Math Club 10/27

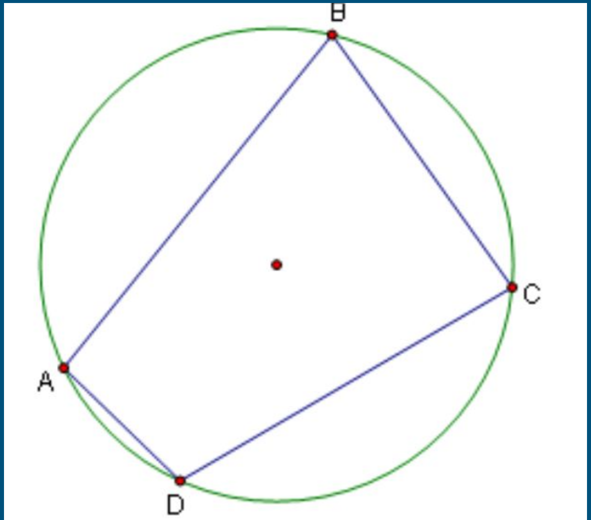
Sign in is on paper :)

AMC Coming up soon!

- The AMC 10/12 A is on November 10th
- The AMC 10/12 B is on November 12th
- We highly recommend **Art of Problem Solving** (aops.com) for practice
  - Alcumus
  - AOPS wiki (List of Math formulas)
- The best way to practice is to just do more problems!

Cyclic Quadrilaterals

- A **Cyclic Quadrilateral** is a quadrilateral that be inscribed in a circle
    - While all triangles are cyclic, this does not hold true for all quadrilaterals



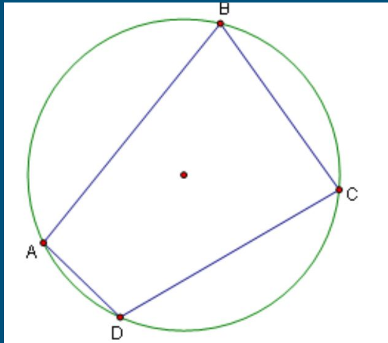
Why do we care?

$$\angle A + \angle C = \angle B + \angle D = 180^o$$

This property is both necessary and sufficient!

$$\angle ABD = \angle ACD$$
$$\angle BCA = \angle BDA$$
$$\angle BAC = \angle BDC$$
$$\angle CAD = \angle CBD$$

Ptolemy's Theorem!!!



Ptolemy's Theorem

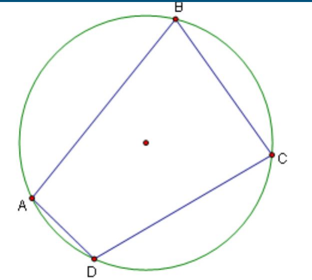
- Ptolemy's Theorem states for cyclic quadrilateral ABCD with sides a, b, c, d and diagonals e and f, **ac + bd = ef**

Given cyclic quadrilateral $ABCD$, extend CD to P such that $\angle BAC = \angle DAP$.

Since quadrilateral $ABCD$ is cyclic, $m\angle ABC + m\angle ADC = 180^\circ$. However, $\angle ADP$ is also supplementary to $\angle ADC$, so $\angle ADP = \angle ABC$. Hence, $\triangle ABC \sim \triangle ADP$ by AA similarity and $\frac{AB}{AD} = \frac{BC}{DP} \implies DP = \frac{(AD)(BC)}{(AB)}$.

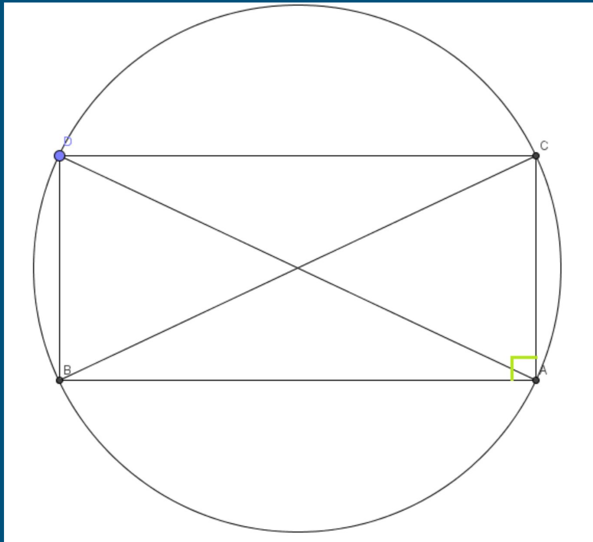
Now, note that $\angle ABD = \angle ACD$ (subtend the same arc) and $\angle BAC + \angle CAD = \angle DAP + \angle CAD \implies \angle BAD = \angle CAP$, so $\triangle BAD \sim \triangle CAP$. This yields $\frac{AB}{AC} = \frac{BD}{CP} \implies CP = \frac{(AC)(BD)}{(AB)}$.

