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Two Coupled Pursuit-Evasion Games in Target-Attacker-Defender Problem

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Abstract— **This paper addresses a differential game with three players: a target, an attacker and a defender, where the Attacker aims to capture the Target while avoiding being captured by the Defender, and the Defender tries to defend the Target from being captured by the Attacker while trying to capture the Attacker. There are two coupled pursuit-evasion problems in this game: Attacker-Target and Defender-Attacker. Firstly, a game of kind in this differential game is considered and a so-called barrier is constructed to divide the space into three disjoint regions: the win region of the Attacker, the win region of the team of the Target and Defender and the uncertain region in which neither of three players can win the game. Secondly, a game of degree is considered by introducing a payoff function. Some explicit expressions of optimal strategies for the players are obtained. Finally, the numerical solutions of optimal strategies and the corresponding optimal trajectories for three players with different initial conditions are provided.**

I. INTRODUCTION

Multi-player pursuit-evasion game is an important tool to deal with the maneuver decision problem arising in cooperative control of multi-agent systems[1], [2], especially the confrontational circumstance[3], [4]. This paper presents a scenario with three players: a Target (T), an Attacker (A) and a Defender (D), where the Attacker chases the Target whilst avoiding being captured by the Defender and the Target cooperates with the Defender in order to escape from the Attacker whilst acting a bait to help the Defender capture the Attacker. This scenario is called as Target-Attacker-Defender (TAD) game. There are two coupled pursuit-evasion problems in this game: Attacker-Target and Defender-Attacker.

The TAD problem has received much attention in recent years. The role switching of the missile is analyzed in [5], [6] where a missile pursues a target, who launches a short-range missile to protect itself from the missile. The switch time, at which the missile ceases to evade the defender and starts pursuing the target is considered but the cooperation between the target and the defending missile is not considered. Different types of the cooperation between the Target and the Defender are considered in [7], [8], [9], [10], [11], [12], [13], [14], where the Target represents an aircraft trying to evade an Attacker and the Defender is a pursuer in order to intercept the Attacker. These articles only consider the interception point between the Defender and the Attacker is closer to the Target's position. In other words, the Attacker only aims to capture the Target but does not take into account the threat from the Defender. Only when the Target is captured by the Attacker, the game is terminated. However, in many scenarios, the Attacker is not only a kind of non-recycling goods similar to the missile, but also is an object which has a certain value to survive, such as a fighter. And the Attacker also need to avoid being intercepted by the Defender.

Therefore, in this paper, we consider two focuses on the TAD problem: one is what cooperations the Target and the Defender should adopt to win the game; the other one is the role balance for the Attacker between pursuer and evader. The authors in [15] also consider the two focuses, but solving the guidance algorithm for three players requires some parameters which are not directly available from sensor measurements.

We analyze the problem from qualitative and quantitative perspective respectively and provide a complete solution. Firstly, we construct a so-called barrier of the problem from the perspective of qualitative analysis. The barrier[3] is a surface with the characteristics of semipermeability which partitions the state space into disjoint regions. Each region is associated with a player that wins or loses the game if the initial state of the players lies in that partition. In the past, researchers have used the concept of barrier to solve zero-sum capture-evasion games in 2-player [16][17][18]as well as multi-player scenarios [19]. Secondly, we quantitatively analyze this problem using the weighted value of two correlation distance as the payoff function and derive the optimal strategies of three players. The solving of the optimal strategies expressions is a two point boundary value problem, we provide the numerical solution of this game using difference method and describe the optimal trajectories for three players with different initial states.

The structure of the paper is as follows. Section II formulates the TAD problem. Section III presents the construction of the barrier and divides the space into different regions. In section IV, optimal control strategies are achieved. In section V, numerical solutions of the differential game are provided and the examples are given. Finally, concluding remarks are made in section VI.

II. THE PROBLEM STATEMENT

In this section, we present the problem formulation of the TAD differential game. Referring to Fig. 1, the Attacker, Target and Defender move in the plane, with speed of V_A,

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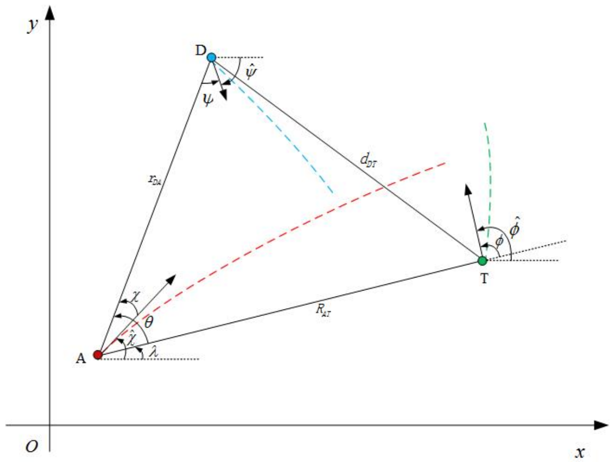


