1 The Serre Spectral Sequence

Here we will introduce the notion of a spectral sequence, as derived from an exact couple, and then give as a specific example the Serre spectral sequence. We will introduce the derivation for the spectral sequence of a filtered topological space, following [Hat04]. For a more in-depth treatment of spectral sequences and their derivation, we direct the reader to [Gal16].

We will begin with the notion of an exact couple and then, following [Hat04], will give an example.

Definition 1.1. *An exact couple consists of abelian groups A and B, and maps $i : A \to A$, $j : A \to E$ and $k : E \to A$ such that the following triangle is exact at each of its three corners*

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

We can define a map $d = j \circ k : A \to A$ and thereby define a *derived couple.*

Definition 1.2. *The derived couple of the exact couple given above consists of abelian groups $A' = i(A) \subset (A)$ and B' the homology of E with respect to d, and maps $i' = i|_{A'}$, $j' : A' \to E'$ defined by $j'(i(a)) = [ja]$ and $k' : E' \to A'$ given by $k'[e] = k(e)$ which form the following commutative triangle*

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

.

Checking that the maps j' and k' are well-defined is a simple exercise of diagram chasing, as is proving the following lemma.

Lemma 1.3. *The derived couple of an exact couple is exact.*

This gives us a sequence of abelian groups $E, E', E'', \ldots$ which may or may not stabilize. This sequence, along with the differentials $d, d', d'', \ldots$ is called a *spectral sequence*. This is generally formulated as a sequences of pages E^r with differentials $d_r : E^r \to E^r$ such that $d_r^2 = 0$. In this sense the spectral sequences is a more complicated (and correspondingly more powerful) analogue to the long exact sequence. Just like long exact sequences are used to express relationships between (co)homology groups of different spaces, spectral sequences are powerful tools for relating the cohomology groups of more complicated structures for which a long exact sequence is insufficient.

For example, the Adams spectral sequence is used for computing stable homotopy groups, the Leray spectral sequence for sheaf cohomology, and the Grothendieck spectral sequence is useful for computing the composition of derived functors. The Serre spectral sequence, which we are most interested in, is useful for expressing the relationship between the (co)homology groups of spaces in a fiber bundle.

In a spectral sequence, the groups E^r are typically expressed as a direct sum of countably many groups which are indexed by $\mathbb{Z}^2$. Because of that, E^r is usually drawn over $\mathbb{R}^2$ with a group at each lattice-point of $\mathbb{Z}^2$. We call E^r the r^{th} *page* of the spectral sequence, where the $(p,q)^{th}$ direct summand of E^r is written $E^r_{p,q}$.

Differentials are also expressed in terms of the summands. Specifically,

$$d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

so one can tell the page by the direction of the differentials.

Typically, for a spectral sequence to be useful, for each (p,q) there should be some $n \in \mathbb{N}$ such that $E^m_{p,q} = E^n_{p,q}$ if $m \geq n$. That is, $E^k_{p,q}$ should stabilize after large enough k. The page with only stable entries is called the *infinity page*, and is written $E^\infty_{p,q}$. In some cases, the infinity page corresponds to an actual page E^r. In others, the infinity page is the limit page as r goes to infinity (hence the name). The relationship on which the spectral sequence sheds light is typically expressed by giving the formula for $E^k_{p,q}$ for some k and then giving the $(p,q)^{th}$ diagonal on the infinity page. For example, the Serre spectral sequence for homology is written

$$E^2_{p,q} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E)$$

while the sequence for cohomology is written

$$E_2^{p,q} = H^p(B, H^q(F)) \Rightarrow H^{p+q}(E)$$

(note the change of the subscripts and superscripts), where $E \to B$ is a fiber bundle with fiber F. What is meant by the notation is that there exists a spectral sequence such that the 2^{nd} page is given by the formula on the left-hand side, and such that

$$\bigoplus_{a+b=p+q} E^\infty_{a,b} = H^{p+q}(E).$$

Just as in the case of results about long exact sequences, despite the amazing power of spectral sequences the differentials are, in general, unknown and very difficult to understand. In the instances which we use the Serre spectral sequence, the pages have enough trivial entries (i.e. enough (p, q) such that, for example, $E^2_{p,q} = 0$) that the calculations become quite straightforward and virtually no information about the nature of the differentials is necessary.

In the remainder of this section we will construct the exact couple which gives rise to the Serre spectral sequence, and then give the main theorem which shows its properties. We will then give some examples to illustrate its power and diversity. There is a version of the Serre spectral sequence for both homology and cohomology; since we are working only with cohomology in this document we will only introduce the theorem with respect to cohomology. The version for homology is very similar; the interested reader can read, for example, in [Hat04].

In order to construct the exact couple, we must first introduce the idea of a *filtered topological space*.

