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Analysis of Role Switch for Cooperative Target Defense Differential Game

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***Abstract*—A Target-Attacker-Defender (TAD) game is a fascinating differential game in both nature and the artificial world. This paper considers a TAD game with a non-suicidal faster Attacker that aims to capture the Target while avoiding being captured by the Defender. The Attacker is both a pursuer (pursues the Target) and an evader (evades the Defender). Naturally, one wonders when and how the Attacker can chase the Target and when and how the Attacker can evade the Defender. In this paper, we first analyze the possibility of winning the game for the Attacker under the framework of games of kind. Then, we present three different stages and corresponding control strategies, as well as the conditions for switching stages. In addition, the Target cooperates with the Defender to compose a team as the opponent of the Attacker. Different capture radii also affect cooperation between the Target and Defender. When the capture radius of the Defender is less than that of the Attacker, the Target and Defender cooperate to avoid being captured by the Attacker or to extend the time to be captured. Conversely, when the capture radius of the Attacker is less than that of the Defender, the Target-Defender team can adopt a rendezvous strategy to end the engagement. We derive the initial state under which the Target can rendezvous with the Defender before the Target is captured by the Attacker.**

***Index Terms*—Target-Attacker-Defender game, Role switch, Cooperative policy, Pursuit-evasion game.**

I. Introduction

It is a common phenomenon in nature that mothers protect their young from attacks and injuries by other animals. In practical engineering applications, there are similar scenarios, e.g., a fired missile defending an evading aircraft against an incoming homing missile [1]–[5], a torpedo safeguarding a naval ship against a submarine [6], and an interceptor defending an asset against an intruder [7].

The above phenomena and scenarios are actually a pursuit-evasion game with three players - Target, Attacker, and Defender - which is called a Target-Attacker-Defender (TAD) game [8], [9]. An active target defense differential game is considered in [10]–[13], where an Attacker missile chases a Target aircraft protected by a Defender missile that aims at intercepting the Attacker before the Attacker captures the Target aircraft. In these papers, the Attacker, regarded as a non-recycled missile, aims only to minimize the distance between itself and the Target. Whether or not the Attacker is captured by the Defender is seldom considered.

However, in many scenarios, the Attacker is not only a kind of non-recyclable item similar to a missile, but also an object with a certain value if it survives, such as a fighter. Moreover, the Attacker also needs to avoid being intercepted by the Defender. This study focuses on solving such a TAD game with a survivable Attacker. The Attacker aims to capture the Target while avoiding being captured by the Defender, and the Defender tries to defend the Target from being captured by the Attacker while trying to capture the Attacker at an opportune moment. The Target and Defender cooperate in a team, and the Attacker is on the opposite side.

This study differs from conventional pursuit-evasion games [14]–[19] because the players' roles have changed. The task of the players is not merely to chase or to escape. In addition to chasing, the Attacker also needs to escape from the Defender. Meanwhile, the Target needs to cooperate with the Defender in addition to escaping from the Attacker. The Defender cooperates with the Target to prevent the Attacker from achieving its goal. The change of the players' roles makes the TAD game more interesting and more difficult to solve.

A lot of studies have been done for the cooperation of multiple players which starts with a three-player pursuit-evasion problem: one pursuer and two evaders [20], or two pursuers and one evader [21]. Then, more cooperation of multiple players [22]–[25] is studied in N-pursuers-one-evader, one-pursuer-M-evaders, and N-pursuers-M-evaders games. However, in these studies, each player only plays a single role.

There is little research on role switch. In [26], the Attacker's cost function is used as a form of indirect control for the defender that attempts to make the Attacker retreat instead of engaging. The role switch of the Attacker is considered in [27], but the cooperation between the Target and defending missile is not the main focus. The authors of [28] consider both the role switch and the cooperation of two players, but solving the guidance algorithm for the three players requires some parameters that are not directly available from sensor measurements.

In this paper, there are two points of focus on the TAD problem: one is what kind of cooperation should the Target and Defender adopt to win the game; the other is the role switch for the Attacker between the pursuer and evader. The goals and main contributions of this work are as follows.

1) The survivability of the Attacker is considered in the problem formulation for a TAD game. The reduced state space is constructed to solve the TAD game, which solves

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the problem that the parameters are not directly available from sensor measurements.

2) By analyzing the usability of the boundary of the terminal set and solving the included angle sub-game, a novel switching policy for the Attacker capturing the Target is presented. The condition under which the Attacker switches roles and the corresponding strategy for the Attacker are derived.
3) Different cooperative policies for the Target and Defender in different cases are presented. The influence of the capture radius on the TAD game is considered. The rendezvous cooperation mode of the Target and Defender is proposed, and the conditions and corresponding strategies for the Target and Defender to win the game by rendezvous mode are obtained.

The rest of the paper is organized as follows. In Section 2, we formulate the TAD game in a reduced state space. In Section 3, we analyze the usability of the boundary of the terminal set for the three players in games of kind. In Section 4, we present the role switch of the Attacker. We give the switching conditions and control strategies for the three players. In Section 5, we obtain the winning conditions and corresponding control strategies for the rendezvous of the Target and Defender. In Section 6, we use some simulation examples to illustrate the results. Finally, Section 7 gives the concluding remarks and a summary of planned future work.

II. Problem Statement

The Target, Attacker, and Defender have simple motions as typically encountered in the games of Isaacs [29], and the three players move in the Euclidean plane. Their control variables are the instantaneous headings $\hat{\phi}$, $\hat{\chi}$, $\hat{\psi} \in \mathbb{R}$, respectively. In addition, the Target, Attacker, and Defender have constant speeds V_T, V_A, $V_D \in \mathbb{R}$, respectively. The states of the Target, Attacker, and Defender are denoted by their Cartesian coordinates $\mathbf{x}_T = (x_T, y_T)$, $\mathbf{x}_A = (x_A, y_A)$, $\mathbf{x}_D = (x_D, y_D) \in \mathbb{R}^2$, respectively.

