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On the Construction of Lyapunov Functions using the Sum of Squares Decomposition[1]

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Abstract

A relaxation of Lyapunov's direct method has been proposed recently that allows for an algorithmic construction of Lyapunov functions to prove stability of equilibria in nonlinear systems, but the search is restricted to systems with polynomial vector fields. In this paper, the above technique is extended to include systems with equality, inequality, and integral constraints. This allows certain non-polynomial nonlinearities in the vector field to be handled exactly and the constructed Lyapunov functions to contain non-polynomial terms. It also allows robustness analysis to be performed. Some examples are given to illustrate how this is done.

1 Introduction

Stability of dynamical systems plays a very important role in control system analysis and design. Unlike the case of linear systems, proving stability of equilibria of nonlinear systems is more complicated. A sufficient condition is the existence of a Lyapunov function [1]: a positive definite function V defined in some region of the state space containing the equilibrium point whose derivative along the system trajectories is negative semi-definite. This is Lyapunov's direct method, which even though addresses exactly and in a simple way the important issue of stability, it does not provide any coherent methodology for constructing such a function. Lyapunov's indirect method that investigates the local stability of the equilibria, is inconclusive when the linearized system has imaginary axis eigenvalues. Other methodologies to determine the stability properties of the equilibria of nonlinear systems (such as exhaustive simulations, Linear Parameter Varying (LPV) techniques, Integral Quadratic Constraint (IQC) formulations [2] etc) are sometimes quite conservative.

A technique developed recently in [3] for constructing Lyapunov functions is based on the following computational relaxation: instead of asking for V to be positive definite, and for its time derivative negated to be positive semidefinite, it requires them to be sums of squares. This relaxation circumvents the NP hardness of proving positivity of a polynomial and uses the computational tractability of proving the existence of a sum of squares decomposition, therefore formulating the Lyapunov function search algorithmically. In this way polynomial nonlinearities can be handled exactly.

In this paper the class of systems for which construction is possible will be extended to nonlinear systems with equality, inequality, and integral constraints. Constraints arise naturally in many physical systems: mechanical systems with equality constraints, systems with parametric uncertainty (which can be accommodated by inequality constraints), and systems containing dynamic uncertainty (which can be modelled by IQCs). Sufficient conditions for proving stability under such constraints will be developed in Section 2. In the same section, the conditions will be computationally relaxed to the existence of a sum of squares decomposition, to allow for their algorithmic verification.

As a side benefit, we will be able to construct Lyapunov functions for nonlinear systems with non-polynomial vector fields, but which can be transformed to equivalent systems with polynomial vector fields under equality and inequality constraints on the state variables; the search for a Lyapunov function will be performed indirectly in terms of these new, non-polynomial terms, and so the resulting Lyapunov functions will not necessarily be polynomial. Examples of this will be given in Section 3. The sum of squares programs corresponding to these examples are converted into semidefinite programs using SOSTOOLS [4], and are solved using SeDuMi [5], a semidefinite programming solver.

2 Constrained Dynamical Systems and Stability

Consider the nonlinear system

$$\dot{x} = f_x(x,u), \tag{1}$$

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with the following constraints:

$$a_{i_1}(x,u) \le 0, \quad \text{for } i_1 = 1,...,N_1, \tag{2}$$

$$b_{i_2}(x,u) = 0, \quad \text{for } i_2 = 1,...,N_2, \tag{3}$$

$$\int_0^T c_{i_3}(x,u)dt \le 0, \text{for } i_3 = 1,...,N_3, \text{ and } \forall T \ge 0. \tag{4}$$

Here $x \in \mathbb{R}^n$ is the state of the system, and $u \in \mathbb{R}^m$ is a collection of auxiliary variables (such as inputs, non-polynomial functions of states, uncertain parameters (for robustness analysis), etc.). We assume that the a_{i_1}'s, b_{i_2}'s, and c_{i_3}'s are polynomial functions in (x,u), and $f_x(x,u)$ is a vector of polynomial or rational functions in (x,u) with no singularity in $\mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^{m+n}$ is defined as

$$\mathcal{D} = \{(x,u) \in \mathbb{R}^{m+n} \mid a_{i_1}(x,u) \le 0, b_{i_2}(x,u) = 0, \text{ for all } i_1 \text{ and } i_2\}.$$

Without loss of generality, it is assumed that $f_x(x,u) = 0$ for $x = 0$ and $u \in \mathcal{D}_u^0$, where

$$\mathcal{D}_u^0 = \{u \in \mathbb{R}^m | (0,u) \in \mathcal{D}\}.$$

The following theorem is an extension of Lyapunov's stability theorem, and can be used to prove that the origin is a stable equilibrium of the above system. It uses a technique reminiscent of the well-known S-procedure [6] in nonlinear and robust control theory.

