Math Club 9/30

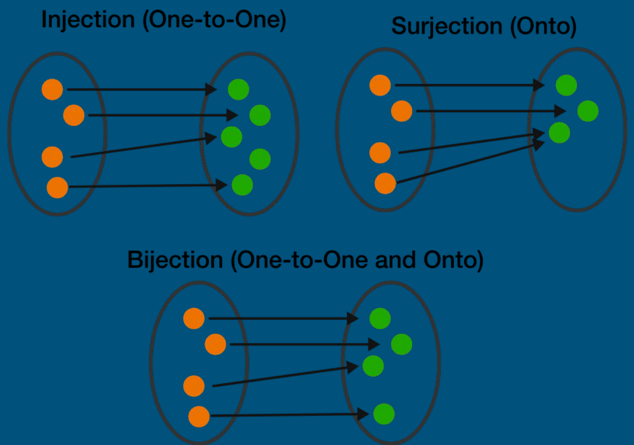
Sign in is on paper :)

Missed AMC Signups?

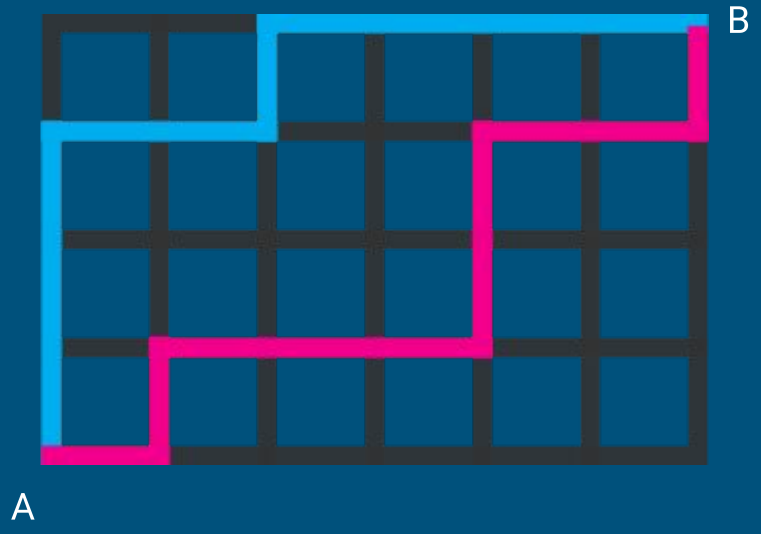
- We do have some extra tests available(for all the tests)
- Tests will be given on a first come, first serve basis
- The form link will also be posted in the discord later

Bijections in Counting

- A function is **bijective** for two sets if it is both **surjective** and **injective**
    - The function f: X -> Y is **injective** if distinct elements of X are mapped to distinct elements of Y
    - The function f: X -> Y is **surjective** if every element of Y is the **image** of at least one element of X.



How is this useful?



How many paths are there from A to B if you can only go up and to the right?

No matter how you move, you always go right 6 times and up 4 times.

We can represent every path as a sequence of steps, such as UUURURRRRR

Each permutation of this sequence is another path! This means the answer is (10!)/(6!4!) = 210

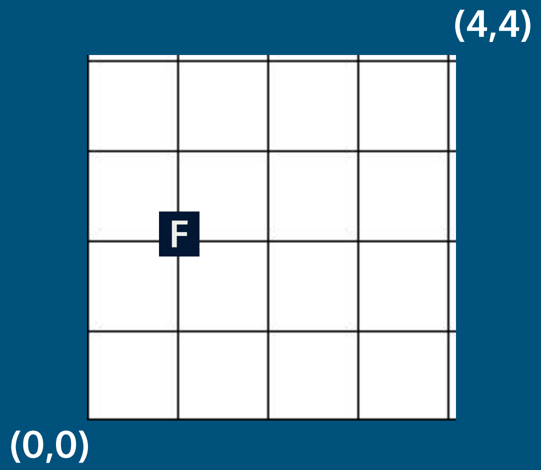
Bijections in the AMC

- In the AMC, almost all the combinatorics (counting) problems **are essentially bijections**
  - A lot of problems come down to just finding a nice way to represent it, with the actual math itself not being hard at all
- A lot of problems also have a lot of **symmetry** to make problems look a lot more time-consuming and challenging than they actually are

