Lab - 3: Modelling Population Growth

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CS-302, Modeling and Simulation

In this lab we analyzed the population growth model using the logistic equation both computationally and numerically. We also modified the logistic equation to incorporate the conditions for constant harvesting, dynamic harvesting and death due to isolation.

I. INTRODUCTION

The simplest model of the population growth is exponential growth model. But in reality, the population does not grow for forever. It has been observed that the population often gets saturated and sometimes even decreases over time. This is due to the fact that resources available to any specie is limited. In addition to that, species are also part of food chain. So, specie is harvested by their predators as well. Sometimes, we observe sudden drop in population due to natural calamities or pandemic. The exponential model is unable to capture this behaviour. So, several other models of the population growth are proposed. We shall study some of them here.

The exponential model considers only birth rate of the specie. Pierre-François Verhulst proposed the logistic equation for the population growth. Other model such as constant harvesting, dynamic harvesting, threshold population model etc. are derived from the logistic equation by minute modifications. These all are discussed here in detail.

II. MODELS

A. Logistic Equation

In this section, we would build a model of population that takes several factors such as birth rate, death rate, food supply, resources, predators etc. into account. Following differential equation defines rate of population in such scenario.

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{1}$$

1. K is the carrying capacity which represents the maximum possible population of an organism in an environment
2. r represents the growth rate of the organism and
3. P(t) represents the population of the organism at time t

To avoid too many parameters in solving the Eq. 1 we can make it dimensionless by changing the variables as follows,

$$\frac{dn}{d\tau} = n\,(1-n) \tag{2}$$

where $n = P/K$ and $\tau = t \cdot r$
Eq. 1 can be modified to consider other factors such as harvesting (reduction of the population due to hunting or capturing individuals) and threshold population.

B. Model with Constant Harvesting Rate h

In this model, we assume that population is being harvested at constant rate h. So, we get the population growth as follows,

$$\frac{dn}{d\tau} = n\,(1-n) - h' \tag{3}$$

where $h' = h/(Kr)$.
Equating Eq.3 to 0 we get the solutions as follows,

$$n = \frac{-1 \pm \sqrt{1-4h'}}{2} \tag{4}$$

From Eq.4 we get 3 different cases depending on the sign of $1 - 4h'$.

1. $1 - 4h' > 0 \implies h' < 0.25$
2. $1 - 4h' = 0 \implies h' = 0.25$
3. $1 - 4h' < 0 \implies h' > 0.25$

Graphical Analysis

1. Initial population < (Carrying Capacity)/2

   Here we consider the initial population to be less than the half the carrying capacity. When the rate of harvesting becomes greater than 0.25 (h = 0.28), the value of the discriminant in Eq.3 becomes -ve

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and hence the population tends to 0. When the rate of harvesting becomes lesser than 0.25 (h = 0.22), the value of the discriminant in Eq.3 becomes +ve and hence the population saturates.

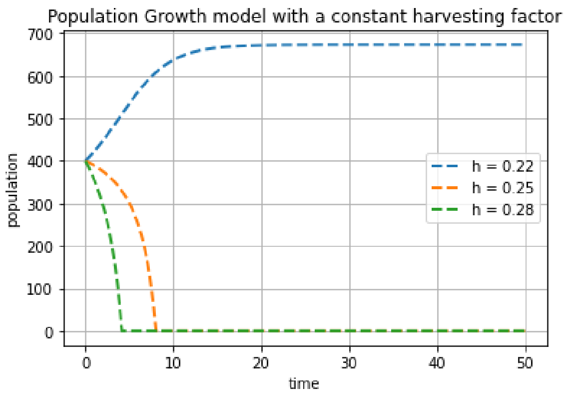

FIG. 1: population vs time for different values of the harvesting factor when initial population is less than half the carrying capacity

However when the rate of harvesting becomes equal to 0.25, the value of the discriminant becomes equal to 0 and the root comes out to be $k/2$.

2. Initial population = (Carrying Capacity)/2

   Here we consider the initial population to be equal to half the carrying capacity. When the rate of harvesting becomes greater than 0.25 (h = 0.28), the population tends to 0 as seen in the above case. When the rate of harvesting becomes lesser than 0.25 (h = 0.22), the value of the discriminant in Eq.3 becomes +ve and hence the population saturates.

