Guidance Laws in 3 body pursuit and evasion games

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1 Problem Statement

The problem consists of three entities: an evading target denoted as T, an attacking missile denoted as A, and a defending missile denoted as D. We assume perfect information and analyze the engagement in two dimensions. R_{DT} denotes the distance between T and D , R_{TA} denotes the distance between A and T. The horizontal axis is assumed as a reference for measuring α_A , α_D , α_T. θ_{DT} and θ_{TA} are the LOS angles measured with respect to the horizontal. a_A, a_D, a_T are the maximum lateral accelerations of A,D and T respectively. Then we have the following state space equations 1:

$$\begin{bmatrix} \dot{R}_{TA} \\ R_{TA}\dot{\theta}_{TA} \\ \dot{R}_{DT} \\ R_{DT}\dot{\theta}_{DT} \\ \dot{\alpha_A} \\ \dot{\alpha_D} \\ \dot{\alpha_T} \end{bmatrix} = \begin{bmatrix} v_A cos(\alpha_A - \theta_{TA}) - v_T cos(\alpha_A - \theta_{TA}) \\ v_A sin(\alpha_A - \theta_{TA}) - v_T sin(\alpha_A - \theta_{TA}) \\ v_D cos(\alpha_D - \theta_{DT}) - v_T cos(\alpha_T - \theta_{DT}) \\ v_D sin(\alpha_D - \theta_{DT}) - v_T sin(\alpha_D - \theta_{DT}) \\ \frac{a_A}{v_A} u_A \\ \frac{a_D}{v_D} u_D \\ \frac{a_T}{v_T} u_T \end{bmatrix}$$

Thus, $|u_A| \leq 1$, $|u_D| \leq 1$ and $|u_T| \leq 1$. The objective of the Defender team is to bring $R_{AD} < \varepsilon$ while maintaining $R_{TA} > \varepsilon$. Similarly the objective of the Attacker is to bring $R_{TA} < \varepsilon$ while maintaining $R_{AD} > \varepsilon$. Where ε is the minimum safe distance. In this paper, we attempt to design a control law for the attacker to capture the target while evading the defender.

2 Safety Critical Control

We now present an overview of safety-critical control systems and a particular control design methodology [1]. For more details refer to .
Consider the control affine system of the form

$$\dot{x} = f(x) + g(x)u \qquad x \in D, u \in U \tag{1}$$

Define the Safe set by

$$C_h := \{x \in D | h(x) \geq 0\} \tag{2}$$

$$\partial C_h := \{x \in D | h(x) = 0\}$$

$$Int(C_h) := \{x \in D | h(x) > 0\}$$

A system is considered to be safe for all time if it never exits the safe set $Int(C_h)$. In order to ensure the safety of the system i.e ensure $Int(C_h)$ is forward invariant we use the ZCBFs.

