Automatica 102 (2019) 58–71

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

A differential game for cooperative target defense[☆]

Li Liang[a,b], Fang Deng[a,*], Zhihong Peng[a,*], Xinxing Li[a], Wenzhong Zha[c]

[a] School of Automation, Beijing Institute of Technology, Beijing, 100081, China
[b] School of Information Engineering, Inner Mongolia University of Science and Technology, Baotou, 014010, China
[c] Information Science Academy, China Electronics Technology Group Corporation, Beijing, 100086, China

ARTICLE INFO

Article history:
Received 31 May 2017
Received in revised form 28 July 2018
Accepted 18 December 2018
Available online xxxx

Keywords:
Pursuit–evasion game
Cooperative policy
Role balance
Target–Attacker–Defender game
Barrier

ABSTRACT

Multi-player pursuit–evasion games are crucial for addressing the maneuver decision problem arising in the cooperative control of multi-agent systems. This work addresses a particular pursuit–evasion game with three players, Target, Attacker, and Defender. The Attacker aims to capture the Target, while avoiding being captured by the Defender and the Defender tries to defend the Target from being captured by the Attacker, while trying to capture the Attacker at an opportune moment. A two-pronged pursuit–evasion problem in this game is considered and we focus on two aspects: the cooperation between the Target and Defender and balancing the roles of the Attacker between pursuer and evader. A barrier based on the explicit policy method and geometric analysis method is constructed to separate the whole state space into two disjoint parts that correspond to two winning regions for the Attacker and Target–Defender team. The main contributions of this work are obtaining the players' winning regions and providing a complete game solution by analyzing the optimal strategies and trajectories of the players based on the barrier.

1. Introduction

The multi-player pursuit–evasion game is an important tool to deal with the maneuver decision problem arising in the cooperative control of multi-agent systems (Bopardikar, Bullo, & Hespanha, 2009; Ding, Rahmani, & Egerstedt, 2010; Ramana & Kothari, 2015), especially in confrontational circumstances (D'Andrea & Murray, 2003; Ge, Ma, & Lum, 2008; Huang, Ding, Zhang, & Tomlin, 2015; Isaacs, 1965; Kraska & Rzymowski, 2011). In many practical applications, agents are required to assist other agents in completing confrontation tasks such as an interceptor defending an asset against an intruder (Li & Cruz, 2011), a torpedo safeguarding a naval ship against a submarine (Boyell, 1976), and a bodyguard protecting a potential victim against a bandit (Rusnak, 2005). The common feature of these application scenarios is that there are three players – Target, Attacker, and Defender – and this is known as a Target–Attacker–Defender (TAD) game.

In a TAD game, the Attacker aims to capture the Target, while avoiding being captured by the Defender, and the Defender tries to defend the Target from being captured by the Attacker, while trying to capture the Attacker at an opportune moment. In other words, the Target and Defender cooperate in a team, and the Attacker is on the opposite side. This study, which focuses on solving such a TAD game, differs from conventional pursuit–evasion games because the players' roles have changed. The task of the players is not merely to chase or to escape; the Attacker also needs to escape from the Defender. Meanwhile, the Target must cooperate with the Defender in addition to escaping from the Attacker. Moreover, the Defender cooperates with the Target to prevent the Attacker from achieving his or her goal. These changes in the players' roles make it more difficult to solve the TAD problem.

The TAD game was first presented by Boyell (1976), who examines the setting in which a moving target launches a missile or torpedo to defend itself against the missile. Subsequently, because of the nature of this problem, much scholarly attention has been paid to the issue. Different types of cooperation mechanisms in the Target–Defender team have been developed by Casbeer, Garcia, Fuchs, and Pachter (2015), Garcia, Casbeer, and Pachter (2015a, 2015b, 2017), Garcia, Casbeer, Pham, and Pachter (2014, 2015), Pachter, Garcia, and Casbeer (2014), Perelman, Shima, and Rusnak (2011), Prokopov and Shima (2013), Shima (2011) and Shaferman and Shima (2010). For example, in Shaferman and Shima (2010), Shima (2011), Prokopov and Shima (2013) and Perelman et al. (2011), where the Defender is a fired missile from an evading aircraft aiming to defend against an incoming homing missile, the authors considered the line-of-sight (LOS) guidance law for the Defender. The implementation of the LOS guidance law requires

☆ This work was supported by the National Natural Science Foundation of China (Grant No. 61203078) and the Key Program of National Natural Science Foundation of China (Grant No. U1613225). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Gurdal Arslan under the direction of Editor Ian R. Petersen.

* Corresponding authors.
E-mail addresses: liangli@bit.edu.cn (L. Liang), dengfang@bit.edu.cn (F. Deng), peng@bit.edu.cn (Z. Peng), lixinxing1006@163.com (X. Li), zhawenzhong@126.com (W. Zha).

https://doi.org/10.1016/j.automatica.2018.12.034
0005-1098/

the Defender to stay on/ride the LOS between the Target and Attacker. In Casbeer et al. (2015), Garcia et al. (2015, 2014), Garcia et al. (2015a, 2015b, 2017), and Pachter et al. (2014), where the Defender missile is launched by the Target or by a Target-friendly platform, the cooperation between the Target and Defender is such that the Defender captures the Attacker before the Attacker captures the Target. In the aforementioned literature, the Attacker, regarded as a non-recycled missile, aims only to minimize the distance between itself and the Target. Whether the Attacker is captured by the Defender is seldom considered. Balancing the roles of the Attacker is considered only in Rubinsky and Gutman (2012, 2014), who present the switch time for the Attacker in the TAD end-game based on the distance between the Target and Attacker for non-cooperation between the Target and Defender. In addition, most of the abovementioned studies focus on quantitative analysis, leaving some fundamental issues unsolved. For example, under what conditions can a player win or lose the game? How should a control scheme for players be designed?

These issues can be solved within a game of kind that constructs a barrier (Isaacs, 1965), namely a semi-permeable surface that partitions the state space into disjoint regions. Each region is associated with a player who wins the game if the initial position of the players lies in that region. Generally, there are two methods of solving the multi-player pursuit–evasion problem in terms of a game of kind. The first method is to divide the game into several sub-games and then analyze the optimal behaviors of the players for each sub-game by using Isaacs' classic approach (Isaacs, 1965). Under this approach, the barrier is constructed by integrating the so-called retrogressive path equations (RPEs) from the points on the boundary of the usable part of the target set (or terminal manifold). For example, Bhattacharya, Basar, and Hovakimyan (2014, 2016) study a visibility-based target tracking game in the presence of a circular obstacle by reducing the dimension of the state space to three and constructing the barrier using Isaacs' techniques according to the symmetry of the environment. The second method is the explicit policy method (Bakolas & Tsiotras, 2012; Bopardikar, Bullo, & Hespanha, 2008; Bopardikar et al., 2009; Isaacs, 1965; Zha, Chen, Peng, & Gu, 2016), in which the strategy is given to the players and then the possibility of winning the game is analyzed. For example, Ramana and Kothari (2017) study a multi-player pursuit–evasion game with one superior evader, using overlapping Apollonius circles around that evader. Oyler, Kabamba, and Girard (2016) consider a P3 game (prey, protector, predator) in which the protector and prey aim to rendezvous before the latter is captured by the predator and the conditions for the two sides dominating the game of kind are presented by means of Apollonius circles.

In some respects, the TAD game is a two-pronged pursuit–evasion problem: Attacker–Target and Defender–Attacker. However, it is difficult to divide the target set in the game into several successive sub-target sets in sequence. Hence, although the TAD game can be described by a three-dimensional state space, solving the RPEs analytically in this case is challenging. Inspired by the abovementioned methods in the literature, we thus decompose the TAD game into sub-problems and demonstrate strategies for players by using the explicit policy method. We then use geometric analysis and the Pontryagin maximum principle to analyze the possibility of the players winning or losing the game.

The main contributions of this work are threefold. First, we simultaneously take into account balancing the roles of the Attacker and cooperation of the Target–Defender team. Second, we combine the explicit policy method with geometric analysis to solve the TAD game and attain the barrier, thereby providing a new notion to solve such a game of kind. Finally, we fuse a game of kind and a game of degree into the TAD game. We can then attain the optimal trajectories associated with the optimal control strategies of the players for every zone, which are the complete solutions of the TAD game. To the best of our knowledge, this is the first study that provides such a complete solution of the TAD game. The results obtained can be employed to solve the maneuvering decision problems arising in the cooperative control of multi-agent systems in adversarial environments such as search and rescue operations and the recovery of military equipment.

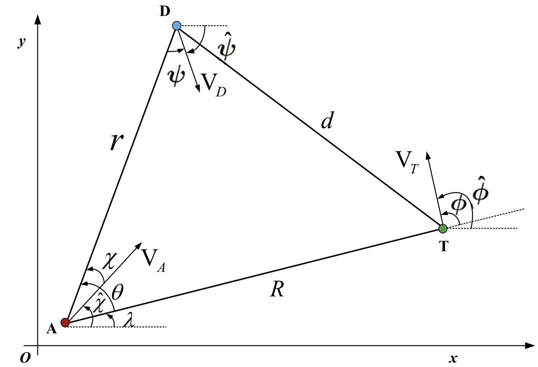


Fig. 1. TAD game in the fixed reference system.

The rest of the paper is organized as follows. Section 2 formulates the TAD game. In Section 3, we briefly introduce some explicit policies as well as provide the winning condition for the players and construct the barrier of the TAD game. In Section 4, the optimal control strategies and corresponding trajectories for the players in different winning regions are obtained. Finally, Section 5 concludes.

2. Problem formulation

In this section, we present the problem formulation of the TAD game. As shown in Fig. 1, the Target, Attacker, and Defender move in the plane at speeds of V_A, V_T, and V_D, respectively. The dynamics of the Target, Attacker, and Defender can be described in the following equations:

$$\dot{x}_T = V_T \cos\hat{\phi},\ \dot{y}_T = V_T \sin\hat{\phi} \tag{1}$$

$$\dot{x}_A = V_A \cos\hat{\chi},\ \dot{y}_A = V_A \sin\hat{\chi} \tag{2}$$

$$\dot{x}_D = V_D \cos\hat{\psi},\ \dot{y}_D = V_D \sin\hat{\psi} \tag{3}$$

where the positions of the Target, Attacker, and Defender are denoted as (x_T, y_T), (x_A, y_A), and (x_D, y_D), respectively and the corresponding control variables are denoted as $\hat{\phi}$, $\hat{\chi}$, and $\hat{\psi}$, respectively.

In the TAD game, we are interested in the following two problems:

(1) What initial states can ensure that the Attacker wins or the Target–Defender team wins when all the players adopt the optimal control strategies?
(2) If the winning conditions have been assured, what strategies should be adopted by the Attacker and Target–Defender team to win the game in the minimum time?

Assume the instantaneous positions and velocities of the players are available to their opponents. Clearly, when $V_T \geq V_A$, the Target–Defender team can always win the game from any given initial position as long as the Target moves away from the Attacker. Therefore, we consider only the case of $V_T < V_A$ and use $\alpha = V_T/V_A$ to denote the speed ratio, that is, $\alpha < 1$.

