Finite state automaton (FSA)

LING 570

Fei Xia

FSA / FST

- It used to be an important technique in NLP.
- Multiple FSAs/FSTs can be combined to form a larger, more powerful FSAs/FSTs.
- Any regular language can be recognized by an FSA.
- Any regular relation can be recognized by an FST.
- It is important to understand formal languages and their connection to (finite state / pushdown) automata.

FST Toolkits

- AT&T: http://www.research.att.com/~fsmtools/fsm
- NLTK: http://nltk.sf.net/docs.html
- ISI: Carmel
- …

Outline

- Deterministic FSA (DFA)
- Non-deterministic FSA (NFA)
- Probabilistic FSA (PFA)
- Weighted FSA (WFA)

DFA

Definition of DFA

An automaton is a 5-tuple = $(\Sigma, Q, q_0, F, \delta)$

- An alphabet input symbols $\sum$
- A finite set of states Q
- A start state q_0
- A set of final states F
- A transition function:

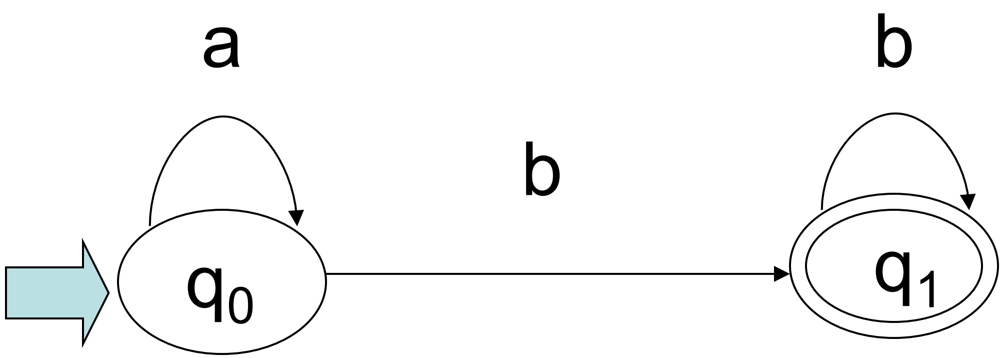
$$\delta : Q \times \Sigma \longrightarrow Q$$

$$\Sigma = \{a, b\}$$

$$S = \{q_0, q_1\}$$

$$F = \{q_1\}$$

$$\delta = \{ q_0 \times a \to q_0, \\ q_0 \times b \to q_1, \\ q_1 \times b \to q_1 \}$$



What about $q_1 \times a$?

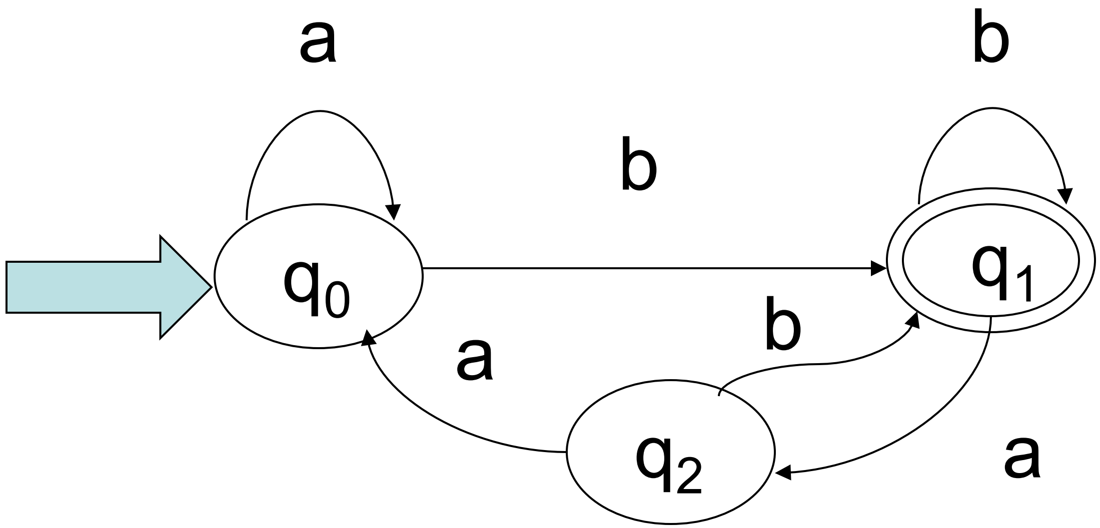
Representing an FSA as a directed graph

- The vertices denote states:
  - Final states are represented as two concentric circles.

