CHAPTER 7

Linear Programming Models: Graphical and Computer Methods

To accompany
Quantitative Analysis for Management, Twelfth Edition,
by Render, Stair, Hanna and Hale
Power Point slides created by Jeff Heyl

LEARNING OBJECTIVES

After completing this chapter, students will be able to:

1. Understand the basic assumptions and properties of linear programming (LP).
2. Graphically solve any LP problem that has only two variables by both the corner point and isoprofit line methods.
3. Understand special issues in LP such as infeasibility, unboundedness, redundancy, and alternative optimal solutions.
4. Understand the role of sensitivity analysis.
5. Use Excel spreadsheets to solve LP problems.

CHAPTER OUTLINE

Introduction

- Many management decisions involve making the most effective use of limited resources
- Linear programming (LP)
  – Widely used mathematical modeling technique
  – Planning and decision making relative to resource allocation
- Broader field of mathematical programming
  – Here programming refers to modeling and solving a problem mathematically

Requirements of a Linear Programming Problem

- Four properties in common
  - Seek to *maximize* or *minimize* some quantity (the **objective function**)
  - Restrictions or **constraints** are present
  - Alternative courses of action are available
  - *Linear* equations or inequalities

LP Properties and Assumptions

TABLE 7.1

PROPERTIES OF LINEAR PROGRAMS
1. One objective function
2. One or more constraints
3. Alternative courses of action
4. Objective function and constraints are linear – proportionality and divisibility
5. Certainty
6. Divisibility
7. Nonnegative variables

Formulating LP Problems

- Developing a mathematical model to represent the managerial problem
- Steps in formulating a LP problem
  1. Completely understand the managerial problem being faced
  2. Identify the objective and the constraints
  3. Define the decision variables
  4. Use the decision variables to write mathematical expressions for the objective function and the constraints

Formulating LP Problems

- Common LP application – **product mix problem**
- Two or more products are produced using limited resources
- Maximize profit based on the profit contribution per unit of each product
- Determine how many units of each product to produce

Flair Furniture Company

- Flair Furniture produces inexpensive tables and chairs
- Processes are similar, both require carpentry work and painting and varnishing
  - Each table takes 4 hours of carpentry and 2 hours of painting and varnishing
  - Each chair requires 3 of carpentry and 1 hour of painting and varnishing
  - There are 240 hours of carpentry time available and 100 hours of painting and varnishing
  - Each table yields a profit of $70 and each chair a profit of $50

Flair Furniture Company

- The company wants to determine the best combination of tables and chairs to produce to reach the maximum profit

TABLE 7.2

DEPARTMENT	HOURS REQUIRED TO PRODUCE 1 UNIT		AVAILABLE HOURS THIS WEEK
	(*T*) TABLES	(*C*) CHAIRS	
Carpentry	4	3	240
Painting and varnishing	2	1	100
Profit per unit	\$70	\$50	

Flair Furniture Company

- The objective is

  Maximize profit

- The constraints are
  - The hours of carpentry time used cannot exceed 240 hours per week
  - The hours of painting and varnishing time used cannot exceed 100 hours per week
- The **decision variables** are

  T = number of tables to be produced per week

  C = number of chairs to be produced per week

Flair Furniture Company

- Create objective function in terms of *T* and *C*

$$\text{Maximize profit} = \$70T + \$50C$$

- Develop mathematical relationships for the two constraints
  - For carpentry, total time used is

    (4 hours per table)(Number of tables produced)
    + (3 hours per chair)(Number of chairs produced)

  - First constraint is

$$\text{Carpentry time used} \le \text{Carpentry time available}$$

$$4T + 3C \le 240 \text{ (hours of carpentry time)}$$

Flair Furniture Company

- Similarly

  Painting and varnishing time used
  ≤ Painting and varnishing time available

$2\,T + 1C \leq 100$ (hours of painting and varnishing time)

This means that each table produced requires two hours of painting and varnishing time

  - Both of these constraints restrict production capacity and affect total profit

Flair Furniture Company

- The values for T and C must be nonnegative

$T \geq 0$ (number of tables produced is greater than or equal to 0)

$C \geq 0$ (number of chairs produced is greater than or equal to 0)

The complete problem stated mathematically

$$\text{Maximize profit} = \$70T + \$50C$$

subject to

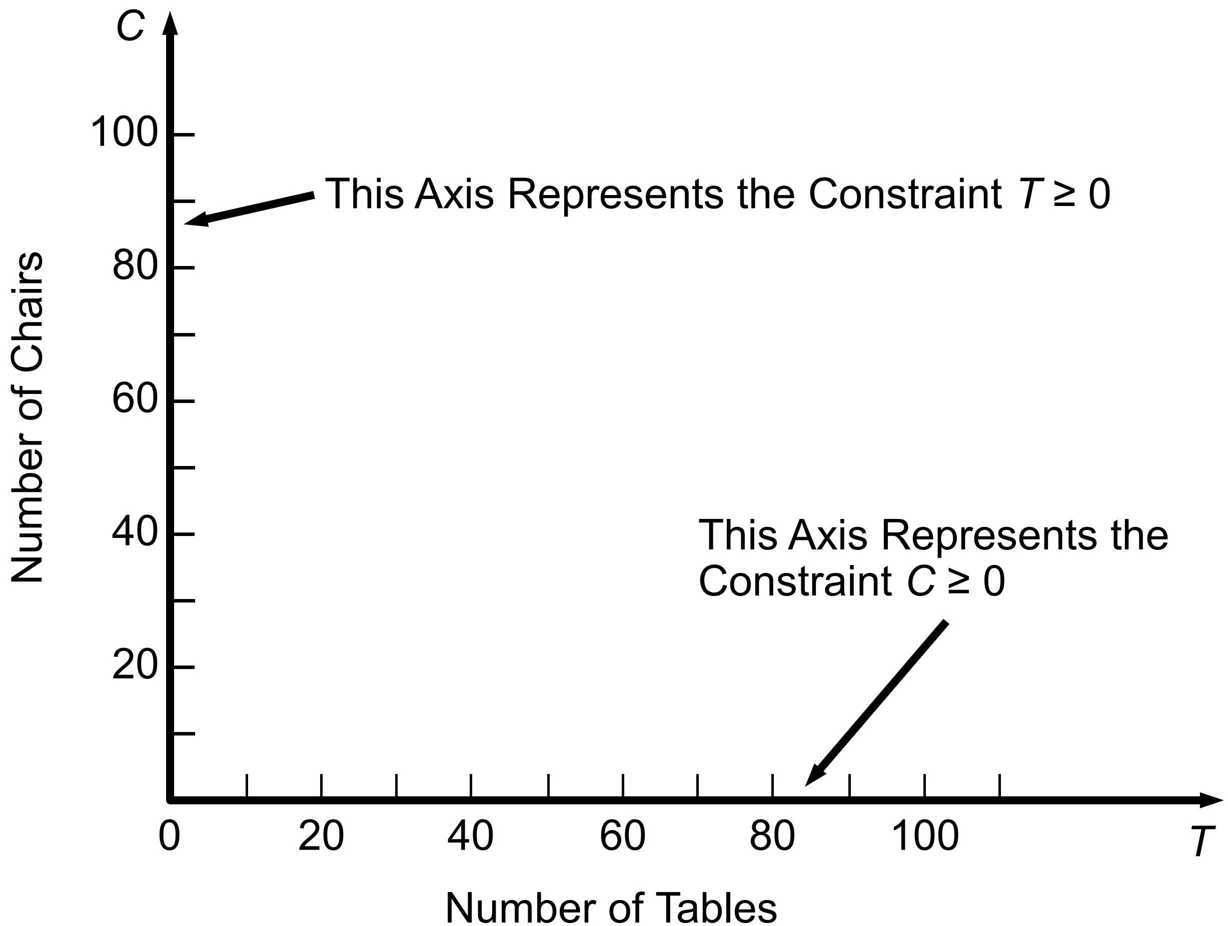
$$\begin{aligned} 4T + 3C &\leq 240 && \text{(carpentry constraint)} \\ 2T + 1C &\leq 100 && \text{(painting and varnishing constraint)} \\ T, C &\geq 0 && \text{(nonnegativity constraint)} \end{aligned}$$

Graphical Solution to an LP Problem

- Easiest way to solve a small LP problems is graphically
- Only works when there are just two decision variables
  – Not possible to plot a solution for more than two variables
- Provides valuable insight into how other approaches work
- **Nonnegativity constraints** mean that we are always working in the first (or northeast) quadrant of a graph

