2019 18th European Control Conference (ECC)
Napoli, Italy, June 25-28, 2019

A computationally efficient feedback solution for a particular case of the game of two cars

Aditya Chaudhari, Rishabh Chourasia and Debraj Chakraborty

***Abstract*— In this paper, a time-optimal feedback solution to the game of two cars is obtained for the case where the pursuer is faster and more agile than the evader. It is shown that the saddle point strategies for both the pursuer and the evader are coincident with one of the common tangents from the minimum radius turning circles of the pursuer to the minimum radius turning circles of the evader when the evader is distant enough from the pursuer. Four valid tangents are identified using geometry and a 2×2 matrix game is formulated corresponding to these tangents. The solution of this game, at each instant, provides feedback strategies for the pursuer and the evader. The computations required to implement this law are insignificant and hence the proposed feedback law is suitable for real time implementation.**

I. Introduction

The game of two cars which was first introduced by Issacs in [1] considers the problem of time optimal pursuit evasion between two Dubins vehicles. Since then variations of the game of two cars have been studied in [2]–[4]. In most of the papers [2], [5], optimal inputs are computed by solving non-linear algebraic equations numerically. This requires significant computation and hence are difficult to implement as instantaneous feedback. In this paper we provide a time optimal feedback law for the special case when the pursuer is faster and more agile than the evader. In this case, capture by the pursuer is always guaranteed for all possible configurations of pursuer and evader [6]. Our law can be implemented in real time as it involves only evaluation of and comparison with closed form algebraic equations.

The optimal paths and control law synthesis for a single Dubins vehicle have been described in [7]–[10]. Also, the reachable sets of the Dubins vehicle are characterized in [11], [12] and we use them extensively in this paper. Reachable sets have also been used to analyze differential games since [6], [13]. Reachable sets have found applications in obtaining feedback strategies [14], [15] and also in multi-player pursuit-evasion games [16]–[18].

In this paper, we analyze a special case of the game of two cars where the pursuer is faster and agile than the evader. We first show that the saddle point strategies are coincident with the common tangent to the minimum turning radius circles if the distance between pursuer and evader is large compared to their turning radius. For this we use Pontryagin's minimum principle and the reachable sets of Dubins vehicle, described in [11], [12]. Since both the vehicles are restricted to have a minimum turning radius, we draw a clockwise minimum radius turning circle and a anti-clockwise minimum radius turning circle for both the pursuer and the evader. This results in four common tangents between each pursuer and evader circle pairs and sixteen tangents in total. Out of these four common tangents for every pair of circles between the evader and the pursuer, we obtain one valid tangent using geometrical arguments. By computing the time to capture along each of the valid tangents a 2×2 matrix game is formulated. The min-max solution of the matrix game gives the strategy for the pursuer while the max-min solution gives the strategy for the evader. The matrix game is solved at each instant of time to obtain the feedback strategies for the differential game.

To verify and validate our control law we solve the game of two cars numerically, using the algorithms described in [19], [20] (based on discretization) using IPOPT [21]. This simulations confirm that the pursuit evasion game takes place along a common tangent to the minimum turning radius circles and the trajectory predicted by the proposed law coincides with those obtained by the numerical computation of [19]. However, obtaining the open-loop solution numerically at every instant on-line is not a feasible option for real time implementations.

II. Problem Formulation and Preliminaries

Consider a pursuer P and an evader E following the equations:

$$\dot{x}_i(t) = v_i(t)\cos\theta_i(t) \quad ; \quad \dot{y}_i(t) = v_i(t)\sin\theta_i(t) \qquad (1)$$
$$\dot{\theta}_i(t) = v_i(t) w_i(t)$$

where $i \in \{p, e\}$. The subscript p corresponds to the pursuer while e corresponds to the evader. The pursuer (evader) can control its velocity $v_i(t)$ in direction $\theta_i(t)$ and the angular velocity $w_i(t)$. We denote by $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^n)$ the set of continuous functions from positive real line $\mathbb{R}^+$ to $\mathbb{R}^n$. Let $\mathbf{z_i}(t) = [x_i(t) \;\; y_i(t)]^\top \in \mathbb{R}^2, \mathbf{z_i} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^2)$ denote the position of the pursuer (evader) in the $x-y$ plane at time t. Also, let $\theta_i(t)$ be the orientation of the pursuer (evader) in the $x-y$ plane, measured in anti-clockwise direction with respect to the $x-$axis at time t. The complete state vector of the pursuer at time t is given by $\mathbf{p}(t) = [x_p(t)\, y_p(t)\, \theta_p(t)] \in \mathbb{R}^3, \mathbf{p} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^3)$ while that of the evader is given by $\mathbf{e}(t) = [x_e(t)\, y_e(t)\, \theta_e(t)]^\top \in \mathbb{R}^3, \mathbf{e} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^3)$. We also denote the restriction of the pursuer's (evader's) trajectory in $x-y$ plane corresponding to the trajectory $\mathbf{p}$ ($\mathbf{e}$) by $\mathbf{z}_i = \mathbf{p}_{|\mathbb{R}^2}$ ($\mathbf{e}_{|\mathbb{R}^2}$). Let the initial state of the pursuer be denoted by $\mathbf{p}(0) = \mathbf{p_0} = [x_{p_0}\, y_{p_0}\, \theta_{p_0}]^\top$ and that of the evader by $\mathbf{e}(0) =$

The authors are with the Department of Electrical Engineering, Indian Institute of Technology Bombay, India. Email: {adityac,dc}@ee.iitb.ac.in, rish033147@gmail.com

