OPTIMAL CONTROL APPLICATIONS & METHODS, VOL. 1, 103–118 (1980)

THE GEOMETRY OF THE BARRIER IN THE 'GAME OF TWO CARS'*

MEIR PACHTER† AND WAYNE M. GETZ‡

National Research Institute for Mathematical Sciences of the CSIR, P.O. Box 395, Pretoria, South Africa

SUMMARY

A complete open barrier surface is constructed in the 'Game of two cars' differential game of kind with a circular target set. Our geometric approach has the advantages, among others, that it yields the explicit barrier termination conditions and that it highlights the geometric meaning of the universal and dispersal curves on the barrier.

KEY WORDS Differential games Game of two cars Barrier surface Universal curve Dispersal curve

1. INTRODUCTION

The pursuit–evasion differential game of two cars was introduced by Isaacs (Reference 1, Section 9.2). Two points or 'players', the 'pursuer' P and 'evader' E, move in a plane with constant speeds w_1 and w_2 respectively. The manoeuvrability of the players is determined by their minimum turning radii R_1 and R_2, respectively, and they steer by selecting at each instant the value of the curvature of their trajectories by choosing an appropriate value of the controls $u_1 \in [-1, 1]$ and $u_2 \in [-1, 1]$, respectively. This curvature then determines their actual instantaneous turning radius.

We suppose further that the 'weapon system' of the pursuer has an all-aspect range of r and that the pursuer strives to bring an uncooperative E within range of his weapons.[1–3] The game (of kind) terminates when the distance between P and E is less than r.

By fixing the origin of the coordinate system (x, y) at the position occupied by P, aligning the y-axis with the direction of motion of P, and considering the relative motion of players P and E, the state space of the game is reduced to $R^2 \times S^1$. The state vector $(x, y, \theta)^{\mathrm{T}}$ designates the position (x, y) of E in the plane and the angle θ of E's velocity vector measured clockwise from the positive y-axis (E's heading).

Specifically, the equations of motion, when these have been transformed to dimensionless form, are

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= -yu_1 + \gamma \sin\theta, \quad x(0) = x_0 \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= xu_1 - 1 + \gamma\cos\theta, \quad y(0) = y_0 \\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= -u_1 + u_2/\beta, \quad \theta(0) = \theta_0, \quad t \geqslant 0
\end{aligned}
\tag{1}
$$

* This work was partially supported by a grant from Control Data.
† Chief Research Officer.
‡ Senior Research Officer.

0143–2087/80/0201–0103$01.00

Received 10 August 1979
Revised 22 October 1979

and the specified capture radius r implies a cylindrical target set

$$T \triangleq \{(x, y, \theta) \mid x^2 + y^2 \leqslant r^2,\ \theta \in [0, 2\pi)\}^*$$

into which the pursuer endeavours to drive the system vector, i.e. the evader, regardless of the u_2 strategy. Note that the transformation to dimensionless variables is : $x \to x/R_1$, $y \to y/R_1$ (and accordingly $x_0 \to x_0/R_1$, $y_0 \to y_0/R_1$, $r \to r/R_1$), $t \to tw_1/R_1$ and $\gamma \triangleq w_2/w_1$, $\beta \triangleq (1/\gamma)(R_2/R_1)$. The notion of the barrier is central to the solution of the differential game. Indeed, parts of the barrier surface may delimit the region of capturability; or, whenever the pursuer has full state space capturability, the barrier surface is indicative of the region of a pursuer swerve, and is the locus of points of discontinuity of the optimal time-to-go function in the min-max time game of degree. Furthermore, in a two-target pursuit–evasion situation,[4–8] the intersection of barrier surfaces gives the partition of the state space into capturability regions for each player and regions in which the outcome is a draw.

In Isaac's approach,[1, 2] the focus is on the pursuit–evasion differential game of degree in that one considers the min-max time-to-capture performance index (or, the min-max of the range $x^2 + y^2$ at zero closing speed[3]).

The Hamilton–Jacobi dynamic programming (H–J–DP) method gives the value function (the value of the performance index) as the solution to the H–J–DP partial differential equation (PDE) associated with equation (1), and the time (or range) performance index. The barrier is then the surface of discontinuity of the value function. When the H–J–DP PDE is being solved, allowance must be made for such possible discontinuities. Yet, whereas the special case of two identical cars and an ensuing bounded region of capturability is considered in Reference 2, the *complete* barrier surface with the barrier boundary is not actually constructed in either Reference 1 or in References 5–7 for target sets :

$$\{(x, y, \theta) \mid x = 0,\ 0 \leqslant y \leqslant r\}, \quad \{(x, y, \theta) \mid x = 0,\ -30^\circ \leqslant \theta \leqslant 30^\circ\}$$

and a fan-shaped target set, respectively. For that matter, the dispersal curve is not constructed in References 1, 3, 5 and 6 either.

However, the barrier surface and its boundary must be known before it is possible to solve the differential game (of degree) for a specific set of problem parameters. It has been observed by Hajek[9] that perhaps the H–J–DP approach does not provide much of the relevant information which is required for the analysis of pursuit–evasion differential games. It is evident though that this surface of discontinuity, the barrier, has rather a 'controllability' meaning and as such is a basic feature of the game of kind since it is divorced from a specific game-of-degree performance index. Thus, in the present paper, we consider a simple circular target set[1–3] and a situation of full state space capturability.

We employ an invariance argument in the reduced state space and the explicit differential geometric expression for the normal to a surface in order to construct explicitly the complete open barrier surface and the related universal and dispersal curves in the game of kind. In this sense, the present work extends our recent paper[4] on a geometric approach to the construction of a solution to the homicidal chauffeur game.

Specifically, in Section 2 we dwell on the differential geometric invariance arguments and derive (two) PDEs which directly determine the barrier surface. In Section 3, we then explicitly construct the barrier surface, the related universal and dispersal curves, and the barrier boundary line for the specific game parameters : $w_2/w_1 = \frac{1}{2}$, $R_2/R_1 = \frac{1}{2}$, $r/R_1 = \frac{1}{3}$. This is followed in Section 4 by an investigation of the dependence on the parameter r. Section 5 is devoted to concluding remarks.

* The target surface T is actually a torus, since $\theta = 0$ and $\theta = 2\pi$ are identified.

Our (game of kind) geometric approach allows us to construct the barrier surface and also yields the explicit termination conditions which delineate the barrier boundary. The geometric meaning of the universal curve on the barrier is thus brought out. The universal curve is the locus of points on the (smooth) barrier surface where $\partial y/\partial\theta = 0$ or $\partial x/\partial\theta = 0$, depending on whether we view the barrier as a surface $y(x,\theta)$ or $x(y,\theta)$ (Propositions 1 and 2). Moreover, we also construct the dispersal curve on the barrier as the intersection of two barrier surface sheets, and comment on the 'dispersal situation' of the flow field there.

2. THE INVARIANCE ARGUMENT

Consider a C^1 surface $\mathscr{S}: y = y(x,\theta) \subset R^2 \times S^2$. Then

$$\mathbf{n} = (-y_x, 1, -y_\theta) \tag{2}$$

where $y_x \triangleq \partial y/\partial x$ and $y_\theta \triangleq \partial y/\partial\theta$ is a normal to this surface with a component in the direction of the positive y-axis.