Graphical Representation of Constraints

FIGURE 7.1 – Quadrant Containing All Positive Values



Graphical Representation of Constraints

- The first step is to identify a set or region of feasible solutions
- Plot each constraint equation on a graph
- Graph the equality portion of the constraint equations

$$4T + 3C = 240$$

- Solve for the axis intercepts and draw the line

Graphical Representation of Constraints

- When Flair produces no tables, the carpentry constraint is:

$$4(0) + 3C = 240$$
$$3C = 240$$
$$C = 80$$

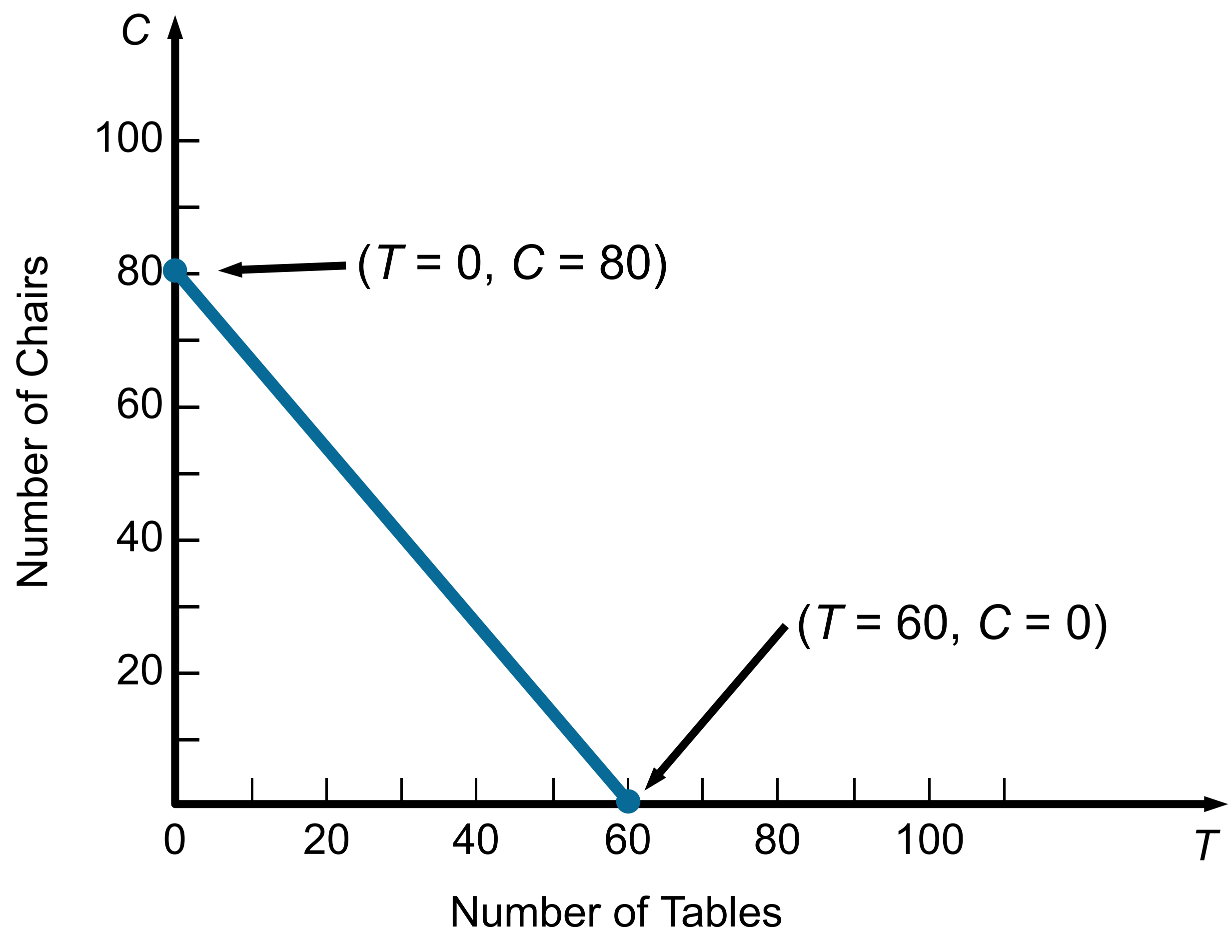
- Similarly for no chairs:

$$4T + 3(0) = 240$$
$$4T = 240$$
$$T = 60$$

  – This line is shown on the following graph

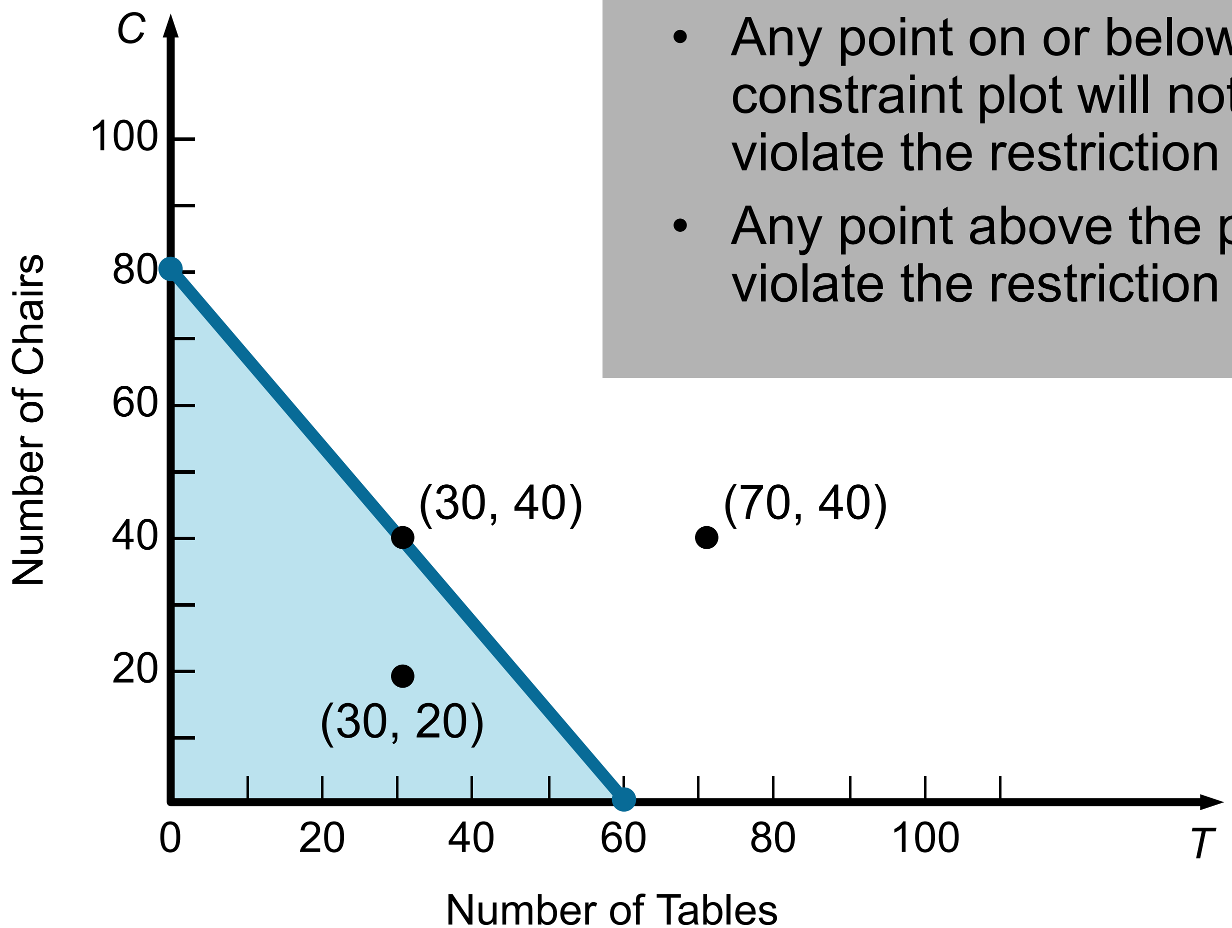
Graphical Representation of Constraints

FIGURE 7.2 – Graph of Carpentry Constraint Equation



Graphical Representation of Constraints

FIGURE 7.3 – Region that Satisfies the Carpentry Constraint



- Any point on or below the constraint plot will not violate the restriction
- Any point above the plot will violate the restriction

