2018 Annual American Control Conference (ACC)
June 27–29, 2018. Wisconsin Center, Milwaukee, USA

Optimal Target Capture Strategies in the Target-Attacker-Defender Differential Game

Eloy Garcia, David W. Casbeer, and Meir Pachter

***Abstract*— A zero-sum differential game with three players, a Target, an Attacker, and a Defender is considered. The Attacker pursues the Target aircraft. The Defender strives to intercept the Attacker before it reaches the aircraft. The two termination set differential game is analyzed in this paper. The players' optimal state feedback strategies for the case where the Attacker is able to capture the Target despite the presence of the Defender are derived. It is also shown how these results mesh with the previously obtained strategies of active target defense in order to obtain the complete solution of this two termination set differential game.**

I. INTRODUCTION

Differential game theory provides the right set of tools to analyze pursuit-evasion problems and combat games. Pursuit-evasion scenarios involving multiple agents represent important and challenging types of problems in aerospace, control, and robotics. They are also useful in order to analyze biologically inspired behaviors. For instance, [1] addressed a scenario where two evaders employ coordinated strategies to evade a single pursuer, but also to keep them close to each other. The authors of [2] discussed a multi-player pursuit-evasion game with line segment obstacles labeled as the Prey, Protector, and Predator Game. Different multi-agent cooperative behaviors have been analyzed in [3]–[6].

The Target-Attacker-Defender (TAD) scenario is a relevant problem in combat and conflict scenarios [7]–[9]. Recent work concerning this problem has addressed optimal control problems where one team assumes a fixed guidance law by the opponent such as in [10]–[13]. The work in [14] considers resource allocations within the TAD problem in the presence of multiple targets, attackers, and defenders.

In this paper the TAD scenario is formulated as a differential game. The TAD differential game provides an illustrative application of differential game theory with all of its possible outcomes and strategies. The focus of the paper is the design of the players' optimal strategies when the state of the control system is located in the Attacker's winning region.

The paper is organized as follows. Section II introduces the TAD differential game in the context of differential games with two termination sets. Section III states the game of capture within the TAD differential game. Basic results and properties of the optimal strategies are presented in Section IV. The Game of Kind is analyzed in Section V. The main results are presented in Section VI where the players' optimal state feedback strategies are obtained for the capture game. Illustrative examples are given in Section VII and concluding remarks are made in Section VIII.

E. Garcia and D. Casbeer are with the Control Science Center of Excellence, Air Force Research Laboratory, Wright-Patterson AFB, OH 45433. eloy.garcia.2@us.af.mil; david.casbeer@us.af.mil

M. Pachter is with the Department of Electrical Engineering, Air Force Institute of Technology, Wright-Patterson AFB, OH 45433. meir.pachter@afit.edu

II. PRELIMINARIES

Consider the players T (Target), A (Attacker), and D (Defender) with constant speed V_T, V_A, V_D, respectively, and simple motion as it is commonly encountered in the games of Isaacs [15]. The states of T, A, and D are respectively specified by their Cartesian coordinates $\mathbf{x}_T = (x_T, y_T)$, $\mathbf{x}_A = (x_A, y_A)$, and $\mathbf{x}_D = (x_D, y_D)$. The complete state of the differential game is defined by $\mathbf{x} := (x_T, y_T, x_A, y_A, x_D, y_D) \in \mathbb{R}^6$. The game set is the entire space $\mathbb{R}^6$. It is assumed that $V_A = V_D$; without loss of generality, the speeds of the players are normalized by V_A.

The Attacker's control variable is his instantaneous heading angle, $\mathbf{u}_A = \{\chi\}$. The T & D team affects the state of the game by cooperatively choosing the instantaneous respective headings, ϕ and ψ, of both the Target and the Defender, so the T & D team's control variable is $\mathbf{u}_B = \{\phi, \psi\}$. The dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}_A, \mathbf{u}_B)$ are defined by the system of linear differential equations

$$\begin{aligned}
\dot{x}_T &= \alpha\cos\phi, & x_T(0) &= x_{T_0}\\
\dot{y}_T &= \alpha\sin\phi, & y_T(0) &= y_{T_0}\\
\dot{x}_A &= \cos\chi, & x_A(0) &= x_{A_0}\\
\dot{y}_A &= \sin\chi, & y_A(0) &= y_{A_0}\\
\dot{x}_D &= \cos\psi, & x_D(0) &= x_{D_0}\\
\dot{y}_D &= \sin\psi, & y_D(0) &= y_{D_0}
\end{aligned} \tag{1}$$

where the speed ratio $\alpha = V_T/V_A < 1$ (T is slower than A) is the problem parameter and the admissible controls are the players' headings $\chi, \phi, \psi \in [-\pi, \pi]$. Both, the state and the controls, are unconstrained. The initial state of the system is

$$\mathbf{x}_0 := (x_{T_0}, y_{T_0}, x_{A_0}, y_{A_0}, x_{D_0}, y_{D_0}) = \mathbf{x}(t_0).$$

