Parametrization of Nonlinear Trajectory for Time Optimal 2D Path Planning for Unmanned Aerial Vehicles

Finding Time Optimal Path in Complex Domain

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Abstract—**This paper presents an effective path decision method to find a two-dimension path that is time optimal and obstacle free using parametrization in Complex domain. Such a path is used only for moving forward vehicles with constant velocity and non-holonomic kinematic constraints such as UAV (Unmanned Aerial Vehicles). Each start and goal point of a vehicle in a planar domain with randomly selected heading angle demonstrates this problem, which is sometimes referred to as Dubins's circle problems. Due to the nonlinearity of problems, researchers have recommended common approaches, including optimal control using cost functions and numerical solutions of nonlinear functions. These approaches do involve additional restrictions that previous research has highlighted. However; the author here suggests a simpler, more effective and more generalized method, based on problems identified in complex domain using parameterization.**

Keywords-parametrization; complex domain; time optimal; unmanned aerial vehicle

I. INTRODUCTION

Path-planning for the forward motion of non-holonomic vehicles is somewhat different from holonomic systems such as holonomic manipulators and omni-directional mobile robots; in other words, simple control and decision schemes cannot be directly applied with success to vehicles with non-holonomic constraints. For example, two-wheel differential mobile robots or UAVs work this way, because constraints such as minimum turning radius or curvature induced from constant velocity cannot be integrated to position level. In addition, its movement is only forward in the cases of UAVs. This geometric problem is so called "*piano-mover problem*" in a configuration space such as a two-dimensional space. In this paper, we confine and solve this path-planning problem with simple kinematics of UAVs using parametrization in Complex domain.

II. BACKGROUIND

Path-planning problems of non-holonomic vehicles could be categorized into three major groups according to approaches taken to solve them. The first one is calculus-based; the second, graph-based, and the last, uses numerical methods. The calculus-based path-planning method requires severe computational loads [1]-[3]. The path-planning based on graphical approaches using piece-wise continuity is naturally sub-optimal due to the discreteness of the UAV workspace [4], [5]. An optimal path that is numerically-solved using the cost function [6] is also computationally overloaded, for even a very simple case composed of only two points must find the shortest possible path between them. In addition if there are more points in the trajectory of a vehicle, computational time resulting from the numerical method will be increased exponentially. The first representative work was done by Dubins [7] and Reeds [8]. Dubins synthesized the shortest path mathematically as also shown in Fig. 1. This analysis categorizes patterns that satisfy the shortest path in a certain condition; so called, *CLC (Circle-Line-Circle), CCC (Circle-Circle-Circle)* in piecewise continuous domain. Reeds and Shepp extended this result for a vehicle that could reverse its directions.

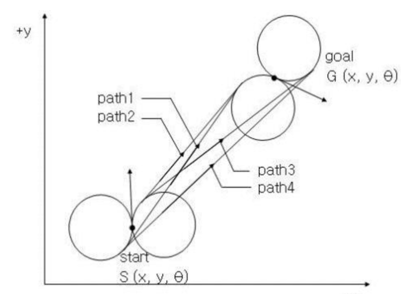


Figure 1. Dubins' circles, their angles, and four possible paths in a situation

As above Fig. 1, there are four possible paths from the starting point to goal point; namely *path 1, 2, 3, and 4*. All of them start from point *S(x)*; then two of them turn in a right direction and take their own straight lines, *path 2, 3*; in contrast, the remaining two turn left to create a path that is almost a circle and then go straight to the tangential circles of the goal point, *path 1, 4*. Instead of using non-linear equations to calculate the length of the each path. Instead, we use the parametrization in complex domain to linearize the equations turned into algebraic equations and solutions.

Reference [9] was first solved by Dubins in the no-wind case using geometric arguments. Reference [10] by Boissonnat et al, later reproduced the Dubins and Reeds-

Shepp results using optimal control methods. These results have also been used for higher-level path- planning through multiple points [11], [12] and to travel around obstacles [13], [14]. Reference [15] solved the sub-optimal problem using the same cost function optimization. Reference [16] used an algorithm called a cell-mapping technique and it is also a kind of cost-solving function. Reference [17] also used the Hamiltonian method to find the minimum cost function and reference [18] used the stochastic algorithm to find the optimal path in multiple via points. Reference [19] used a geometry-based algorithm in the presence of wind and reference [20] included dynamic constraints in the same situation but also used cost functions to minimize and find the optimal path.

