数理逻辑第四次作业

1

用自然推理规则证明下列断言：

1. $\neg\exists x A(x) \vdash\dashv \forall x \neg A(x)$

证"$\vdash$" 方向：$\neg\exists x A(x) \vdash \forall x \neg A(x)$

$$
\begin{aligned}
A(z) &\vdash A(z) && (\in)\\
A(z) &\vdash \exists x A(x) && (\exists^+)\\
&\vdash A(z) \to \exists x A(x) && (\to^+)\\
\neg\neg A(z) &\vdash A(z) \to \exists x A(x) && (+)\\
\neg\neg A(z), \neg A(z) &\vdash \neg\neg A(z) && (\in)\\
\neg\neg A(z), \neg A(z) &\vdash \neg A(z) && (\in)\\
\neg\neg A(z) &\vdash A(z) && (\neg^-)\\
\neg\neg A(z) &\vdash \exists x A(x) && (\to^-)\\
\neg\exists x A(x), \neg\neg A(z) &\vdash \exists x A(x) && (+)\\
\neg\exists x A(x), \neg\neg A(z) &\vdash \neg\exists x A(x) && (\in)\\
\neg\exists x A(x) &\vdash \neg A(z) && (\neg^-)\\
\neg\exists x A(x) &\vdash \forall x \neg A(x) && (\forall^+)
\end{aligned}
$$

证 $\dashv$ 方向：$\forall x \neg A(x) \vdash \neg\exists x A(x)$

$$
\begin{aligned}
&(1) & \forall x \neg A(x) &\vdash \forall x \neg A(x) && (\in)\\
&(2) & \forall x \neg A(x) &\vdash \neg A(z) && (\forall^-)\\
&(3) & \forall x \neg A(x), A(z) &\vdash \neg A(z) && (+)\\
&(4) & \forall x \neg A(x), A(z) &\vdash A(z) && (\in)\\
&(5) & \forall x \neg A(x), \exists x A(x) &\vdash \neg A(z) && (\exists^-\ (3))\\
&(6) & \forall x \neg A(x), \exists x A(x) &\vdash A(z) && (\exists^-\ (4))\\
&(7) & \forall x \neg A(x) &\vdash \neg\exists x A(x) && (\neg^+)
\end{aligned}
$$

