Lab - 7: Random Walk

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CS-302, Modeling and Simulation

In this lab we numerically, analytically and visually analyze the random-walk model for 1-D and 2-D by performing Monte-Carlo simulation. We considered two different type of random walk namely biased random walk and unbiased random walk and produced visual simulation of the same.

I. INTRODUCTION

Random walk refers to the apparently random movement of an entity[1]. Random walk is an event that is function of time in which we have an entity that moves on a grid of cell. The movement of the entity is constrained such that it can only move to neighbouring cell. This grid can be one-dimensional, two dimensional or n-dimensional. Then we perform Monte Carlo simulation of random walk to observe final outcome. Random walk is stochastic process, so we study the outcomes in terms of statistical quantities such as average, variance, probability, standard deviation etc.

II. MODEL

A. Random Walk in 1 Dimension

We start with the simplest problem of 1D random walk. We assume that the entity starts at origin of the number line. It can only moves forward and backward by 1 unit of distance with equal probability. We wants to study the results after entity took N steps. Suppose, d_i represents the step taken by entity after i^{th} step. d_i will either be +1 or -1. -1 represents that entity moves one step forward and -1 represents 1 step backward. Then we can write total displacement d_{tot} of the entity after N step as

$$d_{tot} = d_1 + d_2 + d_3 + d_4 + ... + d_N \tag{1}$$

Obviously, d_{tot} will vary every time we repeat the simulation. If we repeats the experiment many times or in other words, if we perform Monte Carlo simulations of random walk, then average total displacement will be zeros which is explained below. Taking average both the side in Eq. (1)

$$< d_{tot} >=< d_1 > + < d_2 > + < d_3 > + < d_4 > +...+ < d_N > \tag{2}$$

As d_i takes values either 1 or -1 randomly, for Monte Carlo simulations, we can consider $< d_i >$ to be zero. Hence, $d_{tot} = 0$. This result does not provide much information about final displacement. So, we look for d_{tot}^2. We can write,

$$< d_{tot}^2 >= \sum_{i=1}^{N} < d_i^2 > +2\sum_{i=1}^{N}\sum_{j=1}^{i-1} < d_i d_j > \tag{3}$$

Now, d_i^2 can only be +1. Hence $< d_i^2 >= 1$. On the other hand, product of d_i adn d_j can have two possible value -1 and 1 with equal probability. Therefore, $< d_i d_j >= 0$. Now, we can write

$$< d_{tot}^2 >= \sum_{i=1}^{N} 1 + 0 \tag{4}$$

$$< d_{tot}^2 >= N \tag{5}$$

Hence, we can say that after N steps, average displacement of entity will approximately be equal to $\pm\sqrt{N}$. Remember the assumption that we made that probability of moving forward and backward is equal. We describes this model as unbiased random walk.

In biased random walk, entity is biased to move toward particular direction, either in forward or backward direction. Let us assume that entity is biased towards moving forwards with probability q. Now, we define $P(x, N)$ as finding entity at position x after N steps. Probability that entity takes k steps forward and $N - k$ steps backward, then using binomial distribution, we write

$$P(k, N) = \binom{N}{k} \cdot q^k \cdot (1 - q)^{N-k} \tag{6}$$

Here, we have

$$x = 2 \cdot (k - N) \implies k = \frac{N + x}{2} \tag{7}$$

As we keep on increasing N, Monte Carlo simulation tends to produce Gaussian distribution. We calculate mean and variance,

$$mean = N \cdot (2q - 1) \tag{8}$$

$$variance = 4Nq(1 - q) \tag{9}$$

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B. Random Walk in 2 Dimensions