Graphical Representation of Constraints

- The point (30, 40) lies on the line and exactly satisfies the constraint

$$4(30) + 3(40) = 240$$

- The point (30, 20) lies below the line and satisfies the constraint

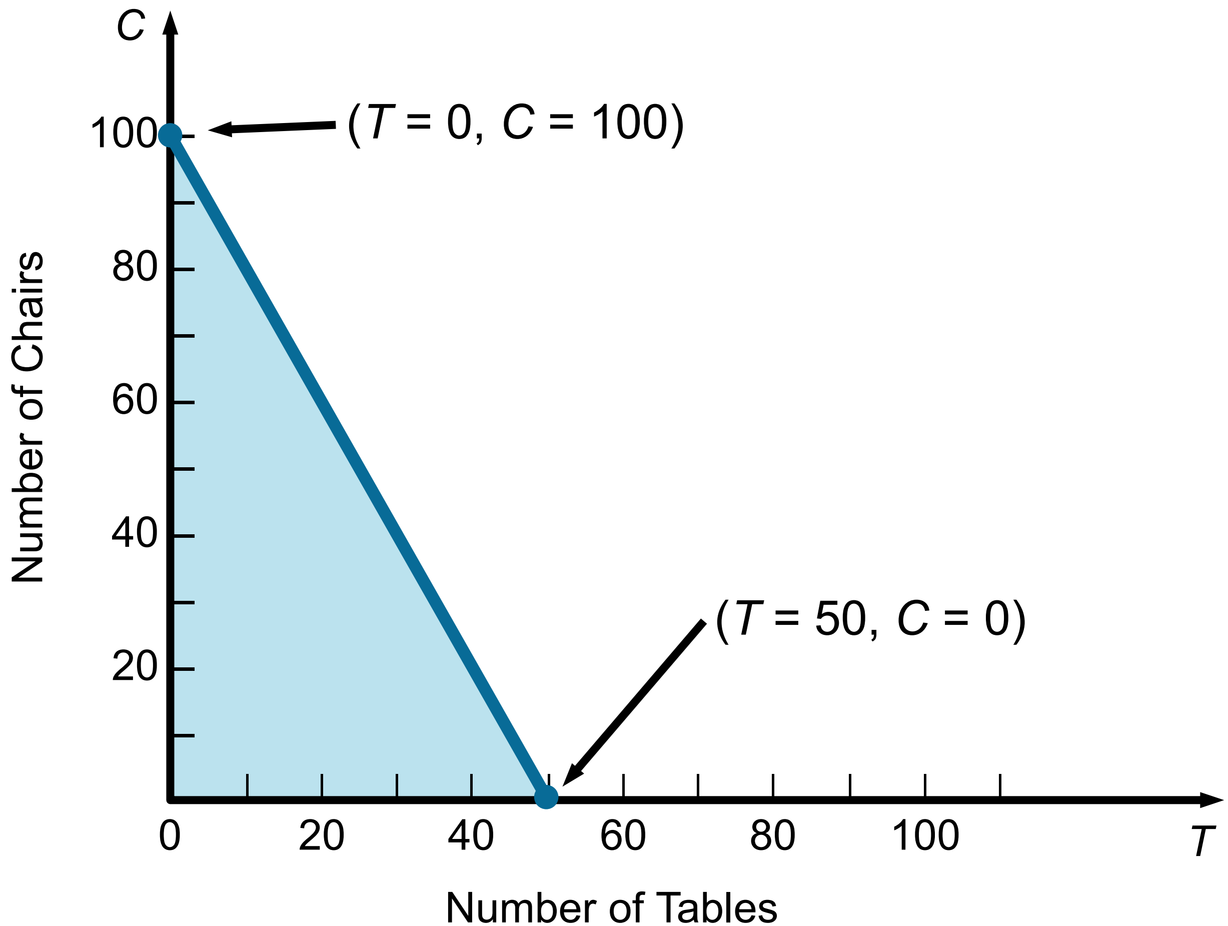
$$4(30) + 3(20) = 180$$

- The point (70, 40) lies above the line and does not satisfy the constraint

$$4(70) + 3(40) = 400$$

Graphical Representation of Constraints

FIGURE 7.4 – Region that Satisfies the Painting and Varnishing Constraint

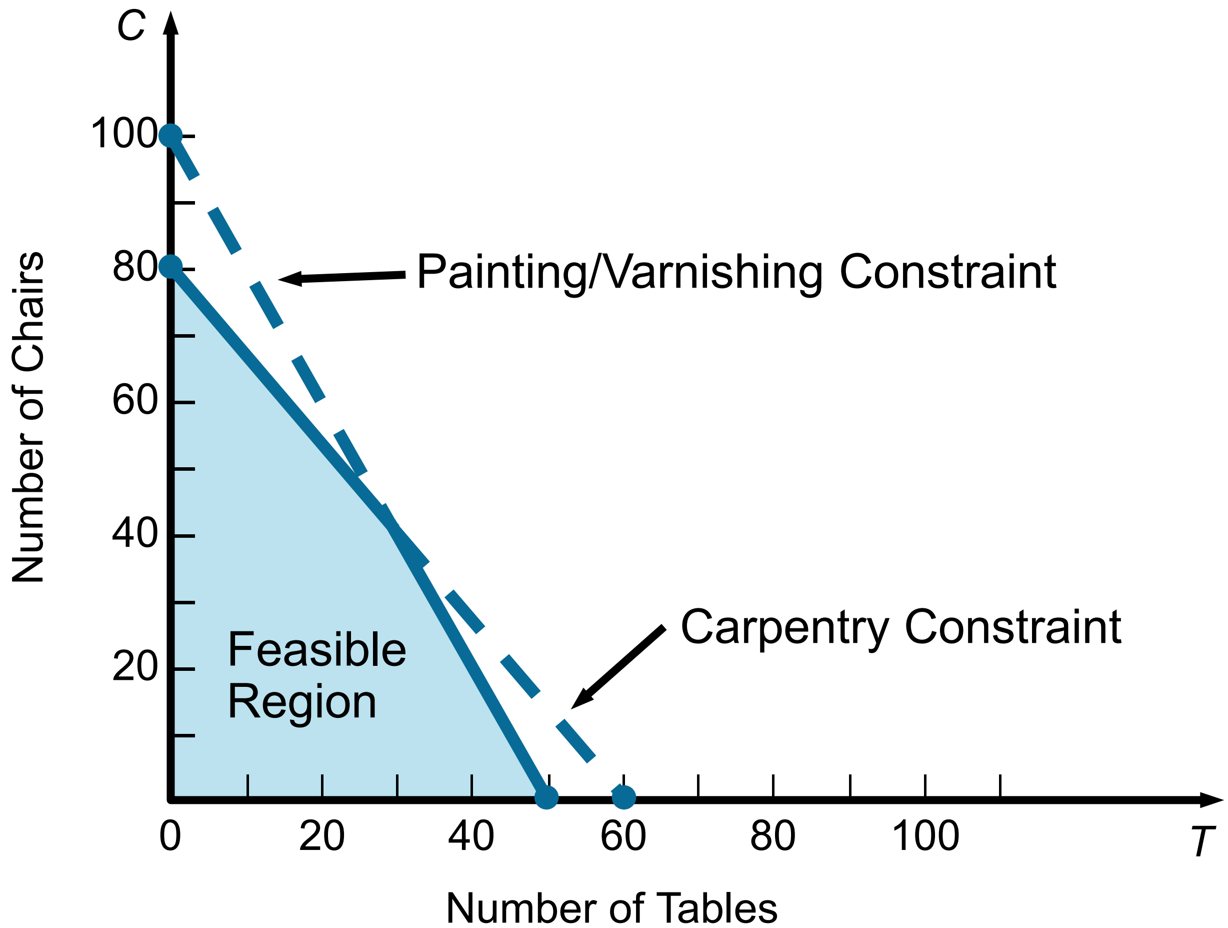


Graphical Representation of Constraints

- To produce tables and chairs, both departments must be used
- Find a solution that satisfies both constraints simultaneously
- A new graph shows both constraint plots
- The feasible region is where all constraints are satisfied
  - Any point inside this region is a feasible solution
  - Any point outside the region is an infeasible solution

Graphical Representation of Constraints

FIGURE 7.5 – Feasible Solution Region



Graphical Representation of Constraints

- For the point (30, 20)

Carpentry constraint	$4T + 3C \le 240$ hours available (4)(30) + (3)(20) = 180 hours used	✓
Painting constraint	$2T + 1C \le 100$ hours available (2)(30) + (1)(20) = 80 hours used	✓

- For the point (70, 40)

Carpentry constraint	$4T + 3C \le 240$ hours available (4)(70) + (3)(40) = 400 hours used	✗
Painting constraint	$2T + 1C \le 100$ hours available (2)(70) + (1)(40) = 180 hours used	✗

Graphical Representation of Constraints

- For the point (50, 5)

Carpentry constraint	$4T + 3C \leq 240$ hours available $(4)(50) + (3)(5) = 215$ hours used	✓
Painting constraint	$2T + 1C \leq 100$ hours available $(2)(50) + (1)(5) = 105$ hours used	✗