To describe the dynamics of the TAD game, the relative distances R, r and included angle θ (see Fig. 1) can form a reduced state space in which the dynamics of the whole system are derived as follows:

$$\dot{R} = \alpha\cos\phi - \cos(\theta - \chi) \qquad R(t_0) = R_0 \tag{4}$$

$$\dot{r} = -\cos\chi - \beta\cos\psi \qquad r(t_0) = r_0 \tag{5}$$

$$\dot{\theta} = -\frac{\alpha}{R}\sin\phi + \frac{1}{R}\sin(\theta-\chi) - \frac{\beta}{r}\sin\psi + \frac{1}{r}\sin\chi \qquad \theta(t_0) = \theta_0 \tag{6}$$

where $\phi = \hat{\phi} - \lambda$, $\chi = \lambda + \theta - \hat{\chi}$, and $\psi = \hat{\psi} - \theta - \lambda + \pi$. λ denotes the included angle of vector $\overrightarrow{AT}$ and the X-axis. ϕ, χ, and ψ are the alternative control variables of the players, defined as the relative headings of Attacker, Defender, and Target from the vectors $\overrightarrow{AD}$, $\overrightarrow{DA}$, and $\overrightarrow{AT}$, respectively. θ is the included angle $\angle DAT$, $\theta = Arg(\overrightarrow{AD}) - Arg(\overrightarrow{AT})$. Arg(.) is the principal value of the argument of a vector, within the range $(-\pi, \pi]$. $\beta = V_D/V_A$ denotes the speed ratio.

In this study, we consider the TAD game with point capture (the capture radius is zero). The objective of the Attacker is to make the distance $R = 0$ on the premise of guaranteeing $r > 0$. The terminal set of the Attacker is $\{R = 0, r > 0\}$. On the contrary, the Target–Defender team aims to prevent the Attacker from achieving his or her goal. The intuitive way in which to do so is for the Defender to capture the Attacker before the Target is captured. In this case, the terminal set of the Target–Defender team is $\{r = 0, R > 0\}$. A non-intuitive way is for the Target to rendezvous with the Defender and then the terminal set is $\{R > 0, r > 0, d = 0\}$. In particular, if the state is indefinitely kept in $\{R > 0, r > 0\}$, the Target–Defender team wins the game.

The full state space is denoted by $\Omega = \{R \geq 0, r \geq 0, -\pi < \theta \leq \pi\} \in \mathbb{R}^3$. Without loss of generality, we assume that the initial space includes the angle $\theta_0 \in [0, \pi]$. For the case of $(-\pi, 0)$, it is easy to obtain a similar result according to symmetry. Thus, at the beginning of the TAD game $\{R_0 > 0, r_0 > 0, 0 \leq \theta_0 < \pi\}$, and the TAD game terminates when one of the following conditions is satisfied:

(a) If $\{R = 0, r > 0, d > 0\}$, the Attacker wins.
(b) If $\{R > 0, r = 0, d > 0\}$, $\{R > 0, r > 0, d = 0\}$, $\{R > 0, r > 0\}$ indefinitely, the Target–Defender team wins.

3. Game of kind

To confirm the conditions under which the players win or lose the game, we turn to the theory of games of kind in this section. The TAD game formulated in the previous section includes three terminal sets, namely $\{R > 0, r = 0, d > 0\}$, $\{R = 0, r > 0, d > 0\}$, and $\{R > 0, r > 0, d = 0\}$. It is difficult to construct the barrier directly by using Isaacs' classic approach because sketching the boundary of the terminal set is a challenge. Instead, we can address this problem by using the explicit policy method that decomposes a complex problem into a few simple sub-problems. In each sub-problem, we then analyze the possibility of winning the game by giving specific strategies to the players. First, we define some of the policies in this study.

Definition 1. Pure policy: This is the pure pursuit–evasion policy between two players without considering any other players. For example, the Attacker adopts the pure pursuit strategy to chase the Target without considering the existence of the Defender. The Defender adopts the pure pursuit strategy to chase the Attacker without the cooperation of the Target. The Target adopts the pure evasion strategy to evade the Attacker without the cooperation of the Defender.

Definition 2. Roundabout policy: The Attacker first evades the Defender and then chases the Target.

Definition 3. Rendezvous policy: The Defender and Target choose a point to rendezvous regardless of the Defender.

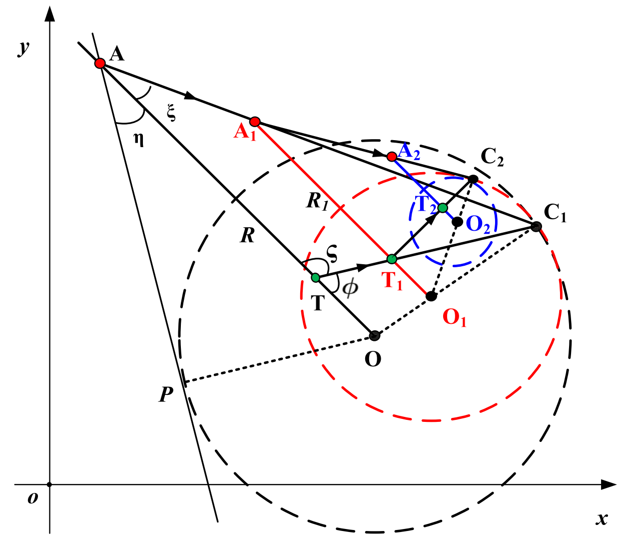


Fig. 2. The Attacker directly captures the Target.

Definition 4. Cooperative evasion policy: The Defender cooperates with the Target to avoid the Attacker evading the Defender. This policy is opposite to the roundabout policy.

3.1. Conditions for the Attacker to win

From the Attacker's perspective, he or she must chase the Target to win the game, while avoid being captured by the Defender. If the initial positions of the three players are favorable for the Attacker (the position of the Defender does not obstruct the Attacker from pursuing the Target), the Attacker can adopt the pure policy to directly chase the Target. If the initial positions of the three players are not favorable for the Attacker (the Attacker is behind the Defender and Target), the Attacker might win the game by using the roundabout policy.

3.1.1. The included angle for the Attacker to win ($\beta < 1$)

Assuming the Attacker directly captures the Target without considering the Defender, this is a typical two-player pursuit–evasion game. We have the following lemma.

Lemma 1. *Without considering the Defender, if the Target chooses the control angle ς (see Fig. 2), the Attacker adopts the parallel strategy*

$$\xi = \arcsin(\alpha\sin(\varsigma)) \tag{7}$$

to capture the Target in or on the Apollonius circle of the Attacker and Target.

Proof. The Apollonius circle (Isaacs, 1965) is a trajectory of a point whereby the ratio of the distances between this point to two fixed points in the plane is constant and does not equal one. If the Target adopts the control angle ς and moves toward point C_1, the Attacker also moves toward point C_1 (see Fig. 2). According to the sine theorem in the $\triangle TAC_1$,

$$\frac{\sin\varsigma}{\sin\xi} = \frac{|AC_1|}{|TC_1|} = \frac{V_A}{V_T} = \frac{1}{\alpha} \tag{8}$$

Then, we can obtain

$$\xi = \arcsin(\alpha\sin(\varsigma)) \tag{9}$$

If the Target moves in a straight line and the Attacker adopts strategy (9), they will reach the corresponding capture point C_1

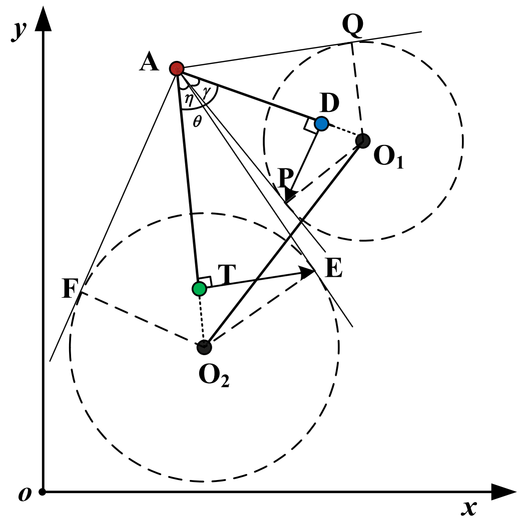


Fig. 3. The included angle for the Attacker adopting the pure policy to win the game.

on the Apollonius circle at the same time; that is, the Attacker can capture the Target on the Apollonius circle.

Moreover, when the Target does not move in a straight line and deviates from the original strategy at any time, we can prove that the Attacker adopts the control strategy (9) to pursue the Target inside the Apollonius circle (see Fig. 2).

The reason is as follows. Assume the Attacker and Target move to points A_1 and T_1, respectively from their initial positions after Δt time. Line AT is parallel to line A_1T_1, and thus we call strategy (9) a parallel strategy.

Hence, we have

$$\frac{|O_1T_1|}{|OT|} = \frac{R_1}{R} = \frac{|O_1C_1|}{|OC_1|} \tag{10}$$

Point O_1 is on the line of OC_1. The new Apollonius circle O_1 based on the new position of the players is tangent to the former Apollonius circle O at point C_1, which is the crossover point of the former movement direction and the former Apollonius circle O. Similarly, if the Target changes direction and moves toward point C_2, the Attacker also moves toward point C_2, and then the newer Apollonius circle O_2 is tangent to the former Apollonius circle O_1 at point C_2. Therefore, no matter what strategy the Target adopts, the Attacker can capture the Target in or on the Apollonius circle based on the initial position of the two players. □

According to Lemma 1, we know that in the TAD game when the included angle θ is larger than a certain value, the Attacker can easily win the game. This leads to the following theorem.

Theorem 1. *If the initial state of the TAD game satisfies*

$$\cos(\theta_0) < \sqrt{(1-\alpha^2)(1-\beta^2)} - \alpha\beta \tag{11}$$

then the Attacker can adopt the ***parallel strategy***

$$\chi^* = \theta_0 - \arcsin(\alpha\sin(\phi)) \tag{12}$$

to capture the Target without considering the pursuit of the Defender.

Proof. In Fig. 3, circle O_1 is the Apollonius circle of the Attacker and Defender and circle O_2 is the Apollonius circle of the Attacker and Target. P, Q and E, F are the intersection points of the tangents of point A to circles O_1 and O_2, respectively. According to the characteristics of Apollonius circles, we have

$$\frac{|DP|}{|AP|} = \frac{V_D}{V_A} = \beta, \frac{|TE|}{|AE|} = \frac{V_T}{V_A} = \alpha \tag{13}$$

Then,

$$\sin\gamma = \beta, \sin\eta = \alpha \tag{14}$$

If condition (11) is true, then

$$\begin{aligned} \cos(\theta_0) &< \sqrt{(1-\alpha^2)(1-\beta^2)} - \alpha\beta \\ &= \cos\gamma\cos\eta - \sin\gamma\sin\eta \\ &= \cos(\gamma+\eta) \end{aligned} \tag{15}$$

Hence, we obtain $\theta_0 > \gamma + \eta$. The Attacker can directly move toward the Target regardless of the presence of the Defender. If the Target adopts arbitrary strategy ϕ, according to Lemma 1, we have the optimal strategy (12) for the Attacker. □

Clearly, when $\cos(\theta_0) \geq \sqrt{(1-\alpha^2)(1-\beta^2)} - \alpha\beta$, that is, $\theta_0 \leq \gamma + \eta$, the Attacker should increase the angle θ to reach $\gamma + \eta$ and simultaneously ensure $r > 0$. Once condition $\{\theta > \gamma + \eta, r > 0\}$ is satisfied, the Attacker not only can escape from the Defender but also can capture the Target as long as the Attacker adopts the parallel strategy (12) to pursue the Target. On the contrary, the Target–Defender team should cooperate to decrease the angle θ. Therefore, this game can be regarded as the included angle game in which the Attacker attempts to increase the angle, while the Target–Defender team hopes to decrease it. In the included angle game, the winning conditions can be described as follows: when $\theta \leq \gamma + \eta$ always holds, the Target–Defender team wins; when $\theta > \gamma + \eta$, the Attacker wins.