Definition 1.4. *A filtration of a topological space X is a collection of subspaces $\{X_\alpha \subset X\}_{\alpha \in I}$ such that I is a totally ordered set, and if $\alpha < \beta$ then $X_\alpha \subset X_\beta$. If X has a filtration, then X is a filtered topological space.*

Examples of filtered topological spaces are CW-complexes, where the filtration is given by the skeleta, as well as simplicial copmlexes. Also, given a continuous map $f : X \to \mathbb{R}$ there exists a natural filtration $X_\alpha = \{f^{-1}(\beta) \mid \beta \leq \alpha\}$.

Now consider a fiber bundle $X \to B$ with fiber F and B a CW-complex. Then B has a natural filtration given by the skeleta of B. We write B_i for the i-skeleton of B, and we can define a filtration for X, given by $X_i = \pi^{-1}(B_i)$. For each i, we can take the long exact sequence associated to the pair (X_{i+1}, X_i), which is

$$\cdots \to H^n(X_{i+1}, X_i) \to H^n(X_{i+1}) \to H^n(X_i) \to H^{n+1}(X_{i+1}, X_i) \to \cdots .$$

By carefully arranging this long exact sequence for each pair (X_i, X_{i-1}), we can fit them together neatly in a *staircase diagram*

$$\begin{array}{ccccccccc}
H^{n-1}(X_i) & \longrightarrow & H^n(X_{i+1},X_i) & \longrightarrow & H^n(X_{i+1}) & \longrightarrow & H^{n+1}(X_{i+2},X_{i+1}) & \longrightarrow & H^{n+1}(X_{i+2}) \\
\downarrow & & & & \downarrow & & & & \downarrow \\
H^{n-1}(X_{i-1}) & \longrightarrow & H^n(X_i,X_{i-1}) & \longrightarrow & H^n(X_i) & \longrightarrow & H^{n+1}(X_{i+1},X_i) & \longrightarrow & H^{n+1}(X_{i+1}) \\
\downarrow & & & & \downarrow & & & & \downarrow \\
H^{n-1}(X_{i-2}) & \longrightarrow & H^n(X_{i-1},X_{i-2}) & \longrightarrow & H^n(X_{i-1}) & \longrightarrow & H^{n+1}(X_i,X_{i-1}) & \longrightarrow & H^{n+1}(X_i)
\end{array}$$

where the red indicates the long exact sequence for the pair (X_{i+1}, X_i). A staircase diagram as given above determines an exact couple by letting A be the direct sum of all the absolute cohomology groups $H^n(X_i)$ and letting E be the the direct sum of all the relative cohomology groups $H^n(X_{i+1}, X_i)$. The maps i, j and k which form the exact couple are the maps forming the long exact sequences in the staircase diagram.

The rather remarkable result is that the spectral sequence derived from this exact couple relates the cohomology of X, B and F in the following way:

Theorem 1.5 (Convergence theorem of the Serre spectral sequence for cohomology)**.** *Let $X \to B$ be a fibration with fiber F such that B is path connected, and let G be an abelian group. If $\pi_1(B)$ acts trivially on $H_*(F;G)$, then there is a spectral sequence $\{E_r^{p,q}, d_r\}$, as defined above, such that:*

(a) $E_2^{p,q} = H^p(B; H_q(F;G))$

(b) $\bigoplus_{p+q=n} E_\infty^{p,q} \cong H^n(X;\mathbb{Z})$

(c) $d_r : E_r^{p,q} \to E_r^{p-(r+1),q+r}$ *where* $E_{r+1}^{p,q}$ *is the homology of* $E_r^{p,q}$ *with respect to* d_r*.*

(d) stable terms $E_\infty^{p,n-p}$ *are isomorphic to the successive quotients* F_p^n/F_{p-1}^n *with respect to a filtration of* $H^*(X)$

The filtration referred to in (d) is given by

$$H^*(X;\mathbb{Z}) = \cdots = F^0 H^*(X;\mathbb{Z}) \supset F^1 H^*(X;\mathbb{Z}) \supset \cdots$$

where $F^i H^*(X;\mathbb{Z}) := \ker(H^*(X;\mathbb{Z}) \to H^*(X_{i-1};\mathbb{Z}))$. We would like to give a few examples which will give the reader an idea of how the spectral sequence might be used.

Example 1.6. *We show that if Y is weakly contractible, then $H^*(X\times Y;G) \cong H^*(X;G)$. This is not difficult to verify via the Künneth theorem, but we include the example nonetheless to give the reader some intuition on how the spectral sequence operates.*

Consider the trivial bundle $X \times Y \to X$ with fiber Y. Using the formula from Theorem 1.5, we have $E_2^{p,q} = H^p(X; H^q(Y;G))$. Since Y is contractible, the E_2-page of the spectral sequence only has nontrivial groups if $q = 0$. This gives us a row of nontrivial groups, namely $E_2^{p,0} = H^p(X;G)$.

