

---

# QUANTUM FIELD THEORY

---

**Mentee: Mahadevan Subramanian**

**Mentor: Som Phene**

## Contents

<b>1</b>	<b>Some Preliminaries</b>	<b>2</b>
1.1	A brief idea for closeness without distance . . . . .	2
1.2	Special Theory of Relativity revision . . . . .	2
1.3	Quantum Mechanics revision . . . . .	3
<b>2</b>	<b>Calculus of Variations</b>	<b>4</b>
2.1	Introduction . . . . .	4
2.2	The Legendre Transformation . . . . .	5
2.3	Hamilton's mechanics . . . . .	6
2.4	Liouville's Theorem and Poincare Recurrence . . . . .	6
<b>3</b>	<b>Manifolds</b>	<b>8</b>
3.1	What is a manifold? . . . . .	8
3.2	Tangent space and the tangent bundle . . . . .	9
3.3	Making physical sense of manifolds . . . . .	10
3.4	Some additional notes . . . . .	11
<b>4</b>	<b>Symmetry</b>	<b>11</b>
4.1	Noether's theorem . . . . .	11
4.2	Stress-energy tensor . . . . .	12
4.3	Symmetry breaking . . . . .	13
<b>5</b>	<b>Klein-Gordon Field</b>	<b>14</b>
5.1	Klein Gordan field as Harmonic Oscillators . . . . .	14
5.2	Solving the causality problem . . . . .	16
5.3	Complex Klein-Gordon Field . . . . .	17
5.4	Detour: Jaynes-Cumming model . . . . .	18
<b>6</b>	<b>Dirac Field</b>	<b>18</b>

## ABSTRACT

This is a report on my explorations in learning Quantum Field Theory in SoS 2021.

## 1 Some Preliminaries

### 1.1 A brief idea for closeness without distance

**Definition 1.1** (Convergence). For a given metric space  $\mathcal{M} = (X, d)$  and a sequence  $\{x_n\}_{n \in \mathbb{N}}$  we say  $x_n$  converges to  $x$  iff  $\forall \epsilon > 0, \exists N_0$  such that  $\forall n > N_0, d(x_n, x) < \epsilon$ .

Here we must use a metric space with a distance metric  $d$  to quantify closeness. However, it is not necessary that we may be able to construct such a metric for any space and so we need to generalize a little more.

Let us define a set  $\mathcal{T}$  where each element is a set of points from a space  $X$  and it satisfies the following properties.

- $X, \emptyset \in \mathcal{T}$
- $O_1, \dots, O_n \in \mathcal{T}$  implies  $O_1 \cap \dots \cap O_n \in \mathcal{T}$  (finite intersections only)
- $O_i, i \in I \in \mathcal{T}$  implies  $\bigcup_i O_i \in \mathcal{T}$  (any unions)

This set  $\mathcal{T}$  will contain all the open sets of  $X$ . Now while we have an  $\epsilon, \delta$  definition of continuity we can obtain a nicer definition of continuity from the following theorem.

**Theorem 1.1** (Open-mapping theorem). For a continuous function  $f : X \rightarrow Y$ , the pre image of any open set in  $Y$  would be an open set in  $X$ .

This directly gives us a very nice definition for continuity which does not invoke the requirement for a metric

### 1.2 Special Theory of Relativity revision

Transformation laws

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \quad (1)$$

$$g_{\mu\nu} = \Lambda^{\rho}_{\mu} g_{\rho\lambda} \Lambda^{\lambda}_{\nu} \quad (2)$$

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \quad (3)$$

$$(4)$$

Maxwell's laws

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \quad (5)$$

$$\partial_{\mu} F^{\mu\nu} = 4\pi j^{\mu} \quad (6)$$

$$\partial_{\mu} F_{\nu\rho} + \partial_{\nu} F_{\rho\mu} + \partial_{\rho} F_{\mu\nu} = 0 \quad (7)$$

Lorentz force

$$\frac{\partial p^{\mu}}{\partial \tau} = q F^{\mu\nu} u_{\nu} \quad (8)$$

Energy momentum tensor

$$T^{\mu\nu} = \sum_n \int d\tau p_n^{\mu} p_n^{\nu} \delta^4(x^{\rho}(\tau) - x_n^{\rho}(\tau)) \quad (9)$$

$$T^{\mu\nu} = -p g^{\mu\nu} + (p + \rho) u^{\mu} u^{\nu} \quad (10)$$

Source flux theorem

$$\partial_{\mu} T^{\mu\nu} = 0 \quad (11)$$

### 1.3 Quantum Mechanics revision

Here I will just list some very useful formulae of quantum mechanics that I am so far familiar.

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(x) \right] \psi(\vec{r}, t) = i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} \quad (12)$$

$$\lim_{\epsilon \rightarrow 0} \left( -\frac{\hbar^2}{2m} \left[ \frac{d\psi(x+\epsilon)}{dx} - \frac{d\psi(x-\epsilon)}{dx} \right] + \int_{x-\epsilon}^{x+\epsilon} V\psi(x) dx \right) = 0 \quad (13)$$

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} \quad (14)$$

$$\frac{d}{dt} \langle \hat{A} \rangle = \frac{1}{i\hbar} \langle [\hat{A}, \hat{H}] \rangle + \left\langle \frac{d\hat{A}}{dt} \right\rangle \quad (15)$$

Dual of vector  $\hat{O} |\xi\rangle$  is  $\langle \xi | \hat{O}^\dagger$  and given an orthogonal basis  $\{|\phi_i\rangle\}$  we have

$$\sum_i |\phi_i\rangle \langle \phi_i| = \mathbb{I} \quad (16)$$

All observables must be hermitian, for example radial momentum is given by

$$\hat{p}_r = \frac{\hat{r} \cdot \vec{p} + \vec{p} \cdot \hat{r}}{2} \quad (17)$$

Spherical harmonics

$$Y_m^l(\theta, \phi) = \Theta(\theta) \Phi(\phi) \quad (18)$$

$$\sin(\theta) \frac{d}{d\theta} \left( \sin(\theta) \frac{d\Theta(\theta)}{d\theta} \right) + [l(l+1)\sin^2(\theta) - m^2] \Theta(\theta) = 0 \quad (19)$$

$$\frac{d^2 \Phi(\phi)}{d\phi^2} = -m^2 \Phi(\phi) \quad (20)$$

Angular momentum algebra

$$L_i = \epsilon_{ijk} x_j p_k \quad (21)$$

$$[L_i, L_j] = \epsilon_{ijk} i\hbar L_k \quad (22)$$

Harmonic oscillator ladder operations

$$[\hat{a}, \hat{a}^\dagger] = \mathbb{I} \quad (23)$$

$$\hat{H} = \hbar\omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \quad (24)$$

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle \quad (25)$$

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \quad (26)$$

$$|\alpha\rangle = e^{-|\alpha|^2/2} D(\alpha) |0\rangle, \quad D(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \quad (27)$$

Addition of angular momentum

$$J_i = L_i + S_i \quad (28)$$

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k \quad (29)$$

$$J_\pm = J_1 \pm J_2, \quad [J_+, J_-] = 2\hbar J_3 \quad (30)$$

$$J^2 |j, m\rangle = j(j+1)\hbar^2 |j, m\rangle, \quad J_3 |j, m\rangle = m\hbar |j, m\rangle \quad (31)$$

$$J_\pm |j, m\rangle = \hbar \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle \quad (32)$$

Rotations in a space

$$U_R(\hat{n}, \theta) = \exp \left( -\frac{i}{\hbar} \hat{n} \cdot \hat{J} \theta \right) \quad (33)$$

Spherical tensors

$$T_q^{(k)} = Y_{l=k}^{m=q}(T) \quad (34)$$

$$[J_z, T_q^{(k)}] = \hbar q T_q^{(k)} \quad (35)$$

$$[J_{\pm}, T_q^{(k)}] = \hbar \sqrt{(k \mp q)(k \pm q + 1)} T_{q \pm 1}^{(k)} \quad (36)$$

Wigner-Eckhart Theorem

$$\langle j', m' | T_q^{(k)} | j, m \rangle = \langle j' || T^{(k)} || j \rangle \langle j, k; m, q | j' m' \rangle \quad (37)$$

Here  $\langle j, k; m, q | j' m' \rangle$  is the CG coefficient for the angular momenta addition of  $|j, m\rangle$  and  $|k, q\rangle$  to get  $|j', m'\rangle$

## 2 Calculus of Variations

The main reference for this section is [12].

### 2.1 Introduction

The calculus of variations concerns itself with the extremals of functions whose domain is the space of curves (hence being infinite dimensional). Such functions are referred to as functionals. An example of one such functional is  $\Phi(\gamma) = \int_{t_0}^{t_1} \sqrt{1 + \dot{x}^2} dt$  where  $\gamma = \{(t, x) : x(t) = x, t_0 \leq t \leq t_1\}$ . An approximation  $\gamma'$  to  $\gamma$  is simply  $\gamma' = \{(t, x) : x = x(t) + h(t), t_0 \leq t \leq t_1\}$

**Definition 2.1** (Differentiable functional).  $\Phi$  is a differentiable functional if  $\Phi(\gamma + h) - \Phi(\gamma) = F + R$  where  $F$  is linear in  $h$  and  $R$  is  $\mathcal{O}(h^2)$ . The linear part  $F(h)$  is referred to as the differential.

The differential of a function is also called its variation and if it exists, it is unique for the given functional. For example the functional  $\Phi(\gamma) = \int_{t_0}^{t_1} L(x, \dot{x}, t)$  the differential can be reached by this definition to be equal to

$$F(h) = \int_{t_0}^{t_1} \left[ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right] h dt + \left( \frac{\partial L}{\partial \dot{x}} h \right) \Big|_{t_0}^{t_1} \quad (38)$$

The proof of this follows from expressing  $L(x + h, \dot{x} + \dot{h}, t)$  in it's multi-variable Taylor's expansion from  $L(x, \dot{x}, t)$ . It follows that seeing first order only we get  $\Phi(\gamma + h) - \Phi(\gamma) = \int \left[ \frac{\partial L}{\partial x} h + \frac{\partial L}{\partial \dot{x}} \dot{h} \right] + \mathcal{O}(h^2)$  which simplifies using integration by parts to the expression in equation 38.

**Definition 2.2** (Extremal). An extremal of a differentiable functional  $\Phi(\gamma)$  is the curve  $\gamma$  where  $F(h) = 0$  for all  $h$ .