Next, we analyze this included angle game qualitatively by using the classical method of Issacs. Based on the assumption that the barrier is a semi-permeable surface (Isaacs, 1965), let $\lambda = [\lambda_R, \lambda_r, \lambda_\theta]^T$ denote the normal to semi-permeable surface passing through the points (R, r, θ). The Hamilton function is given by

$$\begin{aligned} &H(x, \phi, \chi, \psi, \lambda) = \\ &\lambda_R(\alpha\cos\phi - \cos(\theta-\chi)) + \lambda_r(-\cos\chi - \beta\cos\psi) \\ &+\lambda_\theta(-\frac{\alpha}{R}\sin\phi + \frac{1}{R}\sin(\theta-\chi) - \frac{\beta}{r}\sin\psi + \frac{1}{r}\sin\chi) \end{aligned} \tag{16}$$

The optimal control headings of the players satisfy the following equation:

$$\begin{aligned} \max_{\chi(\cdot)} \min_{\phi(\cdot)\psi(\cdot)} H(x, \phi, \chi, \psi, \lambda) &= \min_{\phi(\cdot)\psi(\cdot)} \max_{\chi(\cdot)} H(x, \phi, \chi, \psi, \lambda) \\ &= H(x, \phi^*, \chi^*, \psi^*, \lambda) \end{aligned} \tag{17}$$

By solving (17), we conclude that the optimal control headings of the players on the barrier are given by the following expressions:

$$\sin\phi^* = \frac{\lambda_\theta}{R\rho_1}, \cos\phi^* = -\frac{\lambda_R}{\rho_1}, \rho_1 = \sqrt{(\lambda_\theta/R)^2 + \lambda_R^2} \tag{18}$$

$$\sin\psi^* = \frac{\lambda_\theta}{r\rho_2}, \cos\psi^* = \frac{\lambda_r}{\rho_2}, \rho_2 = \sqrt{(\lambda_\theta/r)^2 + \lambda_r^2} \tag{19}$$

$$\begin{aligned} &\sin\chi^* = -\frac{a}{\rho_3}, \cos\chi^* = -\frac{b}{\rho_3}, \rho_3 = \sqrt{a^2+b^2}, \\ &a = \lambda_R\sin\theta + \frac{\lambda_\theta}{R}\cos\theta - \frac{\lambda_\theta}{r}, \\ &b = \lambda_R\cos\theta + \lambda_r - \frac{\lambda_\theta}{R}\sin\theta \end{aligned} \tag{20}$$

In this included angle game, the terminal set of the Attacker is denoted by $\mathscr{D}_{A1}^1 = \{R, r, \theta \mid R > 0, r > 0, \theta > \eta + \gamma\}$ and the terminal set of the Target–Defender team is denoted by $\mathscr{D}_{TD1}^1 = \{R, r, \theta \mid R > 0, r > 0, \theta \leq \eta + \gamma\}$. The boundary of the terminal set is denoted by $\mathscr{B} = \{R, r, \theta \mid R > 0, r > 0, \theta = \eta + \gamma\}$. $\mathscr{B}$ is defined by

$$R(\tilde{t}) = R, r(\tilde{t}) = r, \theta(\tilde{t}) = \eta + \gamma \tag{21}$$

and the outward normal vector λ by

$$\lambda_R(\tilde{t}) = 0, \lambda_r(\tilde{t}) = 0, \lambda_\theta(\tilde{t}) \text{ is free.} \tag{22}$$

where $\tilde{t}$ is the time to reach boundary $\mathscr{B}$. Thus, the optimal control strategies of the three players can be rewritten as follows:

$$\phi^* = \frac{\pi}{2} \qquad \psi^* = \frac{\pi}{2} \tag{23}$$

$$\cos\chi^* = \frac{r\sin\theta}{d} \qquad \sin\chi^* = \frac{R - r\cos\theta}{d} \tag{24}$$

where $d = \sqrt{R^2 + r^2 - 2Rr\cos\theta}$ is the distance between the Target and Defender. Strategy (23) is called the **narrow angle strategy** for the Target–Defender team and Strategy (24) is called the **expanded angle strategy** for the Attacker.

According to Isaacs (1965), the point that satisfies

$$H(x, \phi^*, \chi^*, \psi^*, \lambda)|_{\partial\mathscr{B}} = 0 \tag{25}$$

is called the boundary of the usable part of the terminal set. Eq. (25) is the first main equation, which can be rewritten as

$$\begin{aligned}&\max_{\chi(\cdot)}\min_{\phi(\cdot)\psi(\cdot)} \{\alpha(\lambda_R\cos\phi - \frac{\lambda_\theta}{R}\sin\phi) - (\lambda_R\cos(\theta-\chi)\\ &+ \lambda_r\cos\chi - \frac{\lambda_\theta}{R}\sin(\theta-\chi) - \frac{\lambda_\theta}{r}\sin\chi)\\ &- \beta(\lambda_r\cos\psi + \frac{\lambda_\theta}{r}\sin\psi)\} = 0\end{aligned} \tag{26}$$

Substituting (23) and (24) into (26) yields the second main equation:

$$-\alpha r - \beta R + d = 0 \tag{27}$$

Then, the parameters of the boundary of the usable part are derived as

$$R/r = \sqrt{(1-\alpha^2)/(1-\beta^2)}, \theta = \eta + \gamma \tag{28}$$

Accordingly, the usable part of $\mathscr{D}^1_{TD1}$ can be determined by the following equation:

$$\begin{aligned}&H(R(0), r(0), \theta(0), \phi^*, \psi^*, \chi^*, \lambda_R(0), \lambda_r(0), \lambda_\theta(0))\\ &= -\alpha r - \beta R + d < 0\end{aligned} \tag{29}$$

that is,

$$d > \alpha r + \beta R \tag{30}$$

Considering the expression of d, from (27), we obtain

$$\cos\Theta = \cos\theta = \frac{(1-\beta^2)R^2 + (1-\alpha^2)r^2 - 2\alpha\beta Rr}{2Rr} \tag{31}$$

Here, we define Θ as the winning angle of the Attacker. The important property of Θ is shown in the following theorem.

Theorem 2. *If the initial state of the TAD game satisfies*

$$\theta_0 > \Theta \tag{32}$$

or equivalent to

$$d_0 > \alpha r_0 + \beta R_0 \tag{33}$$

then the Attacker wins the game as long as he or she adopts the roundabout policy, that is, first adopts the expanded angle strategy (24) to evade the Defender and then adopts the parallel strategy (12) to chase the Target.

Proof. As condition (32) is equivalent to condition (33), we prove only the latter. First, if $d_0 > \alpha r_0 + \beta R_0$, it follows from dynamic equations (4)–(6) and the optimal control strategies (23)–(24) that

$$\dot{\theta}(t_0) = -\frac{\alpha}{R_0} - \frac{\beta}{r_0} + \frac{d_0}{R_0 r_0} > 0 \tag{34}$$

$$\dot{R} = -\frac{R\sin\theta}{d}, \dot{R} < 0 \tag{35}$$

$$\dot{r} = -\frac{r\sin\theta}{d}, \dot{r} < 0 \tag{36}$$

From (35) and (36), it is clear that

$$\frac{\dot{R}}{R} = \frac{\dot{r}}{r} = -\frac{\sin\theta}{d} \tag{37}$$

Since the game continues when $R > 0, r > 0, 0 < \theta < \pi$, we have

$$\dot{b} = \frac{r\dot{R} - R\dot{r}}{r^2} = 0 \tag{38}$$

where $b = R/r$.

Then, Eq. (31) can be rewritten as

$$\cos\Theta = (1-\beta^2)b/2 + (1-\alpha^2)/2b - \alpha\beta \tag{39}$$

Since $\cos\Theta$ is a continuous function of $(0, \pi]$ and the value is constant, the value of Θ is constant and does not change over time. From the combination of the condition $\theta_0 > \Theta$ and Eq. (34) $\dot{\theta}(t_0) > 0$, we know that the value of $\theta(t)$ increases monotonically over time. Further, Eqs. (35) and (36) indicate that the values of $R(t)$ and $r(t)$ decrease monotonically over time. From (38), $b = R/r$ is invariant. Hence, it is clear that $R(t)$ and $r(t)$ decrease proportionally and asymptotically converge to zero, but do not reach zero.

Thus, based on the concept of the Nash equilibrium, $\theta(t)$ will reach $\gamma + \eta$ before $r(t)$ decreases to zero as long as the Attacker adopts strategy (24) no matter what strategies the Target and Defender choose. In other words, the Attacker can round the Defender and then directly pursue the Target. □

Remark 1. The optimal strategies of the three players have a geometric interpretation. The Defender and Target are moving perpendicular to their individual LOSs. The angle with which the Attacker deviates from LOS DA is equal to the angle with which the Target approaches LOS DT. □

Remark 2. When

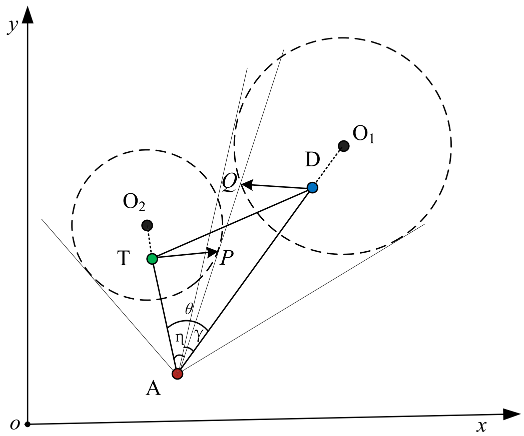
$$\Theta < \theta_0 < arc\cos(\sqrt{(1-\alpha^2)(1-\beta^2)} - \alpha\beta) \tag{40}$$

that is, $\Theta < \theta_0 \leq \gamma + \eta$, there are two cases. Fig. 4a shows the first case ($R_0/r_0 < \sqrt{(1-\alpha^2)/(1-\beta^2)}$) in which the Attacker can adopt the parallel strategy (12) to pursue the Target because the Attacker does not pass through the capture region of the Defender.

