Q1: Modeling the Temporary Impact Function $g_t(x)$

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Project Code: Github

Introduction

Temporary market impact refers to the immediate slippage an order incurs when it "walks" the limit order book (LOB). A market order consumes liquidity from the best bid/ask and, if large enough, from deeper levels. This pushes the execution price away from the mid-price, creating a measurable **cost**.

Historically, models like **Almgren & Chriss (2001)** assumed a **linear impact**:

$$g_t(x) = \beta_t x$$

where $g_t(x)$ is slippage at time t for trade size x, and β_t is cost per share.

While mathematically convenient, the **linear assumption is unrealistic** for real-world order books: doubling trade size rarely just doubles slippage — large trades often "eat" multiple price levels, causing **nonlinear cost growth**.

Why Linear is Too Simple

- **Order book depth isn't infinite**: Small trades only lift the top of book; bigger trades push through deeper levels.
- **Empirical evidence**: Gatheral (2010) and Tóth et al. (2011) show **temporary impact follows a concave "square-root" law**:

$$g(x) \propto x^{\alpha} \qquad (0.5 \leq \alpha \leq 1)$$

Bershova & Rakhlin (2013) confirm that institutional trades exhibit $\alpha \approx 0.5$–0.7.

This means **impact grows sublinearly**: a 4× larger order doesn't cost 4× more; it costs roughly 2× more.

Proposed Model

We adopt a **time-varying power-law model**:

$$\boxed{g_t(x) = \eta_t x^{\alpha}}$$

- η_t: *impact coefficient* — reflects book liquidity at time t (high η_t means thin liquidity; low η_t means deep book).
- α: *concavity exponent* (we fix $\alpha \approx 0.6$, the empirically validated square-root law).

Why fix α? Estimating α minute-by-minute can be unstable (limited data). By fixing $\alpha = 0.6$ (from Bershova & Rakhlin (2013)), we obtain a **stable** η_t series — the key liquidity signal.

Estimation Steps

1. **Data Input**: The tradebook has nanosecond level data for the three scrips, we aggregate them by using the last value at each minute and get **390 one-minute snapshots** of all the scrips, with the top 10 bids and asks each minute.
2. **Simulate Market Orders**: For a range of sizes (e.g., 50–1,000 shares):
   - Walk the book: fill from best ask/bid downward (for buys) or upward (for sells).
   - Compute **VWAP** (Volume Weighted Average Price).

   Calculate slippage vs. mid-price:

$$g(x) = \text{VWAP} - p_{mid} \text{ (buy side)}$$

$$g(x) = p_{mid} - \text{VWAP} \text{ (sell side)}$$

3. **Fit the Power-Law**: Using SciPy's `curve_fit`, estimate η_t from:

$$g(x) \approx \eta_t x^{0.6}$$

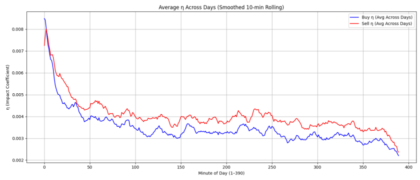
4. **Output**: For each minute we get:

$$\eta_t^{buy}, \eta_t^{sell}$$

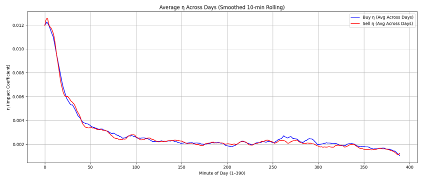
Interpreting η_t

- η_t **is a liquidity gauge**:
  - **Low** η_t → you can trade more cheaply (deep book).
  - **High** η_t → even small trades move the market (thin book).

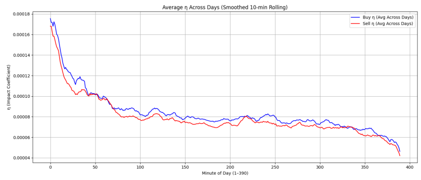
- **Expected intraday pattern**: Research shows η_t typically follows a **U-shape**: higher around **open & close** (liquidity thins), lower **mid-day** (high liquidity).
- For our experiment we calculate a at each minute and then finally plot the smoothed data and receive the following graphs



(a) CRWV



(b) FROG



(c) SOUN

Figure 1: The η v minute graphs obtained for each scrip after optimizing η based on our power law equation

Interpreting Graphs

- We observe that market open is in general a time of low liquidity pointing towards higher η and correspondingly higher temporaryy impact
- The famous U-shape in generally is observed in high liquidity large-cap stocks, not here
- The drop in η towards the closing time could be corresponded to high liquidity due to higher depth because of increasing limit orders
- Another reason for this could be the auction effect, that is closing auction equities and market makers flattening the position

References

- Almgren & Chriss (2001) – Optimal Execution of Portfolio Transactions
- Gatheral (2010) – No-Dynamic-Arbitrage and Market Impact
- Tóth, Eisler & Bouchaud (2011) – Why is Market Impact Concave?
- Bershova & Rakhlin (2013) – The Non-Linear Market Impact of Large Trades

Q2: Mathematical Framework for Optimal Allocation

Mathematical Setup

We aim to execute S shares over 390 minutes, minimizing temporary market impact. From Q1, we model impact as:

$$g_t(x) = \eta_t x^\alpha$$

with minute-level liquidity η_t and concavity $\alpha \approx 0.6$.

We solve:

$$\min_{x_t} \sum_{t=1}^{390} \eta_t x_t^\alpha \quad \text{s.t.} \quad \sum x_t = S.$$

Lagrangian Derivation

The Lagrangian:

$$L = \sum_{t=1}^{390} \eta_t x_t^\alpha + \lambda \left(\sum_{t=1}^{390} x_t - S \right)$$

First-order condition:

$$\alpha \eta_t x_t^{\alpha-1} = \lambda$$

Optimal trade per minute:

$$x_t = \left(\frac{\lambda}{\alpha \eta_t} \right)^{\frac{1}{\alpha-1}}$$

Because $\alpha < 1$, the exponent is negative — trading shifts toward periods with lower η_t.

Computational Steps

1. Estimate η_t (from Q1).
2. Assume $\alpha = 0.6$.
3. Guess λ and compute x_t.
4. Adjust λ (via binary search or root-finding using scipy) until $\sum x_t = S$.

Techniques & Tools

Techniques:

- Convex optimization via Lagrangian multipliers.
- Closed-form x_t expression; numerical solution only for λ.

Tools:

- Python (NumPy/Pandas for data; SciPy `fsolve` for λ).
- Matplotlib/Seaborn to visualize the trading schedule.

Extensions

- **Participation caps**: limit x_t to a
- **Risk term**: add $\phi \sum \sigma_t^2 x_t^2$ to balance market impact and volatility risk.
- **Permanent impact**: include a linear term γx_t if trades shift the reference price.

Outcome

This framework dynamically allocates volume:

$$x_t \propto \eta_t^{\frac{1}{1-\alpha}}$$

Trading more in liquid periods (low η_t) and slowing down in thin periods. If η_t follows a U-shape intraday, the schedule will naturally reflect that.

References

- Almgren & Chriss (2001) – Optimal Execution of Portfolio Transactions
- Gatheral (2010) – No-Dynamic-Arbitrage and Market Impact
- Tóth et al. (2011) – Why is Market Impact Concave?
- Obizhaeva & Wang (2013) – Optimal Trading Strategy and Supply/Demand Dynamics