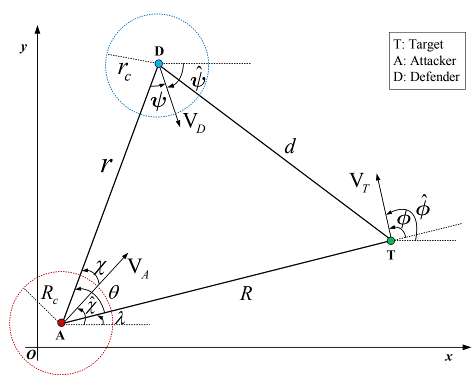

Fig. 1: A TAD game with capture radius in the fixed reference system.

The dynamics of the Target-Attacker-Defender in the realistic game space can be described by the following equations:

$$\dot{x}_T(t) = V_T \cos\hat{\phi}(t), \qquad \dot{y}_T(t) = V_T \sin\hat{\phi}(t), \tag{1}$$

$$\dot{x}_A(t) = V_A \cos\hat{\chi}(t), \qquad \dot{y}_A(t) = V_A \sin\hat{\chi}(t), \tag{2}$$

$$\dot{x}_D(t) = V_D \cos\hat{\psi}(t), \qquad \dot{y}_D(t) = V_D \sin\hat{\psi}(t). \tag{3}$$

In the TAD game with a survivable Attacker, we are interested in the following two questions:

Q1) When does the Attacker switch the roles?
Q2) What strategies should be adopted by the Attacker or the Target–Defender team to achieve success?

It is assumed that each player knows the dynamics (1)-(3) and the speed parameters. Since the Target can always evade the Attacker from any given initial position for $V_T \geq V_A$, we only consider the case in which $V_T < V_A$. In addition, when $V_D > V_A$, the Defender can always adopt the corresponding strategies to reduce the distance from itself to the Attacker. The Attacker can only adopt the direct pursuit strategy to capture the Target before being captured by the Defender. The Attacker cannot switch its strategy in order to win the game in an unbounded domain. The other case in which $V_D > V_A$ will be considered in a bounded domain in the future research. In this paper, we focus on the case in which $V_D < V_A$. Let $\alpha = V_T/V_A < 1$ and $\beta = V_D/V_A < 1$ denote the speed ratios, respectively.

As shown in Fig. 1, the variables R and r represent the distance between the Attacker and Target and the distance between the Attacker and Defender, respectively. θ is the included angle $\angle DAT$, $\theta = Arg(\overrightarrow{AD}) - Arg(\overrightarrow{AT})$. Arg(.) is the principal value of the argument of a vector, within the range $(-\pi, \pi]$. To describe the dynamics of the TAD game more compactly, variables R, r, and θ can form a reduced state space in which the TAD game is specified by $\mathbf{x} = (R, r, \theta) \in \mathbb{R}^3$. The game set is the entire space $\mathbb{R}^3$.

Without loss of generality, the players' speeds are normalized so that $V_A = 1$. The distances R, r, and the included angle θ are also normalized with respect to the velocity V_A. Thus, the dynamics $\dot{\mathbf{x}}(t) = f(\mathbf{x}(\mathbf{t}), \phi(t), \chi(t), \psi(t))$ are specified by the following equations:

$$\dot{R}(t) = \alpha\cos\phi(t) - \cos(\theta(t) - \chi(t)), \quad R(t_0) = R_0, \tag{4}$$

$$\dot{r}(t) = -\cos\chi(t) - \beta\cos\psi(t), \qquad r(t_0) = r_0, \tag{5}$$

$$\begin{aligned}\dot{\theta}(t) = {} & (\sin(\theta(t) - \chi(t)) - \alpha\sin\phi(t))/R \\ & + (\sin\chi(t) - \beta\sin\psi(t))/r, \qquad \theta(t_0) = \theta_0,\end{aligned} \tag{6}$$

where $\phi = \hat{\phi} - \lambda$, $\chi = \lambda + \theta - \hat{\chi}$, and $\psi = \hat{\psi} - \theta - \lambda + \pi$. λ denotes the included angle of the vector $\overrightarrow{AT}$ and the x axis. ϕ, χ, and ψ are the alternative control variables of the players, defined as the relative headings of the Target, Attacker, and Defender from the vectors $\overrightarrow{AT}$,$\overrightarrow{AD}$, and $\overrightarrow{DA}$, respectively.

The positive constants R_c and r_c represent the capture radii of the Attacker and Defender, respectively. The full state space is denoted by $\Omega = \{R \geq R_c, r \geq r_c, -\pi < \theta \leq \pi\} \subseteq \mathbb{R}^3$. We assume that the initial space includes the angle $\theta_0 \in [0, \pi)$. As for the case of $(-\pi, 0)$, it is easy to obtain a similar result according to symmetry. At the beginning of the TAD game $\{R_0 > R_c, r_0 > r_c, 0 \leq \theta_0 < \pi\}$, if $\{R < R_c, r > r_c\}$, the TAD game terminates and the Attacker wins. If $\{R > R_c, r < r_c\}$, the Target–Defender team wins. However, the Attacker is faster than the Defender. The Attacker can switch its strategy and the Defender cannot capture the Attacker. Therefore, if the state stays in $\{R > R_c, r > r_c\}$, the Attacker cannot achieve its goals. Then, the Target-Defender team is defined as winning the game.

III. Boundary Analysis

In order to answer the questions Q1-Q2, we first need to analyze the boundary of the terminal set in games of kind. According to the nature of the payoff function, Isaacs divides differential games into games of degree and games of kind [29]. In games of degree, the players strive to maximize and minimize a certain payoff capable of assuming a continuum of numerical values. In games of kind, which are considered in this paper, the achievement of termination itself is the quintessence of the problem. We are interested in what conditions make the capture possible for the pursuer or the escape for the evader. Isaacs introduces the barrier which divides the whole state space Ω into a capture zone (CZ) and an escape zone (EZ).

When the state is on the barrier $\mathscr{B}$, the pursuer must choose the optimal strategy μ^* to prevent the state from entering the EZ, and the evader must also choose the optimal strategy ν^* to ensure that the state is outside the CZ. That is, on the barrier, two sides will choose the optimal strategy (μ^*, ν^*) to achieve a Nash equilibrium, which is the neutral of the game. The neutral is regarded as the delineating case between the capture and the escape. Retaining the concept of the payoff as it appears in the theory of zero-sum games, the payoff is assigned formally as

$$J(\mu,\nu) = \begin{cases} +1 & \text{for no termination or Escape} \\ -1 & \text{for termination or Capture} \\ 0 & \text{neutral (neither Capture nor Escape)} \end{cases}$$

Consequently, when the initial state is on the barrier, the optimal trajectory neither enters the CZ nor the EZ.

