6

Calculus-Based Optimization

To accompany
Quantitative Analysis for Management, Twelfth Edition,
by Render, Stair, Hanna and Hale
Power Point slides created by Jeff Heyl

LEARNING OBJECTIVES

After completing this module, students will be able to:

1. Find the slope of a curve at any point.
2. Find derivatives for several common types of functions.
3. Find the maximum and minimum points on curves.
4. Use derivatives to maximize total revenue and other functions.

MODULE OUTLINE

M6.1 Introduction
M6.2 Slope of a Straight Line
M6.3 Slope of a Nonlinear Function
M6.4 Some Common Derivatives
M6.5 Maximum and Minimum
M6.6 Applications

Introduction

- Calculus and derivatives are helpful in finding the best solution to some business problems

Slope of a Straight Line

- Equation for a line

$$Y = a + bX$$

where b is the slope of the line

- Given any two points (X_1, Y_1) and (X_2, Y_2)

$$b = \frac{\text{Change in } Y}{\text{Change in } X} = \frac{\Delta Y}{\Delta X} = \frac{Y_2 - Y_1}{X_2 - X_1}$$

- For the points (2,3) and (4,7)

$$b = \frac{\Delta Y}{\Delta X} = \frac{7-3}{4-2} = \frac{4}{2} = 2$$

Slope of a Straight Line

- Finding the intercept using point (2,3)