Theorem 1 *Suppose that for the above system there exist polynomial functions*[1] $V(x)$, $w(x,u)$, $p_{i_1}(x,u)$, $q_{i_2}(x,u)$, *and constants* $r_{i_3} \ge 0$ *such that*

- $V(x)$ *is positive definite*[2] *in a neighborhood of the origin.*
- $w(x,u) > 0$ *and* $p_{i_1}(x,u) \ge 0$ *in* $\mathcal{D}$.

Then

$$-\frac{\partial V}{\partial x} f_x(x,u) + \sum p_{i_1}(x,u) a_{i_1}(x,u) + \sum q_{i_2}(x,u) b_{i_2}(x,u) + \sum r_{i_3} c_{i_3}(x,u) \ge 0 \tag{5}$$

or

$$-w(x,u)\frac{\partial V}{\partial x} f_x(x,u) + \sum p_{i_1}(x,u) a_{i_1}(x,u) + \sum q_{i_2}(x,u) b_{i_2}(x,u) \ge 0 \tag{6}$$

will guarantee that the origin of the state space is a stable equilibrium of the system.

Proof: If condition (5) is fulfilled, then the inequality can simply be integrated from time $t = 0$ to $t = T$ to obtain

$$V(0) - V(T) \ge -\sum \int_0^T \{p_{i_1}(x,u) a_{i_1}(x,u) - r_{i_3} c_{i_3}(x,u)\} dt \ge 0,$$

where we have used the fact that $a_{i_1}(x,u)$, $b_{i_2}(x,u)$ and $c_{i_3}(x,u)$ satisfy (2)–(4). On the other hand, if we have condition (6) fulfilled, then the inequality can be divided by $w(x,u)$ and integrated from 0 to T to obtain

$$V(0) - V(T) \ge -\sum \int_0^T \frac{p_{i_1}(x,u)}{w(x,u)} a_{i_1}(x,u) dt \ge 0.$$

The rest of the proof is similar to the proof of Lyapunov's theorem, which can be found in many standard textbooks, e.g., [1]. ∎

Remark 2 *For the case in which* $f_x(x,u)$ *is a rational vector field such as* $\frac{n(x,u)}{d(x,u)}$*, the multiplier* $w(x,u)$ *should be chosen such that* $w(x,u)\frac{\partial V}{\partial x} f_x(x,u)$ *is a polynomial. However,* $w(x,u)$ *cannot be used if we want to make use of the integral constraints, in which case we have to use condition (5). In the latter case, we introduce a set of auxiliary variables* v *defined to be equal to the derivative of* x*, and we rewrite the system* $\dot{x} = f_x(x,u)$ *as a system with a polynomial vector field and some polynomial equality constraints as follows:* $\dot{x} = v$; $d(x,u)v = n(x,u)$.

Remark 3 *We will term a Lyapunov function* V *for which* V *is positive definite and* $-\frac{dV}{dt} \ge 0$ global *if no restrictions on the state-space have been made. Otherwise, we term it* local.

To construct Lyapunov functions for systems with non-polynomial vector fields, the vector field is rendered polynomial using auxiliary variables u that encompass the non-polynomial terms. Let $u = g(x)$, where $g(x)$ is a vector of non-polynomial functions that appear in the vector field. Then u will have its own dynamics,

$$\dot{u} = \frac{\partial g}{\partial x}\dot{x} \triangleq f_u(x,u).$$

Generally, this transformation will induce some equality and inequality constraints, as we will see in Section 3. To prove stability in this case, we search for a Lyapunov function $\tilde{V}(x,u)$, such that $V(x) = \tilde{V}(x,u)|_{u=g(x)}$ is positive definite around the origin. Theorem 1 can be applied directly in this case, by replacing all occurrence of $\frac{\partial V}{\partial x} f_x(x,u)$ in (5) and (6) by $\frac{\partial \tilde{V}}{\partial x} f_x(x,u) + \frac{\partial \tilde{V}}{\partial u} f_u(x,u)$. Note that $V(x)$ contains non-polynomial terms, so this technique can also be used to construct non-polynomial Lyapunov functions, even when the original system has a polynomial vector field.