We can see from equation 38 that under certain boundary conditions of taking  $x(t_0) = x_0$  and  $x(t_1) = x_1$ , the extremal condition arises for the curve  $\gamma : x = x(t)$  if and only if

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0 \quad (39)$$

This is exactly the Euler-Lagrange equation which shows that it follows from principle of least action. If we take  $x$  to be a  $n$  dimensional vector in  $\mathbb{R}^n$ , we get  $\gamma$  to be a  $n + 1$  dimensional curve in  $\mathbb{R} \times \mathbb{R}^n$  and similarly  $L$  is a function in  $2n + 1$  variables. Hence the extremal now comes to be  $n$  second order equations with the  $2n$  conditions from  $x(t_0) = x_0$  and  $x(t_1) = x_1$ .

The extremal for the functional with  $L(x, \dot{x}, t) = \sqrt{1 + \dot{x}^2}$  comes out to be a straight line connecting the two points in the  $(x, t)$  space. Similarly if one were to find the extremal for distance along a spherical surface it comes to be the great circle connecting the two points. This is generalized as geodesic equations and are used for the equations of motion in general relativity. Using the function  $L(x, \dot{x}, t) = \sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}$  the equations of motion arise to be

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0 \quad (40)$$

Here  $\Gamma_{\beta\gamma}^\alpha = g^{\alpha\mu} (g_{\beta\mu, \gamma} + g_{\gamma\mu, \beta} - g_{\beta\gamma, \mu})$  which is the affine connection. This equation motion does infact arise from Hamilton's principle of least action.

**Definition 2.3** (Lagrangian). We define  $L(q, \dot{q}, t) = T - U$  as the lagrangian where  $q_i$  are the generalized coordinates and  $\dot{q}_i$  are the generalized velocities. We define  $p_i = \partial L / \partial \dot{q}_i$  as the generalized momenta and  $\partial L / \partial q_i$  as the generalized force. The extremal of  $\int_{t_0}^{t_1} L dt$  which is the action is arrived at the solution of the Euler-Lagrange equations.

## 2.2 The Legendre Transformation

Let  $f : \mathbb{I} \rightarrow \mathbb{R}$ ,  $\mathbb{I} \subseteq \mathbb{R}$  be a convex function. The legendre transformation is a new function  $g : \mathbb{I}^* \rightarrow \mathbb{R}$ ,  $\mathbb{I}^* \subseteq \mathbb{R}$  defined as follows

$$g(p) = \sup_{x \in \mathbb{I}} (xp - f(x)) \quad (41)$$

The farthest vertical distance between the line  $y = px$  and  $y = f(x)$  comes out to be the expression of  $g(p)$  which is exactly it's definition. More solidly, let  $F(x, p) = px - f(x)$ , and  $F(x, p)$  has a maximum w.r.t  $x$  at some  $x(p)$ , then  $g(p) = F(p, x(p))$ . This extremal occurs at  $f'(x) = p$  hence we get the expression in equation 41.

If the function  $f$  is convex over some domain we know that it's derivative will have an inverse over that same domain hence guaranteeing that  $x(p)$  must be unique if it exists. Also if for a value of  $p$  the supremum is unbounded, that cannot be in  $\mathbb{I}^*$ .

An example for this is the function  $f = cx$  which is convex but not strictly convex. If  $p \neq c$  the supremum of  $F(x, p)$  is non finite hence the domain is only  $\{c\}$  and  $g(c) = 0$ . We can similarly find that for  $f(x) = mx^2/2$ ,  $g(p) = p^2/2m$ . More generally for  $f(x) = x^\alpha/\alpha$  we get  $g(p) = p^\beta/\beta$  for  $\alpha + \beta = \alpha\beta$  and both  $\alpha, \beta > 1$ .

Importantly the Legendre transform is involutive hence it's square is the identity. This becomes readily apparent on writing the transform in terms of derivatives as shown below.

$$g(p) = p(f')^{-1}(p) - f((f')^{-1}(p)) \quad (42)$$

This form of writing the transform is not entirely correct since it brings up issues for non convex functions.

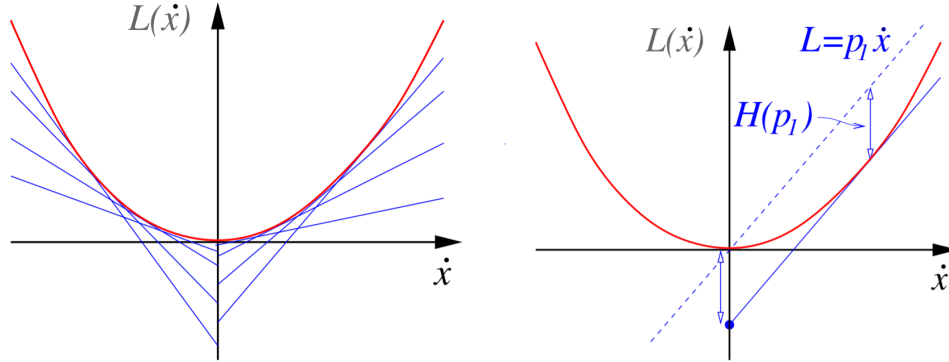


Figure 1: Left panel: envelope of the family of lines would give back the Legendre transform of the intercept value of a certain slope. Right panel: the supremum of the  $px - L(\dot{x})$  can be geometrically represented as the  $y$  intercept of the tangent of same slope. Image credit: [2].

**Corollary 2.0.1.** Consider a given family of straight lines  $y = px - g(p)$ , the envelope of these lines has the equation  $y = f(x)$  where  $f(x)$  is the Legendre transform of  $g$

This above corollary follows from the fact that the enveloping function for a family of lines has these lines to be it's tangents. Hence this gives  $p = \partial f / \partial x$  and at this value  $g(p) = \frac{\partial f}{\partial x} x - f(x)$  which is the supremum of the expression  $px - f(x)$  and hence  $g$  and  $p$  are Legendre transforms of each other.

**Definition 2.4.** Two functions,  $f$  and  $g$ , which are Legendre transforms of each other are called *dual in the sense of Young*.

Since  $g(p)$  is defined as the supremum over  $x$  of  $px - f(x)$ , we get the following inequatlity referred to as the *Young's inequality*

$$px \leq f(x) + g(p) \quad (43)$$

Now let's suppose we have some convex function  $f(\mathbf{x})$  where  $\mathbf{x} = (x_1, \dots, x_n)$  and we now define a  $\mathbf{p} = (p_1, \dots, p_n)$ . We define the legendre transform as follows

$$g(\mathbf{p}) = \sup_{\mathbf{x}} (\langle \mathbf{p}, \mathbf{x} \rangle - f(\mathbf{x})) \quad (44)$$

Since one would reach an maxima when the gradient of the expression is zero, this gives us the condition  $\mathbf{p} = \partial f / \partial \mathbf{x}$ . If we take  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  as a convex function and denote the dual space by  $\mathbb{R}^{n*}$ , we essentially have  $f' : \mathbb{R}^n \rightarrow \mathbb{R}^{n*}$  where  $f'(\mathbf{x}) = \partial f / \partial \mathbf{x}$ . The  $\mathbf{p}$  are in the dual vector space.

**Definition 2.5** (Convex function). A multi-variable function is convex over a convex set iff it's Hessian matrix of second derivatives is positive semidefinite over this convex set.

The hessian matrix is defined as  $(\mathbf{H}_f)_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ . We must note that this Hessian matrix is the Jacobian for the function  $\partial f / \partial \mathbf{x}$ . Using the inverse function theorem, a positive semidefinite Hessian matrix implies that the derivative function has an invertible Jacobian and since this is over the whole domain of  $\mathbb{R}^n$ , the mapping of  $g : \mathbb{R}^{n*} \rightarrow \mathbb{R}^n$  is well defined.

As an example we can pick the function  $f(\mathbf{x}) = f_{ij}x_i x_j$  (repeated indices are summed over). The gradient of this function is  $f'(\mathbf{x})_i = (f_{ki} + f_{ik})x_k$  so putting in the expression we get  $g(\mathbf{p}) = x_i(f_{ki} + f_{ik})x_k - x_i f_{ik}x_k = f_{ij}x_j x_i$ . Since  $x_i x_j = x_j x_i$  this shows that  $g(\mathbf{p}(\mathbf{x})) = f(\mathbf{x})$  and since these are Legendre transforms of each other, we also have  $f(\mathbf{x}(\mathbf{p})) = g(\mathbf{p})$ .

### 2.3 Hamilton's mechanics

**Theorem 2.1.** The system of Lagrange's equations are equivalent to the system of  $2n$  first order equations which are referred to as the Hamilton's equations.

$$\dot{\mathbf{p}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}} \quad (45)$$

$$\dot{\mathbf{q}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \quad (46)$$

Here  $\mathcal{H}(\mathbf{p}, \mathbf{q}, t) = \langle \mathbf{p}, \dot{\mathbf{q}} \rangle - L(\mathbf{q}, \dot{\mathbf{q}}, t)$  which is the Legendre transformation of the lagrangian with respect to  $\dot{\mathbf{q}}$ .

*Proof.* These equations arise from writing the differential of  $\mathcal{H}$  which can expressed simply by chain rule and also in terms of the Legendre transform of  $L$  hence

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \cdot d\mathbf{p} + \frac{\partial \mathcal{H}}{\partial \mathbf{q}} \cdot d\mathbf{q} + \frac{\partial \mathcal{H}}{\partial t} dt = \dot{\mathbf{q}} \cdot d\mathbf{p} - \frac{\partial L}{\partial \mathbf{q}} \cdot d\mathbf{q} - \frac{\partial \mathcal{H}}{\partial t} dt \quad (47)$$

From Lagrange's equations  $\dot{\mathbf{p}} = \partial L / \partial \mathbf{q}$  and this gives us the Hamilton's equations since variations in all directions should be equal.  $\square$

In the case of mechanical equations we have  $L = T - U$  and in the situation that the kinetic energy is in a quadratic form with respect to  $\dot{\mathbf{q}}$  and we can see that the hamiltonian comes out to become  $\mathcal{H} = T + U$  in this case.

**Corollary 2.1.1.**  $d\mathcal{H}/dt = \partial \mathcal{H} / \partial t$ , hence a system whose hamiltonian has no explicit time dependence, will have a law of conservation  $\mathcal{H}(\mathbf{p}(t), \mathbf{q}(t)) = \text{constant}$ .

