5

Mathematical Tools: Determinants and Matrices

To accompany
Quantitative Analysis for Management, Twelfth Edition,
by Render, Stair, Hanna and Hale
Power Point slides created by Jeff Heyl

LEARNING OBJECTIVES

After completing this module, students will be able to:

1. Understand how matrices and determinants are used as mathematical tools in QA.
2. Compute the value of a determinant.
3. Solve simultaneous equations with determinants.
4. Add, subtract, multiply, and divide matrices.
5. Transpose and find the inverse of matrices.
6. Use matrices to represent a system of equations.

MODULE OUTLINE

Introduction

- Matrices and determinants
- Useful in
  - Markov analysis
  - Game theory
  - Linear programming
  - Other quantitative analysis problems

Matrices and Matrix Operations

- A **matrix** is an array of numbers arranged in rows and columns
  - Usually enclosed in parentheses or brackets
  - Effective means of presenting or summarizing business data

AUDIENCE SWITCHING PROBABILITIES, NEXT MONTH'S ACTIVITY			
CURRENT STATION	CHANNEL 6	CHANNEL 8	STOP VIEWING
Channel 6	0.80	0.15	0.05
Channel 8	0.20	0.70	0.10

2 x 3 matrix

Matrix Addition and Subtraction

- *Matrix addition* and *subtraction* are the easiest operations

$$\text{Matrix } A = \begin{pmatrix} 5 & 7 \\ 2 & 1 \end{pmatrix}$$

$$\text{Matrix } B = \begin{pmatrix} 3 & 6 \\ 3 & 8 \end{pmatrix}$$

- To find the sum

$$\text{Matrix } C = \text{Matrix } A + \text{Matrix } B$$

$$= \begin{pmatrix} 5 & 7 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 3 & 8 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ 5 & 9 \end{pmatrix}$$

$$+ \qquad =$$

Matrix Addition and Subtraction

- *Matrix addition* and *subtraction* are the easiest operations

$$\text{Matrix } A = \begin{pmatrix} 5 & 7 \\ 2 & 1 \end{pmatrix}$$

$$\text{Matrix } B = \begin{pmatrix} 3 & 6 \\ 3 & 8 \end{pmatrix}$$

- To subtract matrix *B* from matrix *A*

$$\text{Matrix } C = \text{Matrix } A - \text{Matrix } B$$

$$\underline{\qquad} - \underline{\qquad} = \underline{\qquad}$$

$$= \begin{pmatrix} 5 & 7 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 6 \\ 3 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -7 \end{pmatrix}$$

Matrix Multiplication

- *Matrix multiplication* can take place only if the number of columns in the first matrix equals the number of rows in the second matrix

MATRIX *A* SIZE	MATRIX *B* SIZE	SIZE OF *A* x *B* RESULTING
3 x 3	3 x 3	3 x 3
3 x 1	1 x 3	3 x 3
3 x 1	1 x 1	3 x 1
2 x 4	4 x 3	2 x 3
6 x 9	9 x 2	6 x 2
8 x 3	3 x 6	8 x 6

Matrix Multiplication

- These matrices may *not* be multiplied

MATRIX *A* SIZE	MATRIX *B* SIZE
3 x 4	3 x 3
1 x 2	1 x 2
6 x 9	8 x 9
2 x 2	3 x 3

Matrix Multiplication

- Compute the value of Matrix *A* x Matrix *B* = Matrix *C*

$$\text{Matrix } A = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \qquad \text{Matrix } B = \begin{pmatrix} 4 & 6 \end{pmatrix}$$

- Symbolically

$$AB = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d & e \end{pmatrix} = \begin{pmatrix} ad & ae \\ bd & be \\ cd & ce \end{pmatrix} = \text{C}$$

Matrix Multiplication

- Compute the value of Matrix *A* x Matrix *B* = Matrix *C*

$$\text{Matrix } A = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \qquad \text{Matrix } B = \begin{pmatrix} 4 & 6 \end{pmatrix}$$