III. Problem Statement

UAV missions are usually executed for tens of hours; therefore, it is desperate to find its optimal path. In this case, it induces a time-optimal and fuel-optimal path. The fuel-optimal path can be interpreted differently; it causes the vehicle to remain on its mission and obtain additional data efficiently. It is not easy to find the optimal path of a non-holonomic constrained vehicle, because the problem is characterized by nonlinear system equations and certain kinematical constraints (e.g., minimum turning radius) as mentioned previously.

A. *Kinematic Constrains*

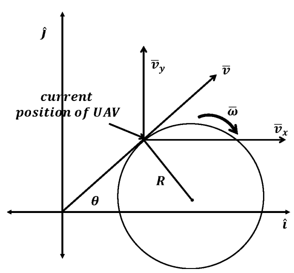


Figure 2. Kinematics of a non-holonomic constrained vehicle

The kinematic constrains of forward-moving vehicles is shown in Fig. 1. R is the minimum turning radius or curvature, $\bar{v}$ is tangential and constant velocity, (x, y) are current coordinates of position, $\bar{\omega}$ is the angular velocity, θ is the current heading angle.

B. *Associated Circles*

Any point with a heading angle in Cartesian coordinates will have two geometric constrained unit circles in Fig. 3.

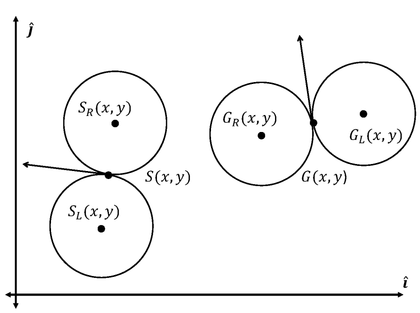


Figure 3. Start/goal points and their associated circles

C. *Number of Cases & Determination of the Optimal Path*

One smooth closed curve (circle) can have infinite numbers of tangential lines along the curve. When there are two circles apart as shown in Fig. 4, we have four (4) tangent lines. Two of the tangent lines are outer lines and the other two are inner lines. When there are four circles in Fig. 5 (two circles at a start point and the other two at a goal point), we can draw 16 tangent lines. The functional relationship between the total number of circles and the total number of tangent lines can be summarized as below.

When n number of circles exists, n^2 number of tangent lines exists where $n = 2, 4, 6, \ldots$ When n number of circles exists, $n^2 - 1$ number of tangent lines exists where $n = 1, 3, 5, \ldots$

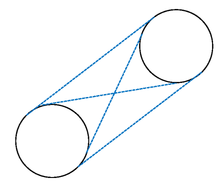


Figure 4. Two smooth closed curves (circles) and their 4 common tangent lines: two are outer and the other two are inner

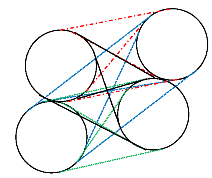


Figure 5. Four smooth closed curves (circles) and their 16 common tangent line: eight outer and eight inner lines

Now we know when two points and associated four circles in 2D plane exist, there will be total 16 common tangential lines and one of them is the shortest path. Earlier mentioned paths are normally *CLC (Circle-Line-Circle)* and arc lengths are nonlinear containing $\sin(\theta)$ and $\cos(\theta)$ terms.

IV. Geometry

This section describes some preliminary geometry analysis in *CLC* cases. In terms of common tangential lines between two circles as shown in Fig. 4 and 5, we can categorized all the tangential lines into two types: (1) inner and (2) outer tangential lines. In other words, (1) inner tangential lines could be considered as pulley; (2) outer ones as belt. When there exists two points: starting and goal with heading and goal angles we can describe some geometrical relationships between two points with associated circles, as shown below.