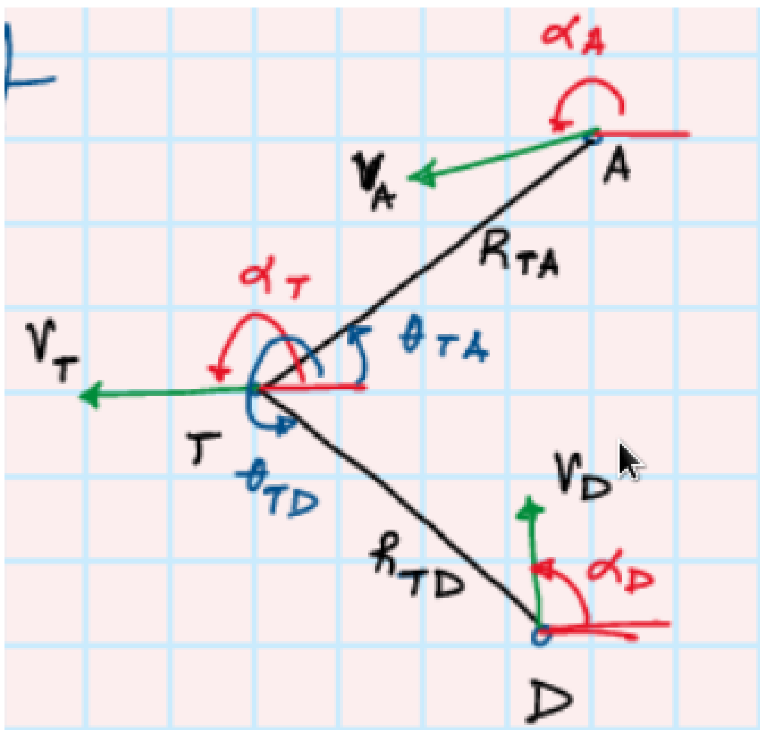


Figure 1: TAD Game

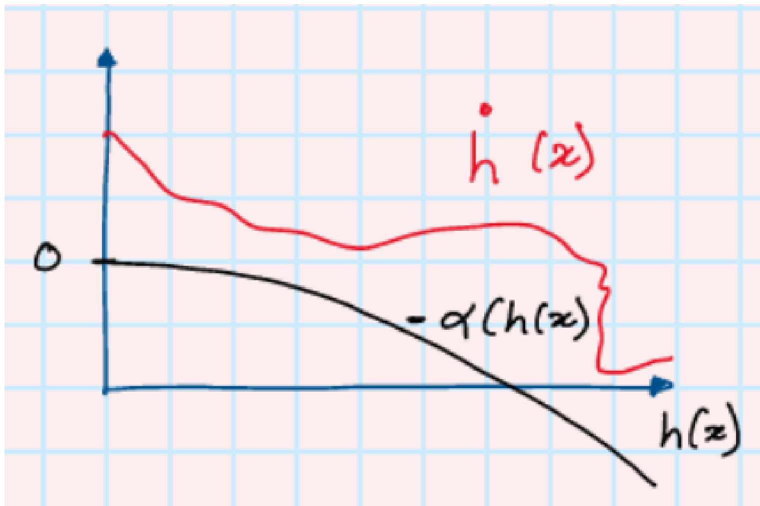


Figure 2: Illustrating the less conservative nature of ZCBFs

2.1 Zeroing Control Barrier Functions

Definition 2.1 ([1])**.** Consider 1 and C_h , h defined by 2. A continuously differentiable function $h \colon Int(C_h) \to \mathcal{R}$ is called a Zeroing Control Barrier function (ZCBF) if there exists class κ function α such that $\forall x \in Int(C_h)$

$$\sup_{u \in U}[L_f h(x) + L_g h(x)u + \alpha(h(x))] \geq 0 \tag{3}$$

Define $K_h(x)$ to be the set of admissible controls at x

$$K_h(x) := \{u \in U | L_f h(x) + L_g h(x)u + \alpha(h(x)) \geq 0\} \tag{4}$$

One must note that the traditional way of ensuring forward invariance of a set is to put $\dot{h} \geq 0$. However this is highly conservative as it ensures that all sub-level sets are forward invariant as well. Hence in 3 the addition of $\alpha(h(x))$ ensures that the derivative of h maybe negative if it is far from the boundary of the safe set 2. This makes the ZCBF h less conservative in nature.

One can use Quadratic Programs (QPs) when one has a control objective and a safety objective that cannot be simultaneously satisfied to choose a control that mediates between the control objective and safety objective [1]. The optimisation problem is formulated as

$$\begin{aligned}
\min_{u} \quad & ||u(x) - \hat{u}(x)|| \\
\text{s.t.} \quad & L_f h(x) + L_g h(x) u(x) + \alpha(h(x)) \geq 0, \\
& u(x) \in U
\end{aligned}$$

where $\hat{u}(x) \in U$ is the nominal control that would ensure that the control objective is satisfied. The idea here is that chooses the control that ensures that the system is safe at all times (Primary Objective) and that the control is as close as possible to $\hat{u}(x)$ so as to satisfy the control objective (Secondary Task). Hence, one can guarantee system safety but one cannot guarantee that the control objective is met.

2.2 A general construction for the ZCBF

Coming up with a closed form expression for the ZCBF which satisfies the properties in 3 is often a challenging task if not impossible for complex systems. Thus, one has to rely on numerical approaches to compute the barrier functions one of which is presented next [2].
Define

$$h(x(t); \rho, \gamma) := \inf_{\tau \in [0,\infty]} \rho(\hat{x}(t+\tau)) \tag{5}$$

where h is the ZCBF, ρ is the safety function i.e $C := \{x \in D | \rho(x) > 0\}$ and γ is some nominal safety control i.e the system can use γ to ensure its safety.

$$\hat{x}(t) = x(t) + \int_0^\tau \gamma(\hat{x}(s)) ds$$

We can see that $h(x(t); \rho, \gamma)$ is a valid ZCBF.

$$\begin{aligned}
\dot{h} &= L_f h(x) + L_g h(x) \gamma(x) \\
&= \lim_{a \longrightarrow 0^+} \frac{1}{a} [\inf_{\tau \in [a,\infty]} \rho(\hat{x}(t+\tau)) - \inf_{\tau \in [0,\infty]} \rho(\hat{x}(t+\tau))] \\
&\geq 0
\end{aligned}$$

Hence $\gamma(x) \in K_h(x) \quad \forall \quad x \in Int(C_h)$ which implies that h is a valid ZCBF. The idea here is to search for a safety ensuring maneuver $\gamma(x)$ which is capable of always ensuring the safety of the system given that the system is initially in $Int(C_h)$. This ensures that h is a valid ZCBF.