- Using actual numbers

$$AB = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 & 6 \end{pmatrix} = \begin{pmatrix} 20 & 30 \\ 8 & 12 \\ 12 & 18 \end{pmatrix} = \text{Matrix } C$$

Matrix Multiplication

- Matrix R = (6 2 5) and Matrix S = $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Matrix R	x	Matrix S	=	Matrix T
(1 x 3)		(3 x 1)		(1 x 1)

$$\begin{pmatrix} a & b & c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} = (ad + be + cf)$$

$$\begin{pmatrix} 6 & 2 & 5 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \left[(6)(3) + (2)(1) + (5)(2)\right] = (30)$$

Matrix Multiplication

$$\text{Matrix } U = \begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix} \qquad \text{Matrix } V = \begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix}$$

Matrix *U*	x	Matrix *V*	=	Matrix *Y*
(2 x 2)		(2 x 2)		(2 x 2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix} = \begin{pmatrix} 18+10 & 24+16 \\ 21+5 & 28+8 \end{pmatrix} = \begin{pmatrix} 28 & 40 \\ 26 & 36 \end{pmatrix}$$

Matrix Multiplication

- An **identity matrix** has 1s on its diagonal and 0s in all other positions

$$\text{Matrix } H = \begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix} \qquad \text{Matrix } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrix *H* x Matrix *I* = Matrix *J*

$$\begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4+0 & 0+7 \\ 2+0 & 0+3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix}$$

Matrix Multiplication

- Blank Plumbing and Heating

PROJECT	DEMAND		
	TOILETS	SINKS	BATHTUBS
Dormitory	5	10	2
Office	20	20	0
Apartments	15	30	15

COST/UNIT	
Toilet	$40
Sink	25
Bathtub	50

Matrix Multiplication

- Blank Plumbing and Heating

Job demand matrix x Fixture cost matrix = Job cost matrix

$$\underset{(3\ \times\ 3)}{\begin{pmatrix} 5 & 10 & 2 \\ 20 & 20 & 0 \\ 15 & 30 & 15 \end{pmatrix}} \times \underset{(3\ \times\ 1)}{\begin{pmatrix} \$40 \\ \$25 \\ \$50 \end{pmatrix}} = \underset{(3\ \times\ 1)}{\begin{pmatrix} \$200 + 250 + 100 \\ \$800 + 500 + 0 \\ \$600 + 750 + 750 \end{pmatrix}} = \begin{pmatrix} \$500 \\ \$1{,}300 \\ \$2{,}100 \end{pmatrix}$$

Matrix Notation for Systems of Equations

- The system

$$2X_1 + 3X_2 = 24$$
$$4X_1 + 2X_2 = 36$$

- Can be written as

$$\begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 24 \\ 36 \end{pmatrix}$$

Matrix Notation for Systems of Equations

- The

In general, express system of equations as

$$AX = B$$

- Can be written as

$$\begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 24 \\ 36 \end{pmatrix}$$

Matrix Transpose

- The **transpose** of a matrix is a means of presenting data in different form
- Interchange rows with columns

$$\text{Matrix } A = \begin{pmatrix} 5 & 2 & 6 \\ 3 & 0 & 9 \\ 1 & 4 & 8 \end{pmatrix}$$

$$\text{Transpose of matrix } A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 0 & 4 \\ 6 & 9 & 8 \end{pmatrix}$$

Matrix Transpose

- The **transpose** of a matrix is a means of presenting data in different form
- Interchange rows with columns

$$\text{Matrix } B = \begin{pmatrix} 2 & 7 & 0 & 3 \\ 8 & 5 & 6 & 4 \end{pmatrix}$$

$$\text{Transpose of matrix } B = \begin{pmatrix} 2 & 8 \\ 7 & 5 \\ 0 & 6 \\ 3 & 4 \end{pmatrix}$$

Determinants, Cofactors, and Adjoints

- A **determinant** is a value associated with a square matrix
- Of value in helping to solve a series of **simultaneous equations**