### 2.4 Liouville's Theorem and Poincare Recurrence

**Definition 2.6** (Phase flow). The phase flow is a one parameter group of transformations of a phase space

$$g^t : (\mathbf{p}(0), \mathbf{q}(0)) \rightarrow (\mathbf{p}(t), \mathbf{q}(t)) \quad (48)$$

Here  $\mathbf{p}(t), \mathbf{q}(t)$  are solutions of Hamilton's equations.

**Theorem 2.2** (Liouville's theorem). Phase flow preserves volume of any region  $D$  in the phase space over which the transformations are applied. Let's say the region evolves with time as  $D(t) = g^t D(0)$  and  $v(t)$  is the volume of  $D(t)$ , this theorem states that  $v(t)$  is constant in  $t$ .

As per the transformation law we get from equation 48 we would express the volume as a multiple variable integral. As shown below it takes the form of a determinant of a matrix. We can think of this making sense by seeing this as an oriented volume as such a thing holds for the volume formed by an object composed of the vectors originating from a point. Intuitively this can be built by the fact that exchanging two of the vectors orientation wise should make the volume flip sign and this fully antisymmetrized nature is the sum represented by the matrix determinant with each vector being a row.

$$v(t) = \int_{D(0)} \det \frac{\partial g^t \mathbf{x}}{\partial \mathbf{x}} d\mathbf{x} \quad (49)$$

For small values of  $t$  we can express the following

$$\frac{\partial g^t \mathbf{x}}{\partial \mathbf{x}} = I + \frac{\mathbf{f}}{\partial \mathbf{x}} t + \mathcal{O}(t^2) \quad (50)$$

**Lemma 2.3.** For any matrix  $A \in \mathbb{R}^{n \times n}$ , we have the following for  $t \rightarrow 0$

$$\det(I + At) = 1 + \text{tr}(A)t + \mathcal{O}(t^2) \quad (51)$$

The above lemma follows from the fact that the only terms first order in  $t$  can be shown to add up to the trace of  $A$ . The trace of  $\partial \mathbf{f} / \partial \mathbf{x} = \nabla \cdot \mathbf{f}$  and so using this we can express volume at small  $t$  as follows

$$v(t) = \int_{D(0)} [1 + t \nabla \cdot \mathbf{f} + \mathcal{O}(t^2)] dx \quad (52)$$

From this we can now write the following theorem

**Theorem 2.4.** If  $\nabla \cdot \mathbf{f} \equiv 0$ ,  $g^t$  is a volume preserving transformation.

*Proof.* From equation 52, it follows that  $(dv/dt)|_{t=0} = \int_{D(0)} \nabla \cdot \mathbf{f} dx$ . This implies that if the divergence of  $\mathbf{f}$  vanishes the derivative of the volume w.r.t time vanishes at  $t = 0$ . Here  $t = 0$  can be extended to any  $t = t_0$  since there is nothing special in choosing 0 hence giving us that the volume is indeed preserved for such a transformation.  $\square$

**Definition 2.7** (Liouville's formula). For the linear system  $\dot{\mathbf{x}} = A(t)\mathbf{x}$ , the Wronskian determinant is given by  $W = W_0 e^{\int \text{tr}(A) dt}$ .

*Proof.* Taken from [13]. Let us define a matrix solution  $\Phi : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ , which satisfies  $\dot{\Phi}(t)A(t) = \Phi(t)$ . Using the notation of determinant as a sum with the levi-civita tensor, it's easy to see that the derivative of  $\det \Phi(t)$  would be the sum of the determinants with each row having been differentiated term-wise. Taking  $A = \{a_{ij}\}$  and  $\Phi = \{\phi_{ij}\}$  we can write the following

$$(\dot{\phi}_{i,1}, \dots, \dot{\phi}_{i,n}) = \sum_{j=1}^n a_{i,j}(\phi_{j,1}, \dots, \phi_{j,n})$$

In the determinant which replaces the  $i$ th row with it's derivative, we can subtract  $\sum_{j=1, j \neq i}^n a_{i,j}(\phi_{j,1}, \dots, \phi_{j,n})$  which is a linear combination of all other rows in the determinant and hence does not change the expression. This gives

$$\det \begin{pmatrix} \phi_{1,1} & \dots & \phi_{1,n} \\ \vdots & \ddots & \vdots \\ \dot{\phi}_{i,1} & \dots & \dot{\phi}_{i,n} \\ \vdots & \ddots & \vdots \\ \phi_{n,1} & \dots & \phi_{n,n} \end{pmatrix} = \det \begin{pmatrix} \phi_{1,1} & \dots & \phi_{1,n} \\ \vdots & \ddots & \vdots \\ a_{i,i}\phi_{i,1} & \dots & a_{i,i}\phi_{i,n} \\ \vdots & \ddots & \vdots \\ \phi_{n,1} & \dots & \phi_{n,n} \end{pmatrix} = a_{i,i} \det(\Phi)$$

Adding the above expression for all  $i$  we get the differential equation  $\dot{W} = \text{tr}(A)W$  which solves to give the expected expression for initial state  $W_0$ .  $\square$

**Theorem 2.5** (Poincaré's recurrence theorem). Let  $g$  be a volume preserving transformation which is continuous and is a one-to-one mapping. Let  $D$  be some bounded region of euclidean space onto itself i.e.,  $gD = D$ . Then in any neighbourhood  $U$  for any point in  $D$ , there exists a point  $x \in U$  such that  $\exists n > 0$  such that  $g^n x \in U$

*Proof.* Let's pick some  $U \subset D$  with  $U \neq \emptyset$ . Since  $g$  preserves volume,  $\forall n, v(g^n U) = v(U)$  where  $v(U)$  is the volume of the set  $U$ . All these images must lie in  $D$  since  $gD = D$ . Since we have an infinite number of such images, assuming none intersect each other this would require  $D$  to have infinite volume which is a contradiction. Hence this requires that  $\exists k, l \geq 0$  with  $k > l$  such that  $g^k U \cap g^l U \neq \emptyset$  which is the same as saying  $g^{k-l} U \cap U \neq \emptyset$  hence proving the statement.  $\square$

Poincaré's statement can be strengthened to state that almost every moving point returns an infinite number of times to it's initial position (explicitly state in it's topological form [1]).

This theorem has some very interesting implications such as the claim that in a two chamber container with gas contained only in the first chamber, all the molecules would return back to their initial point in the first chamber after some time. The resolution of this seemingly paradoxical claim lies in the fact that this time scale may be well longer than the duration of the solar system's existence.

Some common examples for application of Poincaré's theorem are related to systems which involve an inherent periodicity such as movement on a circle or on a surface of a torus.

On a circle a transformation of an angle  $2\pi\alpha$  if  $\alpha$  is irrational is dense on the circle since  $\forall \delta > 0, \exists n : |g^n x - x| < \delta$

due to the irrationality of  $\alpha$  causing it never to be able to perform a proper loop.

Similarly for a system with two angles  $\phi_1, \phi_2$  which are transforming as  $g^t : (\phi_1, \phi_2) \rightarrow (\phi_1 + \alpha_1 t, \phi_2 + \alpha_2 t)$  one can fill up the phase space of the two variables which would be winding lines on a torus if  $\alpha_1/\alpha_2$  is irrational. If their ratio is rational, there is some time where it would exactly return to the initial state but otherwise the winding lines would densely fill up the surface.

### 3 Manifolds



Figure 2: A nice and cozy manifold we happen to inhabit. [Picture credits.](#)

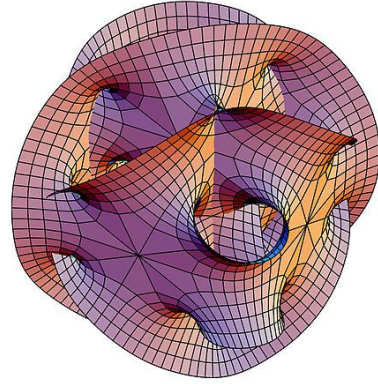


Figure 3: A complex four-space representation of the 'Calabi-Yau Manifold' by Stewart Dickson, which we also possibly inhabit. [Picture credits.](#)

#### 3.1 What is a manifold?

Before we see what a manifold is, let's build toward some examples of similar such objects which are not. This section is in reference to [11], [12], [14], [8]. Let us first consider the 2 dimensional euclidean plane. Let's make a circle, a square and some  $n$  number of non intersecting circles on this flat plane. For whatever point we pick on the circle, we can take some open neighbourhood of points which also lie on that circle and map these to a normal open line segment. This map is onto and continuous and will exist for any such portion picked from the circle. Similarly this can be shown for the square too even around the sharp right angled corners since the orientation of the points in this manner does not matter and it can be parametrized and mapped. The  $n$  circles similarly can also be mapped since none intersect and hence every point would have some open neighbourhood which only has points from the same circle.

Things however get dicey if we try finding such mappings for two intersecting line segments. The point of intersection has no neighbourhood which has a continuous onto map to the open line segment. Since no matter how small the neighbourhood is, there will always be four directions of approach to the intersection, hence making continuous onto mapping to a open line segment (which can only have two directions) impossible. This follows for any kind of intersection of objects that could have been mapped to the open line segment.

In 3 dimensions we have a similar mapping idea. Given a surface in 3D, we pick a point on it and want it's neighbourhood to have a continuous onto map to the open disc. We can see that analogous to the 2D case, a sphere, a torus, a cube and any  $n$  number of spheres will be able to satisfy this property. Again in the case of two spheres that touch, the point of contact will be problematic since no neighbourhood around it can be mapped as we want to an open disc. This mapping property is something which in a way is an important concept of what makes a manifold what it is.

We can now proceed to get into the actual definitions

**Definition 3.1** (Manifold). A  $n$ -manifold is a topological space in which each point has a neighbourhood which is homeomorphic (having a bijective continuous map) to the Euclidean space of dimension  $n$ .

This definition is essentially that of a *topological* manifold since this allows for the manifold to lack a certain structure which would be required for a differentiable manifold. To avoid certain pathological cases a topological manifold is also required to be hausdorff and second countable.



**Definition 3.2** (Hausdorff space). A Hausdorff space is a topological space where any two distinct points in this space there exists a neighbourhood for each point such that these neighbourhoods are disjoint.

**Definition 3.3** (Second countable). A topological space  $T$  is second-countable if there exists some countable collection  $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$  of open subsets of  $T$  such that any open subset of  $T$  can be written as a union of elements of some subfamily of  $\mathcal{U}$ .

Before we dive into differentiable manifolds we need to understand what charts are.

**Definition 3.4** (Chart). A chart is an open set  $U$  in  $\mathbb{R}^n$  with an onto mapping  $\phi : U \rightarrow \phi U \subset M$  where  $M$  is a  $n$  manifold. It is represented by this pair as  $(U, \phi)$ .