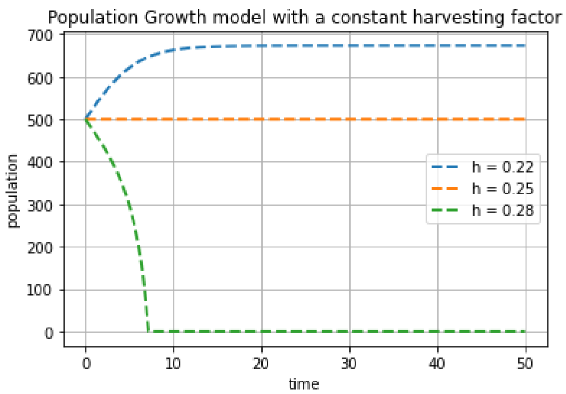

FIG. 2: population vs time for different values of the harvesting factor when initial population is equal to half the carrying capacity

However when the rate of harvesting becomes equal to 0.25, the value of the discriminant becomes equal to 0 and the root comes out to be $k/2$, which in this case is also the starting population and hence it remains constant.

3. Initial population > (Carrying Capacity)/2

   Here we consider the initial population to be greater than half the carrying capacity. When the rate of harvesting becomes greater than 0.25 (h = 0.28), the population tends to 0 as seen in the above case. When the rate of harvesting becomes lesser than 0.25 (h = 0.22), the value of the discriminant in Eq.3 becomes +ve and hence the population saturates as seen in the above cases.

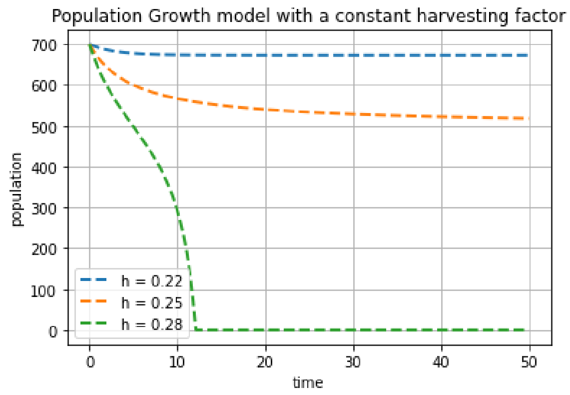

FIG. 3: population vs time for different values of the harvesting factor when initial population is greater than half the carrying capacity

However when the rate of harvesting becomes equal to 0.25, the value of the discriminant becomes equal to 0 and the root comes out to be $k/2$, which is a stable equilibrium and hence the values tends to $k/2$.

C. Model with Dynamic Harvesting Rate

Instead of keeping the harvesting rate constant, we can assume the harvesting rate to be proportional to current population. With slight modification in Eq. 3, we can write

$$\frac{dn}{d\tau} = n\,(1-n) - \epsilon n \tag{5}$$

where ϵ represents rate of harvesting.
Equating Eq.5 to 0 we get the solutions as follows,

$$n = 0 \text{ and } n = 1 - \epsilon$$

From the above solution we get three different possibilities for ϵ i.e. > 1, < 1 and $= 1$.

Graphical Analysis

1. For the first case (Fig.4) we considered the value of the initial population as 400 and we found out that for the cases when $epsilon \geq 1$ the population tends to 0. However when the value of $\epsilon < 1$, the population saturates to $(1-\epsilon) \cdot k$

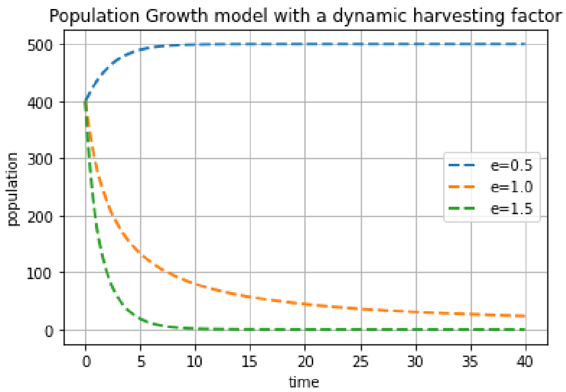

FIG. 4: population vs time for different values of the harvesting factor (> 1,< 1 and $= 1$)

2. For the second case (Fig.5) we considered the value of the initial population as 800 and we found out a similar behaviour as compared to the first case with the only difference being that for $\epsilon < 1$ the population decreases since $(1-\epsilon) \cdot k < 800$

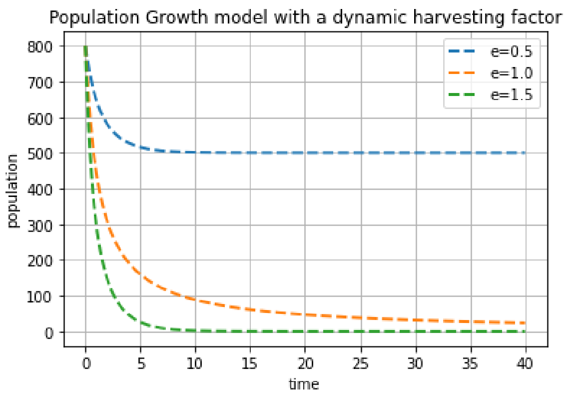

FIG. 5: population vs time for different values of the harvesting factor

D. Death by Isolation

Logistic equation is a very simplified form that predicts the population growth. However a more practical situation would be that a non-zero population results only if the initial population is greater than a particular threshold population. The logistic equation for the above condition can be modified by multiplying it with $(x - A)$, where A is the threshold population lying between 0 and k