$$Y = a + bX$$

$$3 = a + 2(2)$$

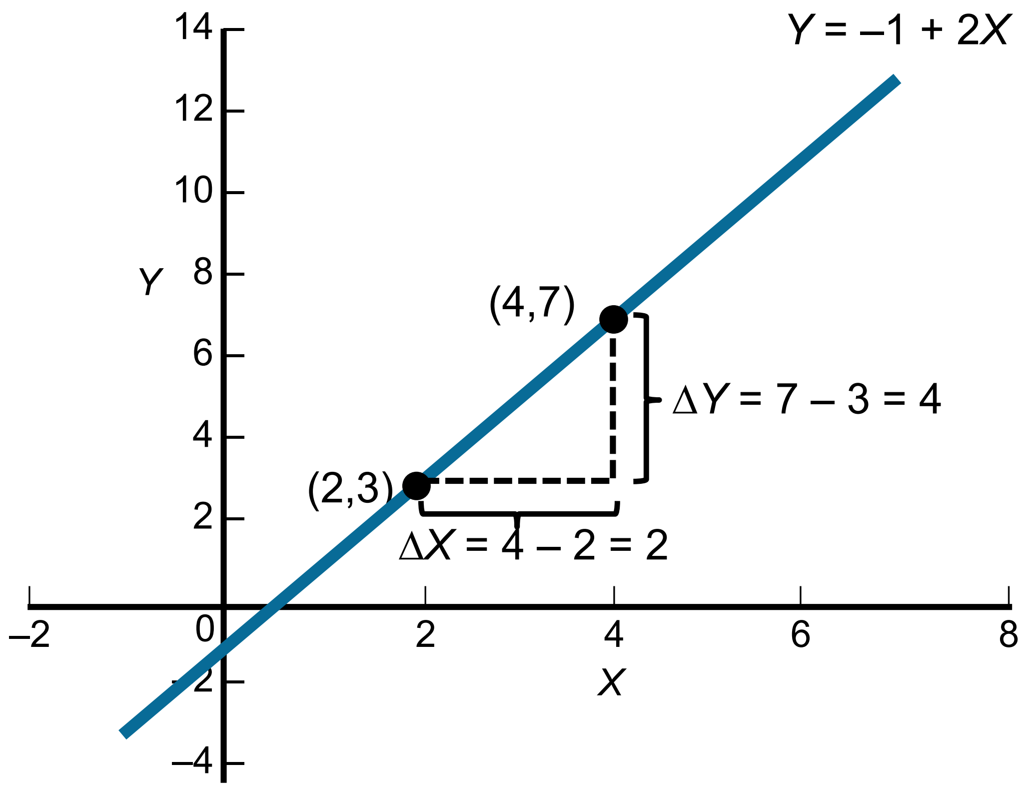
$$a = -1$$

- The equation of the line

$$Y = -1 + 2X$$

Slope of a Straight Line

FIGURE M6.1 – Graph of Straight Line

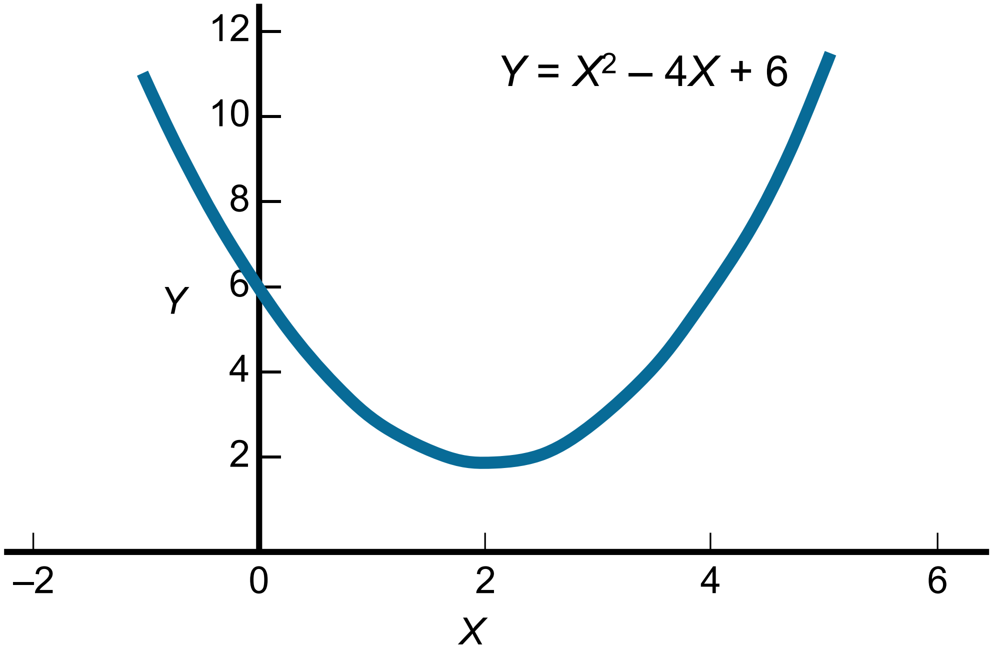


Slope of a Nonlinear Function

- For the function

  $Y = X^2 - 4X + 6$

FIGURE M6.2 – Graph of Quadratic Function



Slope of a Nonlinear Function

- Determine the slope of a curve at any point by finding the slope of a line tangent to the curve at this point
  - Find the *slope* using two points and this equation

$$b = \frac{\text{Change in } Y}{\text{Change in } X} = \frac{\Delta Y}{\Delta X} = \frac{Y_2 - Y_1}{X_2 - X_1}$$

For $X_1 = 3$, $Y_1 = (3)^2 - 4(3) + 6 = 3$ point (3,3)

Choosing $X_2 = 5$, $Y_2 = (5)^2 - 4(5) + 6 = 11$ point (5,11)

$$b = \frac{\Delta Y}{\Delta X} = \frac{11 - 3}{5 - 3} = \frac{8}{2} = 4$$

Slope of a Nonlinear Function

- Determine the slope o
  the slope of a line tang
  - Find the *slope* using tw

$$b = \frac{\text{Change i}}{\text{Change i}}$$

For a closer point, (4,6)

