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# Control of Nonlinear Systems with Full State Constraints Using Integral Barrier Lyapunov Functionals

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***Abstract*—In this paper, we present controller design for strict feedback nonlinear systems with full state constrains. An Integral Barrier Lyapunov Functionals (iBLF) is employed to obtain the adaptation law and the controllers. Compared with existing methods, the unknown parameters are considered in the system. Under the proposed iBLF-based control, we show that tracking errors are achieved without violation of any constraint and the closed loop signals remain bounded. The stability of the closed-loop system is proven by using the Lyapunov theorem.**

***Keywords*—Adaptive control, strict feedback, backstepping, state constrains, nonlinear systems**

## I. INTRODUCTION

Driven by practical and theoretical challenges, the rigorous dealing with constraints in the control design stage has become one important subject in recent decades [1]. Invariance control [2] has been extended to the nonlinear setting by switching between a nominal controller in the interior of the admissible set and an intervention control at the boundary [3], using the idea of barrier certificates to ensure invariance. Some notable methods for the control of constrained nonlinear systems include model predictive control (MPC) [4] and reference governors (RG) [5], extremum seeking control [6], nonovershooting control [7], adaptive variable structure control and error transformation [8]. However these articles ignore the constraints for the output or states. In practice, constraints problems often occur in many nonlinear systems.

The use of barrier Lyapunov function(BLF) for the control of nonlinear systems with output and state constraints has been proposed which involves the construction of a control Lyapunov function that grows to infinity whenever its arguments approaches some limits. Then, by keeping the BLF bounded in the closed-loop system, it is thus guaranteed that the limits are never transgressed. The BLF based design structure accommodates adaptive control design for handling uncertainty parametric, it has also been used for state constraints systems in Brunovsky form [9] and strict feedback form [10], [11], [12].

The current work explores the use of Barrier Lyapunov Functions for SISO nonlinear systems in strict feedback form with an output constraint. By designing the control to render the time derivative of the Barrier Lyapunov Function negative semidefinite, we keep the Barrier Lyapunov Function bounded in the closed loop and make sure the constraints are not transgressed. In [13], [14], the control problem was addressed for nonlinear systems with full state constraints and partial state constraints. The stability is guaranteed without violation of any constraint.

In [15], an Integral Barrier Lyapunov Functionals(iBLFs) was employed for designing a control for a class of nonlinear systems with state constraints. It is shown that output tracking is achieved without violation of any constraint, due to the new integral Lyapunov functional allow the original state constraints to be mixed with the error terms. However, [15] needs the system function is fully certainty, and this result cannot be used to deal with the control problem of uncertain nonlinear systems. So we need to design a control approach to stabilize the uncertain nonlinear systems with full state constraints.

In this paper, we shall consider a class of uncertain nonlinear systems with full state constraints, and presents iBLF-based control design for strict feedback nonlinear systems and the backstepping control design [16]. To deal with parametric uncertainty, we present an adaptive control that ensures constraint satisfaction and asymptotic output tracking. By avoiding the formulation of error constraints that indirectly enforces the state constraints, we overcome the aforementioned limitation and achieve significant simplification and relaxation of the feasibility conditions. It is proven that the system output tracks a desired signal to a bounded compact set and all the signals in the closed-loop system are bounded.

## II. PROBLEM FOR MULATION AND PRELIMINARIES

Consider the strict feedback nonlinear system:

$$\begin{cases} \dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, i = 1, \cdots, n-1 \\ \dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u \\ y = x_1 \end{cases} \tag{1}$$

where $x_i$ , $u \in R$ and $y \in R$ are the state variable, the input and the output of the systems, respectively; $\bar{x}_i = [x_1, ..., x_i]^T$ ; $f_i(\bar{x}_i)$ and $g_i(\bar{x}_i)$ , $i = 1, ..., n$ are smooth nonlinear functions. In the system(1), all the states are constrained in the compact sets, i.e., $x_i$ is required to remain in the set $|x_i| < k_{c_i}$ with $k_{c_i}$ being a positive constant. We define the set $\Omega_x := \{x \in R^n : |x_i| < k_{c_i}, i = 1, \cdots, n\} \subset R^n$ ; The functions

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$g_i(\bar{x}_i), i = 1, \cdots, n$, are known, and there exists a positive constant $g_0$ such that $0 < g_0 \le |g_i(\bar{x}_i)|$.