The boundary of terminal set $\partial\mathscr{D}$ can also be divided into two parts. One is the usable part (UP), on which the pursuer can force the state to penetrate into the interior of the target set despite the evader's strategy; the other one is the nonuseable part (NUP), on which the evader can frustrate the penetration regardless of the pursuer's strategy. Thus, there will be a boundary between the UP and NUP, which is called the boundary of the usable part (BUP). Obviously, the pursuer and the evader will also try their best to choose the optimal strategy (μ^*, ν^*) on the BUP.

Let $\mathbf{x}$ be a point of $\partial\mathscr{D}$ and let $\lambda = (\lambda_R, \lambda_r, \lambda_\theta)$ be a nonzero vector at $\mathbf{x}$, normal to $\partial\mathscr{D}$ and extending into $\mathscr{D}$. The Hamiltonian of the TAD differential game is

$$\begin{aligned} H(\mathbf{x},\phi,\chi,\psi,\lambda) &= \sum_{i=1}^{3} \lambda_i f_i(\mathbf{x},\phi,\chi,\psi) = \lambda_R(\alpha\cos\phi \\ &- \cos(\theta-\chi)) + \lambda_r(-\cos\chi - \beta\cos\psi) + \lambda_\theta((\sin(\theta(t) \\ &- \chi(t)) - \alpha\sin\phi(t))/R + (\sin\chi(t) - \beta\sin\psi(t))/r). \end{aligned}$$

$H(\mathbf{x},\phi,\chi,\psi,\lambda)$ denotes the change rate of the system state $\mathbf{x}$. If $H(\mathbf{x},\phi,\chi,\psi,\lambda) < \mathbf{0}$, $\mathbf{x}$ penetrates $\partial\mathscr{D}$ and the game terminates. If $H(\mathbf{x},\phi,\chi,\psi,\lambda) > 0$, $\mathbf{x}$ cannot penetrate $\partial\mathscr{D}$.

Based on the problem statement of the TAD game, the terminal set of the Attacker denoted by $\mathscr{D}_A$ can be described as

$$\mathscr{D}_A = \{R, r, \theta \,|\, R < R_c, r > r_c\}.$$

The terminal set of the Target-Defender team denoted by $\mathscr{D}_{TD}$ can be described as

$$\mathscr{D}_{TD} = \{R, r, \theta \,|\, R > R_c, r < r_c\}.$$

Then, obviously, three boundaries of the terminal set exist:

$$\begin{aligned} \partial\mathscr{D}_1 &= \{R, r, \theta | R = R_c \wedge r > r_c\}, \\ \partial\mathscr{D}_2 &= \{R, r, \theta | R > R_c \wedge r = r_c\}, \\ \partial\mathscr{D}_3 &= \{R, r, \theta | R = R_c \wedge r = r_c\}. \end{aligned}$$

Next, we first analyze whether the TAD game can terminate on these three boundaries and give the corresponding strategies for the Attacker.

A. Boundary $\partial\mathscr{D}_1$

The boundary $\partial\mathscr{D}_1$ is the boundary of the terminal set for the Attacker. On this boundary, the Attacker aims to decrease R and penetrate $\partial\mathscr{D}_1$. The purpose of the Target-Defender team is the opposite that of the Attacker.

Lemma 3.1: When $\mathbf{x}$ is on the boundary $\partial\mathscr{D}_1$, if the Attacker adopts the optimal strategy

$$\chi^* = \theta, \tag{7}$$

then it can successfully capture the Target.

Proof: Let $\mathbf{x} = (R, r, \theta)$ be a point of $\partial\mathscr{D}_1$ and let $\lambda = (\lambda_R, \lambda_r, \lambda_\theta)$ be a nonzero vector at $\mathbf{x}$, normal to $\partial\mathscr{D}_1$ and extending into $\mathscr{D}_A$. The condition

$$\min_{\chi(\cdot)} \max_{\phi(\cdot)\psi(\cdot)} H(\mathbf{x},\phi,\chi,\psi,\lambda) < 0, \mathbf{x} \in \partial\mathscr{D}_1 \tag{8}$$

expresses the fact that the Attacker can force $\mathbf{x}$ to penetrate $\partial\mathscr{D}_1$ despite all oppositions. The subset of $\partial\mathscr{D}_1$ for which (8) is true is the useable part for the Attacker.

It is noted that the Hamiltonian and the dynamics are decoupled in controls ϕ, ψ and χ. Hence, the Isaacs's condition holds [29], and the optimal strategies of the players satisfy the following equation:

$$\begin{aligned} H(\mathbf{x},\phi^*,\chi^*,\psi^*,\lambda) &= \min_{\chi(\cdot)} \max_{\phi(\cdot)\psi(\cdot)} H(\mathbf{x},\phi,\chi,\psi,\lambda) \\ &= \max_{\phi(\cdot)\psi(\cdot)} \min_{\chi(\cdot)} H(\mathbf{x},\phi,\chi,\psi,\lambda). \end{aligned} \tag{9}$$

By solving (9), the optimal control headings of the three players in terms of the co-state variables are obtained:

$$\sin\phi^* = -\lambda_\theta/(R\rho_1), \cos\phi^* = \lambda_R/\rho_1, \rho_1 = \sqrt{(\lambda_\theta/R)^2 + \lambda_R^2}, \tag{10}$$

$$\sin\psi^* = -\lambda_\theta/(r\rho_2), \cos\psi^* = -\lambda_r/\rho_2, \rho_2 = \sqrt{(\lambda_\theta/r)^2 + \lambda_r^2}, \tag{11}$$

$$\begin{aligned} &\sin\chi^* = a/\rho_3, \quad \cos\chi^* = b/\rho_3, \quad \rho_3 = \sqrt{a^2+b^2}, \\ &a = \lambda_R\sin\theta + \lambda_\theta\cos\theta/R - \lambda_\theta/r, \\ &b = \lambda_R\cos\theta + \lambda_r - \lambda_\theta\sin\theta/R. \end{aligned} \tag{12}$$

We parameterize $\partial\mathscr{D}_1$ by

$$R(\tilde{t}) = R_c, r(\tilde{t}) = s_2, \theta(\tilde{t}) = s_3, s_2 > r_c, 0 < s_3 < \pi, \tag{13}$$

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where $\tilde{t}$ is the time that $\mathbf{x}(\tilde{t})$ reaches the boundary $\partial\mathscr{D}_1$.

