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# Constructive Barrier Certificates With Applications to Fixed-Wing Aircraft Collision Avoidance

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*Abstract*— **In this paper we discuss how to construct a barrier certificate for a control affine system subject to actuator constraints. We motivate this discussion examining collision avoidance for fixed-wing aircraft. In particular, we show the theoretical development in this paper can be used to create a barrier certificate that can ensure that vehicles will not collide. We demonstrate in simulation how the constructed barrier certificate keeps vehicles at a safe distance from each other even though their nominal paths would otherwise cause collisions.**

## I. INTRODUCTION

Fixed-wing unmanned autonomous vehicles (UAVs) often have to satisfy inherent constraints (e.g., never deplete energy or fuel, do not collide with other agents, maintain network connectivity) in order to operate in real-world settings. However, because fixed-wing UAVs have a limited turning radius and non-zero minimum velocity, ensuring that these constraints are always satisfied can be difficult. For instance, two approaching vehicles must avoid each other without stopping or changing direction faster than the vehicles can turn.

As a result, a number of solutions to this problem have been proposed, and we provide a brief summary of methods related to UAV collision avoidance here (see [1] and references therein for a more complete review of robot obstacle avoidance methods). For example, navigation functions rely on following the gradient of a function that promotes attraction to a goal and repulsion away from other vehicles, and have been applied to UAV dynamics in [2]. Velocity obstacles [3] provide a geometric framework to select safe velocities. The dynamic window approach, originally introduced in [4] for static obstables and adapted to moving obstacles in [5], uses circular arcs for trajectories and limits the set of allowable velocities to enable a quick optimization of the control input. Optimal control strategies have also been formulated, as in [6] where safe control inputs are derived by solving an optimal control problem posed as a two player game. Similarly, Model Predictive Control (MPC) is a computationally cheaper method that solves an optimal control problem over a limited horizon. This has been applied to decentralized fixed wing UAV collision avoidance in [7].

In some cases, these approaches can provide guarantees that two UAVs will not collide. For instance, [8] analyzes an algorithm similar to velocity obstacles and shows that when the vehicles start collision free they will remain collision free even when there are more than two vehicles. [2] also provides a collision avoidance guarantee in the limited case when there is no path following goal and the vehicles begin at a precise distance from each other. [9] analyzes a set of traffic rules that ensure neighboring vehicles never enter the circle of another vehicle described by a constant speed right turn plus a safety radius. The authors elegantly show that this approach maintains safety while ensuring vehicles can reach a goal configuration. However, this safety condition may lead to somewhat conservative safe regions and significantly restricts allowable controls. In a more general case, the optimal control formulation in [6] allows for collision avoidance guarantees, but it is calculation intensive as it requires numerically solving the Hamilton-Jacobi-Bellman equations over an infinite horizon. [10] discusses sufficient conditions for guaranteeing collision avoidance by using Lyapunov-like functions. Because we would like a formal guarantee of collision avoidance that is computationally feasible and minimally invasive we discuss in this paper how to apply barrier certificates (e.g., [11], [12]) to the UAV collision avoidance problem.

Barrier certificates (also referred to as barrier functions) provide guarantees that a system will stay safe for all future times. Further, under some assumptions detailed in Section II, they can be formulated as a Quadratic Program for fast online computation of safe control inputs [12]. Given such safety guarantees, barrier certificates have been applied to a set of problems including collision avoidance for autonomous agents ([13], [14]), bipedal robots ([15], [16]), adaptive cruise control and lane following ([17], [12], [18], [19]), and in mobile communication networks [20].

However, barrier certificates rely on being able to find a function satisfying certain properties in order for safety set invariance to be guaranteed. For systems like a fixed wing UAV with actuator constraints, nonlinear dynamics, and nonlinear safety constraints, generating such a function can be difficult. In this respect they are similar to Lyapunov functions. They provide guarantees when a system designer can find appropriate functions but they may be difficult to construct. Thus, this paper describes a general method for barrier certificate construction that will be used to show how to ensure that two UAVs do not collide. It should be noted that although the collision avoidance problem for fixed wing aircraft is used as a motivating example in this paper, the theoretical developments are not specific to this problem.

This paper is organized as follows: Section II introduces the mathematical background for barrier certificates. Section III presents a motivating example, where it is difficult to come up with a barrier certificate. Section IV develops a

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general solution to this problem and discusses how to apply it to the fixed-wing UAV collision avoidance scenario in a computationally efficient way. Section V concludes.