Fig. 1. TAD game in the fixed reference system, where $\hat{\chi}$, $\hat{\psi}$ and $\hat{\phi}$ are the moving directions of Attacker, Defender and Target in the realistic game space, respectively. χ, ψ, ϕ are the alternative control variables of three players, defined as the relative headings of Attacker, Defender and Target from the vectors, $\overrightarrow{AD}$,$\overrightarrow{DA}$,$\overrightarrow{AT}$, respectively. θ is the included angle $\angle DAT$, and λ is the angle of vector $\overrightarrow{AT}$. R_{AT},r_{AD} and d_{TD} are the lengths of segments AT, AD and TD, respectively.

V_T and V_D, respectively. Their control variables are the directions of their velocity vectors respectively.

The dynamics of the Target-Attacker-Defender in the realistic game space can be described in the following equations:

$$\dot{x}_T = V_T \cos\hat{\phi}, \dot{y}_T = V_T \sin\hat{\phi} \tag{1}$$

$$\dot{x}_A = V_A \cos\hat{\chi}, \dot{y}_A = V_A \sin\hat{\chi} \tag{2}$$

$$\dot{x}_D = V_D \cos\hat{\psi}, \dot{y}_D = V_D \sin\hat{\psi} \tag{3}$$

where $\hat{\phi} = \phi + \lambda, \hat{\chi} = \lambda + \theta - \chi$, and $\hat{\psi} = \psi + \theta + \lambda - \pi$. The positions of the Target, the Attacker and the Defender are denoted as (x_T, y_T), (x_A, y_A) and (x_D, y_D). Since the Target and the Defender can always win the game from any given initial state for $V_T, V_D > V_A$, we only consider the case $V_T, V_D < V_A$ and use $\alpha = V_T/V_A$ and $\beta = V_D/V_A$ to denote the speed ratio, $0 < \alpha, \beta < 1$. The positive constant R_c and r_c represent the capture radius of the Attacker and the Defender, respectively.

This game contains two coupled pursuit-evasion games: the Attacker pursues the Target and the Defender pursues the Attacker. The Attacker plays two roles, purser for the Target and evader for the Defender. Naturally, in order to win the game, the Target and the Defender should cooperate as a team to ensure that the Defender can intercept the Attacker before the Attacker capture the Target. We are thus interested in the following two problems:

1) What conditions of the initial state can ensure the win of the Attacker or the team of Target and Defender when they play optimally?
2) What strategies should be adopted by the Attacker or the team of Target and Defender to achieve success?

To describe the dynamics of the TAD game more compactly, the relative distances R_{AT}, r_{DA} and the included angle θ (see Fig.1) can form a reduced state space in which the dynamics of the whole system are derived as follows:

$$\dot{R}_{AT} = \alpha\cos\phi - \cos(\theta - \chi) \tag{4}$$

$$\dot{r}_{DA} = -\cos\chi - \beta\cos\psi \tag{5}$$

$$\dot{\theta} = -\frac{\alpha}{R}\sin\phi + \frac{1}{R}\sin(\theta - \chi) - \frac{\beta}{r}\sin\psi + \frac{1}{r}\sin\chi \tag{6}$$

For the convenience of notation, we denote R_{AT} as R and denote r_{DA} as r in the following parts.

III. THE QUALITATIVE DIFFERENTIAL GAME

The first problem described in the previous section is a game of kind, we borrow existing notions of capture and escape set [20], and modify their definition slightly to address our current scenario. We define the Attacker winning region as the set of initial conditions from which the Attacker can capture the Target without being captured by the Defender. Also we define the team of Target and Defender winning region as the set of initial conditions from which the Defender can capture the Attacker and the Target can escape from the Attacker. Due to the dual identity of the Attacker, there is an uncertain region in this game. In this uncertain region, neither the Attacker captures the Target nor the Defender captures the Attacker.