However, $CP = CD + DP$. Substituting in our expressions for CP and DP, $\frac{(AC)(BD)}{(AB)} = CD + \frac{(AD)(BC)}{(AB)}$. Multiplying by AB yields $(AC)(BD) = (AB)(CD) + (AD)(BC)$.



Proving the Pythagorean Theorem with Ptolemy’s

- Taking our right triangle, we can rotate it and create a rectangle
    - Note that rectangles are always cyclic!
- Let AB = CD = a and BD = CA = b
    - Let the length of the diagonal equal c
- Per Ptolemy’s Theorem, we get that a^2 + b^2 = c^2

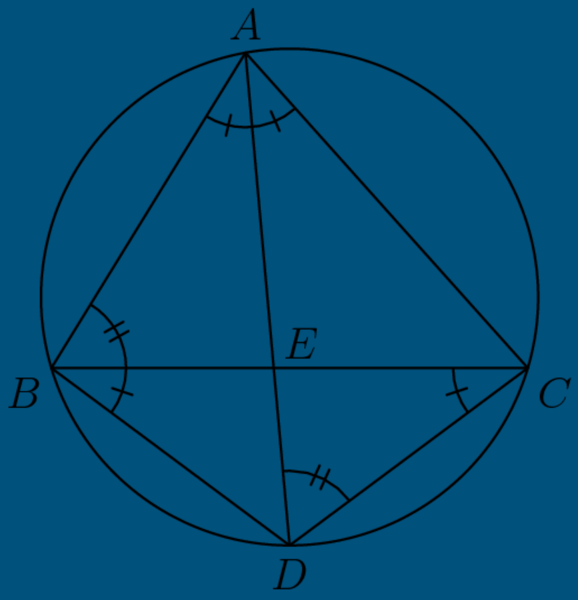


2004 AMC10B #24

In triangle ABC we have $AB = 7$, $AC = 8$, $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects angle BAC. What is the value of $\frac{AD}{CD}$?

(A) $\frac{9}{8}$ (B) $\frac{5}{3}$ (C) 2 (D) $\frac{17}{7}$ (E) $\frac{5}{2}$

$AB = 7$
$AC = 8$
$BC = 9$
We want to find AD/CD



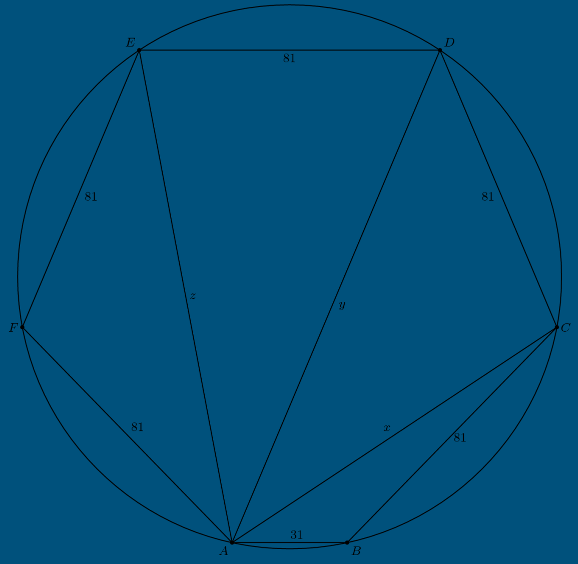
Let $BD = x$. We see that CD must also equal to x because they intercept the same arc.

Setting up Ptolemy's Theorem, we get that $7x + 8x = 9(AD)$
This means that $AD/CD = 15/9 =$ **5/3**

1991 AIME #14

A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.

As with all geometry problems, the first thing we want to do is draw a diagram!



Since we want to find three different lengths, we will probably need three different equations

Luckily for us, we see that we can actually split this hexagon into multiple cyclic quadrilaterals.

Let $x = AC = BF$, $y = AD = BE$, and $z = AE = BD$.

Ptolemy's Theorem on $ABCD$ gives $81y + 31 \cdot 81 = xz$, and Ptolemy on $ACDF$ gives $x \cdot z + 81^2 = y^2$. Subtracting these equations give $y^2 - 81y - 112 \cdot 81 = 0$, and from this $y = 144$. Ptolemy on $ADEF$ gives $81y + 81^2 = z^2$, and from this $z = 135$. Finally, plugging back into the first equation gives $x = 105$, so $x + y + z = 105 + 144 + 135 = \boxed{384}$.

Brahmagupta's Formula

- Essentially Heron's formula for cyclic quadrilaterals
- Area equals $K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ where a,b,c,d are side lengths and s is the semiperimeter
- May be useful if you have most of the side lengths!

Thanks for coming!