A Qualitative Study of Proportional Navigation

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Abstract

This paper represents a study of the trajectories of an ideal missile homing on a target according to the proportional navigation law. A qualitative study is performed and conditions are determined which enable one to demonstrate that: 1) the missile always reaches the target *regardless of the initial conditions* at launch; 2) the rotational rate of the line of sight is decreasing at the pursuit end.

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Introduction

Proportional navigation is a commonly utilized method of navigation to control the trajectory of interceptor missiles. Recently there has been considerable interest in the subject. In [1], the basic principles of proportional navigation were reviewed, and [2] and [3] dealt with the topic in terms of optimality.

These works, as well as previous ones in the same field [4], were based on strong assumptions, in order to reduce the system of nonlinear differential equations which represent the pursuit of a target into a unique linear time-varying differential equation which is solvable in closed form.

The purpose of this paper is to show that qualitative methods can be applied to obtain the *general* solution when planar pursuit is considered and the assumption is made that the target is nonmaneuvering.

A geometric description of the general solution will be presented and a general condition to assure the boundedness of the rotational rate of the line of sight will be demonstrated.

System Equations

Consider a target T and missile M as points in a plane moving with constant velocities V_T and V_M, respectively, as shown in Fig. 1. The system can be described in a relative system of coordinates with its center at T and axis Tx along the straight-line trajectory of the target. The entire system is a combination of the dynamics of the missile, the geometry of the pursuit, and the laws of kinematics, as indicated in Fig. 2.

From Fig. 1,

$$\alpha = \theta - \gamma. \tag{1}$$

Differentiating with respect to time,

$$\dot{\alpha} = \dot{\theta} - \dot{\gamma}. \tag{2}$$

$\dot{\gamma}$ is related to the missile normal acceleration by

$$\dot{\gamma} = a_M / V_M \tag{3}$$

where, in proportional navigation, a_M is given by

$$a_M = N V_M \dot{\theta} \tag{4}$$

where N is the navigation constant. For an ideal missile this represents all its dynamics.

Replacing (3) and (4) into (2),

$$\dot{\alpha} = (1 - N)\dot{\theta}. \tag{5}$$

If (5) is now integrated with initial conditions α_0, θ_0,

$$\alpha = (1 - N)\theta + \alpha_0 + (N - 1)\theta_0. \tag{6}$$

Defining

$$k = N - 1 \tag{7}$$

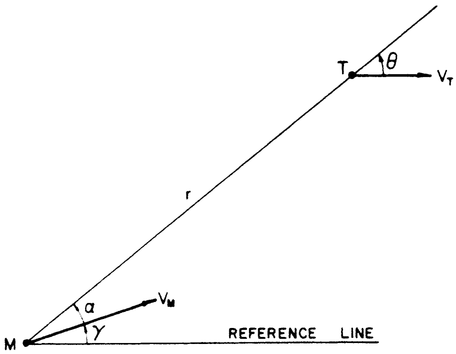
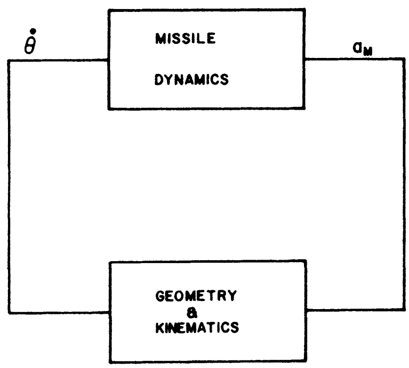


Fig. 1. Pursuit geometry.