In this paper, we confine our attention to point capture. This means that the game ends when the $A-T$ separation becomes zero, that is, A captures T. An additional instance of termination entails interception of A by D, that is, the game terminates if the $A-D$ separation becomes zero. Thus, the target (or termination) set is

$$\mathcal{C} := \mathcal{C}_1 \bigcup \mathcal{C}_2 \tag{2}$$

where

$$\mathcal{C}_1 := \left\{ \mathbf{x} \;|x_A = x_T,\ y_A = y_T \right\} \tag{3}$$

represents capture of the Target by the Attacker, and

$$\mathcal{C}_2 := \left\{ \mathbf{x} \;|x_A = x_D,\ y_A = y_D \right\} \tag{4}$$

978-1-5386-5428-6/$31.00 ©2018 AACC

represents the opposite outcome where the Defender intercepts the Attacker and the Target survives. The TAD differential game with termination set given in (2) belongs to the class of two termination set differential games [16]–[18]. The two termination set differential game concept was introduced in order to extend classical pursuit-evasion games where only one termination set is contemplated. For instance, the pursuer tries to minimize the cost to reach the termination set whereas the evader wants to maximize the payoff to reach the termination set or, when possible, to avoid reaching that set at all. Naturally, the two termination set differential game is useful in the analysis of combat games [18] where the roles of pursuer and evader are not designated ahead of time; instead each player wants to defeat the opponent by terminating the game in its own capture set.

The two termination conditions in (2) give rise to the Game of Kind. The solution of the Game of Kind in the TAD differential game characterizes the Barrier surface which divides the state space into two regions $\mathcal{R}_c$ and $\mathcal{R}_e$ - see Fig. 2 below. If the initial state satisfies $\mathbf{x}_0 \in \mathcal{R}_c$ then, under optimal play, A captures T before D can reach A; in this case the game will terminate when the state $\mathbf{x}$ enters the target set $\mathcal{C}_1$. On the other hand, if $\mathbf{x}_0 \in \mathcal{R}_e$ then, under optimal play, D intercepts A and T escapes; the game will now terminate when the state $\mathbf{x}$ enters the target set $\mathcal{C}_2$.

Two Games of Degree are formulated. The Active Target Defense Differential Game (ATDDG) was formulated and solved in [19], [20]; It solves the TAD differential game for the case where $\mathbf{x}_0 \in \mathcal{R}_e$. It provides the saddle point strategies for all three players: T, A, and D such that the $A-T$ separation at the time of interception of A by D (condition (4)) is maximized by the T & D team and minimized by A.

The focus of this paper is the Game of Degree in $\mathcal{R}_c$, also called the TAD differential game of capture. Solving the game of capture is an important contribution because, together with the solution of the ATDDG, provides the saddle point strategies everywhere in the state space $\mathbb{R}^6$. One can then provide a complete analysis of the problem where it is also necessary that the optimal strategies from the two different Games of Degree agree on the boundary that separates the two regions. For these reasons, it is important first to define the capture game. We introduce, in the following section, the corresponding cost/payoff function for the case where $\mathbf{x}_0 \in \mathcal{R}_c$ which not only provides a logical and practical strategy for the players but it is shown to satisfy the necessary properties of the two termination set differential game.

III. The TAD Differential Game of Capture

Considering the Game of Degree in $\mathcal{R}_c$ or TAD differential game of capture, the termination condition is $\mathcal{C}_1$ in (3). The terminal time t_f is defined as the time instant when the state of the system satisfies (3), at which time the terminal state is $\mathbf{x}_f := (x_{T_f}, y_{T_f}, x_{A_f}, y_{A_f}, x_{D_f}, y_{D_f}) = \mathbf{x}(t_f)$. The terminal cost/payoff functional is

$$J(\mathbf{u}_A(t), \mathbf{u}_B(t); \mathbf{x}_0) = \Phi(\mathbf{x}_f) \tag{5}$$

where

$$\Phi(\mathbf{x}_f) := \sqrt{(x_{D_f} - x_{T_f})^2 + (y_{D_f} - y_{T_f})^2}. \tag{6}$$

The cost/payoff functional depends only on the terminal state - the capture game is a terminal cost/Mayer type game. Its Value is given by

$$V(\mathbf{x}_0) := \max_{\mathbf{u}_A(\cdot)} \min_{\mathbf{u}_B(\cdot)} J(\mathbf{u}_A(\cdot), \mathbf{u}_B(\cdot); \mathbf{x}_0) \tag{7}$$

subject to (1) and (3), where $\mathbf{u}_A(\cdot)$ and $\mathbf{u}_B(\cdot)$ are the teams' state feedback strategies. T and D cooperate in order to minimize their terminal separation; the opponent, player A, strives to capture T and maximize the terminal $T-D$ separation.

Knowing that T will be captured by A, the logical strategy for D is to try to reach a point which is as close as possible to T. From a practical point of view, the strategy of T & D cooperating to minimize their terminal separation provides the best strategy if player A were to err and pursue T using a different guidance law other than the optimal strategy prescribed by the solution of the TAD differential game of capture. In such a case, where A does not play optimally and the T & D team does, the terminal $T-D$ distance will decrease with respect to the Value of the game, which is attained when all players act optimally. Under further non-optimal play by the Attacker, the state can cross the Barrier surface and it will hold that $\mathbf{x} \in \mathcal{R}_e$ where the strategies of the ATDDG, the Game of Degree in $\mathcal{R}_e$, will be used by T and D, and now the Target can escape.