We deal with uncertainty in linearly parameterized nonlinearities

$$f_i(\bar{x}_i) = \theta_i^T \varphi_i(\bar{x}_i), i = 1, ..., n \tag{2}$$

where $\varphi_1 \in R^l$ is a regressor, and $\theta_i \in R$ is a vector of uncertain parameters satisfying $\theta_i \in \Omega_\theta$ with known compact set $\Omega_\theta$. Let $\hat{\theta}_i$ be an estimate of $\theta_i$, $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$.

The control objective is to design an adaptive output feedback controller $u$ such that $y$ tracks a desired trajectory $y_d(t)$ to a bounded compact set, all the signals in the closed-loop system are bounded, and the full state constraints are not violated.

***Assumption 1:*** For $\forall\ k_{c_1} > 0$, there exist positive constants $K_0, Y_i, i = 1, ..., n$, such that the desired trajectory $y_d(t)$ and its time derivatives satisfy

$$|y_d(t)| \le K_0 < k_{c_1},\ |y_d^{(i)}(t)| < Y_i$$

for all $t \ge 0$ and $i = 1, ..., n$.

***Lemma 1***: The function $\gamma_i(z_i, \alpha_{i-1})$ is $C^{n-i}$ in the set $\Omega = \{z_i \in R, \alpha_{i-1} \in R : |\alpha_{i-1}| < k_{c_i}, |z_i + \alpha_{i-1}| < k_{c_i}\}$.

## III. CONTROL SYNTHESIS USING AN INTEGRAL BARRIER LYAPUNOV FUNCTIONAL

Choose the following Integral Barrier Lyapunov Functional candidate

$$V(z, \alpha) = \sum_{i=1}^{n} V_i(z_i, \alpha_{i-1}) \tag{3}$$

$$V_i(z_i, \alpha_{i-1}) = \int_0^{z_i} \frac{\tau k_{c_i}^2}{k_{c_i}^2 - (\tau + \alpha_{i-1})^2} d\tau + \frac{1}{2}\tilde{\theta}_i^T \tilde{\theta}_i, i = 1, ..., n \tag{4}$$

Define $z_i = x_i - \alpha_{i-1}$ where $\alpha_0 = y_d$ and $\alpha_i, i = 1, ..., n-1$ are continuously differentiable funtions satisfying $|\alpha_i| \le K_i < k_{c_{i+1}}$ for positive constants $K_i, i = 0, 1, ..., n-1$.

Define

$$\Phi_i = \int_0^{z_i} \frac{\tau k_{c_i}^2}{k_{c_i}^2 - (\tau + \alpha_{i-1})^2} d\tau \tag{5}$$

The functional $\Phi_i$ is positive definite, continuously differentiable, and satisfies the decrescent condition in the set $|x_i| < k_{c_i}$ for $i = 1, ..., n$:

$$\frac{z_i^2}{2} \le \Phi_i \le z_i^2 \int_0^1 \frac{\omega k_{c_i}^2}{k_{c_i}^2 - (\omega z_i + \operatorname{sgn}(z_i) K_{i-1})^2} d\omega$$

(6)

***Step 1:*** Consider the functional (4) for $i = 1$. The time-derivative is given by

$$\begin{aligned} \dot{V}_1 &= \frac{\partial \Phi_1}{\partial z_1}\dot{z}_1 + \frac{\partial \Phi_1}{\partial y_d}\dot{y}_d + \tilde{\theta}_1^T \dot{\hat{\theta}}_1 \\ &= \frac{k_{c_1}^2 z_1 \dot{z}_1}{k_{c_1}^2 - x_1^2} + \frac{\partial \Phi_1}{\partial y_d}\dot{y}_d + \tilde{\theta}_1^T \dot{\hat{\theta}}_1 \\ &= \frac{k_{c_1}^2 z_1}{k_{c_1}^2 - x_1^2}\big(f_1(x_1) + g_1(x_1) z_2 \\ &\quad + g_1(x_1)\alpha_1 - \dot{y}_d\big) + \frac{\partial \Phi_1}{\partial y_d}\dot{y}_d + \tilde{\theta}_1^T \dot{\hat{\theta}}_1 \end{aligned} \tag{7}$$