Isoprofit Line Solution Method

- Find the optimal solution from the many possible solutions
- Speediest method is to use the *isoprofit line*
- Starting with a small possible profit value, graph the objective function
- Move the objective function line in the direction of increasing profit while maintaining the slope
- The last point it touches in the feasible region is the optimal solution

Isoprofit Line Solution Method

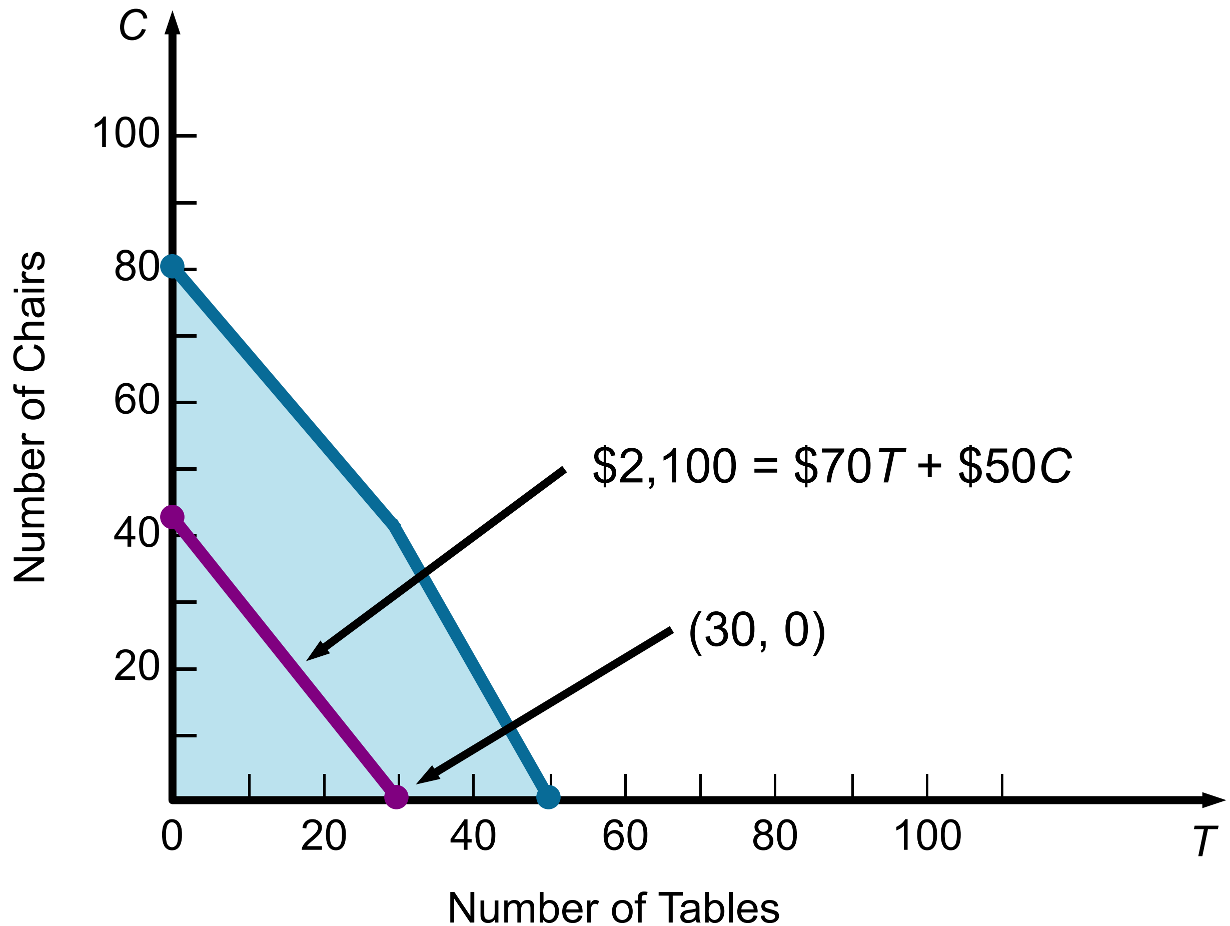
- Choose a profit of $2,100
- The objective function is

$$\$2{,}100 = 70T + 50C$$

- Solving for the axis intercepts, draw the graph
- Obviously not the best possible solution
- Further graphs can be created using larger profits
  - The further we move from the origin, the larger the profit
- The highest profit ($4,100) will be generated when the isoprofit line passes through the point (30, 40)

