n/m \quad (44)$$

and y_0 and x_0 are the initial values of y and x, respectively, and are obtained from (35) and (44) for $t=0$:

$$x_0=n/m=b/a \quad (45)$$

$$y_0=r_0-n. \quad (46)$$

Once the solution for y is obtained, r is obtained as shown in Appendix II. The result is

$$r=r_0(\mu_1 z^{-m/k}+\mu_2 z) \quad (47)$$

where

$$z=y/y_0 \quad (48)$$

is defined by

$$pz^{-m/k}-gz=x. \quad (49)$$

Note that

$$z(x_0)=z_0=y_0/y_0=1. \quad (50)$$

μ_1, μ_2, p, and q are all real constants, respectively, defined by

$$\mu_1=(\nu+\sqrt{\nu^2+1})/(2\sqrt{\nu^2+1}) \quad (51)$$

$$\mu_2=(-\nu+\sqrt{\nu^2+1})/(2\sqrt{\nu^2+1}) \quad (52)$$

where

$$\nu=(V_{r_0}+c)/|V_{\theta_0}| \quad (53)$$

$$p=y_0/(m+k)+x_0=r_0\mu_1/m \quad (54)$$

$$g=y_0/(m+k)=r_0\mu_2/k. \quad (55)$$

Note that from (51) and (52)

$$0\leqslant\mu_1\leqslant 1 \quad (56)$$

$$0\leqslant\mu_2\leqslant 1. \quad (57)$$

Once the solution for r is obtained, $\dot\theta$ and θ are obtained as shown in Appendix III. The result is

$$\dot\theta=\dot\theta_0 z^{(3m/k+1)2}/(\mu_1+\mu_2 z^{m/k+1})^2 \quad (58)$$

$$\theta=\theta_f-2\,\mathrm{sign}\,(\dot\theta_0)\arctan[(\mu_2/\mu_1)\,z^{m/k+1}]^{1/2} \quad (59)$$

where

$$\theta_f=\theta_0+2\,\mathrm{sign}\,(\dot\theta_0)\arctan(\mu_2/\mu_1)^{1/2}. \quad (60)$$

IV. Analysis of the Solution

In order to analyze the solution, the signs of a and b will first be determined.

From the expression for a (27), it follows that

1) if $(V_{r_0}, V_{\theta_0})\in$ a circle C_c defined by $(V_{r_0}+c)^2+V_{\theta_0}^2=c^2$ with center at $(-c, 0)$ and radius c, then $a<0$;
2) if $(V_{r_0}, V_{\theta_0})\in$ the circumference of C_c, then $a=0$;
3) if $(V_{r_0}, V_{\theta_0})\notin C_c$, then $a>0$.

From the definition of b (34),

1) $b<0$ for $V_{r_0}<-2c$;
2) $b>0$ for $V_{r_0}>-2c$.

In Fig. 3 we have depicted, in the plane V_{r_0}, V_{θ_0}, the circle C_c and the straight line $V_{r_0}=-2c$.

The preceding results are shown in Table I and are discussed in the following paragraphs.

Case A) $a>0, b>0$: Rearranging (33),

$$r(\dot r+2c)=at+b. \quad (61)$$

From (61) it follows that a necessary condition for the missile M to reach the target T $(r=0)$ is

$$at+b=0 \quad \text{for} \quad t=t_1, \qquad 0<t_1<\infty.$$

On the other hand,

$$at+b=0 \quad \text{for} \quad t=t_1$$

if and only if

$$ab<0. \quad (62)$$

Theorem 1: A missile M pursuing a nonmaneuvering target T according to the TPN law and starting its course at $M=M_0(r_0,\theta_0)$, where $(V_{r_0}, V_{\theta_0})\notin C_c$, $V_{r_0}>-2c$, will not reach the target for any real t.

Case B) $a<0, b>0$: From (41), with $a<0$, it follows that

$$m=m_1=-c+\sqrt{c^2+a}<0. \quad (63)$$

Substituting this value of m into (40) yields

$$k=m+2c=c+\sqrt{c^2+a}>0 \quad (64)$$

and

$$m+k=2\sqrt{c^2+a}>0. \quad (65)$$

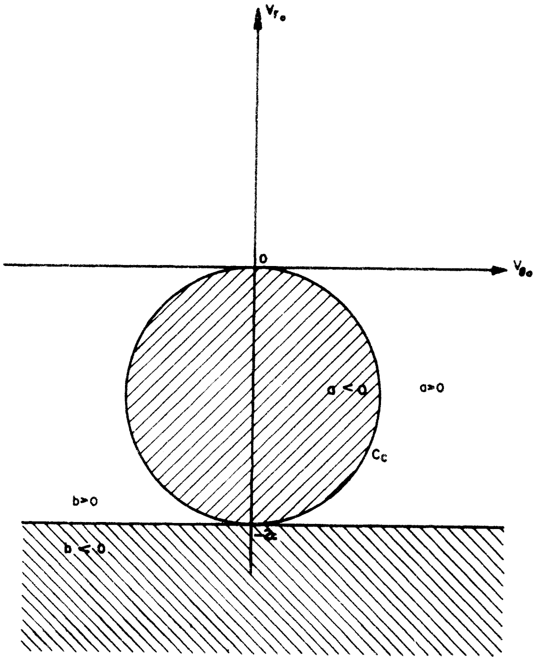


Fig. 3. Signs of a and b.