We can show, using the substitution $\tau = \omega z_1$ [17], [18], that

$$\begin{aligned} \frac{\partial \Phi_1}{\partial y_d} &= \int_0^{z_1} \tau d\left[\frac{k_{c_1}^2}{k_{c_1}^2 - (\tau + y_d)^2}\right] \\ &= \frac{\tau k_{c_1}^2}{k_{c_1}^2 - (\tau + y_d)^2}\Big|_0^{z_1} - \int_0^{z_1} \frac{k_{c_1}^2}{k_{c_1}^2 - (\tau + y_d)^2} d\tau \\ &= \frac{z_1 k_{c_1}^2}{k_{c_1}^2 - x_1^2} - \int_0^{z_1} \frac{k_{c_1}^2}{k_{c_1}^2 - (\omega z_1 + y_d)^2} d\omega z_1 \\ &= z_1\left[\frac{k_{c_1}^2}{k_{c_1}^2 - x_1^2} - \int_0^1 \frac{k_{c_1}^2}{k_{c_1}^2 - (\omega z_1 + y_d)^2} d\omega\right] \\ &= z_1\left[\frac{k_{c_1}^2}{k_{c_1}^2 - x_1^2} - \gamma_1(z_1, y_d)\right] \end{aligned} \tag{8}$$

where

$$\begin{aligned} \gamma_1(z_1, y_d) &= \int_0^1 \frac{k_{c_1}^2}{k_{c_1}^2 - (\omega z_1 + y_d)^2} d\omega \\ &= \frac{k_{c_1}}{z_1}\left[\tanh^{-1}\left(\frac{z_1 + y_d}{k_{c_1}}\right) - \tanh^{-1}\left(\frac{y_d}{k_{c_1}}\right)\right] \\ &= \frac{k_{c_1}}{2z_1}\ln\frac{(k_{c_1} + z_1 + y_d)(k_{c_1} - y_d)}{(k_{c_1} - z_1 - y_d)(k_{c_1} + y_d)} \end{aligned} \tag{9}$$

Using L'Hˆopital's rule, we have

$$\lim_{z_1 \to 0} \gamma_1(z_1, y_d) = \frac{k_{c_1}^2}{k_{c_1}^2 - y_d^2} \tag{10}$$

where it is straightforward to show, $\gamma_1(z_1, y_d)$ is well-defined in a neighborhood of $z_1 = 0$.

Choose the virtual controller $\alpha_1$ as

$$\alpha_1 = \frac{1}{g_1}\left[-\hat{\theta}_1^T\varphi_1 - \lambda_1 z_1 + \frac{\left(k_{c_1}^2 - x_1^2\right)\dot{y}_d\gamma_1}{k_{c_1}^2}\right] \tag{11}$$

where $\lambda_1$ is a positive control gain

Using (8) and (11), we obtain

$$\dot{V}_1 = -\frac{\lambda_1 k_{c_1}^2 z_1^2}{k_{c_1}^2 - x_1^2} + \frac{k_{c_1}^2 z_1}{k_{c_1}^2 - x_1^2}\left(f_1(x_1) - \hat{\theta}_1^T\varphi_1\right) + \frac{k_{c_1}^2 g_1 z_1 z_2}{k_{c_1}^2 - x_1^2} + \tilde{\theta}_1^T\dot{\hat{\theta}}_1$$

$$= -\frac{\lambda_1 k_{c_1}^2 z_1^2}{k_{c_1}^2 - x_1^2} + \frac{k_{c_1}^2 g_1 z_1 z_2}{k_{c_1}^2 - x_1^2} - \tilde{\theta}_1^T\left(\frac{k_{c_1}^2 z_1\varphi_1}{k_{c_1}^2 - x_1^2} - \dot{\hat{\theta}}_1\right) \tag{12}$$

***Step*** $i$ $(i = 2, ..., n-1)$: Consider integral-type functions

$$V_i(z_i, \alpha_{i-1}) = \Phi_i + \frac{1}{2}\tilde{\theta}_i^T\tilde{\theta}_i, i = 1, ..., n-1$$

where time-derivatives are given by

$$\dot{V}_i = \frac{\partial\Phi_i}{\partial z_i}\dot{z}_i + \frac{\partial\Phi_i}{\partial\alpha_{i-1}}\dot{\alpha}_{i-1} + \tilde{\theta}_i^T\dot{\hat{\theta}}_i$$

