We go through the details of how portfolio level risk profile is derived in a portfolio that satisfies the following conditions:

- No external cash flows (self-financing)
- No leverage change
- No rebalancing intraday
- Constituent assets and portfolio are priced at the end of each day.

In this document we will consider both simple return and log return and their differences in deriving the conclusion.

1 Definition of Asset/Portfolio Return

Concepts defined in this section apply equally to both single financial asset and portfolio made of these assets.

For convenience, we use the following notation.

Let $P_{i,t}$ denote price of asset i at day t.

Let n_i denote number of shares of asset i. This stays constant for each asset respectively.

Name $\frac{P_{i,t}}{P_{i,t-1}}$ as price change factor, $X_{i,t}$ for convenience

We can obtain the following two type of returns:

- Simple return: $R_{i,t}^{(S)} = \frac{P_{i,t}}{P_{i,t-1}} - 1$

  which implies that $P_{i,t} = (1 + R_{i,t}^{(S)})P_{i,t-1}$

- log return: $R_{i,t}^{(L)} = \ln \frac{P_{i,t}}{P_{i,t-1}}$

  which impies that $P_{i,t} = e^{R_{i,t}^{(L)}} P_{i,t-1}$

In exact form, we hvae

$$1 + R_{i,t}^{(S)} = e^{R_{i,t}^{(L)}} = X_{i,t}$$

Notice that when $\frac{P_{i,t}}{P_{i,t-1}} \approx 1$, the above two returns are equivalent, that is $R_{i,t}^{(S)} \approx R_{i,t}^{(L)}$ when $P_{i,t} \approx P_{i,t-1}$ for small daily price changes.

2 Portfolio Daily Return

2.1 Exact Form

let V_t be the asset value of day t, which can be written as

$$V_t = \sum_i^N n_i P_{i,t} = \sum_i^N n_i P_{i,t-1} e^{R_{i,t}^{(L)}}$$

where N denotes the total number of constituent assets.

Then the daily change of the portfolio would be deconstructed by using weights from the previous day

$$\begin{aligned}
X_t &= \frac{V_t}{V_{t-1}} && \text{by definition} \\
&= \frac{\sum_i^N n_i P_{i,t-1} X_{i,t}}{V_{t-1}} = \sum_i^N \frac{n_i P_{i,t-1}}{V_{t-1}} X_{i,t} && \text{by expansion} \\
&= \sum_i^N w_{i,t-1} X_{i,t} && \text{weight definition} \\
&= \sum_i^N w_{i,t-1} \frac{P_{i,t}}{P_{i,t-1}} && \text{using price change}
\end{aligned} \tag{1}$$

where $w_{i,t-1}$ is the weight ratio of asset i in the entire portfolio. Interestingly this demonstrates the exact relationship between daily change ratio of the portfolio and that of the constituent assets where $V_{i,t}$ is the total value of asset i on day t. Though this is an exact relationship between constituent price change and portfolio value change.

2.2 Practial Approximations

Approximately, when daily changes are small, simple return and log return are close. Then (1) can be written as

$$1 + R_t = \sum_i^N w_{i,t-1}(1 + r_{i,t})$$

where R_t is portfolio return and $r_{i,t}$ is asset i return on day t.

Given that $\sum_1^N w_{i,t-1} = 1$, we have

$$R_t = \sum_i^N w_{i,t-1} r_{i,t} \tag{2}$$

This can be also written in a vector form

$$R_t = \boldsymbol{w}_{t-1}^T \boldsymbol{r}_t \tag{3}$$

where $\boldsymbol{w}_{t-1} = (w_{1,t-1}, w_{2,t-1}, ..., w_{N,t-1})^T$ and $\boldsymbol{r}_t = (r_{1,t}, r_{2,t}, ..., r_{N,t})^T$ are both $N \times 1$ vectors.

2.3 Is weight vector approximately constant?

Let us understand how $w_{i,t}$ evolves over time. Denote $V_{i,t}$ the value of asset i on day t

$$w_{i,t} = \frac{V_{i,t}}{V_t} = \frac{V_{i,t}/V_{t-1}}{V_t/V_{t-1}} = \frac{\dfrac{V_{i,t}}{V_{i,t-1}} \dfrac{V_{i,t-1}}{V_{t-1}}}{\dfrac{V_t}{V_{t-1}}}$$

where:

$\dfrac{V_{i,t}}{V_{i,t-1}}$ is the growth of asset i on day t, $X_{i,t}$.