$$b = \frac{\Delta Y}{\Delta X} = \frac{6-3}{4-3} = \frac{3}{1} = 3$$

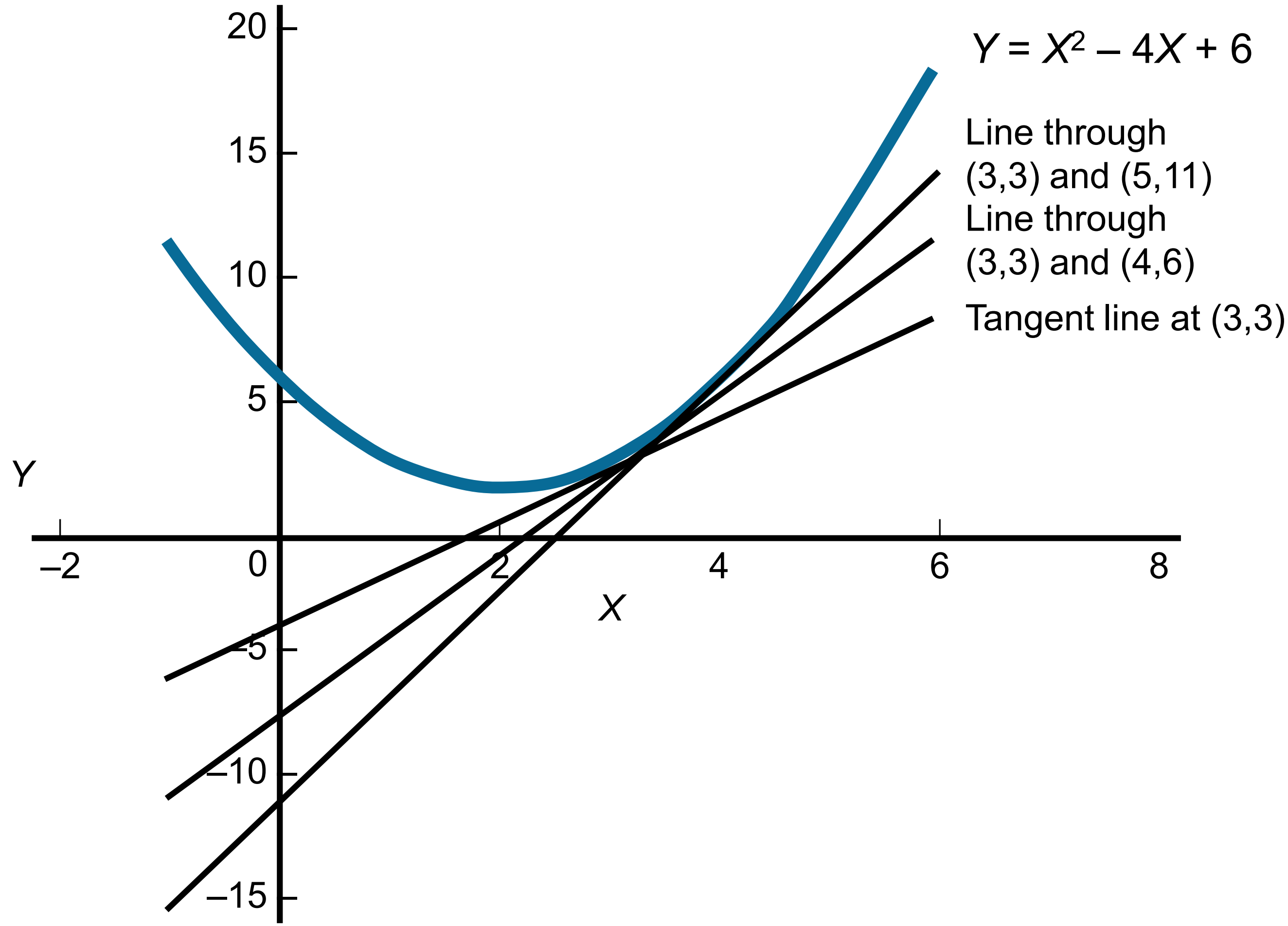
For $X_1 = 3$, $Y_1 = (3)^2 - 4(3) + 6 = 3$ point (3,3)