System (1) will travel along the surface $\mathscr{S}$ (in the state space) for $t \in [t_1, t_2]$ if u_1 and u_2 can be so chosen that

$$\dot{\mathbf{n}} = (-y_x, 1, -y_\theta)\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = 0 \quad \text{for } t \in [t_1, t_2] \tag{3}$$

Equation (3), written out explicitly, is the partial differential equation (for the surface)

$$(yy_x + x + y_\theta)u_1 - (y_\theta/\beta)u_2 - \gamma\sin\theta\, y_x + \gamma\cos\theta - 1 = 0 \tag{4}$$

which depends on the two parameters u_1 and u_2 and whose solution, subject to appropriate boundary conditions, is the surface $\mathscr{S}$.

System (1), by construction of equation (4), is the system of differential equations for the characteristic curves of $\mathscr{S}$ (see Reference 10) and has, for fixed $u_1 \neq 0$ and $u_2 \neq 0$, the reverse time solutions (let $s = -t$):

$$\begin{aligned} x(\theta_0, s) = {} & x_0(\theta_0)\cos su_1 + y_0(\theta_0)\sin su_1 \\ & + (\gamma\beta/u_2)[\cos su_1 - \cos(\theta_0 + s(u_1 - u_2/\beta)] + (1 - \cos su_1)/u_1 \end{aligned} \tag{5}$$

$$\begin{aligned} y(\theta_0, s) = {} & y_0(\theta_0)\cos su_1 - x_0(\theta_0)\sin su_1 \\ & + (\gamma\beta/u_2)[\sin(\theta_0 + s(u_1 - u_2/\beta)) - \sin(\theta_0 + su_1)] + (\sin su_1)/u_1 \end{aligned} \tag{6}$$

and

$$\theta(\theta_0, s) = \theta_0 + s(u_1 - u_2/\beta) \tag{7}$$

where the boundary conditions are

$$x(\theta_0, 0) = x_0(\theta_0) \quad \text{and} \quad y(\theta_0, 0) = y_0(\theta_0)$$

In addition, for $u_1 \neq 0$, $u_2 = 0$, the solution is

$$\begin{aligned} x(\theta_0, s) &= x_0(\theta_0)\cos su_1 + y_0(\theta_0)\sin su_1 - \gamma s\sin(\theta_0 + su_1) + (1 - \cos su_1)/u_1 \\ y(\theta_0, s) &= y_0(\theta_0)\cos su_1 - x_0(\theta_0)\sin su_1 - \gamma s\cos(\theta_0 + su_1) + (\sin su_1)/u_1 \\ \theta(\theta_0, s) &= \theta_0 + su_1 \end{aligned} \tag{8}$$

while similar solutions are obtained for $u_1 = 0$, $u_2 \neq 0$ and $u_1 = 0$, $u_2 = 0$.

The surface $\mathscr{S}$, with characteristics given by the two-parameter family of curves (5)–(7), is a barrier if the controls u_1 and u_2 are so chosen as to satisfy an appropriate saddle point condition; this, in the case where the pursuer P wants to remain on $\mathscr{S}$ or above it (in the direction of the positive y-axis) and the evader wants to remain on $\mathscr{S}$ or below it, is

$$\max_{|u_1|\leqslant 1}\min_{|u_2|\leqslant 1} \dot{\mathbf{n}} = 0 \tag{9}$$

In addition, $\mathscr{S}$ emanates from the boundary of the usable part of the target T.

From equations (3) and (4), it is now clear that the controls that satisfy (9) are

$$\bar{u}_1 = \operatorname{sign}(yy_x + x + y_\theta) \tag{10}$$

and

$$\bar{u}_2 = \operatorname{sign} y_\theta \tag{11}$$

Sometimes it may be more convenient to view $\mathscr{S}$ as a surface $x(y,\theta)$. In this case, equation (2) becomes

$$\mathbf{n} = (1, -x_y, -x_\theta), \tag{12}$$

where $x_y \triangleq \partial x/\partial y$ and $x_\theta \triangleq \partial x/\partial\theta$, and $\mathbf{n}$ is a normal to $\mathscr{S} \triangleq x(y,\theta)$ with a component in the direction of the positive x-axis. The corresponding partial differential equation for $\mathscr{S}$ is then (cf. equation (4))

$$(-y - xx_y + x_\theta)u_1 - (x_\theta/\beta)u_2 + \gamma\sin\theta - \gamma\cos\theta x_y + x_y = 0 \tag{13}$$

As before, condition (9) holds in case $\mathscr{S}$ is a barrier on or above which (in the direction of the positive axis) the pursuer P endeavours to remain, and on or below which the evader E endeavours to remain. From equation (13) it now follows that for equation (9) to hold, it is necessary that the controls, instead of having to satisfy equations (10) and (11), must now satisfy

$$\bar{u}_1 = \operatorname{sign}(-y - xx_y + x_\theta) \tag{14}$$

and

$$\bar{u}_2 = \operatorname{sign} x_\theta \tag{15}$$

respectively. Note that equations (5)–(7) again specify the characteristics of the solution $x(y,\theta)$ to equation (13).

The essential problem in generating the characteristics of the surface $\mathscr{S}$, as given in equations (5)–(7), in terms of the parameters θ_0 and s, is the proper choice of $\bar{u}_1$ and $\bar{u}_2$ as formulated in equations (10) and (11), or equations (14) and (15). Our approach is to choose nominally $\bar{u}_i = 1$ or $-1, i = 1, 2$, using kinematical geometrical considerations as a guide, and for given θ_0 in equations (5)–(7) to let s increase from zero until one of the conditions in equations (10) and (11), or equations (14) and (15) (whichever is applicable) is violated.

To do this, the partial derivatives of $y(x,\theta)$ or $x(y,\theta)$ must be known. In References 1 and 11 these partial derivatives were evaluated by integrating a set of 'adjoint equations' (the second set of equations (8.3.6) on p. 207 of Reference 1 and eqs. (6.4) on p. 115 of Reference 11) obtained by differentiating equation (9) with respect to the state variables and rearranging the terms. These derivatives can, however, be evaluated in all cases except $\beta = 1, u_1 = u_2$, if s is solved for in equation (7) and the result substituted in equation (6) to obtain x and y in terms of θ_0 and θ. For given θ, the partial derivatives are then evaluated by regarding $\mathscr{S}$ as either $y(x,\theta)$ or $x(y,\theta)$. This technique will be demonstrated for the particular barrier constructed in Section 3 below. Indeed, in this game of two cars our geometrical method explicitly yields the curves which delimit the barrier surface (i.e. $x+yy_x+y_\theta = 0, y_\theta = 0$ or $y+xx_y-x_\theta = 0, x_\theta = 0$) so that we are able to obtain the complete open barrier. Then one obtains the dispersal line on the barrier surface which is given by the intersection of two barrier surface sheets, and one also obtains a geometrical characterization for the universal curve.

For the case $\beta = 1$ and $u_1 = u_2$, equation (7) implies that the characteristics are independent of θ and lie in a plane parallel to the x and y axes. Thus from equation (1), the characteristic curve can be obtained directly by integrating

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\bar{u}_1+1+\gamma\cos\theta}{-y\bar{u}_1+\gamma\sin\theta} \tag{16}$$

for a particular value of $\theta = \theta_0$.

In addition, in order to generate the barrier surface $\mathscr{S}$, we need to obtain boundary conditions for system (5)–(7) for the simple cylindrical target T introduced in Section 1, i.e. the boundary of the usable part of the target.