A vector $\lambda(\tilde{t}) = (\lambda_R(\tilde{t}), \lambda_r(\tilde{t}), \lambda_\theta(\tilde{t}))$ normal to $\partial\mathscr{D}_1$ and

$$\lambda_R(\tilde{t})\frac{\partial R(\tilde{t})}{\partial s_j} + \lambda_r(\tilde{t})\frac{\partial r(\tilde{t})}{\partial s_j} + \lambda_\theta(\tilde{t})\frac{\partial \theta(\tilde{t})}{\partial s_j} = 0, j = 2, 3, \quad (14)$$

is also satisfied on $\partial\mathscr{D}_1$. Owing to the length of the normal vector is positive, we can take λ as a unit normal vector. Thus, substituting (13) into (14), we obtain

$$\lambda_R(\tilde{t}) = 1, \lambda_r(\tilde{t}) = 0, \lambda_\theta(\tilde{t}) = 0. \quad (15)$$

From (15), the optimal strategy (12) on the boundary $\partial\mathscr{D}_1$ can be rewritten as (7).

Substituting (10), (11), (12) and (15) into (8), then, it can be rewritten as

$$H(\mathbf{x}, \phi^*, \chi^*, \psi^*, \lambda) = (\alpha - 1)\lambda_R < 0,$$

that is, the condition (8) always holds. The entire $\partial\mathscr{D}_1$ is usable for the Attacker. On the boundary $\partial\mathscr{D}_1$, the Attacker can force the state $\mathbf{x}$ to penetrate into the interior of the terminal set $\mathscr{D}_A$ and successfully capture the Target. ■

B. Boundary $\partial\mathscr{D}_2$

The boundary $\partial\mathscr{D}_2$ is the boundary of the terminal set for the Target-Defender team. On this boundary, the Target and Defender aim to decrease r and penetrate $\partial\mathscr{D}_2$.

Lemma 3.2: When $\mathbf{x}$ is on the boundary $\partial\mathscr{D}_2$, if the Attacker adopts the optimal strategy

$$\chi^* = \pi, \quad (16)$$

then it can successfully evade the Defender.

Proof: Let $\mathbf{x} = (R, r, \theta)$ be a point of $\partial\mathscr{D}_2$ and let $\lambda = (\lambda_R, \lambda_r, \lambda_\theta)$ be a nonzero vector at $\mathbf{x}$, normal to $\partial\mathscr{D}_2$ and extending into $\mathscr{D}_{TD}$. The condition

$$\max_{\chi(\cdot)} \min_{\phi(\cdot)\psi(\cdot)} H(\mathbf{x}, \phi, \chi, \psi, \lambda) < 0, \mathbf{x} \in \partial\mathscr{D}_2 \quad (17)$$

expresses the fact that the Target-Defender team can force $\mathbf{x}$ to penetrate $\partial\mathscr{D}_2$ despite all opposition.

Similarly to (9), the optimal strategies of the players satisfy the following equation:

$$\begin{aligned} H(\mathbf{x}, \phi^*, \chi^*, \psi^*, \lambda) &= \max_{\chi(\cdot)} \min_{\phi(\cdot)\psi(\cdot)} H(\mathbf{x}, \phi, \chi, \psi, \lambda) \\ &= \min_{\phi(\cdot)\psi(\cdot)} \max_{\chi(\cdot)} H(\mathbf{x}, \phi, \chi, \psi, \lambda). \end{aligned} \quad (18)$$

Solving (18), we obtain:

$$\sin\phi^* = \lambda_\theta/(R\rho_1), \cos\phi^* = -\lambda_R/\rho_1, \rho_1 = \sqrt{(\lambda_\theta/R)^2 + \lambda_R^2}, \quad (19)$$

$$\sin\psi^* = \lambda_\theta/(r\rho_2),\ \cos\psi^* = \ \lambda_r/\rho_2,\ \rho_2 = \sqrt{(\lambda_\theta/r)^2 + \lambda_r^2}, \quad (20)$$

$$\sin\chi^* = -a/\rho_3,\ \cos\chi^* = -b/\rho_3,\ \rho_3 = \sqrt{a^2 + b^2}. \quad (21)$$

We parameterize $\partial\mathscr{D}_2$ by

$$R(\tilde{t}) = s_1, r(\tilde{t}) = r_c, \theta(\tilde{t}) = s_3, s_1 > R_c, 0 < s_3 < \pi.$$

Thus, the normal vectors $\lambda_\theta(\tilde{t}) = 0$, $\lambda_R(\tilde{t}) = 0$, and $\lambda_r(\tilde{t}) = 1$. Then, the optimal strategy (21) can be rewritten as (16).

Substituting (19), (20), and (21) into (17), we have

$$H(\mathbf{x}, \phi^*, \chi^*, \psi^*, \lambda) = (1 - \beta)\lambda_r > 0.$$

Therefore, the condition (17) is always false, and the entire $\partial\mathscr{D}_2$ is unusable for the Target–Defender team. On the boundary $\partial\mathscr{D}_2$, the Defender cannot force the state $\mathbf{x}$ to penetrate into the interior of the terminal set $\mathscr{D}_{TD}$ and the Attacker can successfully evade the Defender. ■

Note that the entire $\partial\mathscr{D}_2$ is a condition that forces the Attacker to switch the roles.