TABLE I

Case A	$a > 0, b > 0$	$(V_{r_0}, V_{\theta_0}) \notin C_c,\ V_{r_0} > -2c$
Case B	$a < 0, b > 0$	$V_{r_0} \in C_c$
Case C	$a > 0, b < 0$	$V_{r_0} < -2c$
Case D	$a < 0, b < 0$	—

It follows then that

$$0 < -m/k < 1 \tag{66}$$

and

$$0 < (m+k)/k < 1. \tag{67}$$

Now, from (54) and (55) we have

$$p < 0 \tag{68}$$

$$g > 0 \tag{69}$$

and from (55), (67), and (57) we have

$$0 < y_0 = r_0(m+k)/k\,\mu_2 < r_0. \tag{70}$$

We shall first study z as a function of x as defined in (49). For $t = 0$, $x = x_0 = b/a$, and $z_0 = 1$. For $x = 0$ it follows that $z = 0$.

Differentiating (49) with respect to x and rearranging,

$$dz/dx = -k\,z^{m/k+1}/r_0(\mu_1 + \mu_2 z^{m/k+1}). \tag{71}$$

Hence with $k > 0$,

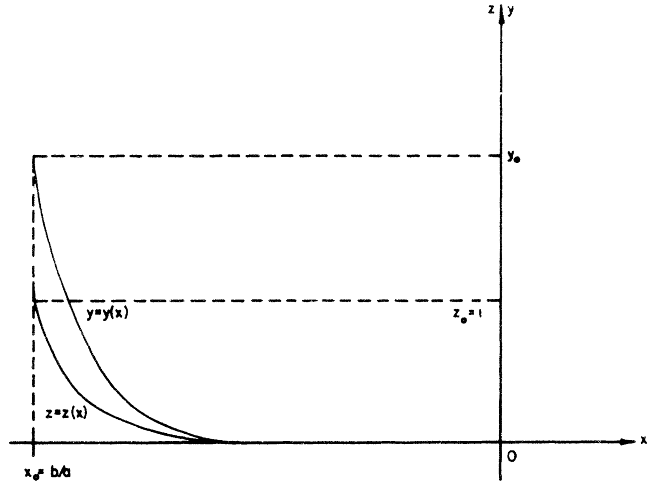


Fig. 4. y and z versus x.

$$dz/dx < 0 \tag{72}$$

and from (66), $m/k + 1 > 0$; thus

$$\lim_{z \to 0} dz/dx = 0. \tag{73}$$

With all these elements z is depicted as a function of x in Fig. 4. y as a function of x is directly obtained multiplying z by y_0 as depicted in Fig. 4. From (44) it follows that y as a function of t is obtained translating the origin along the x axis by $n/m = b/a$. This is depicted in Fig. 5. Finally, recalling (35), r is obtained by adding $mt + n$ to y. This is also depicted in Fig. 5.

It results in consequence that $r = 0$ for $t = -b/a$. The missile reaches the target in this case.

The value of t,

$$t = t_f = -b/a = (-r_0/V_{r_0})\{1 - \{V^2_{\theta_0}/[V_{r_0}(V_{r_0} + 2c) + V^2_{\theta_0}]\}\} \tag{74}$$

is the final time of the pursuit.

It is worthwhile to remark that when $V^2_{\theta_0} = 0$, $t_f = -r_0/V_{r_0}$ as we should expect.

In what concerns the closing velocity, differentiating (35) with respect to time t,

$$\dot{r} = \dot{y} + m. \tag{75}$$

Now, from (44) and (48) it follows that

$$dy/dt = dy/dx = y_0(dz/dx). \tag{76}$$

From (73), for $z = 0$ $(t = t_f)$,

$$dy/dt(t_f) = 0. \tag{77}$$

Hence

$$V_{r_f} = \dot{r}_f = m = -c + \sqrt{(V_{r_0} + c)^2 + V_{\theta_0}^2}. \tag{78}$$

The angle θ is obtained from (59). For $z = 0$,

$$\theta = \theta_f = \theta_0 + 2 \operatorname{sign}(\dot{\theta}_0) \arctan (\mu_2/\mu_1)^{1/2}. \tag{79}$$

