Math Club

Code: second

tinyurl.com/dhsmath2021

Problem 1

Patty is an avid cookie baker. She bakes chocolate chip cookies, snickerdoodle cookies, oatmeal raisin cookies, and double nut cookies. Right now, she has 100 of each type of cookie. How many different combinations of 30 cookies can she make?

Attempt #1: Casework

We could try to bash out every combination. Since there are 4 types of cookies, we have to see how many ways $x + y + z + a = 30$. Starting this naively will quickly prove to be extremely tedious and time consuming

How can we do better?

Concept: Stars and Bars

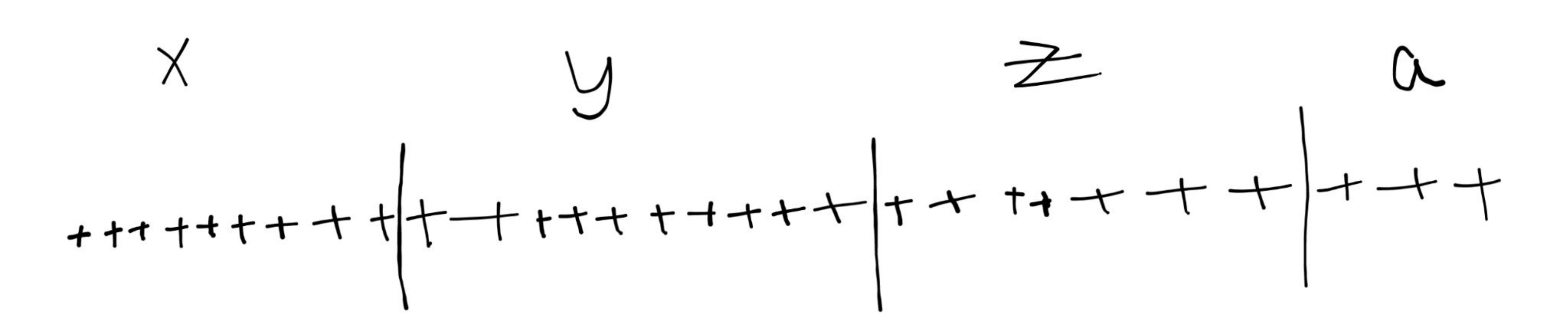
(also known as sticks and stones, balls and urns, chopsticks and dumplings, etc.)

According to Brilliant: stars and bars [is where]... indistinguishable objects are represented by stars and the separation into groups is represented by bars.

Math Lingo: There exists a **bijection**(1 to 1 + onto) from possible solutions to the possible arrangements of stars and bars.

How does this help in our situation?

Since our current goal is to find the amount of ways x + y + z + a = 30 can be satisfied, we can apply stars and bars.



How many different ways can we place these bars?

If you have taken precalculus, this quickly becomes familiar. It's a standard combinatorics problem!

We want to find how many ways we can place 3 bars among 33(3 bars + 30 stars) objects, leading us to ${}_{33}C_3$

Reminder: the choose formula(try proving this yourself!)

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Answer: 5456

General Formula for Stars and Bars (though you should understand through the intuition!)

$$\binom{n+k-1}{k}$$

Problem 2

Source: 2001 Korean Math Olympiad

Let

$$f(x) = \frac{2}{4^x + 2}$$

for real numbers x. Evaluate

$$f\left(\frac{1}{2001}\right) + f\left(\frac{2}{2001}\right) + \cdots + f\left(\frac{2000}{2001}\right).$$

Looks Scary... Why?

- lots of numbers to add
- fractions
- exponents
- fractions in exponents in fractions...
- and you have to add them all up

Sums of Sequences

Sequence — a list of numbers (often follows a pattern)

<u>Summing Sequence</u> — helps to find a pattern, often by pairing terms (things you add) up

Pairing the First and Last Terms

Plug them into f(x) and combine fractions with a common denominator...

$$
\begin{aligned}
f\left(\frac{1}{2001}\right)+f\left(\frac{2000}{2001}\right) &= \frac{2}{4^{1/2001}+2}+\frac{2}{4^{2000/2001}+2} \\
&= \frac{2(4^{2000/2001}+2)}{(4^{1/2000}+2)(4^{2000/2001}+2)}+\frac{2(4^{1/2001}+2)}{(4^{1/2000}+2)(4^{2000/2001}+2)} \\
&= \frac{2(4^{2000/2001})+4+2(4^{1/2001})+4}{(4^{1/2000}+2)(4^{2000/2001}+2)} \\
&= \frac{4+2(4^{2000/2001})+2(4^{1/2001})+4}{4+2(4^{2000/2001})+2(4^{1/2001})+4} = \boxed{1}
\end{aligned}
$$

Pairing the First and Last Terms

Replace **1/2001** and **2000/2001** with any **x** and **1-x**...

$$\begin{aligned} f(x) + f(1-x) &= \frac{2}{4^x + 2} + \frac{2}{4^{1-x} + 2} \\ &= \frac{2}{4^x + 2} + \frac{2(4^x)}{4^1 + 2(4^x)} \\ &= \frac{2}{4^x + 2} + \frac{4^x}{2 + 4^x} \\ &= \frac{2 + 4^x}{2 + 4^x} = \boxed{1} \end{aligned}$$

This means...

$$f\left(\frac{1}{2001}\right)+f\left(\frac{2}{2001}\right)+\cdots+f\left(\frac{2000}{2001}\right)$$

You can pair up the terms starting from the front and the back and they'll always add up to 1.

Since there are 2000 terms, there will be 1000 pairs.

Each pair sums to 1, so the total sum is **1000**.

Takeaway

- Scary things in fractions cancel often
    - Problem creators design it to be like this
- When you’re given a long sum, try to pair them up from the front and back
- Don’t be afraid to get a little messy with your algebra

Code: amc

tinyurl.com/dhsmath2021

Problem 3

Source: 2020 AMC 12A

There is a unique positive integer n such that

$$\log_2(\log_{16} n) = \log_4(\log_4 n).$$

What is the sum of the digits of n?

(A) 4 **(B)** 7 **(C)** 8 **(D)** 11 **(E)** 13

Log Identities

$$\log_{p^k} q^k = \log_p q,$$