## II. BARRIER CERTIFICATES BACKGROUND

This paper is motivated by collision avoidance for fixed-wing aircraft. We model the dynamics using planar equations, as was done for instance in [21]. Consider two vehicles, indexed with $a$ and $b$, with states $x_a$ and $x_b$, respectively. The state of vehicle $a$ is $x_a = \begin{bmatrix} p_{a,x} & p_{a,y} & \theta_a \end{bmatrix}^T$, where $p_{a,x}$ and $p_{a,y}$ are the position and $\theta_a$ is the orientation of vehicle $a$, respectively. Vehicle $b$ is defined similarly. The dynamics of vehicle $a$ are given by

$$\dot{x}_a(t) = \begin{bmatrix} v_a(t)\cos(\theta_a(t)) \\ v_a(t)\sin(\theta_a(t)) \\ \omega_a(t) \end{bmatrix}, \tag{1}$$

where $v_a$ and $\omega_a$ are the translational and angular velocities subject to actuator constraints $v_{min} \leq v_a \leq v_{max}$, $\omega_a \leq |\omega_{max}|$. We represent the overall system state as the concatenation of the two vehicle's states:

$$x = \begin{bmatrix} x_a^T & x_b^T \end{bmatrix}^T. \tag{2}$$

The dynamics of the system are then

$$\dot{x} = \begin{bmatrix} \dot{x}_a^T & \dot{x}_b^T \end{bmatrix}^T. \tag{3}$$

The nominal control inputs for vehicles $a$ and $b$ (perhaps chosen for other goals such as navigation or stability) may result in the vehicles colliding. In this paper, we discuss how to use a barrier certificate to minimally alter the nominal control input $v_a$, $\omega_a$, $v_b$, and $\omega_b$ in order to ensure that the vehicles maintain some minimal separation $D_s$ for all times.

For generality, we can represent the above problem using the formulation from [12]. In particular, (3) is a control affine system

$$\dot{x} = f(x) + g(x)u, \tag{4}$$

where $f$ and $g$ are locally Lipschitz, $x \in \mathbb{R}^n$, and $u \in U \subseteq \mathbb{R}^m$. We assume that solutions are forward complete, meaning that solutions are well defined for all $t \geq 0$. To map (4) to (3) we can let $u = \begin{bmatrix} v_a & \omega_a & v_b & \omega_b \end{bmatrix}^T$ and $f(x) = 0_{6\times 1}$. $g(x)$ has zero entries except for $[g(x)]_{1,1} = \sin(\theta_a)$, $[g(x)]_{2,1} = \cos(\theta_a)$, $[g(x)]_{3,2} = [g(x)]_{6,4} = 1$, $[g(x)]_{4,3} = \sin(\theta_b)$, and $[g(x)]_{5,3} = \cos(\theta_b)$ where $[g(x)]_{i,j}$ is the entry in the $i^{th}$ row and $j^{th}$ column of $g(x)$.

We will represent collision avoidance using a safe set, which is a superlevel set of a continuously differentiable function, denoted by $h$. The set and its complement are defined by

$$\mathcal{C}_h = \{x \in \mathbb{R}^n : h(x) \geq 0\} \tag{5}$$

$$\partial\mathcal{C}_h = \{x \in \mathbb{R}^n : h(x) = 0\} \tag{6}$$

$$\mathcal{C}_h^C = \{x \in \mathbb{R}^n : h(x) < 0\} \tag{7}$$

A barrier certificate (also called a barrier function) can be used to show that the safe set is forward invariant, meaning that any state that starts in the safe set remains in the safe set for all future time. In the following definition $\mathcal{L}_f h$ and $\mathcal{L}_g h$ denote the Lie derivatives.

**Definition 1.** [12] Given a set $\mathcal{C} \subseteq \mathbb{R}^n$ defined in (5)-(7) for a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, the function $h$ is called a *zeroing control barrier function (ZCBF)* defined on a set $\mathcal{D}$ with $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$, if there exists an extended class $\mathcal{K}$ function $\alpha$ such that

$$\sup_{u \in U}[L_f h(x) + L_g h(x)u + \alpha(h(x))] \geq 0. \tag{8}$$

The admissable control space is defined as

$$K_h = \{u \in U : L_f h(x) + L_g h(x)u + \alpha(h(x)) \geq 0\}. \tag{9}$$

The following result gives sufficient conditions for the forward invariance of $\mathcal{C}_h$.