The challenge lies in how the above conditions can be verified algorithmically. To this end, we take advantage of the computational tractability of the sum of squares decomposition in order to avoid the NP hardness of proving that a polynomial function is positive definite or positive semi-definite. The condition that $p(x)$ is a

[1] Although not written explicitly here, we assume that we keep track of the indices.

[2] Strictly speaking, it is enough to require V to have a local minimum at the origin.

sum of squares is more strict, yet more verifiable, than $p(x) \geq 0$.

A multivariate polynomial $p(x_1, ..., x_n) \triangleq p(x)$ is a sum of squares (SOS, for brevity), if there exist polynomials $f_1(x), ..., f_m(x)$ such that

$$p(x) = \sum_{i=1}^{m} f_i^2(x).$$

This in turn is equivalent to the existence of a positive semidefinite matrix Q [3], and a properly chosen vector of monomials $Z(x)$ such that

$$p(x) = Z^T(x) Q Z(x). \tag{7}$$

Notice that $p(x)$ being an SOS implies $p(x) \geq 0$. However, we note that he converse is not true, except for some special cases. In fact there are famous counterexamples to this effect (such as the Motzkin form). For a more extensive account on SOS polynomials and their applications, the readers should consult [3] and the references therein.

We can search for such a Q by solving a semidefinite program. This is done using SOSTOOLS, a software developed for this purpose [4] which uses SeDuMi [5] as the semidefinite programming solver.

Therefore, by deliberately opting to work with polynomial and rational functions, the positive definite conditions in Theorem 1 can be relaxed to the existence of an SOS decomposition, and the problem can be cast as an SOS program [4]. Under this relaxation, the search for a bounded degree Lyapunov function $V(x)$ and multipliers $p_{i_1}(x, u)$, $q_{i_1}(x, u)$, and r_{i_3} can be efficiently performed. The SOS program can therefore be formulated as follows.

Program 4 *Suppose that we are given the system (1)–(4). For a polynomial function $W(x)$ with a predetermined form that is locally positive definite, find bounded degree polynomials $V(x)$, $p_{i_1}(x, u)$'s, $q_{i_2}(x, u)$'s and constants r_{i_3}'s such that*

1. *$V(x) - W(x)$ is a sum of squares (implying ≥ 0),*
2. *The left-hand side of inequality (5) or (6) is a sum of squares,*
3. *$p_{i_1}(x, u)$ are sums of squares,*
4. *$r_{i_3} \geq 0$.*

In this SOS program, Condition 2 is a computational relaxation to inequality (5) or (6) in Theorem 1, whereas Condition 1 is required to impose strict positive definiteness on $V(x)$, as required by Theorem 1. Using $W(x)$ restricts $V(x)$ to be positive definite, or at least have a local minimum at the origin. $W(x)$ may be parameterized by some decision variables, on which restrictions may be applied to render it positive definite. When dealing with non-polynomial terms, it may be a function of both x and u. This will be made clearer in the examples in Section 3.

If such a $V(x)$ that fulfills the conditions of Program 4 is not found, one of higher order will be sought. Failure to find a Lyapunov function does not necessarily mean that the equilibrium is unstable, as all the above conditions are sufficient.

Using equality and inequality constraints, the class of systems for which Lyapunov function synthesis is possible has been extended dramatically: in many cases it is possible to transform systems that originally contained radical, trigonometric and other terms to systems with polynomial vector fields under equality and inequality constraints. Furthermore, Lyapunov functions that can be constructed in this way can be extended to include non-polynomial terms. This will be seen in the next section.

3 Examples

Three examples will be presented to illustrate how Lyapunov functions can be constructed using the method described in the previous section.