Fig. 2. Block diagram.



and

$$\varphi_0 = \alpha_0 + (N-1)\theta_0, \tag{8}$$

(6) takes the form

$$\alpha = \varphi_0 - k\theta. \tag{9}$$

The components of the relative velocity V_C, from missile to target, are in polar coordinates,

$$V_r = \dot{r} \tag{10}$$

$$V_\theta = r\dot{\theta}. \tag{11}$$

With signs defined as usual (radial component, positive in the outward direction; transverse component, positive for θ increasing counterclockwise), the radial and transverse components of the velocity are, in terms of the projections of the missile and target velocities,

$$V_r = V_T \cos\theta - V_M \cos\alpha \tag{12}$$

$$V_\theta = V_M \sin\alpha - V_T \sin\theta. \tag{13}$$

Replacing α from (9),

$$V_r = V_r(\theta) \tag{14}$$

$$V_\theta = V_\theta(\theta) \tag{15}$$

where

$$V_r(\theta) = V_T \cos\theta - V_M \cos(\varphi_0 - k\theta) \tag{16}$$

$$V_\theta(\theta) = V_M \sin(\varphi_0 - k\theta) - V_T \sin\theta. \tag{17}$$

The second-order system of nonlinear differential equations

$$(I)\begin{cases} \dot{r} = V_r(\theta) & (18) \\ r\dot{\theta} = V_\theta(\theta) & (19) \end{cases}$$

completely defines the pursuit, and its solutions will provide the trajectories of the missile in the relative system of coordinates previously defined.

Analytic Solution of the System Equations

In this section analytic methods are applied to solve system (I), a case of a nonlinear system of differential equations.

The general solution can be obtained in the following form [4]. Dividing (18) by (19), time t can be eliminated, obtaining

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{V_r(\theta)}{V_\theta(\theta)}. \tag{20}$$

Integrating,

$$r = r_0 \exp\left[\int_{\theta_0}^{\theta} \frac{V_r}{V_\theta} d\theta\right]. \tag{21}$$

(21) provides r as a function of θ. Unfortunately, the integral appearing here with V_r and V_θ replaced from (17) and (18) is not solvable in the general case. It can be performed only in the two particular cases: $N = 1$ and $N = 2$. In these cases, replacing $r = r(\theta)$ in (19),

$$\dot{\theta} = \frac{V_\theta(\theta)}{r(\theta)} \tag{22}$$

whence,

$$\int_{\theta_0}^{\theta} \frac{r}{V_\theta} d\theta = t - t_0. \tag{23}$$

The integral appearing here can be performed only for $N = 1$. θ can then be obtained as a function of t, which, replaced in (21), gives r as a function of time.

In conclusion, system (I) has no general solution in closed form. The complete analytic solution of system (I) can only be obtained in the particular case $N = 1$, and is partially solvable in the case $N = 2$ (r as a function of θ, but not as a function of time).

In the literature concerning proportional navigation this is the final point of rigorous analysis. From here on, only the final phase of the pursuit is considered and approximations are made to obtain a linearized time-varying system, solvable in analytic terms. A general picture of the missile's trajectory is not available. The purpose of the following sections is to provide this general picture.

Qualitative Study

Since analytic methods fail to provide the solution in the general case, a qualitative study will be made to obtain a geometric description of the general solution.

Let us define normalized velocities

$$V_r^T = \frac{V_r}{V_T} = \cos\theta - \nu\cos(\varphi_0 - k\theta) \qquad (24)$$

$$V_\theta^T = \frac{V_\theta}{V_T} = \nu\sin(\varphi_0 - k\theta) - \sin\theta \qquad (25)$$

where

$$\nu = V_M / V_T \qquad (26)$$

To simplify the notation we shall continue to write V_r and V_θ instead of V_r^T and V_θ^T.

Lemma 1: If $\nu > 1, k\nu > 1$, then the roots of

$$\nu\cos(\varphi_0 - k\theta) - \cos\theta = 0 \qquad (27)$$

$$\nu\sin(\varphi_0 - k\theta) - \sin\theta = 0 \qquad (28)$$

alternate along the θ axis.

The proof can be found in Appendix I.

Let us now determine

$$\frac{dV_\theta}{d\theta}(\theta) = -k\nu\cos(\varphi_0 - k\theta) - \cos\theta. \qquad (29)$$

Lemma 2: If $\nu > 1, k\nu > 1$, then

$$V_r(\theta_\theta)\cdot\frac{dV_\theta}{d\theta}(\theta_\theta) > 0 \qquad (30)$$

where θ_θ is a root of $V_\theta(\theta)$.