In this work, our primary goal is to construct such a barrier which separates the whole space into three disjoint regions, and subsequently solve the optimal feedback strategies for three players. In a general game of kind, target set refers to a set of the states satisfying a termination condition of the game. The pursuer wishes to steer the state into the target set, while the evader hopes to keep it out of that set. The boundary of target set, i.e., terminal manifold, is composed of two parts[21]: one is called as usable part (UP), on which the pursuer can force the state to penetrate into the interior of target set despite the evader's strategy; the other one is called as nonuseable part (NUP), on which the evader can frustrate the penetration regardless of the pursuer's strategy. Thus, each player exerts its optimal endeavor to make the state moving tangentially to the boundary of target set, where the points of tangency constitute the boundary of usable part (BUP). Obviously, the BUP of target set is semipermeable and can be used as the initial curve for the barrier within the reversed time. Starting from a point of the BUP, the barrier can be constructed by integrating the retrogressive path equations (RPEs).

A. the construction of the barrier

Based on our assumption that the barrier is a semipermeable surfaces [3], we analyze the optimal strategies of the players on the barrier. Let $\lambda = [\lambda_R, \lambda_r, \lambda_\theta]^T$ denote the normal to the semi-permeable surface passing through the point (R, r, θ). The Hamilton function is given by

$$\begin{aligned} H = &\lambda_R(\alpha\cos\phi - \cos(\theta - \chi)) + \lambda_r(-\cos\chi - \beta\cos\psi) \\ &+\lambda_\theta(-\tfrac{\alpha}{R}\sin\phi + \tfrac{1}{R}\sin(\theta-\chi) - \tfrac{\beta}{r}\sin\psi + \tfrac{1}{r}\sin\chi) \end{aligned} \tag{7}$$

Then, the first main equation is

$$\begin{aligned}&\min_{\chi(\cdot)}\max_{\phi(\cdot)\psi(\cdot)}\{\alpha(\lambda_R\cos\phi-\tfrac{\lambda_\theta}{R}\sin\phi)-(\lambda_R\cos(\theta-\chi)\\&+\lambda_r\cos\chi-\tfrac{\lambda_\theta}{R}\sin(\theta-\chi)-\tfrac{\lambda_\theta}{r}\sin\chi)-\beta(\lambda_r\cos\psi\\&+\tfrac{\lambda_\theta}{r}\sin\psi)\}=0\end{aligned} \tag{8}$$

Solving the first main equation, we can conclude that the optimal control of the players on the barrier are given by the following expressions

$$\sin\phi^*=-\frac{\lambda_\theta}{R\rho_1},\cos\phi^*=\frac{\lambda_R}{\rho_1},\rho_1=\sqrt{(\lambda_\theta/R)^2+\lambda_R^2} \tag{9}$$

$$\sin\psi^*=-\frac{\lambda_\theta}{r\rho_2},\cos\psi^*=-\frac{\lambda_r}{\rho_2},\rho_2=\sqrt{(\lambda_\theta/r)^2+\lambda_r^2} \tag{10}$$

$$\begin{aligned}&\sin\chi^*=-\tfrac{a}{\rho_3},\cos\chi^*=\tfrac{b}{\rho_3},\rho_3=\sqrt{a^2+b^2},\\&a=\lambda_R\sin\theta+\tfrac{\lambda_\theta}{R}\cos\theta-\tfrac{\lambda_\theta}{r},\\&b=\lambda_R\cos\theta+\lambda_r-\tfrac{\lambda_\theta}{R}\sin\theta\end{aligned} \tag{11}$$

Thus, the second main equation is described as follow:

$$\alpha\rho_1+\beta\rho_2-\rho_3=0 \tag{12}$$

For a point on the hypersurface, there can be two normal that point in opposite directions. In this work, the normal to the barrier is chosen in the direction which points towards the capture set. Note that the expression for the optimal controls of each player at a point on the barrier is a function of the components of the normal vector. In order to compute the optimal control as a function of the state, we obtain the retrogressive path equations for the components of the normal to the barrier along the trajectory of the players on the barrier.