3.1 Example 1: A Simple System with a Polynomial Vector Field

As a first example, consider the system

$$\begin{aligned}
\dot{x}_1 &= -x_1^3 + 4x_2^3 - 6x_3x_4 \\
\dot{x}_2 &= -x_1 - x_2 + x_5^3 \\
\dot{x}_3 &= x_1x_4 - x_3 + x_4x_6 \\
\dot{x}_4 &= x_1x_3 + x_3x_6 - x_4^3 \\
\dot{x}_5 &= -2x_2^3 - x_5 + x_6 \\
\dot{x}_6 &= -3x_3x_4 - x_5^3 - x_6
\end{aligned}$$

which has an equilibrium at the origin. As a first attempt, we will try to construct a quadratic Lyapunov function of the form $V = \sum_{i=1}^{6} \sum_{j=i}^{6} a_{ij} x_i x_j$ where the a_{ij}'s are the unknowns. To guarantee positive definiteness of V, we impose the condition

$$(V - W) \text{ is a sum of squares,}$$

where $W = \sum_{k=1}^{6} \epsilon_k x_k^2$, with ϵ_k's positive decision variables that satisfy $\epsilon_k \geq 0.1 \ \forall \ k$.

A Lyapunov function of this form is not found by SOSTOOLS, so we will aim for a 4th order Lyapunov function; in this case, V will contain all monomials that have order between 2 and 4, and $W = \sum_{k=1}^{6} \epsilon_{1k} x_k^2 + \sum_{k=1}^{6} \epsilon_{2k} x_k^4$ with the ϵ_{1k}'s and ϵ_{2k}'s positive decision variables that satisfy $\epsilon_{1k} + \epsilon_{2k} \geq 0.1 \ \forall \ k$.

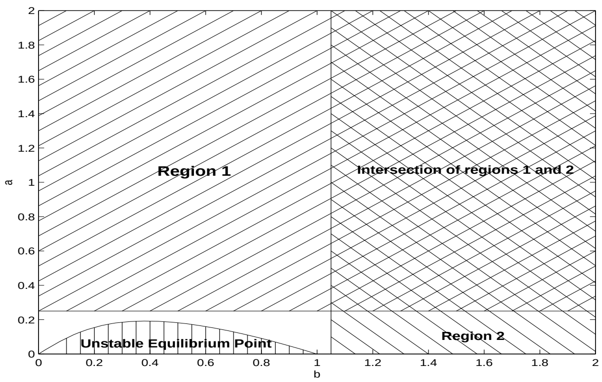

Figure 1: Stability region for the chemical oscillator problem, Regions 1 and 2 defined by equations (14) and (15)

In this case, SOSTOOLS returns

$$V = 0.7257x_1^2 + 1.3x_2^4 + 2.325x_3^2 + 1.575x_4^2 + 0.65x_5^4 + 1.3x_6^2$$

as a Lyapunov function for this system.

3.2 Example 2: A Chemical Oscillator

Two species models of interacting populations can exhibit limit cycle periodic oscillations [7]. The simplest, but chemically plausible tri-molecular reaction that admits periodic solutions is

$$X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} A, \quad B \xrightarrow{k_2} Y, \quad 2X + Y \xrightarrow{k_3} 3X,$$

in which species X is in dynamical equilibrium with species A with a forward rate of reaction k_1 and a backward rate of reaction k_{-1}, and so on. Using the law of mass action, and non-dimensionalising the equations, we get

$$\begin{aligned} \dot{u} &= a - u + u^2 v, && (8) \\ \dot{v} &= b - u^2 v, && (9) \end{aligned}$$

where u, v are the non-dimensional concentrations of X and Y, and a, b are non-negative constant parameters that depend on the concentrations of A and B. It is known that for a and b satisfying

$$(b - a) \geq (b + a)^3,$$

the system exhibits a stable limit cycle and the equilibrium point is unstable (see Figure 1).

A Lyapunov function will be constructed for a region of the rest of the parameter space, to prove robust stability of the equilibrium.

