Qudits for efficiently generating higher moments of Hermitian operators

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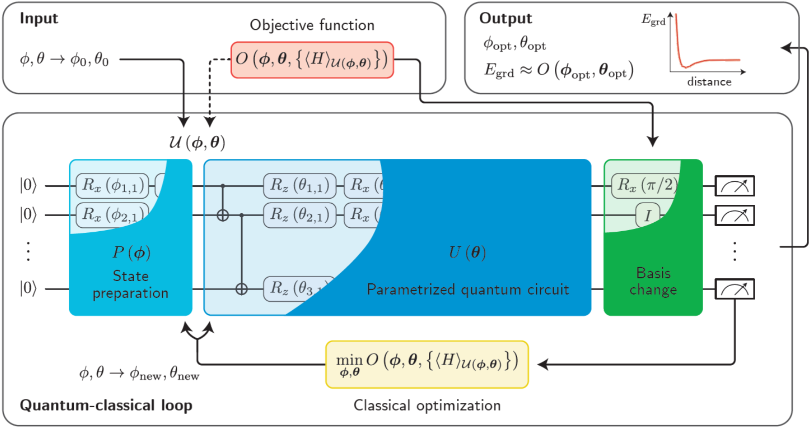


Figure: The workflow of a variational quantum algorithm. Image taken from *K. Bharti et al. 2021* [4]

Motivation for H^k

Quantum assisted eigensolver

Alternate approach: quantum assisted eigensolver
K. Bharti & T. Haug, Phys. Rev. A104 (2021) [2, 3]

$$H = \sum_{i=1}^{n} \beta_i U_i \tag{1}$$

We now pick a basis of m states $\{|\phi_j\rangle\}_{j=1}^{m}$ that satisfies the condition that $\langle\phi_j|\phi_k\rangle = 1$ iff $j = k$ and $|\phi_j\rangle = V^j\,|0^{\otimes N}\rangle$.

$$|\psi(\boldsymbol{\alpha})\rangle = \sum_{j=1}^{m} \alpha_j\,|\phi_j\rangle\,, \quad \boldsymbol{\alpha} \in \mathbb{C}^m \tag{2}$$

Task: find $\boldsymbol{\alpha}$ such that it minimizes $\langle H(\boldsymbol{\alpha})\rangle = \langle\psi(\boldsymbol{\alpha})|\,H\,|\psi(\boldsymbol{\alpha})\rangle$

$$D_{j,k} = \sum_i \beta_i\,\langle\phi_j|\,U_i\,|\phi_k\rangle\,, \quad E_{j,k} = \langle\phi_j|\phi_k\rangle \tag{3}$$

This lets us write $\langle H(\boldsymbol{\alpha})\rangle = \sum_{j,k} \alpha_j^* D_{j,k}\alpha_k = \boldsymbol{\alpha}^\dagger D\boldsymbol{\alpha}$

$$\text{minimize } \boldsymbol{\alpha}^\dagger D\boldsymbol{\alpha} \quad \text{subject to } \boldsymbol{\alpha}^\dagger E\boldsymbol{\alpha} = 1 \tag{4}$$

Motivation for H^k
Quantum assisted eigensolver

The benefits of the quantum assisted eigensolver:

- There is no quantum-classical feedback loop and so the tasks are completely separated hence no need to keep running the circuit.
- Barren plateaus (*J.R. McClean et al., Nature Comm. 2018* [11]) are avoided by not requiring high depth circuit ansatz.

A main requirement: efficiently construct the ansatz. One form of ansatz called Krylov subspace.

$$Kr_K = \text{span}\{|\psi\rangle, H|\psi\rangle, \ldots, H^K|\psi\rangle\} \tag{5}$$

Hence we need a way to efficiently get H^k.

Motivation for H^k
Quantum computed moments

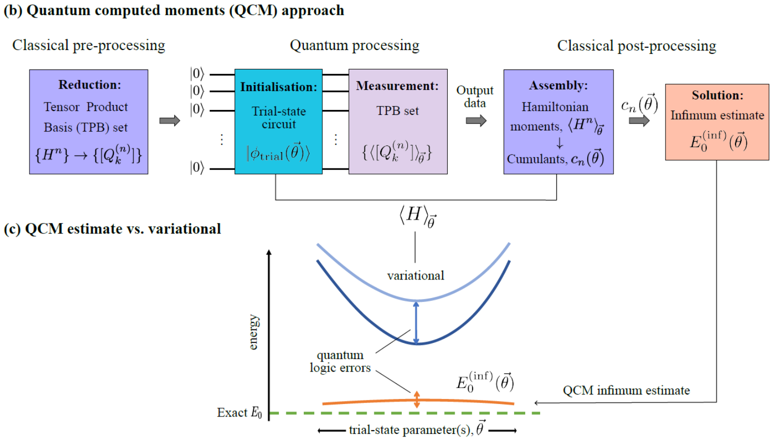


Figure: Depiction of the quantum computed moments method to aid in VQEs. Figure taken from *H. Vallury et al., Quantum (2020)* [13]

Motivation for H^k
Quantum computed moments

Finding these higher moments aid in many-body energy estimations shown by Lanczos expansion theory (*L. Hollenberg & N. Witte, PRB (1996)*) [8].