**Theorem 1.** *[12] Given a set $\mathcal{C}_h \subseteq \mathbb{R}^n$ defined in (5)-(7) for a continuously differentiable function $h$, if $h$ is a ZCBF on $\mathcal{D}$, then any Lipschitz continuous controller $u : \mathcal{D} \to U$ such that $u(x) \in K_h(x)$ will render the set $\mathcal{C}$ forward invariant.*

One way to make sure that $u \in K_h(x)$ is to use quadratic programming (QP) [12]. In particular, assume a nominal controller $\hat{u}$ is available that may be the result of, for instance, path planning or other non safety-critical performance goals. Note that $\hat{u}$ may not necessarily be in $K_h$. Further, assume $U$ can be represented as a system of linear constraints $Au \geq b$. Then at every timestep we can then choose a control input $u$ that is close to $\hat{u}$ in a least squares sense but maintains system safety with the following:

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \ & \frac{1}{2}\|u - \hat{u}\|^2 \\ \text{s.t.} \ & L_f h(x) + L_g h(x)u + \alpha(h(x)) \geq 0 \\ & Au \geq b. \end{aligned} \tag{10}$$

The advantage of this formulation is that it is a QP problem. In other words, the cost is quadratic and the constraints are linear in the decision variable $u$. As such, it supports fast, online computation.

## III. MOTIVATING EXAMPLE

In this section we discuss how to apply Theorem 1 to the fixed-wing UAV collision avoidance problem and find that it can be difficult to generate a ZCBF. Without a ZCBF we will not be able to guarantee that the fixed-wing aircraft will not collide. While this example provides a concrete problem, we note that the theoretical developments in this paper are not constrained to this particular scenario.

Given that we want to guarantee that two vehicles will not collide, consider a candidate ZCBF, $h$, that is the squared distance between the two vehicles in excess of some minimum safety distance $D_s$:

$$h(x(t)) = (p_{a,x}(t) - p_{b,x}(t))^2 + (p_{a,y}(t) - p_{b,y}(t))^2 - D_s^2 \tag{11}$$

To show why $h$ defined in (11) does not work as a ZCBF, let the vehicles be on the $x$-axis pointing toward the origin with a minimum separation distance of $D_s > 0$ so that $x_a = \begin{bmatrix}-D_s/2 & 0 & 0\end{bmatrix}^T$ and $x_b = \begin{bmatrix}D_s/2 & 0 & \pi\end{bmatrix}^T$. Note that $x = \begin{bmatrix}x_a^T & x_b^T\end{bmatrix}^T \in C_h$. For the given state, we have $h(x) = 0$, and

$$\begin{aligned} &L_f h(x) + L_g h(x)u + \alpha(h(x)) \\ &\quad = 2(p_{a,x}(t) - p_{b,x}(t))(v_a \cos\theta_a(t) - v_b \cos\theta_b(t)) \\ &\qquad + 2(p_{a,y}(t) - p_{b,y}(t))(v_a \sin\theta_a(t) - v_b \sin\theta_b(t)) \\ &\quad = -2D_s(v_a + v_b). \end{aligned}$$

We can then check whether $h$ satisfies (8):

$$\begin{aligned} &\sup_{u\in U}[L_f h(x) + L_g h(x)u + \alpha(h(x))] \\ &\quad = \sup_{u\in U}[-2D_s(v_a + v_b) + \alpha(0)] \\ &\quad = -4D_s v_{min}. \end{aligned} \tag{12}$$

Since $v_{min} > 0$ and $D_s > 0$, equation (12) does not satisfy (8) so $h$ defined in (11) is not a ZCBF. The intuitive problem with this candidate ZCBF is that it does not account for the fact that by the time the vehicles are close to colliding, it may be too late to avoid each other due to the limited turning radius and positive minimum velocity.

An alternative candidate ZCBF could be the Euclidean distance, not at the current time, but at some time in the future. This is the approach taken for instance in [13], which considers the collision avoidance problem with double integrator dynamics. However, in the case of unicycle dynamics it is difficult to calculate such an input that minimizes the Euclidean distance for all future time.

Another approach for generating a ZCBF is discussed for instance in ([17], [22], [11]), where a sum of squares decomposition [23] is used. In these cases an initially conservative estimate for a barrier certificate is found and the associated safe set is iteratively enlarged. The sum of squares decomposition applies when the system dynamics are polynomials (i.e., $f$ and $g$ are vectors of polynomials) so it does not readily apply to the dynamics in (3).