$$= \frac{k_{c_i}^2 z_i}{k_{c_i}^2 - x_i^2}\left(f_i(\bar{x}_i) + g_i(\bar{x}_i)z_{i+1}\right.$$

$$\left. + g_i(\bar{x}_i)\alpha_i - \dot{\alpha}_{i-1}\right) + \frac{\partial\Phi_i}{\partial\alpha_{i-1}}\dot{\alpha}_{i-1} + \tilde{\theta}_i^T\dot{\hat{\theta}}_i \tag{13}$$

where

$$\frac{\partial\Phi_i}{\partial\alpha_{i-1}} = \int_0^{z_i}\tau d\left[\frac{k_{c_i}^2}{k_{c_i}^2 - (\tau + \alpha_{i-1})^2}\right]$$

$$= \frac{\tau k_{c_i}^2}{k_{c_i}^2 - (\tau + \alpha_{i-1})^2}\Big|_0^{z_i} - \int_0^{z_i}\frac{k_{c_i}^2}{k_{c_i}^2 - (\tau + \alpha_{i-1})^2}d\tau$$

$$= \frac{z_i k_{c_i}^2}{k_{c_i}^2 - x_i^2} - \int_0^{z_i}\frac{k_{c_i}^2}{k_{c_i}^2 - (\omega z_i + \alpha_{i-1})^2}d\omega z_i$$

$$= z_i\left[\frac{k_{c_i}^2}{k_{c_i}^2 - x_i^2} - \int_0^1\frac{k_{c_i}^2}{k_{c_i}^2 - (\omega z_i + \alpha_{i-1})^2}d\omega\right]$$

$$= z_i\left(\frac{k_{c_i}^2}{k_{c_i}^2 - x_i^2} - \gamma_i(z_i, \alpha_{i-1})\right) \tag{14}$$

where

$$\gamma_i(z_i, \alpha_{i-1}) = \int_0^1\frac{k_{c_i}^2}{k_{c_i}^2 - (\omega z_i + \alpha_{i-1})^2}d\omega$$

$$= \frac{k_{c_i}}{2z_i}\ln\frac{\left(k_{c_i} + z_i + \alpha_{i-1}\right)\left(k_{c_i} - \alpha_{i-1}\right)}{\left(k_{c_i} - z_i - \alpha_{i-1}\right)\left(k_{c_i} + \alpha_{i-1}\right)} \tag{15}$$

The partial derivatives of $\gamma_i(z_i, \alpha_{i-1})$, $i = 1, \cdots, n$, are given by:

$$\frac{\partial\gamma_i}{\partial z_i} = \int_0^1\left(\partial\frac{k_{c_i}^2}{k_{c_i}^2 - (\omega z_i + \alpha_{i-1})^2}/\partial z_i\right)d\omega$$

$$= \frac{1}{z_i}\int_0^1\omega\left(\partial\frac{k_{c_i}^2}{k_{c_i}^2 - (\omega z_i + \alpha_{i-1})^2}/\partial\omega z_i\right)d\omega z_i$$

$$= \frac{1}{z_i}\left[\frac{k_{c_i}^2}{k_{c_i}^2 - (z_i + \alpha_{i-1})^2} - \gamma_i\right] \tag{16}$$

$$\frac{\partial\gamma_i}{\partial\alpha_{i-1}} = \frac{k_{c_i}^2(z_i + 2\alpha_{i-1})}{\left[k_{c_i}^2 - (z_i + \alpha_{i-1})^2\right]\left(k_{c_i}^2 - \alpha_{i-1}^2\right)} \tag{17}$$

Using L'Hˆopital's rule, we obtain

$$\lim_{z_i\to 0}\gamma_i(z_i, \alpha_{i-1}) = \frac{k_{c_i}^2}{k_{c_i}^2 - \alpha_{i-1}^2}$$

$$\lim_{z_i\to 0}\frac{\partial\gamma_i}{\partial\alpha_{i-1}} = \frac{2k_{c_i}^2\alpha_{i-1}}{\left(k_{c_i}^2 - \alpha_{i-1}^2\right)^2}$$

Thus $\gamma_i$, $\frac{\partial\gamma_i}{\partial z_i}$ and $\frac{\partial\gamma_i}{\partial\alpha_{i-1}}$ are well-defined in a neighborhood of $z_i = 0$, in the set $|\alpha_{i-1}| < k_{c_i}$.