Higher moments of Hermitian operators also have uses in imaginary time evolution [1] and many tasks such as the HHL Algorithm [7] rely on Hamiltonian simulation subroutines.

Proposed solution using qudits
Block Encoding

Definition (Block encoding)

Given $\hat{H} : \mathcal{H}_S \to \mathcal{H}_s$ with $\|\hat{H}\| \leq 1$, and $\hat{G}\,|0\rangle_a = |G\rangle_a \in \mathcal{H}_a$, we define an encoding for $\hat{H}$ as the unitary $\hat{U} : \mathcal{H}_a \otimes \mathcal{H}_s \to \mathcal{H}_a \otimes \mathcal{H}_s$. This encoding satisfies the property $(\langle G|_a \otimes \hat{\mathcal{I}}_s)\hat{U}(|G\rangle_a \otimes \hat{\mathcal{I}}_s)) = \hat{H}$.

An explicit encoding for LCU

$$\hat{H} = \sum_{j=1}^{d} \alpha_j \hat{U}_j, \qquad \|\hat{H}\| \leq \|\vec{\alpha}\|_1 = \sum_{j=1}^{d} |\alpha_j| \tag{6}$$

The upper bound on the spectral norm, for a certain choice of decomposition is a tight bound.

$$\hat{G} = \sum_{j=1}^{d} \sqrt{\frac{\alpha_j}{\|\vec{\alpha}\|_1}} |j\rangle\langle 0|_a, \qquad \hat{U} = \sum_{j=1}^{d} |j\rangle\langle j|_a \otimes \hat{U}_j \tag{7}$$

$\langle G|\hat{U}|G\rangle = \hat{H}/\|\vec{\alpha}\|_1$ where $|G\rangle = \hat{G}\,|0\rangle$

Proposed solution using qudits

Block Encoding

To get higher powers: iterate approach using qubitization
G. H. Low & I. L. Chuang, Quantum (2019) [10]
For some eigenvector $\hat{H}\,|\lambda\rangle = \lambda\,|\lambda\rangle$ we can define the subspace $\mathcal{H}_\lambda = \mathrm{span}\{|G_\lambda\rangle\,, \hat{W}\,|G_\lambda\rangle\}$. $|G_\lambda\rangle = |G\rangle\,|\lambda\rangle$.

$$\hat{W}\,|G\rangle\,|\lambda\rangle = \lambda\,|G_\lambda\rangle + \sqrt{1-|\lambda|^2}\,|G_\lambda^\perp\rangle\,, \quad |G_\lambda^\perp\rangle = \frac{(\hat{W}-\lambda)\,|G_\lambda\rangle}{\sqrt{1-|\lambda|^2}} \tag{8}$$

$$\hat{X}_\lambda\,|G_\lambda\rangle = |G_\lambda^\perp\rangle\,, \quad \hat{Y}_\lambda\,|G_\lambda\rangle = \iota\,|G_\lambda^\perp\rangle\,, \quad \hat{Z}_\lambda\,|G_\lambda\rangle = |G_\lambda\rangle \tag{9}$$

For each eigenvalue the iterate acts as per the following

$$\hat{W} = \begin{matrix} \lambda|G_\lambda\rangle\langle G_\lambda| & -\sqrt{1-|\lambda|^2}|G_\lambda\rangle\langle G_\lambda^\perp| \\ +\sqrt{1+|\lambda|^2}|G_\lambda^\perp\rangle\langle G_\lambda| & +\lambda|G_\lambda^\perp\rangle\langle G_\lambda^\perp| \end{matrix} \tag{10}$$

Define the iterate to act as shown for each separate $\mathcal{H}_\lambda$ subspace

$$\hat{W}^n = \bigoplus_\lambda e^{-\iota \hat{Y}_\lambda n \cos^{-1}(\lambda)} \tag{11}$$

This means that $\langle G| \hat{W}^n |G\rangle |\psi\rangle = f_n(\hat{H}) |\psi\rangle$ ($f_n(\cos(\theta)) = \cos(n\theta)$).