$\mathbf{e_0} = [x_{e_0}\, y_{e_0}\, \theta_{e_0}]^\top$. Also, let $d_{pe}(t) = ||\mathbf{z_p}(t) - \mathbf{z_e}(t)||_2$ be the distance between pursuer and evader at time instant t and $d_{pe}(0) := d^0_{pe}$. We denote the input of the pursuer (evader) at time t by $\mathbf{u_i}(t) = [v_i(t)\ w_i(t)]^\top \in \mathbb{R}^2$ where $i \in \{p, e\}$. Also, $v_i(t) \in V_i$ where $V_i = \{v_i(t) \in \mathbb{R} : 0 \le v_i(t) \le v_{i_m}\}$ and $w_i(t) \in W_i$ where $W_i = \{w_i(t) \in \mathbb{R} : |w_i(t)| \le w_{i_m}\}$ for $i \in \{p, e\}$. These restrictions limit the maximum forward velocity with which the pursuer and evader can move and also limits the rate at which the vehicle can change direction. We define the set of feasible inputs for the pursuer and the evader as $\mathbf{U_i} := \{\mathbf{u}_i(t) : v_i(t) \in V_i \,\text{and}\, w_i(t) \in W_i\}$ where $i \in \{p, e\}$. If the input of the pursuer (evader) $\mathbf{u}_i(t) \in \mathbf{U_i}$ for all t then we write $\mathbf{u}_i \in \mathcal{U}_i$. If the pursuer (evader) sets $w_i(t) = +w_{i_m}$, then it moves along a circle of radius $r_i = 1/w_{i_m}$ in anti-clockwise direction while if it applies an input of $w_i(t) = -w_{i_m}$ then it moves in clockwise direction along the circle of radius r_i.

Definition 1. Clockwise pursuer-circle (evader-circle), $C_i(t_1)$, at time t_1 is the clockwise circle of radius $r_i = \frac{1}{w_{i_m}}$ and center $(x_c(t_1) = x(t_1) + \sin(\theta(t_1))/w_{i_m}, y_c(t_1) = y(t_1) - \cos(\theta(t_1))/w_{i_m})$ that the pursuer (evader) follows when $w_i(t) = -w_{i_m}\ \forall\, t \ge t_1$ where $i \in \{p, e\}$.

Definition 2. Anti-clockwise pursuer-circle (evader-circle), $A_i(t_1)$, at time t_1 is the anti-clockwise circle of radius $r_i = \frac{1}{w_{i_m}}$ and center $(x_c(t_1) = x(t_1) - \sin(\theta(t_1))/w_{i_m}, y_c(t_1) = y(t_1) + \cos(\theta(t_1))/w_{i_m})$ that the pursuer (evader) follows when $w_i(t) = +w_{i_m}\ \forall\, t \ge t_1$ where $i \in \{p, e\}$.

These circles for a particular position of pursuer (evader) are shown in Figure 1 and will be called the pursuer-circles (evader-circles). If the input $w_i(t) = 0$ and $v_i(t) \in (0, v_{i_m}]$ for $i \in \{p, e\}$ then the pursuer (evader) moves in a straight line. In order to design feedback laws we make use of the common tangents from the pursuer-circles to the evader-circles. Note that the pursuer-circles (evader-circles) at time t depend only on the position and orientation of the pursuer (evader) at time instant t. Let $OC(t) := \{A_p(t), C_p(t), A_e(t), C_e(t)\}$ denote the set of pursuer-circles and evader-circles at time t. A PE-pair is a pair of circles with one circle belonging to the pursuer-circles and the other circle belonging to the evader-circles. Thus in total we have four PE-pairs. Let, $PE(t) := \{\{C_p(t), C_e(t)\},\ \{C_p(t), A_e(t)\},\ \{A_p(t), C_e(t)\},$ $\{A_p(t), A_e(t)\}\}$ be the set of PE-pairs at time t. Between the two circles belonging to an PE-pair whose centers are at least $r_p + r_e$ away from each other there will be four tangents. These tangents have been shown in between the PE-pair $\{A_p(t), C_e(t)\}$ in Figure 1. We assign direction to the tangents from the pursuer to evader and we call them directed common tangents.

Definition 3. Valid common tangent for a $PE-$pair is a directed common tangent whose orientation matches with both the pursuer circle and the evader circle in the $PE-$pair.

In Figure 1 only the tangent shown in red color is a valid tangent for pair $\{A_p(t), C_e(t)\}$.

Definition 4. If a pursuer's (evader's) trajectory up to time T is such that it traverses one of the pursuer-circles (evader-circles) in time interval $[0, t')$ where $t' < T$ and then traverses one of the tangent to that pursuer-circle (evader-circle) in time interval $[t', T]$ then such a trajectory is of the type CS (circle and straight line).

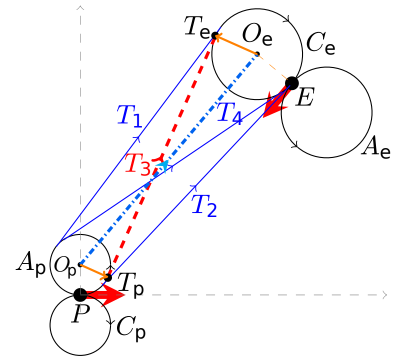

Fig. 1: Common Tangents $\{A_p(t), C_e(t)\}$

In order to guarantee the capture of the evader by the pursuer we impose following restriction on evaders input.