Theorem 2: A missile M pursuing a nonmaneuvering target T according to TPN, starting at $M_0(r_0, \theta_0)$ such that

$$(V_{r_0}, V_{\theta_0}) \in C_c \tag{80}$$

reaches the target at a finite time

$$t = t_f = (-r_0/V_{r_0})\{1 - \{V_{\theta_0}^2/[V_{r_0}(V_{r_0} + 2c) + V_{\theta_0}^2]\}\}. \tag{81}$$

Moreover M arrives to T with a closing speed

$$V_r = V_{r_f} = -c + \sqrt{(V_{r_0} + c)^2 + V_{\theta_0}^2} \tag{82}$$

at an aspect angle

$$\theta = \theta_f = \theta_0 + 2 \operatorname{sign}(\dot{\theta}_0) \arctan \{[-(V_{r_0} + c) + \sqrt{(V_{r_0} + c)^2 + V_{\theta_0}^2}\,]/[(V_{r_0} + c) + \sqrt{(V_{r_0} + c)^2 + V_{\theta_0}^2}\,]\}^{1/2}. \tag{83}$$

Remark 1) The conditions to capture the target depend on the initial conditions and their relations with c.

No conditions at all are imposed on $\nu = V_M/V_T$, the ratio of velocities, as was the case in PPN. Even when $\nu < 1$ capture is possible.

The rate of rotation of the line of sight plays a fundamental role in missile design. For a stable functioning of the missile it is essential that $\dot{\theta}$ should be bounded.

From the expression of $\dot{\theta}$ in (58),

$$\dot{\theta} = \dot{\theta}_0 z^{(3m/k+1)/2}/(\mu_1 + \mu_2 z^{m/k+1})^2$$

it follows that if $k > 0$ and

1) if $3m + k < 0$, then $\lim_{z\to 0} |\dot{\theta}| = \infty$; (84)

2) if $3m + k = 0$, then $\lim_{z\to 0} \dot{\theta} = \dot{\theta}_0/\mu_1^2$; and (85)

3) if $3m + k > 0$, then $\lim_{z\to 0} \dot{\theta} = 0$. (86)

Substituting for m and k their values given in (63) and (64) into (84), (85), and (86) and rearranging, we obtain, respectively, that

1) if $(V_{r_0}, V_{\theta_0}) \in C_s$, where C_s is a circle defined by $(V_{r_0} + c)^2 + V_{\theta_0}^2 = (c/2)^2$, then $\lim_{t\to t_f} |\dot{\theta}| = \infty$;
2) if $(V_{r_0}, V_{\theta_0}) \in$ the circumference of C_s then $\lim_{t\to t_f} \dot{\theta} = \dot{\theta}_0/\mu_1^2$;

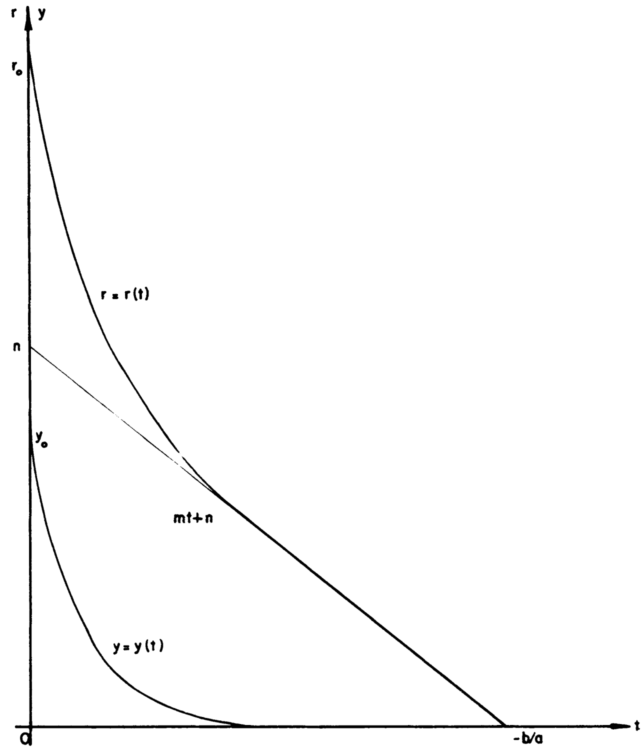


Fig. 5. r and y versus time t.

3) if $(V_{r_0}, V_{\theta_0}) \notin C_s$, then $\lim_{t\to t_f} \dot{\theta} = 0$.