The proof is given in Appendix II.

Lemma 1 implies that the successive roots of V_θ encounter alternate signs for V_r. From Lemma 2, it results, therefore, that

$$V_r(\theta_{\theta_i}) < 0 \Rightarrow \frac{dV_\theta}{d\theta}(\theta_{\theta_i}) < 0 \Rightarrow V_\theta(\theta)$$

is decreasing at θ_{θ_i}

$$V_r(\theta_{\theta_{i+1}}) > 0 \Rightarrow \frac{dV_\theta}{d\theta}(\theta_{\theta_{i+1}}) > 0 \Rightarrow V_\theta(\theta)$$

is increasing at $\theta_{\theta_{i+1}}$.

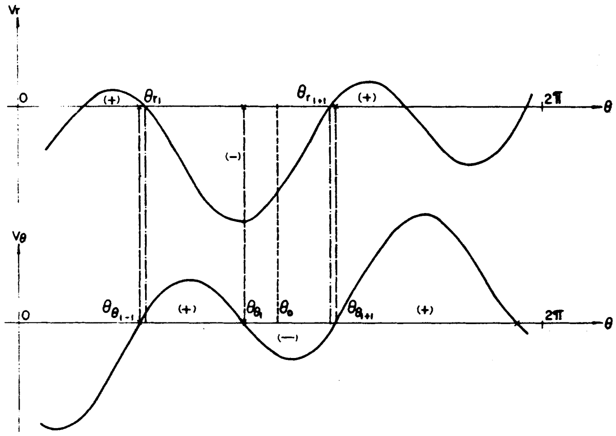


Fig. 3. V_r and V_θ versus θ.

Consequently, it appears that $V_r(\theta)$ and $V_\theta(\theta)$ have, in all generality, the characteristics depicted in Fig. 3.

In general, the study of the trajectories of a system described by

$$r\dot{\theta} = V_\theta(\theta) \qquad (31)$$

$$\dot{r} = V_r(\theta) \qquad (32)$$

can be made by its replacement (for a nonvanishing V_r) with the equation

$$r\frac{d\theta}{dr} = \frac{V_\theta(\theta)}{V_r(\theta)}. \qquad (33)$$

Any constant θ_c which satisfies

$$V_\theta(\theta_c) = 0 \qquad (34)$$

and

$$V_r(\theta_c) \neq 0 \qquad (35)$$

is a solution of (33). These straight-line trajectories adhering to the origin for $t \to \infty$ or for $t \to -\infty$ single out polar directions. These "critical" directions play a fundamental role in the geometric study of differential equations. Furthermore, the behavior of trajectories in each sector containing a single straight-line trajectory determines the overall behavior of integral curves. The critical directions and the sectors that contain them will now be defined in a precise form [5] for more general systems described by

$$r\frac{d\theta}{dr} = \frac{V_\theta(r,\theta)}{V_r(r,\theta)}. \qquad (36)$$

Definition 1: Consider a ray $r > 0$, $\theta = \theta_c$. Its direction, determined by θ_c, is "critical" if

$$\lim_{\substack{r\to 0\\ \theta\to\theta_c}} \frac{V_\theta(r,\theta)}{V_r(r,\theta)} = 0.$$

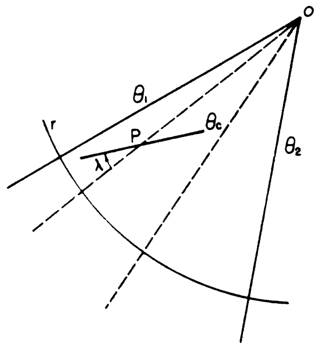


Fig. 4. Normal domain.

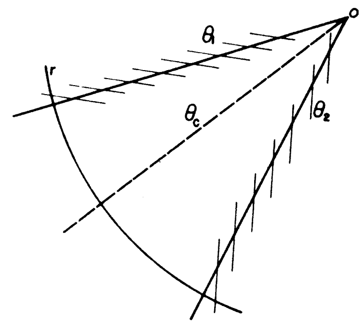


Fig. 5. Normal domain of type I.