In our non-dimensional form these boundary conditions are

$$\begin{aligned} x_0(\theta_0) &= \pm r(1/\gamma-\cos\theta_0)/\surd(1-2\cos\theta_0/\gamma+1/\gamma^2) \\ y_0(\theta_0) &= \pm r\sin\theta_0/\surd(1-2\cos\theta_0/\gamma+1/\gamma^2) \end{aligned} \tag{17}$$

Finally, bearing in mind that the barrier surface is periodic (with period 2π) and in view of symmetry conditions, namely $y(x,\theta) = y(-x, 2\pi-\theta)$, we need only consider the game for $x \geqslant 0$ (whence for $\gamma < 1$ the boundary corresponding to the expressions with positive sign in equation (17) will be selected).

3. CONSTRUCTING THE BARRIER FOR A SPECIFIC EXAMPLE

In this section we construct the barrier surface for the specific example $w_2/w_1 = \frac{1}{2}$ and $R_2/R_1 = \frac{1}{2}$ (see also Getz and Pachter[4]), or in the notation of equation (1), $\gamma = \frac{1}{2}$ and $\beta = 1$. Indeed, our method applies to any choice of $0<\gamma\leqslant 1$ and $\beta \geqslant 0$. If however $\gamma > 1$, i.e. the evader E is faster than the pursuer P, the analysis will follow exactly the case $\gamma \leqslant 1$, provided E is fixed at the origin with the coordinates (x, y, θ) which are now the coordinates of P relative to E, and provided the target is the cylindrical one specified above. We have purposely chosen $\beta = 1$ here so that we can demonstrate the solution for $\mathscr{S}$ in both the general case where system (5)–(7) is applicable and $\bar{u}_1 = -\bar{u}_2$, and in the special case when equation (16) is applicable (and $\bar{u}_1 = \bar{u}_2$). It should be noted that, for this choice of parameters, P is twice as fast as E whereas E is twice as manoeuvrable as P. Thus, the trade-off between speed and manoeuvrability can be examined. For the moment the parameter r will be left unspecified since its influence on the construction of the barrier will be of special interest.

Specifically, equation (1) now becomes

$$\begin{aligned} \dot{x} &= -yu_1+(\sin\theta)/2 \\ \dot{y} &= xu_1-1+(\cos\theta)/2 \\ \dot{\theta} &= -u_1+u_2 \end{aligned} \tag{18}$$

From equation (17), the boundary of the usable part is now

$$\begin{aligned} x_0(\theta_0) &= r(2-\cos\theta_0)/\surd(5-4\cos\theta_0) \\ y_0(\theta_0) &= r\sin\theta_0/\surd(5-4\cos\theta_0) \end{aligned} \tag{19}$$

For the parameter values considered in equation (18), in case $\bar{u}_1 = -\bar{u}_2$, the solution (5)–(7) can, by eliminating s, be expressed (with a slight abuse of notation) as

$$\begin{aligned} x(\theta_0,\theta) &= [x_0(\theta_0)-1/\bar{u}_1]\cos\tfrac{1}{2}(\theta-\theta_0)+y_0(\theta_0)\sin\tfrac{1}{2}(\theta-\theta_0)+[2+\cos\theta-\cos\tfrac{1}{2}(\theta+\theta_0)]/2\bar{u}_1 \\ y(\theta_0,\theta) &= [1/\bar{u}_1-x_0(\theta_0)]\sin\tfrac{1}{2}(\theta-\theta_0)+y_0(\theta_0)\cos\tfrac{1}{2}(\theta-\theta_0)+(\sin\tfrac{1}{2}(\theta+\theta_0)-\sin\theta) \end{aligned} \tag{20}$$

For $\bar{u}_1 = \bar{u}_2$, equation (16) can be solved to yield

$$(y\bar{u}_1-\tfrac{1}{2}\sin\theta_0)^2+(x\bar{u}-1+\tfrac{1}{2}\cos\theta_0)^2 = c(\theta_0) \tag{21}$$

where the actual value of $c(\theta_0)$ depends on the initial condition (19).

Since we are considering the barrier in the half-plane $x \geqslant 0$, it follows, from the consideration that the pursuer should be turning towards the evader at point of capture, that $\bar{u}_1 = 1$. This result also applies to the homicidal chauffeur game (see Reference 1, Section 9.1) for which an initial guess of $\bar{u}_1 = -1$ would not satisfy conditions (10) or (11). Thus, putting $\bar{u}_1 = 1$ in equation (21) and using equation (19) we have

$$(y-\tfrac{1}{2}\sin\theta_0)^2+(x-1+\tfrac{1}{2}\cos\theta_0)^2 = [r-\tfrac{1}{2}\sqrt{(5-4\cos\theta_0)}] \tag{22}$$

If equation (19) is used and $\bar{u}_1$ is equated to 1, equation (20) reduces to

$$\begin{aligned} x &= \frac{r}{\sqrt{(5-4\cos\theta_0)}}2\cos\tfrac{1}{2}(\theta-\theta_0)-\cos\tfrac{1}{2}(\theta+\theta_0)-\cos\tfrac{1}{2}(\theta-\theta_0)-\tfrac{1}{2}\cos\tfrac{1}{2}(\theta+\theta_0)+\tfrac{1}{2}\cos\theta+1 \\ y &= \frac{r}{\sqrt{(5-4\cos\theta_0)}}\sin\tfrac{1}{2}(\theta+\theta_0)-2\sin\tfrac{1}{2}(\theta-\theta_0)+\sin\tfrac{1}{2}(\theta-\theta_0)+\tfrac{1}{2}\sin\tfrac{1}{2}(\theta-\theta_0)-\tfrac{1}{2}\sin\theta. \end{aligned} \tag{23}$$

Henceforth, we shall consider the barrier to be given by a surface $y(x,\theta)$. (If the barrier is regarded as a surface $x(y,\theta)$, the analysis is pursued in an analogous way.)

In view of equations (10) and (11), we need to generate the expressions y_θ and yy_x+x+y_θ and especially the curves on the barrier generated by setting these expressions equal to zero. We shall first generate these curves in terms of x and θ, i.e. as they appear projected on the x,θ plane. The corresponding y values can be found very easily once this has been done. Also of importance in constructing the projection of the barrier on the x,θ plane are the envelope curves. These curves consist of points on the surface for which y_x is infinite or, equivalently, points at which the surface folds over or under itself. These points represent extreme values of x on the surface. Hence, the terminology 'envelope curves' is used. Two separate cases need to be examined : (1) $u_1 = 1$, $u_2 = 1$ and $y(x,\theta)$ is determined by equation (22), and (2) $u_1 = 1$, $u_2 = -1$ and $y(x,\theta)$ is determined by equation (23).

Case 1 : $u_1 = 1$, $u_2 = 1$, target boundary

From equation (22), it is clear that the characteristics (and hence system trajectories) are arcs of circles centred at $(1-\tfrac{1}{2}\cos\theta_0, \tfrac{1}{2}\sin\theta_0)$ with values $|r-\tfrac{1}{2}\sqrt{(5-4\cos\theta)}|$. Bearing in mind that for this case $\theta = \theta_0$, the following can be derived :

$$\begin{aligned} y &= \tfrac{1}{2}\sin\theta\pm\sqrt{\{[r-\tfrac{1}{2}\sqrt{(5-4\cos\theta)}]^2-(x-1+\tfrac{1}{2}\cos\theta)^2\}} \\ y_x &= \frac{2-2x-\cos\theta}{2y-\sin\theta} \\ y_\theta &= \frac{y\cos\theta+\sin\theta[x-2r/\sqrt{(5-4\cos\theta)}]}{2y-\sin\theta} \\ x+yy_x+y_\theta &= \frac{2y-2r\sin\theta/\sqrt{(5-4\cos\theta)}}{2y-\sin\theta} \end{aligned} \tag{24}$$