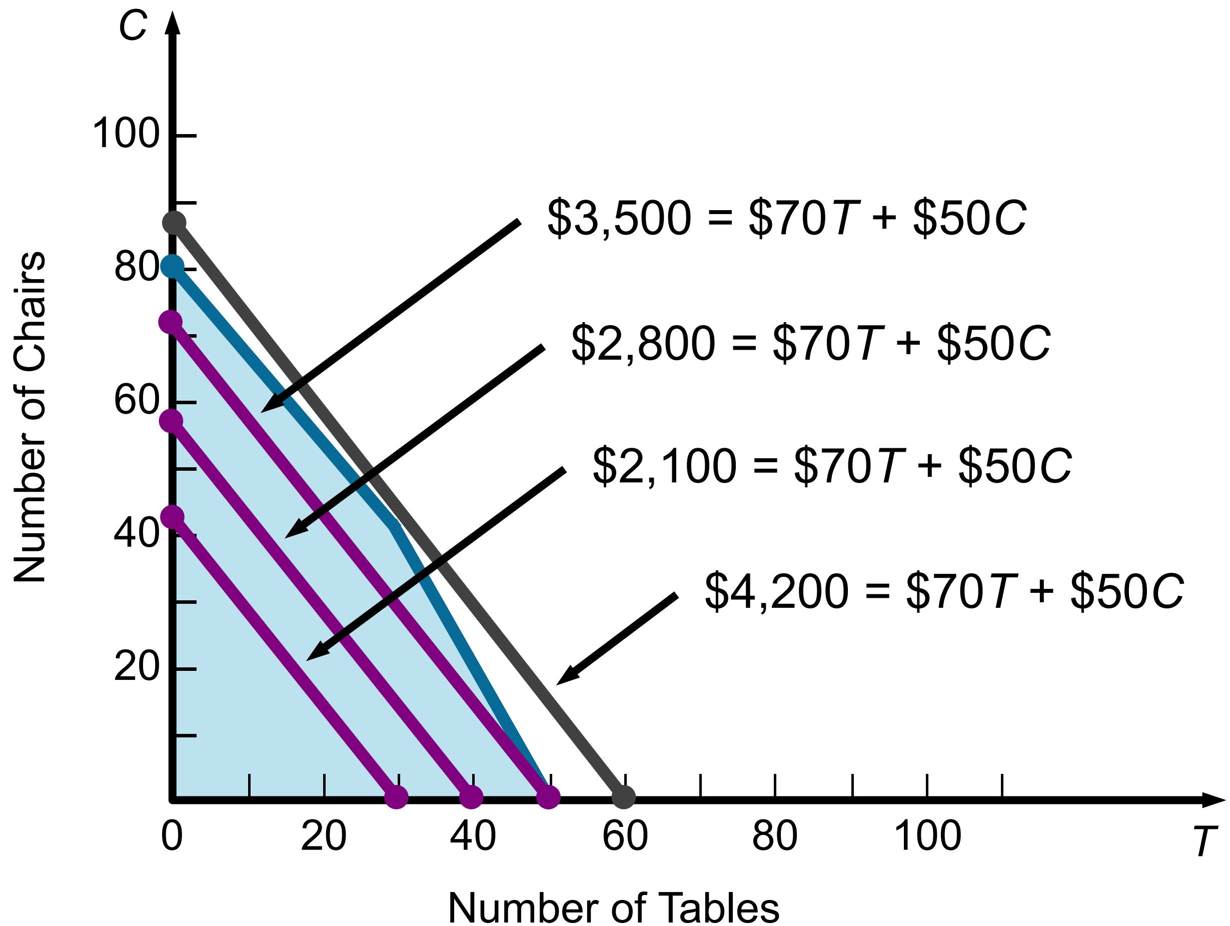
Isoprofit Line Solution Method

FIGURE 7.6 – Profit line of $2,100



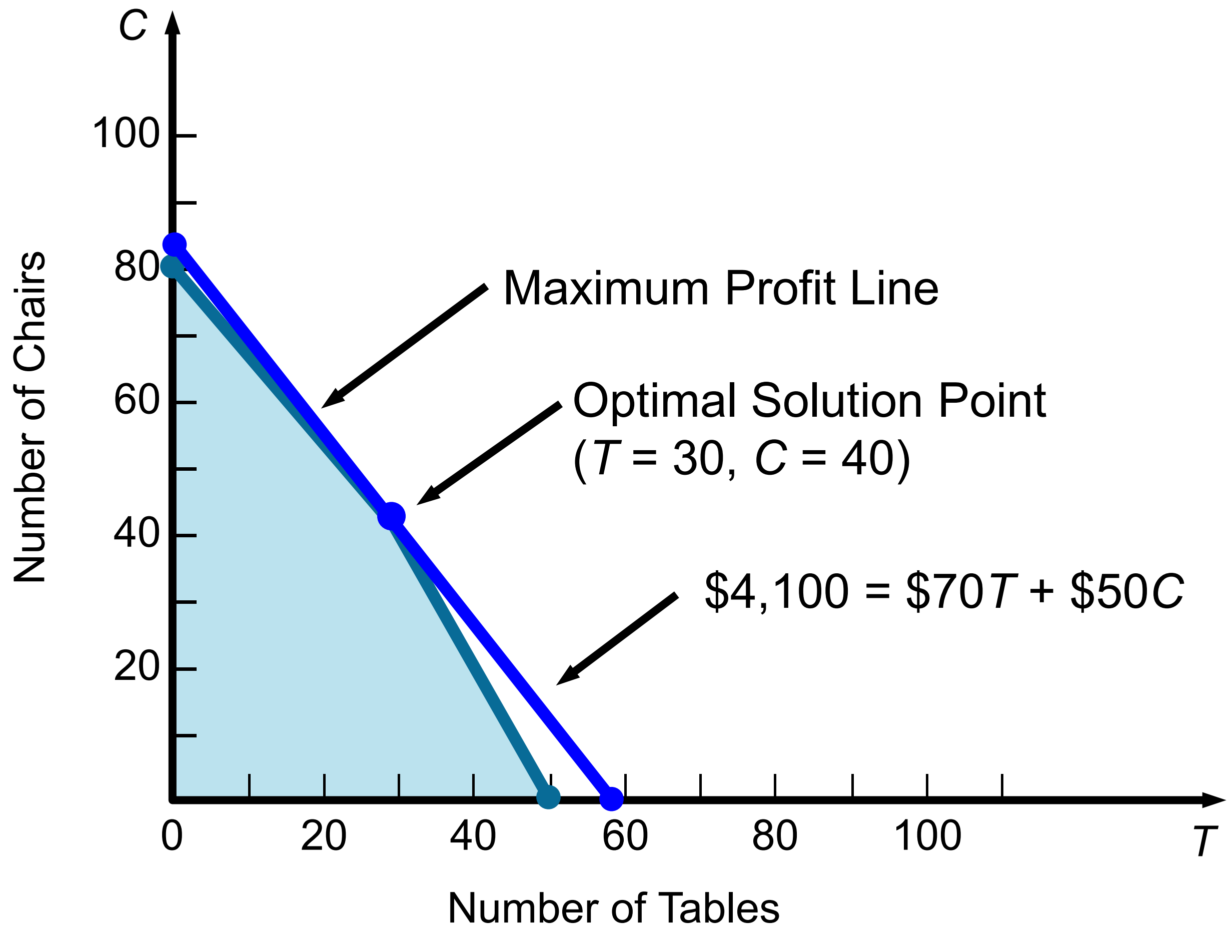
Isoprofit Line Solution Method

FIGURE 7.7 – Four Isoprofit Lines



Isoprofit Line Solution Method

FIGURE 7.8 – Optimal Solution

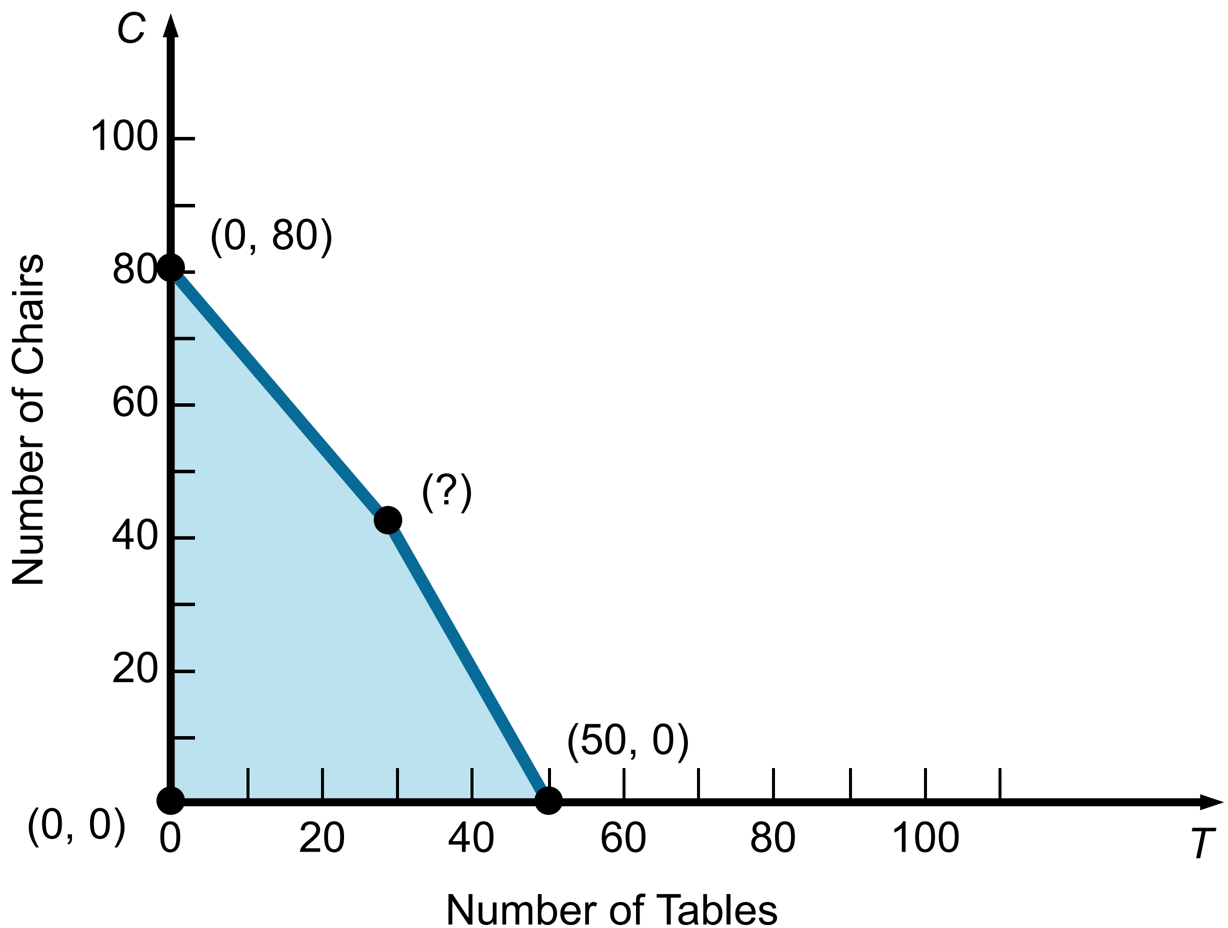


Corner Point Solution Method

- The **corner point method** for solving LP problems
- Look at the profit at every corner point of the feasible region
- Mathematical theory is that an optimal solution must lie at one of the **corner points** or **extreme points**

Corner Point Solution Method

FIGURE 7.9 – Four Corner Points of the Feasible Region



Corner Point Solution Method

- Solve for the intersection of the two constraint lines
- Using the elimination method to solve simultaneous equations method, select a variable to be eliminated
- Eliminate T by multiplying the second equation by –2 and add it to the first equation

$$-2(2T + 1C = 100) = -4T - 2C = -200$$

$$\begin{array}{rrl} 4T + 3C = & 240 & \quad \text{(carpentry)} \\ -4T - 2C = & -200 & \quad \text{(painting)} \\ \hline C = & 40 & \end{array}$$

Corner Point Solution Method

- Substitute C = 40 into either equation to solve for T

$$\begin{aligned} 4T + 3(40) &= 240 \\ 4T + 120 &= 240 \\ 4T &= 120 \\ T &= 30 \end{aligned}$$

Thus the corner point is (30, 40)

TABLE 7.3 – Feasible Corner Points and Profits

NUMBER OF TABLES (T)	NUMBER OF CHAIRS (C)	PROFIT = \$70T + \$50C
0	0	\$0
50	0	\$3,500
0	80	\$4,000
30	40	\$4,100

Highest profit – Optimal Solution

Slack and Surplus

- **Slack** is the amount of a resource that is not used
  - For a less-than-or-equal constraint

    Slack = (Amount of resource available) – (Amount of resource used)