Δ_P: *gradient between start and goal points in pulley*
L_P: *distance of tangential straight line in pulley*
Δ_B: *gradient between start and goal points in belt*
L_B: *distance of tangential straight line in belt*
$arc_{b,p}^{S}$: arc length associated with start point
$arc_{b,p}^{G}$: arc length associated with goal *point*
α: angle difference between start heading angle and $\Delta_{b,P}$
β: angle difference between goal heading angle and $\Delta_{b,P}$

$$\Delta_P = \tan^{-1}\left(\frac{G(y)-S(y)}{G(x)-S(x)}\right) \quad (1)$$

$$L_P = \sqrt{\left(S(x)-G(x)\right)^2 + \left(S(y)-G(y)\right)^2} \quad (2)$$

$$\Delta_B = \cos^{-1}\left(\frac{2R}{L_P}\right) \tag{3}$$

$$L_B = \sqrt{\left(\frac{L_P}{2}\right)^2 - R^2} \tag{4}$$

$$arc_b = R \cdot \cos^{-1}(\frac{2R}{L}) \tag{5}$$

$$arc_p = R \cdot \theta = R \cdot 2\cos^{-1}(\frac{R_1 - R_2}{L}) \tag{6}$$

$$\alpha = \left|\theta_S - \Delta_{b,P}\right| \tag{7}$$

$$\beta = \left|\theta_G - \Delta_{b,P}\right| \tag{8}$$

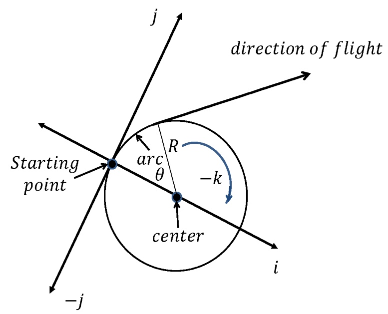


Figure 6. Geometry related to arc length, θ in a circle

To find the shortest path in conventional manner, we consider four possible paths and find the shortest one. In equation form, the shortest path is described as below.

$$shortest\ path = \min(arc_{b,p} + L_{b,p} + arc_{b,p})$$

V. Parameterization in Complex Domain

This section explains how nonlinear equations describing *CLC (Circle-Line-Circle)* paths are converted into Complex functions in its domain. When we have a unit circle with radius *R,* centered at (a, b) in Cartesian coordinates, it is described $(x-a)^2 + (y-b)^2 = R^2$

In complex domain, it could be mapped into a complex function $f(z)$ where,

$$|z - z_0| = R, \text{ where } z_0 = a + bi \tag{9}$$

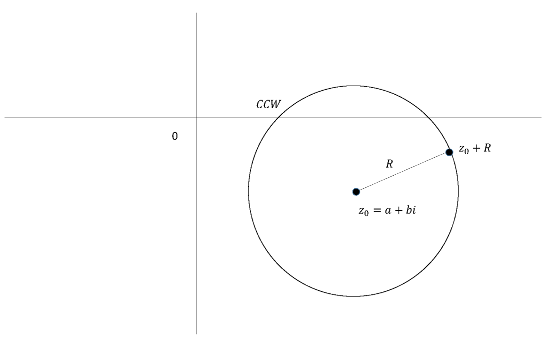


Figure 7. A circle in complex domain with CCW rotation

Any point on the unit circle centered at the origin can be written in the form $e^{i\theta}, 0 \le \theta \le 2\pi$; therefore, an admissible parametrization for this smooth closed curve is constructed by interpreting θ as the parameter: $z_0(\theta) = e^{i\theta}, 0 \le \theta \le 2\pi$. Time t Parametrized function of $f(z)$ is now

$$z(t) = z_0 + Re^{it}, 0 \le t \le 2\pi \tag{10}$$

Setting $f(z) = (z - z_0)^n$,

$$f(z(t)) = (z_0 + Re^{it} - z_0)^n = R^n e^{int} \tag{11}$$

and

$$z'(t) = iRe^{it} \tag{12}$$

where n is the number of traversing in the counterclockwise direction.

For example, if there is a circle centered at (2,-5) with radius of 3, the equation in Cartesian coordinates is $(x-2)^2 + (y+5)^2 = 9$. The mapped equation in Complex domain would be $f(z)$: $|z - (2 - 5i)| = 3$

$$z(t) = (2 - 5i) + 3e^{it}, 0 \le t \le 2\pi$$

In the case of straight lines, given any two distinct points z_1 and z_2, every point on the line segment joining z_1 and z_2 is of the form of $z_1 + t(z_2 - z_1)$, where $0 \le t \le 1$.