The curves $y_\theta = 0$, $yy_x + x + y_\theta = 0$ and $y_x = \infty$, as they appear projected on x, θ plane, can now be explicitly derived from equation (24). These curves are

$$y_x = \infty \Rightarrow \begin{array}{l} x^+ = 1 - \frac{1}{2}\cos\theta + [r - \frac{1}{2}\sqrt{(5-4\cos\theta)}] \\ \text{or} \\ x^- = 1 - \frac{1}{2}\cos\theta - [r - \frac{1}{2}\sqrt{(5-4\cos\theta)}] \end{array} \tag{25}$$

$$y_\theta = 0 \Rightarrow \begin{array}{l} x = r(2-\cos\theta)/\sqrt{(5-4\cos\theta)} \\ \text{or} \\ x = 2\cos^2\theta - \cos\theta + r(2+\cos\theta - 4\cos^2\theta)/\sqrt{(5-4\cos\theta)} \end{array} \tag{26}$$

and

$$yy_x + x + y_\theta = 0 \Rightarrow \begin{array}{l} x = r(2-\cos\theta)/\sqrt{(5-4\cos\theta)} \\ \text{or} \\ x = (2-\cos\theta)[1 - r/\sqrt{(5-4\cos\theta)}] \end{array} \tag{27}$$

Case 2 : $u_1 = 1$, $u_2 = -1$, target boundary

This case is more complicated than the previous one since x and y depend on both θ and θ_0 in equation (23). In this case, the characteristics and hence system trajectories are not confined to a plane parallel to the x and y axes but instead depend on θ. In fact, from equation (18), $\dot\theta = -2$. Thus, θ actually decreases along trajectories as they approach the target T.

From equation (23),

$$y_x = a(\theta, \theta_0)/b(\theta, \theta_0)$$

where

$$\begin{aligned} a(\theta,\theta_0) &= -\tfrac{1}{2}\cos\tfrac{1}{2}(\theta-\theta_0) + \tfrac{1}{4}\cos\tfrac{1}{2}(\theta+\theta_0) + \frac{r}{\sqrt{(5-4\cos\theta_0)}} \\ &\quad \times [\tfrac{1}{2}\cos\tfrac{1}{2}(\theta+\theta_0) + \cos\tfrac{1}{2}(\theta-\theta_0)] - \frac{2r\sin\theta_0}{[5-4\cos\theta_0]^{3/2}}[\sin\tfrac{1}{2}(\theta+\theta_0) - 2\sin\tfrac{1}{2}(\theta-\theta_0)] \\ b(\theta,\theta_0) &= -\tfrac{1}{2}\sin(\theta-\theta_0) + \tfrac{1}{4}\sin\tfrac{1}{2}(\theta+\theta_0) + \frac{r}{\sqrt{(5-4\cos\theta_0)}} \\ &\quad \times [\sin\tfrac{1}{2}(\theta-\theta_0) + \tfrac{1}{2}\sin\tfrac{1}{2}(\theta+\theta_0)] - \frac{2r\sin\theta_0}{(5-4\cos\theta_0)^{3/2}}[2\cos\tfrac{1}{2}(\theta-\theta_0) - \cos\tfrac{1}{2}(\theta+\theta_0) \end{aligned} \tag{28}$$

and

$$y_\theta = \tfrac{1}{2} - \tfrac{1}{2}x - \tfrac{1}{4}\cos\theta - y_x(\tfrac{1}{2}y + \tfrac{1}{4}\sin\theta)$$

$$yy_x + x + y_\theta = \tfrac{1}{2} + \tfrac{1}{2}x - \tfrac{1}{4}\cos\theta + y_x(\tfrac{1}{2}y + \tfrac{1}{4}\sin\theta)$$

In this case, it turned out to be too difficult to derive explicitly the curves $y_x = \infty$ (for the envelope), $y_\theta = 0$, and $yy_x + x + y_\theta = 0$. Instead, the values of the functions (23) and (28) were generated for a grid of points in the (θ, θ_0) plane, and the information was then extracted from the data so obtained.

Before we proceed with a detailed discussion of cases 1 and 2, the following important point should be noted. The barrier is tangent to the target at the boundary of the usable part, since equation (9) holds both on the barrier and by definition on the boundary of the usable part. Thus, in both cases, y_x will be negative for $\theta \in (0, \pi)$ and positive for $\theta \in (\pi, 2\pi)$. In the former case, that part of the barrier emanating from $(0, \pi)$ is a lower surface of the barrier, below or on which player 1 endeavours to remain and above or on which player 2 endeavours to remain (see Fig. 1).

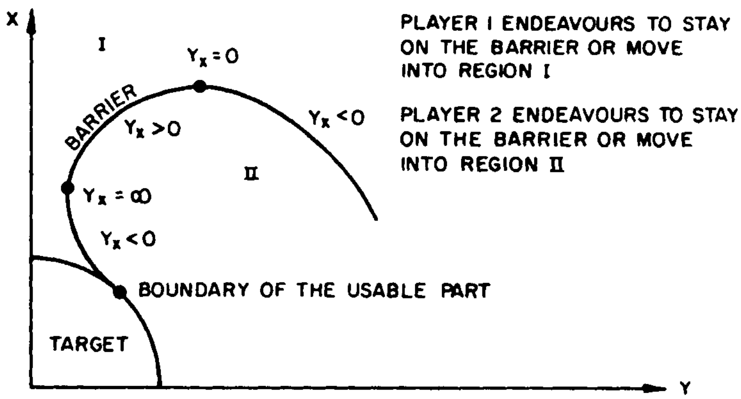

Figure 1. Sign of y_x on the barrier for $\theta \in (0, \pi)$

On a lower surface of the barrier, therefore, the roles of u_1 and u_2 are reversed so that the sign is reversed in equations (10) and (11). As the surface passes through $y_x = \infty$, however, the value of y_x changes from being very large and positive to very large and negative, or vice versa, and hence from equations (24) and (28) so do the values of y_θ and $yy_x + x + y_\theta$. Thus, a passage from a lower surface to an upper surface, and vice versa, does not change the actual control strategies unless y_θ or $yy_x + x + y_\theta$ are actually zero at $y_x = \infty$.

From equations (19), (26) and (27), we see that, in case 1, y_θ and $yy_x + x + y_\theta$ are zero along the boundary curve. This can be shown to be true for case 2 as well. Hence, it is not immediately obvious which part of the boundary corresponds to case 1 and which part to case 2. The second equation in equation (26) has two branches, however, one emanating from $\theta = \frac{1}{3}\pi$ and one emanating from $\theta = \frac{5}{3}\pi$. The same is true for case 2. Furthermore, from the data generated in case 2, it was found that y_θ in equation (25) has the correct sign, close to the boundary, for case 2 to lie between $\frac{1}{3}\pi$ and $\frac{5}{3}\pi$. Hence from $\frac{1}{3}\pi$ to $\frac{5}{3}\pi$, the surface satisfies case 2 as it emanates from the boundary of the usable part.

Similarly, from 0 to $\frac{1}{3}\pi$ and from $\frac{5}{3}\pi$ to 2π, the surface satisfies case 1 as it emanates from the boundary of the usable part. Alternatively, the same conclusion can be reached by examining the time derivative of y_θ along trajectories as they reach the boundary of the usable part, i.e.

$$(\mathrm{d}/\mathrm{d}t)\, y_\theta[x(t), y(t), \theta(t)] \quad \text{evaluated at } x_0(\theta_0),\ y_0(\theta_0), \theta_0.$$

Since the surface is periodic with period 2π (note that equations (24)–(28) are all periodic with period 2π), we can regard the latter as the single region $[-\frac{1}{3}\pi, \frac{1}{3}\pi]$.