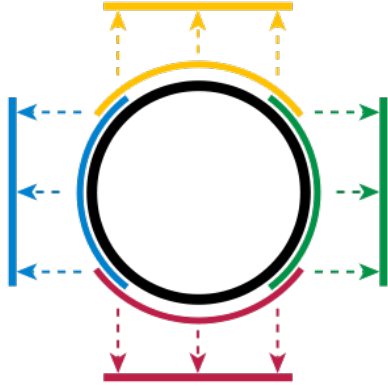


Figure 4: Mapping portions of a circle to open line segments. Picture taken from [14].

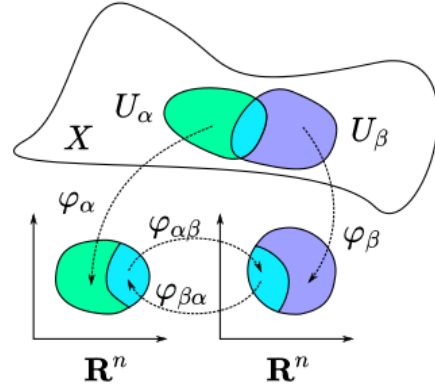


Figure 5: Compatible charts. Picture taken from [8]

Let's assume we have two points  $p$  and  $p'$  in charts  $U$  and  $U'$  respectively such that their image is the same point in  $M$ , then these points have neighbourhoods  $V \subset U$ ,  $V' \subset U'$  such that they have the same image in  $M$ . Hence this gives us a mapping  $\phi'^{-1}\phi : V \rightarrow V'$  and the charts are called compatible if this function and its inverse mapping are differentiable (see figure 5). An atlas is a union of compatible charts. Two atlases are referred to as equivalent if their union is also an atlas. A differentiable manifold is a class of equivalent atlases.

We can see that the euclidean space  $\mathbb{R}^n$  is a manifold which has an atlas with only one chart. The  $n$  sphere given by  $S^n = \{(x_1, \dots, x_{n+1}) : \sum x_i^2 = 1\}$  has an atlas with two charts which are its stereographic projections. One can see various examples of configuration spaces which come out to be manifolds such as a spherical pendulum is essentially having a configuration space of  $S^2$ . Similarly the configuration space of two pendulums which are both planar is the direct product  $S^1 \times S^1 = T^2$  which is a torus.

**Definition 3.5** (Embedded manifold). We say  $M$  is an embedded  $k$  dimensional sub-manifold of  $\mathbb{R}^n$  if in some neighbourhood  $U$  for each point  $\mathbf{x} \in M$  there are  $n - k$  functions  $f_1 : U \rightarrow \mathbb{R}, \dots, f_{n-k} : U \rightarrow \mathbb{R}$  such that  $f_i = 0$  and the gradients of all these functions are linearly independent at  $\mathbf{x}$

It can be shown that every manifold is embedded in some euclidean space. We can show that  $SO(3)$ , the special orthogonal group of  $3 \times 3$  matrices are embedded in  $\mathbb{R}^9$ . To show this we first note that any member of  $SO(3)$  satisfies the equation

$$O \in SO(3) \implies OO^T = I_{3 \times 3} \quad (53)$$

More importantly  $OO^T$  is a  $3 \times 3$  symmetric matrix and so has six independent parameters and hence making it equal identity gives us six independent equations. This shows that the  $O$  matrix should have only 3 independent variables hence showing that it is 3 dimensional. However more importantly the six equations which constrain it are nothing but mappings from a neighbourhood of the point in the manifold to the set of real numbers and shows that we have  $f_1 : U \rightarrow \mathbb{R} \dots f_6 : U \rightarrow \mathbb{R}$  and it's easy to see that they have linearly independent gradients at the solution hence showing that the 3 dimensional manifold  $SO(3)$  is embedded in  $\mathbb{R}^9$ . It must be noted that the implies in equation 53 becomes an iff when the manifold is  $O(3)$ . The  $SO(3)$  has an additional condition which makes it a transformation which preserves handedness and so is just the connected form of  $O(3)$ .

### 3.2 Tangent space and the tangent bundle

We have already seen what an embedded manifold is. Now let's say we have a  $k$  manifold embedded in  $\mathbb{R}^n$ . At any given point we have a  $k$  dimensional tangent space  $TM_{\mathbf{x}}$  which is orthogonal to all the vectors  $\nabla f_i$  where  $i$  goes from

1 to  $n - k$ . The vectors in this space are called tangent vectors to  $M$  at  $\mathbf{x}$  and can also be defined as velocity vectors of curves in  $M$

$$\dot{\mathbf{x}} = \lim_{t \rightarrow 0} \frac{\varphi(t) - \varphi(0)}{t}, \varphi : \mathbb{R} \rightarrow M, \varphi(0) = \mathbf{x} \quad (54)$$

Two curves  $\varphi(t)$  and  $\psi(t)$  are called equivalent if  $\psi(0) = \varphi(0) = \mathbf{x}$  and  $\lim_{t \rightarrow 0} (\varphi(t) - \psi(t))/t = 0$  in some chart. The tangent relationship will hold in all charts for the manifold since compatibility forces that they map to each other using a differentiable map and so the composition of the map on top of the curve would still obey the tangent relation.

**Definition 3.6.** A tangent vector to a manifold  $M$  at the point  $\mathbf{x}$  is an equivalence class of curves  $\varphi(t)$ , with  $\varphi(0) = \mathbf{x}$ .

The tangent vector is essentially just  $d\varphi/dt|_{t=0}$ . Now one thing which happens here is that the union  $\bigcup_{\mathbf{x} \in M} TM_{\mathbf{x}}$  has a natural differential manifold structure which has twice the dimension of  $M$ . This new manifold has twice as many dimensions since any point on it is defined by a  $k$  dimensional tangent vector  $\dot{\mathbf{x}}$  on a  $k$  dimensional point  $\mathbf{x}$  in  $M$ .

This manifold  $TM$  is referred to as the tangent bundle. The mapping  $p : TM \rightarrow M$  takes the tangent vector to the point in  $M$  where it is tangent to it and is called the natural projection. The inverse image of a point in  $M$  is it's tangent space  $TM_{\mathbf{x}}$  and this space is called the fiber of the tangent bundle over that point.

**Definition 3.7** (Reimannian manifold). A differentiable manifold with a fixed positive definite quadratic form  $\langle \zeta, \zeta \rangle$  on every tangent space  $TM_{\mathbf{x}}$  is called as a Reimannian manifold and the quadratic form is referred to as the Reimannian metric.

The concept of the Reimannian metric is extensively used for manifolds in general relativity where one writes  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ .

One can create say some mapping  $f : M \rightarrow N$  where  $M$  and  $N$  are manifolds and in local coordinates of  $M$  and  $N$  we have  $f$  to be differentiable. We can proceed to define a new map between the tangent bundles of these manifolds.

**Definition 3.8.** The derivative of a differentiable mapping  $f : M \rightarrow B$  at some point  $\mathbf{x} \in M$  is the linear map

$$f_{*\mathbf{x}} : TM_{\mathbf{x}} \rightarrow TN_{f(\mathbf{x})} \quad (55)$$

If we take some  $v \in TM_{\mathbf{x}}$  and  $\varphi : \mathbb{R} \rightarrow M$  with  $\varphi(0) = \mathbf{x}$  with the velocity vector for this curve being  $v$ , we would define the velocity vector of the curve  $f \circ \varphi : \mathbb{R} \rightarrow N$  as

$$f_{*\mathbf{x}}v = \left. \frac{df(\varphi(t))}{dt} \right|_{t=0} \quad (56)$$

We can check that the value of the RHS has no dependence on  $\varphi$  since on applying chain rule and the constraints that  $d\varphi/dt|_{t=0} = v$  and  $\varphi(0) = \mathbf{x}$  we get  $f_{*\mathbf{x}}v = \nabla_{\mathbf{x}} f v$ .

We could take the union of all these mappings  $F_{*\mathbf{x}}$  and just denote  $F_* TM \rightarrow TN$  by keeping  $F_* v = f_{*\mathbf{x}}v$  for  $v \in TM_{\mathbf{x}}$ . We can see that this is well defined as each point in  $TM$  would have a point in  $M$  along with a vector in  $TM_{\mathbf{x}}$  associated with it. Hence all this map does is take a point  $(\mathbf{x}, \dot{\mathbf{x}})$  to  $(f(\mathbf{x}), \nabla_{\mathbf{x}} f \dot{\mathbf{x}})$  and so is clearly differentiable.

### 3.3 Making physical sense of manifolds

While we were building the tangent space and tangent bundle, we actually were getting to the very familiar Lagrangian dynamics which actually is defined on a tangent bundle. If  $M$  is a differentiable manifold and  $TM$  is the tangent bundle,  $L : TM \rightarrow \mathbb{R}$  is the lagrangian (assuming it has no time dependence). The extremal defined for the functional  $\Phi(\gamma) = \int_{t_0}^{t_1} L(\dot{\gamma}) dt$  where  $\dot{\gamma}(t) \in TM_{\gamma(t)}$ .

The evolution of these coordinates is defined by the Euler-Lagrange equations in equation 39.

**Definition 3.9.** A lagrangian system on a Reimannian manifold is called *natural*  $L = T_U$

One can now have certain holonomic constraints in the system. Since every manifold can be embedded in a certain Euclidean space and one can define the potential as something of the form  $U' = U + N\mathbf{q}_2^2$  where the coordinate  $\mathbf{q}_2$  is chosen in a way that it is non zero only outside the manifold hence taking  $N \rightarrow \infty$  the potential rapidly grows outside of  $M$ .

Now we must note that this is all for only autonomous lagrangian systems since  $L$  has no explicit time dependence. More generally we would denote  $L : TM \times \mathbb{R} \rightarrow \mathbb{R}$ . The time dependent manifold is mapped to the euclidean space using  $i : M \times \mathbb{R} \rightarrow E^{3n}$  for a system of  $n$  particles. One can have situations where constraints itself are time dependent such as a bead on a rotating circle. For this case  $i(q, t) = (r \sin(q) \cos(\omega t), r \sin(q) \sin(\omega t), r \cos(q))$  and so it effectively is just a one dimensional problem now and the kinetic energy from the rotating circle can be grouped into the potential to make it  $V = mgr \cos(q) - m\omega^2 r^2 \sin^2(q)/2$ .