From a theoretical point of view, when the state of the system $\mathbf{x}$ is exactly on the Barrier surface, both Games of Degree have feasible solutions and joint capture is attained. The solution of the Game of Degree in $\mathcal{R}_c$ needs to provide the same strategies and the same Value of the game as the solution of the Game of Degree in $\mathcal{R}_e$ in order to achieve joint capture, *i.e.* A captures T at the same time instant when D intercepts A. This property will be corroborated in the sequel.

IV. Nature of Saddle Point Strategies

Let the co-state be represented by

$$\lambda^T = (\lambda_{x_A}, \lambda_{y_A}, \lambda_{x_D}, \lambda_{y_D}, \lambda_{x_T}, \lambda_{y_T}) \in \mathbb{R}^6. \tag{8}$$

The Hamiltonian of the differential game is

$$\begin{aligned} \mathcal{H} &= \lambda_{x_A} \cos\chi + \lambda_{y_A} \sin\chi + \lambda_{x_D} \cos\psi \\ &\quad + \lambda_{y_D} \sin\psi + \alpha\lambda_{x_T} \cos\phi + \alpha\lambda_{y_T} \sin\phi \end{aligned} \tag{9}$$

where the parameter α is the V_T/V_A speed ratio.

Theorem 1: Consider the game of capture in the TAD differential game (1), (3), (5)-(7). The headings of the players T, A, and D are constant under optimal play and the optimal trajectories are straight lines.

Proof. Proof is given in [21].

Having characterized the optimal strategies in Theorem 1, namely, that the optimal trajectories are straight lines, we can proceed to determine the optimal headings as a function of the current state $\mathbf{x}$. Determining the optimal, *state feedback* strategies for the game of capture in the TAD differential game is the main objective in this paper. To accomplish this objective we first introduce the concept of the Apollonius circle $-$ an important tool in pursuit-evasion games.

The Apollonius circle is the locus of points P with a fixed ratio of distances to two given points which are called foci. For instance, the foci are A and T, where the fixed ratio is $\alpha = \frac{\overline{TP}}{\overline{AP}}$. Consider players A and T flying at constant speeds and with constant heading. In this scenario A aims at capturing T at a point $I = (x_I, y_I)$ on the circumference of the Apollonius circle.

When both agents aim at the same point on the circle the distance traveled by T is equal to α times the distance traveled by A, where $\alpha = \frac{V_T}{V_A}$ is the speed ratio parameter and V_A and V_T are the speeds of the Attacker and the Target. It is important to note that in a differential game the aimpoint of a player is not guessed by the adversary but the solution of the differential game determines the optimal strategies of each player. This means that each player, by solving the differential game, obtains its own and its opponent's optimal heading. When a state feedback solution is obtained, actual implementation of non-optimal strategies is in detriment to the player which does not implement its optimal strategy and this benefits the adversary. This saddle point property will be illustrated for the TAD differential game in Section VII.

In the TAD differential game we define an Apollonius circle using the instantaneous separation between A and T and the speed ratio parameter α. Let C denote the center of the circle. We have that the three points: A, T, and C are collinear. Let $\overline{TC}$ denote the distance between T, the Target position, and C, the center of the Apollonius circle. The distance $\overline{TC}$ is given by

$$\overline{TC} = \frac{\alpha^2}{1-\alpha^2} d \tag{10}$$

where $d = \sqrt{(x_A - x_T)^2 + (y_A - y_T)^2}$ is the distance between agents A and T. Also, let r be the radius of the Apollonius circle. Then, r is given by

$$r = \frac{\alpha}{1-\alpha^2} d. \tag{11}$$

Also note that, in order to minimize the terminal $D - T$ separation, the Defender heads straight to point I. Thus, the points D, D', and I are collinear.

V. Game of Kind

In order to facilitate the design and analysis of the saddle point strategies we consider, without loss of generality, the rotating reference frame illustrated in Fig. 1. In Fig. 1 the points A and D represent the instantaneous positions of the Attacker and the Defender, respectively. The X-axis of this frame goes from D to A which results in $y_A = y_D = 0$. The Y-axis is given by the orthogonal bisector of the segment $\overline{AD}$. Hence, we have that $x_D = -x_A$.

The illustration of the problem as shown in Fig. 1 provides important insight into the outcome of the differential game. If the Apollonius circle intersects the Y-axis, then there exists an strategy for D to intercept A and for T to escape. The ATDDG Game of Degree is played in $\mathcal{R}_e$, where $\mathbf{x} \in \mathcal{R}_e$. On the other hand if the Apollonius circle does no intersect the Y-axis, then A is able to capture T before D can reach

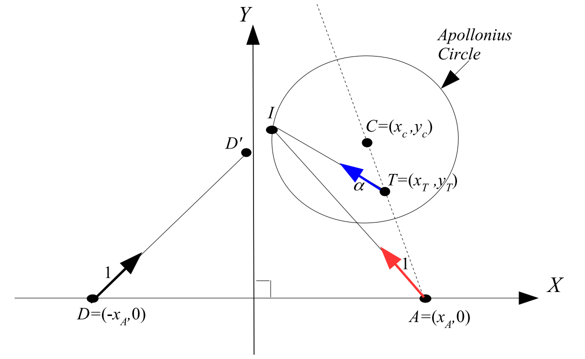

Fig. 1. Coordinate transformation to reduce the dimension of the state space

A; in this case a different Game of Degree, the capture game in the TAD scenario, is played in $\mathcal{R}_c$, where $\mathbf{x} \in \mathcal{R}_c$; this Game of Degree is addressed in this paper. Finally, if the Apollonius circle is only tangent to the Y-axis, then $\mathbf{x} \in \mathcal{B}$ where $\mathcal{B}$ denotes the Barrier surface that divides the state space into the two regions $\mathcal{R}_e$ and $\mathcal{R}_c$.