The path pursued in this paper has similarities to [16], which also examines how to generate a barrier function by ensuring the admissable control space is non-empty for any $x$ in the safe set. It does this with a backstepping approach but does not consider constraints on the input. This paper also has similarities to [19], where unicycle dynamics are considered in the context of lane keeping and speed constraints. In that context the dynamics are simplified by considering a point slightly in front of the vehicle.

## IV. BARRIER CERTIFICATE CONSTRUCTION

### A. BARRIER CONSTRUCTION

In this section we discuss a method for constructing a barrier certificate that can be applied to the UAV collision avoidance problem. First, we introduce a *safety function* $\rho : \mathcal{D} \to \mathbb{R}$ that is used to represent the safety objective we want to guarantee. In the example from Section III,

$$\rho(x(t)) = (p_{a,x}(t) - p_{b,x}(t))^2 + (p_{a,y}(t) - p_{b,y}(t))^2 - D_s^2. \tag{13}$$

Second, we introduce a *nominal evading maneuver* $\gamma : \mathcal{D} \to U$ that represents the control input associated with an evading maneuver. We allow $\gamma$ to be a function of the state to allow it to be a feedback law. Section IV-B discusses specific examples of $\gamma$ for the UAV collision avoidance problem. For now, assuming $\gamma$ has been selected, we introduce a candidate ZCBF, paramerized by $\rho$ and $\gamma$, as

$$h(x(t);\rho,\gamma) = \inf_{\tau\in[0,\infty)} \rho(\hat{x}(t+\tau)), \tag{14}$$

where the semicolon indicates $h$ is a function of $x$ but parameterized by $\rho$ and $\gamma$. We define $\hat{x}$ by

$$\hat{x}(t+\tau) = x(t) + \int_0^\tau \dot{\hat{x}}(t+\eta)d\eta, \tag{15}$$

where

$$\dot{\hat{x}}(t+\tau) = f(\hat{x}(t+\tau)) + g(\hat{x}(t+\tau))\gamma(\hat{x}(t+\tau)). \tag{16}$$

Here, $h$ measures how close the state will get to the boundary of the safe set assuming $\gamma$ is used as the control input for all future time. We first need to establish the conditions under which $h$ differentiable. To do this, we assume that $h$ has a unique $x$ minimizer. In other words, there is a unique $x_{\min} \in \mathcal{D}$ such that $h(x) = \rho(x_{\min})$ where $x_{\min} = \hat{x}(t+\tau)$ for at least one $\tau \geq 0$. See the appendix for the proof.

**Lemma 1.** *Assume $h$ is defined in (14) and is parameterized by $\rho : \mathcal{D} \to \mathbb{R}$ and $\gamma : \mathcal{D} \to U$. Let $h$ have a unique $x$ minimizer for all $x \in \mathcal{D}$, $\rho$ be continuously differentiable, and $\gamma$ be such that $f(x)+g(x)\gamma(x)$ is continuously differentiable. Then $h$ is continuously differentiable.*

*Remark* 1. For cases where $h$ has multiple $x$ minima at $x_{\min_1}, \ldots, x_{\min_k}$ the derivative will not necessarily be smooth. See [24] for handling this case.

In Section III we saw that we could not use the Euclidean distance for a ZCBF. The central problem with using $h$ defined in (11) was that $K_h$ could be empty even though the candidate ZCBF was non-negative. In other words, $h$ could be non-negative but there was no control input available to keep the system safe. The following Lemma shows that with $h$ defined in (14), this situation is impossible.

**Lemma 2.** *Assume $h$ is defined in (14) and is parameterized by $\rho : \mathcal{D} \to \mathbb{R}$ and $\gamma : \mathcal{D} \to U$. Let $h$ have a unique $x$ minimizer for all $x \in \mathcal{D}$, $\rho$ be continuously differentiable, and $\gamma$ be such that $f(x)+g(x)\gamma(x)$ is continuously differentiable. Then $\gamma(x) \in K_h(x)$ for all $x \in \mathcal{C}_h$.*

*Proof.* Because $x \in \mathcal{C}_h$, $h(x) \geq 0$ so $\alpha(h(x)) \geq 0$. Further, from Lemma 1, $h$ is differentiable. Further, note that $L_f h(x) + L_g h(x)\gamma$ is the derivative along the trajectory of $\hat{x}$. In other words,

$$\begin{aligned} &L_f h(x(t)) + L_g h(x(t))\gamma(x(t)) \\ &= \lim_{a\to 0^+} \frac{1}{a}\left(\inf_{\tau\in[a,\infty)} \rho(\hat{x}(t+\tau)) - \inf_{\tau\in[0,\infty)} \rho(\hat{x}(t+\tau))\right). \end{aligned} \tag{17}$$