$\dfrac{V_{i,t-1}}{V_{t-1}}$ is the weight of asset i on day $t-1$, $w_{i,t-1}$. We obtain approximately

$\dfrac{V_t}{V_{t-1}}$ is the growth of portfolio, see (1)

when returns are small:

$$\begin{aligned}
w_{i,t} &= \frac{w_{i,t-1}X_{i,t}}{\sum_j^N w_{j,t-1}X_{j,t}} \\
&\approx \frac{w_{i,t-1}(1+r_{i,t})}{1+\sum_j^N w_{j,t-1}r_{j,t}} && \text{replacing } X \text{ by } 1+r \\
&\approx w_{i,t-1}(1+r_{i,t})(1-\sum_j^N w_{j,t-1}r_{j,t}) && \text{first order expansion} \\
&= w_{i,t-1}(1+r_{i,t})(1-\boldsymbol{w}_{t-1}^T\boldsymbol{r}_t) && \text{use vector form} \\
&= w_{i,t-1}(1+r_{i,t}-\boldsymbol{w}_{t-1}^T\boldsymbol{r}_t) && \text{ignore second order terms of } r
\end{aligned} \tag{4}$$

The crux is that the difference of weights between neighboring days is at the first order of r.

$$w_{i,t} \approx w_{i,t-1} + O(r) \quad \text{where r is small} \tag{5}$$

which would lead (3) to be rewritten as

$$R_t = (\boldsymbol{w}_0^T + O(r))\boldsymbol{r}_t = \boldsymbol{w}_0^T\boldsymbol{r}_t + O(r^2) \approx \boldsymbol{w}^T\boldsymbol{r}_t \tag{6}$$

The intuition is that though weight vector changes slowly over time, however as far as the return of the portfolio is concerned, the impact of drift is at second-order. That's why in practice, the initial weight vector is used to perform the risk modeling

2.4 Portfolio Risk

The model is now equivalently a linear model if we write (6) using a generic $\boldsymbol{w} = (w_1, w_2, w_3, ..., w_N)^T$. Let's consider the random variable for each asset i

as r_i (dropping t). When it is small, it is the same as the log return which is often modeled by normal distribution, $r_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for analytical convenience. Let R be the random varible for the portfolio return. On day t, it takes the value of R_t. Similarily, let r_i be the random variable for the daily return of asset i. On day t, it takes the value of $r_{i,t}$. And let $\boldsymbol{r}$ be the random vector $(r_1, r_2, ..., r_N)^T$. So for any given day, $R = \boldsymbol{w}^T\boldsymbol{r}$. Across different days, $\boldsymbol{r}$ is considered i.i.d.

Now we have estabblished the relationship between the random variables, we can compute the expected value and variance of R.

$$\begin{aligned} \mathrm{E}(R) &= \boldsymbol{w}^T\,\mathrm{E}(\boldsymbol{r}) \\ \mathrm{Var}(R) &= \mathrm{Var}(\boldsymbol{w}^T\boldsymbol{r}) = \boldsymbol{w}^T\boldsymbol{\Sigma}_r\boldsymbol{w} \end{aligned} \tag{7}$$

where $\boldsymbol{\Sigma}_r$ is the covariance matrix of r. We can also write that in terms of summation for variance of a linear combination of random variables, as follows

$$\begin{aligned} \mathrm{Var}(R) &= \mathrm{Var}\left(\sum_i^N w_i r_i\right) \\ &= \sum_i^N w_i^2\,\mathrm{Var}(r_i) + 2\sum_{i,j,i<j}^N w_i w_j\,\mathrm{Cov}(r_i, r_j) \\ &= \begin{bmatrix} w_1 & w_2 & \cdots & w_N \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,N} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,N} \\ \vdots & \vdots & & \vdots \\ \sigma_{N,1} & \sigma_{N,2} & \cdots & \sigma_N^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} \\ &= \boldsymbol{w}^T\boldsymbol{\Sigma}_r\boldsymbol{w} \end{aligned} \tag{8}$$

where $\mathrm{Var}(r_i) = \sigma_i^2$ and $\mathrm{Cov}(r_i, r_j) = \sigma_{i,j}$.

Apparently we observe that risk (variance of return) is not additive given a linear relationship between portfolio return and asset returns. Correlation between assets matters.