The Game of Degree in $\mathcal{R}_e$ was solved in [19] and [20] where the optimal headings are constant and the terminal condition is given by (4). The solution in [19] and [20] provided the optimal saddle point strategies for the T & D team to maximize the terminal $A-T$ separation and for A to minimize the same distance. State feedback strategies were obtained in [19] which provide the optimal interception point y^* on the Y-axis. Because $V_A = V_D$ the Y-axis divides the reachable regions of A and D; then, A is intercepted by D on a point located at the Y-axis. The Barrier surface $\mathcal{B}$ that divides the state space into the two regions $\mathcal{R}_e$ and $\mathcal{R}_c$ is characterized as follows.

Proposition 1: For given speed ratio α the Barrier surface $\mathcal{B}$ in the reduced state space (x_A, x_T, y_T) that separates the state space into the two regions $\mathcal{R}_e$ and $\mathcal{R}_c$ is given by

$$\mathcal{B} = \{(x_A, x_T, y_T) | x_A > 0, x_T > 0, x_A^2 + \tfrac{y_T^2}{1-\alpha^2} - \tfrac{x_T^2}{\alpha^2} = 0\}. \tag{12}$$

The Barrier surface (12) is obtained from the condition $x_c = r$, that is, the Apollonius circle is tangent to the Y-axis; such condition can be written as

$$x_T - \alpha^2 x_A - \alpha\sqrt{(x_A - x_T)^2 + y_T^2} = 0. \tag{13}$$

When $x_T < 0$ then $\mathbf{x} \in \mathcal{R}_e$ since T is closer to D than it is to A. The x_A-cross section of (12) is shown in Fig. 2. For fixed x_A the Barrier surface (12) is an (right-hand branch) hyperbola in the $X-Y$ plane. This figure helps to visualize the possible locations of the Target and the associated outcomes. If T is in the right side of the hyperbola, it will be captured, provided that A plays optimally. On the other hand, if T is in the left side of the hyperbola, then it will survive, provided that T/D play optimally.

We place especial emphasis on the Barrier surface. Since the optimal strategies in $\mathcal{R}_c$ will be derived in this paper, then, the complete solution for any state $\mathbf{x} \in \mathbb{R}^6$ can be obtained (where the state space dimension is reduced to 3

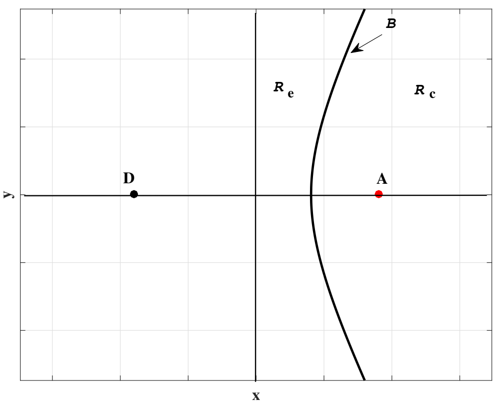

Fig. 2. x_A-cross section of Barrier surface $\mathcal{B}$

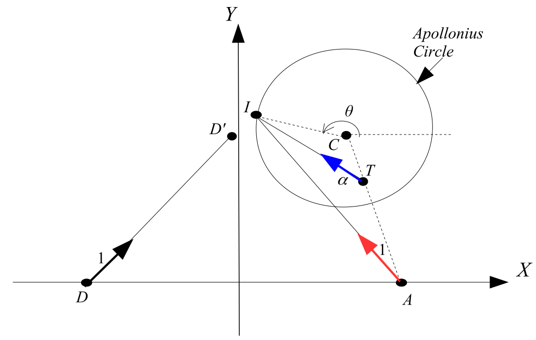

Fig. 3. Game of Capture in relative frame where $I = (r\cos\theta + x_c,\ r\sin\theta + y_c)$

after applying the corresponding coordinate transformation). Because the TAD differential game is a two termination set problem (2)-(4) we need to first determine in which region of the state space the current state is located, that is, use the solution of the Game of Kind given by (12). From that we are able to determine which strategy to implement: employ the solution of the corresponding Game of Degree.

VI. Game of Degree in $\mathcal{R}_c$

The coordinates of the center of the Apollonius circle are denoted by $C = (x_c, y_c)$ and they are given by

$$x_c = \tfrac{1}{1-\alpha^2}x_T - \tfrac{\alpha^2}{1-\alpha^2}x_A, \qquad y_c = \tfrac{1}{1-\alpha^2}y_T. \tag{14}$$

Let $Re(\xi)$ and $Im(\xi)$ be the real and the imaginary part of the complex number ξ. We now provide the main result of the paper which determines the optimal interception point, *i.e.* the aimpoint of the players.