Assumption 5. *Maximum velocities of the pursuer and the evader satisfy $v_{p_m} > v_{e_m}$, while the maximum turning rates are such that $w_{p_m} > w_{e_m}$.*

In this paper, the pursuers objective is to intercept the evader in minimum possible time, while that of the evader is to avoid interception by the pursuer for as long as possible. The complete state of the game is described by $\mathbf{x}(t) = [\mathbf{p}(t)^\top\, \mathbf{e}(t)^\top]^\top \in \mathbb{R}^6, \mathbf{x} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^6)$. For capture we require that only the x and y coordinates of both the pursuer and evader must match. We do not impose the restriction that the final orientation of the pursuer and the evader must be the same. Thus, the condition at the time of capture T is

$$\psi(\mathbf{x}(T)) := \left.\begin{pmatrix} x_p(t) - x_e(t) \\ y_p(t) - y_e(t) \end{pmatrix}\right|_{t=T} = 0 \tag{2}$$

Thus the time of capture T at which the game terminates i.e. the cost function in the game of two cars, is defined by $T = \inf\{t \in R^+ : \psi(\mathbf{x}(t)) = 0\}$. The pursuer tries to minimize T while the evader tries to maximize it using feedback strategies $\mathbf{u_p} := \gamma_\mathbf{p}(\mathbf{x}) \in \mathcal{U}_p$ and $\mathbf{u_e} := \gamma_\mathbf{e}(\mathbf{x}) \in \mathcal{U}_e$. The time to capture is a function of feedback strategy pair $(\gamma_\mathbf{p}(\mathbf{x}), \gamma_\mathbf{e}(\mathbf{x}))$ and we denote it by $T(\gamma_\mathbf{p}(\mathbf{x}), \gamma_\mathbf{e}(\mathbf{x}))$. The pursuer must guard against the worst-case strategies of the evader. Hence, the minimum time capture problem for the pursuer is a $\min - \max$ time-optimal problem. So the pursuer's problem is:

Problem 6. Find $\mathbf{u_p} = \gamma^*_\mathbf{p}(\mathbf{x})$ which solves $\gamma^*_\mathbf{p}(\mathbf{x}) = \underset{\gamma_\mathbf{p}}{\text{argmin}}\, \underset{\gamma_\mathbf{e}}{\max}\, T(\gamma_\mathbf{p}(\mathbf{x}), \gamma_\mathbf{e}(\mathbf{x}))$

Similarly, the evader must guard against every possible strategy of the pursuer. Thus, the maximum time evasion problem for the evader is to find the $\max - \min$ strategy of the evader.

Problem 7. Find $\mathbf{u_e}(t) = \gamma^*_{\mathbf{e}}(\mathbf{x})$ which solves $\gamma^*_{\mathbf{e}}(\mathbf{x}) = \underset{\gamma_{\mathbf{e}}}{\text{argmax}} \min_{\gamma_{\mathbf{p}}} T(\gamma_{\mathbf{p}}(\mathbf{x}), \gamma_{\mathbf{e}}(\mathbf{x})$

Such problems have been studied extensively in the theory of differential games where the solutions of both Problem 6 and Problem 7 are characterized in terms of saddle-point strategies and the value of the game [1], [22], [23].

Definition 8. A feedback strategy pair $(\gamma^*_{\mathbf{p}}, \gamma^*_{\mathbf{e}})$ is said to be a saddle-point equilibrium if $T(\gamma^*_{\mathbf{p}}, \gamma_{\mathbf{e}}) \le T(\gamma^*_{\mathbf{p}}, \gamma^*_{\mathbf{e}}) \le T(\gamma_{\mathbf{p}}, \gamma^*_{\mathbf{e}}) \quad \forall\ \gamma_e(\mathbf{x}) \in \mathbf{U_e}, \gamma_p(\mathbf{x}) \in \mathbf{U_p}$ and the value of the game, if it exists, is $T^* = T(\gamma^*_{\mathbf{p}}, \gamma^*_{\mathbf{e}})$.

In the game of two cars considered above, under Assumption 5, the following result holds.

Theorem 9. *[6] If Assumption 5 holds, there exists a pursuer input $\mathbf{u_p} := \gamma_p(\mathbf{x})$ such that for all $\mathbf{u_e} := \gamma_e(\mathbf{x})$, capture is guaranteed i.e. $\psi(\mathbf{x}(T)) = 0$ for some $T < \infty$.*

Since the capture is guaranteed and the Hamiltonian (4) is separable in pursuer's input and evader's input, the existence of saddle point strategies $\left(\gamma^*_{\mathbf{p}}, \gamma^*_{\mathbf{e}}\right)$ follows. This implies that $\mathbf{u}^*_{\mathbf{p}} := \gamma^*_p(\mathbf{x})$ and $\mathbf{u}^*_{\mathbf{e}} := \gamma^*_e(\mathbf{x})$ are the solutions of Problem 6 and 7 respectively with $T^* = T(\gamma^*_{\mathbf{p}}, \gamma^*_{\mathbf{e}}) < \infty$.

III. Min-Max time trajectories

In this section we show that the trajectories generated by the saddle-point strategies of the pursuer and the evader must coincide with one of the common tangents from pursuer-circles to the evader-circles.

A. Reachable Sets

First we define the reachable sets of pursuer and evader and also list some properties of reachable sets which will be used later.

Definition 10. [11] The reachable set of an pursuer, denoted by $R_p(\mathbf{p_0}, \bar{t}) \subset \mathbb{R}^2$, at time $\bar{t}$ from initial state $\mathbf{p}(0) = \mathbf{p_0}$, is the set of all points that can be reached in time $t \le \bar{t}$ by applying inputs $\mathbf{u_p} \in \mathcal{U}_p$.

$$\begin{aligned} R_p(\mathbf{p_0}, \bar{t}) \quad = \quad & \{\mathbf{z} \in \mathbb{R}^2 : \exists\ \mathbf{u_p} \in \mathcal{U}_p \text{ and corresponding} \\ & \text{trajectory } \mathbf{p} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^3) \text{ s.t.} \\ & \mathbf{p}_{|\mathbb{R}^2}(t) = \mathbf{z} \text{ for } t \le \bar{t} \ \text{ and } \mathbf{p}(0) = \mathbf{p_0}\} \end{aligned}$$