3 Approach 1: TAD as a Safety Critical QP

3.1 Problem Formulation

Formulating TAD as a safety critical control problem is a natural way to move ahead because the control objective (Capturing the Target) and safety objective (Evading the Defender) are clear.
Let $\hat{u}_A(x)$ be some guidance law that enables pure-pursuit of the target by the attacker.
Let $h(x)$ be some appropriately defined ZCBF.

$$\begin{aligned} \min_{u} \quad & ||u(x) - \hat{u}(x)|| \\ \text{s.t.} \quad & L_f h(x) + L_g h(x) u(x) + \alpha(h(x)) > 0, \\ & |u(x)| \leq 1 \end{aligned}$$

Note: This is a pointwise optimisation problem.

$$h(x(t); \rho, \gamma) := \inf_{\tau \in [0, \infty]} \rho(\hat{x}(t + \tau))$$

we need to chose ρ and γ appropriately.

One strategy is to chose γ in a greedy way that drives the system towards the direction of maximum increase in $h(x)$ i.e in the direction of $\nabla h(x)$.
A way to do this would be to point the tangent vector of the system in the direction closest to that of $\nabla h(x)$.

However in our case, the tangent vector has the form $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_A \\ 0 \\ 0 \end{bmatrix}$

Clearly this wont give good results as the control does not directly affect the states but does it through some integrator layers.
We need a different approach to choose γ.

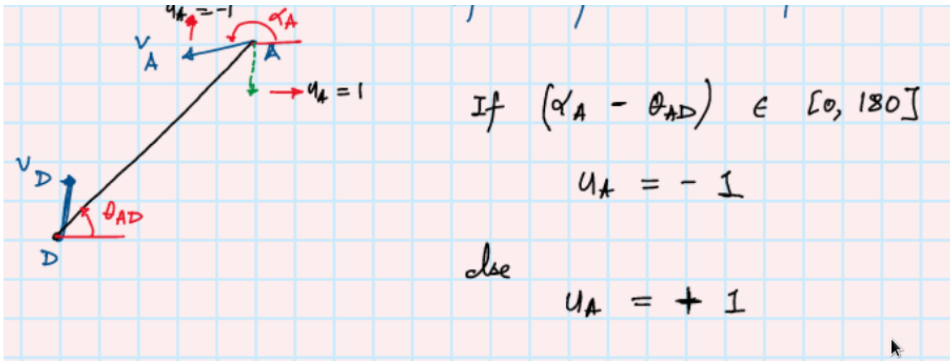


Figure 3: Choice of γ for $\rho(x) = ((x_A - x_D)^2 + (y_A - y_D)^2)^{1/2} - \varepsilon$

3.2 Choice of γ using physical understanding of ρ

We use two choices of ρ in our simulations.
1] $\rho(x) = ((x_A - x_D)^2 + (y_A - y_D)^2)^{1/2} - \varepsilon$
This is the natural choice for the safety function. We use a bang-bang control 3 which ensures that the velocity vector of the attacker is always aligned to the LOS joining A and D as the γ for this safety function. The choice of γ in this case is heuristic based but simple to implement as we do not want to the computational burden in calculating the ZCBF to be very high.

2] $\rho(x) = \Delta\theta(x) = |\theta_{TA} - \theta_{DT}|$
As shown in [4] if the defender is able to position itself on the LOS joining A and T, then the defender has won. Thus, this safety function tries to ensure that $\Delta\theta(x)$ which is the angle between TA and TD does not fall below a threshold. This is ensured by choosing another bang-bang control γ 4 which ensures that the velocity vector of the attacker is always aligned with the line perpendicular to the LOS joining A and T. Intuitively, this would be the natural choice to increase $\Delta\theta(x)$ as quickly as possible.