By designing the stabilizing function as

$$\alpha_i = \frac{1}{g_i}\left[-\hat{\theta}_i^T\varphi_i - \lambda_i z_i + \frac{\left(k_{c_i}^2 - x_i^2\right)\gamma_i}{k_{c_i}^2}\left(\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}g_j x_{j+1}\right.\right.$$

$$+\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}}y_d^{(j+1)} - \frac{1}{2}z_i\gamma_i\left(\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right)^2$$

$$\left.\left. + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j\right) - \frac{k_{c_{i-1}}^2\left(k_{c_i}^2 - x_i^2\right)g_{i-1}z_{i-1}}{k_{c_i}^2\left(k_{c_{i-1}}^2 - x_{i-1}^2\right)}\right] \tag{18}$$

Using (18), we have

$$\dot{\alpha}_i = \sum_{j=1}^{i}\frac{\partial\alpha_i}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j + \sum_{j=1}^{i}\frac{\partial\alpha_i}{\partial x_j}\dot{x}_j + \sum_{j=0}^{i}\frac{\partial\alpha_i}{\partial y_d^{(j)}}y_d^{(j+1)} \tag{19}$$

Similar to (19), we can obtain

$$\dot{\alpha}_{i-1}=\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j+\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}}y_d^{(j+1)} \tag{20}$$

Substituting (18) and (20) into (13) leads to

$$\dot{V}_i=-\frac{\lambda_i k_{c_i}^2 z_i^2}{k_{c_i}^2-x_i^2}-\tilde{\theta}_i^T\left[\frac{k_{c_i}^2 z_i}{k_{c_i}^2-x_i^2}\varphi_i-\dot{\hat{\theta}}_i\right]-\frac{1}{2}z_i^2\gamma_i^2\left(\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right)^2$$
$$-\frac{k_{c_{i-1}}^2 g_{i-1}z_{i-1}z_i}{k_{c_{i-1}}^2-x_{i-1}^2}+\frac{k_{c_i}^2 g_i z_i z_{i+1}}{k_{c_i}^2-x_i^2}-z_i\gamma_i\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\theta_j^T\varphi_j \tag{21}$$

Using the following inequality

$$-z_i\gamma_i\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\theta_j^T\varphi_j\le\frac{1}{2}z_i^2\gamma_i^2\left(\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right)^2+\frac{1}{2}\sum_{j=1}^{i-1}\theta_j^{T2} \tag{22}$$

Then, (21) becomes

$$\dot{V}_i\le-\frac{\lambda_i k_{c_i}^2 z_i^2}{k_{c_i}^2-x_i^2}+\frac{k_{c_i}^2 g_i z_i z_{i+1}}{k_{c_i}^2-x_i^2}-\frac{k_{c_{i-1}}^2 g_{i-1}z_{i-1}z_i}{k_{c_{i-1}}^2-x_{i-1}^2}$$
$$-\tilde{\theta}_i^T\left[\frac{k_{c_i}^2 z_i}{k_{c_i}^2-x_i^2}\varphi_i-\dot{\hat{\theta}}_i\right]+\frac{1}{2}\sum_{j=1}^{i-1}\theta_j^{T2} \tag{23}$$

***Step*** $n$: The time derivative of $z_n=x_n-\alpha_{n-1}$ is

$$\dot{z}_n=\dot{x}_n-\dot{\alpha}_{n-1}=f_n(\bar{x}_n)+g_n(\bar{x}_n)u-\dot{\alpha}_{n-1} \tag{24}$$

where

$$\dot{\alpha}_{n-1}=\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j+\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\dot{x}_j+\sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial y_d^{(j)}}y_d^{(j+1)} \tag{25}$$

Define the integral-type functions

$$V_n(z_n,\alpha_{n-1})=\Phi_n+\frac{1}{2}\tilde{\theta}_n^T\tilde{\theta}_n \tag{26}$$

Design the controller $u$ as

$$u=\frac{1}{g_n}\Bigg[-\hat{\theta}_n^T\varphi_n-\lambda_n z_n+\frac{\left(k_{c_n}^2-x_n^2\right)\gamma_n}{k_{c_n}^2}\Bigg(\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}g_j x_{j+1}$$
$$+\sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial y_d^{(j)}}y_d^{(j+1)}-\frac{1}{2}z_n\gamma_n\left(\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\varphi_j\right)^2$$
$$+\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j\Bigg)-\frac{k_{c_{n-1}}^2\left(k_{c_n}^2-x_n^2\right)g_{n-1}z_{n-1}}{k_{c_n}^2\left(k_{c_{n-1}}^2-x_{n-1}^2\right)}\Bigg] \tag{27}$$