From the above, we see that the trajectories corresponding to case 1 emanate (in reverse time) from the boundary between $[-\frac{1}{3}\pi, \frac{1}{3}\pi]$ in a direction parallel to the x-axis (as projected on the x, y plane). In the interval $[-\frac{1}{3}\pi, 0)$, these trajectories emanate from the target with positive slope and are terminated by the branch of (26), second equation, that emanates from $-\frac{1}{3}\pi$. In the interval $(0, \frac{1}{3}\pi)$, these trajectories emanate from the target with negative slope and thus form an undersurface that rolls over at the left envelope, again rolls over at the right envelope, and then terminates at the curve given by the second equation in (27). This is illustrated for the case $r = \frac{1}{3}$ in

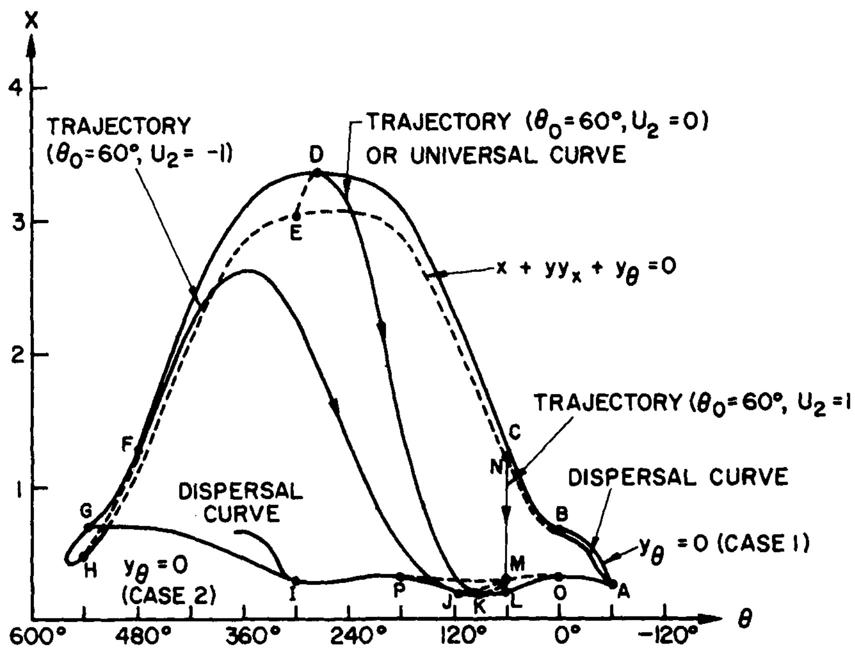

Figure 2. The projection of the barrier surface onto the (x, θ) plane

Figure 2 and in the cross-sections in Figure 3; see also Figure 4 for a three-dimensional computer drawing of the barrier surface.

At the point $\frac{1}{3}\pi$ separating cases 1 and 2, we thus have two trajectories meeting : the trajectories from the surface generated by cases 1 and 2 that terminate at $\theta_0 = \frac{1}{3}\pi$. Between these two trajectories, however, there exists a void (see Figure 2). In order to fill the void by continuing the surfaces of cases 1 and 2 beyond the $\theta = \frac{1}{3}\pi$ trajectories, the existence will therefore be required of a universal curve (see also Reference 1, Chapter 7). This curve would provide a new set of boundary conditions for equations (5)–(7) and equation (16) and is itself a trajectory of the system with $u_1 = 1$ and $u_2(t)$ assuming values in $[-1, 1]$. In order that $\mathscr{S}$ should satisfy equation (9) in the neighbourhood and on the case 1 side of the universal curve, we require that $y_\theta > 0$ be satisfied [see equation (11)]. Similarly, on the case 2 side of the universal curve, we require that $y_\theta < 0$.

We now prove the following proposition.

Proposition 1

If the curve along which cases 1 and 2 meet is to be a trajectory (universal curve), it is necessary that $y_\theta = 0$ across the curve (i.e. the derivative y_θ taken from both sides of the curve must be zero).

Proof. Let the universal curve be given by $\hat{x}(\theta)$, $\hat{y}(\theta)$. Then, in case 1, trajectories terminating at the universal curve will satisfy

$$(y-\tfrac{1}{2}\sin\theta)^2+(x-1+\tfrac{1}{2}\cos\theta)^2 = (\hat{y}-\tfrac{1}{2}\sin\theta)^2+(\hat{x}-1+\tfrac{1}{2}\cos\theta)^2 \tag{29}$$

for all applicable values of θ [cf. equation (21)]. Differentiating equation (31) with respect to θ, we obtain on the universal curve, i.e. at $x = \hat{x}$, $y = \hat{y}$

$$(\hat{y}-\tfrac{1}{2}\sin\theta)\frac{d\hat{y}}{d\theta}+(\hat{x}-1+\tfrac{1}{2}\cos\theta)\frac{d\hat{x}}{d\theta} = y_\theta(\hat{y}-\tfrac{1}{2}\sin\theta) \tag{30}$$