Definition 2: A sector σ $(\theta_1, \theta_2, r_0)$ is called a normal domain whenever

1) it contains only one critical direction θ_c, $\theta_1 < \theta_c < \theta_2$
2) the direction of the field element $[\lambda = \arctan (V_\theta/V_r)]$ at every point P (see Fig. 4) either on the boundary or in the interior of the sector σ is not orthogonal to the direction of the radius vector $0P$; i.e., $r(t)$ is strictly monotonic in σ.

Definition 3: A normal domain σ is of type I if all the trajectories cutting across the boundary segments enter σ (with increasing t if $V_r < 0$ in σ, and with decreasing t if $V_r > 0$ in σ). (See Fig. 5.)

We shall now recall an important result obtained for systems described by (36).

Theorem 1: If as t increases (decreases) a trajectory enters a normal domain of type I, then

1) it remains in σ, and
2) it tends to the origin in such a way that $\theta \to \theta_c$, where θ_c is the critical direction of σ.

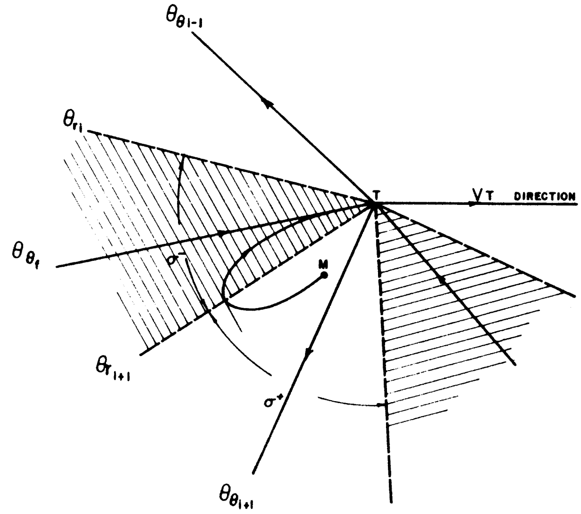
All the necessary elements are now available to prove the following theorem.

Theorem 2: An ideal missile, pursuing a target following a proportional navigation law, with $NV_M > V_M + V_T$ $(kv > 1)$ and $V_M > V_T$ $(v > 1)$, will reach the target for all but a finite number of possible initial conditions at launch. Moreover, the missile will arrive at the target along a straight line whose direction $\theta = \theta_{\theta_f}$ is determined by the initial conditions.

Proof: Lemmas 1 and 2, with $kv > 1$ and $v > 1$, enable us to divide the polar plane r, θ as depicted in Fig. 6. The full lines represent the roots of V_θ, positive in the outward direction, negative in the inward one. The shadowed surfaces correspond to zones of negative V_r. The missile is located at r_0, θ_0.

The directions $\theta = \theta_{\theta_i}$, corresponding to the roots of V_θ, are critical directions (see Definition 1). The sections σ: $\theta_{r_i} < \theta_1 \leqslant \theta \leqslant \theta_2 < \theta_{r_{i+1}}$ with critical directions θ_{θ_i} are normal domains.

Fig. 6. Relative trajectory.



Given the fact that, as seen from Fig. 3,

$$V_\theta/V_r < 0 \qquad \text{for } \theta_{r_i} < \theta < \theta_{\theta_i}$$

and

$$V_\theta/V_r > 0 \qquad \text{for } \theta_{\theta_i} < \theta < \theta_{r_{i+1}},$$

the result is that the normal domains σ are of type I.

These normal domains can be subdivided into two subclasses, denoted σ^+ and σ^-, respectively. σ^+ is a normal domain of type I with $V_r > 0$. A trajectory starting at σ^+ will leave σ^+ for increasing t. σ^- is a normal domain of type I with $V_r < 0$. A trajectory starting at σ^- will remain in σ^- for increasing t.

Consequently, the result is that in the general case, given initial conditions $r_0, \theta_0 \in \sigma^+$, the trajectory will leave σ^+ and, after crossing normally to $\theta = \theta_r$, will enter σ^-.