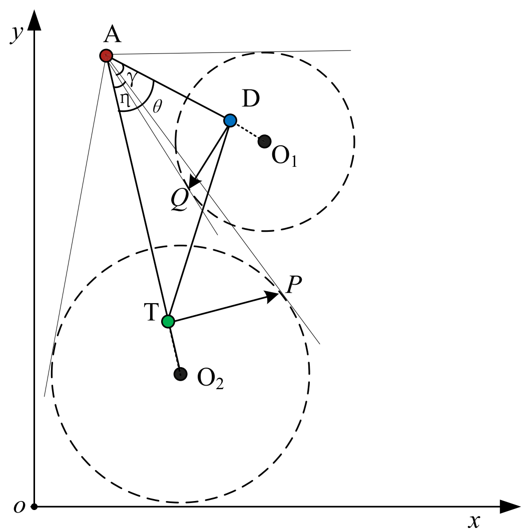
Fig. 4b shows the second case ($R_0/r_0 \geq \sqrt{(1-\alpha^2)/(1-\beta^2)}$). Assuming the Target is bait to escape to point P, if the Attacker adopts the parallel strategy (12) to pursue the Target, then the Attacker will go straight through the Apollonius circle O_1 and the Defender will successfully intercept the Attacker. Thus, in this case, the Attacker should adopt the roundabout policy. □

3.1.2. The distance for the Attacker to win ($\beta \geq 1$)

When $\beta \geq 1$, the Attacker is slower than the Defender. Hence, to win the game, the Attacker cannot adopt the roundabout policy; only adopting the pure policy allows him or her to capture the Target before being captured by the Defender.



(a) $R_0/r_0 < \sqrt{(1-\alpha^2)/(1-\beta^2)}$.



(b) $R_0/r_0 \geq \sqrt{(1-\alpha^2)/(1-\beta^2)}$.

Fig. 4. Two cases when $\Theta < \theta_0 < arc\cos(\sqrt{(1-\alpha^2)(1-\beta^2)} - \alpha\beta)$.

Based on Lemma 1, we have the following theorem.

Theorem 3. *When $\beta \geq 1$, if the initial state of the TAD game satisfies*

$$d_0 > \alpha r_0 + \beta R_0 \tag{41}$$

then the Attacker can adopt the parallel strategy to chase the Target and win the game.

Proof. When $\beta > 1$, as shown in Fig. 5, O_1 is the Apollonius circle of the Attacker and Defender and O_2 is the Apollonius circle of the Attacker and Target. The centers of circles O_1 and O_2 are

$$\begin{aligned} O_1 &= (a_1, b_1), a_1 = \frac{x_D - \beta^2 x_A}{\beta^2 - 1}, b_1 = \frac{y_D - \beta^2 y_A}{\beta^2 - 1} \\ O_2 &= (a_2, b_2), a_2 = \frac{x_T - \alpha^2 x_A}{1 - \alpha^2}, b_2 = \frac{y_T - \alpha^2 y_A}{1 - \alpha^2} \end{aligned} \tag{42}$$

and the corresponding radiuses are

$$r_{d1} = \frac{\beta r}{\beta^2 - 1}, r_{d2} = \frac{\alpha R}{1 - \alpha^2} \tag{43}$$

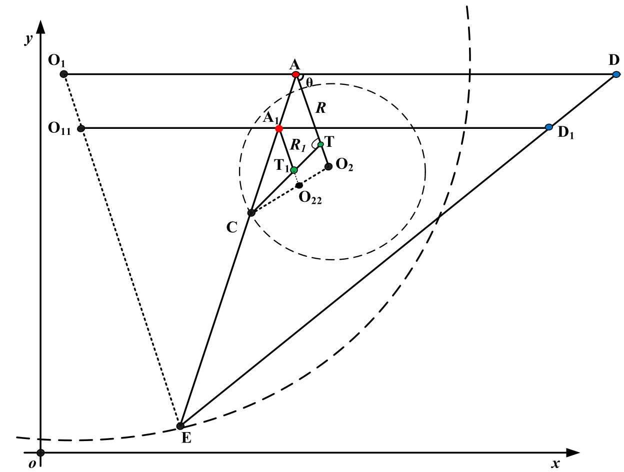


Fig. 5. The distance for the Attacker to win.

Condition (41) is equivalent to condition

$$\|O_1 - O_2\| < r_{d1} - r_{d2} \tag{44}$$

That is, when condition (41) is true, the Apollonius circle of the Attacker and Target is included in the Apollonius circle of the Attacker and Defender.

If condition (41) is true, then

$$\cos\theta < \frac{(1-\alpha^2)r^2 + (1-\beta^2)R^2 - 2\alpha\beta Rr}{2Rr} \tag{45}$$

Assume the three players move to points A_1, T_1, and D_1 from their respective initial positions after Δt time. Suppose

$$\frac{|AA_1|}{|AE|} = \frac{1}{k_1}, \frac{|AA_1|}{|AC|} = \frac{1}{k_2}, k_1 > k_2 \tag{46}$$

then

$$\begin{aligned} |O_{11}E| &= \frac{k_1 - 1}{k_1}|O_1E| = \frac{k_1 - 1}{k_1} \cdot \frac{\beta r}{\beta^2 - 1} \\ |O_{22}C| &= \frac{k_2 - 1}{k_2}|O_2C| = \frac{k_2 - 1}{k_2} \cdot \frac{\alpha R}{1 - \alpha^2} \end{aligned} \tag{47}$$

$$\begin{aligned} |O_{11}A_1| &= \frac{k_1 - 1}{k_1} \cdot \frac{r}{\beta^2 - 1} \\ |O_{22}A_2| &= \frac{k_2 - 1}{k_2} \cdot \frac{R}{1 - \alpha^2} \end{aligned} \tag{48}$$

$$\begin{aligned} &\|O_{11} - O_{22}\|^2 = |O_{11}A_1|^2 + |O_{22}A_1|^2 \\ &- 2|O_{11}A_1| \cdot |O_{22}A_1| \cdot \cos(\pi - \theta) \end{aligned} \tag{49}$$

By substituting (45), (46), and (48) into (49), we obtain

$$\|O_{11} - O_{22}\|^2 < \left(\frac{k_1 - 1}{k_1} \cdot \frac{\beta r}{\beta^2 - 1} - \frac{k_2 - 1}{k_2} \cdot \frac{\alpha R}{1 - \alpha^2}\right)^2 \tag{50}$$

that is,

$$\|O_{11} - O_{22}\| < |O_{11}E| - |O_{22}C| \tag{51}$$

Thus, the new two Apollonius circles are also internally included at any time.

When $\beta = 1$, the Apollonius circle O_1 no longer exists. The vertical bisector of the Attacker and Defender is described as the following equation:

$$A_0 x + B_0 y + C_0 = 0 \tag{52}$$

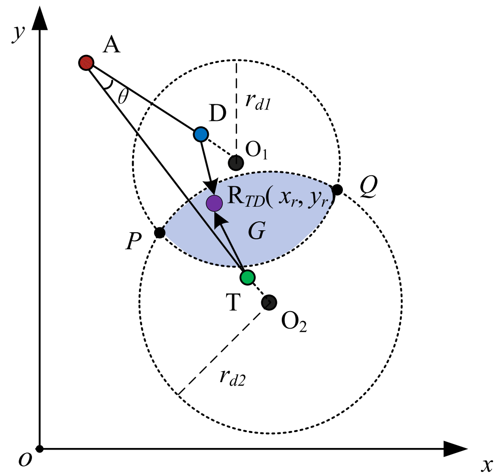


Fig. 6. The distance for the Target's survival . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where $A_0 = 2(x_A - x_D)$, $B_0 = 2(y_A - y_D)$, $C_0 = (y_D^2 - y_A^2) + (x_D^2 - x_A^2)$. When the vertical bisector of the Attacker and Defender is disjoint to the Apollonius circle O_2 of the Target and Attacker, the Attacker can win the game. That is,

$$r_{d2} < \left| \frac{A_0 \frac{x_T - \alpha^2 x_A}{1-\alpha^2} + B_0 \frac{y_T - \alpha^2 y_A}{1-\alpha^2} + C_0}{\sqrt{A_0^2 + B_0^2}} \right| \tag{53}$$

By substituting the parameters and simplifying, we obtain

$$d_0 > R_0 + \alpha r_0 \tag{54}$$

Therefore, if condition (41) is true, the Attacker can adopt the parallel strategy to capture the Target before being captured by the Defender. □

Remark 3. When $\beta \geq 1$, if the initial state of the TAD game satisfies

$$d_0 \leq \alpha r_0 + \beta R_0 \tag{55}$$

two Apollonius circles are intersecting or disjoint. The Target can adopt the **pure evasion strategy** $\phi^* = 0$ (see the detailed description in Section 4.2) to evade the Attacker or the Target can act as bait to move toward the outside of the intersecting region of the two Apollonius circles. Then, no matter what strategy the Attacker chooses, if the Defender adopts the **pure pursuit strategy**

$$\psi^* = \arcsin(\beta \sin(\chi)) \tag{56}$$

to chase the Attacker, the Defender will capture the Attacker before the Attacker captures the Target. Clearly, the pure pursuit strategy (56) is then a parallel strategy. □

3.2. Distance for the Target's survival

From the perspective of the Target–Defender team, three policies can be adopted to win the game: (1) the Defender can adopt the pure pursuit policy to capture the Attacker, (2) the Target–Defender team can adopt the rendezvous policy to win the game, and (3) the Target–Defender team can adopt the cooperative evasion policy to avoid the Target being captured. The pure pursuit policy of the Defender is the parallel strategy (56), which is described in Remark 3. The cooperative evasion policy of the Target–Defender team is the narrow angle strategy of Eq. (24), which is opposite to the expanded strategy (23) of the Attacker. In this subsection, we describe the rendezvous policy for the Target–Defender team.

As shown in Fig. 6, O_1 is the Apollonius circle of the Attacker and Defender ($\beta < 1$) and O_2 is the Apollonius circle of the Attacker and Target.

To ensure that the Target and Defender rendezvous before the Attacker arrives, the two Apollonius circles should intersect. The intersection region is denoted by G, as shown in the dashed area of Fig. 6. Points A, T, O_2 and points A, D, O_1 are collinear. The centers of circles O_1 and O_2 are

$$\begin{aligned} O_1 &= (a_1, b_1),\ a_1 = \frac{x_D - \beta^2 x_A}{1-\beta^2},\ b_1 = \frac{y_D - \beta^2 y_A}{1-\beta^2} \\ O_2 &= (a_2, b_2),\ a_2 = \frac{x_T - \alpha^2 x_A}{1-\alpha^2},\ b_2 = \frac{y_T - \alpha^2 y_A}{1-\alpha^2} \end{aligned} \tag{57}$$

and the corresponding radiuses are

$$r_{d1} = \frac{\beta r}{1-\beta^2},\ r_{d2} = \frac{\alpha R}{1-\alpha^2} \tag{58}$$