Theorem 2: Consider the Target, Attacker, Defender differential game (1), (3), (5)-(7) and assume that $\mathbf{x} \in \mathcal{R}_c$. Then, the optimal interception point in the reduced state space is $I^* = (r\cos\theta^* + x_c, r\sin\theta^* + y_c)$ where $\theta^* = \arccos Re(\nu^*) = \arcsin Im(\nu^*)$ and ν^* is the solution of the quartic equation

$$\begin{aligned}&[x_cy_c + \tfrac{i}{2}(x_c^2 - x_A^2 - y_c^2)]\nu^4 + r(y_c + ix_c)\nu^3\\ &+r(y_c - ix_c)\nu + x_cy_c - \tfrac{i}{2}(x_c^2 - x_A^2 - y_c^2) = 0\end{aligned} \tag{15}$$

which minimizes the function

$$\begin{aligned}J(\theta) &= \sqrt{(r\cos\theta + x_c + x_A)^2 + (r\sin\theta + y_c)^2}\\ &\quad-\sqrt{(r\cos\theta + x_c - x_A)^2 + (r\sin\theta + y_c)^2}.\end{aligned} \tag{16}$$

Proof. Because the Attacker and the Defender have the same speed, the distance traveled by player D, denoted by $\overline{DD'}$, is the same as the distance traveled by player A, which is represented by $\overline{AI}$, the cost/payoff can be written

$$\begin{aligned}J(\theta) &= \overline{D'I} = \overline{DI} - \overline{DD'} = \overline{DI} - \overline{AI}\\ &= \sqrt{(r\cos\theta + x_c - x_D)^2 + (r\sin\theta + y_c)^2}\\ &\quad-\sqrt{(r\cos\theta + x_c - x_A)^2 + (r\sin\theta + y_c)^2}\end{aligned} \tag{17}$$

where the cost/payoff J has been written only in terms of θ, the angle at C, see Fig. 3. This means, that the interception point coordinates $I = (x_I, y_I)$ have been expressed in terms of only one variable by using the constraint

$$(x_I - x_c)^2 + (y_I - y_c)^2 = r^2 \tag{18}$$

that is, the interception point is located on the Apollonius circle. In order to obtain the optimal angle θ (and the associated optimal interception point) we compute $\frac{dJ(\theta)}{d\theta}$ and set the derivative equal to zero

$$\begin{aligned}\frac{dJ(\theta)}{d\theta} &= \frac{r[-(x_c+x_A)\sin\theta + y_c\cos\theta]}{\sqrt{(r\cos\theta + x_c + x_A)^2 + (r\sin\theta + y_c)^2}}\\ &\quad+\frac{r[(x_c-x_A)\sin\theta - y_c\cos\theta]}{\sqrt{(r\cos\theta + x_c - x_A)^2 + (r\sin\theta + y_c)^2}} = 0\end{aligned} \tag{19}$$

where $x_D = -x_A$ has been used. The previous equation can also be written as follows

$$\begin{aligned}&[(r\cos\theta + x_c - x_A)^2 + (r\sin\theta + y_c)^2]\\ &\times[y_c\cos\theta - (x_c + x_A)\sin\theta]^2\\ &= [(r\cos\theta + x_c + x_A)^2 + (r\sin\theta + y_c)^2]\\ &\times[(x_c - x_A)\sin\theta - y_c\cos\theta]^2\end{aligned} \tag{20}$$

where both sides of the equation were raised to the second power. Expanding the terms in each side of (20) we obtain

$$\begin{aligned}&-4x_Ax_cy_c^2\cos^2\theta + 4(r^2 + y_c^2)x_Ax_c\sin^2\theta\\ &-4y_c[x_A(r^2 + y_c^2) - x_A(x_c^2 - x_A^2)]\sin\theta\cos\theta\\ &-4rx_Ay_c^2\cos^3\theta - 8rx_Ay_c^2\sin^2\theta\cos\theta\\ &+4r[x_A(x_c^2 - x_A^2)\cos\theta + 2x_Ax_cy_c\sin\theta]\sin^2\theta = 0.\end{aligned}$$

Simplifying terms we obtain the equivalent expression

$$\begin{aligned}&-x_cy_c^2 - y_c(r^2 + x_A^2 + y_c^2 - x_c^2)\sin\theta\cos\theta - ry_c^2\cos\theta\\ &+[x_c(r^2 + 2y_c^2) + r(x_c^2 - x_A^2 - y_c^2)\cos\theta\\ &\quad+2rx_cy_c\sin\theta]\sin^2\theta = 0.\end{aligned} \tag{21}$$

In order to solve eq. (21) we use the complex exponential form of the angle θ, that is, $\nu = e^{i\theta} = \cos\theta + i\sin\theta$. Thus, eq. (21) can be written as it is shown next

$$\begin{aligned}&-x_cy_c^2 - \tfrac{y_c}{4i}(r^2 + x_A^2 + y_c^2 - x_c^2)(\nu^2 - \nu^{-2}) - \tfrac{ry_c^2}{2}(\nu + \nu^{-1})\\ &-\tfrac{1}{4}[x_c(r^2 + 2y_c^2) + \tfrac{r}{2}(x_c^2 - x_A^2 - y_c^2)(\nu + \nu^{-1})\\ &\quad+\tfrac{r}{i}x_cy_c(\nu - \nu^{-1})](\nu^2 + \nu^{-2} - 2) = 0.\end{aligned}$$