- **Qubitization:** to obtain an encoding that can be used as an iterate such as $\hat{W}$.
- **Conditions for qubitization:** for all unitary $\hat{U}$ that encode $\hat{H}$, there exists a $\hat{U}'$, which uses one extra qubit and queries a controlled-$\hat{U}$ and controlled-$\hat{U}^\dagger$ once to implement an encoding that satisfies the properties required [10].

Hence we need a way to make arbitrary controlled gates efficiently.

Making a controlled version of an unknown unitary using 4-level qudits.
Zhou et al., Nature comm. (2011) [14]
We first define the internal swap gate called X_a as follows

$$X_a = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{12}$$

$$X_a |0\rangle = |2\rangle , \quad X_a |1\rangle = |3\rangle \tag{13}$$
$$X_a |2\rangle = |0\rangle , \quad X_a |3\rangle = |1\rangle \tag{14}$$

Proposed solution using qudits

Arbitrary controlled gates

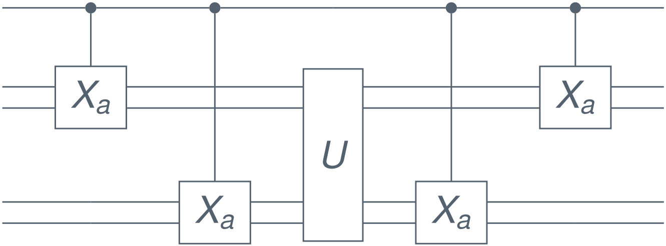


Figure: The channels with two lines represent 4 level qudits. Controlled X_a here has a normal qubit as control.

Can be used in conjunction to qubitization to efficiently make H^k using existence of qubitization.

Definition (Diamond norm & distance)

Given $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ which is a linear map and $X \in M_{n^2}(\mathbb{C})$ where $M_n(\mathbb{C})$ is the set of $n \times n$ complex valued matrices.

$$\|\Phi\|_\diamond := \max_{X; \|X\|_1 \leq 1} \|(\Phi \otimes \mathbb{I}_n)X\|_1, \quad (15)$$

Here $\|A\|_1 = \mathrm{Tr}\sqrt{A^\dagger A}$. Using the above equation we define the diamond distance for two CPTP maps Φ_1 and Φ_2 as follows for density matrices ρ

$$d_\diamond(\Phi_1, \Phi_2) = \|\Phi_1 - \Phi_2\|_\diamond := \max_\rho \|(\Phi_1 \otimes \mathbb{I}_n)\rho - (\Phi_2 \otimes \mathbb{I}_n)\rho\|_1, \quad (16)$$

Φ is a general quantum gate represented as a map from one space of density matrices to another. $\Phi_2 \circ \Phi_1$ is the gate obtained on applying Φ_1 first and then Φ_2.

Lemma

Let T_1, T_2 and T_1', T_2' be super-operators such that they all have diamond norm less than 1 and $d_\diamond(T_i, T_i') \leq \epsilon_i$. Then it follows that $d_\diamond(T_2 \circ T_1, T_2' \circ T_1') \leq \epsilon_1 + \epsilon_2$.

Lets define cX_a' as cX_a for with some noise such that $\|cX_a' - cX_a\|_\diamond \leq \epsilon$. If we have a qudit channel with n qudits being controlled by 1 qubit just using a single controlled-U (the case of figure 3), the error of noisy controlled-Xa will add up as

$$
\begin{aligned}
d_\diamond(cU', cU) &\leq \sum_i d_\diamond(cX_a', cX_a) + d_\diamond(U, U) + \sum_{i=1}^{n} d_\diamond(cX_a', cX_a) \\
&\leq 2n d_\diamond(cX_a', cX_a) \\
&= \mathcal{O}(n\epsilon)
\end{aligned}
$$

Hence error scaling is linear and so this is feasible to implement.

- **Stabilizer code** written as $[[n, k, d]]_q$. q is levels of qudit, k is number of logical qudits and n is the number of qudits these k qudits are encoded within. d is minimum hamming distance between valid messages. This corrects a maximum of t errors where $2t + 1 \leq d$
- **Stabilizers** are operators $\hat{P}_i$ such that a valid message is the eigenvector of $\hat{P}_i$ with eigenvalue 1. There are $n - k$ stabilizers. These are used to obtain syndrome measurements and identify the error. All stabilizers commute with each other.