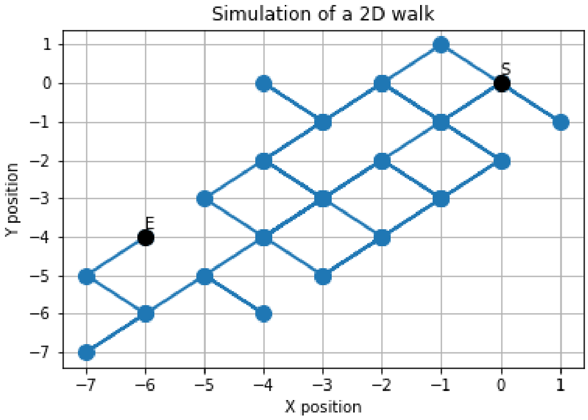


FIG. 1: Simulation of a 2D random walk with 50 steps

In 2D random walk the entity moves on 2D grid. We can thought 2D random walk as independent 1D random walk in two direction. The entity starts at origin and can move along North, East, West and South direction at every step. Going East and West can be considered as 1D random walk on x axis and going North and South can be considered as 1D random walk on y axis. We can show same calculation as shown in 1D random walk to prove that net average displacement is zero. Also, we calculate Pythagorean distance $\sqrt{x^2+y^2}$ from the origin. 2D random walk has also two variants, Unbiased and biased random walk. In next section, we have shown simulations results for both model.

III. RESULTS

A. 1D Random Walk

1. **Unbiased Random Walk:** For the first case we consider an unbiased random walk i.e. the events of the walker going in +ve x direction or -ve x direction are equally probable.
   Mean Distance travelled: On implementing the case of unbiased random walk for 1D we found that as the number of steps increases the mean distance travelled also increases.

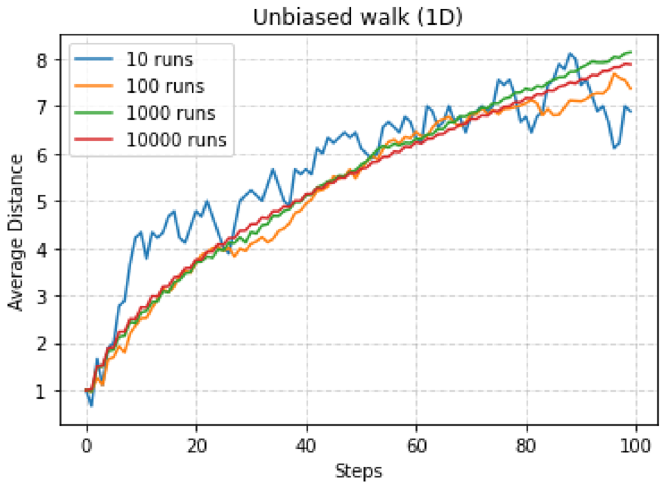


FIG. 2: Average Distance travelled

Average position: From Fig. 3 we can see that as the number of steps increases the average position of the walker tends towards 0. This should be the case because for very large number of steps the number of moves in the +ve x direction would be almost equal to the number of moves in the -ve x direction.

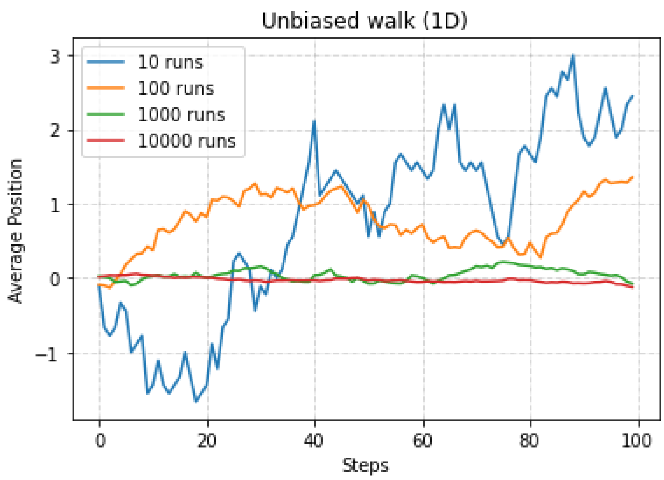


FIG. 3: Average Position of the walker for different number of simulations

Variance in distance: From the Fig 5 we can see that for more number of simulations the plot of variance tends towards the line y=x, because with increase in number of steps the walker has more possible options to travel and hence the variance increases.