Multiplying the previous equation by $-4i\nu^3$ and rearranging common terms we obtain

$$\begin{aligned}&r[x_cy_c + \tfrac{i}{2}(x_c^2 - x_A^2 - y_c^2)]\nu^6\\ &+[ix_c(r^2 + 2y_c^2) + y_c(r^2 + x_A^2 + y_c^2 - x_c^2)]\nu^5\\ &+r[-3x_cy_c - \tfrac{i}{2}(x_c^2 - x_A^2 - 5y_c^2)]\nu^4 - 2ir^2x_c\nu^3\\ &+r[3x_cy_c - \tfrac{i}{2}(x_c^2 - x_A^2 - 5y_c^2)]\nu^2\\ &+[ix_c(r^2 + 2y_c^2) - y_c(r^2 + x_A^2 + y_c^2 - x_c^2)]\nu\\ &r[-x_cy_c + \tfrac{i}{2}(x_c^2 - x_A^2 - y_c^2)] = 0\end{aligned} \tag{22}$$

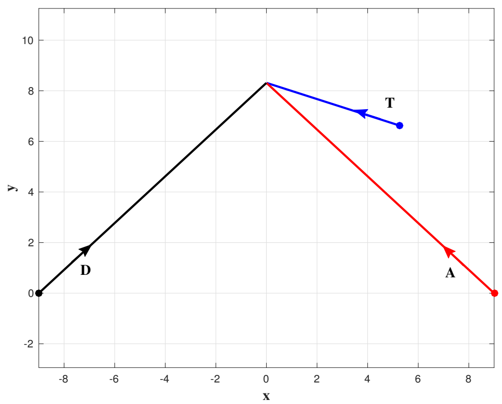

Fig. 4. Optimal trajectories in Example 1. All players reach the interception point at the same time

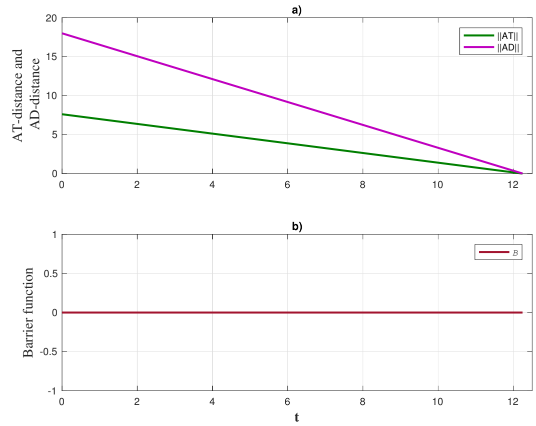

Fig. 5. Example 1. a) $\overline{AT}(t)$ and $\overline{AD}(t)$. b) $\mathcal{B}(t)$

which is a sixth-order polynomial equation in ν. Eq. (22) can be further simplified in order to obtain a quartic equation. This simplification is obtained by writing the left-hand-side of (22) as the product of two polynomials in ν, that is, the previous equation is equivalent to

$$\begin{aligned}&\left(\nu^2 + 2\tfrac{i}{r}y_c\nu - 1\right)\times\\ &\left(r[x_cy_c + \tfrac{i}{2}(x_c^2 - x_A^2 - y_c^2)]\nu^4 + r^2(y_c + ix_c)\nu^3\right.\\ &\left.+r^2(y_c - ix_c)\nu + r[x_cy_c - \tfrac{i}{2}(x_c^2 - x_A^2 - y_c^2)]\right) = 0\end{aligned} \tag{23}$$

where the roots of the quadratic polynomial represent strictly imaginary roots since $\left(\nu^2+2\frac{i}{r}y_c\nu-1\right) = (\nu+ip)(\nu+iq)$ and the variables p and q are real. The imaginary solutions of (22) contained on the quadratic polynomial of (23) correspond, after normalization, to values of $\theta = \pm\frac{\pi}{2}$ and they can be disregarded. One can show that, in general, these roots do not represent the desired solution by substituting the corresponding values of $\theta = \pm\frac{\pi}{2}$, $\sin\theta = \pm 1$, and $\cos\theta = 0$ in the equivalent equation, eq. (21)

$$x_c(r^2 + 2y_c^2) \pm 2rx_cy_c - x_cy_c^2 = x_c(r \pm y_c)^2 \tag{24}$$

which is different than zero in general. Therefore, the optimal solution is obtained from the solution of eq. (15) which minimizes the function (16). □

Let us now focus on the case where the state of the system satisfies $\mathbf{x} \in \mathcal{B}$.

Corollary 1: Assume that $\mathbf{x} \in \mathcal{B}$, then the optimal solution is given by $\theta^* = \pi$ and the optimal interception point in the relative coordinate frame is $I^* = (0, y_c)$ which is the same interception point obtained by solving the $\mathcal{R}_e$-Game of Degree.

Proof. Proof is given in [21].