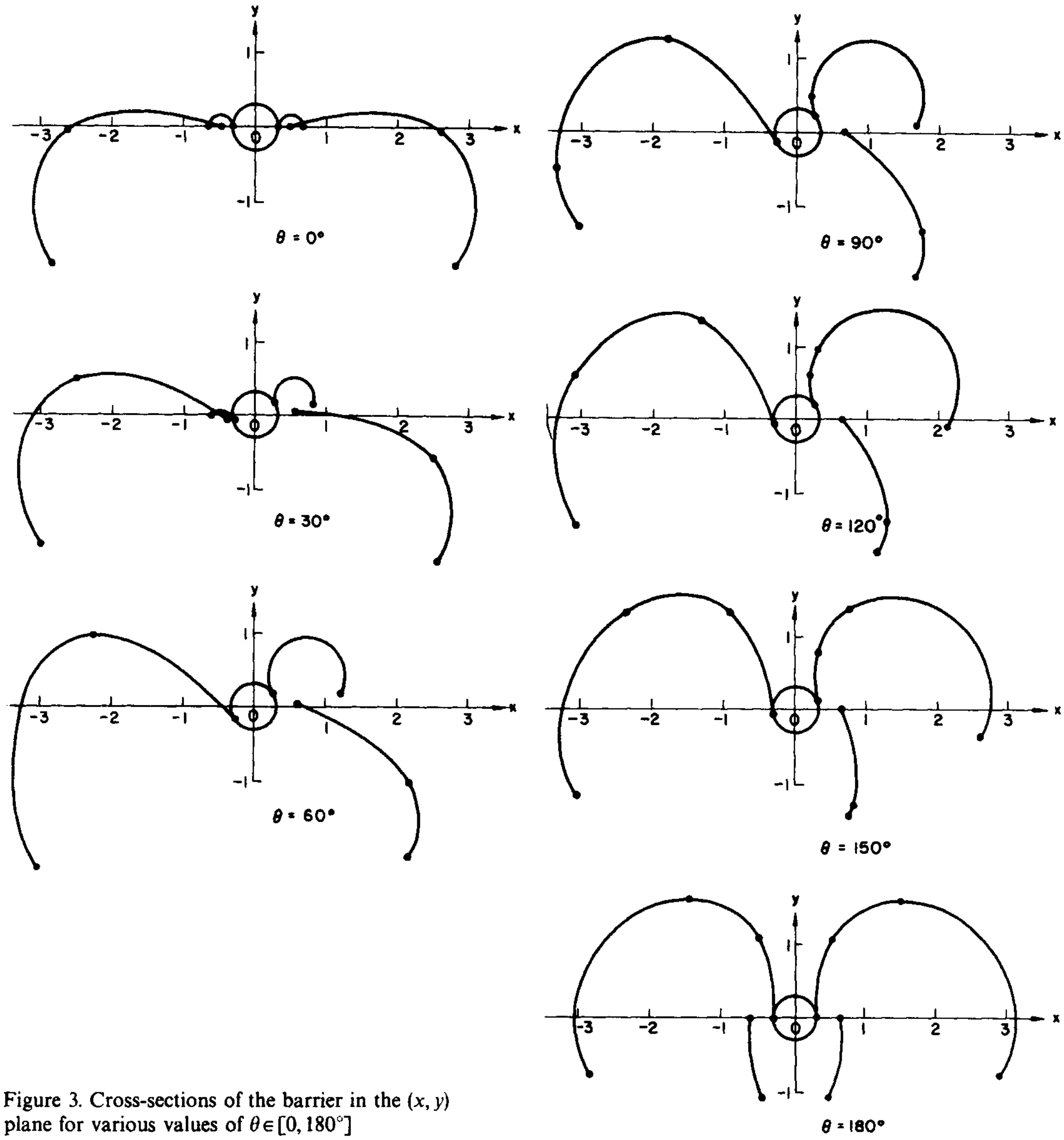

Figure 3. Cross-sections of the barrier in the (x, y) plane for various values of $\theta \in [0, 180°]$

Now, since we require $\hat{x}(\theta)$, $\hat{y}(\theta)$ to be a trajectory, it follows from equation (18) with $u_1 = 1$ that

$$\begin{aligned} \frac{d\hat{x}}{d\theta} &= (-\hat{y}+\tfrac{1}{2}\sin\theta)/(u_2-1) \\ \frac{d\hat{y}}{d\theta} &= (\hat{x}-1+\tfrac{1}{2}\cos\theta)/(u_2-1) \end{aligned} \tag{31}$$

which, when substituted in equation (30), implies

$$y_\theta(\hat{y}-\tfrac{1}{2}\sin\theta) = 0.$$

This in turn implies $y_\theta = 0$ since $\hat{y} = \frac{1}{2}\sin\theta$ only on the surface envelope; and, from the equations of motion (18), the surface envelope cannot be a trajectory. A similar but more involved argument leads to the same conclusion for case 2. □

Although Proposition 1 has been proved for the specific case of $\gamma = \frac{1}{2}, \beta = 1$, it is obvious that the same reasoning will apply to general γ and β.

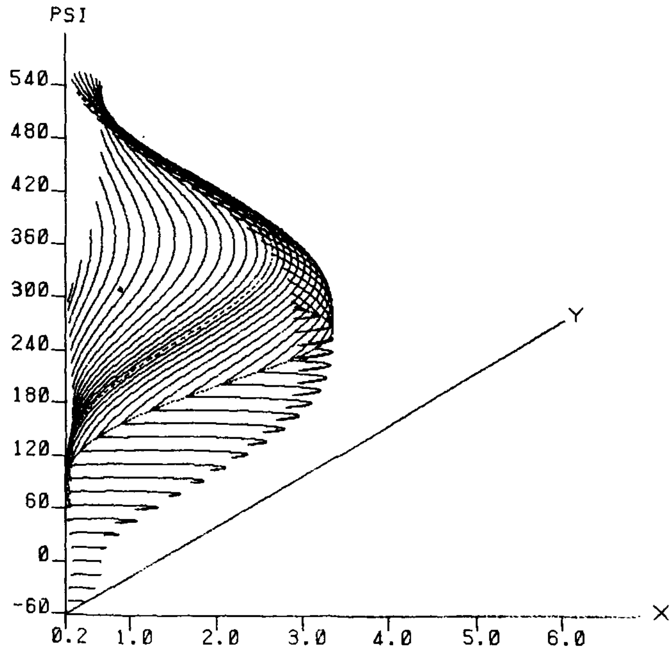

Figure 4. A three-dimensional view of the barrier surface

Remark

If y_θ is set equal to zero in the invariance argument (4), then equation (4), regarded as an equation for the curve $\hat{x}(\theta)$, $\hat{y}(\theta)$, in the (x, y) state space, is just the differential equation for the solution to the homicidal chauffeur game [cf. equations (8) and (9) with $\gamma = \frac{1}{2}$ in Reference 3]. That is, the universal curve projected on the (x, y) plane is the barrier for the homicidal chauffeur game. This simple corollary is in line with the conclusion reached by Isaacs in his analysis in Section 9.2 of Reference 1.

Using Proposition 1, we now prove the following proposition, which allows for construction of the universal curve as a solution to equation (1).

Proposition 2

On the universal curve, $u_2 = 0$.

Proof. From Proposition 1, $y_\theta = 0$ along the universal curve $\hat{x}(\theta)$, $\hat{y}(\theta)$. Hence, along the universal curve, we have

$$\left.\frac{\mathrm{d}^2 y_\theta}{\mathrm{d}t^2}\right|_{x = \hat{x}(\theta),\, y = \hat{y}(\theta)} = 0$$

This implies

$$(1-u_2)[\hat{y}\cos\theta-(1-\hat{x})\sin\theta] = \hat{y}\cos -(1-\hat{x})\sin\theta$$

so that either

$$u_2 = 0 \quad \textit{or} \quad \hat{y}\cos\theta = (1-\hat{x})\sin\theta$$

Since the latter solution satisfies the equations of motion (1) only when $\hat{y} = \gamma\sin\theta$, $\hat{x}-1 = \gamma\cos\theta$ and since this is the case when $(\hat{x}, \hat{y})$ is an equilibrium point (not a curve), the result follows. □

It should again be noted that Proposition 2 holds for general γ and β.

Thus, from Proposition 2, the universal curve is the solution to (1) with $u_1 = 1, u_2 = 0$. Also, from equation (8) for $\gamma = \frac{1}{2}$, $\beta = 1$ and general r, this solution, after a certain amount of simplification and after recalling that in this case $\theta_0 = \frac{1}{3}\pi$, becomes

$$\begin{aligned}\hat{x}(\theta) &= r\sin\theta+1-\cos(\theta-\tfrac{1}{3}\pi)-(\theta-\tfrac{1}{3}\pi)\sin\theta\\ \hat{y}(\theta) &= r\cos\theta+\sin(\theta-\tfrac{1}{3}\pi)-(\theta-\tfrac{1}{3}\pi)\cos\theta\end{aligned} \tag{32}$$

As in constructing surfaces emanating from the boundary, we now construct surfaces emanating from the universal curve for the two cases $u_2 = 1$ and $u_2 = -1$. The surface in case $u_2 = 1$ lies between the universal curve and the trajectory emanating from $\theta_0 = \frac{1}{3}\pi$ for $u_2 = 1$ (see Figure 2) and will for the same reason join smoothly. Of course, we need to check that y_θ has the correct signs for equation (9) to hold. As it turns out in our case, the sign of y_θ is correct: this is actually a consequence of Proposition 1. Specifically, using the universal curve as a boundary condition, we can as before derive equations describing some of the characteristics of the surface.