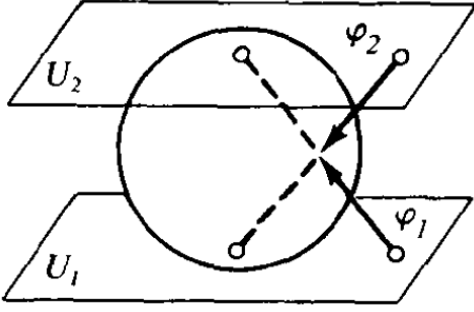


Figure 6: Stereographic projections of a sphere. Picture taken from [12]

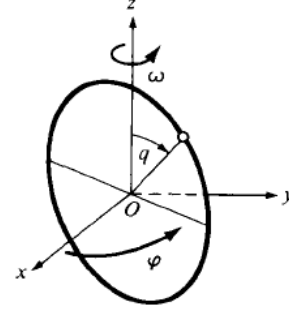


Figure 7: A bead constrained to move on a rotating circle. Picture taken from [12]

### 3.4 Some additional notes

So one may have noted that the topological manifold which is supposed to be the most general definition for any manifold. This definition however included two important points to eliminate pathological cases which were being Hausdorff and second countable. As we saw in section 1.1, closeness can be claimed without concerning oneself with a distance metric for that space.

There are some interesting non-Hausdorff spaces which show why it's best to have this as an additional condition. The most common example is that of a real line with two origins. It is defined with matching the real line at all points except at the origin which is left to two distinct points. One cannot picture it physically but the broken definition makes the two points for origin indistinguishable since their neighbourhoods will always have some non empty intersection. This line is represented by the quotient space of  $\mathbb{R} \times \{a\}$  and  $\mathbb{R} \times \{b\}$  with the equivalence relation  $(x, a) \sim (x, b)$ ,  $x \neq 0$  and we can see that  $0_a$  has no disjoint neighbourhood from  $0_b$ . More of such examples are termed as non-hausdorff manifolds and so here are our pathological examples.

## 4 Symmetry

We see symmetry arise in lots of situations in nature such as rotational invariance for wavefunctions when they have a potential only depending on radius. One thing to note is that every symmetry is associated with some conserved quantity and interestingly symmetry breaking can cause some interesting things too.

### 4.1 Noether's theorem

**Definition 4.1.** A lagrangian  $L$  on a manifold  $M$  admits the mapping  $h : M \rightarrow M$  if  $\forall v \in TM, L(h_*v) = L(v)$

For example the transformation  $h : (x_1, x_2, x_3) \rightarrow (x_1 + s, x_2, x_3)$  on  $L = (m/2)(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - U(x_2, x_3)$  does not change it and hence it admits  $h$ . Using this we state Noether's theorem

**Theorem 4.1** (Noether's theorem). If a lagrangian  $L$  on  $M$  admits the on-parameter group of diffeomorphisms  $h^s : M \rightarrow M$ ,  $s \in \mathbb{R}$  then the lagrangian system of equations corresponding to  $L$  has a first integral  $I : TM \rightarrow \mathbb{R}$  which in local coordinates  $\mathbf{q}$  on  $M$  can be written as

$$I(\mathbf{q}, \dot{\mathbf{q}}) = \left. \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{dh^s(\mathbf{q})}{ds} \right|_{s=0} \quad (57)$$

*Proof.* Let  $\varphi : \mathbb{R} \rightarrow M$ ,  $\mathbf{q} = \varphi(t)$  be a solutions to the Lagrange's equations. Since  $L$  admits  $h^s$ , this means  $h^s \varphi : \mathbb{R} \rightarrow M$  also satisfies the Lagrange's equations for all  $s \in \mathbb{R}$ . Since  $s$  is a continuous variable, this means that  $L$  does not depend explicitly on  $s$  hence we can write the following for  $\mathbf{q} = h^s(\varphi(t))$

$$\frac{\partial L(\Phi, \dot{\Phi})}{\partial s} = \frac{\partial L}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial s} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \dot{\mathbf{q}}}{\partial s} = 0 \quad (58)$$

However it is known that  $\mathbf{q}$  is a solution to the Lagrange equations and hence we can write  $\frac{\partial L}{\partial \mathbf{q}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}}$  and so we would write the following from the previous equation

$$\left( \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \frac{\partial \mathbf{q}}{\partial s} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \dot{\mathbf{q}}}{\partial s} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \mathbf{q}}{\partial s} \right) = 0 \quad (59)$$

Hence giving us the first integral to be the claimed expression.  $\square$

Now let us first understand what exactly a first integral by referring to it's definition [9].

**Definition 4.2** (First integral). Given a system of differential equations  $\Phi(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) = 0$ , the first integral is a non constant function  $I : TM \rightarrow \mathbb{R}$  such that  $dI(\mathbf{x}(t), \dot{\mathbf{x}}(t))/dt = 0$  for any solution of  $\Phi(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) = 0$ .

The reason it is referred to as an integral actually is due to the fact that in earlier times mathematicians used to integrate ODEs to get solutions and so the term "integral" is associated with solution in this sense. While one may see this quantity as just the conserved quantity due to symmetry (which it infact really is), there is more interesting physical observations to it. For autonomous systems, one can think of the first integral to represent a scalar potential and solutions represent the equipotential surfaces.

If a system admits rotations around an axis say  $e_1$ , the angular momentum with respect to that axis is conserved. We can see this by observing that if  $h^s$  is the rotation,  $(dh^s \mathbf{x}_i / ds)|_{s=0} = e_1 \times \mathbf{x}_i$  and so from this it follows that  $I = \sum_i \partial L / \partial \dot{\mathbf{x}}_i \cdot (e_1 \times \mathbf{x}_i)$  which is simply the total angular momentum about that axis since  $\partial L / \partial \dot{\mathbf{x}}_i = m_i \dot{\mathbf{x}}_i$  (assuming the system is natural).

While our proof dealt with an autonomous system, it can be extended to non autonomous systems too. If we define a new manifold  $M_1 = M \times \mathbb{R}$ , we can also define a new lagrangian for this system as  $L_1 : TM_1 \rightarrow \mathbb{R}$  by  $L_1 = L \frac{dt}{d\tau}$  where  $\tau$  is the parameter which we are using to parametrize everything including the actual time  $t$ . Hence we would write

$$L_1 \left( \mathbf{q}, t, \frac{d\mathbf{q}}{d\tau}, \frac{dt}{d\tau} \right) = L \left( \mathbf{q}, \frac{d\mathbf{q}}{d\tau}, t \right) \frac{dt}{d\tau} \quad (60)$$

We can now apply Noether's on  $L_1$  to get the first integral  $I_1 : TM_1 \rightarrow \mathbb{R}$  with no issues and since  $\int L dt = \int L_1 d\tau$  this integral reduces to  $I : TM \times \mathbb{R} \rightarrow \mathbb{R}$ . A particular interesting case here is when  $L$  has no explicit time dependence in which case translation of the time coordinate is a transformation that does not change the lagrangian and the quantity that is conserved here is the energy (a very common case in dynamical systems).

## 4.2 Stress-energy tensor

So far we have abstained from using the field theory notation with tensors and indices but now we will introduce the stress energy tensor (or energy momentum tensor) from the lagrangian density motivation. Previously we used continuous transformations on tangent vectors on a manifold, we will now instead deal with continuous transformations on fields  $\phi(x)$ . Let's write any continuous infinitesimal transformation as  $\phi'(x) = \phi(x) + \alpha \Delta \phi(x)$  where  $\alpha$  is an infinitesimal parameter and  $\Delta \phi$  is a deformation of the field. The lagrangian would have to transform invariant upto a 4-divergence to be conserved as we would have  $\mathcal{L}(x) = \mathcal{L}(x) + \alpha \partial_\mu \mathcal{J}^\mu(x)$ . Similar to how we proved Noether's theorem, we use the Euler Lagrange equations and we get the following current

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi - \mathcal{J}^\mu \quad (61)$$

This current follows the conservation law  $\partial_\mu j^\mu = 0$  which implies that there is some kind of conserved charge which can be expressed as  $Q = \int j^0 d^3x$ . In case the Lagrangian is invariant to the transformation we have  $\mathcal{J}^\mu = 0$  and we can see that  $j^\mu$  is nothing but the first integral in equation 57.

Now let's say that we are transforming the coordinates by some translation which effectively changes the field too. We would represent the transformation as follows

$$\phi(x + a) = \phi(x) + a^\mu \partial_\mu \phi(x) \quad (62)$$

Given that the lagrangian is scalar we would have it transform as  $\mathcal{L}' = \mathcal{L} + a^\nu \partial_\mu (\delta_\nu^\mu \mathcal{L})$ . On comparing with the previous case this is essentially a non zero  $\mathcal{J}^\mu$  and to take this into account one can obtain four separately conserved currents by the following tensor

$$T_\nu^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta_\nu^\mu \quad (63)$$

This gives us the conserved charge to be the Hamiltonian which is conserved with time translations.

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x \quad (64)$$

The other spatial translations are associated with a physical momentum carried by the field

$$P^i = \int T^{0i} d^3x = - \int \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} \partial_i \phi d^3x \quad (65)$$

### 4.3 Symmetry breaking

A symmetric world would be dull [15]

Let us consider the following lagrangian density

$$\mathcal{L} = \frac{1}{2} [(\partial) \vec{\varphi}]^2 - \mu^2 \vec{\varphi}^2] - \frac{\lambda}{4} (\vec{\varphi}^2)^2 \quad (66)$$

Here  $\vec{\varphi}$  is an  $N$  component vector. We can see that  $L$  admits all transformations which rotate  $\vec{\varphi}$  which is essentially  $O(N)$  symmetry. Here  $\vec{\varphi}$  is essentially a field. One can manually remove the symmetry present by adding terms which simply depend on certain components alone however there is a way for spontaneous symmetry breaking if we define our lagrangian slightly differently.

$$\mathcal{L}' = \frac{1}{2} [(\partial) \vec{\varphi}]^2 + \mu^2 \vec{\varphi}^2] - \frac{\lambda}{4} (\vec{\varphi}^2)^2 \quad (67)$$

This is essentially what looks like a double well potential's lagrangian where we have

$$V(\varphi) = -\frac{1}{2} \mu^2 \vec{\varphi}^2 + \frac{\lambda}{4} (\vec{\varphi}^2)^2 \quad (68)$$

