

We go through the details of how portfolio level risk profile is derived. In this document we will consider both simple return and logarithmic return and their differences in deriving the conclusion.

1 Definition of Asset/Portfolio Return

Concepts defined in this section apply equally to both single financial asset and portfolio made of these assets.

For convenience, we use the following notation.

Let $P_{i,t}$ denote price of asset i at day t .

Let n_i denote number of shares of asset i . This stays constant for each asset respectively.

We can obtain the following two type of returns:

- Simple return: $R_{i,t}^{(S)} = \frac{P_{i,t}}{P_{i,t-1}} - 1$

which implies that $P_{i,t} = (1 + R_{i,t}^{(S)})P_{i,t-1}$

- logarithmic return: $R_{i,t}^{(L)} = \ln \frac{P_{i,t}}{P_{i,t-1}}$

which impies that $P_{i,t} = e^{R_{i,t}^{(L)}} P_{i,t-1}$

In exact form, we hvae

$$1 + R_{i,t}^{(S)} = e^{R_{i,t}^{(L)}}$$

Notice that when $\frac{P_{i,t}}{P_{i,t-1}} \approx 1$, the above two returns are equivalent, that is $R_{i,t}^{(S)} \approx R_{i,t}^{(L)}$ when $P_{i,t} \approx P_{i,t-1}$ for small daily price changes.

2 Portfolio Daily Return

Here we work our portfolio daily return using both simple return and logarithmic return at asset level. Given that simple return is a first order approximation of logarithmic return at asset level, we will try to derive using logarithmic first:

let V_t be the asset value of day t , which can be written as

$$V_t = \sum_i^N n_i P_{i,t} = \sum_i^N n_i P_{i,t-1} e^{R_{i,t}^{(L)}}$$

where N denotes the total number of constituent assets.

Then the daily change of the portfolio would be deconstructed by using weights from the previous day

$$\begin{aligned}
e^{R_t^{(L)}} &= \frac{V_t}{V_{t-1}} && \text{by definition} \\
&= \frac{\sum_i^N n_i P_{i,t-1} e^{R_{i,t}^{(L)}}}{V_{t-1}} = \sum_i^N \frac{n_i P_{i,t-1}}{V_{t-1}} e^{R_{i,t}^{(L)}} && \text{by expansion} \quad (1) \\
&= \sum_i^N w_{i,t-1} e^{R_{i,t}^{(L)}} && \text{weight definition}
\end{aligned}$$

where $w_{i,t-1}$ is the weight ratio of asset i in the entire portfolio. Interesting this demonstrate the exact relationship between daily change ratio of the portfolio and that of the constituent assets if we replace the exponentials:

$$\frac{V_t}{V_{t-1}} = \sum_i^N w_{i,t-1} \frac{V_{i,t}}{V_{i,t-1}} = \sum_i^N w_{i,t-1} \frac{P_{i,t}}{P_{i,t-1}}$$

where $V_{i,t}$ is the total value of asset i on day t . Though this is an exact relationship between constituent price change and portfolio value change.

Approximately, when daily changes are small, simple return and logarithmic return are close. Then (1) can be written as

$$1 + R_t = \sum_i^N w_{i,t-1} (1 + r_{i,t})$$

where R_t is portfolio return and $r_{i,t}$ is asset return.

Given that $\sum_1^N w_{i,t-1} = 1$, we have

$$R_t = \sum_i^N w_{i,t-1} r_{i,t} \quad (2)$$

This can be also written in a vector form

$$R_t = \mathbf{w}_{t-1}^T \mathbf{r}_t \quad (3)$$

where \mathbf{w}_{t-1} and \mathbf{r}_t are both $N \times 1$ vectors.