The second part of Theorem 1 assures that the trajectory will tend to the origin along the critical direction $\theta = \theta_{\theta_f}$.

Remarks:

1) For each set of initial conditions there is a different but equivalent division of the polar plane. The value $\theta =$

$\theta_{\theta_i}(\alpha_0, \theta_0)$ such that

a) $V_r(\theta_{\theta_i}) < 0$
b) θ_0 and θ_{θ_i} belong to the same sector, $\theta_{\theta_{i-1}}, \theta_{\theta_{i+1}}$ $(V_r(\theta_{\theta_{i-1}}) > 0, V_r(\theta_{\theta_{i+1}}) > 0)$

defines the direction θ_{θ_f} (see Fig. 3).

2) The sets of initial conditions that satisfy

$$V_\theta(\theta_0, \alpha_0) = 0$$

$$V_r(\theta_0, \alpha_0) > 0$$

result in straight-line trajectories such that $t \to \infty$ implies $r \to \infty$; i.e., the missile does not reach the target.

3) This theorem demonstrates rigorously the validity of the basic assumption made in previous works [1],[6] on proportional navigation. At the pursuit end the missile approaches a straight-line trajectory, the so-called collision course.

4) Theorem 2 shows that theoretically the missile can reach the target even if initially it moves away from it. A real missile possesses a tracking system that strongly limits the lead angle α (see Fig. 1); this prevents the practical utilization, in the present time, of this significant but theoretical result.

Boundedness of Line of Sight Rate of Rotation

In the classical theory of proportional navigation, the behavior of $\dot{\theta}$, the rate of rotation of the line of sight (LOS), plays a fundamental role. In this section a condition for boundedness for $\dot{\theta}$ will be derived.

The following theorem will first be demonstrated.

Theorem 3: If

$$\frac{dV_\theta}{d\theta}(\theta) < V_r(\theta) \tag{37}$$

along the missile's trajectory, then $\dot{\theta}$ is a decreasing function of time.

Proof: Condition (37) can be rewritten as

$$\frac{dV_\theta}{dt}\frac{1}{\dot{\theta}} < V_r.$$

Multiplying by $1/r > 0$,

$$\frac{dV_\theta}{dt}\frac{1}{r\dot{\theta}} < \frac{\dot{r}}{r}.$$

Integrating,

$$\int_{t_0}^{t} \frac{\dot{V}_\theta}{V_\theta}\, dt < \int_{t_0}^{t} \frac{\dot{r}}{r}\, dt.$$

Hence,

$$\frac{V_\theta}{V_{\theta_0}} < \frac{r}{r_0}.$$

Replacing V_θ by $r\dot{\theta}$,

$$\frac{\dot{\theta}}{\dot{\theta}_0} < 1. \qquad \text{Q.E.D.}$$

Of particular interest is the behavior of $\dot{\theta}$ when the missile approaches the target. The following result can be obtained as a corollary.

Corollary 1: If $V_M > V_T$ and

$$\left(\frac{N-2}{2}\right) V_M > V_T \tag{38}$$

the rate of rotation of the line of sight decreases in the final phase of the pursuit.

The proof follows directly from the fact that (38) is a sufficient condition for the existence of (37) along $\theta = \theta_{\theta_f}$. This proof is given in Appendix III.

If $N > 4$, (38) is always fulfilled ($V_M > V_T$). If $N < 4$, condition (38) serves to determine the minimum value for the gain of the control system:

$$NV_M > 2(V_M + V_T). \tag{39}$$

Remarks:

1) Condition (37) is strictly positive when condition (38) is fulfilled. Consequently, it appears that, due to the continuity of V_r and $(dV_\theta/d\theta)$, (37) is verified not only for $\theta = \theta_{\theta_f}$, but for all θ in the neighborhood of θ_{θ_f}. It follows that during the final phase of the pursuit $\dot{\theta}$ is a decreasing function of time.