2. $\exists x A(x) \to B \vdash\dashv \forall x(A(x) \to B)$ ，其中 x 不在 B 中

证 $\vdash$ 方向：$\exists x A(x) \to B \vdash \forall x(A(x) \to B)$

$$A(z) \vdash A(z) \quad (\in)$$

$$A(z) \vdash \exists x A(x) \quad (\exists^+)$$

$$(\exists x A(x)) \to B, A(z) \vdash \exists x A(x) \quad (\exists^+)$$

$$(\exists x A(x)) \to B, A(z) \vdash (\exists x A(x)) \to B \quad (\in)$$

$$(\exists x A(x)) \to B, A(z) \vdash B \quad (\to^-)$$

$$(\exists x A(x)) \to B \vdash A(z) \to B \quad (\to^+)$$

$$(\exists x A(x)) \to B \vdash \forall x(A(x) \to B) \quad (\forall^+)$$

证 $\dashv$ 方向：$\forall x(A(x) \to B) \vdash (\exists x A(x)) \to B$

$$\forall x(A(x) \to B) \vdash \forall x(A(x) \to B) \quad (\in)$$

$$\forall x(A(x) \to B) \vdash A(z) \to B \quad (\forall^-)$$

$$\forall x(A(x) \to B), A(z) \vdash A(z) \to B \quad (\forall^-)$$

$$\forall x(A(x) \to B), A(z) \vdash A(z) \quad (\in)$$

$$\forall x(A(x) \to B), A(z) \vdash B \quad (\to^-)$$

$$\forall x(A(x) \to B), \exists x A(x) \vdash B \quad (\exists^-)$$

$$\forall x(A(x) \to B) \vdash (\exists x A(x)) \to B \quad (\to^+)$$

2

证明下列断言：

1. $\vdash \forall x \forall y r(x, y) \to \forall x r(x, x)$

$$\forall x \forall y r(x, y) \vdash \forall x \forall y r(x, y) \quad (\in)$$

$$\forall x \forall y r(x, y) \vdash \forall y r(u, y) \quad (\forall^-)$$

$$\forall y r(u, y) \vdash \forall y r(u, y) \quad (\in)$$

$$\forall y r(u, y) \vdash r(u, u) \quad (\forall^-)$$

$$\forall x \forall y r(x, y) \vdash r(u, u) \quad (\mathrm{Tr})$$

$$\forall x \forall y r(x, y) \vdash \forall x r(x, x) \quad (\forall^+)$$

$$\vdash \forall x \forall y r(x, y) \to \forall x r(x, x) \quad (\to^+)$$

其中 u 不在 $\forall x \forall y r(x, y)$ 中出现。

2. $\vdash \forall x A(x) \to \forall x(A(x) \lor B(x))$

$$\forall x A(x) \vdash \forall x A(x) \qquad (\in)$$

$$\forall x A(x) \vdash A(z) \qquad (\forall^-)$$

$$(\forall x A(x)) \vdash A(z) \lor B(z) \qquad (\lor^+)$$

$$(\forall x A(x)) \vdash \forall x(A(x) \lor B(x)) \qquad (\forall^+)$$

$$\vdash (\forall x A(x)) \to \forall x(A(x) \lor B(x)) \qquad (\to^+)$$

3. $\vdash (\exists x A(x) \to \forall x B(x)) \to \forall x(A(x) \to B(x))$

$$A(z) \vdash A(z) \qquad (\in)$$

$$A(z) \vdash \exists x A(x) \qquad (\exists^+)$$

$$(\exists x A(x)) \to (\forall x B(x)), A(z) \vdash \exists x A(x) \qquad (+)$$

$$(\exists x A(x)) \to (\forall x B(x)), A(z) \vdash (\exists x A(x)) \to (\forall x B(x)) \qquad (\in)$$

$$(\exists x A(x)) \to (\forall x B(x)), A(z) \vdash \forall x B(x) \qquad (\to^-)$$

$$(\exists x A(x)) \to (\forall x B(x)), A(z) \vdash B(z) \qquad (\forall^-)$$

$$(\exists x A(x)) \to (\forall x B(x)) \vdash A(z) \to B(z) \qquad (\to^+)$$

$$(\exists x A(x)) \to (\forall x B(x)) \vdash \forall x(A(x) \to B(x)) \qquad (\forall^+)$$

$$\vdash ((\exists x A(x)) \to (\forall x B(x))) \to \forall x(A(x) \to B(x)) \qquad (\to^+)$$

4. $\vdash \exists y(r(y) \to \forall y r(y))$

借助了定理 $\neg\exists x A(x) \vdash \forall x \neg A(x)$ ，$\neg(A \to B) \vdash A$ 和 $\neg(A \to B) \vdash \neg B$ 。

(1)	$\neg\exists y(r(y) \to \forall y r(y)) \vdash \forall y \neg(r(y) \to \forall y r(y))$	$(\neg\exists x A(x) \vdash \forall x \neg A(x))$
(2)	$\neg\exists y(r(y) \to \forall y r(y)) \vdash \neg(r(z) \to \forall y r(y))$	$(\forall^-)$
(3)	$\neg(r(z) \to \forall y r(y)) \vdash r(z)$	$(\neg(A \to B) \vdash A)$
(4)	$\neg(r(z) \to \forall y r(y)) \vdash \neg\forall y r(y)$	$(\neg(A \to B) \vdash \neg B)$
(5)	$\neg\exists y(r(y) \to \forall y r(y)) \vdash r(z)$	(Tr (2)(3))
(6)	$\neg\exists y(r(y) \to \forall y r(y)) \vdash \neg\forall y r(y)$	(Tr (2)(4))
(7)	$\neg\exists y(r(y) \to \forall y r(y)) \vdash \forall y r(y)$	($\forall^+$ (5))
(8)	$\vdash \exists y(r(y) \to \forall y r(y))$	$(\neg^-)$