A 2 x 2 determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

A 3 x 3 determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Determinants

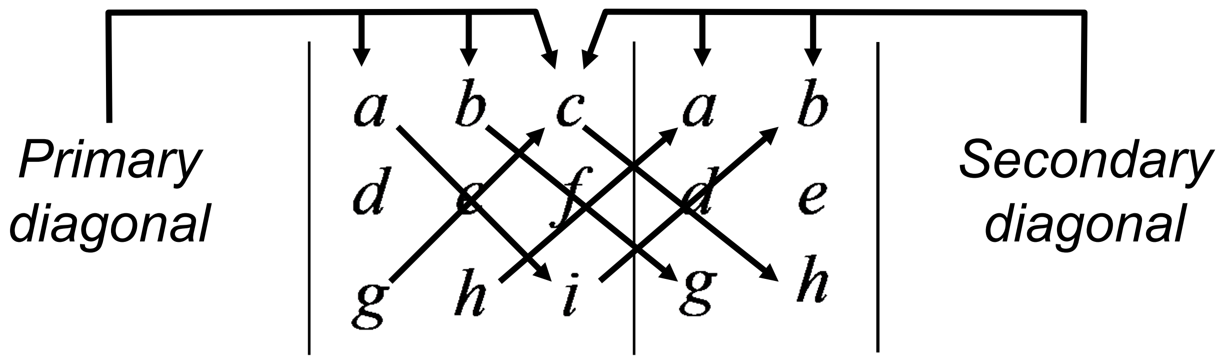
- Find the determinant by drawing primary and secondary diagonals

$$\text{Value} = (a)(d) - (c)(b)$$

Primary diagonal $\longrightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} \longleftarrow$ *Secondary diagonal*

Determinants

- For a 3 x 3 matrix, redraw the first two columns



$$\text{Value} = \begin{pmatrix} 1^{st} \text{ primary diagonal product } (aei) + \\ 2^{nd} \text{ primary diagonal product } (bfg) + \\ 3^{rd} \text{ primary diagonal product } (cdh) \end{pmatrix}$$

$$- \begin{pmatrix} 1^{st} \text{ secondary diagonal product } (gec) + \\ 2^{nd} \text{ secondary diagonal product } (hfa) + \\ 3^{rd} \text{ secondary diagonal product } (idb) \end{pmatrix}$$

$$= aei + bfg + cdh - gec - hfa - idb$$

Determinants

(a) $\begin{vmatrix} 2 & 5 \\ 1 & 8 \end{vmatrix}$ (b) $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix}$

(a) $\begin{vmatrix} 2 & 5 \\ 1 & 8 \end{vmatrix}$ $\text{Value} = (2)(8) - (1)(5) = 11$

(b) $\begin{vmatrix} 3 & 1 & 2 & 3 & 1 \\ 2 & 5 & 1 & 2 & 5 \\ 4 & -2 & 1 & 4 & -2 \end{vmatrix}$

$$\begin{aligned} \text{Value} &= (3)(5)(-1) + (1)(1)(4) + (2)(2)(-2) \\ &\quad - (4)(5)(2) - (-2)(1)(3) - (-1)(2)(1) \\ &= -15 + 4 - 8 - 40 + 6 + 2 = -51 \end{aligned}$$

Determinants

- Solving *simultaneous equations*

$$\begin{aligned} 2X + 3Y + 1Z &= 10 \\ 4X - 1Y - 2Z &= 8 \\ 5X + 2Y - 3Z &= 6 \end{aligned}$$

Coefficients for right-hand-side

Coefficients for *Y*

Coefficients for *Z*

Numerator determinant, in which column with *X*s is replaced by column of numbers to the right-hand-side of the equal sign

$$X = \frac{\begin{vmatrix} 10 & 3 & 1 \\ 8 & -1 & -2 \\ 6 & 2 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 4 & -1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}$$

Denominator determinant, in which coefficients of all unknown variables are listed (all columns to the left of the equal sign)