Choosing $X_2 = 5$, $Y_2 = (5)^2 - 4(5) + 6 = 11$ point (5,11)

$$b = \frac{\Delta Y}{\Delta X} = \frac{11-3}{5-3} = \frac{8}{2} = 4$$

Slope of a Nonlinear Function

FIGURE M6.3 – Graph of Tangent Line and Other Lines Connecting Points



Slope of a Nonlinear Function

- For a closer point, add ΔX to $X = 3$

For $X_1 = 3$, $\quad Y_1 = (3)^2 - 4(3) + 6 = 3$

and $\quad Y_2 = (3 + \Delta X)^2 - 4(3 + \Delta X) + 6$

$= (9 + 6\Delta X + \Delta X^2) - 12 - 4\Delta X + 6$

$= \Delta X^2 + 2\Delta X + 3$

The limit as ΔX approaches 0 is used to find the slope of a tangent line

$$\lim_{\Delta X \to 0} (\Delta X + 2) = 2$$

Slope of a Nonlinear Function

- The general form

$$Y_1 = aX^2 + bX + c$$

$$Y_2 = a(X + \Delta X)^2 + b(X + \Delta X) + c$$

- Expanding and simplifying

$$\Delta Y = Y_2 - Y_1 = b(\Delta X) + 2aX(\Delta X) + c(\Delta X)^2$$

$$\frac{\Delta Y}{\Delta X} = \frac{b(\Delta X) + 2aX(\Delta X) + c(\Delta X)^2}{\Delta X}$$

$$= \frac{\Delta X(b + 2aX + c\Delta X)}{\Delta X} = b + 2aX + c\Delta X$$

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Taking the limit as ΔX approaches 0

$$\lim_{\Delta X \to 0} (b + 2aX + c\Delta X) = b + 2aX$$

The slope of the function at point X

The **derivative** of Y denoted as Y' or dY/dX

- E…

$$\Delta Y = Y_2 - Y_1 = \ldots$$

$$\frac{\Delta Y}{\Delta X} = \frac{b(\Delta X) + \ldots}{\ldots}$$

$$Y' = \frac{dY}{dX} = \lim_{\Delta X \to 0} \left\{ \frac{\Delta Y}{\Delta X} \right\}$$

$$= \frac{\Delta X (b + 2aX + c\Delta X)}{\Delta X} = b + 2aX + c\Delta X$$

Some Common Derivatives

TABLE M6.1 – Some Common Derivatives

FUNCTION	DERIVATIVE
$Y = C$	$Y' = 0$
$Y = X^n$	$Y' = nX^{n-1}$
$Y = cX^n$	$Y' = cnX^{n-1}$
$Y = \frac{1}{X^n}$	$Y' = \frac{-n}{X^{n+1}}$
$Y = g(x) + h(x)$	$Y' = g'(x) + h'(x)$
$Y = g(x) - h(x)$	$Y' = g'(x) - h'(x)$

Some Common Derivatives

1. If $Y = c$, then $Y' = 0$
   c = constant

2. If $Y = X^n$, then $Y' = nX^{n-1}$

   if $Y = X^2$, then $Y' = 2X^{2-1} = 2X$
   if $Y = X^3$, then $Y' = 3X^{3-1} = 3X^2$
   if $Y = X^9$, then $Y' = 9X^{9-1} = 9X^8$

Some Common Derivatives

3. If $Y = cX^n$, then $Y' = cnX^{n-1}$

   if $Y = 4X^3$, then $Y' = 4(3)X^{3-1} = 12X^2$

   if $Y = 2X^4$, then $Y' = 2(4)X^{4-1} = 8X^3$

4. If $Y = \dfrac{1}{X^n}$, then $Y' = -nX^{-n-1} = \dfrac{-n}{X^{n+1}}$

   if $Y = \dfrac{1}{X^3}\left(\text{or } Y = X^{-3}\right)$, then $Y' = -3X^{-3-1} = -3X^{-4} = \dfrac{-3}{X^4}$

   if $Y = \dfrac{2}{X^4}$, then $Y' = 2(-4)X^{-4-1} = \dfrac{-8}{X^5}$

Some Common Derivatives

5. If $Y = g(x) + h(x)$, then $Y' = g'(x) + h'(x)$

if $Y = 2X^3 + X^2$, then $Y' = 2(3)X^{3-1} + 2X^{2-1} = 6X^2 + 2X$
if $Y = 5X^4 + 3X^2$, then $Y' = 5(4)X^{4-1} + 3(2)X^{2-1} = 20X^3 + 6X$