定理 $\neg(A\to B)\to A$ 证明:

$$
\begin{array}{lcr}
(1) & A, \neg A, \neg B \vdash A & (\in) \\
(2) & A, \neg A, \neg B \vdash \neg A & (\in) \\
(3) & A, \neg A \vdash B & (\neg^-) \\
(4) & \neg A \vdash A \to B & (\to^+) \\
(5) & (A\to B), \neg(A\to B), \neg A \vdash (A\to B) & (\in) \\
(6) & (A\to B), \neg(A\to B), \neg A \vdash \neg(A\to B) & (\in) \\
(7) & (A\to B), \neg(A\to B) \vdash A & (\neg^-) \\
(8) & \neg(A\to B) \vdash (A\to B)\to A & (\to^+) \\
(9) & \neg(A\to B), \neg A \vdash A\to B & (+\ (4)) \\
(10) & \neg(A\to B), \neg A \vdash (A\to B)\to A & (+\ (8)) \\
(11) & \neg(A\to B), \neg A \vdash A & (\to^-\ \ (9)(10)) \\
(12) & \neg(A\to B), \neg A \vdash \neg A & (\in) \\
(13) & \neg(A\to B) \vdash A & (\neg^-\ (11)(12))
\end{array}
$$

定理 $\neg(A\to B)\to \neg B$ 证明:

$$
\begin{array}{cr}
A, B \vdash B & (\in) \\
B \vdash A\to B & (\to^+) \\
\neg(A\to B), B \vdash A\to B & (+) \\
\neg(A\to B), B \vdash \neg(A\to B) & (\in) \\
\neg(A\to B) \vdash \neg B & (\neg^+)
\end{array}
$$

定理 $\neg\exists x A(x) \to \forall x \neg A(x)$ （书上命题 5.3.3 第 2 条）证明:

$$
\begin{aligned}
\neg A(z) &\vdash \neg A(z) && (\in)\\
\neg A(z) &\vdash \forall x \neg A(x) && (\forall^+)\\
\neg\forall x\neg A(x), \neg A(z) &\vdash \forall x \neg A(x) && (+)\\
\neg\forall x\neg A(x), \neg A(z) &\vdash \neg\forall x \neg A(x) && (\in)\\
\neg\forall x\neg A(x) &\vdash A(z) && (\neg^-)\\
\neg\forall x\neg A(x) &\vdash \exists x A(x) && (\exists^+)\\
\neg\exists x A(x), \neg\forall x\neg A(x) &\vdash \exists x A(x) && (+)\\
\neg\exists x A(x), \neg\forall x\neg A(x) &\vdash \neg\exists x A(x) && (\in)\\
\neg\exists x A(x) &\vdash \forall x \neg A(x) && (\neg^-)
\end{aligned}
$$

推理规则 $(\neg^+)$ 的证明:

$$
(\neg^+) : \frac{\begin{array}{c}\Sigma, A \vdash B\\ \Sigma, A \vdash \neg B\end{array}}{\Sigma \vdash \neg A}
$$

$$
\begin{array}{lcr}
(1) & \Sigma, \neg\neg A \vdash \Sigma & (\in)\\
(2) & \neg\neg A, \neg A \vdash \neg\neg A & (\in)\\
(3) & \neg\neg A, \neg A \vdash \neg A & (\in)\\
(4) & \neg\neg A \vdash A & (\neg^-)\\
(5) & \Sigma, \neg\neg A \vdash A & (+\ (4))\\
(6) & \Sigma, A \vdash B & (\text{假设})\\
(7) & \Sigma, \neg\neg A \vdash B & (\mathrm{Tr}\ (5)(6))\\
(8) & \Sigma, \neg\neg A \vdash \neg B & (\mathrm{ibid})\\
(9) & \Sigma \vdash \neg A & (\neg^-\ (7)(8))
\end{array}
$$