- The transitions forms the edges.

- The edges are labeled with symbols.

An example



a b b a a

q_0 q_0 q_1 q_1 q_2 q_0

a b b a b

q_0 q_0 q_1 q_1 q_2 q_1

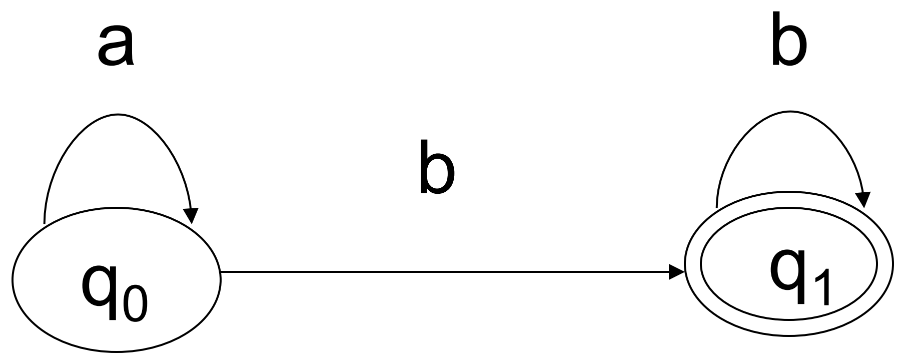
DFA as an acceptor

- A string is said to be accepted by an FSA if the FSA is in a final state when it stops working.
  – that is, there is a path from the initial state to a final state which yields the string.
  – Ex: does the FSA accept "abab"?

- The set of the strings that can be accepted by an FSA is called the language accepted by the FSA.

An example

FSA:



Regular language: {b, ab, bb, aab, abb, …}

Regular expression: $a^* \, b^+$

Regular grammar:

$q_0 \rightarrow a \; q_0$
$q_0 \rightarrow b \; q_1$
$q_1 \rightarrow b \; q_1$
$q_1 \rightarrow \epsilon$

NFA

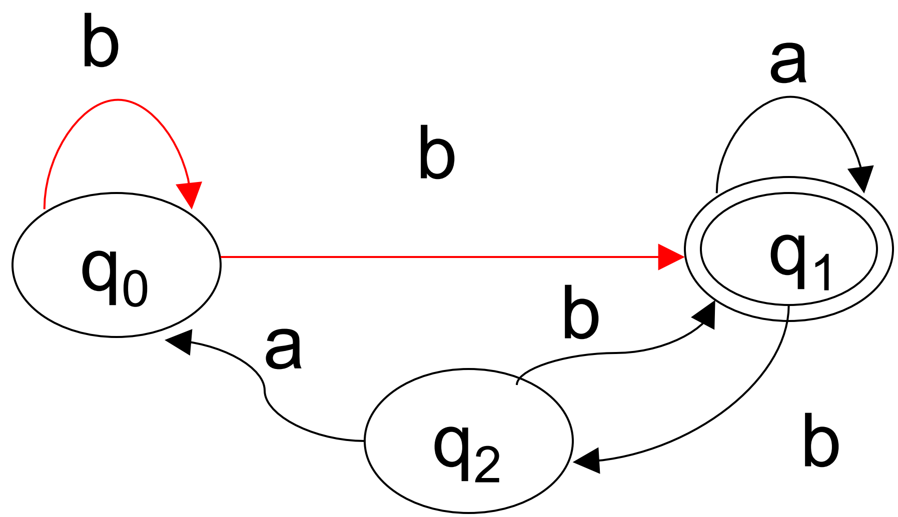
NFA

- A transition can lead to more than one state.
- There could be multiple start states.
- Transitions can be labeled with ϵ, meaning states can be reached without reading any input.