According to ϕ^*, ψ^*, χ^*, the retrogressive path equations are written as follows:

$$\mathring{R}=-\alpha\frac{\lambda_R}{\rho_1}+\frac{b}{\rho_3}\cos\theta-\frac{a}{\rho_3}\sin\theta \tag{13}$$

$$\mathring{r}=\frac{b}{\rho_3}-\beta\frac{\lambda_r}{\rho_2} \tag{14}$$

$$\mathring{\theta}=-\frac{\alpha\lambda_\theta}{R^2\rho_1}-\frac{b\sin\theta+a\cos\theta}{R\rho_3}-\frac{\beta\lambda_\theta}{r^2\rho_2}+\frac{a}{r\rho_3} \tag{15}$$

$$\mathring{\lambda}_R=\frac{\partial H}{\partial R}=-\frac{\lambda_\theta}{R^2}(\frac{b\sin\theta+a\cos\theta}{\rho_3}+\frac{\alpha\lambda_\theta}{R\rho_1}) \tag{16}$$

$$\mathring{\lambda}_r=\frac{\partial H}{\partial r}=-\frac{\lambda_\theta}{r^2}(-\frac{a}{\rho_3}+\frac{\beta\lambda_\theta}{r\rho_2}) \tag{17}$$

$$\mathring{\lambda}_\theta=\frac{\partial H}{\partial\theta}=\lambda_R\frac{b\sin\theta+a\cos\theta}{\rho_3}+\frac{\lambda_\theta}{R}\frac{b\cos\theta-a\sin\theta}{\rho_3} \tag{18}$$

where $^\circ$ denotes the temporal derivative in retrogressive time.

Based on the problem definition, the target set of the Attacker denoted by $\mathscr{D}_p^1$ can be described as follows

$$\mathscr{D}_p^1=\{R,r,\theta\,|\,R\le R_c,r>r_c\} \tag{19}$$

The target set of the team of Target and Defender denoted by $\mathscr{D}_p^2$ can be described as follows

$$\mathscr{D}_p^2=\{R,r,\theta\,|\,R>R_c,r\le r_c\} \tag{20}$$

The boundary of the target set

$$\mathscr{B}=\{R,r,\theta\,|\,R=R_c\wedge r=r_c\} \tag{21}$$

We parameterize $\mathscr{B}$ by

$$R(0)=R_c,r(0)=r_c,\theta(0)=s. \tag{22}$$

The outward normal vector $\lambda_\theta(0)=0$, the $\lambda_R(0)$ and $\lambda_r(0)$ are free. Let $\lambda_R(0)$ and $\lambda_r(0)$ be an external normal vector with the same value in the opposite direction. Accordingly, the usable part of the $\mathscr{D}$ can be determined by the following equation

$$\begin{aligned}&H(R(0),r(0),\theta(0),\phi^*,\psi^*,\chi^*,\lambda_R(0),\lambda_r(0),\lambda_\theta(0))\\&=\alpha\rho_1+\beta\rho_2-\rho_3<0\end{aligned} \tag{23}$$

Define $0\le S\le\pi$, $\cos S=\frac{(\alpha+\beta)^2-2}{2}$. The corresponding parameter s satisfied (23) is $|s|<S$, the BUP satisfy $s=\pm S$. So the Boundary of the Usable Part (BUP) of the target set $\mathscr{B}$ can be denoted by $\mathscr{B}_1$,

$$\mathscr{B}_1=\{R,r,\theta\,|\,R=R_c,r=r_c,\theta=\pm\arccos(\frac{(\alpha+\beta)^2}{2}-1)\} \tag{24}$$

We can obtain the barrier by integrating (13)-(18) from the Boundary of the Usable Part (BUP) of the target set.

B. the Discussion

1) When the speed of players is constant, the size of the winning regions changes with the initial distance between players. When the value of r is constant, the winning region of the team of Target and Defender increase with the raise of the value of R, correspondingly, when the value of R is constant, the winning region of the Attacker increases with the raise of the value of r.
2) The smaller capture radius is, the smaller corresponding winning region will be.
3) When the distance between players is constant, the size of the winning regions also changes with the ratio of speed. The winning region of the team of Target and Defender increases with the raise of α and β .
4) The uncertain region reduces with the decrease of the speed of the Target and Defender.

IV. The Quantitative Differential Game

In this section, we will discuss the optimal strategies for the players in a game of degree, where the two sides of the game have clear goals which are minimizing or maximizing the value of payoff function. In this game, the Attacker should adopt an optimal control angle, denoted by χ, to minimize the distance R and to maximize the distance r at the terminal time t_f, where t_f is free. On the contrary, the objective of the team of Target and Defender is to adopt the optimal heading angles, denoted by ϕ and ψ respectively, to maximize the R and to minimize the r at the terminal time.

When the distance $R(t_1) = R_c$ or the distance $r(t_2) = r_c$, the differential game terminates. The t_f, denotes the game terminal time, is equal to $\min\{t_1, t_2\}$.