Similar to Step $i$ and using (23) with $n=i-1$, we can obtain

$$\dot{V}_n\le-\frac{\lambda_n k_{c_n}^2 z_n^2}{k_{c_n}^2-x_n^2}-\frac{k_{c_{n-1}}^2 g_{n-1}z_{n-1}z_n}{k_{c_{n-1}}^2-x_{n-1}^2}$$
$$-\tilde{\theta}_n^T\left[\frac{k_{c_n}^2 z_n}{k_{c_n}^2-x_n^2}\varphi_n-\dot{\hat{\theta}}_n\right]+\frac{1}{2}\sum_{j=1}^{n-1}\theta_j^{T2} \tag{28}$$

Based on (12), (23) and (28), (3) can be rewritten as

$$\dot{V}(z,\alpha)=\sum_{i=1}^{n}\dot{V}_i(z_i,\alpha_{i-1})$$
$$\le-\sum_{i=1}^{n}\frac{\lambda_i k_{c_i}^2 z_i^2}{k_{c_i}^2-x_i^2}-\sum_{i=1}^{n}\tilde{\theta}_i^T\left[\frac{k_{c_i}^2 z_i}{k_{c_i}^2-x_i^2}\varphi_i-\dot{\hat{\theta}}_i\right]$$
$$+\frac{n-1}{2}\theta_1^{T2}+\frac{n-2}{2}\theta_2^{T2}+\ldots+\frac{1}{2}\theta_{n-1}^{T\,2} \tag{29}$$

Design the adaptation laws for $\hat{\theta}_j, j=1,\cdots,n$ as

$$\dot{\hat{\theta}}_j=\frac{k_{c_j}^2 z_j}{k_{c_j}^2-x_j^2}\varphi_j \tag{30}$$

Based on (30), (29) can be rewritten as

$$\dot{V}(z,\alpha)\le-\sum_{i=1}^{n}\frac{\lambda_i k_{c_i}^2 z_i^2}{k_{c_i}^2-x_i^2}$$
$$+\frac{n-2}{2}\theta_2^{T2}+\ldots+\frac{1}{2}\theta_{n-1}^{T\,2} \tag{31}$$

Define $\Psi_i(\tau,\alpha_{i-1})=\frac{\tau k_{c_i}^2}{k_{c_i}^2-(\tau+\alpha_{i-1})^2}$, we can show that: $\frac{\partial\Psi_i}{\partial\tau}=\frac{k_{c_i}^2\left(k_{c_i}^2+\tau^2-\alpha_{i-1}^2\right)}{\left[k_{c_i}^2-(\tau+\alpha_{i-1})^2\right]^2}$ is positive, $\Psi_i(\tau,\alpha_{i-1})>\Psi_i(0,\alpha_{i-1})=0$, in the set $|\tau+\alpha_{i-1}|<k_{c_i}$.

$$\int_0^{z_i}\Psi_i(\tau,\alpha_{i-1})d\tau=\frac{\tau k_{c_i}^2}{k_{c_i}^2-(\tau+\alpha_{i-1})^2}\Big|_0^{z_i}-\int_0^{z_i}\tau d\Psi_i$$
$$=z_i\Psi_i(z_i,\alpha_{i-1})-\int_0^{z_i}\tau d\Psi_i \tag{32}$$

It can be shown that

$$\int_0^{z_i}\Psi_i(\tau,\alpha_{i-1})d\tau\le z_i\Psi_i(z_i,\alpha_{i-1}) \tag{33}$$

Substituting for $\Psi_i$ we can leads to $\Phi_i\le\frac{k_{c_i}^2 z_i^2}{k_{c_i}^2-x_i^2}$ in $|z_i+\alpha_{i-1}|<k_{c_i}$. Consequently, we have the functional $\Phi_i=\int_0^{z_i}\frac{\tau k_{c_i}^2}{k_{c_i}^2-(\tau+\alpha_{i-1})^2}d\tau$ satisfies $\Phi_i\le\frac{k_{c_i}^2 z_i^2}{k_{c_i}^2-x_i^2}$.