  - Flair decides to produce 20 tables and 25 chairs

4(20) + 3(25)	=	155	(carpentry time used)
240	=		(carpentry time available)
240 – 155	=	85	(Slack time in carpentry)

Slack and Surplus

At the optimal solution, slack is 0 as all 240 hours are used

- **Slack** is the amount of a … used
  - For a less-than-or-equal constraint

    Slack = (Amount of resource available) – (Amount of resource used)

  - Flair decides to produce 20 tables and 25 chairs

$4(20) + 3(25)$	=	155	(carpentry time used)
240	=		(carpentry time available)
$240 - 155$	=	85	(Slack time in carpentry)

Slack and Surplus

- **Surplus** is used with a greater-than-or-equal-to constraint to indicate the amount by which the right-hand side of the constraint is exceeded

  Surplus = (Actual amount) – (Minimum amount)

  – New constraint

  $$T + C \geq 42$$

  – If T = 20 and C = 25, then

  $$20 + 25 = 45$$

  $$\text{Surplus} = 45 - 42 = 3$$

Summaries of Graphical Solution Methods

TABLE 7.4

ISOPROFIT METHOD
1. Graph all constraints and find the feasible region.
2. Select a specific profit (or cost) line and graph it to find the slope.
3. Move the objective function line in the direction of increasing profit (or decreasing cost) while maintaining the slope. The last point it touches in the feasible region is the optimal solution.
4. Find the values of the decision variables at this last point and compute the profit (or cost).
CORNER POINT METHOD
1. Graph all constraints and find the feasible region.
2. Find the corner points of the feasible reason.
3. Compute the profit (or cost) at each of the feasible corner points.
4. Select the corner point with the best value of the objective function found in Step 3. This is the optimal solution.

Solving Flair Furniture's LP Problem

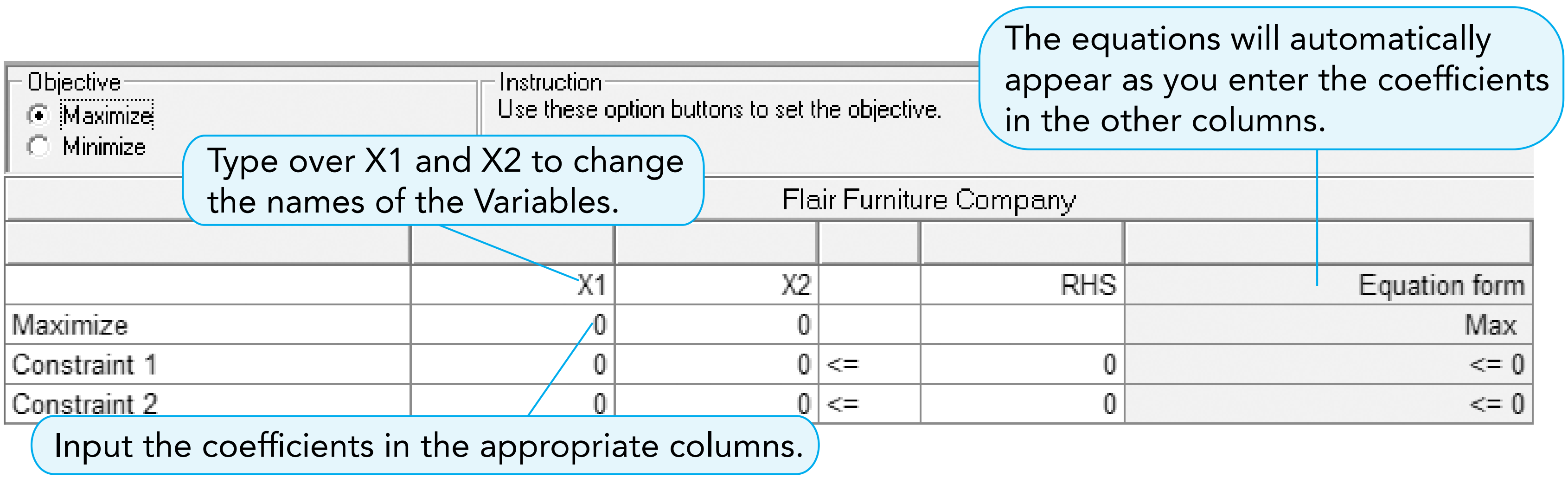
- Most organizations have access to software to solve big LP problems
- There are differences between software implementations, the approach is basically the same
- With experience with computerized LP algorithms, it is easy to adjust to minor changes

Using QM for Windows

- Select the Linear Programming module
- Specify the number of constraints (non-negativity is assumed)
- Specify the number of decision variables
- Specify whether the objective is to be maximized or minimized
- For Flair Furniture there are two constraints, two decision variables, and the objective is to maximize profit