➔ now the transition function is:

$$S \times (\Sigma \bigcup \{\epsilon\}) \rightarrow 2^S$$

NFA example



b	b	a	b	b

q_0 q_0 q_1 q_1 q_2 q_1

q_0 q_1 q_2 q_0 q_0 q_0

b	b	a	b	b

q_0 q_1 q_2 q_0 q_0 q_1

q_0 q_1 q_2 q_0 q_1 q_2

Relation between DFA and NFA

- DFA and NFA are equivalent.

- The conversion from NFA to DFA:
  – Create a new state for each equivalent class in NFA
  – The max number of states in DFA is 2^N, where N is the number of states in NFA.

- Why do we need both?

Regular grammar and FSA

- Regular grammar: (N, Σ, P, S)
- FSA: $(\Sigma, Q, q_0, F, \delta)$
- Conversion between the two

Common algorithms for FSA packages

- Converting regular expressions to NFAs
- Converting NFAs to regular expressions
- Determinization: converting NFA to DFA
- Other useful *closure* properties: union, concatenation, Kleene closure, intersection

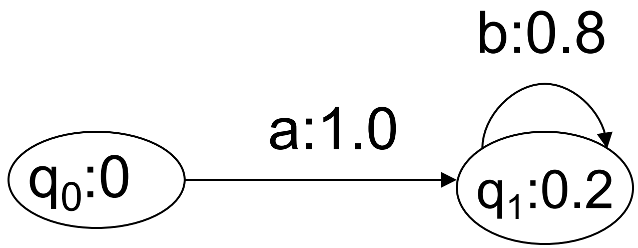
So far

- A DFA is a 5-tuple: $(\Sigma, Q, q_0, F, \delta)$
- A NFA is a 5-tuple: $(\Sigma, Q, I, F, \delta)$
- DFA and NFA are equivalent.
- Any regular language can be recognized by an FSA.
  - Reg lang ⇔ Regex ⇔ NFA ⇔ DFA ⇔ Reg grammar

Outline

- Deterministic finite state automata (DFA)
- Non-deterministic finite state automata (NFA)
- **Probabilistic finite state automata (PFA)**
- Weighted Finite state automata (WFA)

An example of PFA



$F(q_0)=0$
$F(q_1)=0.2$

$I(q_0)=1.0$
$I(q_1)=0.0$

$P(ab^n)=I(q_0)*P(q_0,ab^n,q_1)*F(q_1)$
$=1.0*(1.0*0.8^n)*0.2$

$$\sum_x P(x) = \sum_{n=0}^{\infty} P(ab^n) = 0.2 * \sum_{n=0}^{\infty} 0.8^n = 0.2 * \frac{0.8^0}{1-0.8} = 1$$

Formal definition of PFA

A PFA is $(Q, \Sigma, I, F, \delta, P)$

- Q: a finite set of N states
- Σ: a finite set of input symbols
- I: Q ➔ R^+ (initial-state **probabilities**)
- F: Q ➔ R^+ (final-state **probabilities**)
- $\delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times Q$: the transition relation between states.
- **P:** $\delta \rightarrow R^+$ **(transition probabilities)**

Constraints on function:

$$\sum_{q \in Q} I(q) = 1$$

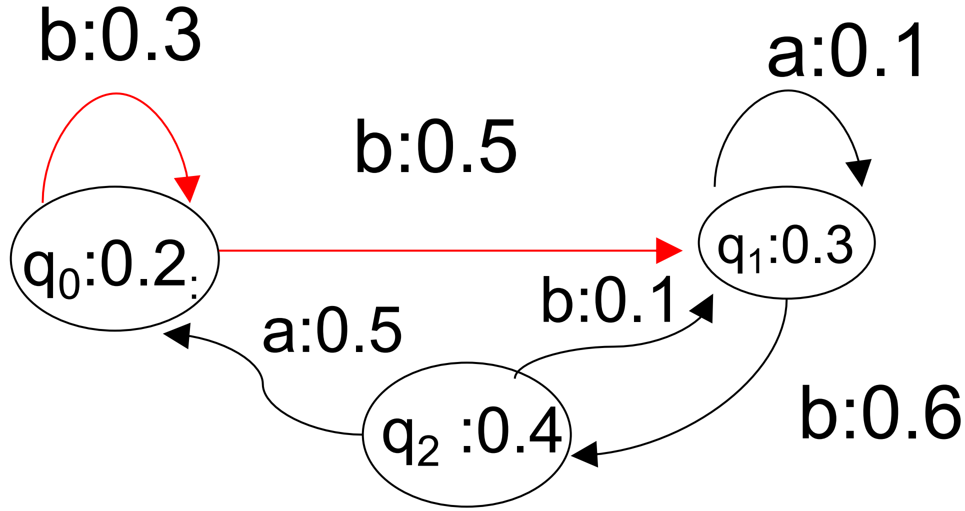
$$\forall q \in Q \quad F(q) + \sum_{\substack{a \in \Sigma \cup \{\varepsilon\} \\ q' \in Q}} P(q, a, q') = 1$$

Probability of a string:

$$P(w_{1,n}, q_{1,n+1}) = I(q_1) * F(q_{n+1}) * \prod_{i=1}^{n} p(q_i, w_i, q_{i+1})$$

$$P(w_{1,n}) = \sum_{q_{1,n+1}} P(w_{1,n}, q_{1,n+1})$$

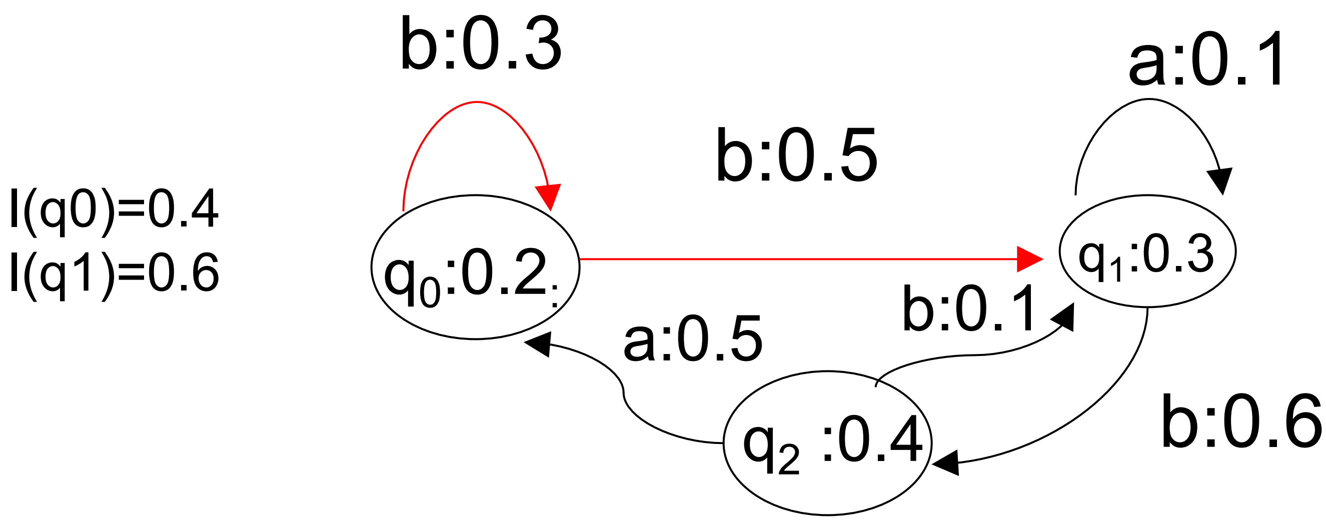
I(q0)=0.4
I(q1)=0.6



Input: $\boxed{b \quad b \quad b}$

Prob(input, path)