Consider the term inside the parenthesis on the right hand side of (17)

$$\inf_{\tau\in[a,\infty)} \rho(\hat{x}(t+\tau)) - \inf_{\tau\in[0,\infty)} \rho(\hat{x}(t+\tau)) \tag{18}$$

and notice that it is the subtraction of an infimum of the same function $\rho$ evaluated on two different intervals. Further, note that the first interval is a subset of the second interval since $a$ approaches 0 from above. Thus, the term inside the parenthesis on the right hand side of (17) is positive so $L_f h(x) + L_g h(x)\gamma(x) \geq 0$. We can then conclude that $L_f h(x)+L_g h(x)\gamma(x)+\alpha(h(x)) \geq 0$ so $\gamma(x) \in K_h(x)$. □

We can now state the main result of the paper:

**Theorem 2.** *Assume $h$ is defined in (14) and is parameterized by $\rho : \mathcal{D} \to \mathbb{R}$ and $\gamma : \mathcal{D} \to U$. Let $h$ have a unique $x$ minimizer for all $x \in \mathcal{D}$, $\rho$ be continuously differentiable, and $\gamma$ be such that $f(x)+g(x)\gamma(x)$ is continuously differentiable. Then $h$ is a ZCBF.*

*Proof.* From Definition 1, $h$ must satisfy two conditions, namely it must be differentiable function and satisfy (8). These follow from Lemmas 1 and 2 respectively. □

*Remark* 2. In Definition 1 there must exist a class $\mathcal{K}$ function $\alpha$ satisfying $\sup_{u\in U}[L_f h(x)+L_g h(x)+\alpha(h(x))] \geq 0$ which implies that an $\alpha$ must also be found to specify a valid ZCBF. The above result holds for all $\alpha$, resolving this ambiguity.

*Remark* 3. The intuitive reason why $h$ is a ZCBF is that whenever $h(x)$ is non-negative, we have by definition a control input $\gamma$ available to keep the system safe. Note that $\gamma$ is not the output control of the Quadratic Program (10). Instead, the role of $\gamma$ is to allow $h$ to be evaluated via (14).

### B. BARRIER COMPUTATION

We now consider how to calculate $h$ defined in (14) for the UAV collision avoidance problem where the state, dynamics, and $\rho$ are given in (2), (3), and (13), respectively. From Theorem 2 the only restriction on $\gamma$ is that it make $f(x) + g(x)\gamma(x)$ continuously differentiable so there is flexibility in choosing it. In particular, we can choose $\gamma$ so that $h$ can be calculated in closed form. In this section, we consider two cases for which we have been able to derive a closed form solution. Let the initial state for vehicle $i$ ($i = a, b$) be given by $\begin{bmatrix} p_{i,x_0} & p_{i,y_0} & \theta_{i,0} \end{bmatrix}^T$.

In the first case, let $\gamma = \begin{bmatrix} v & \omega & v & \omega \end{bmatrix}$ with $\omega \neq 0$. In other words, $\gamma$ uses the same speed and turn rate for both vehicles. Letting $b_{i,0} = p_{i,x_0} - \frac{v}{\omega}\sin(\theta_{i,0})$ and $c_{i,0} = p_{i,y_0} + \frac{v}{\omega}\cos(\theta_{i,0})$, $\Delta b_0 = b_{a,0} - b_{b,0}$, and $\Delta c_0 = c_{a,0} - c_{b,0}$, we get

$$\begin{aligned} &h(x) = \\ &\inf_{\tau\in[0,\infty)} \left(\Delta b_0 + \frac{v}{\omega}\sin(\omega\tau + \theta_{a,0}) - \frac{v}{\omega}\sin(\omega\tau + \theta_{b,0})\right)^2 \\ &+ \left(\Delta c_0 - \frac{v}{\omega}\cos(\omega\tau + \theta_{a,0}) + \frac{v}{\omega}\cos(\omega\tau + \theta_{b,0})\right)^2 - D_s^2. \end{aligned}$$

After expanding the square terms and applying two trigonometric identities,[1]

$$\begin{aligned} h(x) = &\inf_{\tau\in[0,\infty)} \Delta b_0^2 + \Delta c_0^2 + 2\frac{v^2}{\omega^2} - 2\frac{v^2}{\omega^2}\cos(\theta_{a,0} - \theta_{b,0}) \\ &+2\Delta b_0 \frac{v}{\omega}\sin(\omega\tau + \theta_{a,0}) - 2\Delta b_0 \frac{v}{\omega}\sin(\omega\tau + \theta_{b,0}) \\ &-2\Delta c_0 \frac{v}{\omega}\cos(\omega\tau + \theta_{a,0}) + 2\Delta c_0 \frac{v}{\omega}\cos(\omega\tau + \theta_{b,0}) - D_s^2. \end{aligned}$$