Let us determine now the value of $\ddot{\theta}$. Rearranging (21),

$$\ddot{\theta} = [-(c + 2\dot{r})/r]\, \dot{\theta}. \tag{87}$$

From (75) and (76) it follows that

$$\dot{r} = y_0(dz/dx) + m. \tag{88}$$

Substituting dz/dx defined in (71) into (88),

$$\dot{r} = -y_0\, k z^{m/k+1}/r_0(\mu_1 + \mu_2 z^{m/k+1}) + m. \tag{89}$$

Substituting r, $\dot{r}$, and $\dot{\theta}$ from (47), (89), and (58) into (87) and rearranging,

$$\ddot{\theta} = -\dot{\theta}_0\, z^{(5m/k+1)/2} [\mu_1(c + 2m) + \mu_2(c - 2k)z^{m/k+1}]/r_0(\mu_1 + \mu_2 z^{m/k+1})^4. \tag{90}$$

Now

$$c + 2m = (3m + k)/2 \tag{91}$$

thus, $c + 2m > 0$ if $(V_{r_0}, V_{\theta_0}) \notin C_s$ and $c + 2m < 0$ for $(V_{r_0}, V_{\theta_0}) \in C_s$.

$$c - 2k = -c - 2\sqrt{c^2 + a} < 0. \tag{92}$$

For $c + 2m > 0$, there exists z

$$z = z_1 = [-\mu_1(c + 2m)/\mu_2(c - 2k)]^{k/(m+k)} \tag{93}$$

such that if $z_1 < 1$, then $\ddot{\theta}(t_1) = 0$ for $0 < t_1 < t_f$, where t_1 is such that $z(t_1) = z_1$.

Substituting μ_1, μ_2, m, and k into z_1 it is readily shown that $z_1 < 1$ if

$$V_{r_0} < -c/2. \tag{94}$$

Moreover, for $z > z_1 (t < t_1)$,

$$\text{sign}\,(\ddot{\theta}) = \text{sign}\,(\dot{\theta}_0) \tag{95}$$

and for $z < z_1$ $(t > t_1)$,

$$\text{sign}\,(\ddot{\theta}) = -\,\text{sign}\,(\dot{\theta}_0). \tag{96}$$

It follows then that for $\dot{\theta}_0 > 0$ $(\dot{\theta}_0 < 0)$, $\dot{\theta}(t)$ has a maximum (minimum) at $t = t_1$.

In what concerns the value of $\ddot{\theta}$ at $t = 0$, it is directly obtained from (87) that

$$\ddot{\theta}_0 = -(c + 2V_{r_0})(\dot{\theta}_0/V_{r_0}). \tag{97}$$

Thus, if $V_{r_0} > -c/2$,

$$\text{sign}\,(\ddot{\theta}_0) = -\,\text{sign}\,(\dot{\theta}_0) \tag{98}$$

and if $V_{r_0} < -c/2$,

$$\text{sign}\,(\ddot{\theta}) = \text{sign}\,(\dot{\theta}_0). \tag{99}$$

For $t = t_f$,

1) if $5m + k < 0$, then $\lim_{z\to 0} |\ddot{\theta}| = \infty$; (100)

2) if $5m + k = 0$, then $\lim_{z\to 0} \ddot{\theta} = -(\dot{\theta}_0/r_0)[(c + 2m)/\mu_1^3]$; (101)

3) if $5m + k > 0$, then $\lim_{z\to 0} \ddot{\theta} = 0$. (102)

Substituting m and k into (100), (101), and (102) we obtain

1) if $(V_{r_0}, V_{\theta_0}) \in C_D$, where C_D is a circle defined by $(V_{r_0} + c)^2 + V_{\theta_0}^2 = (2c/3)^2$, then $\lim_{t\to t_f} |\ddot{\theta}| = \infty$;
2) if $(V_{r_0}, V_{\theta_0}) \in$ the circumference of C_D, then $\lim_{t\to t_f} \ddot{\theta} = -(\dot{\theta}_0/r_0)[(c + 2m)/\mu_1^3]$;
3) if $(V_{r_0}, V_{\theta_0}) \notin C_D$, then $\lim_{t\to t_f} \ddot{\theta} = 0$.

In Fig. 6 all the three circles C_c, C_s, and C_D and the straight line $V_{r_0} = -c/2$ are depicted.

For case 1), $(V_{r_0}, V_{\theta_0}) \in C_D$, we can distinguish between three subcases:

a) if $(V_{r_0}, V_{\theta_0}) \in C_s$, then $\lim_{t\to t_f} \ddot{\theta} = [\text{sign}\,(\dot{\theta}_0)]\infty$;
b) if $(V_{r_0}, V_{\theta_0}) \in$ the circumference of C_s, then $\lim_{t\to t_f} \ddot{\theta} = 0$;
c) if $(V_{r_0}, V_{\theta_0}) \notin C_s$, then $\lim_{t\to t_f} \ddot{\theta} = -[\text{sign}\,(\dot{\theta}_0)]\infty$.

We can distinguish now five different zones.