Case 3 : $u_1 = 1$, $u_2 = 1$, *universal curve boundary*

As in deriving equations (24)–(27), we can similarly derive

$$\begin{aligned}y &= \tfrac{1}{2}\sin\theta\pm\sqrt{\{[r-\tfrac{1}{2}(\sqrt{3}+\theta-\tfrac{1}{3}\pi)]^2-(x-1+\tfrac{1}{2}\cos\theta)^2\}}\\ y_x &= \frac{2-2x-\cos\theta}{2y-\sin\theta}\\ y_\theta &= \frac{(x-1)\sin\theta+y\cos\theta-[r-\tfrac{1}{2}(\sqrt{3}+\theta-\tfrac{1}{3}\pi)]}{2y-\sin\theta}\\ x+yy_x+y_\theta &= \frac{2y-\sin\theta-r+\tfrac{1}{2}(\sqrt{3}+\theta-\tfrac{1}{3}\pi)}{2y-\sin\theta}\end{aligned} \tag{33}$$

which results in the envelope curves

$$y_x = \infty \Rightarrow \quad \begin{array}{c} x^+ = 1-\tfrac{1}{2}\cos\theta+r-\tfrac{1}{2}(\sqrt{3}+\theta-\tfrac{1}{3}\pi)\\ \text{or}\\ x^- = 1-\tfrac{1}{2}\cos\theta-r+\tfrac{1}{2}(\sqrt{3}+\theta-\tfrac{1}{3}\pi)\end{array} \tag{34}$$

and the stopping criteria

$$yy_x+x+y_\theta = 0 \Rightarrow \quad \begin{array}{c} x = \tfrac{1}{4}-\tfrac{1}{2}\cos\theta-\tfrac{1}{2}\sqrt{(3)}\,r-\tfrac{1}{4}\sqrt{3}(\tfrac{1}{3}\pi-\theta)\\ \text{or}\\ x = \tfrac{1}{4}-\tfrac{1}{2}\cos\theta+\tfrac{1}{2}(\sqrt{3})\,r+\tfrac{1}{4}\sqrt{3}(\tfrac{1}{3}\pi-\theta).\end{array} \tag{35}$$

In equations (33), we have omitted $y_\theta = 0$, and the resulting curves are not effective in determining this part of the surface. In fact, $y_\theta > 0$ on this part of the surface. It can also be shown that the first alternative in equation (35) is the relevant one.

Case 4 : $u_1 = 1$, $u_2 = -1$, *universal curve boundary*

As in deriving equations (23) and (28), we can derive

$$\begin{aligned} x &= 1+\tfrac{1}{2}\cos\theta-\cos\tfrac{1}{2}(\theta-\theta_0)-\tfrac{1}{2}\cos\tfrac{1}{2}(\theta+\theta_0)+\hat{x}\cos\tfrac{1}{2}(\theta-\theta_0)+\hat{y}\sin\tfrac{1}{2}(\theta-\theta_0) \\ y &= -\tfrac{1}{2}\sin\theta+\sin\tfrac{1}{2}(\theta-\theta_0)+\tfrac{1}{2}\sin\tfrac{1}{2}(\theta+\theta_0)-\hat{x}\sin\tfrac{1}{2}(\theta-\theta_0)+\hat{y}\cos\tfrac{1}{2}(\theta-\theta_0) \end{aligned} \tag{36}$$

and

$$y_x = c(\theta,\theta_0)/d(\theta,\theta_0) \tag{37}$$

where

$$\begin{aligned} c(\theta,\theta_0) &= -\tfrac{1}{2}y-\tfrac{1}{4}\sin\theta+\tfrac{1}{2}\sin\tfrac{1}{2}(\theta+\theta_0)+\frac{d\hat{x}(\theta_0)}{d\theta}\cos\tfrac{1}{2}(\theta-\theta_0)+\frac{d\hat{y}(\theta_0)}{d\theta}\sin\tfrac{1}{2}(\theta-\theta_0) \\ d(\theta,\theta_0) &= \tfrac{1}{2}x-\tfrac{1}{2}-\tfrac{1}{4}\cos\theta+\tfrac{1}{2}\cos\tfrac{1}{2}(\theta+\theta_0)-\frac{d\hat{x}(\theta_0)}{d\theta}\sin\tfrac{1}{2}(\theta-\theta_0)+\frac{d\hat{y}(\theta_0)}{d\theta}\cos\tfrac{1}{2}(\theta-\theta_0) \end{aligned} \tag{38}$$

Also

$$y_\theta = \tfrac{1}{2}-\tfrac{1}{2}x-\tfrac{1}{4}\cos\theta+\tfrac{1}{2}y_x[\tfrac{1}{2}(\sin\theta)-y]$$

and (39)

$$x+yy_x+y_\theta = \tfrac{1}{2}+\tfrac{1}{2}x-\tfrac{1}{4}\cos\theta+\tfrac{1}{2}y_x[\tfrac{1}{2}(\sin\theta)+y]$$

As before, the expressions in equations (36)–(39) are evaluated for a grid of points in the (θ,θ_0) plane, and the surface is constructed to satisfy equations (10) and (11). Thus, for example in case 4, $y_\theta<0$ and $x+yy_x+y_\theta>0$ as the surface emanates from the universal curve, but the surface is terminated when the condition $x+yy_x+y_\theta\leqslant 0$ is first encountered.

Finally, the structure of the barrier depicted in Figure 2 is as follows. AB is the curve $y_\theta = 0$ for case 1. BC, CD, DF and FG are, respectively, the right envelope curves for cases 1, 3, 4 and 2. BN, NE, EH and HG are, respectively, the curves $x+yy_x+y_\theta = 0$ for cases 1, 3, 4 and 2 which terminate that portion of the surface which at the envelope curve folds under the main body of the surface (see cross-sections in Figure 3). OL, LK, KJ and JP are, respectively, the left envelope curves for cases 1, 3, 4 and 2, whereas AOMPI is the boundary of the usable part of the target. It should be noted that the portion OMPJKLO of the surface is a fold under the main surface (see right hand cross-sections in Figure 3 for $x>0$). Finally, IG is the curve $y_\theta = 0$ for case 2.

The surface is periodic with period 2π. Surfaces identical to the one depicted in Figure 2 adjoin it on either side with the points A and I translated by 360° to the left and to the right, respectively. The dispersal curves at I and A represent the intersection of the surface with its neighbouring surface (see cross-sections 0° and 30° in Figure 3). Since the dispersal curve lies in those parts of the surface generated by both case 1 and case 2, we have the option of playing either $u_2 = -1$ or $u_2 = 1$ from any point on the dispersal curve. Note that at I the dispersal curve terminates well within the interior of the surface, while at AB the dispersal curve terminates at the edge of the surface. Also note that from the initial state H on the barrier (H corresponds to the lowest point of the inner right-hand surface of the cross-section $\theta = 180^\circ$ in Figure 3) the evader, travelling along the $\theta_0 = 60^\circ$ trajectory, will undergo a comparatively large change of relative heading (namely 480°) before brushing past the pursuers target.

4. DEPENDENCE ON r

4.1. *The existence of the barrier surface*

We comment on the existence of a barrier surface for a variable capture radius parameter r. To do this, recall that the barrier surface is made up of four different parts depending upon whether the surface emanates from the boundary of the usable part (b.u.p.) or from the universal curve (u.c.) and also upon whether the parameter u_2 has the values $u_2 = +1$ or $u_2 = -1$ ($u_1 = 1$ throughout).