Grouping the constant terms and applying phasor addition then yields

$$h(x) = \inf_{\tau\in[0,\infty)} A_1 + A_2\cos(\omega\tau + \Theta) - D_s^2, \tag{19}$$

where $A_1$ results from grouping constant terms, while $A_2$ and $\Theta$ are the amplitude and phase resulting from the phasor addition. Then in this case $h(x) = A_1 - A_2 - D_s^2$.

For a second case where we can solve (14) in closed form, let $\gamma = \begin{bmatrix} v_1 & 0 & v_2 & 0 \end{bmatrix}$. In other words, $\gamma$ uses a 0 turn rate while allowing the vehicles to have different speeds. In this case we have

$$\begin{aligned} &h(x) = \\ &\inf_{\tau\in[0,\infty)} (p_{a,x_0} + tv_1\cos(\theta_{a,0}) - p_{b,x_0} - tv_2\cos(\theta_{b,0}))^2 \\ &\quad + (p_{a,y_0} + tv_1\sin(\theta_{a,0}) - p_{b,y_0} - tv_2\sin(\theta_{b,0}))^2 - D_s^2, \end{aligned} \tag{20}$$

which is quadratic in $t$ so the minimum can be calculated in closed form.

When an arbitrarily chosen $\gamma$ does not allow for a closed-form solution, it is still possible to numerically compute the desired barrier function. In this case however, since we limit the value of $\tau$ to a finite interval (i.e., $\tau \in [0, T]$), safety of the system can only be guaranteed for a finite horizon of time. In cases where the dynamics are simple or can be solved in closed form it may be possible to make $T$ relatively large.[2]

### C. SIMULATION OF TWO VEHICLES WITHOUT COLLISIONS

We now apply the above developments to the UAV collision avoidance problem with $v_{min} = 0.1$, $v_{max} = 1$ and $\omega_{max} = 1$. We setup two vehicles around the origin at positions $(10, 0)$ and $(-10, 0)$ with goal positions of $(-10, 0)$ and $(10, 0)$, respectively. This ensures that the nominal controller $\hat{u}$ will cause a collision at the origin. The nominal controller is the same as that described in [25]. In each case the vehicles evaluate (10) using the CGAL library [26]. We compare the performance of $h$ based on the nominal evading maneuvers described in equations (19) and (20). For (19) we let $\gamma_{turn} = \begin{bmatrix} v_{min} & \omega_{max} & v_{min} & \omega_{max} \end{bmatrix}$. For (20) we let $\gamma_{straight} = \begin{bmatrix} v_{min} & 0 & v_{min} & 0 \end{bmatrix}$. For $\gamma_{straight}$ we start the vehicles offset by two degrees since starting with

[1]The identities are $\sin^2(\alpha) + \cos^2(\alpha) = 1$ and $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$.

[2]i.e., a horizon much larger than a barrier certificate would normally be applied anyway. This can be compared to MPC where many integrations of a trajectory may be necessary to calculate a solution.

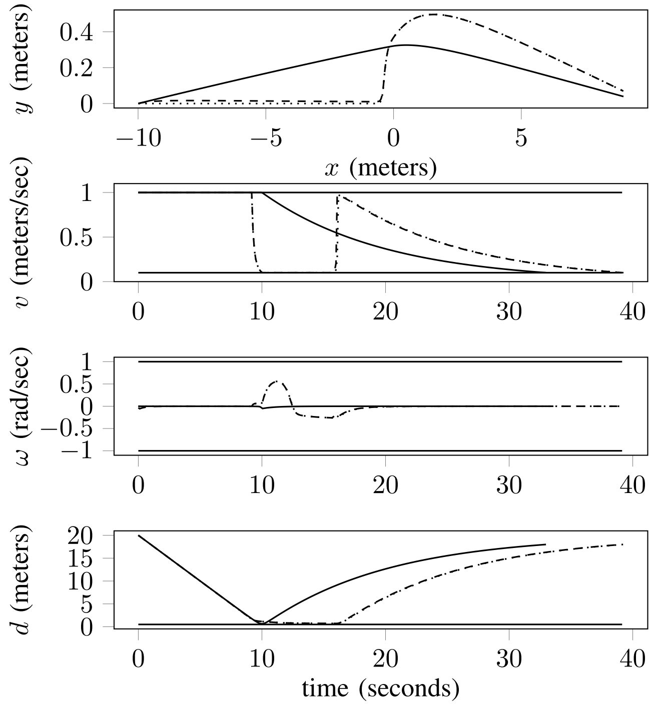
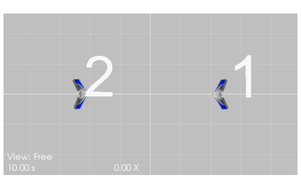
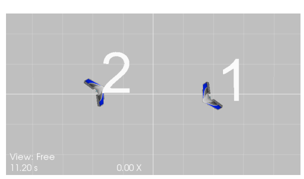