C. Boundary $\partial\mathscr{D}_3$

Lemma 3.3: When $\mathbf{x}$ is on the set $\{R, r, \theta | R = R_c, r = r_c, \eta < \theta < \pi\} \in \partial\mathscr{D}_3$, if the Attacker adopts the strategy

$$\chi^* = \begin{cases} \arccos(-\beta) & \eta < \theta < \arccos(-\beta) \quad (22) \\ \theta & \arccos(-\beta) \leqslant \theta < \pi \quad (23) \end{cases}$$

where $\eta = \arccos(-\beta) - \arccos(\alpha)$, the Attacker can successfully capture the Target while avoiding being captured by the Defender.

Proof: When $\mathbf{x}$ is on the boundary $\partial\mathscr{D}_3$, the Attacker wins the game if the state variables satisfy the following conditions at the same time:

$$\dot{R} < 0 \quad \text{and} \quad \dot{r} \geqslant 0. \quad (24)$$

We parameterize $\partial\mathscr{D}_3$ by

$$R(\tilde{t}) = R_c, r(\tilde{t}) = r_c, \theta(\tilde{t}) = s_3, 0 < s_3 < \pi.$$

When $\mathbf{x}$ reaches the boundary $\partial\mathscr{D}_3$, we have

$$\dot{R} = \alpha\cos\phi^* - \cos(s_3 - \chi^*),$$
$$\dot{r} = -\cos\chi^* - \beta\cos\psi^*.$$

The Target adopts the strategy $\phi^* = 0$ to maximize R, and the Defender adopts the strategy $\psi^* = 0$ to minimize r. According to (24), if the Attacker wins the game on the boundary $\partial\mathscr{D}_3$, it follows that

$$\cos(s_3 - \chi^*) > \alpha \quad \text{and} \quad \cos\chi^* \leqslant -\beta. \quad (25)$$

Combining the two formulas of (25), we have

$$\eta < s_3 < \pi. \quad (26)$$

That is, the set $\{R, r, \theta | R = R_c, r = r_c, \eta < \theta < \pi\}$ is the useable part for the Attacker.

In order to capture the Target in the shortest time, the Attacker adopts the control strategy to maximize $|\dot{R}|$, that is, to minimize $|\theta - \chi^*|$. Thus, on the premise of guaranteeing $\chi^* \geqslant \arccos(-\beta)$, when $\eta < \theta < \arccos(-\beta)$, the optimal strategy for the Attacker is (22); when $\arccos(-\beta) \leqslant \theta < \pi$, the optimal strategy for the Attacker is (23). ■

According to the analysis of the above three boundaries, the Attacker plays a pursuer on the boundary $\partial\mathscr{D}_1$, and an evader on the boundary $\partial\mathscr{D}_2$. The boundary $\partial\mathscr{D}_3$ is a particular boundary on which the Attacker plays both the pursuer and the evader. In the useable part of $\partial\mathscr{D}_3$, the Attacker can adopt the strategy to decrease R and increase r at the same time. In the next section, we extend the analysis of boundary $\partial\mathscr{D}_3$ to the whole system state space, and qualitatively analyze whether and how the Attacker can win the game.

IV. Strategy switch of the Attacker

From the analysis of the boundary $\partial\mathscr{D}_3$, when $\theta > \eta$, the Attacker can capture the Target while avoid being captured by the Defender. When $\theta < \eta$, the Attacker should maneuver around the Defender to increase θ, and then chase the Target. On the contrary, the Target-Defender team should cooperate to decrease the angle θ. Therefore, this game can be regarded as an included angle game [8], in which the Attacker attempts to increase the angle while the Target-Defender team hopes to decrease it. At the beginning of the TAD game $\{R_0 > R_c, r_0 > r_c, 0 \le \theta_0 < \pi\}$, and the included angle game terminates when the system state satisfies one of the following conditions:

1) If $\{R > R_c, r > r_c, \theta > \eta\}$, the Attacker wins the included angle game.
2) If $\{R > R_c, r > r_c, \theta < \eta\}$ indefinitely, the Target–Defender team wins the included angle game.

In this included angle game, the boundary of the terminal set can be written as

$$\partial\mathscr{D}_4 = \{R, r, \theta \,|\, R > R_c, r > r_c, \theta = \eta\}.$$

According to the theory of barrier [29], the point that satisfies the following condition,

$$H(\mathbf{x}, \phi^*, \chi^*, \psi^*, \lambda)|_{\partial\mathscr{B}} = 0, \tag{27}$$

is called the BUP of the terminal set.

Equation (27) is the first main equation, which can be rewritten as

$$\begin{aligned} &\max_{\chi(\cdot)} \min_{\phi(\cdot)\psi(\cdot)} \lambda_R(\alpha\cos\phi - \cos(\theta-\chi)) + \lambda_r(-\cos\chi \\ &- \beta\cos\psi) + \lambda_\theta((\sin(\theta(t)-\chi(t)) - \alpha\sin\phi(t))/R \\ &+ (\sin\chi(t) - \beta\sin\psi(t))/r) = 0. \end{aligned} \tag{28}$$

Solving (28), the optimal control headings of the three players on the barrier are the same as (19), (20), and (21).

Furthermore, $\partial\mathscr{D}_4$ can be parameterized as

$$R(\tilde{t}) = s_1, r(\tilde{t}) = s_2, \theta(\tilde{t}) = \eta.$$

Then, we can obtain that

$$\lambda_R(\tilde{t}) = 0, \lambda_r(\tilde{t}) = 0, \lambda_\theta(\tilde{t}) \text{ is free.}$$

Thus, the optimal control strategies of the three players can be rewritten as follows:

$$\phi^* = \pi/2, \qquad \psi^* = \pi/2, \tag{29}$$

$$\cos\chi^* = r\sin\theta/d, \qquad \sin\chi^* = (R - r\cos\theta)/d, \tag{30}$$

where $d = \sqrt{R^2 + r^2 - 2Rr\cos\theta}$.

Substituting (29) and (30) into (28) yields the second main equation:

$$-\alpha r - \beta R + d = 0. \tag{31}$$

From the expression of d in (31), we have

$$\cos\theta = \cos\Theta_1 = \frac{(1-\alpha^2)r^2 + (1-\beta^2)R^2 - 2\alpha\beta Rr}{2Rr}. \tag{32}$$

Based on Θ_1, we give the following theorem.