I) For $V_{r_0} > -c/2, (V_{r_0}, V_{\theta_0}) \notin C_D, \in C_c$,
$\dot{\theta}(t_f) = 0, \quad \ddot{\theta}(t_f) = 0$
$\text{sign}\,(\ddot{\theta}) = -\,\text{sign}\,(\dot{\theta}_0)$.
II) For $V_{r_0} > -c/2, (V_{r_0}, V_{\theta_0}) \in C_D, \in C_c$,
$\dot{\theta}(t_f) = 0, \quad \ddot{\theta}(t_f) = -[\text{sign}\,(\dot{\theta}_0)]\infty$
$\text{sign}\,(\ddot{\theta}) = -\,\text{sign}\,(\dot{\theta}_0)$.
III) For $V_{r_0} \in C_s, \in C_c$,
$\dot{\theta}(t_f) = \infty, \quad \ddot{\theta}(t_f) = [\text{sign}\,(\dot{\theta}_0)]\infty$
$\text{sign}\,(\ddot{\theta}) = \text{sign}\,(\dot{\theta}_0)$.
IV) For $V_{r_0} < -c/2, (V_{r_0}, V_{\theta_0}) \notin C_s, \in C_D, \in C_s$,
$\dot{\theta}(t_f) = 0, \quad \ddot{\theta}(t_f) = -[\text{sign}\,(\dot{\theta}_0)]\infty$.
$\dot{\theta}$ has an extremum (maximum for $\dot{\theta}_0 > 0$) for $t = t_1$,
$\text{sign}\,(\ddot{\theta}_0) = \text{sign}\,(\dot{\theta}_0)$.
V) For $V_{r_0} < -c/2, (V_{r_0}, V_{\theta_0}) \in C_D, \in C_c$
$\dot{\theta}(t_f) = 0, \quad \ddot{\theta}(t_f) = 0$;
$\dot{\theta}$ has an extremum for $t = t_1$,
$\text{sign}\,(\ddot{\theta}_0) = \text{sign}\,(\dot{\theta}_0)$.

Representing now $\dot{\theta}$ as a function of t we obtain the five different curves depicted in Fig. 7.

Theorem 3: The commanded acceleration of a missile M pursuing a nonmaneuvering target T according to TPN is $a_M = c\dot{\theta}$ for M starting its course at M_0 such that $(V_{r_0}, V_{\theta_0}) \notin C_s, \in C_c$ is a bounded function of time (zones I, II, IV, V). For $(V_{r_0}, V_{\theta_0}) \in C_s$, (zone III) a_M becomes unbounded at the pursuit end.

Case C) $a > 0, b < 0$: For this case,

$$m > 0 \tag{103}$$

$$k > 0. \tag{104}$$

From the expression (47) of r it is readily seen that for $m/k > 0$, there does not exist any real t for which

$$r = 0. \tag{105}$$

In consequence the following result is obtained.

Theorem 4: A missile M pursuing a nonmaneuvering target T according to TPN, starting at M_0, where

$$V_{r_0} < -2c \tag{106}$$

will not reach the target for any real t.

V. A Particular Case

The system of differential equations (20), (21) has the particular solution

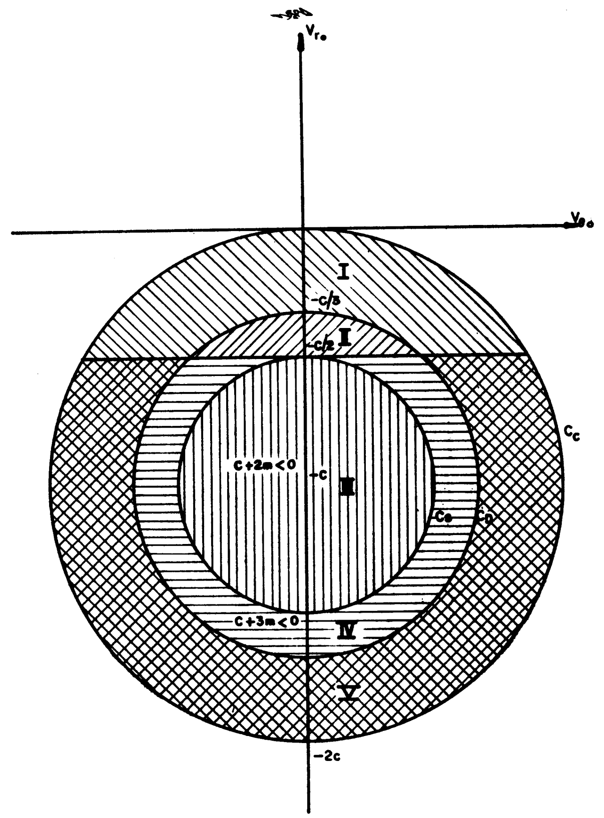


Fig. 6. Zones *I* to *V*.

$$\dot{\theta} = 0 \tag{107}$$

$$\ddot{r} = 0 \tag{108}$$

an can be directly proved by substitution.

In terms of V_r and V_θ this solution corresponds to the case

$$V_\theta = 0 \tag{109}$$

$$V_r = \text{cte.} \tag{110}$$

In the plane V_r, V_θ, depicted in Fig. 8, this particular case corresponds to the points belonging to the straight line $V_\theta = 0$.