FIG. 4: Variance in the mean distance travelled for different number of simulations

Histogram:

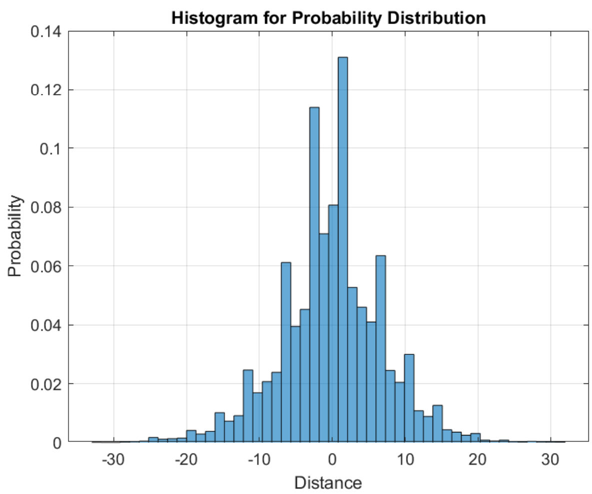


FIG. 5: Histogram showing the probability distribution for unbiased 1D walk.

2. **Biased Random Walk:** Now we look at the case of biased random walk, where the walker is not equally likely to go in any direction. As the biasing probability increases, walker tends to move towards positive +ve x direction.

   Mean Distance travelled: On implementing the case of biased random walk for 1D with probability

   Average position: From Fig. 7 we can see that this time as the number of steps increases the average position of the walker is not close to 0 but is in the +ve x direction. This is because the probability of the walker going in the +ve x direction is more as compared to that going in the -ve x direction.

FIG. 6: Average Distance travelled

FIG. 7: Average Position of the walker for different number of simulations and $p = 0.4$

Variance in distance: From the Fig 5 we can see that for more number of simulations the plot of variance tends towards the line y=x, because with increase in number of steps the walker has more possible options to travel and hence the variance increases.

FIG. 8: Variance in the mean distance travelled for different number of simulations

Histogram:

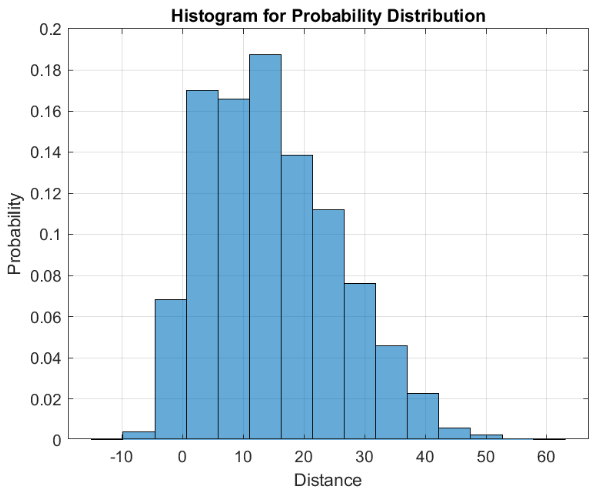


FIG. 9: Histogram showing the probability distribution for biased 1D walk.

B. 2D Random Walk

1. **Unbiased Random Walk:** For the first case we consider unbiased random walk i.e. the walker has an equal probability of going in any of the 4 directions.
   Mean Distance travelled: The following figure shows the average distance travelled by the hiker with increasing number of steps. It can be seen that as the number of steps increases the mean distance travelled also increases.