Theorem 4.1: If the initial state of the TAD game satisfies

$$\theta_0 > \Theta_1, \tag{33}$$

then the Attacker, which adopts the control strategy (30), wins the included angle game.

Proof: (33) can be equivalently written as

$$d_0 > \alpha r_0 + \beta R_0. \tag{34}$$

Substituting (29) and (30) into system equations (4)–(6), we obtain

$$\begin{aligned} \dot\theta &= -\alpha/R - \beta/r + d/(Rr), \quad && \dot\theta > 0, \\ \dot R &= -R\sin\theta/d, && \dot R < 0, \\ \dot r &= -r\sin\theta/d, && \dot r < 0. \end{aligned}$$

Since the game continues when $R > R_c$, $r > r_c$, and $0 < \theta < \pi$, letting $b = R/r$, we have

$$\dot b = (r\dot R - R\dot r)/r^2 = 0.$$

Then, Θ_1 can be described as

$$\cos\Theta_1 = (1-\beta^2)b/2 + (1-\alpha^2)/(2b) - \alpha\beta.$$

Therefore, the value of Θ_1 is constant when the three players play optimally, while the value of $\theta(t)$ will increase monotonically. Thus, the Attacker can adopt the strategy (30) to increase θ to η. That is, the Attacker can win the included angle game. ■

In the TAD game, whether the Attacker can capture the Target and win the game depends on $R < R_c$. This included angle game is a sub-game of the TAD game. When the Attacker wins the included angle game, the Attacker also needs to adopt the strategy to reduce the distance R. It is worth noting that there is a coupling relationship between the included angle and the distances R and r. Thus, we give the following remark.

Remark 4.1: When $\theta_0 > \Theta_1$, the three players adopt control strategies (29) and (30) in the included angle game. In this process, $R(t)$ and $r(t)$ decrease monotonically.

If the initial state satisfies $R_0/r_0 > R_c/r_c$, the TAD game may reach the boundary $\partial\mathscr{D}_2$ before the included angle game ends. That is, the Attacker needs to switch its strategy to evade the Defender.

Similarly, when the initial state satisfies $R_0/r_0 < R_c/r_c$, the TAD game may reach the boundary $\partial\mathscr{D}_1$ before the included angle game ends. That is, the Attacker may win the TAD game before the included angle game ends. □

Based on Lemma 3.1, Lemma 3.2, Lemma 3.3, and Theorem 4.1, we give the following definition.

Definition 4.1: Stage I: the system state satisfies $\{R > R_c, r \ge r_c, \Theta_1 < \theta \leqslant \eta\}$.

Stage II: the system state satisfies $\{R > R_c, r > r_c, \eta < \theta < \arccos(-\beta)\}$ or $\{R > R_c, r = r_c\}$.

Stage III: the system state satisfies $\{R > R_c, r \ge r_c, \arccos(-\beta) \leqslant \theta < \pi\}$.

The Attacker has different roles in different stages. Now, based on the above definition, we summarize how can the Attacker switch its strategies to capture the Target. First, according to the current data of the state, the parameters λ,

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R, r, d and θ are computed and the stage in which the system state lies can be determined. If the state lies in stage I, the Attacker adopts the strategy (30) to increase θ and decrease R. The Attacker plays two roles at the same time and creates opportunities to capture the Target directly. If the state lies in stage II, the Attacker adopts the strategy (22) to decrease R while guaranteeing $\dot{r} = 0$. The Attacker plays an evader while trying its best to pursue the Target. If the state lies in stage III, the Attacker adopts the strategy (7) to directly pursue the Target. The Attacker plays a pure pursuer.

In the process of the Attacker switching roles, it also implies the transformation of the cooperation of the Target–Defender team. In stage I, the Target and Defender cooperate to decrease the included angle θ. In stage II, the Defender plays a pursuer to capture the Attacker. In stage III, the Target plays an evader to escape from the Attacker.

V. Rendezvous strategy of the Target and Defender

According to the boundary analysis and the included angle game described above, the TAD game terminates only when the Target is captured. The Target–Defender team cannot end the TAD game, and they can only stay in $\{R > R_c, r > r_c, \theta < \eta\}$ indefinitely to avoid the Attacker winning the game. Because the Attacker has the speed advantage, and the Target and Defender are incapable of directly preventing an engagement by capturing the Attacker.

However, there is a special case in which $R_c < r_c$, the Target–Defender team maybe can end the TAD game. When $R_c < r_c$, the Target and Defender can come close to each other (to a distance of less than $d_c = r_c - R_c$) and stay in such a position. After this, the Attacker is unable to capture the Target since it would be captured by the Defender if it tries to get closer to the Target. Therefore, from the perspective of the Target and Defender, they can adopt a rendezvous strategy to win the game.

In fact, the rendezvous of the Target and Defender can also be regarded as a pursuit-evasion game of the Target and Defender. Thus, there are two pursuit-evasion games: Target–Defender and Target–Attacker. The Target plays a dual role. It needs to make a balance between rendezvousing with the Defender and avoiding the pursuit of the Attacker.

First, we consider the pursuit-evasion game with a pursuer and an evader. According to the barrier theory [29] and the isochrones principle [30], we have the following lemma.

Lemma 5.1: In a pursuit-evasion game with a pursuer (P) and an evader (E), if E moves along an arbitrary trajectory, and P aims to capture E in the shortest time, then the optimal strategy of P is the parallel strategy [29], [30]

$$\xi = \arcsin(\alpha \sin \varsigma),$$

where ς is the included angle between the motion direction of E and the line of sight of PE, ξ is the included angle between the motion direction of P and the line of sight of PE. □

Lemma 5.1 can be directly extended in the Target–Defender game and the Target–Attacker game. As shown in Fig. 2, ϕ_1, χ_1, and ψ_1 are the control variables of the players, defined as the relative headings of the Target, Attacker, and Defender from the vectors $\overrightarrow{TA}$,$\overrightarrow{AT}$, and $\overrightarrow{DT}$, respectively. γ is the included angle $\angle DTA$, $\gamma = Arg(\overrightarrow{TA}) - Arg(\overrightarrow{TD})$. The initial value is expressed in γ_0. If the Target chooses the control angle ϕ_1 to move along an arbitrary trajectory, the Attacker adopts the strategy $\chi_1 = \arcsin(\alpha \sin \phi_1)$ to pursue the Target. The Defender adopts the strategy $\psi_1 = \arcsin((\alpha/\beta) \sin \phi_1)$ to rendezvous with the Target. The rates of change of the distances R and d are expressed as follows:

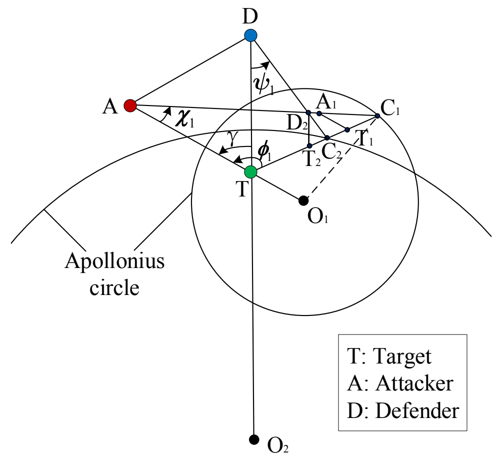

Fig. 2: The Target rendezvous with the Defender.

$$\begin{aligned} \dot{R} &= -\cos \chi_1 - \alpha \cos \phi_1, \\ \dot{d} &= -\beta \cos \psi_1 - \alpha \cos(\phi_1 - \gamma). \end{aligned} \tag{35}$$

In order to win the game, the Target and Defender should come close to each other before the Attacker minimizes the distance R to R_c. Therefore, we give the following theorem.

Theorem 5.1: When $R_c < r_c$, if the initial state of the TAD game satisfies

$$\cos \gamma_0 < (a_1^2 + a_2^2 - k_3(d_0^2 - R_0^2))/(2a_1 a_2), \tag{36}$$

where

$$\begin{aligned} k_1 &= \alpha/\beta, \\ k_2 &= \alpha d_0 (R_0 - R_c)/(k_1 R_0 (d_0 - d_c)), \\ k_3 &= 1 - \alpha^2 - {k_2}^2 + {k_2}^2 k_1^2, \\ a_1 &= \alpha R_0, \\ a_2 &= k_1 d_0 k_2, \end{aligned}$$

then the Target and Defender can rendezvous to win the game by adopting the following strategies

$$\begin{aligned} \cos \phi_1^* &= (a_1 - a_2 \cos \gamma_0)/\rho, \quad \sin \phi_1^* = -a_2 \sin \gamma_0 / \rho, \\ \rho &= \sqrt{(a_1 - a_2 \cos \gamma_0)^2 + (a_2 \sin \gamma_0)^2}, \end{aligned} \tag{37}$$

$$\psi_1^* = \arcsin(k_1 \sin(\phi_1 - \gamma_0)). \tag{38}$$

Proof: We define t_a as the time that the Attacker forces the Target into its capture range and t_d as the time that the distance between Target and Defender is less than the rendezvous range d_c. Since the three players adopt parallel strategies, their control angles are constant. Thus, t_a and t_d can be described as follows:

$$t_a = (R_0 - R_c)/\dot{R}, \quad t_d = (d_0 - d_c)/\dot{d}. \tag{39}$$

If $t_a > t_d$, the Attacker is unable to capture the Target, and the Target and Defender win the game, and vice versa.

By (35) and (39), the Target and Defender can rendezvous to win the game when the initial state satisfies the following inequality:

$$\frac{\dot{R}}{\dot{d}} = \frac{\cos\chi_1 + \alpha\cos\phi_1}{\beta\cos\psi_1 + \alpha\cos(\phi_1 - \gamma_0)} < \frac{R_0 - R_c}{d_0 - d_c}. \tag{40}$$

Obviously, it is difficult to obtain an analytical solution to this inequality, so we use a geometric method to find the solution. As shown in Fig. 2, the circle O_1 is an Apollonius circle of the Attacker and Target. The circle O_2 is an Apollonius circle of the Defender and Target.

Assume that the Target adopts the control angle ϕ_1 to move towards the point C_1 which is on the circle O_1, and the Attacker adopts a corresponding control angle χ_1 to move towards the point C_1.

Let $|AC_1| = l_1$. According to the definition of the Apollonius circle, we have $|TC_1| = \alpha l_1$. For the triangle $\triangle TAC_1$,

$$R_0^2 + (\alpha l_1)^2 - l_1^2 = 2\alpha R_0 l_1 \cos\phi_1. \tag{41}$$

Assuming $|A_1T_1| = R_c$, and according to $A_1T_1/AT = T_1C_1/TC_1 = R_c/R_0$, we have

$$TT_1 = TC_1 - T_1C_1 = \alpha l_1(1 - R_c/R_0).$$

Similarly, let $|DC_2| = l_2$, then $|TC_2| = k_1 l_2$. For the triangle $\triangle TDC_2$,

$$d_0^2 + (k_1 l_2)^2 - l_2^2 = 2d_0 k_1 l_2 \cos(\phi_1 - \gamma_0). \tag{42}$$

Assuming $|D_2T_2| = d_c$, we have

$$TT_2 = TC_2 - T_2C_2 = k_1 l_2(1 - d_c/d_0).$$

The inequality (40) is equivalent to $|TT_2| < |TT_1|$, that is,

$$k_1 l_2(1 - d_c/d_0) \leqslant \alpha l_1(1 - R_c/R_0). \tag{43}$$

Substituting (41) and (42) into (43), we obtain an inequality in terms of l_1:

$$k_3 l_1^2 + 2(a_1\cos\phi_1 - a_2\cos(\phi_1 - \gamma_0))l_1 - (R_0^2 - d_0^2) < 0. \tag{44}$$

The capture radius is much smaller than the initial distance, that is, $R_c << R_0, d_c << d_0$. Thus, $k_2 \approx \beta$ and $k_3 > 0$. Therefore, if (44) has a solution, it must be satisfied that

$$(a_1\cos\phi_1 - a_2\cos(\phi_1 - \gamma_0))^2 > k_3 * (d_0^2 - R_0^2). \tag{45}$$

The left-hand side of the inequality (45) can be considered a function of ϕ_1. It has a maximum $(a_1 - a_2\cos\gamma_0)^2 + (a_2\sin\gamma_0)^2$, when the Target and Defender adopt strategies (37) and (38), respectively.