6. If $Y = g(x) - h(x)$, then $Y' = g'(x) - h'(x)$

if $Y = 5X^3 - X^2$, then $Y' = 5(3)X^{3-1} - 2X^{2-1} = 15X^2 - 2X$
if $Y = 2X^4 - 4X^2$, then $Y' = 2(4)X^{4-1} - 4(2)X^{2-1} = 8X^3 - 8X$

Second Derivatives

- The **second derivative** of a function is the derivative of the first derivative
- Denoted as Y'' or d^2Y/dX^2

If

$$Y = 64X^4 + 4X^3$$

$$Y' = \frac{dY}{dX} = 6(4)X^{4-1} + 4(3)X^{3-1}$$

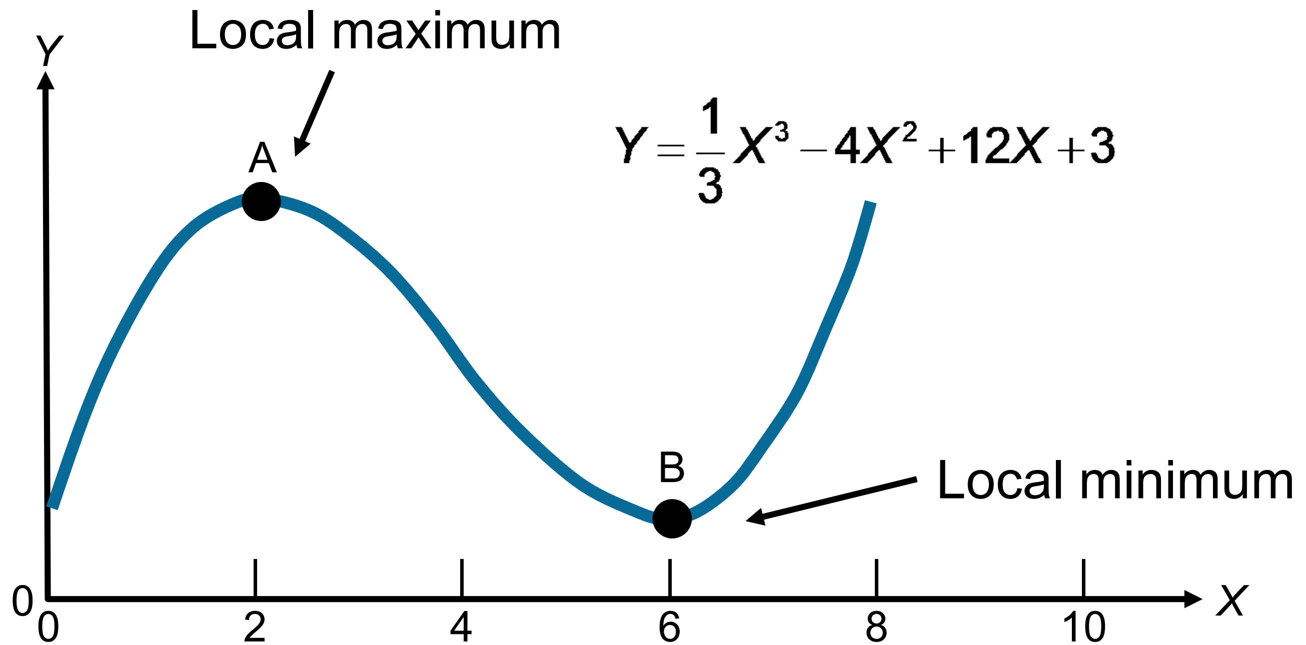
$$= 24X^3 + 12X^2$$

$$Y'' = \frac{d^2Y}{dX^2} = 24(3)X^{3-1} + 12(2)X^{2-1}$$

$$= 72X^2 + 24X$$

Maximum and Minimum

FIGURE M6.4 – Graph of Curve with Local Maximum and Local Minimum



Maximum and Minimum

- Find a local optimum by taking the first derivative of the function, set it equal to 0, and solve for X
- **Critical point**