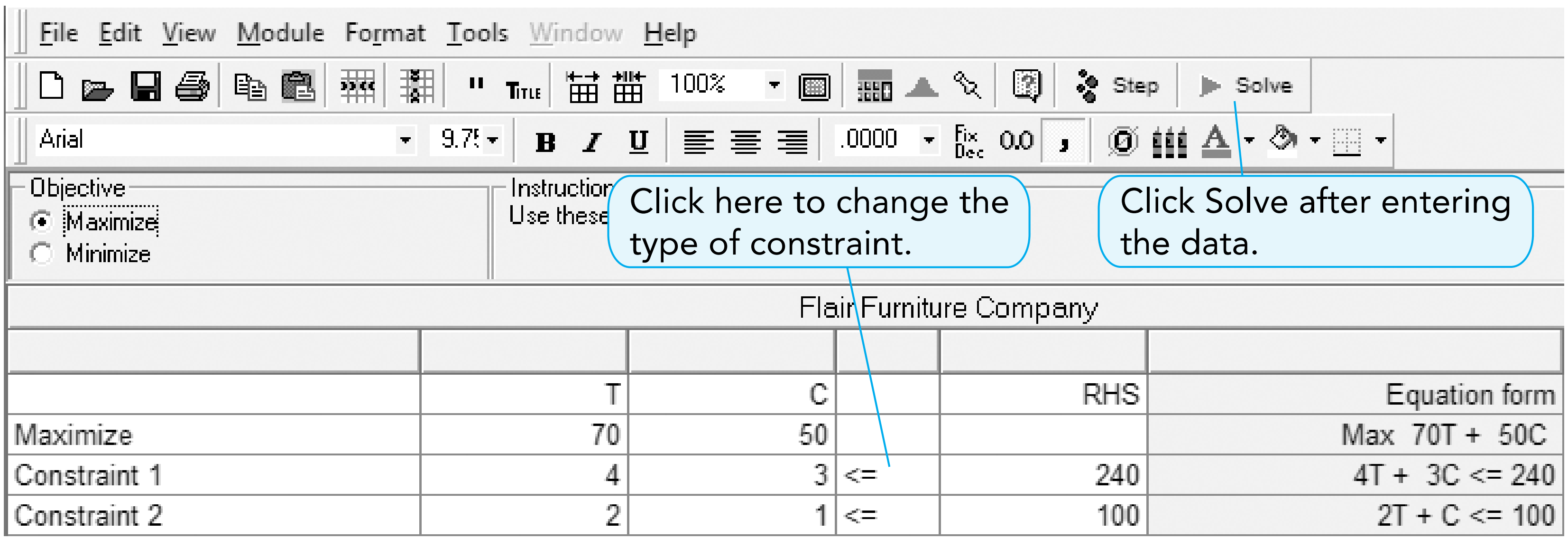
Using QM for Windows

PROGRAM 7.1A – QM for Windows Linear Programming Computer Input Screen



Using QM for Windows

PROGRAM 7.1B – QM for Windows Data Input



Using QM for Windows

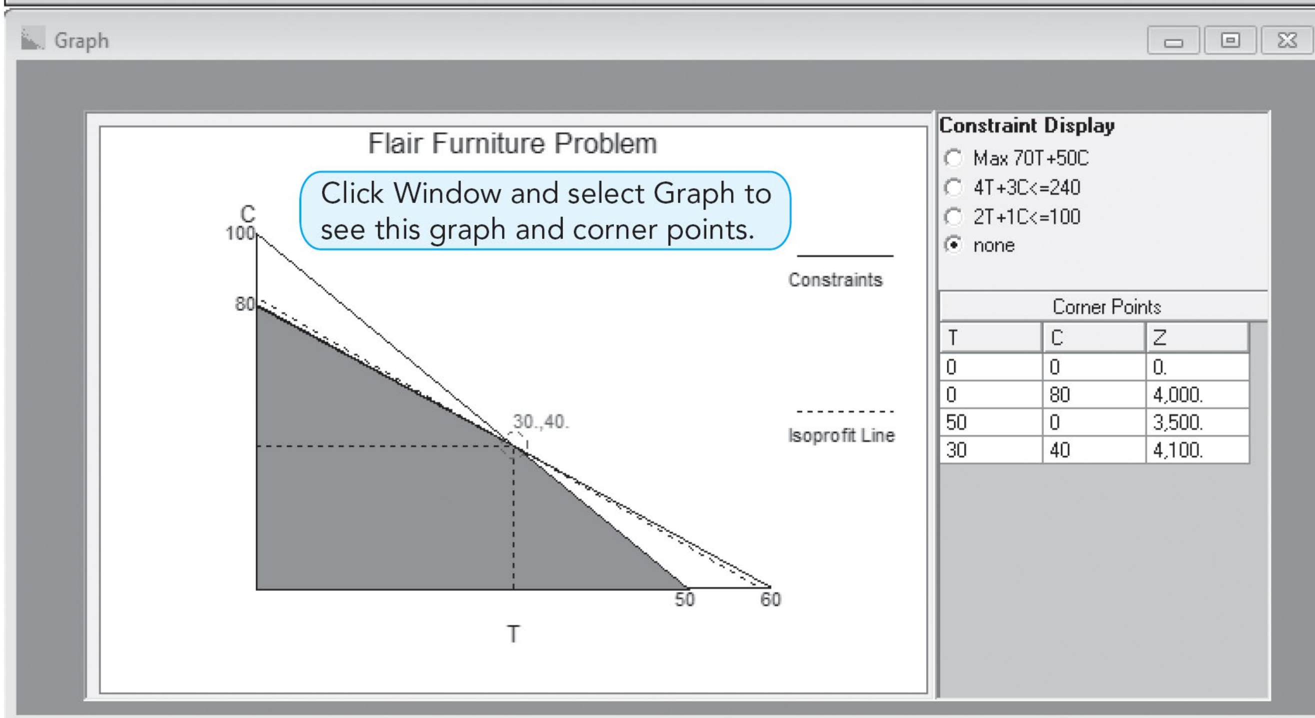
PROGRAM 7.1C – QM for Windows Output and Graph

Linear Programming Results

The values of the variables are shown here.

The objective function value (profit) is shown here.

	T			RHS	Dual
Maximize	70	50			
Carpentry	4	3	<=	240	15
Painting	2	1	<=	100	5
Solution->	30	40		4,100	



Using Excel's Solver

- The Solver tool in Excel can be used to find solutions to
  - LP problems
  - Integer programming problems
  - Noninteger programming problems
  - Solver is limited to 200 variables and, in some situations, 100 constraints

Using Solver

- Recall the model for Flair Furniture is

$$\begin{aligned}
\text{Maximize profit} = \quad & \$70T + \$50C \\
\text{Subject to} \quad & 4T + 3C \le 240 \\
& 2T + 1C \le 100
\end{aligned}$$

- To use Solver, it is necessary to enter data and formulas

Using Solver

1. Enter problem data
   – Variable names, coefficients for the objective function and constraints, RHS values for each constraint
2. Designate specific cells for the values of the decision variables
3. Write a formula to calculate the value of the objective function
4. Write a formula to compute the left-hand sides of each of the constraints

Using Solver

PROGRAM 7.2A – Excel Data Input

These cells are selected to contain the values of the decision variables. Solver will enter the optimal solution here, but you may enter numbers here also.

The signs for the constraints are entered here for reference only.

	A	B	C	D	E	F
1	**Flair Furniture**					
2						
3	**Variables**	**T (Tables)**	**C (Chairs)**			
4	**Units Produced**			**Profit**		
5	**Objective function**	**70**	**50**	**0**		
6						
7	**Constraints**			**LHS (Hours used)**		**RHS**
8	**Carpentry**	**4**	**3**	**0**	$\leq$	**240**
9	**Painting**	**2**	**1**	**0**	$\leq$	**100**

Using Solver

PROGRAM 7.2B – Formulas

	D
4	Profit
5	=SUMPRODUCT($B$4:$C$4,B5:C5)
6	
7	LHS (Hours used)
8	=SUMPRODUCT($B$4:$C$4,B8:C8)
9	=SUMPRODUCT($B$4:$C$4,B9:C9)

Using Solver

PROGRAM 7.2C – Excel Spreadsheet

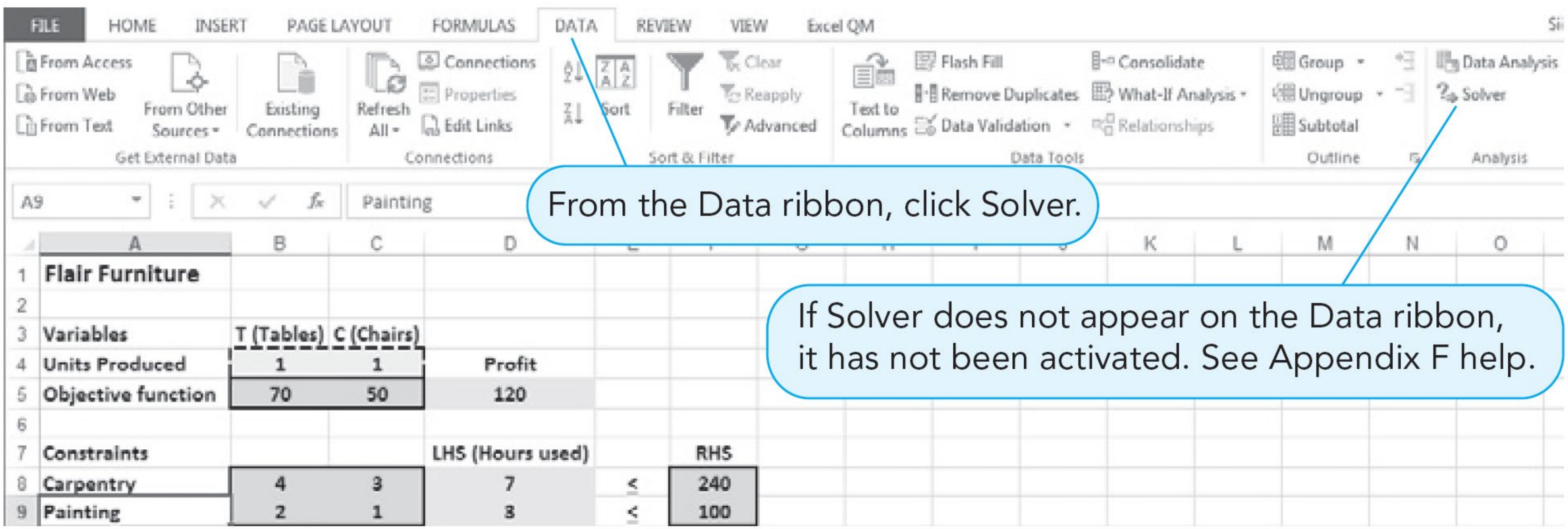
	A	B	C	D	E	F
1	**Flair Furniture**					
2						
3	**Variables**	**T (Tables)**	**C (Chairs)**			
4	**Units Produced**	**1**	**1**	**Profit**		
5	**Objective function**	**70**	**50**	**120**		
6						
7	**Constraints**			**LHS (Hours used)**		**RHS**
8	**Carpentry**	**4**	**3**	**7**	≤	**240**
9	**Painting**	**2**	**1**	**3**	≤	**100**

Because there is a 1 in each of these cells, the LHS values can be calculated very easily to see if a mistake has been made.

You can change these values to see how the profit and resource utilization change.