The differentials from the E_2-page onward are not horizontal, and so all the differentials are trivial and we have that the E_2-page is the E_∞-page and thus $H^p(X\times Y;G) \cong H^p(X;G) = H^p(X;H^0(Y;G))$, as desired.

Example 1.7 (Products of spheres)**.** *In this example we will compute $H^*(S^d\times S^{d+1};\mathbb{Z})$ for all d. Note that $S^d \times S^{d+1}$ fits into the trivial fiber bundle*

$$\begin{tikzcd} S^d \arrow[r, hook] & S^d \times S^{d+1} \arrow[d] \\ & S^{d+1}. \end{tikzcd}$$

This gives us an E_2 page with only four nontrivial entries, being $E_2^{0,0}$, $E_2^{0,d}$, $E_2^{d+1,0}$ and $E_2^{d+1,d}$, all of which are $\mathbb{Z}$. In this case, the E_2 page is the E_∞ page because nowhere do the differentials map between any two of these four nontrivial entries. Thus $H^k(S^d \times S^{d+1}) = \mathbb{Z}$ if $k = 0, d, d+1, 2d+1$ and 0 otherwise.

There is an additional structure on the Serre spectral sequence for cohomology which is extremely important. This is that it is *multiplicative*, in the sense that there are bilinear maps

$$\cup : E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$$

induced by the cup product on cohomology. The differential follows the Leibniz rule with regards to this multiplication. This is very useful, as we will see here in Example 1.8.

Example 1.8. *This example is considerably more interesting than the previous two: we will show the cup product structure of $\mathbb{C}P^n$, which cannot be done using exact sequences. Recall that $H^i(\mathbb{C}P^n) = \mathbb{Z}$ if i is even, $i \leq 2n$, and 0 otherwise. We first start with the fibration*

$$\begin{tikzcd} S^1 \arrow[r, hook] & S^{2n+1} \arrow[d] \\ & \mathbb{C}P^n \end{tikzcd}$$

which gives us, by the formula from Theorem 1.5, the only nontrivial rows are 0 and 1. The first two rows of the first quadrant have the form

$$\begin{array}{ccccccc} \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z} \end{array}$$

where the only differentials which could be nonzero are of the form $d_2 : E_2^{p,1} \to E_2^{p+2,0}$*, mapping* $\mathbb{Z} \to \mathbb{Z}$*. Since we know the cohomology of* S^{2n+1} *is* $\mathbb{Z}$ *if* $i = 0$ *or* $2n+1$*, and* 0 *otherwise, each of these maps must be isomorphisms for* $0 \leq p \leq 2n-2$ *because taking the homology of the* E_2*-page with respect to* d_2 *must leave nontrivial groups only in* $E_3^{0,0}$ *and* $E_3^{2n,1}$*. This gives us the following image of the* E_2*-page.*

$$\begin{tikzcd}
\mathbb{Z} \arrow[drr, "\cong"] & 0 & \mathbb{Z} \arrow[drr, "\cong"] & 0 & \cdots & \mathbb{Z} \arrow[drr, "\cong"] & 0 & \mathbb{Z} \\
\mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & \mathbb{Z} & 0 & \mathbb{Z}
\end{tikzcd}$$

We will use the cup product structure on the spectral sequence to deduce the cup product structure on $\mathbb{C}P^n$*, whose cohomology groups lie on the bottom row of the diagram.*

First, take $1 \in E_2^{0,0} = H^0(\mathbb{C}P^n;\mathbb{Z})$ *and choose a generator* $a \in E_2^{0,1} = H^0(\mathbb{C}P^n;\mathbb{Z})$*. Let* $x \in E_2^{2,0} = H^2(\mathbb{C}P^n;\mathbb{Z})$ *be the image of* a *under* d_2*. Now consider* $xa \in E_2^{2,1} = H^2(\mathbb{C}P^n;\mathbb{Z})$ *and note that by the Leibniz rule*

$$d_2(xa) = (d_2x) \cdot a + x \cdot (d_2a).$$

Then $d_2x = d_2(d_2a) = 0$ *and* $d_2a = x$ *so*

$$d_2(xa) = (d_2x) \cdot a + x \cdot (d_2a) = x^2,$$

which generates $E_2^{4,0} = H^4(\mathbb{C}p^n;\mathbb{Z})$*. Likewise,*

$$d_2(x^2a) = (d_2x^2)a + x^2d_2(a) = x^3,$$

and so on. By continuing this process we deduce that

$$H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{2n+1}).$$

References

[Gal16] Søren Galatius. Notes on spectral sequences. *Available at http://math.stanford.edu/ galatius/282B16/spectral.pdf*, 1 2016.

[Hat04] Allen Hatcher. Spectral sequences in algebraic topology. 1 2004.