Example #1: Symmetries everywhere

A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0, 0)$, $(0, 4)$, $(4, 4)$, and $(4, 0)$. What is the probability that the sequence of jumps ends on a vertical side of the square?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{7}{8}$



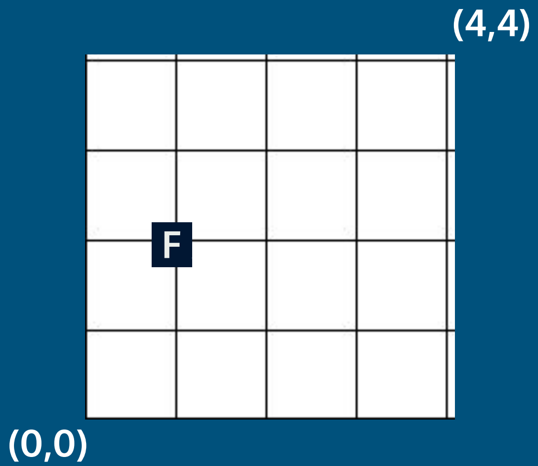
At first glance, this problem may seem rather challenging! The frog could literally go in circles forever and never hit any sides.
However, symmetry allows to basically hand wave away most of these problems.

If the frog chooses the ¼ chance of going up, then its equidistant from the two closer and further sides, meaning there is a ½ chance of hitting a vertical wall. This gives us the overall chance of ½ * ¼ = ⅛

If the frog chooses the ¼ chance of going left, then it has hit the wall, giving us a 1 x ¼ = ¼ chance of hitting the vertical wall

Same as going up :)

If the frog chooses the ¼ chance of going right, it is equidistant from all walls, meaning there is a ½ chance of hitting a vertical wall. This gives us a ½ * ¼ = ⅛ chance



Overall probability = ⅛ + ¼ + ⅛ + ⅛ = ⅝

Example #2: Finding a nice way to represent the problem

A fair die is rolled four times. The probability that each of the final three rolls is at least as large as the roll preceding it may be expressed in the form m/n where m and n are relatively prime positive integers. Find m + n.

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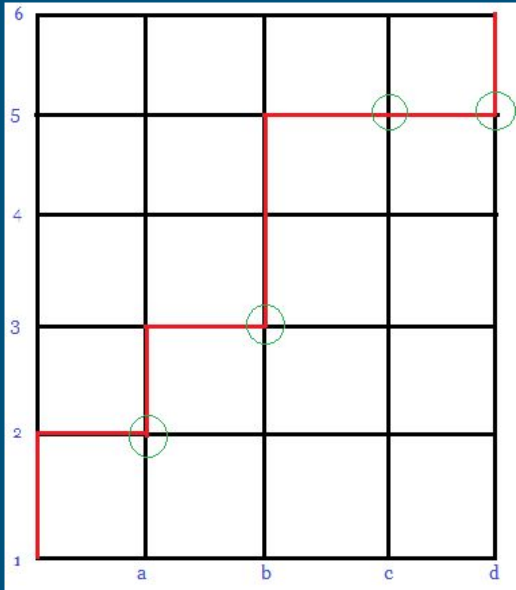
We can treat this just like the block walking problem!
Since the coins must always be greater than or equal to the previous one, that means we can only go “up” and “right”
This established a **bijection** between the dice sequences and our block walking diagram.

Now to do the math!
The amount of valid paths = 9C4
= (9!)/(5!4!) = 126
Total outcomes possible for dice
= 6^4 = 1296
126/1296 = 7/72
7+72 = **79**



Some Useful Theorems

- Hockey Stick Identity
  - This is probably **the most** common combinatorial identity in the AMC/AIME
- Vandermonde's Identity
- Burnside's Lemma

Want to do a presentation at Math Club?