$q_0 \; q_0 \; q_0 \; q_0$	I(q0)*P(q0, b, q0) * P(q0,b,q0)*P(q0,b,q0)*F(q0)
$q_0 \; q_0 \; q_0 \; q_1$	I(q0)*P(q0, b, q0) * P(q0,b,q0)*P(q0,b,q1)*F(q1)
$q_1 \; q_2 \; q_1 \; q_2$	I(q1)*P(q1, b, q2) * P(q2,b,q1)*P(q1,b,q2)*F(q2)
…	



Input: b b b

Prob(input, path)

q_0 q_0 q_0 q_0 I(q0)*P(q0, b, q0) * P(q0,b,q0) *P(q0,b,q0)*F(q0)

q_0 q_0 q_0 q_1 I(q0)*P(q0, b, q0) * P(q0,b,q0) *P(q0,b,q1)*F(q1)

q_1 q_2 q_1 q_2 I(q1)*P(q1, b, q2) * P(q2,b,q1)*P(q1,b,q2)*F(q2)

…

PFA

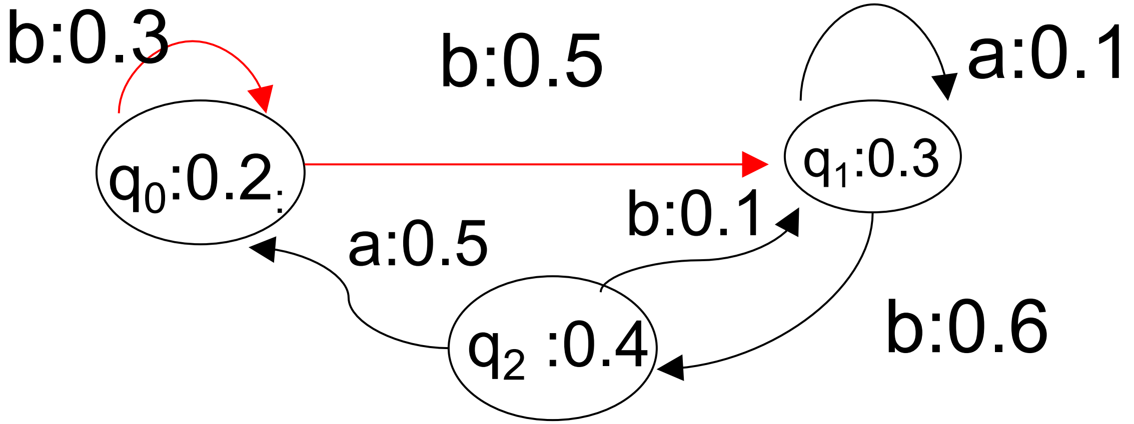
- Informally, in a PFA, each arc is associated with a probability.