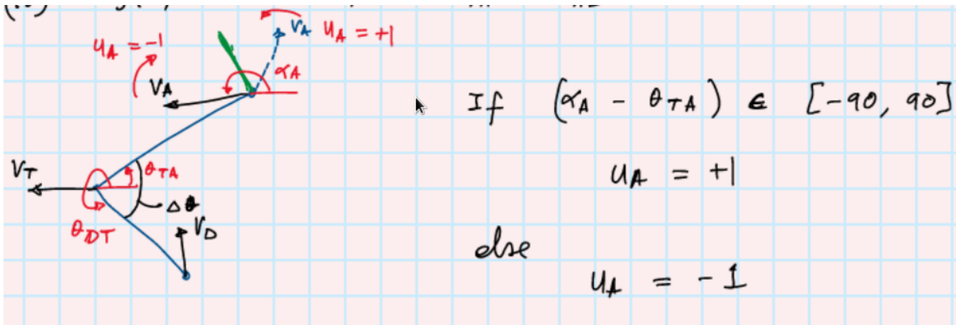


Figure 4: Choice of γ for $\rho(x) = \Delta\theta(x)$

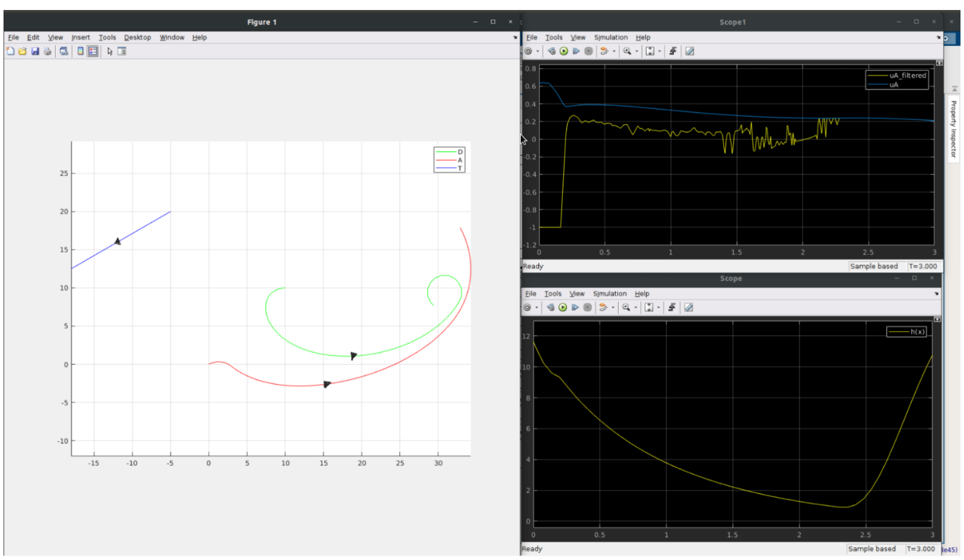


Figure 5: Attacker is successfully able to evade the Defender while pursuing the Target using 1st safety function