Thus, the payoff function can be indicated as

$$\max_{\phi(\cdot),\psi(\cdot)} \min_{\chi(\cdot)} J = \max_{\phi,\psi} \min_{\chi} \int_{t_0}^{t_f} (w_1\dot{R} - w_2\dot{r})dt \tag{25}$$

where the parameters w_1 and w_2 are the weighting coefficients.

Then from the differential game theory [3], the Hamiltonian function is given by

$$\begin{aligned} &H(\lambda,\phi,\psi,\chi,t) = w_1\dot{R} - w_2\dot{r} + \lambda_R\dot{R} + \lambda_r\dot{r} + \lambda_\theta\dot{\theta} \\ &= w_1(\alpha\cos\phi - \cos(\theta-\chi)) - w_2(-\cos\chi - \beta\cos\psi) \\ &+\lambda_R(\alpha\cos\phi - \cos(\theta-\chi)) + \lambda_r(-\cos\chi - \beta\cos\psi) \\ &+\lambda_\theta(-\tfrac{\alpha}{R}\sin\phi + \tfrac{1}{R}\sin(\theta-\chi) - \tfrac{\beta}{r}\sin\psi + \tfrac{1}{r}\sin\chi) \end{aligned} \tag{26}$$

and the co-state dynamics are given by

$$\dot{\lambda}_R = \frac{\lambda_\theta}{R^2}(\sin(\theta-\chi) - \alpha\sin\phi) \tag{27}$$

$$\dot{\lambda}_r = \frac{\lambda_\theta}{r^2}(\sin\chi - \beta\sin\psi) \tag{28}$$

$$\dot{\lambda}_\theta = -w_1\sin(\theta-\chi) - \lambda_R\sin(\theta-\chi) - \frac{\lambda_\theta}{R}\cos(\theta-\chi) \tag{29}$$

The game will end in two different conditions. In Case I, we define the team of the Target and Defender win the game and the Attacker is captured by the Defender. The terminal state $r(t_f)$ is fixed and equal to r_c. The terminal states $R(t_f)$ and $\theta(t_f)$ are free, so we have $\lambda_R(t_f) = 0, \lambda_\theta(t_f) = 0$. The best solution for this problem requires $H(x^*(t_f), u^*(t_f), \lambda^*(t_f), t_f) = 0$ at the terminal condition. In summary, the terminal conditions can be described as:

$$\begin{aligned} &r(t_f) = r_c \\ &\lambda_R(t_f) = 0 \\ &\lambda_\theta(t_f) = 0 \\ &w_1\alpha + \beta(w_2 - \lambda_r(t_f)) = \\ &\sqrt{w_1^2 + (w_2 - \lambda_r(t_f))^2 - 2w_1(w_2 - \lambda_r(t_f))\cos\theta(t_f)} \end{aligned} \tag{30}$$

In Case II, the Attacker wins the game, $R(t_f) = R_c$, the Attacker captures the Target while avoiding being captured by the Defender $r(t_f) > r_c$, the terminal conditions can be described as:

$$\begin{aligned} &R(t_f) = R_c \\ &\lambda_r(t_f) = 0 \\ &\lambda_\theta(t_f) = 0 \\ &(w_1 + \lambda_R(t_f))\alpha + w_2\beta = \\ &\sqrt{(w_1 + \lambda_R(t_f))^2 + {w_2}^2 - 2w_2(w_1 + \lambda_R(t_f))\cos\theta(t_f)} \end{aligned} \tag{31}$$

We have the following theorem.

Theorem 4.1: In the quantitative differential game, the Target and Defender optimal control angles are given by

$$\begin{aligned} \sin\phi^* &= -\frac{\lambda_\theta/R}{\sqrt{(\lambda_\theta/R)^2 + (w_1+\lambda_R)^2}} \\ \cos\phi^* &= \frac{w_1+\lambda_R}{\sqrt{(\lambda_\theta/R)^2 + (w_1+\lambda_R)^2}} \end{aligned} \tag{32}$$

$$\begin{aligned} \sin\psi^* &= -\frac{\lambda_\theta - r}{\sqrt{(\lambda_\theta/r)^2 + (w_2-\lambda_r)^2}} \\ \cos\psi^* &= \frac{w_2-\lambda_r}{\sqrt{(\lambda_\theta/r)^2 + (w_2-\lambda_r)^2}} \end{aligned} \tag{33}$$

The optimal control angle of the Attacker is given by Eqs (35) (shown on the top of the next page).