The reachable set of evader $R_e(\mathbf{e_0}, \bar{t}) \subset \mathbb{R}^2$ is defined analogously. The reachable set for the Dubins vehicle has been studied in [11], [12]. The points inside the pursuer (evader) circles can be reached in minimum time by CC (Circle-Circle) types of curves. The points external to the pursuer (evader) circles can be reached in minimum time by CS type of curves. The external boundary of the reachable set of pursuer and evader at time $\bar{t}$ is denoted by $\partial R_p(\mathbf{p_0}, \bar{t})$ and $\partial R_e(\mathbf{e_0}, \bar{t})$ respectively. It is known [12] that the points on $\partial R_p(\mathbf{p_0}, \bar{t})$ $(\partial R_e(\mathbf{e_0}, \bar{t}))$ at time $\bar{t}$ can be reached only by the trajectories of the type CS for $\bar{t} \ge 2\pi r_p/v_{p_m}$ $(\bar{t} \ge 2\pi r_e/v_{e_m})$. Thus $\partial R_p(\mathbf{p_0}, \bar{t})$ $(\partial R_e(\mathbf{e_0}, \bar{t}))$ at $\bar{t} \ge 2\pi r_p/v_{p_m}$ $(\bar{t} \ge 2\pi r_e/v_{e_m})$ is comprised of two portions. The first portion is characterized by trajectories which begin on the anti-clockwise circle and then follow straight line. The second portion is characterized by trajectories which begin on the clockwise circle and then follow straight line. Consider the pursuer with initial state vector $\mathbf{p}(0) = [x_0\ y_0\ \theta_0]$. The trajectories $\mathbf{p} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^3)$ corresponding to input

$$\begin{aligned} w_p(t) \quad &= \quad +w_{p_m} \quad t \in [0, t_1) \\ &= \quad 0 \quad t \in [t_1, \bar{t}] \end{aligned}$$

and $v_p(t) = v_{p_m} \quad t \in [0, \bar{t}]$ will initially follow the anti-clockwise circle and then travel on a tangent to the anti-clockwise circle. The trajectory can be parameterized by the switching time t_1 and can be obtained by integrating (1) as

$$\begin{aligned} x_{fl}(\bar{t}) \quad &= \quad x_0 + (\sin(\theta_0 + v_{p_m} w_{p_m} t_1) - \sin(\theta_0))/w_{p_m} \\ &\qquad + v_{p_m} \cos(\theta_0 + v_{p_m} w_{p_m} t_1)(\bar{t} - t_1) \\ y_{fl}(\bar{t}) \quad &= \quad y_0 - (\cos(\theta_0 + v_{p_m} w_{p_m} t_1) - \cos(\theta_0))/w_{p_m} \\ &\qquad + v_{p_m} \sin(\theta_0 + v_{p_m} w_{p_m} t_1)(\bar{t} - t_1) \end{aligned} \tag{3}$$

Using these expressions the left reachable set of pursuer is defined as $R^l_p(\mathbf{p_0}, \bar{t}) = \{[x\ y]^\top \in \mathbb{R}^2 | x = x_{fl}(\bar{t}) \text{ and } y = y_{fl}(\bar{t})\ \forall\ t_1 \le \bar{t}\}$. The left reachable set for the evader $R^l_e(\mathbf{e_0}, \bar{t})$ is defined analogously. The left reachable set is shown in Figure 2. The right reachable sets for the pursuer $R^r_p(\mathbf{p_0}, \bar{t})$ and evader $R^r_e(\mathbf{e_0}, \bar{t})$ are defined similarly by the trajectories which first travel on the clockwise circle and then on the tangent to the clockwise circle. The right reachable set is shown in Figure 3. The left reachable sets and right reachable sets of the pursuer (evader) are the subsets of the reachable set $R_p(\mathbf{p_0}, \bar{t})$ $(R_e(\mathbf{e_0}, \bar{t}))$. The union of $R^l_p(\mathbf{p_0}, \bar{t})$ and $R^r_p(\mathbf{p_0}, \bar{t})$ is shown in Figure 4. The boundary of right reachable set of pursuer (evader) is denoted by $\partial R^r_p(\mathbf{p_0}, \bar{t})$ $(\partial R^r_e(\mathbf{e_0}, \bar{t}))$ and that of the left reachable set by $\partial R^l_p(\mathbf{p_0}, \bar{t})$ $(\partial R^l_e(\mathbf{e_0}, \bar{t}))$. Define, $R^b_p(\mathbf{p_0}, \bar{t}) := \partial R^r_p(\mathbf{p_0}, \bar{t}) \cup \partial R^l_p(\mathbf{p_0}, \bar{t})$ and $R^b_e(\mathbf{e_0}, \bar{t}) := \partial R^r_e(\mathbf{e_0}, \bar{t}) \cup \partial R^l_e(\mathbf{e_0}, \bar{t})$.

Theorem 11. *Let T and $\mathbf{z}$ be the time of capture and point of capture respectively if the pursuer and evader use saddle point strategies. If the initial distance between pursuer and evader $d^0_{pe} \ge 2r_e + 2\pi r_e(v_{p_m}/v_{e_m})$, then $\mathbf{z} \in R^b_e(\mathbf{e_0}, T) \cap R^b_p(\mathbf{p_0}, T)$.*

Remark 12. The proof of Theorem 11 is not given due to lack of space. The detailed proof can be found here[1]. The idea of the proof is as follows. For capture, the containment of evader's reachable set inside the pursuer's reachable set is a necessary condition. However, since we are using feedback strategies we need to consider containment of evaders reachable set in some special subsets of pursuers reachable set. Both the left reachable set and the right reachable set satisfy the conditions that characterize these subsets. And hence the capture occurs at boundary of these sets. The distance condition is added to avoid complications arising from possible capture inside the pursuer circles.