For $V_r < 0$ the missile reaches the target without maneuvering

$$a_M = c\dot{\theta} = 0. \tag{111}$$

In previous works [1] the analysis of TPN was restricted to the neighborhood of this particular case

$$V_\theta \simeq 0 \tag{112}$$

$$V_r \simeq \text{cte.} \tag{113}$$

In this case, as previously mentioned, c is defined as

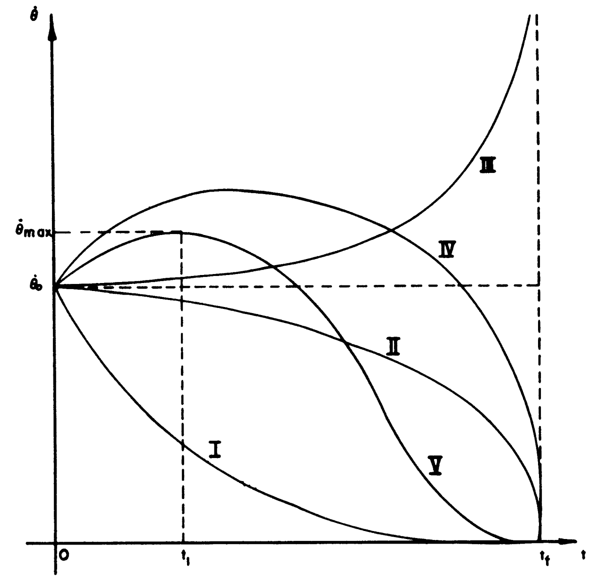
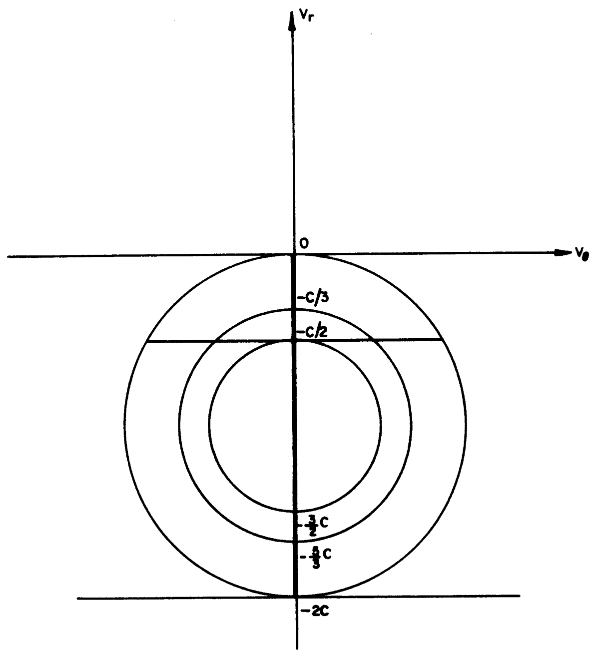


Fig. 7. Line of sight rotational rate versus time for (V_{r_0}, V_{θ_0}) belonging to zones *I* to *V*.

Fig. 8. The case $V_\theta \simeq 0$, $V_r \simeq$ cte.



$$c = -\lambda V_r \tag{114}$$

where λ is the navigation constant.

It follows from the results of Section IV that the missile reaches the target for $V_\theta \cong 0$ if

$$V_r < 0 \tag{115}$$

and if

$$V_r > -2c = 2\lambda V_r \tag{116}$$

that is,

$$\lambda > 1/2. \tag{117}$$

In what concerns the rate of rotation of the line of sight we can distinguish between five different cases:

I) $3 < \lambda$;
II) $2 < \lambda < 3$;
III) $2/3 < \lambda < 2$;
IV) $3/5 < \lambda < 2/3$;
V) $1/2 < \lambda < 3/5$.

These five different cases are depicted in Fig. 7.

VI. Summary and Conclusions

In this study the closed-form solution of the differential equations describing the trajectories of a missile pursuing a nonmaneuvering target according to the TPN law was derived.

The solution was analyzed and a circle was defined where capture can be demonstrated. For the case of initial conditions belonging to the circle of capture the rate of rotation of the line of sight was analyzed and new results were found concerning its boundedness at the pursuit end.

The point of greatest interest in this study is the fact that the analysis of the closed-form solution of TPN enabled us to demonstrate the basic differences existing between the two most utilized forms of proportional navigation. Essentially, when capture of the target can be assured for the entire plane of initial conditions in PPN, except for a well defined particular case, in TPN capture is restricted to the circle of capture defined in this paper.