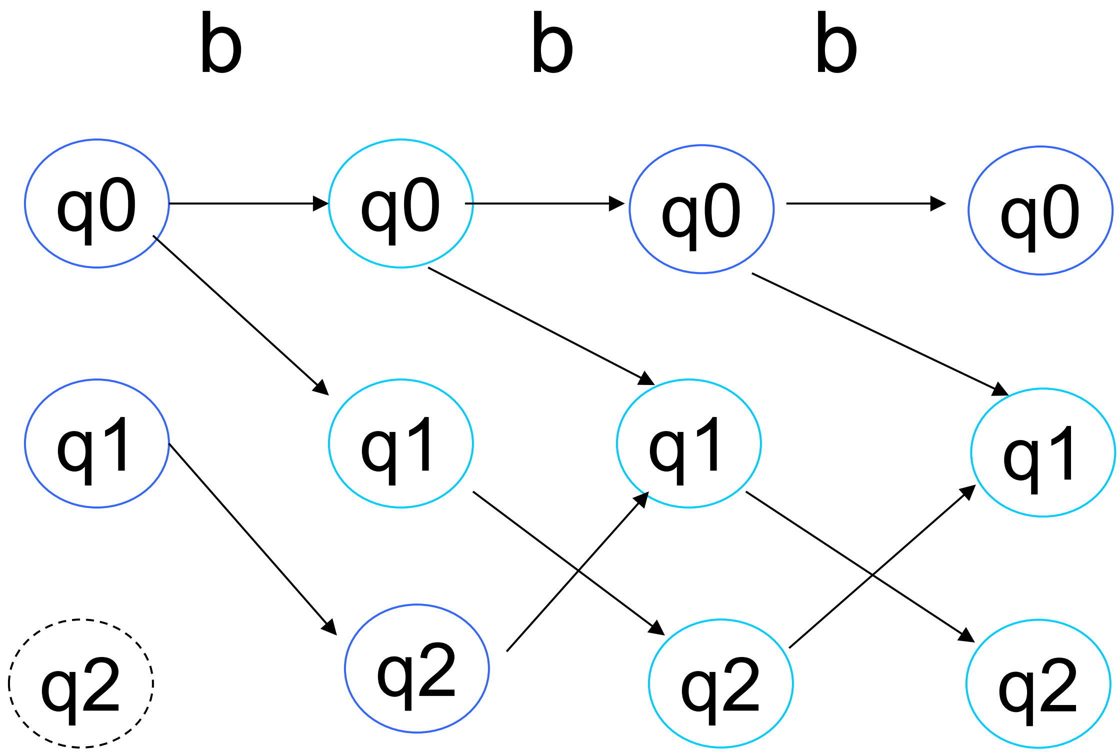
- The probability of <u>a path</u> is the multiplication of the arcs on the path.

- The probability of <u>a string</u> x is the sum of the probabilities of all the paths for x.

- Tasks:
  - Given a string x, find the best path for x.
  - Given a string x, find the probability of x in a PFA.
  - Find the string with the highest probability in a PFA
  - …

I(q0)=0.4
I(q1)=0.6


b:0.3
b:0.5
a:0.1
q_0:0.2
q_1:0.3
b:0.1
a:0.5
q_2 :0.4
b:0.6


b
b
b
q0
q0
q0
q0
q1
q1
q1
q1
q2
q2
q2
q2

Finding the best path for input x

- Read Section 3.2 of the 2005 PFA paper.

$$\tilde{\theta} = \underset{\theta \in \Theta_{\mathcal{A}}(x)}{\operatorname{argmax}} \ \Pr_{\mathcal{A}}(\theta)$$

The probability of this *optimal path* $\tilde{\theta}$ will be denoted as $\widetilde{\Pr}_{\mathcal{A}}(x)$.

The computation of $\widetilde{\Pr}_{\mathcal{A}}(x)$ can be efficiently performed by defining a function $\gamma_x(i, q)$ $\forall q \in Q$, $0 \leq i \leq |x|$, as the probability of generating the prefix $x_1 \ldots x_i$ through the best path and reaching state q:

$$\gamma_x(i,q) = \max_{(s_0,s_1,\ldots,s_i)\in\Theta_{\mathcal{A}}(x_1\ldots x_i)} I(s_0)\cdot\prod_{j=1}^{i} P(s_{j-1},x_j,s_j)\cdot 1(q,s_i)$$

where $1(q,q') = 1$ if $q = q'$ and 0 if $q \neq q'$.

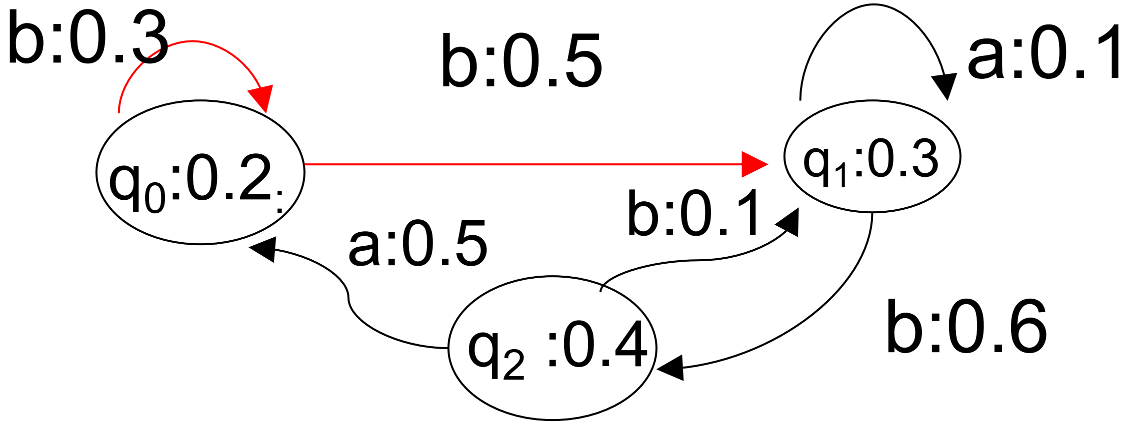
Viterbi algorithm:

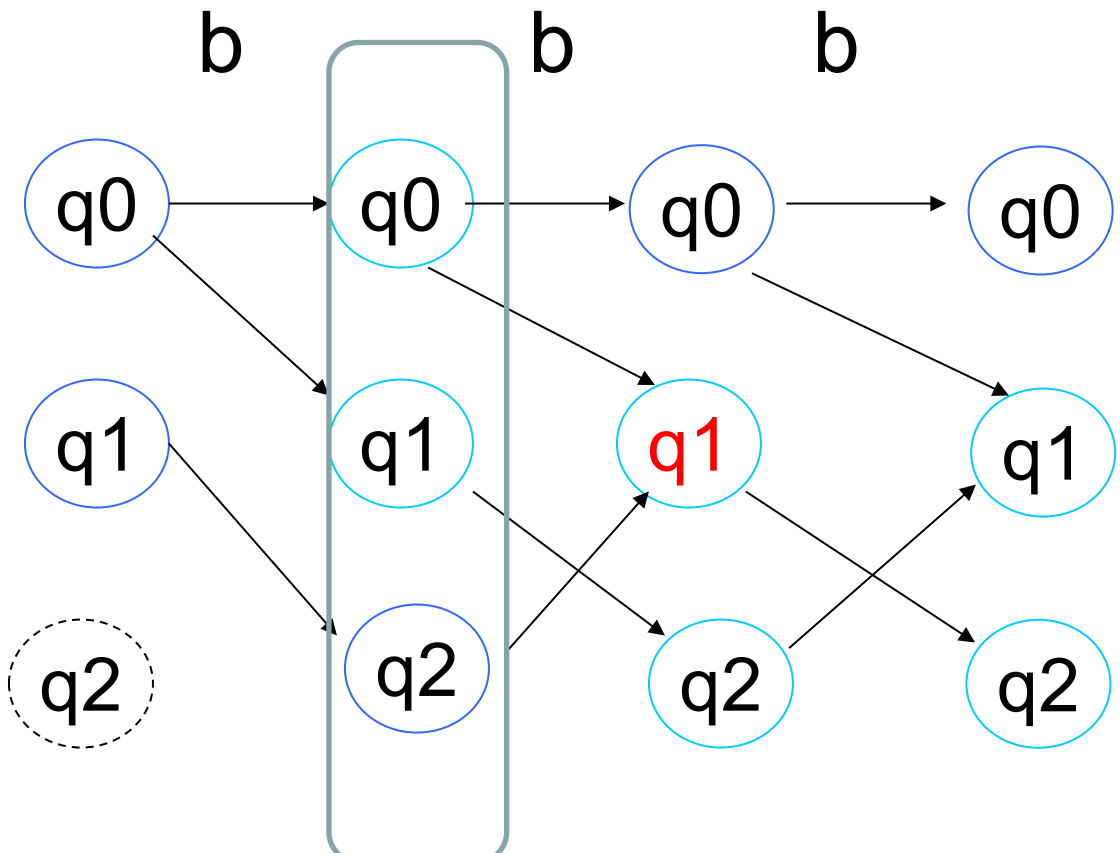
$$\gamma_x(0,q) = I(q),$$
$$\gamma_x(i,q) = \max_{q'\in Q} \gamma_x(i-1,q')\cdot P(q',x_i,q), \quad 1 \leq i \leq |x|.$$