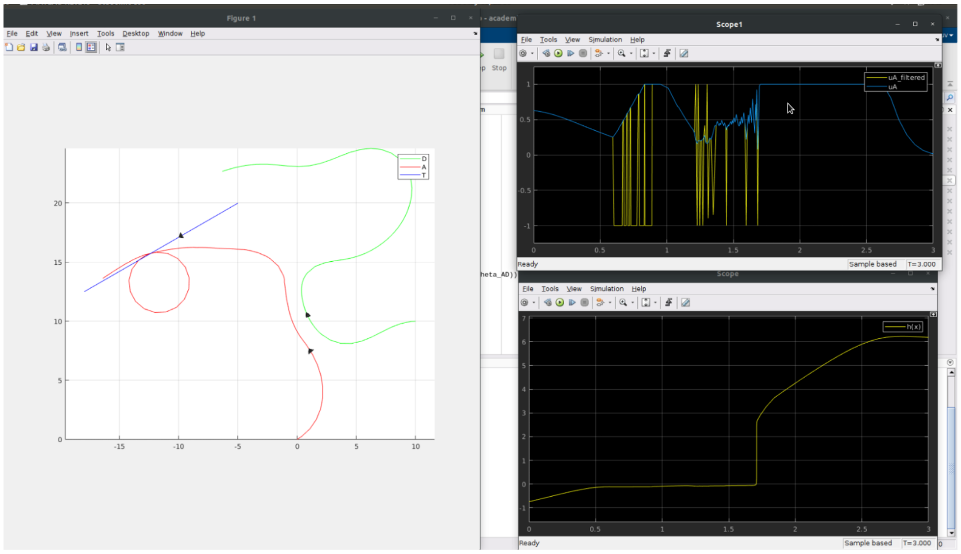


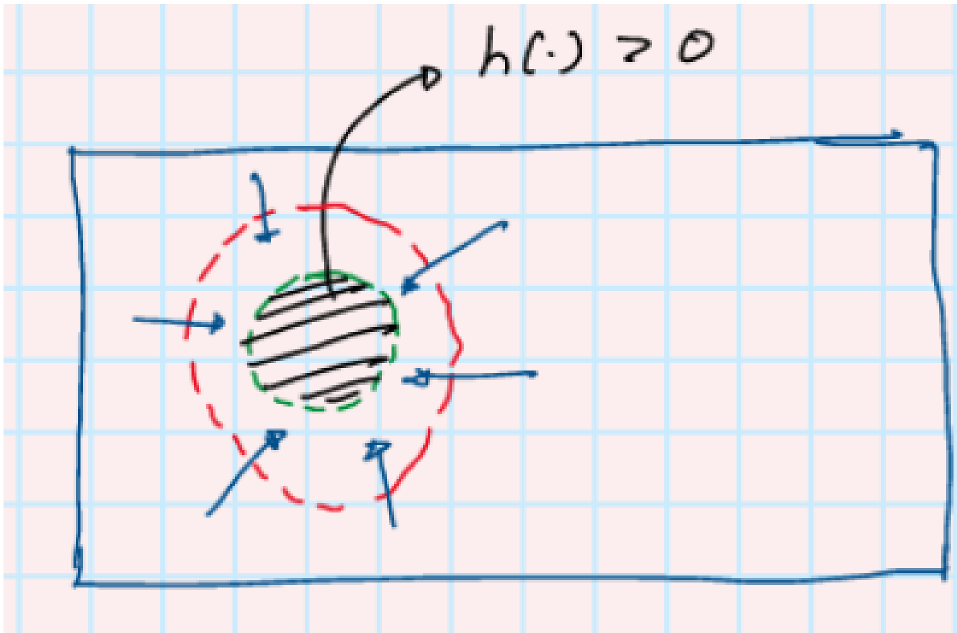
Figure 6: Attacker is successfully able to evade the Defender while pursuing the Target using 2nd safety function

4 Approach 2 : Using Time of Engagement

Let $T(x)$ be the time required for the attacker to capture the target with the initial state being x.

$$h(x(t); \rho, \gamma) := \inf_{\tau \in [0, T(x)]} \rho(\hat{x}(t+\tau)) \tag{6}$$

where γ is some guidance law where the attacker pursues the target ignoring the defender and h is the minimum separation between the attacker and defender under γ over the finite time horizon.



Case 1] $h(x(t); \rho, \gamma) > \varepsilon$

Capture is ensured and we are done

Case 2] $h(x(t); \rho, \gamma) \leq \varepsilon$

In this case we need to design a maneuver say $E(x)$ for the attacker to move in a way that increases the value of h(x) until $h(x) > \varepsilon$.
One way to do this would be to steer the system using the tangent vector in the direction of $\nabla h(x)$. However as described above, the system under consideration is not steerable using the tangent vector.
As done in the previous case (using the physical understanding of h(x)), one could use a guidance law where the attacker purely evades the defender. However there is no guarantee that this is the optimal choice. Thus, this remains a further area of research.

5 Future Work

5.1 Future Work : Numerical Studies

Under Approach 1, simulations need to be conducted with different choices of nominal guidance strategies and varying initial conditions.
Furthermore, these simulations need to be compared with the existing literature for the TAD problem (Specifically those implementing cooperative guidance against the attacker).
Under Approach 2, various heuristic based maneuvers E(x) need to be evaluated using simulations and their performance compared with that of Approach 1.

5.2 Future Work: Theoretical Investigations

An open challenge to obtain a closed-form expression of the barrier function for the system under consideration still prevails. One could also characterize the capture regions numerically if a closed-form expression for a ZCBF is available.
Designing the maneuver $E(x)$ with a theoretically sound justification remains to be explored. One could use ideas from backstepping to steer the system in the direction of $\nabla h(x)$ or explore something completely different.

References

[1] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs with application to automotive safety systems," 2016.

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[3] D. G. Luenberger, Optimization by vector space methods. John Wiley and Sons, 1969

[4] Ratnoo and Shima, "Line-of-Sight Interceptor Guidance for Defending an Aircraft",2011 Journal of Guidance, Control, and Dynamics

[5] Github Repository