Appendix I

The solution of

$$(y + mt + n)(dy/dt) + ky = 0 \tag{118}$$

is obtained as follows. Let

$$t = x - n/m. \tag{119}$$

Substituting (119) into (118) and rearranging,

$$dy/dx = -ky/(y + mx). \tag{120}$$

Defining a new variable u instead of y,

$$y = ux. \tag{121}$$

Differentiating (121) with respect to x,

$$dy/dx = u + x(du/dx). \tag{122}$$

Substituting (121) and (122) into (120),

$$x(du/dx) + u = -ku/(u + m). \tag{123}$$

Rearranging,

$$dx/x = \{-(u + m)/[u + (m + k)]u\}\, du. \tag{124}$$

Integrating (124) with initial conditions $u = u_0$, $x(t = 0) = x_0 = n/m$,

$$\log\,[(u/u_0)^{-m/(m+k)}[(u + m + k)/(u_0 + m + k)]^{-k/(m+k)}] = \log(x/x_0) \tag{125}$$

thus

$$(u/u_0)^{-m/(m+k)}[(u + m + k)/(u_0 + m + k)]^{-k/(m+k)} = x/x_0. \tag{126}$$

Substituting u from (121) into (126) and rearranging, with $y_0 = u_0x_0$,

$$(x/x_0)(y/y_0)^{-m/(m+k)}\{[y + (m + k)x]/[y_0 + (m + k)x_0]\}^{-k/(m+k)} = x/x_0. \tag{127}$$

Eliminating x/x_0 and elevating to $-(m + k)/k$,

$$(y/y_0)^{m/k}\{[y + (m + k)x]/[y_0 + (m + k)x_0]\} = 1. \tag{128}$$

Appendix II

The solution for r is obtained as follows. Let

$$z = y/y_0. \tag{129}$$

Rearranging (128) and substituting y by z yields

$$x = pz^{-m/k} - qz \tag{130}$$

where

$$p = y_0/(m + k) + x_0 \tag{131}$$

and

$$q = y_0/(m + k). \tag{132}$$

Substituting x from (130) into (119), and the corresponding value of t so obtained into (35) gives us

$$r = mpz^{-m/k} + qkz \tag{133}$$

where $kq = y_0 - mq$.

From (27) and (41),

$$m = -c + \sqrt{(V_{r_0} + c)^2 + V_{\theta_0}^2}. \tag{134}$$

thus from (40),

$$k = c + \sqrt{(V_{r_0} + c)^2 + V^2_{\theta_0}} \tag{135}$$

and

$$m + k = 2\sqrt{(V_{r_0} + c)^2 + V^2_{\theta_0}}\ . \tag{136}$$

Substituting now x_0 from (45) and y_0 from (46) into (131) and (132) and rearranging,

$$p = (r_0 m + nk)/[m(m + k)] \tag{137}$$

and

$$q = (r_0 - n)/(m + k). \tag{138}$$

Substituting k from (40) into (37) and the value of a so obtained into (38) it follows that

$$nk = b. \tag{139}$$

Multiplying (131) by m and replacing nk from (139) gives

$$mp = (mr_0 + b)/(m + k). \tag{140}$$

Substituting now m, b, and $m + k$ by their values given in (134), (34), and (136), respectively, and rearranging,

$$mp = r_0\{[(V_{r_0} + c) + \sqrt{(V_{r_0} + c)^2 + V^2_{\theta_0}}]\}/ [2\sqrt{(V_{r_0} + c)^2 + V^2_{\theta_0}}]. \tag{141}$$

Let us define

$$\nu = V_{r_0} + c/|V_{\theta_0}|. \tag{142}$$

Substituting into (141),

$$mp = r_0\mu_1 \tag{143}$$

where

$$\mu_1 = (\nu + \sqrt{\nu^2 + 1})/(2\sqrt{\nu^2 + 1})\ . \tag{144}$$

Substituting now (139) into (138) and rearranging,

$$qk = (kr_0 - b)/(m + k) \tag{145}$$

from which it follows that

$$kq = r_0\mu_2 \tag{146}$$

where

$$\mu_2 = (-\nu + \sqrt{\nu^2 + 1})/(2\sqrt{\nu^2 + 1}). \tag{147}$$

Substituting now mp and qk from (143) and (146), respectively, into (138)

$$r = r_0(\mu_1 z^{-m/k} + \mu_2 z). \tag{148}$$

This is the solution for r as a function of z. z is implicity defined in (130) with x defined in (119).

