Radioactive Chain Reactions

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In this paper we analyzed a system of radioactivity containing 3 elements A, B and C where element A (decay rate a) disintegrates to element B (decay rate b) which further disintegrates to element C. Computationally we observed that the amount of A always decreases exponentially with time irrespective of the decay constant whereas the amount of C always increases exponentially as time progresses. However the amount of substance B increases up to a certain maximum value and then approaches to zero as time increases.

I. INTRODUCTION

The radioactive chain reaction is a series of reactions in which different radioactive elements transform sequentially through radioactive decay. From the perspective of modeling and simulation, these reactions fall under the category of dynamic systems with rates proportional to the amount. These reactions are very common in nature. For example, thorium series, neptunium series, uranium series, etc. For the scope of this paper, we are building a model to study the dynamics of the reaction that involves three radioactive elements say A, B, C. A will decay into B and further B will decay into C. The general idea is that rate of decay is proportional to the amount of the radioactive element present.

II. MODEL

In this model of radioactive decay we assumed that at time $t = 0$ only substance A is present i.e. $m_b = m_c = 0$ and $m_a = m_0$. Secondly we have assumed that the system is isolated i.e. there is no effect of external factors on the system. Using the above assumptions we arrived at the following differential equations,

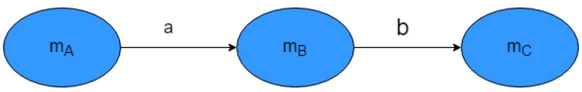

FIG. 1: Schematic diagram to understand radioactive chain reaction

$$\frac{dm_A}{dt} = -a \cdot m_A(t) \tag{1}$$

$$\frac{dm_B}{dt} = a \cdot m_A(t) - b \cdot m_B(t) \tag{2}$$

$$\frac{dm_C}{dt} = b \cdot m_B(t) \tag{3}$$

Eq. (1) represents the rate of change of mass of A having a decay rate of a as the product of the decay rate a and the mass at time t i.e. $m_A(t)$. The negative sign indicates a decrease in the mass of A is time progresses.

Eq. (2) represents the rate of change of mass of B having a decay rate of b as the product of the decay rate a and the mass at time t i.e. $m_A(t)$ minus the product of the decay rate b and the mass at time t i.e. $m_B(t)$. This is because substance B is replenished at a rate a and it disintegrates at a rate b.

Eq. (3) represents the rate of change of mass of C as the product of the decay rate b and the mass of B at time t i.e. $m_B(t)$. This is because the mass of C increases at the rate at which B disintegrates.

$$m_A[t+1] = m_A[t] - \Delta T \cdot a \cdot m_A[t] \tag{4}$$

$$m_A[t+1] = m_B[t] + (\Delta T \cdot a \cdot m_A[t] - \Delta T \cdot b \cdot m_B[t]) \tag{5}$$

$$m_A[t+1] = m_C[t] + \Delta T \cdot b \cdot m_B[t] \tag{6}$$

Eq. (4), (5) and (6) represent difference equations counterparts of Eq. (1), (2) and (3).

III. RESULTS

(a) Fig. (2) shows the decay of three elements A, B and C with the decay rates of A and B being $0.02/da$ and $0.004/da$ respectively. The initial amount of A is assumed to be 500g and that of B and C is assumed to be 0.

(b) The m_A should decrease over time as per Eq. (1). As the m_A is decreasing, dm_A/dt should increase

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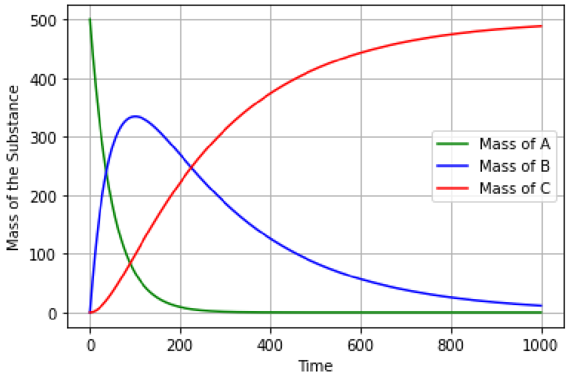

FIG. 2: Mass (in gm) vs Time (in days)

over time. i.e. graph should be concave. We can clearly see that graph of A is concave and decreasing.

Decay of A increases m_B which was initially assumed to be zero. An increase in m_B contributes to a decrease in dm_B/dt. Hence, the initial graph should be increasing and convex. At some point dm_B/dt should be zero and at this point, B_m should be maximum. After that m_B will start to decrease.

Decay of B will result in an increase of m_C which should saturate at some point as m_C can not be more than the initial value of the m_A. So we should get a convex increasing graph.

(c) From Fig (3) we can see that as the value of a is increased from 0.1 to 1 (keeping b constant at 0.3) the time for maximum total radioactivity decreases up to a certain point (where a becomes equal to b) and after that it almost becomes constant. This is because as the value of a increases, element A decays much faster and hence maximum total radioactivity is achieved much earlier.