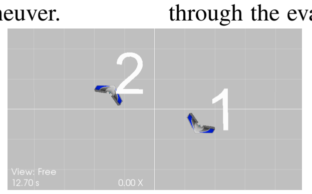

Fig. 1: The solid, dashed, and dotted lines represent cases of parameterizing $h$ with $\gamma_{straight}$ starting offset by 2 degrees, $\gamma_{turn}$ starting offset by 2 degrees, and $\gamma_{turn}$ starting offset by 0 degrees, respectively. The first plot shows the actual path taken by vehicle 2 when two vehicles start pointing at the origin from positions $(10, 0), (-10, 0)$. Notice that when the vehicle starts 2 degrees offset from the origin, $\gamma_{turn}$ allows the vehicle to initially move more directly toward at the goal position than with $\gamma_{straight}$. Also notice the effect of the nominal evading maneuver on the avoiding maneuver. The second and third plot show that $v$ and $\omega$ satisfy the actuator constraints (horizontal black lines). The last plot shows the distance between the vehicles stays above $D_s$ (black line). The dashed and dotted lines are largely the same.




(a) The vehicles right before the evasive maneuver.




(b) The vehicles midway through the evasive maneuver.




(c) The vehicles after they have safely avoided each other.

Fig. 2: Screenshots of the simulation of two fixed-wing aircraft using $\gamma_{turn}$. The gridlines are 0.25 meters apart.

orientation pointing at the origin does not start the vehicles in the safe set. We start $\gamma_{turn}$ pointing directly at the origin and also offset by 2 degrees so it is easier to compare to $\gamma_{straight}$. The simulation is done in SCRIMMAGE,[3] a multi-agent simulator capable of modeling varying levels of UAV dynamics fidelity, from unicycle dynamics (1) to 6 degree of freedom models, using a simple plugin interface.

The results in Figure 1 show that the vehicles are able to maintain a safe distance of $D_s = 0.5$ from each other for either choice of $\gamma$. Further, it indicates that the choice of $\gamma$ can significantly affect the final control value. In particular, $\gamma_{turn}$ allows the vehicles to get much closer before overriding the nominal control value. Regardless of the choice of $\gamma$ however, when we start the vehicles in the safe set, they remain in the safe set for the entire simulation. Figure 2 shows screenshots of the simulation in SCRIMMAGE.

## V. CONCLUSION

In this paper we discussed a collision avoidance scenario for fixed wing aircraft where applying barrier certificates is not straightforward. In particular, when the dynamics of a system is constrained by a finite set of control inputs, a Euclidean-based barrier function can actually lead to unsafe configurations. We then derived a constructive technique that yielded a method for constructing barrier certificates. The proposed method for generating barrier certificates was applied to multi-agent scenarios and generalizes beyond fixed wing dynamics or collision avoidance scenarios.

## APPENDIX

*Proof of Lemma 1.* We note that $x(t+\tau)$ is a continously differentiable function of the initial condition $x(t)$ by Theorem 6.1 of [27].

Let $\nu_k$ be a sequence of vectors in $\mathbb{R}^n$ approaching $x(t)$ where each $k$ is a positive integer. Let $(x+\delta x_k)(t+\tau)$ be the trajectory resulting from starting at $\nu_k$. For each $k$ let $\tau_{1,k} \geq 0$ be such that $\rho((x+\delta x_k)(t+\tau_{1,k}))$ is a minimum.