The crossover points of circles O_1 and O_2 are denoted by $P(x_P, y_P)$ and $Q(x_Q, y_Q)$, which satisfy the following equations:

$$\begin{aligned} &(x_i - a_1)^2 + (y_i - b_1)^2 = {r_{d1}}^2 \\ &(x_i - a_2)^2 + (y_i - b_2)^2 = {r_{d2}}^2 \end{aligned} \tag{59}$$

where index $i = P, Q$. The length of segment O_1O_2 is denoted by L

$$L = \|O_1 - O_2\| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

The slopes of segments O_1O_2 and PQ are k_1 and k_2, respectively:

$$k_1 = \frac{b_2 - b_1}{a_2 - a_1},\ k_2 = -1/k_1 \tag{60}$$

Define $L_2 = r_{d1}^2 - (1 + k_1^2)(\frac{(a_2-a_1)(r_{d1}^2 - r_{d2}^2 + L^2)}{2L^2})^2$, and then

$$\begin{aligned} x_P &= a_1 + \frac{(a_2 - a_1)(r_{d1}^2 - r_{d2}^3 + L^2)}{2L^2} - \sqrt{L_2/(1+k_2^2)} \\ y_P &= b_1 + \frac{k_1(a_2 - a_1)(r_{d1}^2 - r_{d2}^3 + L^2)}{2L^2} - k_2\sqrt{L_2/(1+k_2^2)} \\ x_Q &= a_1 + \frac{(a_2 - a_1)(r_{d1}^2 - r_{d2}^3 + L^2)}{2L^2} + \sqrt{L_2/(1+k_2^2)} \\ y_Q &= b_1 + \frac{k_1(a_2 - a_1)(r_{d1}^2 - r_{d2}^3 + L^2)}{2L^2} + k_2\sqrt{L_2/(1+k_2^2)} \end{aligned} \tag{61}$$

Remark 4. Any point in region G can be used as the rendezvous point of the Target and Defender, which is denoted by $R_{TD}(x_r, y_r)$, $\forall R_{TD}(x_r, y_r) \in G$. □

Remark 5. The rendezvous point R_{TD}, at which the Target and Defender arrive at the same time, should satisfy condition $d_{RD}/V_D = d_{RT}/V_T$, that is, $\alpha d_{RD} = \beta d_{RT}$, where $d_{RD} = \sqrt{(x_D - x_r)^2 + (y_D - y_r)^2}$ is the distance between the rendezvous R_{TD} and Defender and $d_{RT} = \sqrt{(x_T - x_r)^2 + (y_T - y_r)^2}$ is the distance between the rendezvous R_{TD} and Target. □

As shown in Fig. 7, the solid black curves are the two Apollonius circles that intersect at points P and Q, where $\alpha = 0.6$, $\beta = 0.7$. The dashed curves are the isochronous circles ($V_A = 1$, $t = 1.9$) of the Attacker, Target, and Defender. The solid green curve is a set of points at which the Target and Defender can simultaneously arrive. Then, the corresponding **rendezvous strategies** of the Target and

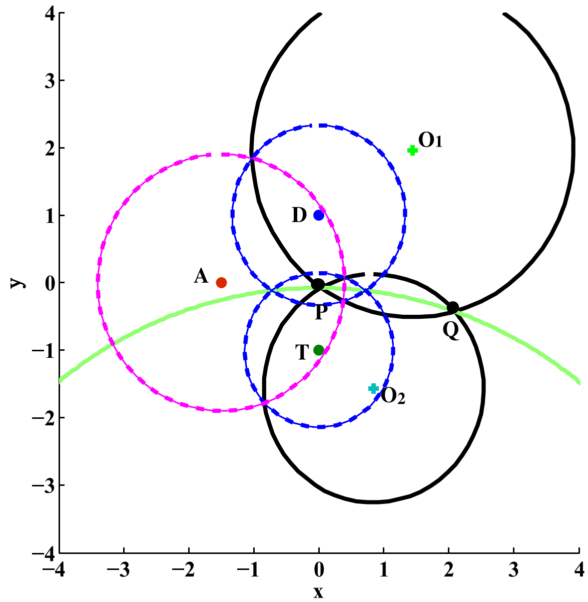


Fig. 7. The Target and Defender choose a point to rendezvous, $\beta = 0.7$.

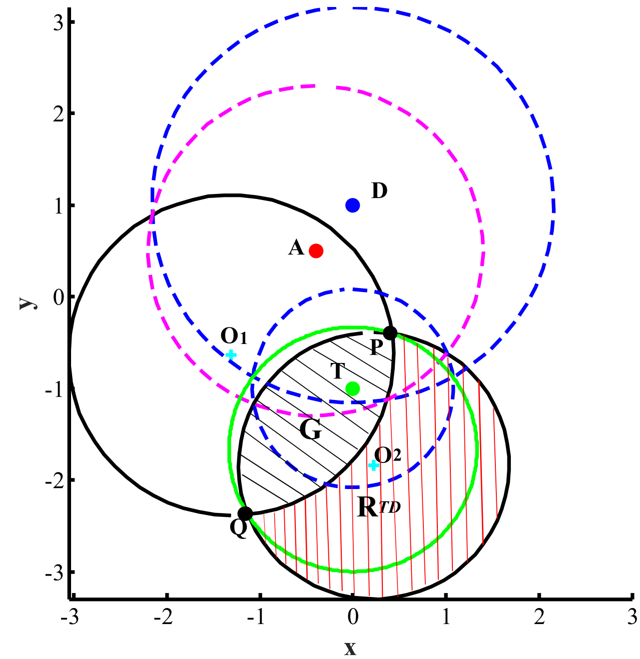


Fig. 8. The Target and Defender choose a point to rendezvous, $\beta = 1.2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Defender are

$$\begin{aligned} \sin\hat{\phi} &= \frac{y_r - y_T}{d_{RT}}, \cos\hat{\phi} = \frac{x_r - x_T}{d_{RT}} \\ \sin\hat{\psi} &= \frac{y_r - y_D}{d_{RD}}, \cos\hat{\psi} = \frac{x_r - x_D}{d_{RD}} \end{aligned} \tag{62}$$

Theorem 4. *When $\beta < 1$, if the initial state of the TAD game satisfies*

$$d_0 \le \alpha r_0 + \beta R_0 \tag{63}$$

then the Target–Defender team can adopt the rendezvous strategies (62) to rendezvous in the intersection region of the two Apollonius circles and win the game.

Proof. If two Apollonius circles intersect, that is,

$$\|O_1 - O_2\| < r_{d1} + r_{d2} \tag{64}$$

the Target–Defender team can rendezvous at any point in intersection region G before the Attacker arrives. Substituting (57) and (58) into condition (64) yields

$$d < \alpha r + \beta R \tag{65}$$

Second, when $\|O_1 - O_2\| = r_{d1} + r_{d2}$, $d = \alpha r + \beta R$, the two Apollonius circles are tangents, and the three players will meet at the same time. Furthermore, the Target–Defender team will win the game. □

When $\beta > 1$ (see Fig. 8), the solid black curves are the two Apollonius circles that intersect at points P and Q, where $\alpha = 0.6$, $\beta = 1.2$. The dotted curves are the isochronous circles ($V_A = 1, t = 1.8$) of the Attacker, Target, and Defender. The solid green curve is a set of points at which the Target and Defender can simultaneously arrive. Similar to Fig. 6, the intersection region is denoted by G (black shaded part). The inside region of the Apollonius circle O_2 is described as $\mathscr{D}_{O_2}$. The Target wants to rendezvous with the Defender before being captured by the Attacker. Therefore, we have following remark.

Remark 6. Any point in region $\mathscr{D}_{O_2} \backslash G$ (red shaded part) can be used as the rendezvous point of the Target–Defender team, which is denoted by $R_{TD}(x_r, y_r)$, $\forall R_{TD}(x_r, y_r) \in \mathscr{D}_{O_2} \backslash G$. The corresponding rendezvous strategies of the Target–Defender team are (62). □

In particular, when $\beta = 1$, the Apollonius circle O_1 becomes a vertical bisector of the Attacker and Defender, and Remark 6 is still established.

3.3. Construction of a barrier

We integrate the winning conditions obtained by using these policies to construct the barrier of the game. According to Theorems 2–4, the winning region of the Attacker denoted by $\mathscr{D}_A$ can be described as

$$\begin{aligned} \mathscr{D}_A &= \{R, r, \theta | \sqrt{R^2 + r^2 - 2Rr\cos\theta} > \alpha r + \beta R\} \\ &= \{R, r, \theta | \cos\theta < \frac{(1-\beta^2)R^2 + (1-\alpha^2)r^2}{2Rr} - 2\alpha\beta\} \end{aligned} \tag{66}$$

The winning region of the Target–Defender team $\mathscr{D}_{TD}$ can be described as

$$\begin{aligned} \mathscr{D}_{TD} &= \{R, r, \theta | \sqrt{R^2 + r^2 - 2Rr\cos\theta} \le \alpha r + \beta R\} \\ &= \{R, r, \theta | \cos\theta \ge \frac{(1-\beta^2)R^2 + (1-\alpha^2)r^2}{2Rr} - \alpha\beta\} \end{aligned} \tag{67}$$

Result (66) is complementary to the result in (67). $\mathscr{D}_{TD} = \Omega \backslash \mathscr{D}_A$, Ω is the full state space. The boundary of $\mathscr{D}_A$ and $\mathscr{D}_{TD}$, denoted by B_1, can be expressed as

$$B_1 = \{R, r, \theta | \cos\theta = \frac{(1-\beta^2)R^2 + (1-\alpha^2)r^2}{2Rr} - 2\alpha\beta\} \tag{68}$$

In other words, boundary B_1 is the barrier that separates the whole space into two disjoint regions: the winning region of the Attacker ($\mathscr{D}_A$) and the winning region of the Target–Defender team ($\mathscr{D}_{TD}$).

Fig. 9 illustrates the barrier (68) of the game in the three-dimensional space (R, r, θ). The state inside the barrier constitutes the winning region of the Target–Defender team, which can adopt suitable strategies to win the game no matter what strategy the

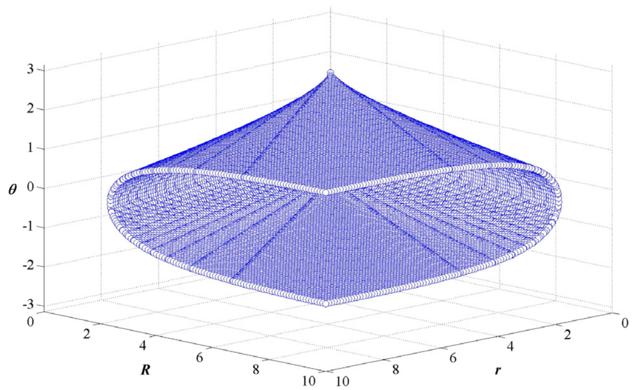


Fig. 9. The barrier of the TAD game in the three-dimensional state space (R, r, θ) with $\alpha = 0.6$ and $\beta = 0.7$.