Coefficients for *Z*

Coefficients for *Y*

Coefficients for *X*

Determinants

- Solving *simultaneous equations*

$$2X + 3Y + 1Z = 10$$
$$4X - 1Y - 2Z = 8$$
$$5X + 2Y - 3Z = 6$$

$$Y = \frac{\begin{vmatrix} 2 & 10 & 1 \\ 4 & 8 & -2 \\ 5 & 6 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 4 & -1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}$$

Numerator determinant, in which column with *Y*s is replaced by right-hand-side numbers

Denominator determinant stays the same, regardless of the variable we are solving for

Determinants

- Solving *simultaneous equations*

$$2X + 3Y + 1Z = 10$$
$$4X - 1Y - 2Z = 8$$
$$5X + 2Y - 3Z = 6$$

$$Z = \frac{\begin{vmatrix} 2 & 3 & 10 \\ 4 & -1 & 8 \\ 5 & 2 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 4 & -1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}$$

Numerator determinant, in which column with *Z*s is replaced by right-hand-side numbers

Denominator determinant, again the same as when solving for *X* and *Y*

Determinants

- Determine the values for *X*, *Y*, and *Z*

$$X = \frac{\text{Numerical value of numerator determinant}}{\text{Numerical value of denominator determinant}}$$

$$= \frac{128}{33} = 3.88$$

$$Y = \frac{-20}{33} = -0.61$$

$$Z = \frac{134}{33} = 4.06$$

Verify the values of *X*, *Y*, and *Z* by substituting into and solving any one of the original three simultaneous equations

- Determine t

$$X = \frac{\text{Nume}}{\text{Numerical value of denominator determinant}}$$

$$= \frac{128}{33} = 3.88$$

$$Y = \frac{-20}{33} = -0.61$$

$$Z = \frac{134}{33} = 4.06$$

$$2X + 3Y + 1Z = 10$$

$$2(3.88) + 3(-0.61) + 1(4.06) =$$

$$7.76 - 1.83 + 4.06 = 10$$

Matrix of Cofactors and Adjoint

- A **cofactor** is defined as the set of numbers that remains after a given row and column have been taken out of a matrix
- An **adjoint** is the transpose of the matrix of cofactors

Matrix of Cofactors and Adjoint

- Six steps in computing a matrix of cofactors
  1. Select an element in the original matrix.
  2. Draw a line through the row and column of the element selected. The numbers uncovered represent the cofactor for that element.
  3. Calculate the value of the determinant of the cofactor.
  4. Add together the location numbers of the row and column crossed out in step 2. If the sum is even, the sign of the determinant's value (from step 3) does not change. If the sum is an odd number, change the sign of the determinant's value.

Matrix of Cofactors and Adjoint

5. The number just computed becomes an entry in the matrix of cofactors; it is located in the same position as the element selected in step 1.
6. Return to step 1 and continue until all elements in the original matrix have been replaced by their cofactor values.

Matrix of Cofactors and Adjoint

- Compute the matrix of cofactors and the adjoint

$$\text{Original matrix} = \begin{pmatrix} 3 & 7 & 5 \\ 2 & 0 & 3 \\ 4 & 1 & 8 \end{pmatrix}$$

From Table M5.1

$$\text{Matrix of cofactors} = \begin{pmatrix} -3 & -4 & 2 \\ -51 & 4 & 25 \\ 21 & 1 & -14 \end{pmatrix}$$

$$\text{Adjoint of matrix} = \begin{pmatrix} -3 & -51 & 21 \\ -4 & 4 & 1 \\ 2 & 25 & -14 \end{pmatrix}$$