This has two minimas which are symmetric about the origin in the one dimensional case. In quantum mechanics, presence in one of the minimas would imply the possibility of being able to tunnel to the other minima too. However the tunneling barrier is now  $[V(0) - V(\pm v)] \int d^D x$  where  $D$  is the dimensions of space (1 in this case) and so tunneling is shut down as this barrier is effectively infinite. Hence the ground state must make a choice between two equivalent situations showing that naturally the symmetry is destroyed. If we choose the ground state to be at  $+v$  we can write  $\varphi = v + \varphi'$  and expanding about  $\varphi'$  we get

$$\mathcal{L}' = \frac{\mu^4}{4\lambda} + \frac{1}{2} (\partial \varphi')^2 - \mu^2 \varphi'^2 - \mathcal{O}(\varphi'^3) \quad (69)$$

The shifted field  $\varphi'$  has a mass of  $\sqrt{2}\mu$  on comparing with equation 68 which is an interesting point to note. On

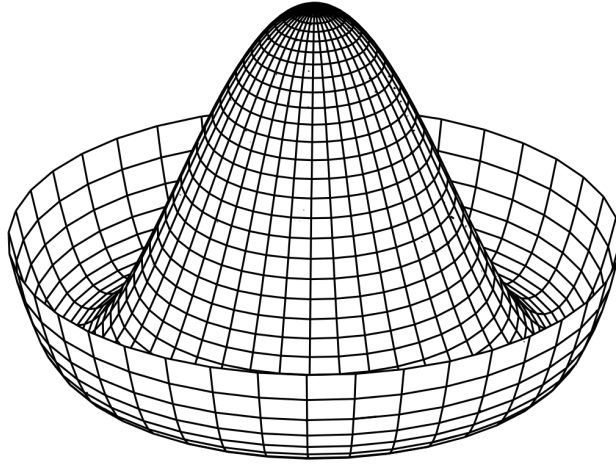


Figure 8: The sombrero potential. [Picture credits.](#)

extending to two dimensions, we get the sombrero potential (figure 8) which has  $O(2)$  symmetry that will be broken

for the ground state since one would have choose an angle and so this is a breaking of continuous symmetry. Since the choice is arbitrary we can pick a point with  $\varphi_1 = v$  and  $\varphi_2 = 0$  and on expanding we get

$$L' = \frac{\mu^4}{4\lambda} + \frac{1}{2}(\partial\vec{\varphi}')^2 - \mu^2\varphi_1'^2 - \mathcal{O}(\varphi'^3) \quad (70)$$

We can see now that the field  $\varphi_2$  seems to have no mass and  $\varphi = \sqrt{2}\mu$ .

**Theorem 4.2** (Goldstone's theorem). Whenever a continuous symmetry is spontaneously broken, massless fields, known as Nambu-Goldstone bosons emerge.

*Proof.* Every continuous symmetry has a conserved charge  $Q$  which generates a symmetry which relates to the Hamiltonian as  $[H, Q] = 0$ . Let's say the ground state is  $|0\rangle$  and the Hamiltonian is shifted by a constant so that  $H|0\rangle = 0$ . Normally the ground state would have to be invariant under symmetry transform however that does not hold here so  $Q|0\rangle \neq |0\rangle$ . So let's say we want the energy of this transformed state which would be  $HQ|0\rangle = (HQ - QH + QH)|0\rangle = 0$ . More importantly from the Fabri-Picasso theorem,  $Q|0\rangle = 0$  as otherwise it would not be able to exist in the Hilbert space. This is due to  $Q(t) = \int dx^D J^\mu(t)$  and the correlation function  $\langle 0|QQ|0\rangle$  will end up going to infinity unless it's zero itself.  $\square$

## 5 Klein-Gordan Field

Before we go on to the actual nature of this field, it is important to understand what makes this necessary in the first place. It is seen that when writing a single particle relativistic wave equation (such as the Klein-Gordan equation or the Dirac equation) there are negative energy states which arise. These are resolved in the Dirac equation using a concept called the Dirac Sea [4].

The necessity of a multiparticle theory arises from having causality. Taking the propagation amplitude of a free particle from one point in space  $\mathbf{x}_0$  to another  $\mathbf{x}$ . We would have the following expression for the same

$$U(t) = \langle \mathbf{x} | e^{-iHt} | \mathbf{x}_0 \rangle \quad (71)$$

One can expand this in the relativistic expression of  $E = \sqrt{p^2 + m^2}$  and on reduction of the integral the following expression is obtained

$$U(t) = \frac{1}{2\pi^2|\mathbf{x} - \mathbf{x}_0|} \int_0^\infty dp \sin(p|\mathbf{x} - \mathbf{x}_0|) p e^{-it\sqrt{p^2 + m^2}} \quad (72)$$

This can be evaluated in terms of Bessel functions [6]. What concerns us is the asymptotic behaviour well outside the light cone  $\mathbf{x}^2 \gg t^2$ . The phase function would have a stationary point at  $p = imx/\sqrt{x^2 - t^2}$  and so the contour integral can be pushed to this point and we get that  $U(t) \sim e^{-m\sqrt{x^2 - t^2}}$  upto some rational function of  $x$  and  $t$ . This shows that there is a propagation amplitude that is vanishing but non zero which goes outside the light cone and that is not a good thing clearly as we no longer have causality.

The Klein Gordan equation is the equations of motion of a very basic lagrangian density shown below

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \quad (73)$$

Which gives us the Klein-Gordan equation below

$$(\partial^\mu\partial_\mu + m^2)\phi = 0 \quad (74)$$

### 5.1 Klein Gordan field as Harmonic Oscillators

Here we will quantize classical field theory by promoting  $\phi$  and  $\pi = \partial\mathcal{L}/\partial\dot{\phi}$  (canonical momenta) into operators much like how we treat  $q$  (position) and  $p$  (momentum) as operators in quantum mechanics. Similar to our very familiar commutation relations in equation 14 we write the following

$$\begin{aligned} [\phi(\mathbf{x}), \pi(\mathbf{y})] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ [\phi(\mathbf{x}), \phi(\mathbf{y})] &= [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0 \end{aligned} \quad (75)$$

Now our aim would be to construct the Hamiltonian that describes the system for the Klein-Gordan field. We can start that off by taking the Fourier transform of the field

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t) \quad (76)$$



The Klein-Gordon equation in equation 74 becomes into

$$\left[ \frac{\partial^2}{\partial t^2} + (|\mathbf{p}|^2 + m^2) \right] \phi(\mathbf{p}, t) = 0 \quad (77)$$

However this really is the same as the equation of a harmonic oscillator with frequency of  $\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$ . Now similar to how we have written the ladder operators for a simple harmonic oscillator, we can do the same for every individual Fourier mode of the field hence treating each mode as an independent oscillator with it's own ladder operations.

$$\phi(\mathbf{x}) = \int \frac{d^3}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}}) \quad (78)$$

$$\pi(\mathbf{x}) = \int \frac{d^3}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}}) \quad (79)$$

We will find it useful to write  $a_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}} = a_{-\mathbf{p}}^\dagger e^{i\mathbf{p} \cdot \mathbf{x}}$  which would make further calculations simpler. We note that we have the following commutation relation hold

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \quad (80)$$

It can be checked using this commutator relation that it equivalently gives  $[\phi(\mathbf{x}), \pi(\mathbf{x}')] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}')$ . Given a  $\phi$  and  $\pi$  we can construct the Hamiltonian from these as follows

$$H = \int d^3x \mathcal{H} = \int d^3x \left[ \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right] \quad (81)$$

Using this we can now finally construct the Hamiltonian for the Klein-Gordon Field.

$$\begin{aligned} H &= \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(\mathbf{p}+\mathbf{p}') \cdot \mathbf{x}} \left\{ -\frac{\sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{p}'}}}{4} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger)(a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) + \frac{m^2 - \mathbf{p} \cdot \mathbf{p}'}{4\sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{p}'}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) \right\} \\ &= \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left( a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2} [a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] \right) \end{aligned} \quad (82)$$

One may notice that the second term is proportional to  $\delta(0)$  which is essentially infinite which while disturbing is not a real issue since energy differences are what matter. This is due to the sum of all the zero point energies being summed over for all the Harmonic oscillators. However one cannot ignore this zero point energy always as one can see in the Casimir effect [3].

$$[H, a_{\mathbf{p}}^\dagger] = \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger \quad [H, a_{\mathbf{p}}] = -\omega_{\mathbf{p}} a_{\mathbf{p}} \quad (83)$$

We now see that we have the state  $|0\rangle$  which we will treat as the ground state for all  $\mathbf{p}$  since  $a_{\mathbf{p}} |0\rangle = 0$  and will be referred to as the vacuum. We can apply creation operators on this to create states  $a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger \dots |0\rangle$  which would have energy  $\omega_{\mathbf{p}} + \omega_{\mathbf{q}} + \dots$  and these are the states that are present in the spectrum. We can on this basis create a total

momentum operator as  $\int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ . We can call these excitations created by the creation operations to simply be particles each with a different momentum wavefunction. The states  $a_{\mathbf{p}} a_{\mathbf{q}} |0\rangle$  and  $a_{\mathbf{q}} a_{\mathbf{p}} |0\rangle$  are equivalent since these operations commute and also a single mode  $\mathbf{p}$  can contain arbitrarily many particles which would be driven by applying any number of  $a_{\mathbf{p}}^\dagger$  and so these particles obey Bose-Einstein statistics.

We would define the vacuum state to be normalized hence  $\langle 0 | 0 \rangle = 1$ . However if we similarly define our states as  $|\mathbf{p}\rangle = a_{\mathbf{p}}^\dagger |0\rangle$  upto a constant and  $\langle \mathbf{p} | \mathbf{q} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$  the normalization does not remain Lorentz invariant. This is due to the fact that  $\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$  and so to correct for this we simply define the states as

$$|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^\dagger |0\rangle \quad \langle \mathbf{p} | \mathbf{q} \rangle = 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \quad (84)$$

Since we have  $E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q})$  as Lorentz invariant. However introducing this extra factor will also require we define the completeness relation for one particle as

$$\mathbb{I}_{\text{particle}} = \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}\rangle \frac{1}{2E_{\mathbf{p}}} \langle \mathbf{p}| \quad (85)$$

In fact the integral  $\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}}$  is a Lorentz invariant integral in the sense that if  $f(p)$  is Lorentz invariant, so is  $\int \frac{d^3p}{(2\pi)^3} \frac{f(p)}{2E_{\mathbf{p}}}$ . This can be seen as the integral over the  $p^0 > 0$  branch of the hyperboloid  $p^2 = m^2$  in the 4-momentum space if written as follows

$$\int \frac{d^3p}{(2\pi)^3} \frac{f(p)}{2E_{\mathbf{p}}} = \int \frac{d^4p}{(2\pi)^3} \frac{f(p)}{\delta} (p^2 - m^2) \Big|_{p^0 > 0} \quad (86)$$

This finally leads us to interpretation of  $\phi(x) |0\rangle$ . As per how  $\phi$  is written in equation 80 we get

$$\phi(\mathbf{x}) |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-i\mathbf{p}\cdot\mathbf{x}} |\mathbf{p}\rangle \quad (87)$$

We will be treating this as  $|\mathbf{x}\rangle$  since we can see for the non relativistic case  $E_{\mathbf{p}}$  is nearly constant hence this simply sums all plane waves hence would give a point particle.