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)(x - A) \tag{6}$$

Normalizing the equation we get,

$$\frac{dn}{d\tau} = n\,(1 - n)\,(n - n_{thresh}) \tag{7}$$

where $n_{thresh} = A/k$

1. For the first case (Fig.6) we considered the threshold population to be 400, the carrying capacity as 1000, and the growth rate as 2. We considered two different values of initial population, one being less than the threshold and the other greater than the threshold.

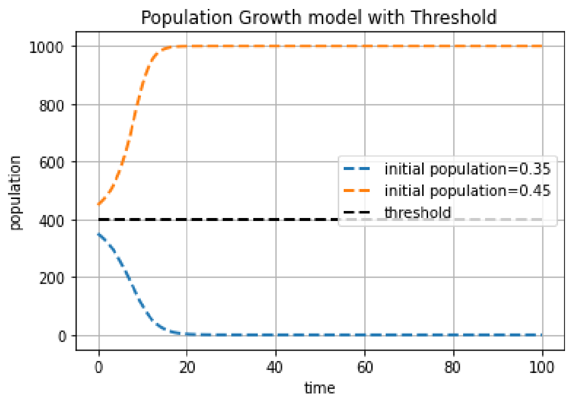

FIG. 6: population vs time for different values of the initial population

We can see that the population dies down if the initial value is lesser than the threshold and saturates to the carrying capacity otherwise.

2. For the second case (Fig.7) we added a dynamic harvesting factor of 0.02 with the same initial conditions as above. Here also we can see that the population dies down if the initial value is lesser than the threshold but saturates to a value lesser than the carrying capacity because of harvesting.

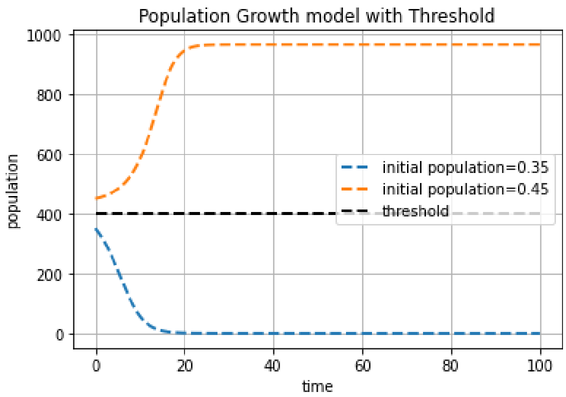

FIG. 7: population vs time for different values of the initial population along with dynamic harvesting.

3. For the third case (Fig.8) we added a dynamic harvesting factor of 0.04 with the same initial conditions as above. Here also we can see that the population dies down if the initial value is lesser than the threshold. However for the other case as well the population decreases because after a certain point the population becomes lesser than the threshold because of harvesting.

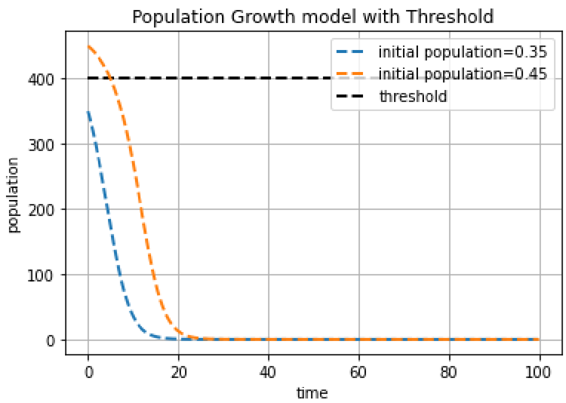

FIG. 8: population vs time for different values of the initial population along with dynamic harvesting.

III. CONCLUSIONS

We started with exponential growth model. We observed that this model does not accurately model the real life scenario. It only considers birth rate not death rate. Next,we analyzed the logistic equation which represents the self limiting population. Then we we modified the model by introducing constant harvesting rate and dynamic harvesting rate. We simulated the model in python and obtained graphical results. We discussed the effect of harvesting rate on population growth. Lastly, we studied death by isolation in which we found that population may extinct if it goes below a certain threshold. In all of the models, simulated results justifies our theory.