Matrix of Cofactors and Adjoint

ELEMENT REMOVED	COFACTORS	DETERMINANT OF COFACTORS	VALUE OF COFACTOR
Row 1, column 1	$\begin{pmatrix} 0 & 3 \\ 1 & 8 \end{pmatrix}$	$\begin{vmatrix} 0 & 3 \\ 1 & 8 \end{vmatrix} = -3$	–3 (sign not changed)
Row 1, column 2	$\begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix}$	$\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} = 4$	–4 (sign changed)
Row 1, column 3	$\begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$	$\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = 2$	2 (sign not changed)
Row 2, column 1	$\begin{pmatrix} 7 & 5 \\ 1 & 8 \end{pmatrix}$	$\begin{vmatrix} 7 & 5 \\ 1 & 8 \end{vmatrix} = 51$	–51 (sign changed)
Row 2, column 2	$\begin{pmatrix} 3 & 5 \\ 4 & 8 \end{pmatrix}$	$\begin{vmatrix} 3 & 5 \\ 4 & 8 \end{vmatrix} = 4$	4 (sign not changed)
Row 2, column 3	$\begin{pmatrix} 3 & 7 \\ 4 & 1 \end{pmatrix}$	$\begin{vmatrix} 3 & 7 \\ 4 & 1 \end{vmatrix} = -25$	25 (sign changed)
Row 3, column 1	$\begin{pmatrix} 7 & 5 \\ 0 & 3 \end{pmatrix}$	$\begin{vmatrix} 7 & 5 \\ 0 & 3 \end{vmatrix} = 21$	21 (sign not changed)
Row 3, column 2	$\begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$	$\begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} = -1$	1 (sign changed)
Row 3, column 3	$\begin{pmatrix} 3 & 7 \\ 2 & 0 \end{pmatrix}$	$\begin{vmatrix} 3 & 7 \\ 2 & 0 \end{vmatrix} = -14$	–14 (sign not changed)

TABLE M5.1 – Matrix of Cofactor Calculations

Finding the Inverse of a Matrix

- The **inverse** of a matrix is a unique matrix of the same dimensions which, when multiplied by the original matrix, produces a *unit* or *identity* matrix
- Inverse of A is denoted as A^{-1}

$$A \times A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{Identity matrix}$$

Finding the Inverse of a Matrix

- Original matrix

$$\begin{pmatrix} 3 & 7 & 5 \\ 2 & 0 & 3 \\ 4 & 1 & 8 \end{pmatrix} = \text{Original matrix}$$

- Value of determinant

$$\left|\begin{array}{ccc|cc} 3 & 7 & 5 & 3 & 7 \\ 2 & 0 & 3 & 2 & 0 \\ 4 & 1 & 8 & 4 & 1 \end{array}\right|$$

Value = 0 + 84 + 10 – 0 – 9 – 112 = –27

Finding the Inverse of a Matrix

$$\text{Inverse} = \begin{pmatrix} -3/-27 & -51/-27 & 21/-27 \\ -4/-27 & 4/-27 & 1/-27 \\ 2/-27 & 25/-27 & -14/-27 \end{pmatrix}$$

$$= \begin{pmatrix} 3/27 & 51/27 & -21/27 \\ 4/27 & -4/27 & -1/27 \\ -2/27 & -25/27 & 14/27 \end{pmatrix}$$

Finding the Inverse of a Matrix

- Verify the inverse

$$\underset{\text{Original matrix}}{\begin{pmatrix} 3 & 7 & 5 \\ 2 & 0 & 3 \\ 4 & 1 & 8 \end{pmatrix}} \times \underset{\text{Inverse}}{\begin{pmatrix} 3/27 & 51/27 & -21/27 \\ 4/27 & -4/27 & -1/27 \\ -2/27 & -25/27 & 14/27 \end{pmatrix}} = \underset{\text{Identity matrix}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Finding the Inverse of a Matrix

- For a 2 x 2 matrix

$$\text{Original matrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\text{Determinant value of original matrix} = ad - cb$$

$$\text{Matrix of cofactors} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$\text{Adjoint of the matrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Finding the Inverse of a Matrix

- The inverse equals the adjoint divided by the determinant

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{d}{ad-cb} & \dfrac{-b}{ad-cb} \\ \dfrac{-c}{ad-cb} & \dfrac{a}{ad-cb} \end{pmatrix}$$

Finding the Inverse of a Matrix

- For example

$$\text{Original matrix} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$$

$$\text{Determinant value} = 1(8) - 3(2) = 2$$

then

$$\text{Inverse} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}^{-1} = \begin{pmatrix} 8/2 & -2/2 \\ -3/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1.5 & 0.5 \end{pmatrix}$$

Copyright