It is obvious that

$$\dot{V}_i \le \frac{k_{c_i}^2 z_i^2}{k_{c_i}^2 - x_i^2} + \frac{1}{2}\tilde{\theta}_i^T \tilde{\theta}_i \tag{34}$$

From (26), (29) can be represented as

$$\dot{V} \le -\varepsilon V \tag{35}$$

where $\varepsilon = 2\min_i |\lambda_i|$.

The closed loop system can be rewritten in the form

$$\dot{z} = \nu(t, z) \tag{36}$$

as

$$\dot{z}_1 = -\tilde{\theta}_1^T \varphi_1 + g_1 z_2 - \lambda_1 z_1 + \left( \frac{\left(k_{c_1}^2 - x_1^2\right)\gamma_1}{k_{c_1}^2} - 1 \right) \dot{y}_d$$

$$\dot{z}_i = -\tilde{\theta}_i^T \varphi_i + g_i z_{i+1} - \lambda_i z_i - \frac{k_{c_{i-1}}^2 \left(k_{c_i}^2 - x_i^2\right) g_{i-1} z_{i-1}}{k_{c_i}^2 \left(k_{c_{i-1}}^2 - x_{i-1}^2\right)}$$

$$+ \frac{\left(k_{c_i}^2 - x_i^2\right)\gamma_i}{k_{c_i}^2} \left( \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} g_j x_{j+1} + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} \right.$$

$$\left. + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}_j} \dot{\hat{\theta}}_j - \frac{1}{2} z_i \gamma_i \left( \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right)^2 \right) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} g_j x_{j+1}$$

$$- \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}_j} \dot{\hat{\theta}}_j - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \theta_j^T \varphi_j \tag{37}$$

where $\dot{z} = \nu(t, z)$ is piecewise continuous in the set $\left(z_i(t), \alpha_{i-1}(t)\right) \in \Omega$.

We define $\overline{z}_i = \left[z_1, z_2, ..., z_i\right]^T$, $\overline{y}_{d_i} = \left[\mathrm{y}_d, y_d^{(1)}, \mathrm{y}_d^{(2)}, ..., y_d^{(i)}\right]^T$, and $\left(\overline{z}_i, \overline{y}_{d_i}\right) \in \Gamma$, $\Gamma = \Big\{ \overline{z}_n \in R^n, \overline{y}_{d_n} \in R^{n+1} : \left|z_j\right| \le \sqrt{2V|_{t=0}}$, $\left|y_d(t)\right| \le K_0, \left|y_d^{(j)}\right| \le Y_j, j = 1, ..., n \Big\}$ where $\left|y_d(t)\right| \le \mathrm{K}_0 < k_{c_1}$, define $K_i = \max\left|\alpha_i\left(\overline{z}_i, \overline{y}_{d_i}\right)\right|, i = 1, ..., n-1$.

***Theorem 1:*** Consider unknown system (1) under Assumptions1, control law (11), (18), (27), and initial condition $x(0) \in \Omega_x := \left\{ x \in R^n : \left|x_i\right| < k_{c_i}, \mathrm{i} = 1, ..., \mathrm{n} \right\}$. If $k_{c_i} > K_{i-1}\left(\lambda_1, ..., \lambda_{i-1}\right), i = 1, ..., n$. Where $\lambda_1, ..., \lambda_{n-1}$ are positive constants. We hold that

**(i)** $\left|z_i(t)\right| \le \sqrt{2V|_{t=0}}\, e^{-\frac{\varepsilon t}{2}}$, $\forall t > 0$.

(ii) $x(t) \in \Omega_x$, for all $t > 0$.

(iii) For all $t > 0$, $\alpha_i(t), i = 1, ..., n-1; u(t)$ are bounded.

**Proof:** The proof process is similar to previous Lyapunov analysis method. Thus, the proof process is omitted here.

## IV. CONCLUSION

We have introduced Integral Barrier Lyapunov Functionals for control design of strict-feedback nonlinear systems with full state constraints. The main advantage is that the initial state constraints are mixed with the error terms, different form the existing methods using BLFs which lead to the relaxation of feasibility conditions for constraint satisfaction. By ensuring boundedness of the Integral Barrier Lyapunov Functionals in the closed loop, we ensure that Provided that the feasibility conditions are satisfied, the closed loop tracking error has an exponentially decreasing bound, the state is guaranteed to remain in the constrained region, and the control input is always bounded.

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