The equilibrium of the above system is a $(\bar{u}, \bar{v})$ pair that satisfies

$$\begin{aligned} 0 &= a - \bar{u} + \bar{u}^2 \bar{v}, && (10) \\ 0 &= b - \bar{u}^2 \bar{v}. && (11) \end{aligned}$$

We translate the equilibrium to the origin using a state transformation $u \to x_1$, $v \to x_2$, $x_1 = u - \bar{u}$, $x_2 = v - \bar{v}$, to get an equivalent system

$$\begin{aligned} \dot{x}_1 &= a - (x_1 + \bar{u}) + (x_1 + \bar{u})^2 (x_2 + \bar{v}), && (12) \\ \dot{x}_2 &= b - (x_1 + \bar{u})^2 (x_2 + \bar{v}), && (13) \end{aligned}$$

whose equilibrium is at the origin. Now suppose that the parameters a and b are not exactly known, but belong to the set $\{a \geq \underline{a}, b \geq \underline{b}\}$. Notice that when the parameters a and b are changed, the equilibrium $(\bar{u}, \bar{v})$ also changes, see Equations (10)–(11). So there are four parameters in the state equations (12)-(13) - namely a, b, $\bar{u}$, and $\bar{v}$ - that are coupled via the algebraic equality constraints (10)-(11). Denote these parameters (which can be regarded as auxiliary variables) by u_1 through u_4, in accordance with the notation in Theorem 1.

Moreover, there exist inherent constraints on the state variables, as the concentrations of the reactants has to be positive. Furthermore, for our purpose, it is enough to find a Lyapunov function that has non-positive derivative in a local region around the equilibrium. In this case, we can impose the inequality constraints $x_1^2 \leq \gamma u_3^2$, $x_2^2 \leq \gamma u_4^2$ where $0 < \gamma \leq 1$. Thus, our system (complete with the equality and inequality constraints) is described by

$$\begin{aligned} \dot{x}_1 &= u_1 - (x_1 + u_3) + (x_1 + u_3)^2 (x_2 + u_4) \\ \dot{x}_2 &= u_2 - (x_1 + u_3)^2 (x_2 + u_4) \\ 0 &\geq x_1^2 - \gamma u_3^2 \\ 0 &\geq x_2^2 - \gamma u_4^2 \\ 0 &\geq \underline{a} - u_1 \\ 0 &\geq \underline{b} - u_2 \\ 0 &= u_1 - u_3 + u_3^2 u_4 \\ 0 &= u_2 - u_3^2 u_4. \end{aligned}$$

For this example, two quartic Lyapunov functions, which prove stability of all the dynamical systems within the following ranges of a and b and which are parameterized by u_1 and u_2 have been constructed.

$$\begin{aligned} &\text{Region 1:} \qquad (\underline{a}, \underline{b}) = (0.25, 0) && (14) \\ &\text{Region 2:} \qquad (\underline{a}, \underline{b}) = (0, 1.05) && (15) \end{aligned}$$

The SOS program is the same as Program 4, and the positive definite function W is similar to the one in the previous example. Each of the resulting Lyapunov functions has more than 30 terms in it, and is therefore not listed here. However, the level curves of these Lyapunov functions are shown for two different parameter values in Figures 2 and 3.

3.3 Example 3: A Whirling Pendulum

Consider the whirling pendulum [8] shown in Figure 4. It is a pendulum of length l_p whose suspension end is attached to a rigid arm of length l_a, with a mass m_b

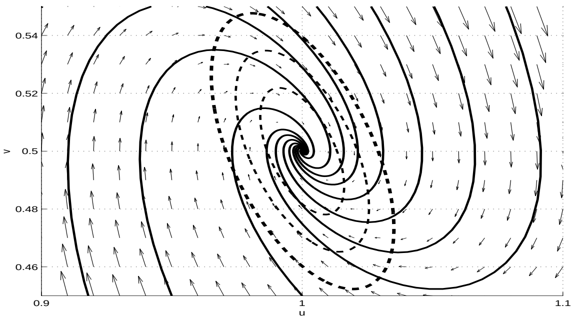

Figure 2: Example 2: $a = 0.5$, $b = 0.5$. Solid lines show trajectories, dotted curves are level curves of the Lyapunov function