Corollary 13. *If the initial distance $d^0_{pe} \ge 2r_e + 2\pi r_e(v_{p_m}/v_{e_m})$, then the saddle-point strategies of the*

[1]https://www.ee.iitb.ac.in/~dc/ECC2019_2.pdf

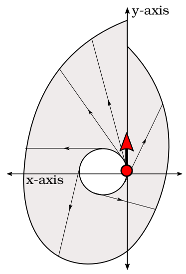

Fig. 2: Left Reachable Set

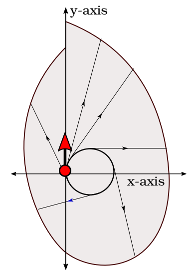

Fig. 3: Right Reachable Set

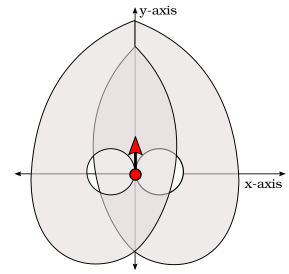

Fig. 4: $R_p^r(\mathbf{p_0}, \bar{t}) \cup R_p^l(\mathbf{p_0}, \bar{t})$

evader and the pursuer are of the type CS that is a circle and straight line.

Proof: All the points in the set $R_e^b(\mathbf{e_0}, T) \cap R_p^b(\mathbf{p_0}, T)$ are characterized by the curves of the type CS for both, the pursuer and the evader. Since the capture point $\mathbf{z} \in R_e^b(\mathbf{e_0}, T) \cap R_p^b(\mathbf{p_0}, T)$ the claim follows.

B. Analysis using Hamiltonian

For the system given by (1) the Hamiltonian is

$$\begin{aligned} H &= 1 + \lambda_{p_x} v_p \cos\theta_p + \lambda_{p_y} v_p \sin\theta_p + \lambda_{p_\theta} v_p w_p \quad (4) \\ &\quad + \lambda_{e_x} v_e \cos\theta_e + \lambda_{e_y} v_e \cos\theta_e + \lambda_{e_\theta} v_e w_e \end{aligned}$$

where $[\lambda_{p_x}\ \lambda_{p_y}\ \lambda_{p_\theta}]^\top$ denotes the adjoint vector corresponding to the pursuer. Also, let $[\lambda_{e_x}\ \ \lambda_{e_y}\ \ \lambda_{e_\theta}]^\top$ denote the adjoint vector corresponding to the evader. Define, $\lambda_p = \sqrt{\lambda_{p_x}^2 + \lambda_{p_y}^2}$, $\lambda_e = \sqrt{\lambda_{e_x}^2 + \lambda_{e_y}^2}$, $\phi_p = \tan^{-1}(\lambda_{p_y}/\lambda_{p_x})$ and $\phi_e = \tan^{-1}(\lambda_{e_y}/\lambda_{e_x})$. Thus we can write (4) as

$$\begin{aligned} H &= 1 + v_p \lambda_p \cos(\theta_p - \phi_p) + \lambda_{p\theta} v_p w_p \\ &\quad + v_e \lambda_e \cos(\theta_e - \phi_e) + \lambda_{e_\theta} v_e w_e \quad (5) \end{aligned}$$

By Theorem 9 the capture time $T \leq \infty$. Let $\mu = [\mu_x\ \mu_y]^\top$ be the undetermined constants and $\psi(\mathbf{x}(T))$ is defined by (2). Thus, the complete final time constraint is $\Phi(\mathbf{x}(T)) = \mu^\top \psi(\mathbf{x}(T))$ (see [23]). The adjoint system for pursuer (evader) is given as

$$\begin{aligned} \dot{\lambda}_{i_x} = 0 \qquad & \dot{\lambda}_{i_y} = 0 \\ \dot{\lambda}_{i_\theta} &= -v_i \left[-\lambda_{i_x} \sin\theta_i + \lambda_{i_y} \cos\theta_i \right] \quad (6) \\ &= v_i \lambda_i \sin(\theta_i - \phi_i) \end{aligned}$$

for $i \in \{p, e\}$.

Lemma 14. *Both the pursuer and evader use input policy $\mathbf{u}_i \in \mathcal{U}_i$ such that $v_i(t) = v_{i_m}\ \forall\ t \in [0, T]$.*

Lemma 15. *Any optimal path for the pursuer (evader) is the concatenation of arcs of circles of radius r_p (r_e) and line segments, all parallel to some fixed direction ϕ_p (ϕ_e).*

Proof: The Hamiltonian is affine in $w_p(t)$ and $w_e(t)$. If $\lambda_{p_\theta}(t) = 0$ for all $t \in [t_1, t_2] \subseteq [0, T]$ then from (6) we must have $\dot{\lambda}_{p_\theta}(t) = 0 = v_p(t)\lambda_p(t)\sin(\theta_p(t) - \phi_p)$. Thus $\theta_p(t) = \phi_p$ or $\theta_p(t) = \phi_p + \pi$ for all $t \in [t_1, t_2] \subseteq [0, T]$ and the path is a line segment with direction ϕ_p. Thus $w_p(t) = 0$ for all $t \in [t_1, t_2] \subseteq [0, T]$. If $|\lambda_{p_\theta}| > 0$, this would imply that $w_p(t) = \pm w_{p_m}$ and the path would be an arc of circle $A_p(t)$ or $C_p(t)$. Thus H will be minimized with respect to $w_p(t)$ only if $w_p(t) = 0$ or $w_p(t) = \pm w_{p_m}$. Similar arguments can be used to prove the claim for the evader, where instead of minimizing H we need to maximize H with respect to $w_e(t)$.

