Part: I

1. prove that $E_{agg} = \frac{1}{M} E_{avg}$

Assumptions:

1. Each of the errors have a 0 mean
   $E(\varepsilon_i(x)) = 0$ for all i
2. Errors are uncorrelated
   $E(\varepsilon_i(x)\,\varepsilon_j(x)) = 0$ for all $i \neq j$

$$E_{avg} = \frac{1}{M} \sum_{i=1}^{M} E(\varepsilon_i(x)^2)$$

where $E(\varepsilon_i(x)^2) = E[(f(x) - h_i(x))^2]$

$$E_{agg}(x) = E\left[\left\{\frac{1}{M} \sum_{i=1}^{M} \varepsilon_i(x)\right\}^2\right]$$

where $\varepsilon_i(x) = f(x) - h_i(x)$

$$\therefore E_{agg}(x) = \frac{1}{M^2} E\left[\left\{\sum_{i=1}^{M} \varepsilon_i(x)\right\}^2\right]$$

$(\because E(aX) = a\,E(X))$

$$= \frac{1}{M^2} E(\varepsilon_1(x) + \varepsilon_2(x) + \dots + \varepsilon_m(x))^2$$

$$= \frac{1}{M^2} E(\varepsilon_1^2(x) + \varepsilon_2^2(x) + \dots + \varepsilon_m^2(x) + 2\,\varepsilon_1(x)\,\varepsilon_2(x) + \dots)$$

$$= \frac{1}{M^2}\left[E\sum_{i=1}^{M}(\varepsilon_i(x)^2) + 2E\sum_{\substack{i=0 \\ j=1}}^{M,M}(\varepsilon_i(x)\,\varepsilon_j(x))\right]$$

$$= \frac{1}{M^2}\left[E \sum_{i=1}^{M} (\varepsilon_i(x)^2\right]$$

$(\because E(\varepsilon_1(x) \cdot \varepsilon_2(x)) = 0$
Error are uncorrelated$)$

$$= \frac{1}{m}\left[\frac{1}{m} E \sum_{i=1}^{m} \epsilon_i(x)^2\right]$$

$$= \frac{1}{m} E_{avg}(X)$$

$$\boxed{\therefore E_{agg}(x) = \frac{1}{m} E_{avg}(x)}$$

2. To prove : $E_{agg} \leq E_{avg}$

Jensen's inequality,

$$f\left(\sum_{i=1}^{m} \lambda_i x_i\right) \leq \sum_{i=1}^{m} \lambda_i f(x_i)$$

Now, Errors are not uncorrelated.

$E(\epsilon_i(x)\,\epsilon_j(x)) \neq 0$ for all $i \neq j$

$$E_{agg} = E\left[\left\{\frac{1}{m}\sum_{i=1}^{m} \epsilon_i(x)\right\}^2\right]$$

$$E_{avg} = \frac{1}{m} E\left[\sum_{i=1}^{m} \epsilon_i(x)^2\right]$$

compare with $\sum_{i=1}^{m} \lambda_i f(x_i)$

$$\lambda_i = \frac{1}{m}, \quad f(x_i) = E\left[\epsilon_i(x)^2\right]$$

$$\because E\left(\sum_{i=1}^{M} \frac{1}{M} \epsilon_i(x)^2\right) \leq \sum_{i=1}^{M} \frac{1}{M} E\left(\epsilon_i(x)^2\right) = E_{avg}$$

We have to prove $E_{agg} \leq E_{avg}$

$$E_{agg} \leq E\left(\sum_{i=1}^{M} \frac{1}{M} \epsilon_i(x)^2\right) \leq E_{avg}$$

$\because$ As per cauchy schwarz inequality,

$$\left(\frac{a+b}{2}\right)^2 \leq a^2+b^2$$

$$\because \left(\frac{\epsilon_1(x) + \epsilon_2(x) + \dots + \epsilon_m(x)}{M}\right)^2 \leq (\epsilon_1(x)^2 + \epsilon_2(x)^2 + \dots + \epsilon_m(x)^2)$$

$$\because E\left[\frac{1}{M}\sum_{i=1}^{M} (\epsilon_i(x))\right]^2 \leq E\left(\sum_{i=1}^{M} \frac{1}{m} \epsilon_i(x)^2\right)$$

$$E\frac{(\epsilon_1(x) + \epsilon_2(x) + \dots + \epsilon_m(x))^2}{M} \leq E(\epsilon_1(x)^2 + \epsilon_2(x)^2 + \dots + \epsilon_m(x)^2)$$

$\because$ hence prove,

$$\boxed{E_{agg} \leq E_{avg}}$$