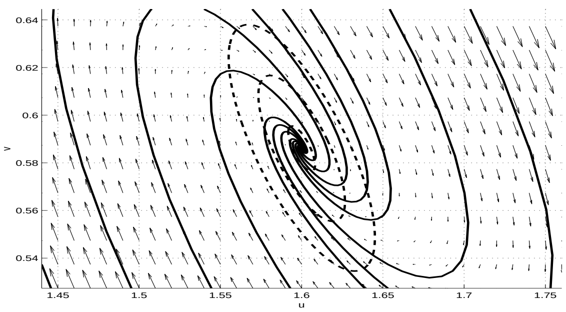

Figure 3: Example 2: $a = 0.1$, $b = 1.5$. Solid lines show trajectories, dotted curves are level curves of the Lyapunov function

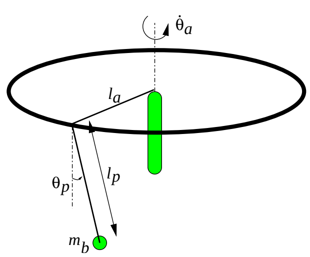

Figure 4: The whirling pendulum

attached to its free end. The arm rotates with angular velocity $\dot{\theta}_a$. The pendulum can oscillate with angular velocity $\dot{\theta}_p$ in a plane normal to the arm, making an angle θ_p with the vertical in the instantaneous plane of motion. We will ignore frictional effects and assume that all links are slender so that their moment of inertia can be neglected.

Using $x_1 = \theta_p$ and $x_2 = \dot{\theta}_p$ as state variables, we obtain the following state equations for the system:

$$\dot{x}_1 = x_2, \tag{16}$$

$$\dot{x}_2 = \dot{\theta}_a^2 \sin x_1 \cos x_1 - \frac{g}{l_p} \sin x_1. \tag{17}$$

The number and stability properties of equilibria in this system depend on the value of $\dot{\theta}_a$. When the condition

$$\dot{\theta}_a^2 < g/l_p \tag{18}$$

is satisfied, the only equilibria in the system are (x_1, x_2) satisfying $\sin x_1 = 0$, $x_2 = 0$. One equilibrium corresponds to $x_1 = 0$, i.e., the pendulum is hanging vertically downward (stable), and the other equilibrium corresponds to $x_1 = \pi$, i.e., the vertically upward position (unstable). As $\dot{\theta}_a^2$ is increased beyond g/l_p, a supercritical pitchfork bifurcation of equilibria occurs [9]. The $(x_1, x_2) = (0, 0)$ equilibrium becomes unstable, and two other equilibria appear. These equilibria correspond to $\cos x_1 = \frac{g}{l_p \dot{\theta}_a^2}$, $x_2 = 0$.

We will now prove the stability of the equilibrium point at the origin for $\dot{\theta}_a$ satisfying (18), by constructing a Lyapunov function. Obviously the energy of this mechanical system can be used as a Lyapunov function but since our purpose is to show that a Lyapunov function can be found using the SOS decomposition, we will assume that our knowledge is limited to the state equations describing the system and that we know nothing about the underlying energy.

Since the vector field (16)–(17) is not polynomial, a transformation to a polynomial vector field must be performed before we are able to construct a Lyapunov function using the SOS decomposition. For this purpose, introduce $u_1 = \sin x_1$ and $u_2 = \cos x_1$ to get:

$$\dot{x}_1 = x_2, \tag{19}$$

$$\dot{x}_2 = \dot{\theta}_a^2 u_1 u_2 - \frac{g}{l_p} u_1, \tag{20}$$

$$\dot{u}_1 = x_2 u_2, \tag{21}$$

$$\dot{u}_2 = -x_2 u_1. \tag{22}$$

In addition, we have the algebraic constraint

$$u_1^2 + u_2^2 - 1 = 0. \tag{23}$$

The whirling pendulum system will now be described by Equations (19)–(23). Notice that all the functions here are polynomial, so that Theorem 1 can be used to prove stability.