(i) Consider that part of the barrier surface which emanates from the b.u.p. and the parameter $u_2 = +1$. From equation (22), we see that this surface is made up of circles with centres at

$$a(\theta) = (1 - \tfrac{1}{2}\cos\theta, \tfrac{1}{2}\sin\theta).$$

Hence, the barrier is 'swallowed up' at the heading θ if and only if $a(\theta) \in T$, which implies that

$$r^2 \geqslant \tfrac{5}{4}\cos\theta$$

Since on this part of the surface $-\tfrac{1}{3}\pi \leqslant \theta \leqslant \tfrac{1}{3}\pi$, we have that

$$\tfrac{1}{2} \leqslant \cos\theta \leqslant 1$$

Thus, if $r \geqslant \tfrac{1}{2}\sqrt{3}$, then this part of the barrier surface does not exist. If $r \geqslant \tfrac{1}{2}$, then the barrier surface is partially 'swallowed up' by the target set. This is not the case if $r < \tfrac{1}{2}$, and we see that this part of the barrier surface is complete.

(ii) Next, consider those parts of the barrier surface which emanate from the u.c. To this end we compute the value of

$$F \triangleq x + yy_x + y_\theta$$

along the u.c. It turns out that at the initial point of the u.c. on the target, we always have

$$F = F' = 0$$

Here, we denoted by F' the total derivative with respect to θ of the expression F along the u.c. at the initial point on the target. Furthermore, since on the u.c. we know that $y_\theta = 0$, we only have to consider $x + yy_x$. This is equivalent to considering $(\mathrm{d}/\mathrm{d}x)(x^2 + y^2)$. Upon computing F'' and requiring that $F'' > 0$, we eventually arrive at the critical target radius of $r = \tfrac{1}{2}\sqrt{3}$. Thus, if $r > \tfrac{1}{2}\sqrt{3}$, then the u.c. is swallowed up by the target set. This in turn implies that the two parts of the barrier surface corresponding to $u_2 = +1$, $u_2 = -1$ which emanate from the u.c. do not exist.

(iii) Consider the barrier surface part which emanates from the b.u.p. and which corresponds to the parameter value $u_2 = -1$. The surface termination criteria are now given by the expression F for $60^\circ \leqslant \theta \leqslant 180^\circ$ or by the value of y_θ for $180^\circ \leqslant \theta \leqslant 300^\circ$. Whereas now on the target set $F = y_\theta = 0$, we also have, however, that $F' \neq 0$ and $y'_\theta \neq 0$ and have the correct values for all r. Thus, it seems that this part of the surface will always exist for all values of r. Hence, if $r < \tfrac{1}{2}$, the barrier surface is complete, while for $r \geqslant \tfrac{1}{2}$ only parts of the surface exist.

Also, for the more general case, the following can easily be shown.

Proposition 3. Assume that $\beta = 1$ and $\gamma < 1$. Then, if $r < 1-\gamma$, the barrier is complete. □

If, however, $\gamma > 1$, then the problem must be reconsidered with the coordinate system fixed at the position of the evader.

4.2. *The minimum radius required for capturability*

For the sake of completeness, we also consider the dependence of full state space capturability on the target radius. This problem has been thoroughly investigated,[3] and here we will confine our attention to only interpreting the results in terms of the barrier surface in the reduced state space $R^2 \times S^1$. This interpretation is contained in the following proposition.

Proposition 4

$$\text{(barrier surface)} \cap \text{(positive } y\text{-axis)} = \emptyset \tag{40}$$

is equivalent to full state space capturability. □

For a general set of parameters r, β and γ, it is apparent from the equations of motion (1) and the boundary conditions (17) that the only parts of the barrier surface of significance in equation (40) are those that are made up of trajectories for which $u_1 = 1$, $u_2 = 1$ and which either terminate on the boundary of the usable part of the target $[\theta_0 \in (-\cos^{-1}\gamma, \cos^{-1}\gamma)]$ or join the universal curve which terminates on the boundary of the usable part of the target at $\theta_0 = \cos^{-1}\gamma$ (see Figure 4). It is thus also apparent from the third equation in (1) that, if $\beta > 1$, then only the former part of the barrier surface is significant in equation (40).

Kinematically, u_1 and u_2 correspond to P and E turning hard in the same direction while the universal curve corresponds to P turning hard towards E and E coasting. Hence, if P is chasing E along a line of sight, E is sufficiently far from P, and equation (40) is violated at $\hat{y}$ say, then E can allow P to close in until the gap is just greater than $\hat{y}$ whereupon a tight turn followed, if applicable, by a coasting motion will allow E to safely skim past P's target.

Once the full barrier has been constructed, Proposition 4, by an examination of the 0° cross-section, provides an immediate check whether full state space capturability holds. For example, in the 0° cross-section in Figure 3, we see that full state space capturability holds for the set of parameters used to construct the specific barrier considered in this paper. Furthermore, it follows from equation (17) that, if $\gamma \leqslant \frac{1}{2}$, then the y coordinate of the boundary of the usable part is non-positive. Also, since the barrier emanates tangentially from the target, it follows from equation (16) that equation (40) holds for all r whenever $\gamma \leqslant \frac{1}{2}$ and $\beta = 1$.

5. CONCLUDING REMARKS

We employ an invariance argument and the differential geometric expression for the normal to a surface in order to obtain a first-order PDE which is easy to solve for the barrier surface. The geometric argument readily yields the explicit termination conditions for the barrier. The geometric meaning of the universal curve is now clarified. It is the locus of $\partial y/\partial\theta = 0$ or $\partial x/\partial\theta = 0$. We also explicitly obtain the dispersal curve. Note also that we have trajectories on the barrier surface along which a large change of relative heading occurs, i.e. a change of more than 360°.

It is now a straightforward matter to actually construct the barrier for general values of the game parameters by noting that, if (the speed ratio) $\gamma > 1$ (see also Reference 12), then the coordinate axes must be fixed relative to the evader's motion. We have applied our geometrical method and constructed the complete closed barrier surface (including the dispersal curve) also in this case.[8] It is interesting though to remark that apparently such a closed ('collision avoidance') barrier surface has a simpler geometry and therefore was easier to construct.

Finally, our method of analysis is also readily adaptable to non-circular pursuer targets.

REFERENCES

1. Isaacs, R., *Differential Games*, Wiley, New York, 1965.
2. Merz, A. W., 'The game of two identical cars', *Journal of Optimization Theory and Applications*, **9**, 324–343 (1972).
3. Breakwell, J. V. and A. W. Merz, 'Minimum required capture radius in a coplanar model of the aerial combat problem', *AIAA J.*, **15**, 1089–1094 (1977).
4. Getz, W. M. and M. Pachter, 'Two-target pursuit–evasion differential games in the plane', *Journal of Optimization Theory and Applications* (to appear).
5. Olsder, G. J. and J. V. Breakwell, 'Role determination in aerial dogfight', *Int. J. of Game Theory*, **3**, 47–66 (1974).
6. Merz, A. W. and D. S. Hague, 'Coplanar tail-chase aerial combat as a differential game', *AIAA J.*, **15**, 1419–1423 (1977).
7. Peng, W. Y. and T. L. Vincent, 'Some aspects of aerial combat', *AIAA J.*, **13**, 7–11 (1975).
8. Getz, W. M. and M. Pachter, 'Capturability in a two-target game of two cars', *TWISK 115*, National Research Institute for Mathematical Sciences of the CSIR, Pretoria, South Africa, Oct. 1979.
9. Hàjek, O., *Pursuit Games*, Academic Press, New York, 1975.
10. Garabedian, P. R., *Partial Differential Equations*, Wiley, New York, 1964, Chapter 2.
11. Blaquière, A., F. Gérard and G. Leitmann, *Quantitative and Qualitative Games*, Academic Press, New York and London, 1969.
12. Vincent, T. L., E. M. Cliff, W. T. Grantham and W. Y. Peng, 'Some aspects of collison avoidance', *AIAA J.*, **12**, 3–4 (1974).