Proposition 16. *The straight line paths that are followed by both pursuer and evader are parallel to each other i.e. $\phi_p = \phi_e$.*

Proof: From Hamiltonian analysis (see [23]) $\lambda_{p_x}(T) = \frac{\partial \Phi}{\partial x_p} = \mu_x$, $\lambda_{p_y}(T) = \frac{\partial \Phi}{\partial y_p} = \mu_y$, $\phi_p(T) = \tan^{-1}(\mu_y/\mu_x)$. Similarly we can show, $\lambda_{e_x}(T) = \frac{\partial \Phi}{\partial x_e} = -\mu_x$, $\lambda_{e_y}(T) = \frac{\partial \Phi}{\partial y_e} = -\mu_y$, $\phi_e(T) = \tan^{-1}(\mu_y/\mu_x)$. Further, from (6), $\lambda_{p_x}, \lambda_{p_y}, \lambda_{e_x}, \lambda_{e_y}$ are constants. Thus we can conclude that $\phi_p(t) = \phi_e(t) = \tan^{-1}(\mu_y/\mu_x)$ for all time t.

Thus from the Hamiltonian formalism we have concluded that the trajectories are either optimal circles or straight lines parallel to some fixed line.

Theorem 17. *If $d_{pe}^0 \geq 2r_e + 2\pi r_e(v_{p_m}/v_{e_m})$ then the saddle-point strategies of the pursuer and the evader result in pursuer and evader trajectories being coincident to circles in an PE-pairs and one of the common tangents of that pair.*

IV. Feedback law using geometry

In this section we design algorithms to select an appropriate tangent which is the open-loop representation of feedback saddle-point strategies. As discussed in Section III each PE-pair has four common tangents. Since there are four such PE-pairs we will have 16 directed tangents in total. First we show that at any time t, only one directed tangent corresponding to each PE-pair in the set $PE(t)$ is a valid tangent along which the saddle-point trajectories may occur. Next we give an algorithm to find the time of capture for the valid tangent on each of the PE-pairs. Finally, we formulate a matrix game at each instant of time to design feedback saddle-point strategies for the pursuit-evasion game.

A. Selection of valid tangent corresponding to each PE-pair

Lemma 18. *At each time t, corresponding to each pair of PE-circles in the set $PE(t)$ there is only one valid common tangent with which saddle point strategies can coincide.*

Algorithm 1 is designed to compute the valid tangent for each SE-pair. The common tangents of all the PE-pairs

Algorithm 1 Valid Tangent

1) For the tangent under consideration let T_p be its intersection point with the pursuer circle under consideration and T_e be the intersection point with the evader circle under consideration.
2) The following observations for the valid tangent can be seen from Figure 1 and the figures for other pairs.
   a) For $A_p(t)$ and $A_e(t)$ the angle between the valid tangent and $\overrightarrow{O_pT_p}$ and $\overrightarrow{O_eT_e}$ translated to T_p and T_e respectively is $\pi/2$ in anti-clockwise direction.
   b) For $C_p(t)$) and $C_e(t)$ the angle between the valid tangent and $\overrightarrow{O_pT_p}$ and $\overrightarrow{O_eT_e}$ translated to T_p and T_e respectively is $-\pi/2$ in anti-clockwise direction.
3) For a given directed tangent $\overrightarrow{T}$ in a PE-pair if the angles with $\overrightarrow{O_pT_p}$ and $\overrightarrow{O_eT_e}$ satisfy the conditions above then it is a valid tangent.

have been shown in Figures 1 and the valid tangent has been shown by a dashed red line.

B. Algorithm for selecting the correct PE-pair

Recall that the clockwise circle $C_p(t)$ is traversed for $w_p(t) = -w_{p_m}$ and anticlockwise circle $A_p(t)$ for $w_p(t) = +w_{p_m}$. Similarly, for the evader the clockwise circle $C_e(t)$ is traversed for $w_e(t) = -w_{e_m}$ and anticlockwise circle $A_e(t)$ for $w_e(t) = +w_{e_m}$. Selecting $w_p(t) = +w_{p_m}$ and $w_e(t) = -w_{e_m}$ is equivalent to selecting the valid tangent of the pair $\{A_p(t), C_e(t)\}$ along which saddle-point strategies for the pursuit-evasion game will occur. The computation of time to capture at time t, $T_{ac}(t)$, for the valid tangent of the pair $\{A_p(t), C_e(t)\}$, shown in Figure 1, is given in Algorithm 2. Similarly, we calculate the times corresponding to each circle pairs and hence each input pairs. At an given time instant, say t, let $A_p(t)$, $C_p(t)$, and $A_e(t)$, $C_e(t)$ be the pursuer and evader circles respectively. Let $T_{aa}(t)$, $T_{ac}(t)$, $T_{ca}(t)$ and $T_{cc}(t)$ be the times corresponding to valid tangents of circle-pairs $\{A_p(t), A_e(t)\}$, $\{A_p(t), C_e(t)\}$, $\{C_p(t), A_e(t)\}$, and $\{C_p(t), C_e(t)\}$ respectively. For example, if the pursuit-evasion saddle-point occurs on the PE-pair $\{A_p(t), C_e(t)\}$ then at t we must have $w_p(t) = +w_{p_m}$ and $w_e(t) = -w_{e_m}$ initially. Similarly we have, $\{A_p(t), A_e(t)\} \Rightarrow w_p(t) = +w_{p_m}$, $w_e(t) = +w_{e_m}$, $\{C_p(t), A_e(t)\} \Rightarrow w_p(t) = -w_{p_m}$, $w_e(t) = +w_{e_m}$, $\{C_p(t), C_e(t)\} \Rightarrow w_p(t) = -w_{p_m}$, $w_e(t) = -w_{e_m}$ until the time that the trajectory leaves the corresponding circle and starts on the straight line path along the common tangent. Thus corresponding to pursuer and evader inputs at time t we obtain times of capture along each of the valid tangents. Using this times we formulate a matrix game as shown in Table I. The valid tangent on which the pursuit-evasion game occurs constitutes the open-loop saddle point strategies for the pursuit-evasion differential game. Thus the saddle-point solution of the matrix game at each instant t will give the common tangent with which the open-loop representation of the feedback saddle-point strategies is coincident. Thus the policy of the evader would

Algorithm 2 Algorithm to compute time to capture along a valid tangent

Let ET_e be the arc subtended between O_eE and O_eT_e in clockwise direction and let PT_p be the arc subtended by O_pP and O_pT_p in anticlockwise direction as shown in Figure 1. Compute the length of the arcs $PT_p = l_{ap}$ and $ET_E = l_{a_e}$ and define $t_p = l_{ap}/v_{p_m}$ and $t_e = l_{ae}/v_{e_m}$. Also, let the distance between T_pT_e be denoted by d.