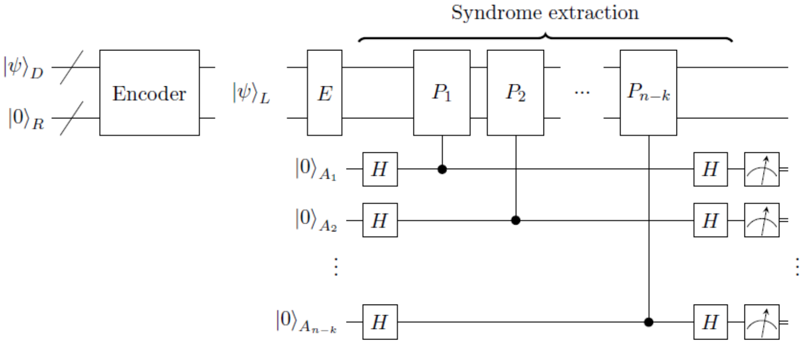


Figure: A general scheme for error correction using stabilizer codes. Here P_1 to P_{n-k} represent the stabilizers and the ancillary qubits $|0\rangle_{A_1}$ to $|0\rangle_{A_{n-k}}$ are measured to extract the error. Figure taken from *J. Roffe, Contemporary Physics (2019)*[12].

The necessary and sufficient encoding condition for a QECC is given here as

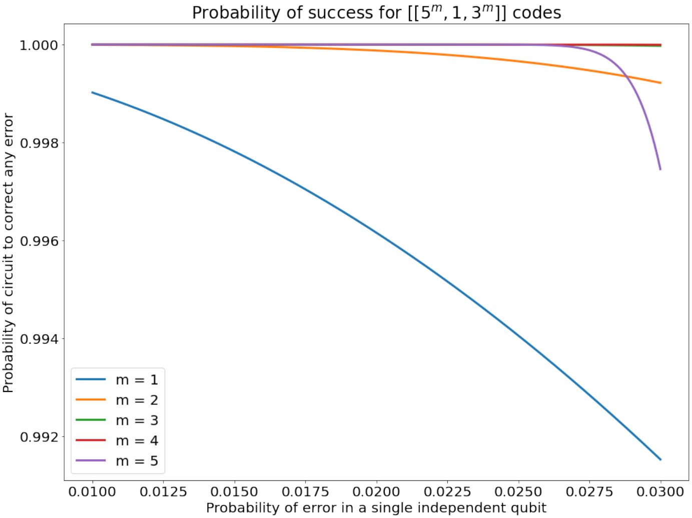
$$\langle i_{\text{Encode}}|A^{\dagger}B|j_{\text{Encode}}\rangle = \lambda_{A,B}\delta_{ij} \tag{17}$$

The 5-qubit code: most optimal code (from Knill-Lafflame bound [9]) capable of correcting any 1 qudit error. Can be shown that $[[5,1,3]]_q$ exists for all $q \geq 2$ by choosing logical qudit as

$$|k_L\rangle = \frac{1}{q^{3/2}} \sum_{l,m,n=0}^{q-1} \omega_q^{k(l+m+n)+ln} |l+m+k, l+n, m+n, l, m\rangle \tag{18}$$

This is proven as a valid encoding in *H. F. Chau, PRA (1997)* [5].

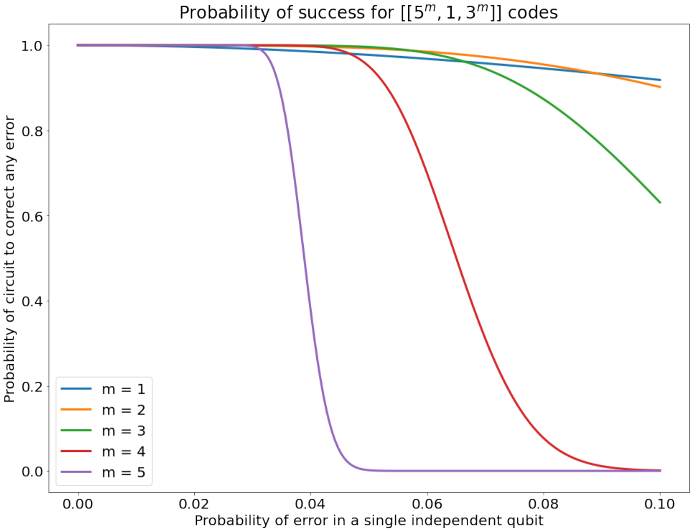
Concatenating codes: studying $[[5, 1, 3]]$ concatenated to itself



Error correction in qudits

Additional results

Concatenating codes: studying $[[5, 1, 3]]$ concatenated to itself



Conclusion

- The aim: illustrate the quantum signal processor approach in conjunction to use of 4-level qudits for arbitrary controlled qubit gates.
- Can make an iterate block encoding if we have access to controlled version of encoding. *G. H. Low & I. L. Chuang, Quantum (2019)* [10]
- Additionally this method can be shown to have linear error scaling and qudits are great choices for quantum error correction too. *M. Grassl et al., International Journal of Quantum Information (2004)* [6]

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Thank you!