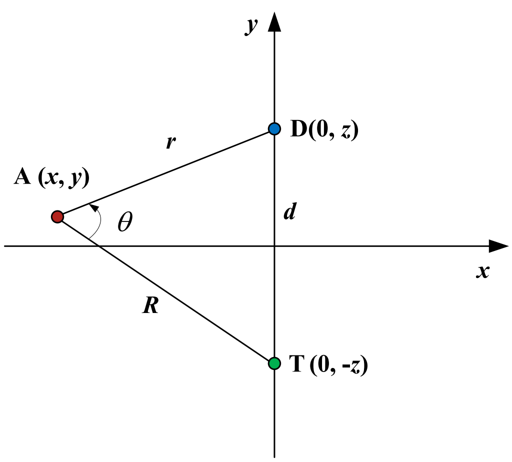
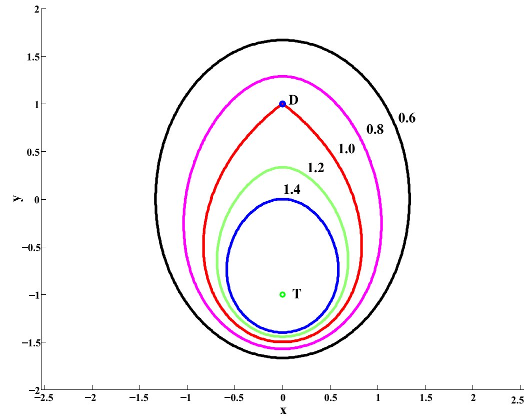
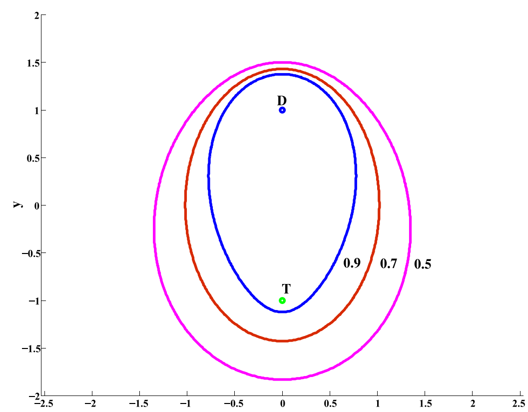


Fig. 10. The TAD game in the relatively fixed state space (x, y).



(a) The barriers in the relatively fixed state space with different β and fixed $z = 1$, $\alpha = 0.6$.



(b) The barriers in the relatively fixed state space with different α and fixed $z = 1$, $\beta = 0.7$.

Fig. 11. The barriers of the TAD game in the relatively fixed state space with different parameters.

Attacker chooses. On the contrary, the state outside the barrier constitutes the winning region of the Attacker, who can choose a suitable strategy to capture the Target. For a more intuitive display, we switch the problem to a relatively fixed two-dimensional space, as shown in Fig. 10. The coordinates of the three players in the plane correspond to $D(0, z)$, $T(0, -z)$, and $A(x, y)$, respectively. Barrier B_1 is given by the following equation:

$$\alpha\sqrt{x^2 + (z - y)^2} + \beta\sqrt{x^2 + (z + y)^2} = 2 * z \tag{69}$$

Fig. 11a illustrates the barriers with different β for fixed z and α. In this relatively fixed space, the inside of the curves (except point D) is the Attacker's winning region, whose size decreases when β increases and when α and z are unchanged. When $\beta > 1$, the Defender is outside the Attacker's winning region, which indicates that the Attacker only adopts the pure policy. Likewise, the Attacker's winning region decreases when α increases and when β and z are unchanged (see Fig. 11b). Furthermore, the Attacker's winning region increases when z increases. The outside of the curves and curves themselves constitute the winning region of the Target–Defender team.

4. Optimal strategies for different regions

We have already analyzed the TAD problem from the perspective of the game of kind and constructed the barrier that divides the space into $\mathscr{D}_A$ and $\mathscr{D}_{TD}$. Hence, in this section, we present the optimal strategy for each player in the different winning areas. When the state lies in the winning region of the Attacker ($\mathscr{D}_A$), the Attacker hopes to capture the Target in the shortest time, but the Target–Defender team tries to extend the length of the game as much as possible. In the same way, when the state lies in the winning region of the Target–Defender team ($\mathscr{D}_{TD}$), the Target–Defender team hopes to intercept the Attacker quickly or rendezvous directly to end the game, but the Attacker will try to extend the termination time. Therefore, in this section, we use the winning time as the payoff to discuss the optimal strategies for the players in the scope of a game of degree.

4.1. Optimal strategies in the winning region of the Target–Defender team

We take the winning time T_w as the payoff of the TAD game of degree. Thus, when the initial positions of the players are in the winning region of the Target–Defender team ($\mathscr{D}_{TD}$), the payoff function is given by

$$\max_{\chi(\cdot)} \min_{\phi(\cdot)\psi(\cdot)} J_1 = \max_{\chi(\cdot)} \min_{\phi(\cdot)\psi(\cdot)} \int_{t_0}^{T_w} 1dt \tag{70}$$

When $\beta < 1$, since the Attacker has an advantage in speed and ability to change roles, the Target–Defender team adopts the narrow angle strategy (23), which can only prevent the Attacker from chasing the Target, but the game will always continue (see Fig. 14a). To end the game in the shortest time, the Target and Defender should adopt the rendezvous strategies (62) regardless of the ratio value of R/r. In this case, the terminal condition is $\{R > 0, r > 0, d = 0\}$.

Theorem 5. *If the Attacker is far away from the Target and Defender, that is, if the initial state satisfies*

$$\cos\theta \geq \frac{(1-\alpha^2)r^2 + (1-\beta^2)R^2}{2(1+\alpha\beta)Rr} \tag{71}$$

then the Target and Defender can move in the opposite direction to terminate the game.

Proof. To win the game in the shortest time, the Target–Defender team can rendezvous at point M (see Fig. 12).

According to the sine law and Pythagorean theorem,

$$|AM| = \frac{1}{\alpha+\beta}\sqrt{\alpha^2r^2 + \beta^2R^2 + 2\alpha\beta Rr\cos\theta} \tag{72}$$

To ensure the Target–Defender team rendezvous before the Attacker arrives at point M, the following inequality should hold:

$$\frac{d}{v_T + v_D} \leq \frac{|AM|}{v_A} \tag{73}$$

Substituting (72) into (73) yields

$$\frac{d^2}{(\alpha+\beta)^2} \leq \frac{1}{(\alpha+\beta)^2}\left(\alpha^2r^2 + \beta^2R^2 + 2\alpha\beta Rr\cos\theta\right) \tag{74}$$

By using the triangle cosine theorem, we can obtain

$$d^2 \leq \frac{\alpha^2+\alpha\beta}{\alpha\beta+1}r^2 + \frac{\beta^2+\alpha\beta}{\alpha\beta+1}R^2 \tag{75}$$

or

$$\cos\theta \geq \frac{(1-\alpha^2)r^2 + (1-\beta^2)R^2}{2(1+\alpha\beta)Rr} \qquad \square \tag{76}$$

By using the geometrical analysis method (see Fig. 12), the control strategies of the three players can be derived as

$$\sin\psi^* = \frac{|TP|}{|TD|} = \frac{R\sin\theta}{d}, \cos\psi^* = \frac{r - R\cos\theta}{d} \tag{77}$$

$$\sin\phi^* = \frac{|DE|}{|TD|} = \frac{r\sin\theta}{d}, \cos\phi^* = \frac{R - r\cos\theta}{d} \tag{78}$$

$$\begin{aligned} \sin\chi^* &= \frac{|MQ|}{|AM|} = \frac{\beta R\sin\theta}{\sqrt{\alpha^2r^2+\beta^2R^2+2\alpha\beta Rr\cos\theta}} \\ \cos\chi^* &= \frac{|AQ|}{|AM|} = \frac{\alpha r + \beta R\cos\theta}{\sqrt{\alpha^2r^2+\beta^2R^2+2\alpha\beta Rr\cos\theta}} \end{aligned} \tag{79}$$

The control strategies (77) and (78) are called the **direct rendezvous strategies** of the Target–Defender team. Condition (71) in Theorem 5 is also regarded as a boundary, denoted by B_3, which can be expressed as

$$B_3 = \{R, r, \theta | \cos\theta = \frac{(1-\beta^2)R^2 + (1-\alpha^2)r^2}{2(1+\alpha\beta)Rr}\} \tag{80}$$

For $R - r < d < \alpha r + \beta R$, we have

$$\begin{aligned} &\frac{(1-\beta^2)R^2 + (1-\alpha^2)r^2}{2(1+\alpha\beta)Rr} - \frac{(1-\beta^2)R^2 + (1-\alpha^2)r^2 - 2\alpha\beta Rr}{2Rr} \\ &= \frac{\alpha\beta((R-r)^2 - (\alpha r+\beta R)^2)}{2(1+\alpha\beta)Rr} \leq 0 \end{aligned} \tag{81}$$

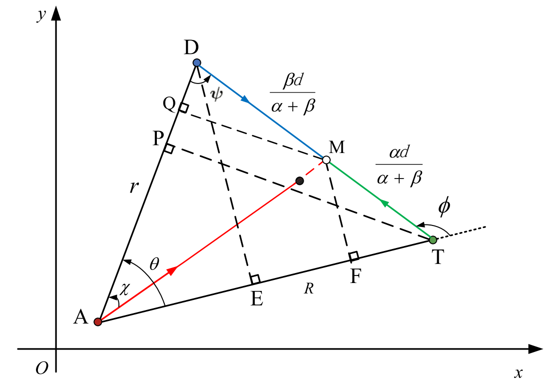


Fig. 12. The Target and Defender go straight toward each other.

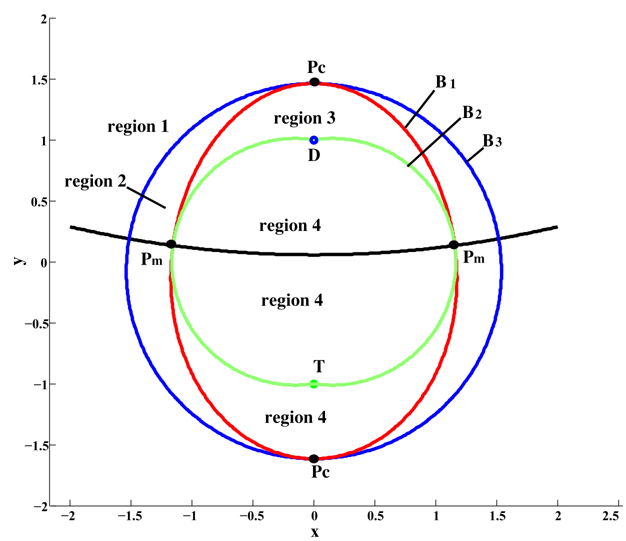
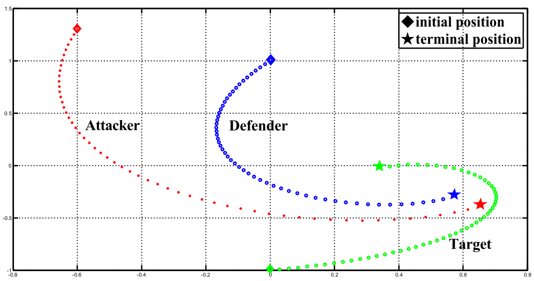


Fig. 13. The separation of the relatively fixed two-dimensional space with $z = 1$, $\alpha = 0.6$, and $\beta = 0.7$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

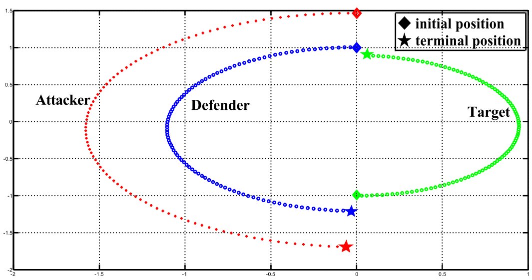
$\mathscr{D}_{TD}$ includes boundary B_3, which separates the winning region ($\mathscr{D}_{TD}$) of the Target–Defender team into region 1 ($\mathscr{D}_1$) and region 2 ($\mathscr{D}_{TD} \backslash \mathscr{D}_1$). In particular, when $R - r = d = \alpha r + \beta R$, boundary B_3 is tangent to $\mathscr{D}_{TD}$ at two points P_c (see Fig. 13), where $\theta = 0$ or π and the three players are located on the same line. If the Target–Defender team adopts the direct rendezvous strategies (77) and (78), they will win the game because the three players will meet at the same time. If the Target–Defender team adopts the narrow angle strategy (23) to decrease the angle θ, the team will still win the game because the Attacker cannot evade the Defender and the optimal trajectories of the three players are circles with the same center (see Fig. 14b).