The length of arc or circle is calculated using the contour integral over time using the theory.

$$\oint f(z)dz = \int_a^b f(z(t))z'(t)dt,\ a \le t \le b \tag{13}$$

$$\int_0^{\alpha} \left[(S(x) + iS(y)) + Re^{it} - (S(x) + iS(y))\right) = arc_{b,p}^S$$

$$arc_{b,p}^G = \int_0^{\beta} \left[(G(x) + iG(y)) + Re^{it} - (G(x) + iG(y))\right)$$

$$L_p = \int_0^1 [S(x) + iS(y) + t(G(x) + iG(y) - (S(x) + iS(y))]dt$$

where $0 \le t \le 1$

Another alternative is directly using the magnitude of distance of two points: start and goal point

$$L_P = \sqrt{(S(x) - G(x))^2 + (S(y) - G(y))^2}$$

$$L_p = \int_0^1 \left[\sqrt{(S(x) - G(x))^2 + (S(y) - G(y))^2}\right] t \cdot dt$$

$$L_B = \int_0^1 \left[\sqrt{\left(\frac{L_P}{2}\right)^2 - R^2}\right] t \cdot dt$$

VI. Optimal Path Decision

A. Step 1: List All Possible Paths

When two points exist, we can draw four associated circles and sixteen tangential lines between two sets of circles, as shown in Fig. 5. Another definition of smooth curves could be made if it is the range of some continuous complex-valued function

B. Step 2: Select four (4) Candidates Using Constraints of Tangential Lines

Based on the geometric restrictions shown in section *D* of the *Problem Statement*, each a starting and a goal point is to have only one $\hat{k}$ and one $-\hat{k}$ directional circle. This geometrical restriction also will now allow only total four tangent lines between two points with two circles attached in each circle as shown in Fig. 8.

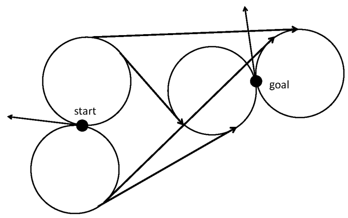


Figure 8. Four tangent lines between two points with two circles attached in each point determined by heading angles

C. Step 3: List 4 Possible Paths using Geometry

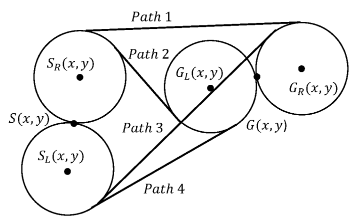
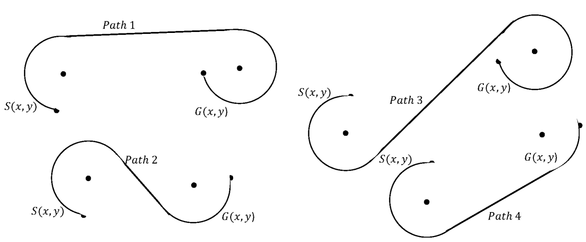


Figure 9. Associated circles in start and goal points and 4 possible tangential lines for paths.



(a) (b)

Figure 10. 4 (four) paths: each has arcs and a straight line, (a) for path 1 and 2 and (b) for path 3 and 4.

In Fig. 10, paths 1 and 4 are pulley paths and paths 2 and 3 are belt paths. We can also define a gradient and a distance between starting and goal points. Each path has one arc associated with a start point, a straight line, and another arc associated with a goal point.

D. Step 4: Parametrization and Find the Shortest Path

As explained in detail from eqn. 10-13 and results in *Parametrization in Complex Domain* in Sec. 5, each case can be linearized using parametrization in complex domain which enables us to calculate complicated nonlinear and geometry-based equations to simple algebraic equations.

VII. Conclusion

This paper presents an effective path decision method to find a two-dimension path that is time optimal and obstacle free using parametrization in Complex domain. Such a path is used only for moving forward vehicles with constant velocity and non-holonomic kinematic constraints such as UAV. The biggest contribution in this paper is transformation of nonlinear trajectory equations to simple integral equations in complex domain.

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