Proof: The analytic solution of differential game can be obtained by:

1) Differentiating the Hamiltonian function in ϕ and setting the derivative to zero:

$$\frac{\partial H}{\partial \phi} = \alpha[-\frac{\lambda_\theta}{R}\cos\phi - (w_1 + \lambda_R)\sin\phi] = 0 \tag{36}$$

Using the trigonometric identity, we can conclude the result in (33).

The second partial derivative of the Hamiltonian function in ϕ :

$$\begin{aligned} \frac{\partial^2 H}{\partial \phi^2} &= \alpha[\tfrac{\lambda_\theta}{R}\sin\phi - (w_1 + \lambda_R)\cos\phi] \\ &= -\alpha[\frac{\lambda_\theta^2 + R^2(w_1+\lambda_R)^2}{R^2\sqrt{(\lambda_\theta/R)^2 + (w_1+\lambda_R)^2}}] < 0 \end{aligned} \tag{37}$$

which means that ϕ^* maximizes the payoff J; That is to say, it maximizes the final distance $R(t_f)$ and minimizes the distance $r(t_f)$.

2) With regard to ψ, we can conclude the result (34) of the Defender in a similar way.

3) The optimal heading χ^* of the Attacker can also be solved by differentiating the Hamiltonian function in χ and setting the derivative to zero:

$$\begin{aligned} &\tfrac{\partial H}{\partial \chi} = \big((w_1 + \lambda_R)\cos\theta - (w_2 - \lambda_r) - \tfrac{\lambda_\theta}{R}\sin\theta\big)\sin\chi \\ &- \big((w_1 + \lambda_R)\sin\theta + \tfrac{\lambda_\theta}{R}\cos\theta - \tfrac{\lambda_\theta}{r}\big)\cos\chi = 0 \end{aligned} \tag{38}$$

We can obtain (35). Setting

$$a = (w_1 + \lambda_R)\sin\theta + \frac{\lambda_\theta}{R}\cos\theta - \frac{\lambda_\theta}{r}$$

$$b = (w_1 + \lambda_R)\cos\theta - (w_2 - \lambda_r) - \frac{\lambda_\theta}{R}\sin\theta$$

Furthermore, we have

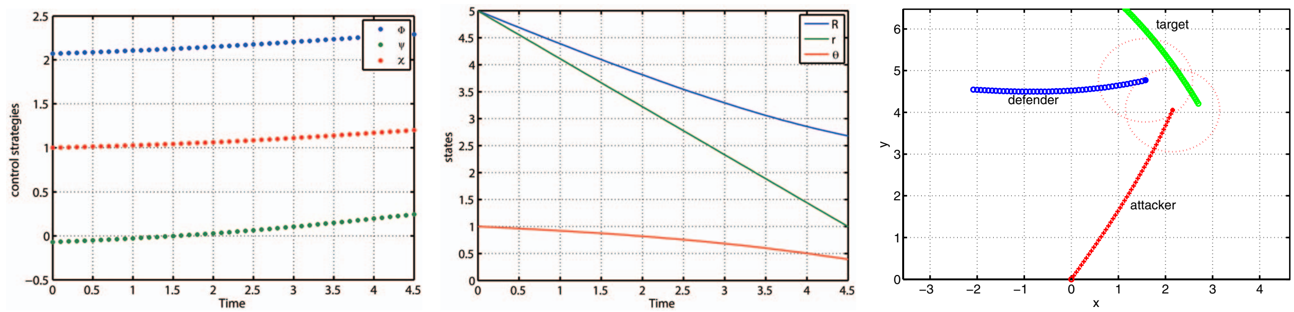
$$\frac{\partial^2 H}{\partial \chi^2} = b\cos\chi + a\sin\chi = \frac{b^2}{\sqrt{a^2+b^2}} + \frac{a^2}{\sqrt{a^2+b^2}} > 0 \tag{39}$$

It shows that χ^* minimizes the payoff J, which is equivalent to minimizing the distance $R(t_f)$ and maximizing the distance $r(t_f)$. ■