When the initial state lies in 1, the connecting line of the Target and Defender (segment TD) definitely passes through region G (see Fig. 6). Therefore, the Target and Defender go straight toward each other to terminate the game, and their rendezvous strategies are (77) and (78), respectively (see Fig. 14c).

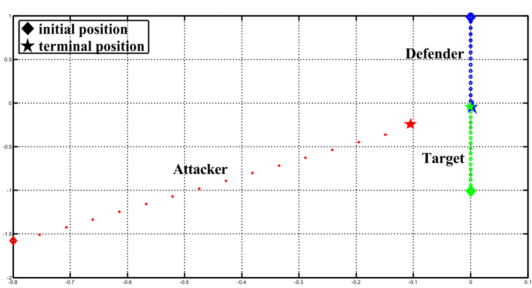
When the initial state lies in region 2, the Target should cooperate with the Defender to choose a point in region G to rendezvous to win the game and the corresponding rendezvous strategies are (62). To win the game in the shortest time, the Target–Defender team should choose an optimum point R_{TD} that is nearest the Target and on the line at which the Target and Defender simultaneously arrive (see Fig. 14d).



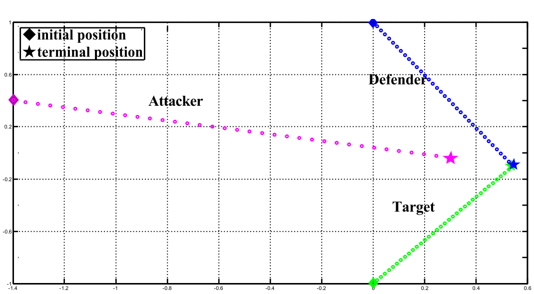
(a) When the initial state is in region 2, $\alpha = 0.6, \beta = 0.7$, the Target and Defender adopt strategy (23).



(b) When the initial state is at points P_c, $\alpha = 0.6, \beta = 0.7$, the Target and Defender adopt strategy (23).



(c) When the initial state is in region 1, $\alpha = 0.6, \beta = 0.7$, the Target and Defender adopt strategies (77) and (78).



(d) When the initial state is in region 2, $\alpha = 0.6, \beta = 0.7$, the Target and Defender adopt strategy (62).

Fig. 14. The trajectories of the three players with different strategies in the winning region of the Target–Defender team.

When $\beta \geq 1$, the Defender has an advantage in speed and ability to capture the Attacker. If the Attacker adopts the **pure evasion strategy**

$$\chi^* = \pi \tag{82}$$

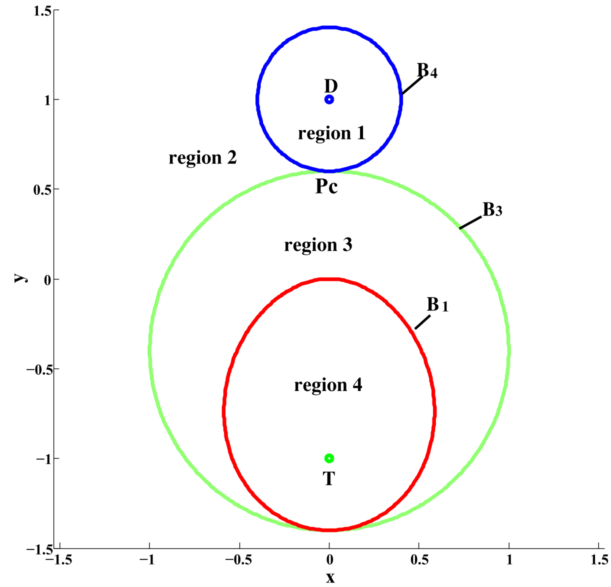


Fig. 15. The separation of the relatively fixed two-dimensional space with $z = 1$, $\alpha = 0.6$, and $\beta = 1.4$.

then the Defender can adopt the parallel strategy to capture the Attacker. t_c indicates the time spent for the Defender to capture the Attacker and t_r indicates the time spent for the Defender to rendezvous directly with the Target.

$$t_c = \frac{r}{V_D - V_A}, t_r = \frac{d}{V_A + V_T} \tag{83}$$

Let $t_c = t_r$; thus, we have the following condition:

$$\frac{r}{\beta - 1} = \frac{d}{\alpha + \beta} \tag{84}$$

Condition (84) can be considered to be boundary B_4. Boundaries B_4 and B_3 separate the winning region of the Target–Defender team into three regions (see Fig. 15). Boundary B_4 is tangent to boundary B_3 at point P_c, where the three players are located on the same line, $\theta = \pi$. The Target–Defender team takes the same time to win the game regardless of whether it adopts the pure policy or rendezvous policy.

In region 1, $t_c < t_r$, the Defender should adopt the parallel strategy (56) to capture the Attacker and the Target–Defender team wins the game in the shortest time.

In region 2, $t_c > t_r$, the Target and Defender adopt the direct rendezvous strategies (77) and (78) to go straight toward each other to terminate the game in the shortest time. In region 3, the Target and Defender can choose a point to rendezvous. The corresponding rendezvous strategies are (62).

4.2. Optimal strategies in the winning region of the Attacker

When the initial positions of the players lie in the Attacker's winning region ($\mathscr{D}_A$), the payoff function of the game of degree changes to

$$\min_{\chi(\cdot)} \max_{\phi(\cdot)\psi(\cdot)} J_2 = \min_{\chi(\cdot)} \max_{\phi(\cdot)\psi(\cdot)} \int_{t_0}^{T_w} 1dt \tag{85}$$

When $\beta \geq 1$, the winning region of the Attacker is region 4 (see Fig. 15). The Attacker can adopt only the parallel strategy (12) to capture the Target.

When $\beta < 1$, in region $\mathscr{D}_A$, the condition in Theorem 1 is regarded as a boundary, denoted by B_2, which can be expressed as

$$B_2 = \{R, r, \theta | \cos\theta = \sqrt{(1-\alpha^2)(1-\beta^2)} - \alpha\beta\} \tag{86}$$

It is clear that

$$\begin{aligned}&\frac{(1-\beta^2)R^2 + (1-\alpha^2)r^2 - 2\alpha\beta Rr}{2Rr} \\ &- (\sqrt{(1-\alpha^2)(1-\beta^2)} - \alpha\beta) \\ &= \frac{(\sqrt{1-\beta^2}R - \sqrt{1-\alpha^2}r)^2}{2Rr} \geq 0\end{aligned} \tag{87}$$

$\mathscr{D}_A$ includes boundary B_2. The Attacker's winning region ($\mathscr{D}_A$) is divided by boundary B_2 and condition $R/r \geq \sqrt{(1-\alpha^2)/(1-\beta^2)}$ into region 3 ($\mathscr{D}_3$) and region 4 ($\mathscr{D}_A \backslash \mathscr{D}_3$) (see Fig. 13). Boundary B_2 is depicted by the green curve outside which the Attacker needs to evade the Defender. Condition $R/r = \sqrt{(1-\alpha^2)/(1-\beta^2)}$ is described by the black curve. In particular, when $R/r = \sqrt{(1-\alpha^2)/(1-\beta^2)}$, $d = \alpha r + \beta R$, boundary B_2 is tangent to B_1 at two points P_m.

When the initial position of the Attacker lies in region 4, the Attacker aims to capture the Target in the shortest time. On the contrary, the goal of the Target–Defender team is to extend the length of the game. The Hamiltonian function is given by

$$\begin{aligned}&H(\lambda, \phi, \psi, \chi, t) = 1 + \lambda_R \dot{R} + \lambda_r \dot{r} + \lambda_\theta \dot{\theta} \\ &= 1 + \alpha(\lambda_R \cos\phi - \frac{\lambda_\theta}{R}\sin\phi) - (\lambda_R \cos(\theta - \chi) + \lambda_r \cos\chi - \\ &\frac{\lambda_\theta}{R}\sin(\theta-\chi) - \frac{\lambda_\theta}{r}\sin\chi) - \beta(\lambda_r \cos\psi + \frac{\lambda_\theta}{r}\sin\psi)\end{aligned} \tag{88}$$

Differentiating the Hamiltonian function in ϕ and setting the derivative to zero:

$$\frac{\partial H}{\partial \phi} = \alpha[-\frac{\lambda_\theta}{R}\cos\phi - \lambda_R \sin\phi] = 0 \tag{89}$$

By using the trigonometric identity, we can conclude

$$\sin\phi^* = -\frac{\lambda_\theta}{R\rho_1}, \cos\phi^* = \frac{\lambda_R}{\rho_1}, \rho_1 = \sqrt{(\lambda_\theta/R)^2 + \lambda_R^2} \tag{90}$$

The second partial derivative of the Hamiltonian function in ϕ is

$$\frac{\partial^2 H}{\partial \phi^2} = \alpha[\frac{\lambda_\theta}{R}\sin\phi - \lambda_R\cos\phi] < 0 \tag{91}$$

which means that ϕ^* maximizes the payoff J_2. Similarly, we can conclude that

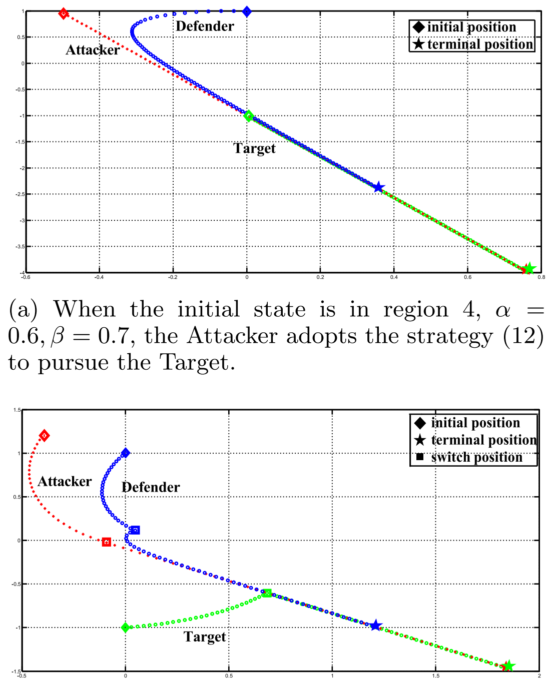
$$\sin\psi^* = -\frac{\lambda_\theta}{r\rho_2}, \cos\psi^* = -\frac{\lambda_r}{\rho_2}, \rho_2 = \sqrt{(\lambda_\theta/r)^2 + \lambda_r^2} \tag{92}$$

$$\begin{aligned}&\sin\chi^* = \frac{a}{\rho_3}, \cos\chi^* = \frac{b}{\rho_3}, \rho_3 = \sqrt{a^2+b^2} \\ &a = \lambda_R \sin\theta + \frac{\lambda_\theta}{R}\cos\theta - \frac{\lambda_\theta}{r} \\ &b = \lambda_R\cos\theta + \lambda_r - \frac{\lambda_\theta}{R}\sin\theta\end{aligned} \tag{93}$$

In this winning region of the Attacker, the terminal condition is $\{R = 0, r > 0\}$. The outward normal vector $\lambda_r(0) = 0$, $\lambda_\theta(0) = 0$, and $\lambda_R(0)$ is free. Thus, the optimal strategies of the Target–Defender team can be rewritten as

$$\phi^* = 0 \tag{94}$$

$$\psi^* = 0 \tag{95}$$



(a) When the initial state is in region 4, $\alpha = 0.6, \beta = 0.7$, the Attacker adopts the strategy (12) to pursue the Target.