## 5.2 Solving the causality problem

A thing to note is that the previous subsection stuck to the Schrödinger picture. We will now proceed to use the Heisenberg picture. Using the Heisenberg equations of motion we can write the following

$$\begin{aligned} i\frac{\partial}{\partial t}\phi(\mathbf{x}, t) &= [\phi(\mathbf{x}, t), \int d^3x' \left\{ \frac{1}{2}\pi^2(\mathbf{x}', t) + \frac{1}{2}(\nabla\phi(\mathbf{x}', t))^2 + \frac{1}{2}m^2\phi^2(\mathbf{x}', t) \right\}] \\ &= \int d^3x' \left( i\delta^3(\mathbf{x} - \mathbf{x}')\pi(\mathbf{x}', t) \right) \\ &= i\pi(\mathbf{x}, t) \end{aligned} \quad (88)$$

$$\begin{aligned} i\frac{\partial}{\partial t}\pi(\mathbf{x}, t) &= [\pi(\mathbf{x}, t), \int d^3x' \left\{ \frac{1}{2}\pi^2(\mathbf{x}', t) + \frac{1}{2}\phi(\mathbf{x}', t)(-\nabla^2 + m^2)\phi(\mathbf{x}', t) \right\}] \\ &= \int d^3x' \left( i\delta^3(\mathbf{x} - \mathbf{x}')(-\nabla^2 + m^2)\phi(\mathbf{x}', t) \right) \\ &= -i(-\nabla^2 + m^2)\phi(\mathbf{x}, t) \end{aligned} \quad (89)$$

Using the commutation relation of  $H$  and  $a_{\mathbf{p}}$  we can write  $Ha_{\mathbf{p}} = a_{\mathbf{p}}(H - E_{\mathbf{p}})$  which can be extended to  $H^n a_{\mathbf{p}} = a_{\mathbf{p}}(H - E_{\mathbf{p}})^n$ . Using this we get the identities

$$e^{iHt} a_{\mathbf{p}} e^{-iHt} = a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} \quad e^{iHt} a_{\mathbf{p}}^\dagger e^{-iHt} = a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t} \quad (90)$$

We would similarly get the same identities using the total momentum operator  $\mathbf{P}$  instead of  $H$  and  $\mathbf{p}$  instead of  $E_{\mathbf{p}}$ . This lets us relate  $\phi(x)$  to  $\phi(0)$  as

$$\begin{aligned} \phi(x) &= e^{i(Ht - \mathbf{P}\cdot\mathbf{x})} \phi(0) e^{-i(Ht - \mathbf{P}\cdot\mathbf{x})} \\ &= e^{iP\cdot x} \phi(0) e^{-iP\cdot x} \end{aligned} \quad (91)$$

Here the notation used is  $P^\mu = (H, \mathbf{P})$ . We must note that both  $e^{-iP^0 t}$  and  $e^{iP^0 t}$  appear in the expressions which shows positive and negative energy modes respectively. We will refer to these as positive and negative frequency modes. The positive frequency solution has its coefficient as the operator which destroys a particle in that single particle wavefunction and the negative mode has one which creates a particle.

As per how the operators have been defined, the amplitude for a particle to propagate to  $y$  from  $x$  is  $\langle 0 | \phi(x) \phi(y) | 0 \rangle$  which we will call as  $D(x-y)$ . Since each operator  $\phi$  is composed of the ladder operators, only the term  $\langle 0 | a_{\mathbf{p}} a_{\mathbf{q}}^\dagger | 0 \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q})$  will be non zero. The expression which is left is

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-iP\cdot(x-y)} \quad (92)$$



We already know that this kind of integral is Lorentz invariant. Let's say that  $x - y$  is timelike separated we get  $D(x - y) \sim e^{-\imath mt}$  taking  $t \rightarrow \infty$ . When they are spatially separated ( $\mathbf{x} - \mathbf{y} = \mathbf{r}$ ) the integral becomes as follows

$$\begin{aligned} D(x - y) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{\imath \mathbf{p} \cdot \mathbf{r}} \\ &= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2E_{\mathbf{p}}} \frac{e^{\imath pr} - e^{-\imath pr}}{\imath pr} \\ &= \frac{-\imath}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \frac{p e^{\imath pr}}{\sqrt{p^2 + m^2}} \end{aligned} \quad (93)$$

There are two poles here which are  $p = \pm \imath m$  so we can push the contours around one of the poles and substitute  $\rho = -\imath p$  to get  $D(x - y) \sim e^{-mr}$  when  $r \rightarrow \infty$ . Seeing these expressions it seems that our aim to solve the causality problem which we made so apparent does not really happen. However in the context of quantum mechanics, one would have to refer to causality as the act of one measurement affecting another. One knows that measurements that have a zero commutator do not affect each other and so we can pick the simplest measurement here which is  $\phi$  and find the value of  $[\phi(x), \phi(y)]$

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{q}}}} [(a_{\mathbf{p}} e^{-\imath p \cdot x} + a_{\mathbf{p}}^\dagger e^{\imath p \cdot x}), (a_{\mathbf{q}} e^{-\imath q \cdot y} + a_{\mathbf{q}}^\dagger e^{\imath q \cdot y})] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} (e^{-\imath p \cdot (x-y)} - e^{\imath p \cdot (x-y)}) \\ &= D(x - y) - D(y - x) \end{aligned} \quad (94)$$

Now if  $(x - y)^2 < 0$ , we can perform a Lorentz transformation on the second term (the integral is Lorentz invariant so this shouldn't change anything) which takes  $(x - y)$  to  $(y - x)$ . Such a transformation is only possible if they have a spacelike separation hence showing that in such a case the commutator is zero so causality is held.

The case of a complex Klein Gordan field happens to show some interesting properties. When the complex scalar field is quantized,  $\phi(x)$  creates positively charged particles and destroys negatively charged particles while  $\phi^\dagger(x)$  does the opposite. As a result the commutator  $[\phi(x), \phi^\dagger(y)]$  would have non zero contributions which must cancel outside the light cone to maintain causality. The term  $\phi(x)\phi^\dagger(y)$  represents the propagation of a negatively charged particle from  $y$  to  $x$  and  $\phi^\dagger(y)\phi(x)$  represents propagation of a positively charged particle from  $x$  to  $y$ . For both of these to exist we would need these particles to have the same mass and so these cancel out. The causality in quantum field theory requires that for every particle there is a corresponding antiparticle which has the same mass but opposite charge. When the Klein Gordan field is real, a particle is it's own antiparticle.

### 5.3 Complex Klein-Gordan Field

We will now look more deeply into the complex valued field. If  $\phi(x)$  is complex we can write it as  $\phi(x) = (\phi_1(x) + \phi_2(x)\imath)/\sqrt{2}$  where  $\phi_1(x)$  and  $\phi_2(x)$  are both real valued functions. Since  $\phi(x)$  is a solution, this would require both  $\phi_1(x)$  and  $\phi_2(x)$  must be solutions since  $(\partial_\mu \partial^\mu + m^2)\phi_i(x)$  would be real hence we equate both to zero. Since  $\phi^\dagger(x) = (\phi_1(x) - \phi_2(x)\imath)/\sqrt{2}$  this means we have

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = 0 \quad (95)$$

$$(\partial_\mu \partial^\mu + m^2)\phi^\dagger(x) = 0 \quad (96)$$

Since  $\phi_1(x)$  and  $\phi_2(x)$  are two separate solutions for the equation that are real valued we can add their contributions to the Lagrangian density as follows

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} (\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2) - \frac{m^2}{2} (\phi_1^2 + \phi_2^2) \\ &= \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \end{aligned} \quad (97)$$

Similarly the Hamiltonian density would add up and we would have

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} (\pi_1^2 + \pi_2^2) + \frac{1}{2} (\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2) + \frac{m^2}{2} (\phi_1^2 + \phi_2^2) \\ &= \pi^\dagger \pi + \partial_\mu \phi^\dagger \partial^\mu \phi + m^2 \phi^\dagger \phi \end{aligned} \quad (98)$$

Due to having two real valued fields as the solutions, we would have two sets of ladder operators for each momentum value. We will call them as  $a_{\mathbf{p}}^{(1)}$  and  $a_{\mathbf{p}}^{(2)}$ . Using them we define  $a_{\mathbf{p}} = (a_{\mathbf{p}}^{(1)} + a_{\mathbf{p}}^{(2)})/\sqrt{2}$  and  $b_{\mathbf{p}} = (a_{\mathbf{p}}^{(1)} - a_{\mathbf{p}}^{(2)})/\sqrt{2}$  which lets us define

$$\phi(\mathbf{x}) = \int \frac{d^3}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}) \quad (99)$$

#### 5.4 Detour: Jaynes-Cumming model

The Jaynes cumming model was introduced to explain the interaction of an atom with an electromagnetic field. The problem of an atom interacting with an electromagnetic had received semi-classical treatments which had however failed to explain all the characteristics of such a system. It received a proper quantum treatment where the field was quantized in the Jaynes-Cummings Hamiltonian [7]. This succesfully explained very important characteristics of the system like Rabi oscillations [10] and the collapse and revival of the system [5].

The field is treated as photons in a cavity where each particle is essentially treated as an excitation in this field. One can think of this as a Klein Gordan field but all the  $\omega_{\mathbf{p}}$  is set to only one value effectively just making this into a normal SHO. The full Jaynes-Cummings Hamiltonian is given as follows: (ignoring the zero-point energy)

$$\begin{aligned} \mathbf{H}_{JC} &= \mathbf{H}_A + \mathbf{H}_F + \mathbf{H}_{A-F} \\ &= \hbar\omega_c \hat{a}_c^\dagger \hat{a}_c + \hbar\omega_{eg} |e\rangle\langle e| + \hbar g_c (\hat{\sigma}_+ \hat{a}_c + \hat{\sigma}_- \hat{a}_c^\dagger) \end{aligned} \quad (100)$$

Here  $\hat{a}_c$  is the ladder operation for the cavity field and  $\sigma_{\pm}$  are the ladder operators of the two level atom ( $|e\rangle, |g\rangle$ ). Shown below are some state evolution simulations of this system done using QuTip.