for

$$Y = \frac{1}{3}X^3 - 4X^2 + 12X + 3$$

$$Y' = X^2 - 8X + 12 = 0$$

$$(X - 2)(X - 6) = 0$$

$$Y'' = 2X - 8$$

Critical points when
X = 2 and X = 6

at X = 2,

$$Y'' = 2(2) - 8 = -4$$

negative number
= local maximum

at X = 6,

$$Y'' = 2(6) - 8 = 4$$

positive number
= local minimum

Maximum and Minimum

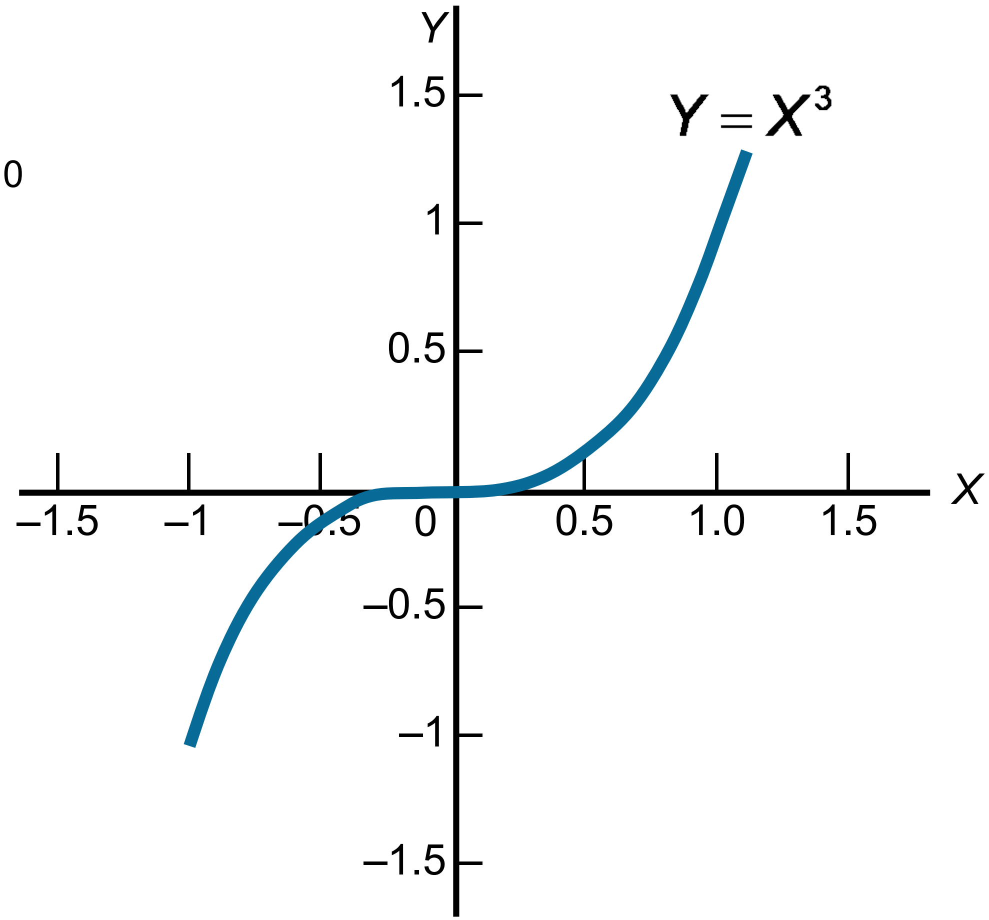


FIGURE M6.5 –
Graph of Function with
Point of Inflection at $X = 0$

Maximum and Minimum

for

$$Y = X^3$$

$$Y' = 3X^2$$

Critical point, $X = 0$

$$Y'' = 3(2)X^{2\text{-}1} = 6X$$

when $X = 0$, $Y'' = 6(0) = 0$

- Neither minimum nor maximum
- Point of inflection

Critical point will be

1. A maximum if the second derivative is negative
2. A minimum if the second derivative is positive
3. A point of inflection if the second derivative is zero