2) In previous works [1], [6] the condition for boundedness for $\dot{\theta}$ was found to be

$$NV_M > 2\left(V_M - V_T\frac{\cos\theta}{\cos\alpha}\right)$$

where $\cos\theta$ and $\cos\alpha$ were assumed to be constant and equal to their final values. Condition (38) is significant since no assumptions were needed to obtain it, and it does not depend on the dynamics of the pursuit. Furthermore, when $N > 4$ ($V_M > V_T$), boundedness for $\dot{\theta}$ is assured independently of the remaining parameters of the pursuit.

Summary and Conclusions

A qualitative study of the trajectories of a missile pursuing a nonmaneuvering target according to the proportional navigation law was performed. This study did not employ the usual assumptions made in the classical theory of proportional navigation.

It was rigorously demonstrated that, first, the missile reaches the target for all but a finite number of initial conditions, and second, that it arrives at the target along a straight line whose direction is determined by the initial conditions.

Conditions were determined under which the rotational rate of the line of sight is decreasing at the pursuit end.

The approach employed here is expected to be powerful enough to be able to provide in the near future a new insight into more complicated systems, in particular, for the case of a maneuvering target.

Appendix I

The proof of Lemma 1 is shown, translating the problem to the complex plane.

Define

$$U = ve^{j(\varphi_0 - k\theta)} - e^{j\theta}$$

U has real and imaginary parts given by

$$\mathrm{Re}(U) = v \cos(\varphi_0 - k\theta) - \cos\theta \tag{39a}$$

$$\mathrm{Im}(U) = v \sin(\varphi_0 - k\theta) - \sin\theta. \tag{39b}$$

In the complex plane, U is the vector AB as indicated in Fig. 7.

When θ changes from 0 to 2π, A travels along the unity circle in the counterclockwise direction and B travels along circle v in the clockwise direction, k times as fast. The vector AB continuously changes its position, and its angular velocity is given by $(d\psi/d\theta)$, where

$$\psi = \arctan \frac{v \sin(\varphi_0 - k\theta) - \sin\theta}{v \cos(\varphi_0 - k\theta) - \cos\theta}.$$

Differentiating,

$$\frac{d\psi}{d\theta} = -\frac{(kv^2 - 1) - v(k-1)\cos[(k+1)\theta - \varphi_0]}{v^2 + 1 - 2v\cos[(k+1)\theta - \varphi_0]}.$$

$(d\psi/d\theta)$ is strictly negative. This follows directly from the inequalities

$$(kv^2 - 1) - v(k-1)\cos[(k+1)\theta - \varphi_0] > \begin{cases} (kv-1)(v+1) > 0 & \text{for } kv > 1 \\ (kv+1)(v-1) > 0 & \text{for } v > 1. \end{cases}$$

This result proves the fact that the vector AB rotates monotonously in one sense only, being successively real, complex, imaginary, negative real, and so on, which implies that the real and imaginary parts of U, given by (39a) and (39b), are successively zero. The roots of (27) and (28) are intercalated.

Appendix II

Proof follows that

$$V_r(\theta_\theta) \cdot \frac{dV_\theta}{d\theta}(\theta_\theta) > 0$$

where θ_θ is such that

$$v \sin(\varphi_0 - k\theta_\theta) = \sin\theta_\theta.$$

Evaluating the product

$$V_r \cdot \frac{dV_\theta}{d\theta} = kv^2 \cos^2(\varphi_0 - k\theta_\theta) - \cos^2\theta_\theta + v(1-k)\cos(\varphi_0 - k\theta_\theta)\cos\theta_\theta \tag{40}$$

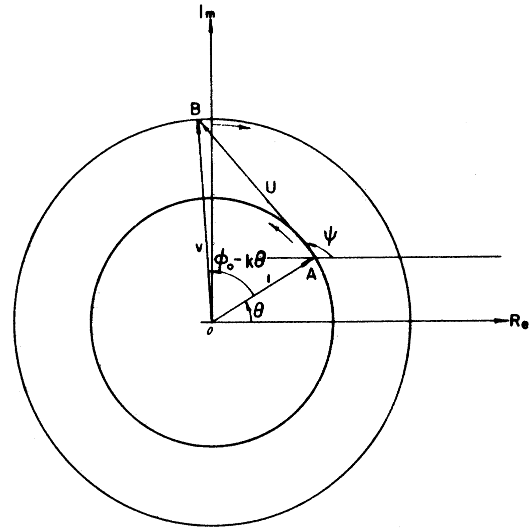


Fig. 7. Complex plane.