(b) When the initial state is in region 3, $\alpha = 0.6, \beta = 0.7$, the Attacker adopts the roundabout policy (24) (12) to win the game.

Fig. 16. Trajectories of the three players with different strategies in the winning region of the Attacker.

The optimal strategy of the Attacker can be rewritten as

$$\chi^* = \theta \tag{96}$$

which is consistent with the parallel strategy (12). Clearly, strategy (94) is a **pure evasion strategy** of the Target and strategy (96) is a **pure pursuit strategy** of the Attacker. Fig. 16a shows the trajectories of the three players.

When the initial position of the Attacker lies in region 3, the Attacker should adopt the roundabout policy. In other words, there are two sub-games: the included angle θ game and the distance game. In the included angle θ game, the Attacker adopts the expanded angle strategy (24) to increase the angle θ until condition (11) is satisfied. Then, in the distance game, the Attacker adopts the parallel strategy (12) to capture the Target (see Fig. 16b). Point $\theta = \eta + \gamma$ is the turning point at which the three players change their strategies. By contrast, the Target–Defender team also chooses the narrow angle strategy (23) and pure evasion strategy (94) to extend the termination time as much as possible.

5. Conclusion

In this study, we investigate a TAD game in which the Attacker tries to capture the Target, while avoiding being intercepted by the Defender and the Defender cooperates with the Target to intercept the Attacker and defend the Target. We consider the cooperation in the Target–Defender team and balancing the role of the Attacker between pursuer and evader. By employing the explicit policy method, we construct a barrier that separates the whole space into the winning region of the Attacker and the winning region of the Target–Defender team.

In these different winning regions, we analyze the optimal strategies for the players, using the winning time as the payoff of the TAD game of degree, and provide a complete solution to the

Table 1
Strategy of the TAD game.

Condition	Region		Attacker	Target	Defender
$\beta < 1$	TD	1	Arbitrary	Direct rendezvous (77) (78)	
		2	Parallel strategy (12)	Rendezvous (62)	
	A	3	Roundabout policy (24) (12)	Cooperative evasion policy (23) (94)	
		4	Pure pursuit strategy (96)	Pure evasion strategy (94)	Arbitrary
$\beta \geq 1$	TD	1	Pure evasion strategy (82)	Pure evasion strategy (94)	Pure pursuit strategy (56)
		2	Arbitrary	Direct rendezvous (77) (78)	
		3	Parallel strategy (12)	Rendezvous (62)	
	A	4	Pure pursuit strategy (96)	Pure evasion strategy (94)	Arbitrary

game (Table 1). Furthermore, the corresponding optimal trajectories are illustrated to confirm the validity of our analysis. This work lays a foundation for research on search and rescue, the recovery of military equipment, and biological behavior.

In future work, we aim to investigate the TAD game with more practical motion models of the players, more complex gaming environments (containing obstacles, limited observation and intercommunication, and jamming), and different rules (ways of cooperation and conditions of ending the game). Investigating a TAD game with more players will be another focus of our future research.

References

Bakolas, E., & Tsiotras, P. (2012). Relay pursuit of a maneuvering target using dynamic Voronoi diagrams . *Automatica*, *48*(9), 2213–2220.

Bhattacharya, S., Basar, T., & Hovakimyan, N. (2014). On the construction of barrier in a visibility based pursuit evasion game. In *European control conference* (pp. 1894–1901).

Bhattacharya, S., Basar, T., & Hovakimyan, N. (2016). A visibility-based pursuit-evasion game with a circular obstacle. *Journal of Optimization Theory and Applications*, *171*(3), 1071–1082.

Bopardikar, S. D., Bullo, F., & Hespanha, J. P. (2008). On discrete-time pursuit-evasion games with sensing limitations. *IEEE Transactions on Robotics*, *24*(6), 1429–1439.

Bopardikar, S. D., Bullo, F., & Hespanha, J. P. (2009). A cooperative Homicidal Chauffeur game. *Automatica*, *45*(7), 1771–1777.

Boyell, R. (1976). Defending a moving target against missile or torpedo attack. *IEEE Transaction on Aerospace and Electronic Systems*, *AES-12*(4), 522–526.

Casbeer, D. W., Garcia, E., Fuchs, Z. E., & Pachter, M. (2015). Cooperative target defense differential game with a constrained-maneuverable Defender. In *IEEE conference on decision and control* (pp. 1713–1718).

D'Andrea, R., & Murray, R. M. (2003). The RoboFlag competition. In *American control conference, Vol. 1* (pp. 650–655).

Ding, X. C., Rahmani, A. R., & Egerstedt, M. (2010). Multi-UAV convoy protection: An optimal approach to path planning and coordination. *IEEE Transactions on Robotics*, *26*(2), 256–268.

Garcia, E., Casbeer, D. W., & Pachter, M. (2015a). Active target defense differential game with a fast defender. In *American control conference* (pp. 3752–3757).

Garcia, E., Casbeer, D. W., & Pachter, M. (2015b). Cooperative strategies for optimal aircraft defense from an attacking missile. *Journal of Guidance Control and Dynamics*, *38*(8), 1–11.

Garcia, E., Casbeer, D. W., & Pachter, M. (2017). Active target defense using first order missile models. *Automatica*, *78*, 139–143.

Garcia, E., Casbeer, D. W., Pham, K., & Pachter, M. (2014). Cooperative aircraft defense from an attacking missile. In *IEEE 53rd annual conference on decision and control* (pp. 2926–2931).

Garcia, E., Casbeer, D., Pham, K. D., & Pachter, M. (2015). Cooperative aircraft defense from an attacking missile using proportional navigation. In *AIAA guidance, navigation, and control conference*.

Ge, S. S., Ma, H., & Lum, K. Y. (2008). Detectability in games of pursuit evasion with antagonizing players. In *IEEE conference on decision and control* (pp. 1404–1409).

Huang, H., Ding, J., Zhang, W., & Tomlin, C. J. (2015). Automation-assisted capture-the-flag: A differential game approach. *IEEE Transactions on Control Systems Technology*, *23*(3), 1014–1028.

Isaacs, R. (1965). *Differential games*. New York: John Wiley and Sons.

Kraska, B., & Rzymowski, W. (2011). Guarding a line segment. *Journal of Convex Analysis*, *18*(3), 645–671.

Li, D., & Cruz, J. B. (2011). Defending an asset: A linear quadratic game approach. *IEEE Transactions on Aerospace and Electronic Systems*, *47*(2), 1026–1044.

Oyler, D. W., Kabamba, P. T., & Girard, A. R. (2016). Pursuit-evasion games in the presence of obstacles. *Automatica*, *65*(C), 1–11.

Pachter, M., Garcia, E., & Casbeer, D. W. (2014). Active target defense differential game. In *Fifty-second annual Allerton conference* (pp. 46–53).

Perelman, A., Shima, T., & Rusnak, I. (2011). Cooperative differential games strategies for active aircraft protection from a homing missile. *Journal of Guidance Control and Dynamics*, *34*(3), 761–773.

Prokopov, O., & Shima, T. (2013). Linear quadratic optimal cooperative strategies for active aircraft protection. *Journal of Guidance, Control and Dynamics*, *36*(3), 753–764.

Ramana, M. V., & Kothari, M. (2015). A cooperative pursuit-evasion game of a high speed evader. In *IEEE conference on decision and control* (pp. 2969–2974).

Ramana, M. V., & Kothari, M. (2017). Pursuit strategy to capture high-speed evaders using multiple pursuers. *Journal of Guidance, Control, and Dynamics*, *40*(1), 139–149.

Rubinsky, S., & Gutman, S. (2012). Three body guaranteed pursuit and evasion. In *AIAA guidance, navigation and control conference* (pp. 1–24).

Rubinsky, S., & Gutman, S. (2014). Three-player pursuit and evasion conflict. *Journal of Guidance, Control and Dynamics*, *37*(1), 98–110.

Rusnak, I. (2005). The lady, the bandits and the body guards–a two team dynamic game. *IFAC Proceedings Volumes*, *38*(1), 441–446.

Shaferman, V., & Shima, T. (2010). Cooperative multiple-model adaptive guidance for an aircraft defending missile. *Journal of Guidance, Control and Dynamics*, *33*(6), 1801–1813.

Shima, T. (2011). Optimal cooperative pursuit and evasion strategies against a homing missile. *Journal of Guidance, Control and Dynamics*, *34*(2), 414–425.

Zha, W., Chen, J., Peng, Z., & Gu, D. (2016). Construction of barrier in a fishing game with point capture. *IEEE Transactions Cybernrtics*, 2168–2267.

Li Liang received the B.E and M.S. degrees in control science and engineering from the Inner Mongolia University, Hohhot, China, in 2003 and 2006, respectively.

She is currently a Ph.D. student with the School of Automation, Beijing Institute of Technology. Her research interests include differential games, multi-agent systems, multi-objective optimization and decision.

Fang Deng received the B.E. and Ph.D. degrees in control science and engineering from the Beijing Institute of Technology, Beijing, China, in 2004 and 2009, respectively.

He is currently a Professor with the School of Automation, Beijing Institute of Technology. His current research interests include multi-objective optimization and decision, intelligent fire control, intelligent information processing and smart wearable devices.

Zhihong Peng received the B.S. degree from the Xiangtan Mining Institute (currently, the Hunan University of Science and Technology), Xiangtan, China, in 1995, and the Ph.D. degree from Central South University, Changsha, China, in 2000.

She held one post-doctoral appointment at the Beijing Institute of Technology, Beijing, China. Since 2012, she has been a Professor with the School of Automation, Beijing Institute of Technology. Her current research interests include intelligent information processing, and multi-agent cooperation, optimization and decision.

Xinxing Li received the M.S. in applied mathematics from Beijing Institute of Technology, Beijing, China, in 2014. Currently, he is working towards his Ph.D. degree in control science and engineering at School of Automation, Beijing Institute of Technology. His research interests include optimal control, game theory and reinforcement learning.

Wenzhong Zha received the B.S. degree in mathematics and the Ph.D. degree in control science and engineering from Beijing Institute of Technology, Beijing, China, in 2008 and 2016, respectively. He is the leader of autonomous intelligence of Key Laboratory of Cognition and Intelligence Technology, Information Science Academy of China Electronics Technology Group Corporation.

His current research interests include differential games, dynamic games, robotics, simultaneous location and mapping, multi-agent systems, and incomplete information processing.