1) If $t_p > t_e$ the evader will come out of the circle and onto the tangent earlier than the evader.
   a) $\tilde{t} = t_p - t_e$. Thus the evader will travel a distance $d_e = v_{e_m}\tilde{t}$ on the straight line before the pursuer comes onto the tangent.
   b) Thus at the time t_p the distance between the pursuer and evader will be $\tilde{d} = d + d_e$. Now the time to capture from this point will be $\bar{t} = \tilde{d}/(v_{p_m} - v_{e_m})$.
   c) Thus the time to capture will be $T_{ac} = t_p + \bar{t}$.
2) If $t_p \leq t_e$ the pursuer will come on straight line earlier.
   a) $\tilde{t} = t_e - t_p$. Thus the pursuer will travel a distance $d_p = v_{p_m}\tilde{t}$ on the straight line before the pursuer comes on the straight line.
   b) Thus at the time t_e the distance between the pursuer and evader will be $\tilde{d} = d - d_p$. Now the time to capture from this point will be $\bar{t} = \tilde{d}/(v_{p_m} - v_{e_m})$.
   c) Thus the time to capture will be $T_{ac} = t_e + \bar{t}$.

P\E	$w_e(t) = +w_{e_m}$	$w_e(t) = -w_{e_m}$
$w_p(t) = +w_{p_m}$	$T_{aa}(t)$	$T_{ac}(t)$
$w_p(t) = -w_{p_m}$	$T_{ca}(t)$	$T_{cc}(t)$

TABLE I: Matrix game at time instant t

be $\max - \min$ solution of the matrix game while that of the evader would be $\min - \max$ solution of the matrix game at each instant t. From this discussion we propose the following theorem.

Theorem 19. *The tangents selected by the saddle-point equilibrium in the matrix game described by Table I will be coincident with the open-loop representation of feedback saddle-point strategies if $d_{pe}^0 \geq 2r_e + 2\pi r_e(v_{p_m}/v_{e_m})$.*

Feedback strategies are of prime importance to make the feedback law robust to external disturbances. If the open-loop solution is computed at each instant of time t and the input value corresponding to time instant t i.e. $w_p(t)$ and $w_e(t)$ is applied then it constitutes a feedback law. To compute feedback an function $\mathrm{F}(\mathbf{x}(t)) : \mathbf{x}(t) \in \mathbb{R}^6 \rightarrow [\mathbf{u_p}^\top(t)\ \mathbf{u_e}^\top(t)]^\top \in \mathbb{R}^4$ is described in Algorithm 3 which gives as an output the inputs to the pursuer and the evader.

V. Comparison with Numerical Simulations

In [20], the authors have proposed an iterative numerical algorithm to solve pursuit-evasion games numerically. We use their algorithm to numerically solve the game of two cars and the $\min$ and the $\max$ problems are solved using direct

Algorithm 3 Feedback $[\mathbf{u_p}^\top(t)\ \mathbf{u_e}^\top(t)]^\top = \mathrm{F}(\mathbf{x}(t))$

Input: Pursuer state $\mathbf{p}(t)$ and Evader state $\mathbf{e}(t)$

1) Given the pursuer and evader states $\mathbf{p}(t)$ and $\mathbf{e}(t)$ at time t compute the circles $C_p(t), C_e(t), A_p(t), A_e(t)$.
2) Calculate the tangents corresponding to each pair of circles in $PE(t)$.
3) Use Algorithm 1 to obtain the four valid tangents out of 16 common tangents.
4) Use Algorithm 2 to compute time to capture for each of the valid tangents and formulate the matrix game at time t as shown above.
5) Find $w_p(t)$ as the $\min-\max$ solution of the matrix game and $w_e(t)$ as the $\max-\min$ solution of the matrix game.
6) At each instant t use $v_p(t) = v_{p_m}$ and $v_e(t) = v_{e_m}$.

Output: $\mathbf{u_p}(t) = [w_p(t)\ v_{p_m}]^\top$ and $\mathbf{u_e}(t) = [w_e(t)\ v_{e_m}]^\top$

numerical optimal control methods [24] using IPOPT [21]. The parameters for the pursuer and evader were $v_{p_m} = 2$, $w_{p_m} = 2$, $v_{e_m} = 1$ and $w_{e_m} = 1$. The initial states of the pursuer and the evader in the two simulations are: 1) $\mathbf{p_0} = [0, 0, \pi/2], \mathbf{e_0} = [-3, -6, \pi/2]$ 2) $\mathbf{p_0} = [0, 0, \pi/2], \mathbf{e_0} = [6, 3, \pi/2]$.