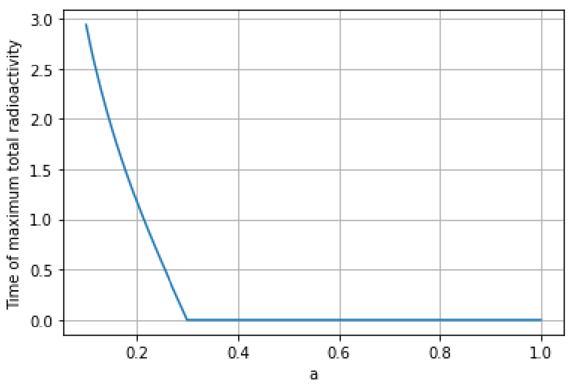

FIG. 3: The time of maximum total radioactivity (in days) vs the decay rate a (in day^{-1})

(d) Continuing from Eq. (1)

$$\frac{dm_A}{A} = -adt \implies m_A(t) = ke^{-at}$$

We assume that $m_A = A_0$ at $t = 0$. Hence

$$m_A(t) = A_0 e^{-at} \tag{7}$$

Now from Eq. (2),

$$\frac{dm_B}{dt} - bm_B - aA = 0$$

Assuming $m_B = 0$ at $t = 0$, by solving first order differential equation we get

$$m_B(t) = \frac{aA_0}{b-a}(e^{-at} - e^{-bt}) \tag{8}$$

Using Eq. (3), we can write

$$m_C(t) = \frac{abA_0}{b-a}(e^{-at} - e^{-bt}) \tag{9}$$

For the case of transient equilibrium we have $a < b$ and hence in Eq. (8) we can ignore the term e^{-bt} with respect to e^{-at} for larger values of t.

$$m_B(t) = \frac{aA_0}{b-a}e^{-at} \implies m_B(t) = \frac{a}{b-a}m_A(t)$$

Hence, we get

$$\frac{m_B(t)}{m_A(t)} = \frac{a}{b-a} \tag{10}$$

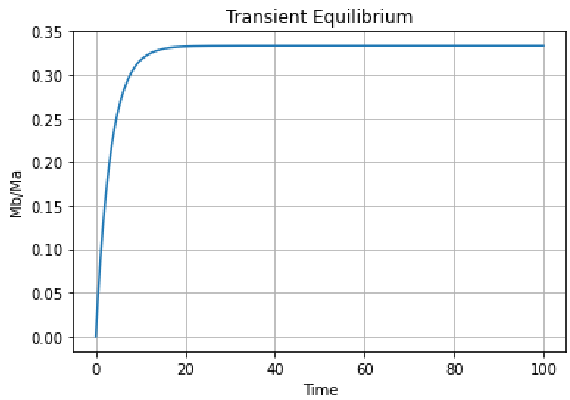

FIG. 4: m_B/m_A vs time (in days) for the case of transient equilibrium

(e) When $a < b$ we do not achieve transient equilibrium as seen from Fig (5) where we have used $a = 0.1$ and $b = 0.05$.

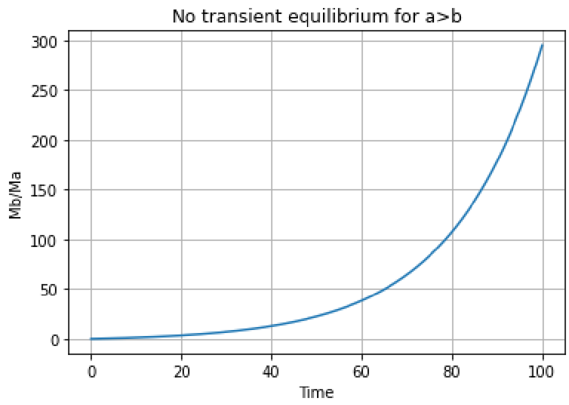

FIG. 5: m_B/m_A vs time (in days) for the case when $a > b$

(f) In order to achieve equilibrium, ratio $\frac{m_B(t)}{m_A(t)}$ must be constant after some finite t.

Combining Eq. (13) and (8), we can write

$$\frac{m_B(t)}{m_A(t)} = \frac{a}{b-a}(1 - e^{(a-b)t}) \tag{11}$$

for $a < b$ the function $f(t) = \frac{m_B(t)}{m_A(t)}$ is strictly increasing over interval $[0,\infty)$. Hence, equilibrium is not possible in this case.

(g) Fig (6) shows the phenomenon of Secular Equilibrium for the radioactive chain of ^{226}Ra $\rightarrow$ ^{222}Rn $\rightarrow$ ^{218}Po with the decay rates of ^{226}Ra and ^{222}Rn being $0.00000117/da$ and $0.181/da$ respectively. The initial value of the mass of ^{226}Ra is 500 g.