with

$$v^2\cos^2(\varphi_0 - k\theta_\theta) = v^2 - v^2\sin^2(\varphi_0 - k\theta_\theta) = v^2 - v\sin\theta_\theta \sin(\varphi_0 - k\theta_\theta)$$

and

$$\cos^2\theta_\theta = 1 - \sin^2\theta_\theta = 1 - v\sin\theta_\theta\sin(\varphi_0 - k\theta_\theta)$$

(40) becomes

$$V_r \cdot \frac{dV_\theta}{d\theta} = (kv^2 - 1) - v(k-1)[\sin\theta_\theta\sin(\varphi_0 - k\theta_\theta) + \cos\theta_\theta\cos(\varphi_0 - k\theta_\theta)]. \tag{41}$$

Therefore,

$$V_r \cdot \frac{dV_\theta}{d\theta} = (kv^2 - 1) - v(k-1)\cos[(k+1)\theta_\theta - \varphi_0]. \tag{42}$$

From (42) it is seen that $V_r \cdot (dV_\theta/d\theta)$ is strictly positive for $kv > 1$ and $v > 1$ (see Appendix I).

Appendix III

Proof follows that

$$\frac{dV_\theta}{d\theta}(\theta_{\theta f}) < V_r(\theta_{\theta f})$$

if

$$\left(\frac{N-2}{2}\right)V_M > V_T$$

and

$$V_M > V_T.$$

Along $\theta = \theta_{\theta_f}$, $V_r < 0$. Multiplying both sides by V_r,

$$\frac{dV_\theta}{d\theta}V_r > V_r^2.$$

Substituting from (24) and (29),

$$kv^2 \cos^2(\varphi_0 - k\theta) - \cos^2\theta + (1-k)v \cos(\varphi_0 - k\theta)\cos\theta$$
$$> \cos^2\theta + v^2 \cos^2(\varphi_0 - k\theta) - 2v\cos\theta\cos(\varphi_0 - k\theta).$$

Rearranging this inequality we finally obtain

$$(k-1)v^2 \cos^2(\varphi_0 - k\theta) - 2\cos^2\theta$$
$$- v(k-3)\cos(\varphi_0 - k\theta)\cos\theta > 0. \quad (43)$$

For $\theta = \theta_{\theta_f}$,

$$v\sin(\varphi_0 - k\theta) = \sin\theta.$$

Consequently,

$$v^2\cos^2(\varphi_0 - k\theta) = v^2 - \sin^2\theta$$
$$= v^2 - v\sin\theta\sin(\varphi_0 - k\theta) \quad (44)$$
$$\cos^2\theta = 1 - v^2\sin^2(\varphi_0 - k\theta)$$
$$= 1 - v\sin\theta\sin(\varphi_0 - k\theta). \quad (45)$$

Replacing (44) and (45) into (43) and rearranging,

$$(k-1)v^2 - 2 - (k-3)v\cos[\varphi_0 - (k+1)\theta] > 0. \quad (46)$$

Finally, (46) is fulfilled if

$$[(k-1)v - 2](v+1) > 0 \quad \text{for } k < 3$$
$$[(k-1)v + 2](v-1) > 0 \quad \text{for } k > 3.$$

In conclusion, the general condition is

$$(k-1)v > 2 \quad (47)$$

for any k, provided $v > 1$.

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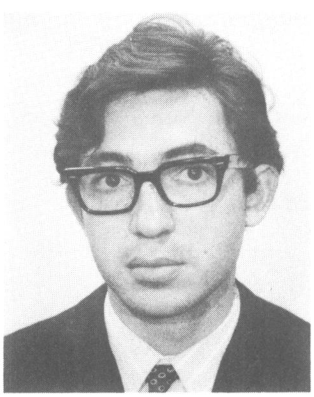

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