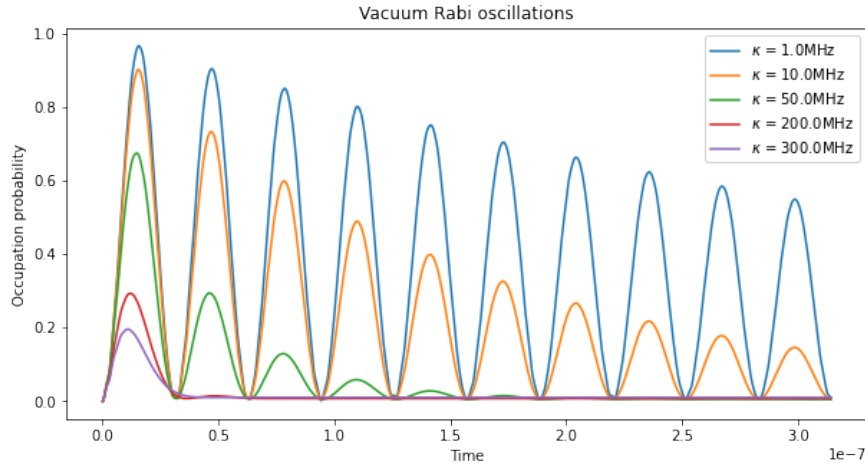


Figure 9: Shown above is a simulation of a two level atom which has been entangled with a cavity with 16 fock states which starts off in the first excited state. QuTip has been used for the simulation. The value of the coupling  $g = 100$  MHz, the decay for the atom is  $\gamma = 3$  MHz and the cavity decay is  $\kappa$  which is varying.

## 6 Dirac Field

### References

- [1] Koro (127). *Poincare recurrence theorem*. [Online; accessed 08-June-2021]. 2013. URL: <https://planetmath.org/poincarerecurrencetheorem>.
- [2] Mark Alford. *Legendre Transforms*. 2019. URL: [https://web.physics.wustl.edu/alford/physics/Legendre\\_introduction.pdf](https://web.physics.wustl.edu/alford/physics/Legendre_introduction.pdf).
- [3] H. B. G. Casimir and D. Polder. “The Influence of Retardation on the London-van der Waals Forces”. In: *Phys. Rev.* 73 (4 Feb. 1948), pp. 360–372. DOI: [10.1103/PhysRev.73.360](https://doi.org/10.1103/PhysRev.73.360). URL: <https://link.aps.org/doi/10.1103/PhysRev.73.360>.

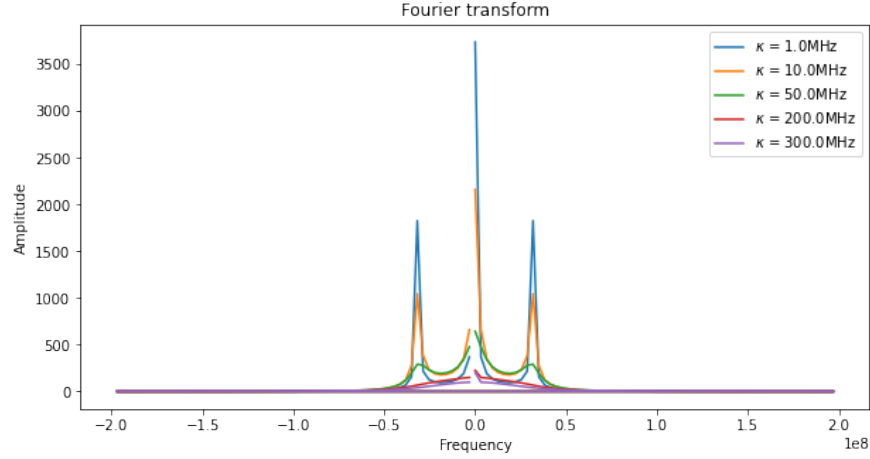


Figure 10: Fourier transform of the Rabi oscillations shown in the previous figure. As we can see, we have three modes arise as shown in [7].

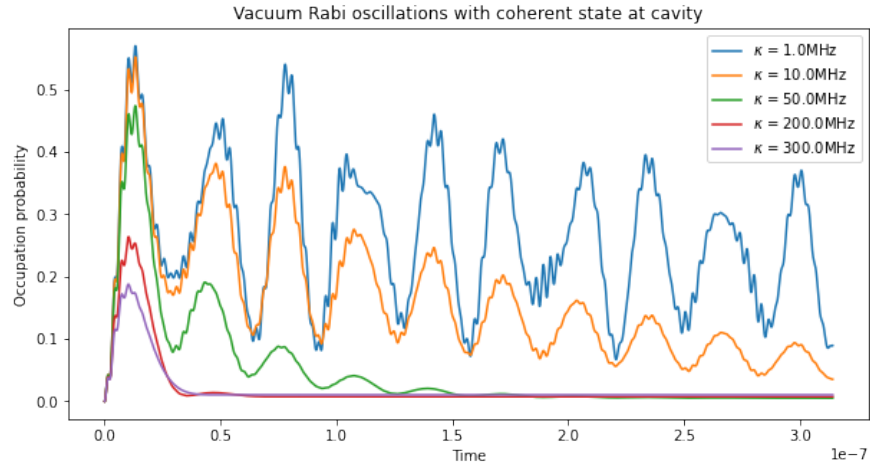


Figure 11: Shown above is a simulation of a two level atom which has been entangled with a cavity with 16 fock states which starts off in the coherent state with  $\alpha = 1$ . The value of the coupling  $g = 100$  MHz, the decay for the atom is  $\gamma = 3$  MHz and the cavity decay is  $\kappa$  which is varying.

- [4] Paul Adrien Maurice Dirac and Ralph Howard Fowler. “A theory of electrons and protons”. In: *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* 126.801 (1930), pp. 360–365. DOI: [10.1098/rspa.1930.0013](https://doi.org/10.1098/rspa.1930.0013). eprint: <https://royalsocietypublishing.org/doi/pdf/10.1098/rspa.1930.0013>. URL: <https://royalsocietypublishing.org/doi/abs/10.1098/rspa.1930.0013>.
- [5] J. H. Eberly, N. B. Narozhny, and J. J. Sanchez-Mondragon. “Periodic Spontaneous Collapse and Revival in a Simple Quantum Model”. In: *Phys. Rev. Lett.* 44 (20 May 1980), pp. 1323–1326. DOI: [10.1103/PhysRevLett.44.1323](https://doi.org/10.1103/PhysRevLett.44.1323). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.44.1323>.
- [6] Ryzhik. Gradshteyn. *Table of integrals, series, and products*. 5th ed. AP, 1996.
- [7] E.T. Jaynes and F.W. Cummings. “Comparison of quantum and semiclassical radiation theories with application to the beam maser”. In: *Proceedings of the IEEE* 51.1 (1963), pp. 89–109. DOI: [10.1109/PROC.1963.1664](https://doi.org/10.1109/PROC.1963.1664).
- [8] Brian Kend. *Manifolds: A Gentle Introduction*. 2018. URL: <https://bjlkeng.github.io/posts/manifolds/>.
- [9] Encyclopedia of Mathematics. *First Integral*. URL: [http://encyclopediaofmath.org/index.php?title=First\\_integral&oldid=36265](http://encyclopediaofmath.org/index.php?title=First_integral&oldid=36265).

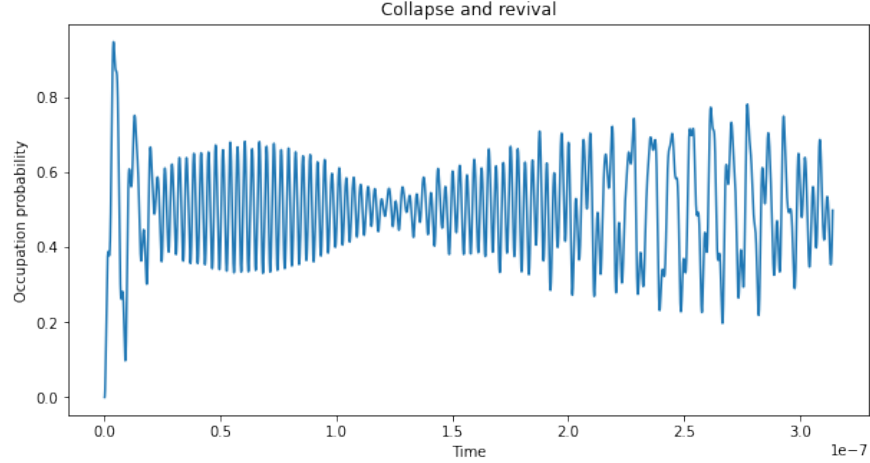


Figure 12: Demonstration of collapse and revival for a cavity with 32 fock states which starts off in the coherent state of  $\alpha = 4$ . As we can see the probability dies down and goes to 0.5 a little after  $0.1\mu\text{s}$  and then again rises and oscillates.

- [10] I. I. Rabi. “Space Quantization in a Gyating Magnetic Field”. In: *Phys. Rev.* 51 (8 Apr. 1937), pp. 652–654. DOI: [10.1103/PhysRev.51.652](https://link.aps.org/doi/10.1103/PhysRev.51.652). URL: <https://link.aps.org/doi/10.1103/PhysRev.51.652>.
- [11] Harish Seshadari. *Short Talk-What is a Manifold-I*. Youtube. 2017. URL: <https://www.youtube.com/watch?v=cLRtp5yytAg>.
- [12] K. Vogtmann V. I. Arnold A. Weinstein. *Mathematical Methods Of Classical Mechanics*. 2nd. Graduate Texts in Mathematics. Springer, 1989. ISBN: 9780387968902,0387968903,3540968903,9783540968900.
- [13] Wikipedia contributors. *Liouville’s formula*. [Online; accessed 08-June-2021]. 2021. URL: [https://en.wikipedia.org/wiki/Liouville%27s\\_formula](https://en.wikipedia.org/wiki/Liouville%27s_formula).
- [14] Wikipedia contributors. *Manifold*. [Online; accessed 09-June-2021]. 2021. URL: <https://en.wikipedia.org/wiki/Manifold>.
- [15] A. Zee. *Quantum Field Theory in a Nutshell*. illustrated edition. Princeton University Press, 2003. ISBN: 9780691010199,0-69-1-01019-6.