Remember to include the final probability:

$$\widetilde{\mathbf{Pr}}_{\mathcal{A}}(x) = \max_{q\in Q} \gamma_x(|x|,q)\cdot F(q).$$

I(q0)=0.4
I(q1)=0.6





gamma(2, q1)

- Calculate the gamma function efficiently:
  - Use a two-dimensional array g[i, q], not a recursive function

- Need to remember the best path, not just the highest probability:
  - For each [i, q], remember q'
  - In other words, you need one array for gamma, another for backpointer: b[i, q] = q'

- To find the best path and the corresponding output sequence given input x and an FST:
  - Since you know the (q', x_i, q) arc on the best path, you can find the corresponding y_i for that arc.

Outline

- Deterministic FSA (DFA)
- Non-deterministic FSA (NFA)
- Probabilistic FSA (PFA)
- Weighted FSA (WFA)

Weighted finite-state automata (WFA)

- Each arc is associated with a weight.

- "Addition" and "Multiplication" can have other meanings.

$$weight(x) = \bigoplus_{s,..,t \in Q} (I(s) \otimes F(t) \otimes P(s,x,t))$$

- In PFA:

$$P(w_{1,n}, q_{1,n+1}) = I(q_1) * F(q_{n+1}) * \boxed{\prod_{i=1}^{n} p(q_i, w_i, q_{i+1})}$$

$$P(w_{1,n}) = \sum_{q_{1,n+1}} P(w_{1,n}, q_{1,n+1})$$

Summary

- DFA and NFA are 5-tuple: $(\Sigma, Q, I, F, \delta)$
  - They are equivalent
  - Algorithm for constructing NFAs for Regexps

- PFA and WFA are 6-tuple: $(Q, \Sigma, I, F, \delta, P)$

- Existing packages for FSA/FSM algorithms:
  - Ex: intersection, union, Kleene closure, difference, complementation, …

Two Views of FSAs

- Recognition: An FSA is a model that, given an input string, accepts the string if it is in the language, and rejects otherwise.
- Generation: An FSA m is a model that can generate all and only the strings in L(m).

Additional slides

Semiring

A semiring is a set R equipped with two binary operations + (i.e., $\oplus$) and · (i.e., $\otimes$), called addition and multiplication, such that:

(1) (R, +) is a commutative monoid with identity element 0:

- ❖ $(a + b) + c = a + (b + c)$
- ❖ $0 + a = a + 0 = a$
- ❖ $a + b = b + a$

(2) (R, ·) is a monoid with identity element 1:

- ❖ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- ❖ $1 \cdot a = a \cdot 1 = a$

(3) Multiplication left and right distributes over addition:

- ❖ $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- ❖ $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$

(4) Multiplication by 0 annihilates R:

- ❖ $0 \cdot a = a \cdot 0 = 0$

Examples of semirings

Set R	$\oplus$	$\otimes$	**0**	**1**	**Arc weight**	**weight (x) is**
[0, 1]	+	x	0	1	prob	Prob of x
[0, 1]	??	??	??	??	prob	Prob of the best path for x
R ∪ {+∞, -∞}	min	+	??	??	distance	Shortest distance
R ∪ {+∞, -∞}	max	+	??	??	distance	Longest distance
N	+	x	0	1	??	Number of paths

$$weight(x) = \bigoplus_{s,..,t \in Q} (I(s) \otimes P(s,x,t) \otimes F(t))$$

x is the input string. Let's ignore I(s) and F(t).

An algorithm for deterministic recognition of DFAs

function D-RECOGNIZE(*tape, machine*) **returns** accept or reject
  index ← Beginning of tape
  current-state ← Initial state of machine
  loop
    if End of input has been reached **then**
      if *current-state* is an accept state **then**
        return accept
      else
        return reject
    elsif *transition-table*[*current-state,tape*[*index*]] is empty **then**
      return reject
    else
      current-state ← *transition-table*[*current-state,tape*[*index*]]
      index ← *index* +1
  end