   Frames of animation

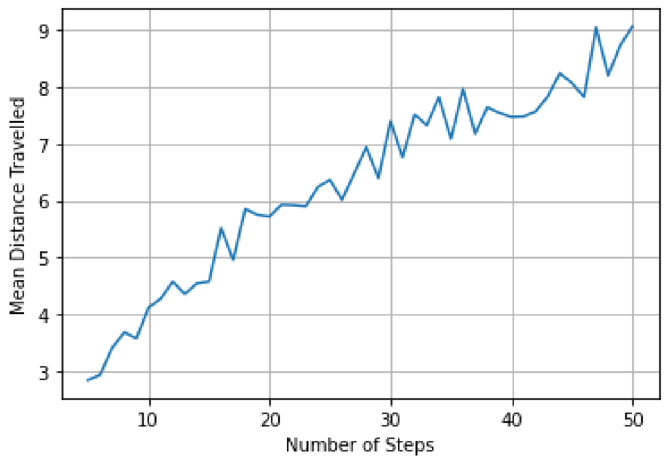


FIG. 10: Mean distance travelled for different number of steps

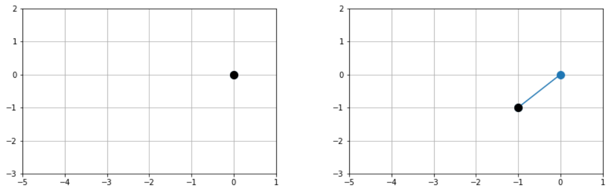
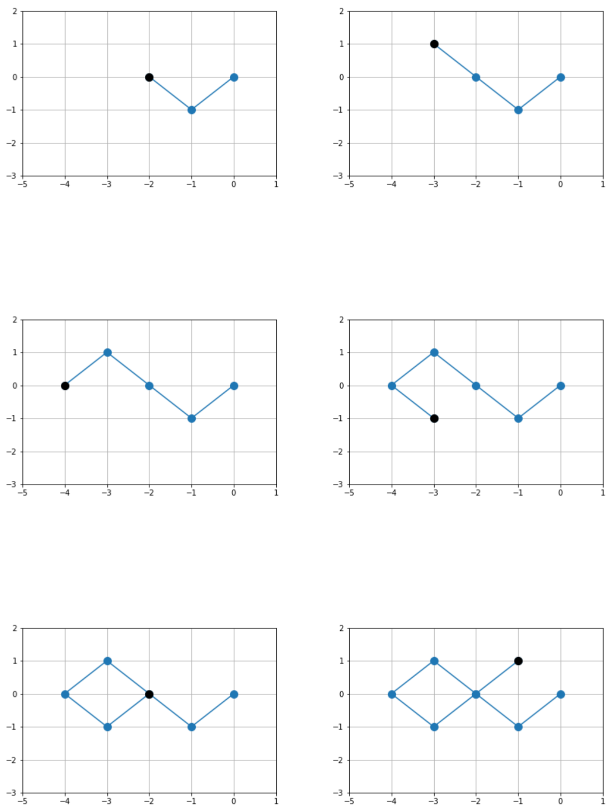
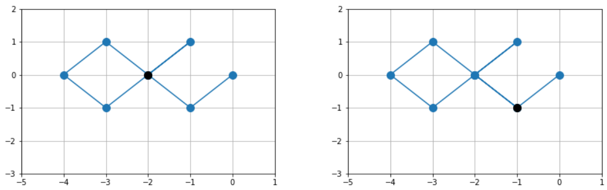


FIG. 11: Start





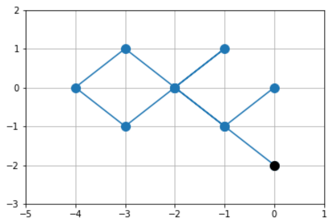


FIG. 12: End

2. **Biased Random Walk:** For the case of biased random walk we consider the case that a hiker is standing in the dark and we are given the probabilities of directions in which he tends to move: N: 19%; NE: 24%; E: 17%; SE: 10%; S: 2%; SW: 3%; W: 10%; NW: 15%. The figure shown below shows the random walk generated by giving the probabilities as input.