V. Numerical Solutions of The Optimal Control Problem

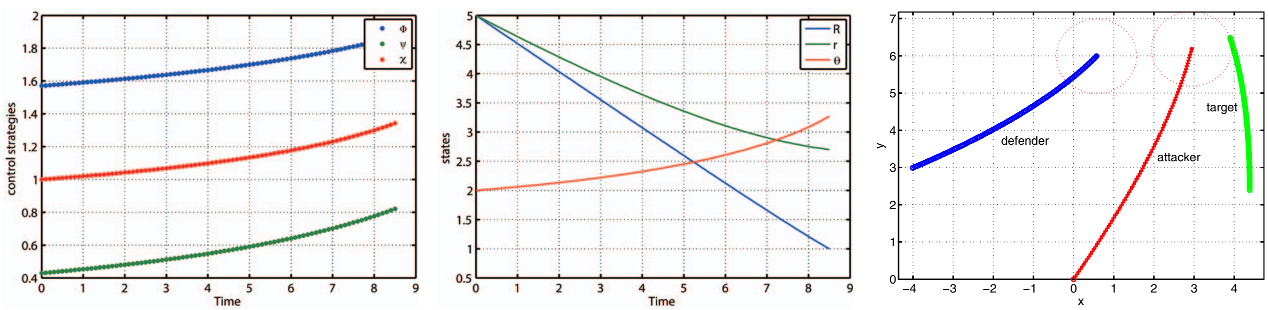
Eqs(4)-(6)and Eqs(26)-(32) represent a two point boundary value problem (TPBVP). In many cases, it is difficult to obtain the analytic solution. At present, there are some methods to solve this problem numerically [22], such as shooting method, difference method and finite element method and so on. Regardless of the methods, it is necessary to estimate the initial values of the TPBVP. If the initial values are uncertain, obtaining an ideal solution will be very difficult. That is the main challenge in solving TPBVP [23].Since the

$$
\begin{aligned}
\sin\chi^* &= \frac{(w_1+\lambda_R)\sin\theta+\frac{\lambda_\theta}{R}\cos\theta-\frac{\lambda_\theta}{r}}{\sqrt{\left((w_1+\lambda_R)\sin\theta+\frac{\lambda_\theta}{R}\cos\theta-\frac{\lambda_\theta}{r}\right)^2+\left((w_1+\lambda_R)\cos\theta-(w_2-\lambda_r)-\frac{\lambda_\theta}{R}\sin\theta\right)^2}} \\
\cos\chi^* &= \frac{(w_1+\lambda_R)\cos\theta-(w_2-\lambda_r)-\frac{\lambda_\theta}{R}\sin\theta}{\sqrt{\left((w_1+\lambda_R)\sin\theta+\frac{\lambda_\theta}{R}\cos\theta-\frac{\lambda_\theta}{r}\right)^2+\left((w_1+\lambda_R)\cos\theta-(w_2-\lambda_r)-\frac{\lambda_\theta}{R}\sin\theta\right)^2}}
\end{aligned} \tag{34}
$$



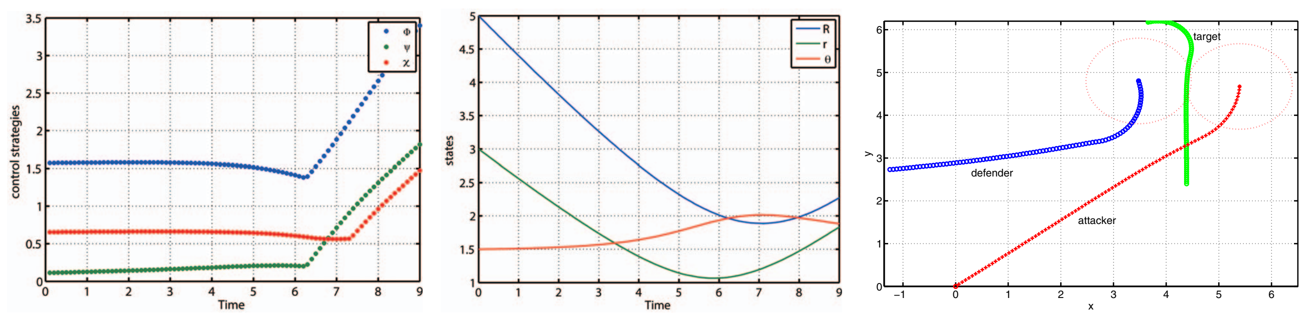
(a) The optimal heading angles of three players. (b) The change curves of relative distances R, r and the included angle θ. (c) The optimal motion trajectories of three players.

Fig. 2. The simulation results of Example 1. The speed ratios are $\alpha = 0.6$, and $\beta = 0.8$. The initial conditions are $R_0 = 5$, $r_0 = 5$, $R_c = 1$, $r_c = 1$, $\theta_0 = 1$ rad, $\lambda_0 = 1$ rad, $w_1 = w_2 = 0.5$. The Attacker initially located at the coordinates $(0, 0)$. R and r decrease monotonously and the game ends at 4.5s, $r = r_c$.