VII. Examples

Example 1. Consider the initial positions $A_0 = (9, 0)$, $D_0 = (-9, 0)$, and $T_0 = (5.25, 6.63)$. The speed ratio parameter is $\alpha = 0.45$. The initial state satisfies $\mathbf{x}_0 \in \mathcal{B}$. Each agent implements its saddle point strategy. Since the initial state is on the Barrier surface, it is expected that the solutions of the two Games of Degree will provide the same solution and all agents reach the interception point at the same time.

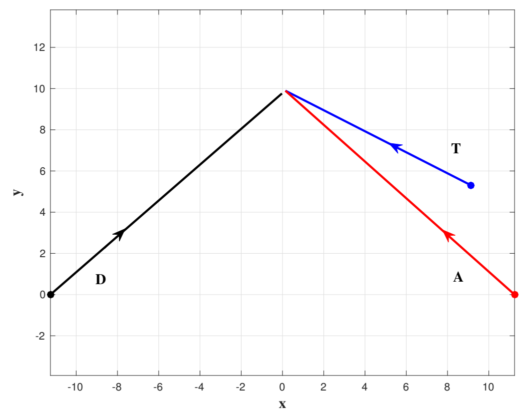

Fig. 6. Optimal trajectories in Example 2. $\mathbf{x}_0 \in \mathcal{R}_c$, under optimal play, A captures T

Optimal trajectories are shown in Fig. 4 where the solution of both Games of Degree is computed at every time instant in a state feedback manner. Both solutions provide the same interception point and, hence, the same headings. Fig. 5.a shows the separations $\overline{AT}(t)$ and $\overline{AD}(t)$ where $\overline{AT}(t_f) = \overline{AD}(t_f) = 0$, that is, D intercepts A at the same time instant that A captures T - the expected outcome when $\mathbf{x}_0 \in \mathcal{B}$ and each agent plays optimally. The function $\mathcal{B}(t)$, the right hand side of eq. (12), is also computed; it can be seen in Fig. 5.b that, under optimal play, $\mathcal{B}(t) = 0$ for all the duration of the engagement. This is expected since the Barrier surface is a semipermeable surface, that is, each player is able to hold the state on this surface by applying its saddle point strategy.

Example 2. Let us now consider the example where $A_0 = (11.25, 0)$, $D_0 = (-11.25, 0)$, $T_0 = (9.15, 5.30)$, and $\alpha = 0.68$. It holds that $\mathbf{x}_0 \in \mathcal{R}_c$ and, under optimal play, the Attacker is able to capture the Target before the Defender can reach the Attacker as it is shown in Fig. 6. The Value of the game is $V(\mathbf{x}) = 0.213$ which represents the terminal distance between T and D. The function $\mathcal{B}(t)$ is shown in Fig. 8.a; note that $\mathcal{B}(t) > 0$ for all the duration of the engagement.

Consider the same initial conditions but now the Attacker implements PN guidance law with navigation constant $N = 3$. Not only the terminal distance between Target and De-

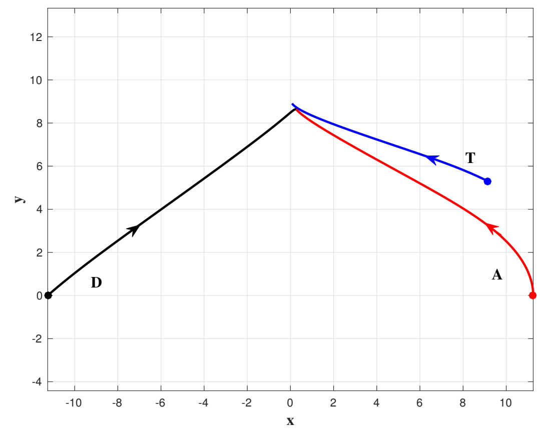

Fig. 7. Example 2. Attacker implements PN. State of the system crosses from $\mathcal{R}_c$ into $\mathcal{R}_e$ and T survives

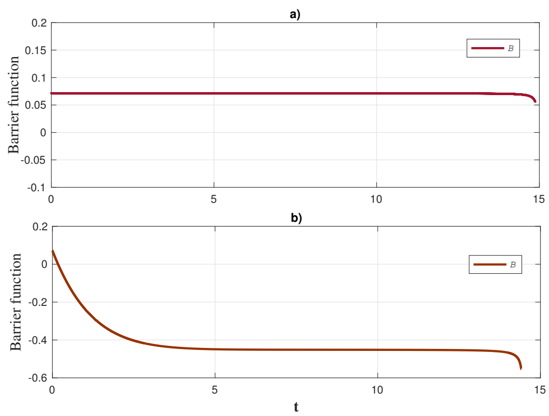

Fig. 8. Example 2. Barrier function. a) Under optimal play. b) When A implements PN

fender decreases, but the Attacker is unable to hold that state in $\mathcal{R}_c$ because it is not using its saddle point strategy. The state transitions into $\mathcal{R}_e$ where the T & D team implements the ATDDG strategy, that is, the saddle point strategy in $\mathcal{R}_e$. The resulting trajectories are shown in Fig. 7 where D actually intercepts A and T escapes.