Since $x + \delta x_k$ is a differentiable function of $\tau$ and the initial condition, for any $\tau_1' \geq 0$, there exists a $\tau_2'$ such that

$$\begin{aligned}
&\rho((x+\delta x_k)(t+\tau_1')) \\
&\quad = \rho(x(t+\tau_2')) + \frac{\partial \rho(x(t+\tau_2'))}{\partial x}\delta x_k(t+\tau_2') \\
&\qquad + \frac{\partial \rho(x(t+\tau_2'))}{\partial x}\dot{x}(t+\tau_2)(\tau_1' - \tau_2') \\
&\qquad + o(\|\delta x_k\|^2) + o(|\tau_1' - \tau_2'|)
\end{aligned}$$

where $o(\|\delta x_k\|^2)$ and $o(|\tau_1' - \tau_2'|^2)$ denote higher order terms.

We claim that for large enough $k$ and for some $\tau_2 \geq 0$ such that $x(t+\tau_2) = \inf_{\tau \in [0,\infty)} \rho(x(t+\tau))$, we have

$$\begin{aligned}
&\rho((x+\delta x_k)(t+\tau_{1,k})) \\
&\quad = \rho(x(t+\tau_2)) + \frac{\partial \rho(x(t+\tau_2))}{\partial x}\delta x_k(t+\tau_2) \\
&\qquad + \frac{\partial \rho(x(t+\tau_2))}{\partial x}\dot{x}(t+\tau_2)(\tau_{1,k} - \tau_2) \\
&\qquad + o(\|\delta x_k\|^2) + o(|\tau_{1_k} - \tau_2|). \qquad (21)
\end{aligned}$$

[3]available at https://www.scrimmagesim.org/

Suppose not. That is, suppose in place of $\tau_2$ in (21) we have some $\tau_2'$. We will consider two cases. First, assume

$$\rho(x(t+\tau_2')) > \rho(x(t+\tau_2)) + \alpha. \tag{22}$$

for some $\alpha > 0$. Noting that $\rho((x+\delta x_k)(t+\tau_{1,k})) \leq \rho((x+\delta x_k)(t+\tau_2))$ we then have for all $k$ large enough we have

$$\begin{aligned}
&\rho(x(t+\tau_2')) + \frac{\partial\rho(x(t+\tau_2'))}{\partial x}\delta x_k(t+\tau_2') \\
&\quad + \frac{\partial\rho(x(t+\tau_2'))}{\partial x}\dot{x}(t+\tau_2')(\tau_{1,k}-\tau_2') \\
&\quad \leq \rho(x(t+\tau_2)) + \frac{\partial\rho(x(t+\tau_2))}{\partial x}\delta x_k(t+\tau_2) - \alpha/2,
\end{aligned}$$

which contradicts (22) since $\delta x_k(t+\tau_2')$, $(\tau_{1,k}-\tau_2')$, and $\delta x_k(t+\tau_2)$ all go to 0 as $k \to \infty$.

For the second case, assume $\tau_2' < 0$ and $\rho(x(t+\tau_2')) \leq \rho(x(t+\tau_2))$. Note that this implies the Taylor expansion around $\tau = 0$ is valid since $\tau_{1,k} \geq 0$. We will show that this case implies that $\tau_2 = 0$ so (21) must hold in this case as well. To see this, suppose not. That is, suppose

$$\rho(x(t+0)) > \rho(x(t+\tau_2)) + \alpha \tag{23}$$

for some $\alpha > 0$. Then for all $k$ large enough,

$$\begin{aligned}
&\rho(x(t+\tau_2')) - \frac{\partial\rho(x(t+\tau_2'))}{\partial x}\delta x_k(t+\tau_2') \\
&\quad - \frac{\partial\rho(x(t+\tau_2'))}{\partial x}\dot{x}(t+\tau_2')(0-\tau_2') \\
&\quad > \rho(x(t+\tau_2)) + \alpha/2,
\end{aligned}$$

which contradicts the claim that $\rho(x(t+\tau_2')) \leq \rho(x(t+\tau_2))$.

Then (21) holds. Then from (21) we have

$$\begin{aligned}
&\lim_{k\to\infty} \frac{h(x+\nu_k)-h(x)}{\nu_k} \\
&= \frac{\rho((x+\delta x_k)(t+\tau_{1,k} - \rho(x(t+\tau_2))}{\nu_k} \\
&= \frac{\frac{\partial\rho(x(t+\tau_2))}{\partial x}\delta x_k(t+\tau_2) + \frac{\partial\rho(x(t+\tau_2))}{\partial x}\dot{x}(t+\tau_2)(\tau_{1,k}-\tau_2)}{\nu_k}
\end{aligned}$$

where the last line is well defined since $x(t+\tau)$ is a continuously differentiable function of the initial condition and $\tau$. □

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