Therefore, if the initial state of the TAD game satisfies the condition (36), the Target and Defender can adopt strategies (37) and (38) to win the game. ■

Remark 5.1: In particular, when the Attacker is far away from the Target and Defender, the Target and Defender can move towards each other to win the game, that is,

$$\phi_1^* = \gamma_0, \tag{46}$$

$$\psi_1^* = 0. \tag{47}$$

Then, $l_2 = d/(1+k_1)$. Substituting (46) and (41) into (43),we have

$$\cos\gamma_0 < \frac{k_2^2 R_0^2 (1+k_1)^2 - d_0^2(1-\alpha^2)}{2\alpha(1+k_1)k_2 R_0 d_0}. \tag{48}$$

In other words, when the initial state satisfies the condition (48), the Target and Defender can adopt strategies (46) and (47) to win the game in the shortest time. □

VI. Simulation Results

The role switch of the Attacker and the rendezvous of the Target-Defender team are illustrated in this section.

Fig. 3(a) illustrates one instance of the Attacker that switches two control strategies. At the beginning, the system state is in stage I. The Attacker adopts the strategy (30) to increase the included angle θ, while the Target-Defender team adopts the strategy (29). When the Attacker reaches the boundary $\partial\mathscr{D}_2$, the system state enters stage II. The Attacker switches its control strategy to (22) to avoid being captured by the Defender. When $\theta = \arccos(-\beta)$, the system state enters stage III. The Defender cannot prevent the Attacker from chasing the Target, and the Attacker switches its strategy to the direct pursuit strategy (7) to pursue the Target.

Fig. 3(b) illustrates one instance of the Attacker that switches one control strategy. When the Attacker reaches the boundary $\partial\mathscr{D}_2$, the system state changes from stage I to stage II. The Attacker switches its control strategy from (30) to (22).

Fig. 3(c) illustrates one instance of the Attacker that does not switch control strategies. In the process of adopting the strategy (30) to increase θ, the Attacker successfully captures the Target.

Fig. 4 illustrates two instances of rendezvous patterns. When the initial state satisfies the condition (36), the Target-Defender team should choose an appropriate point to rendezvous. The corresponding strategies are (37) and (38) (see Fig. 4(a)). Fig. 4(b) shows that the Target directly rendezvouses with the Defender when the condition (48) holds. The Target and Defender can adopt strategies (46) and (47) to move towards each other and to win the game.

VII. Conclusions

In this paper, we have formulated a Target-Attacker-Defender differential game with a non-suicidal faster Attacker. First, we have shown the influence of the terminal set boundary on the role switch of the Attacker. Then, 1) we have characterized the role switch of the Attacker and derived the switching conditions and corresponding control strategies for the Attacker, and further 2) we have characterized the winning conditions under which the Target-Defender team can win the game by rendezvousing with each other regardless of the Attacker. The results may disclose some natural phenomena

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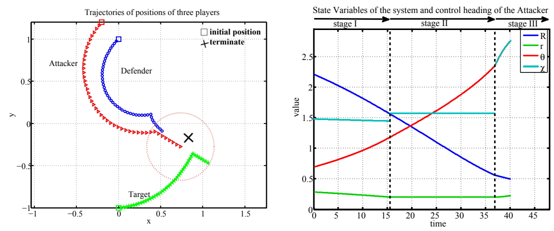
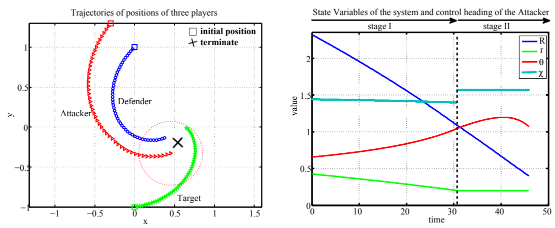
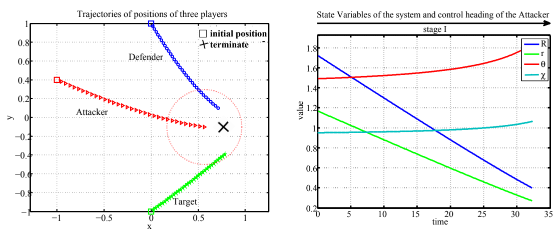


(a) $x_A = \quad 0.2, y_A = 1.2$.

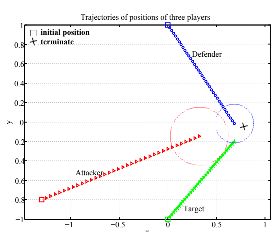


(b) $x_A = \quad 0.3, y_A = 1.3$.

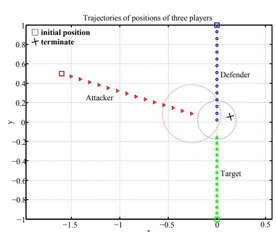


(c) $x_A = \quad 1, y_A = 0.4$.

Fig. 3: Illustration of dynamics and characteristic variables. $x_T = 0$, $y_T = -1$, $x_D = 0$, $y_D = 1$, $\alpha = 0.6$, $\beta = 0.7$, $R_c = 0.4$, and $r_c = 0.2$, except values specified.



(a) $x_A = \quad 1.3, y_A = \quad 0.8$.



(b) $x_A = \quad 1.6, y_A = \quad 0.5$.

Fig. 4: Trajectories of the three players with different rendezvous strategies. $x_T = 0$, $y_T = -1$, $x_D = 0$, $y_D = 1$, $\alpha = 0.6$, $\beta = 0.7$, $R_c = 0.3$, $r_c = 0.5$, except values specified.

and suggest some applications in the competition of multiple agents.

In future, we will investigate a TAD game with more practical motion models of the players, more complex game environments (containing obstacles, limited observation and intercommunication, jamming and measurement noise), and different rules (ways of cooperation, conditions of ending the game). In addition, a TAD game with more players is another focus forward in the direction of our research.

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