Fig. 8.b shows $\mathcal{B}(t)$ for the case when A uses PN guidance law. Note that the agents, in this case only T & D (since A foolishly implemented a non-optimal strategy), track the sign of $\mathcal{B}(t)$ in order to decide which Game of Degree to play. If $\mathcal{B}(t) > 0$ then $\mathbf{x} \in \mathcal{R}_c$ and the T & D team implements the optimal strategies of the game of capture derived in this paper. On the other hand when $\mathcal{B}(t) < 0$ then $\mathbf{x} \in \mathcal{R}_e$ and the T & D team switches to the optimal strategies of the ATDDG.

VIII. Conclusions

The game of capture in the Target-Attacker-Defender differential game was introduced and solved in this paper. The TAD differential game belongs to the class of two termination set differential games where each team strives at ending the game in its preferred termination set. The Game of Degree in the region of win of the Attacker provides the optimal state feedback strategy for the Attacker to capture the Target and maximize the terminal distance between Target and Defender. It also provides the optimal state feedback strategy for the T & D team to minimize the same distance. This strategy provides the important property in two termination set differential games where the players' optimal strategies of the Game of Degree in each region are the same when the state of the system is on the Barrier surface that separates the winning regions of each team. Several examples illustrate the results.

References

[1] W. Scott and N. E. Leonard, "Pursuit, herding and evasion: A three-agent model of caribou predation," in *American Control Conference*, 2013, pp. 2978–2983.

[2] D. W. Oyler, P. T. Kabamba, and A. R. Girard, "Pursuit–evasion games in the presence of obstacles," *Automatica*, vol. 65, pp. 1–11, 2016.

[3] S.-Y. Liu, Z. Zhou, C. Tomlin, and K. Hedrick, "Evasion as a team against a faster pursuer," in *American Control Conference*, 2013, pp. 5368–5373.

[4] M. Chen, Z. Zhou, and C. J. Tomlin, "Multiplayer reach-avoid games via pairwise outcomes," *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1451–1457, 2017.

[5] E. Bakolas and P. Tsiotras, "Optimal pursuit of moving targets using dynamic Voronoi diagrams," in *49th IEEE Conference on Decision and Control*, 2010, pp. 7431–7436.

[6] J. Selvakumar and E. Bakolas, "Evasion with terminal constraints from a group of pursuers using a matrix game formulation," in *American Control Conference*, 2017, pp. 1604–1609.

[7] S. Rubinsky and S. Gutman, "Three-player pursuit and evasion conflict," *Journal of Guidance, Control, and Dynamics*, vol. 37, no. 1, pp. 98–110, 2014.

[8] I. Rusnak, H. Weiss, and G. Hexner, "Guidance laws in target-missile-defender scenario with an aggressive defender," in *Proceedings of the 18th IFAC World Congress*, 2011, pp. 9349–9354.

[9] M. Robb, A. Tsourdos, and B. A. White, "Earliest intercept line guidance using a game theory approach," in *AIAA Guidance, Navigation, and Control Conference and Exhibit*, 2006.

[10] E. Garcia, D. W. Casbeer, K. Pham, and M. Pachter, "Cooperative aircraft defense from an attacking missile," in *53rd IEEE Conference on Decision and Control*, 2014, pp. 2926–2931.

[11] E. Garcia, D. W. Casbeer, K. Pham, and M. Pachter, "Cooperative aircraft defense from an attacking missile using proportional navigation," in *2015 AIAA Guidance, Navigation, and Control Conference*, 2015.

[12] M. Weiss, T. Shima, D. Castaneda, and I. Rusnak, "Combined and cooperative minimum-effort guidance algorithms in an active aircraft defense scenario," *Journal of Guidance, Control, and Dynamics*, vol. 40, no. 5, pp. 1241–1254, 2017.

[13] E. Garcia, D. W. Casbeer, and M. Pachter, "Active target defense using first order missile models," *Automatica*, vol. 78, pp. 139–143, 2017.

[14] V. Shalumov and T. Shima, "Weapon-target-allocation strategies in multiagent target-missile-defender engagement," *Journal of Guidance, Control, and Dynamics*, vol. 40, no. 10, pp. 2452–2464, 2017.

[15] R. Isaacs, *Differential Games*. New York: Wiley, 1965.

[16] W. M. Getz and G. Leitmann, "Qualitative differential games with two targets," *Journal of Mathematical Analysis and Applications*, vol. 68, no. 2, pp. 421–430, 1979.

[17] W. M. Getz and M. Pachter, "Capturability in a two-target game of two cars," *Journal of Guidance, Control, and Dynamics*, vol. 4, no. 1, pp. 15–21, 1981.

[18] M. D. Ardema, M. Heymann, and N. Rajan, "Combat games," *Journal of Optimization Theory and Applications*, vol. 46, no. 4, pp. 391–398, 1985.

[19] M. Pachter, E. Garcia, and D. W. Casbeer, "Active target defense differential game," in *52nd Annual Allerton Conference on Communication, Control, and Computing*, 2014, pp. 46–53.

[20] E. Garcia, D. W. Casbeer, and M. Pachter, "Cooperative strategies for optimal aircraft defense from an attacking missile," *Journal of Guidance, Control, and Dynamics*, vol. 38, no. 8, pp. 1510–1520, 2015.

[21] E. Garcia, D. W. Casbeer, and M. Pachter, "Pursuit in the presence of a defender," *To appear*, 2018.