Applications

- **Economic Order Quantity**

Total cost = (Total ordering cost) + (Total holding cost) + (Total purchase cost)

$$TC = \frac{D}{Q}C_o + \frac{Q}{2}C_h + DC$$

where

Q = order quantity
D = annual demand
C_o = ordering cost per order
C_h = holding cost per unit per year
C = purchase (material) cost per unit

Applications

- **Economic Order Quantity**

$$\frac{dTC}{dQ} = \frac{-DC_o}{Q^2} + \frac{C_h}{2}$$

$$Q = \pm\sqrt{\frac{2DC_o}{C_h}}$$

$$\frac{d^2TC}{dQ^2} = \frac{DC_o}{Q^3}$$

Applications

- **Total Revenue**

Demand = $Q = 6{,}000 - 500P$

where

Q = quantity demanded (or sold)
P = price in dollars

Total revenue function

Total revenue = Price x Quantity

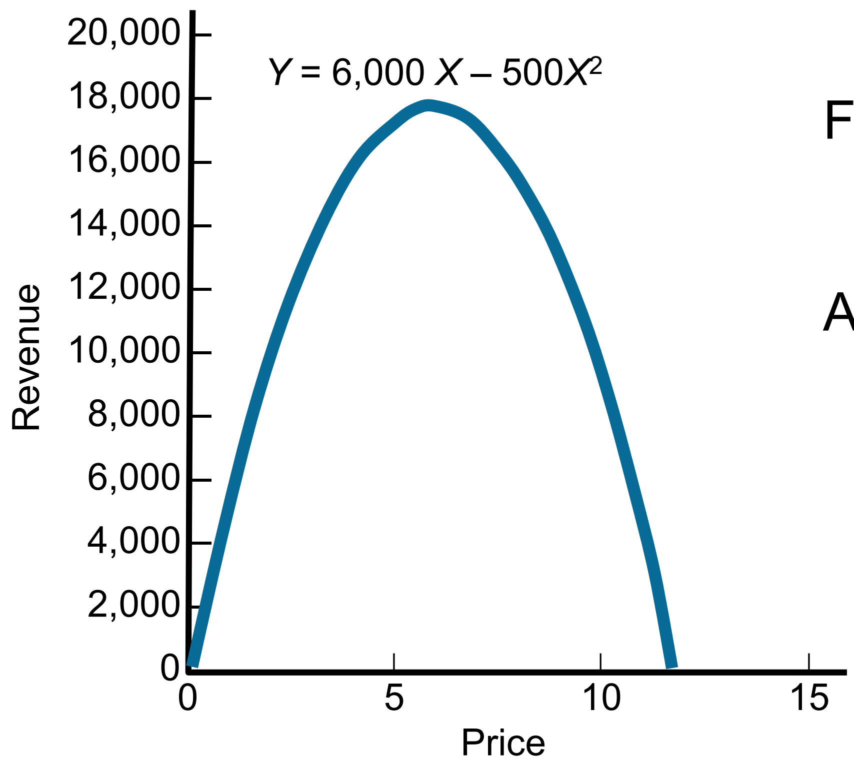
$$TR = PQ$$

$$TR = P(6{,}000 - 500P)$$

$$TR = 6{,}000P - 500P^2$$

Applications

FIGURE M6.6 – Total Revenue Function



$$TR' = 6{,}000 - 1{,}000P$$

For $TR' = 0$

$$P = 6$$

At this price

$$Q = 6{,}000 - 500P = 3{,}000 \text{ units}$$

$$TR = PQ = 6(3{,}000) = \$18{,}000$$

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