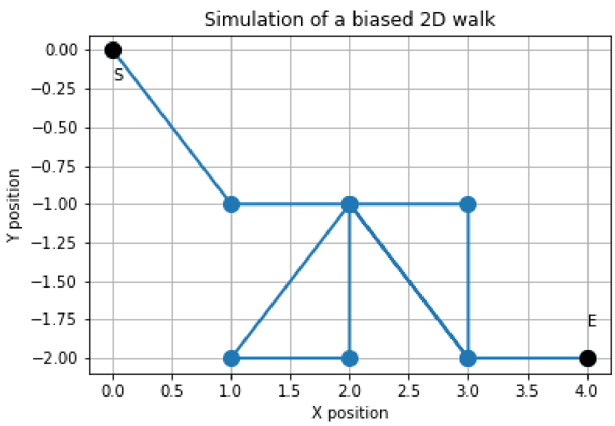


FIG. 13: Simulation of 2D random walk using the above mentioned probabilities

Frames of animation

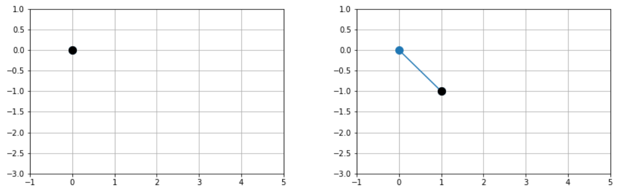
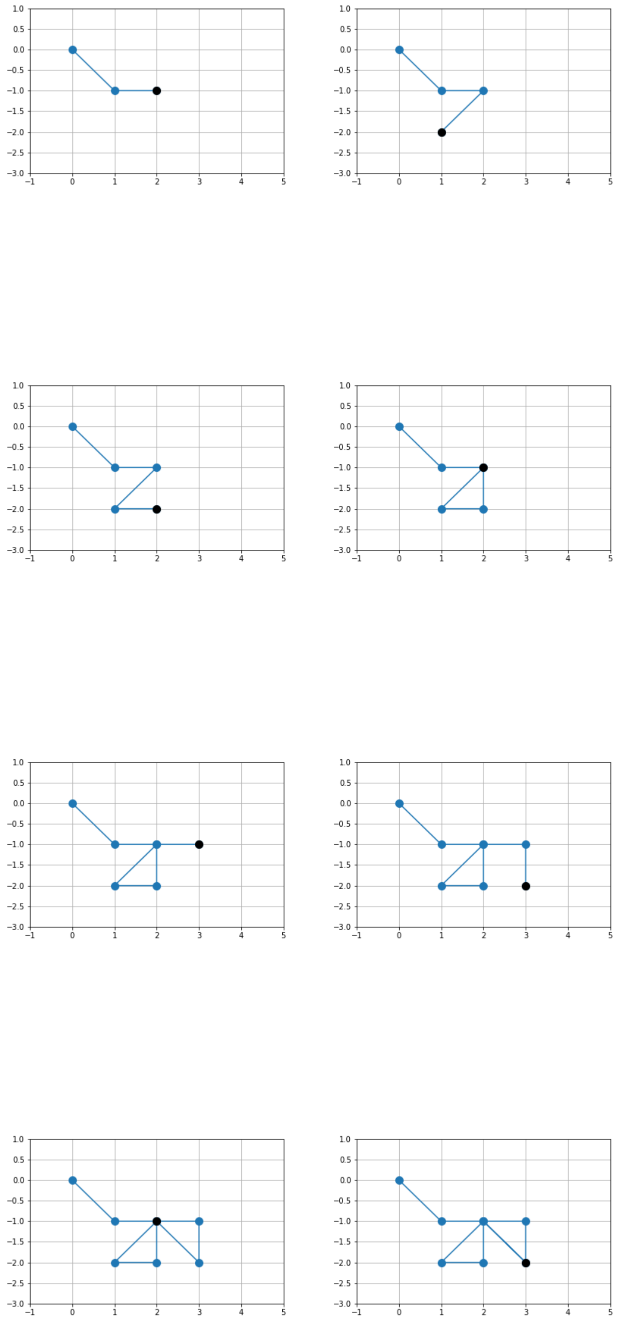


FIG. 14: Start



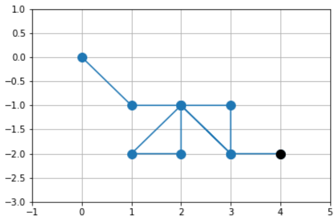


FIG. 15: End

IV. CONCLUSIONS

We have observed that unbiased random has zero average displacement. Variance is linear function of number of steps. In other words, Average distance after N the step is approximately $\pm\sqrt{N}$. When N is significantly increased, distribution tends to be Gaussian distribution following Central Limit Theorem. Hence, we see stationary graph. On the other hand, biased random walk has non zero average displacement. The slop of variance decreases. We observe similar thing in 2D random walk. The average displace is zero for 2D unbiased random walk and non-zero in case of biased random walk.

[1] A. Shiflet and G. Shiflet, *Introduction to Computational Science: Modeling an Simulation for the Sciences*, Princeton University Press.3, 276 (2006).