(a) The optimal heading angles of three players. (b) The change curves of relative distances R, r and the included angle θ. (c) The optimal motion trajectories of three players.

Fig. 3. The simulation results of Example 2. The speed ratios are $\alpha = 0.6$, and $\beta = 0.8$. The initial conditions are $R_0 = 5$, $r_0 = 3$, $R_c = 1$, $r_c = 1$, $\theta_0 = 2$ rad, $\lambda_0 = 0.5$ rad, $w_1 = w_2 = 0.5$. The Attacker initially located at the coordinates $(0, 0)$. R decreases and r increases monotonously and the game ends at 8.5s, $R = R_c$.



(a) The optimal heading angles of three players. (b) The change curves of relative distances R, r and the included angle θ. (c) The optimal motion trajectories of three players.

Fig. 4. The simulation results of Example 3. The speed ratios are $\alpha = 0.6$, and $\beta = 0.8$. The initial conditions are $R_0 = 5$, $r_0 = 3$, $R_c = 1$, $r_c = 1$, $\theta_0 = 1.5$ rad, $\lambda_0 = 0.5$ rad, $w_1 = w_2 = 0.5$. The Attacker initially located at the coordinates $(0, 0)$. Neither R nor r reaches their capture radius, no sides can win the game.

co-state variable has no obvious physical meaning, in this paper, we utilize difference method to solve this problem and indirectly estimate the initial value of the co-state variables by setting the control variables. Considering the cooperation of the Target and the Defender, we assume that the Target and the Defender move towards each other at the initial time.

The Attacker, aimed to capture the Target, moves towards the Target initially.In this section, we give some examples to illustrate the theoretical results.

Example 1: The initial conditions are $R_0 = 5$, $r_0 = 5$, and $\theta_0 = 1$ rad. The initial line-of-sight angle is given by $\lambda_0 = 1$ rad. The Attacker initially located at the coordinates $(0, 0)$. The capture radiuses are $R_c = 1$, and $r_c = 1$. The speed ratios are $\alpha = 0.6$, and $\beta = 0.8$. We obtain the initial value of the co-state variables using the initial value of the control angles: $\lambda_R(0) = 0.2235$, $\lambda_r(0) = -0.2235$, and $\lambda_\theta(0) = -6.6215$. The optimal control strategies of three players are shown in Fig. 2(a) and the optimal states trajectories of three players are shown in the Fig. 2(b). The optimal motion trajectories of three players are shown in Fig. 2(c).

We can note that R and r continue to decrease and $r = r_c$ at 4.5s. That is to say, the Defender captures the Attacker at 4.5s and the team of Defender and Target win the game.

Example 2: We give the initial conditions $r_0 = 3$, $\theta_0 = 2$ rad, $\lambda_0 = 0.5$ rad, and the other parameters are same as the parameters in Example 1. The optimal control strategies and the optimal states trajectories are shown in Fig. 3(a) and Fig. 3(b), respectively. The optimal motion trajectories of three players are shown in Fig. 3(c).

We can note that R continue to decrease and r continue to increase. At 8.5 seconds, $R = R_c$, the Attacker captures the Target. That is to say, the Attacker win the game.

Example 3: Based on the initial conditions of Example1, we change the initial conditions $r_0 = 3$, $\theta_0 = 1.5$ rad, and $\lambda_0 = 0.5rad$. The results of game are shown in Fig. 4(a), Fig. 4(b) and Fig. 4(c).

We can note that R and r respectively reach their minimum at 7 and 6 seconds, and then begin to increase. Their minimum are both greater than their capture radius, so the game continues at all times and no side can win the game.

VI. Conclusions

In this paper, we have formulated an Attacker-Target-Defender differential game and obtain the optimal solutions. The characteristic of this game is that there are two coupled pursuit-evasion games and two target sets. The Attacker plays two roles, and the Target and the Defender cooperate with each other. A game of kind in this differential game is considered. In addition to the traditional winning and non winning region, an uncertain region is first proposed to describe the scenario in which no players can win the game. Then, the players optimal feedback strategies are solved in terms of the game of degree. By estimating the initial value of the control variables, we obtain the numerical solutions of this problem and illustrate the simulation results.

In the future work, we will investigate the TAD game with more practical motion models of the players, more complex gaming environment (containing obstacles, limited observation and intercommunication, jamming), and different rules (ways of cooperation, conditions of ending game). Also, the TAD game with more players is another focus forward in the direction of our research.

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