Appendix III

The solution for $\dot{\theta}$, the rate of rotation of the line of sight, and θ, the aspect angle, is obtained as follows:

$$dy/dt = dy/dx = (dy/dz)(dz/dx). \tag{149}$$

Differentiating (129),

$$dy/dt = y \cdot dz/dx. \tag{150}$$

Differentiating once again,

$$d^2y/dt^2 = y_0(d^2z/dx^2). \tag{151}$$

Differentiating now (35) twice with respect to t,

$$d^2r/dt^2 = d^2y/dt^2. \tag{152}$$

Substituting (151) into (152),

$$\ddot{r} = y_0(d^2z/dx^2). \tag{153}$$

Differentiating now (130) with respect to x and rearranging,

$$dz/dx = -kz^{m/k+1}/r_0(\mu_1 + \mu_2 z^{m/k+1}). \tag{154}$$

Differentiating once again with respect to x and operating,

$$\begin{aligned} d^2z/dx^2 &= d/dz\,(dz/dx) \cdot dz/dx \\ &= k(m + k)\mu_1 z^{2m/k+1}/r_0^2(\mu_1 + \mu_2 z^{m/k+1})^3. \end{aligned} \tag{155}$$

Substituting (155) into (153),

$$\ddot{r} = V^2_{\theta_0}/r_0^2\, z^{2m/k+1}/(\mu_1 + \mu_2 z^{m/k+1})^3. \tag{156}$$

Rearranging (20),

$$\dot{\theta}^2 = \ddot{r}/r. \tag{157}$$

Substituting r from (148) and $\ddot{r}$ from (156), respectively, into (157) and squaring the root,

$$\dot{\theta} = \dot{\theta}_0\, z^{3(m/k)/2+1}/(\mu_1 + \mu_2 z^{m/k+1})^2. \tag{158}$$

This is the solution for the rate of rotation of the line of sight as a function of z defined in (130).

θ is obtained as follows:

$$\theta - \theta_0 = \int_0^t \dot{\theta}(t)\,dt = \int_{x_0}^{x} \dot{\theta}(x)\,dx. \quad (159)$$

Changing the variable x by z, where for $x = x_0$, $y = y_0$ and consequently $z = 1$,

$$\theta - \theta_0 = \int_1^z [\dot{\theta}(z)\,(dx/dz)]\,dz. \quad (160)$$

Rearranging (154),

$$dx/dz = -r_0(\mu_1 + \mu_2 z^{m/k+1})/kz^{m/k+1}. \quad (161)$$

Substituting (158) and (161) into (160),

$$\theta - \theta_0 = -V_{\theta_0}/k \int_1^z z^{(m/k-1)/2}/\mu_1 + \mu_2 z^{m/k+1}\,dz. \quad (162)$$

Let

$$s = z^{m/k+1} \quad (163)$$

thus

$$dz = ks^{-m/(m+k)}/(m+k)\,ds. \quad (164)$$

Substituting into (162),

$$\theta - \theta_0 = -V_{\theta_0}/(m+k) \int_1^s s^{-1/2}/(\mu_1 + \mu_2 s)\,ds. \quad (165)$$

Hence

$$\theta - \theta_0 = -2\,\mathrm{sign}\,(V_{\theta_0})\arctan\left[(\mu_2/\mu_1)\,s\right]^{1/2}\Big|_1^s. \quad (166)$$

Substituting s by z as defined in (163),

$$\theta = -2\,\mathrm{sign}\,(V_{\theta_0})\arctan\left[(\mu_2/\mu_1)\,z^{m/k+1}\right]^{1/2} + \theta_f \quad (167)$$

where

$$\theta_f = \theta_0 + 2\,\mathrm{sign}\,(V_{\theta_0})\arctan(\mu_2/\mu_1)^{1/2}. \quad (168)$$

Acknowledgment

The author would like to express appreciation for helpful and stimulating discussions of this topic with A. Davidovitch and Dr. O. Jacusiel.

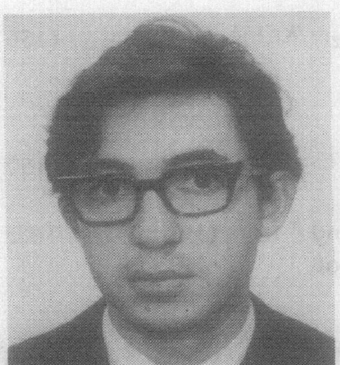
References

[1] S.A. Murtaugh, and H.E. Criel, "Fundamentals of proportional navigation," *IEEE Spectrum*, vol. 3, pp. 75-85, Dec. 1966.

[2] M. Guelman, "A qualitative study of proportional navigation," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-7, pp. 637-643, July 1972.

[3] J.J. Jerger, *Systems Preliminary Design.* Princeton, N.J.: Van Nostrand, 1960.

[4] E. Goursat, *Cours d'Analyse Mathématique*, vol. 2. Paris: Gauthier Villars, 1949.



Mauricio Guelman was born in Montevideo, Uruguay, on March 5, 1942. He was educated at the Facultad de Ingenieria de Montevideo and received the D.E.A. and the Dr. en Electronique degrees from the University of Paris, France, in 1967 and 1968, respectively.

In 1966 he was granted a French Government fellowship. From 1966 to 1967 he was at Saclay, France, and from 1967 to 1968 at the Laboratoire d'Automatique Theorique of the Faculty of Sciences, Paris, where he did research on automatic control systems. Since 1969 he has been with the Armament Development Authority, Ministry of Defence, Tel-Aviv, Israel.