We will perform the analysis with the parameters of the system set at some fixed values. Assume that all the parameters except g are equal to 1, and g itself is equal to 10, for which condition (18) is satisfied. For a mechanical system like this, we expect that some trigonometric terms will be needed in the Lyapunov function. Thus we will try to find a Lyapunov function of the following form:

$$\begin{aligned} V &= a_1 x_2^2 + a_2 u_1^2 + a_3 u_2^2 + a_4 u_2 + a_5, \qquad (24) \\ &= a_1 x_2^2 + a_2 \sin^2 x_1 + a_3 \cos^2 x_1 + a_4 \cos x_1 + a_5 \end{aligned}$$

where the a_i's are the unknown coefficients. These coefficients must satisfy

$$a_3 + a_4 + a_5 = 0, \tag{25}$$

for V to be equal to zero at $(x_1, x_2) = (0, 0)$. To guarantee that V is positive definite, we search for Vs that satisfy

$$V - \epsilon_1(1 - u_2) - \epsilon_2 x_2^2 \geq 0, \tag{26}$$

where ϵ_1 and ϵ_2 are positive constants (we set $\epsilon_1 \geq 0.1$, $\epsilon_2 \geq 0.1$). Positive definiteness holds as

$$\epsilon_1(1 - u_2) + \epsilon_2 x_2^2 = \epsilon_1(1 - \cos x_1) + \epsilon_2 x_2^2$$

is a positive definite function in the (x_1, x_2)–space (assuming all x_1 that differ by 2π are in the same equivalence class).

An example of Lyapunov function for this whirling pendulum system is given by

$$V = 0.33445x_2^2 + 1.4615u_1^2 + 1.7959u_2^2 - 6.689u_2 + 4.8931.$$

In fact, we can take things a bit further, and assume that the parameters of the system are fixed at the same values as before except for $\dot{\theta}_a$, which is assumed to be unknown, aiming to construct a Lyapunov function that is parameterized by $\dot{\theta}_a$. It is expected that the function will be a Lyapunov function when $\dot{\theta}_a$ satisfies condition (18), and not otherwise. Such a candidate Lyapunov function has been found:

$$V = 7.3601 + 0.33033x_2^2 - 6.6066u_2 + 7.0698\dot{\theta}_a^2 + 0.3304u_2^2\dot{\theta}_a^2$$

In this case, we cannot impose a condition like (25) to make $V = 0$ at $(x_1, x_2) = (0, 0)$, because the value of V at that point also depends on $\dot{\theta}_a$. Nevertheless, notice that $(x_1, x_2) = (0, 0)$ will always be a stationary point of V.

By construction, the derivative of this function along the trajectories of the system is nonpositive, but its shape near the origin of the state space depends on the value of $\dot{\theta}_a$. The origin will be a minimum of V (at least locally) when the Hessian of V evaluated at the origin,

$$\begin{bmatrix} \frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 x_2} \\ \frac{\partial^2 V}{\partial x_1 x_2} & \frac{\partial^2 V}{\partial x_2^2} \end{bmatrix}_{x_1=0, x_2=0} = \begin{bmatrix} 6.6066 - 0.6608\dot{\theta}_a^2 & 0 \\ 0 & 0.33033 \end{bmatrix},$$

is positive definite, i.e., when $\dot{\theta}_a^2 < 9.9979 \approx 10$. If this condition holds, then V will qualify as a Lyapunov function for the system. Notice that the condition we obtain here agrees exactly with condition (18): When $\dot{\theta}_a^2 \geq 10$, the equilibrium point undergoes a pitchfork bifurcation and becomes unstable; at the same point, the origin changes from being a minimum of V to a saddle point for it, and hence V stops being a Lyapunov function.

4 Conclusions

Constructing Lyapunov functions has always been a challenging task and an important problem in dynamical systems and control theory. An algorithmic approach was developed recently to construct Lyapunov functions for dynamical systems with polynomial vector fields. This was based on a relaxation of the condition that a function is positive semidefinite to the condition that it is a sum of squares.

The class of dynamical systems for which Lyapunov functions can be synthesized with the above method has been extended in this paper to accommodate systems with equality, inequality and integral constraints. In doing so, certain nonpolynomial nonlinearities can be handled, as shown in the examples in Section 3, and Lyapunov functions that are nonpolynomial in the state variables can be constructed.

Future research directions in this area include characterizing the terms that are needed in the Lyapunov function, and choosing the shaping function W. Such a characterization is crucial, as in most cases the size of the resulting semidefinite program can be reduced drastically by using a proper choice of W and searching for Lyapunov functions containing only a minimal set of terms that are really needed.

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