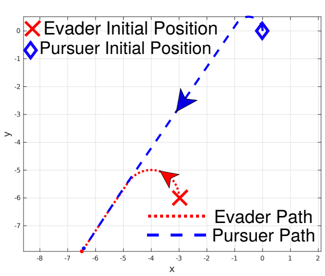

Fig. 5: Feedback law-1

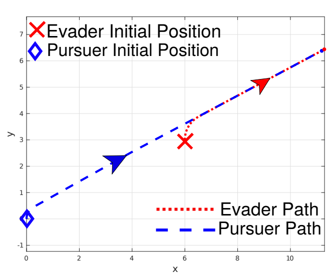

Fig. 6: Feedback law-2

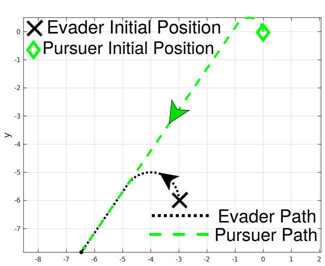

Fig. 7: Numerical [19]-1

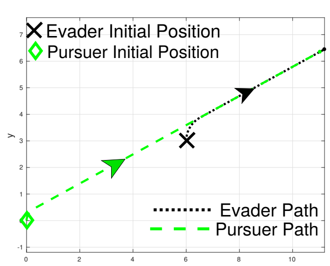

Fig. 8: Numerical [19]-2

The results obtained using the matrix law are shown in Figure 5 and 6. The pursuer trajectory is shown in blue while the evader trajectory in red. The results of numerical simulation using the algorithm in [19] are shown in Figure 7 and 8. The pursuer trajectory is shown in green color while the evader trajectory in black. The comparison of matrix law and the numerical simulation show that the trajectories are identical.

References

[1] R. Isaacs, *Differential Games*. Dover Publications, 1999.

[2] A. W. Merz, "The game of two identical cars," *Journal of Optimization Theory and Applications*, vol. 9, no. 5, pp. 324–343, 1972.

[3] I. Exarchos and P. Tsiotras, "An asymmetric version of the two car pursuit-evasion game," in *53rd IEEE Conference on Decision and Control*, 2014.

[4] I. Exarchos, P. Tsiotras, and M. Pachter, "On the suicidal pedestrian differential game," *Dynamic Games and Applications*, vol. 5, no. 3, pp. 297–317, 2015.

[5] R. Bera, V. R. Makkapati, and M. Kothari, "A comprehensive differential game theoretic solution to a game of two cars," *Journal of Optimization Theory and Applications*, vol. 174, no. 3, pp. 818–836, Sep 2017.

[6] E. Cockayne, "Plane pursuit with curvature constraints," *SIAM Journal on Applied Mathematics*, vol. 15, no. 6, pp. 1511–1516, 1967.

[7] L. E. Dubins, "On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents," *American Journal of mathematics*, vol. 79, no. 3, pp. 497–516, 1957.

[8] H. J. Sussmann and G. Tang, "Shortest paths for the reeds-shepp car: a worked out example of the use of geometric techniques in nonlinear optimal control," *Rutgers Center for Systems and Control Technical Report*, vol. 10, pp. 1–71, 1991.

[9] J.-D. Boissonnat, A. Cérézo, and J. Leblond, "Shortest paths of bounded curvature in the plane," *Journal of Intelligent and Robotic Systems*, vol. 11, no. 1-2, pp. 5–20, 1994.

[10] P. Soueres, A. Balluchi, and A. Bicchi, "Optimal feedback control for route tracking with a bounded-curvature vehicle," *International Journal of Control*, vol. 74, no. 10, pp. 1009–1019, 2001.

[11] E. Cockayne and G. Hall, "Plane motion of a particle subject to curvature constraints," *SIAM Journal on Control*, vol. 13, no. 1, pp. 197–220, 1975.

[12] X.-N. Bui and J.-D. Boissonnat, "Accessibility region for a car that only moves forwards along optimal paths," Ph.D. dissertation, INRIA, 1994.

[13] K. Mizukami and K. Eguchi, "A geometrical approach to problems of pursuit-evasion games," *Journal of the Franklin Institute*, vol. 303, no. 4, pp. 371–384, 1977.

[14] W. Sun and P. Tsiotras, "Pursuit evasion game of two players under an external flow field," in *American Control Conference*, 2015.

[15] W. Sun, P. Tsiotras, T. Lolla, D. N. Subramani, and P. F. J. Lermusiaux, "Pursuit-evasion games in dynamic flow fields via reachability set analysis," in *American Control Conference*, 2017.

[16] J. S. Jang and C. J. Tomlin, "Control strategies in multi-player pursuit and evasion game," *AIAA Guidance, Navigation, and Control Conference and Exhibit*, vol. 6239, pp. 15–18, 2005.

[17] H. Huang, W. Zhang, J. Ding, D. M. Stipanović, and C. J. Tomlin, "Guaranteed decentralized pursuit-evasion in the plane with multiple pursuers," in *50th IEEE Conference on Decision and Control and European Control Conference*, 2011.

[18] S. Pan, H. Huang, J. Ding, W. Zhang, C. J. Tomlin *et al.*, "Pursuit, evasion and defense in the plane," in *American Control Conference*, 2012.

[19] T. Raivio and H. Ehtamo, *On the Numerical Solution of a Class of Pursuit-Evasion Games*. Birkhäuser Boston, 2000, pp. 177–192.

[20] H. Ehtamo and T. Raivio, "On applied nonlinear and bilevel programming or pursuit-evasion games," *Journal of Optimization Theory and Applications*, vol. 108, no. 1, pp. 65–96, Jan 2001.

[21] A. Wachter and L. T. Biegler, "On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming," *Mathematical Programming*, vol. 106, no. 1, pp. 25–57, Mar 2006.

[22] T. Başar and G. J. Olsder, *Dynamic noncooperative game theory*. SIAM, 1998.

[23] A. E. Bryson, *Applied optimal control: optimization, estimation and control*. CRC Press, 1975.

[24] J. Betts, *Practical Methods for Optimal Control and Estimation Using Nonlinear Programming*, 2nd ed. Society for Industrial and Applied Mathematics, 2010.