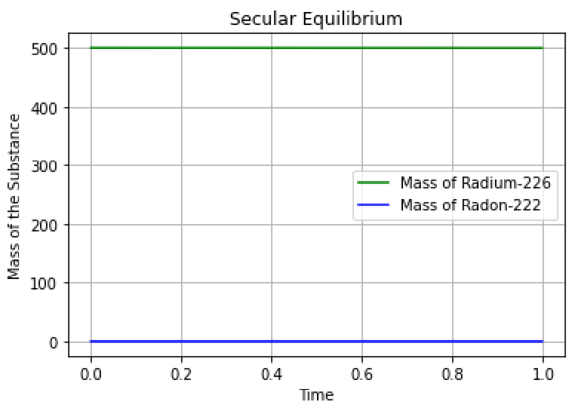

FIG. 6: Mass (in gm) of Radium and Radon vs Time (1 year)

(h) Consider Eq. (13). For very small values of a, we can write $e^{-at} \approx 1$. Hence, we get

$$m_A(t) = A_0 \tag{12}$$

Combining equation (10) and (12), we get

$$m_B(t) = \frac{a}{b-a} A_0 \tag{13}$$

(i) Fig (7) shows the decay of substances for the radioactive chain ^{210}Bi $\rightarrow$ ^{210}Po $\rightarrow$ ^{206}Pb with the decay rates of ^{210}Bi and ^{210}Po being $0.0137/da$ and $0.0051/da$ respectively. The initial mass of ^{210}Bi is 10^{-8}g.

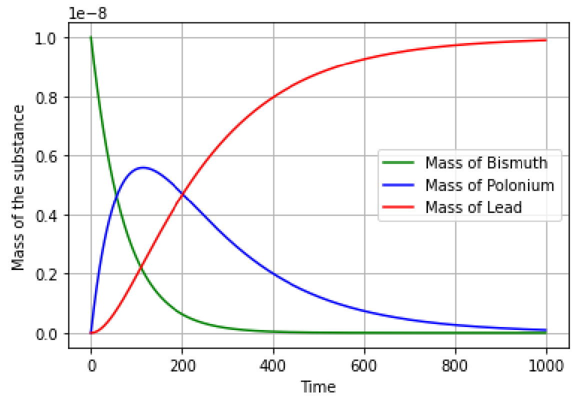

FIG. 7: Mass (in gm) of Bismuth, Polonium and Lead vs Time (in days)

Computationally the maximum value of the mass ^{210}Po is 5.56×10^{-9}g and it occurs at $t = 114.9$ days.

(j) When $m_B(t)$ is maximum dm_B/dt must be zero. Using Eq (8),
$\frac{dm_B}{dt} = \frac{aA_0}{b-a}(be^{-bt} - ae^{-at}) = 0 \implies be^{-bt} = ae^{-at}$
Hence, we get t for which $m_B(t)$ is maximum.

$$t_{max} = \frac{\ln(a) - \ln(b)}{a-b} \tag{14}$$

Now, we will find maximum $m_B(t)$ using Eq. (14) and (8).

$$(m_B)_{max} = \frac{aA_0}{b-a}(e^{-at_{max}} - e^{-bt_{max}}) \tag{15}$$

(k) Analytically, the maximum value of ^{210}Po occurs at $t = 114.901778$ days and the corresponding value of the mass is $(m_B) = 5.5655 \times 10^{-9}$g.

(l) Analytically the maximum value of the mass of ^{222}Rn occurs at $t = 66.0184$ days and the corresponding value of the mass is 0.0032g.

(m) Computationally the maximum value of the mass of ^{222}Rn occurs at $t = 65.96$ days and the corresponding value of the mass is 0.00323g. The computational values show very negligible error as compared to the analytical values.

IV. CONCLUSIONS

We analyzed the dynamics of radioactive chain reaction involving three elements by building a mathematical model and solving it analytically and computationally. We found that the total maximum radioactivity decreases as the decay constant of the parent element increases. In addition to that, we studied an important phenomenon in radioactivity called *Equilibrium.* In the state of Equilibrium, the ratio of mass of the parent and the daughter element becomes constant as time progresses. Equilibrium can be further classified into two category: *Transient Equilibrium* and *Secular Equilibrium.* We derived the conditions for *Transient Equilibrium* $(a < b)$, *Secular Equilibrium*$(a << b)$ and *No Equilibrium* $(a > b)$. We can say that secular equilibrium is a special case of transient equilibrium when the parent element's half life is much longer as compared to the daughter element. Further we discussed about the occurrence of the maxima of $m_B(t)$ and the time taken to reach that maxima. The time to reach the maximum mass of daughter element depends only on the decay constants of parent and daughter. Lastly, using a similar approach as discussed in section II can be used to build a model that involves more than three radioactive elements.