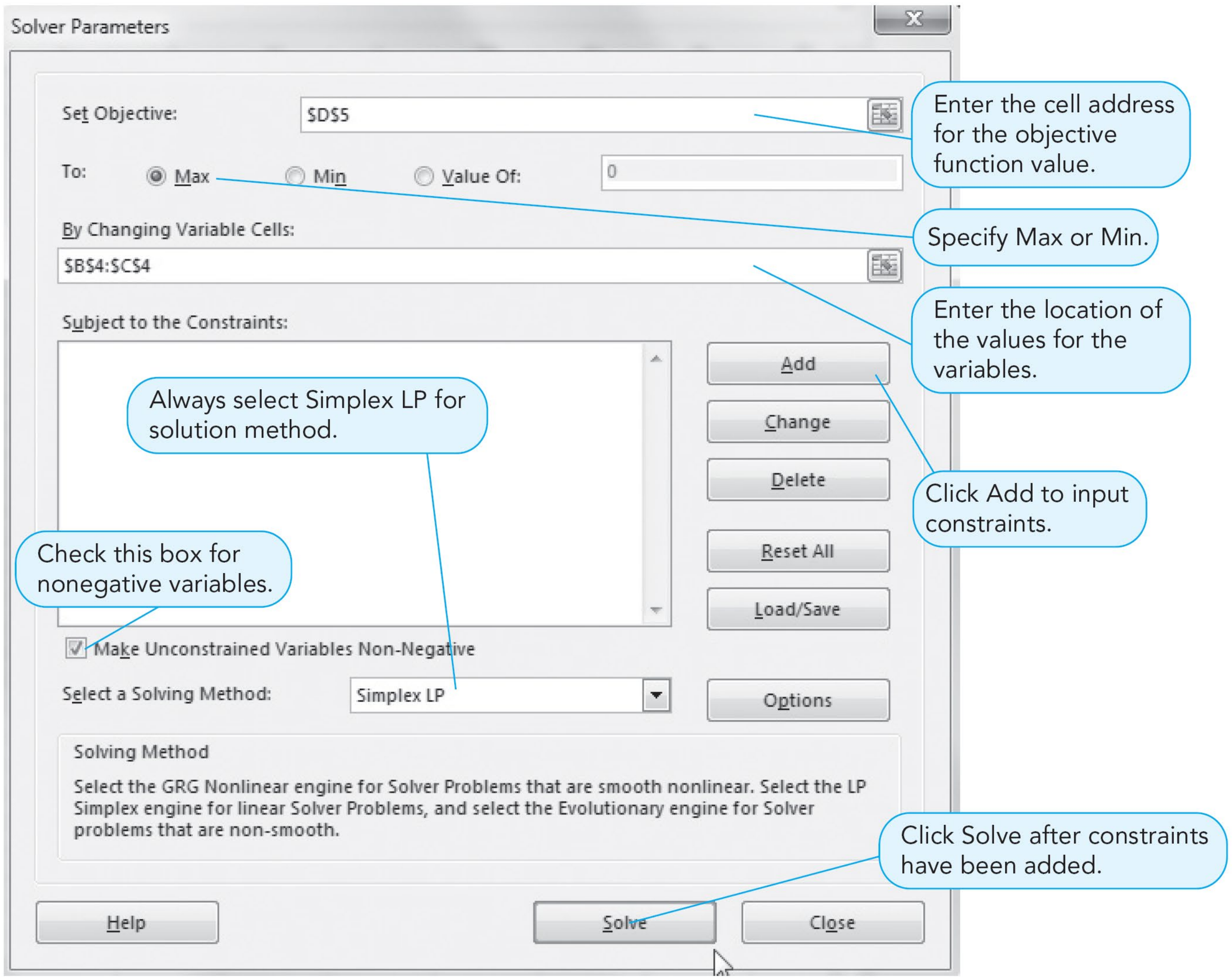
Using Solver

PROGRAM 7.2D – Starting Solver



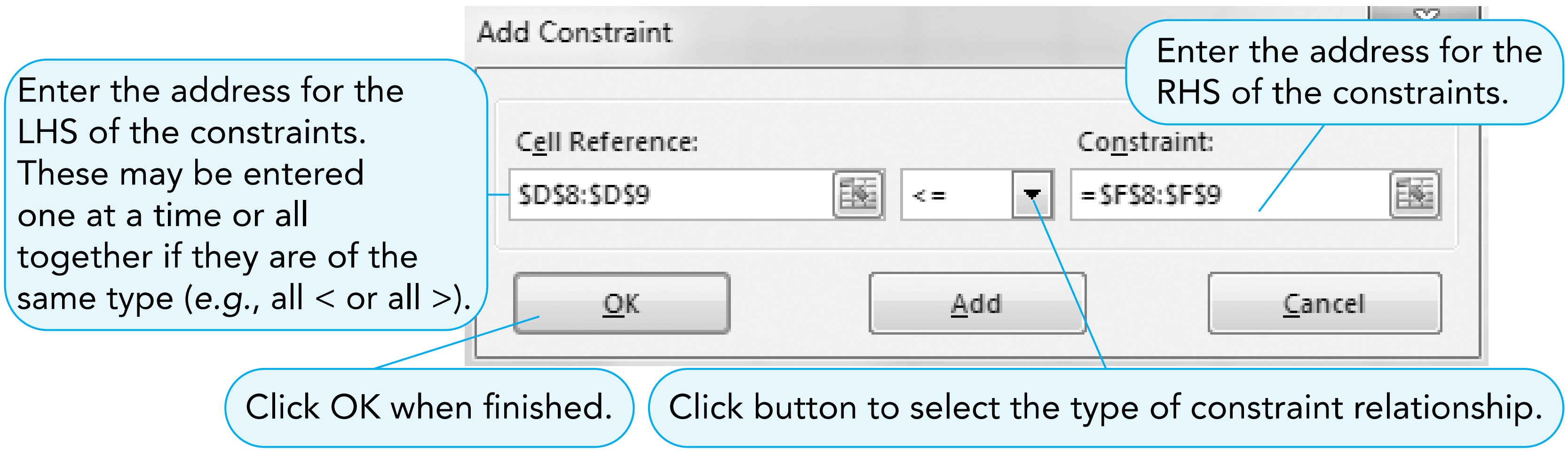
Using Solver

PROGRAM 7.2E – Solver Parameters Dialog Box



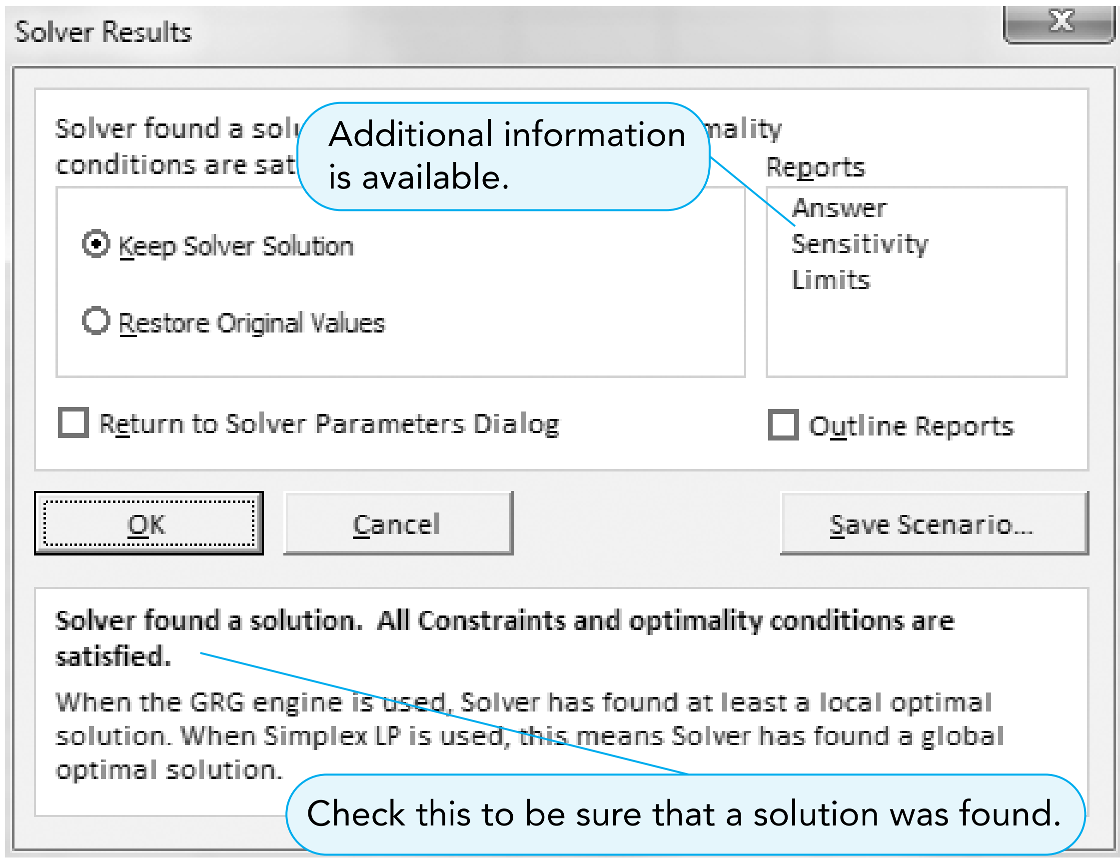
Using Solver

PROGRAM 7.2F – Solver Add Constraint Dialog Box



Using Solver

PROGRAM 7.2G – Solver Results Dialog Box



Using Solver

PROGRAM 7.2H – Solution

	A					
1	**Flair Furniture**					
2						
3	Variables	T (Tables)	C (Chairs)			
4	Units Produced	30	40	Profit		
5	Objective function	70	50	4100		
6						
7	Constraints			LHS (Hours used)		
8	Carpentry	4	3	240	≤	240
9	Painting	2	1	100	≤	100

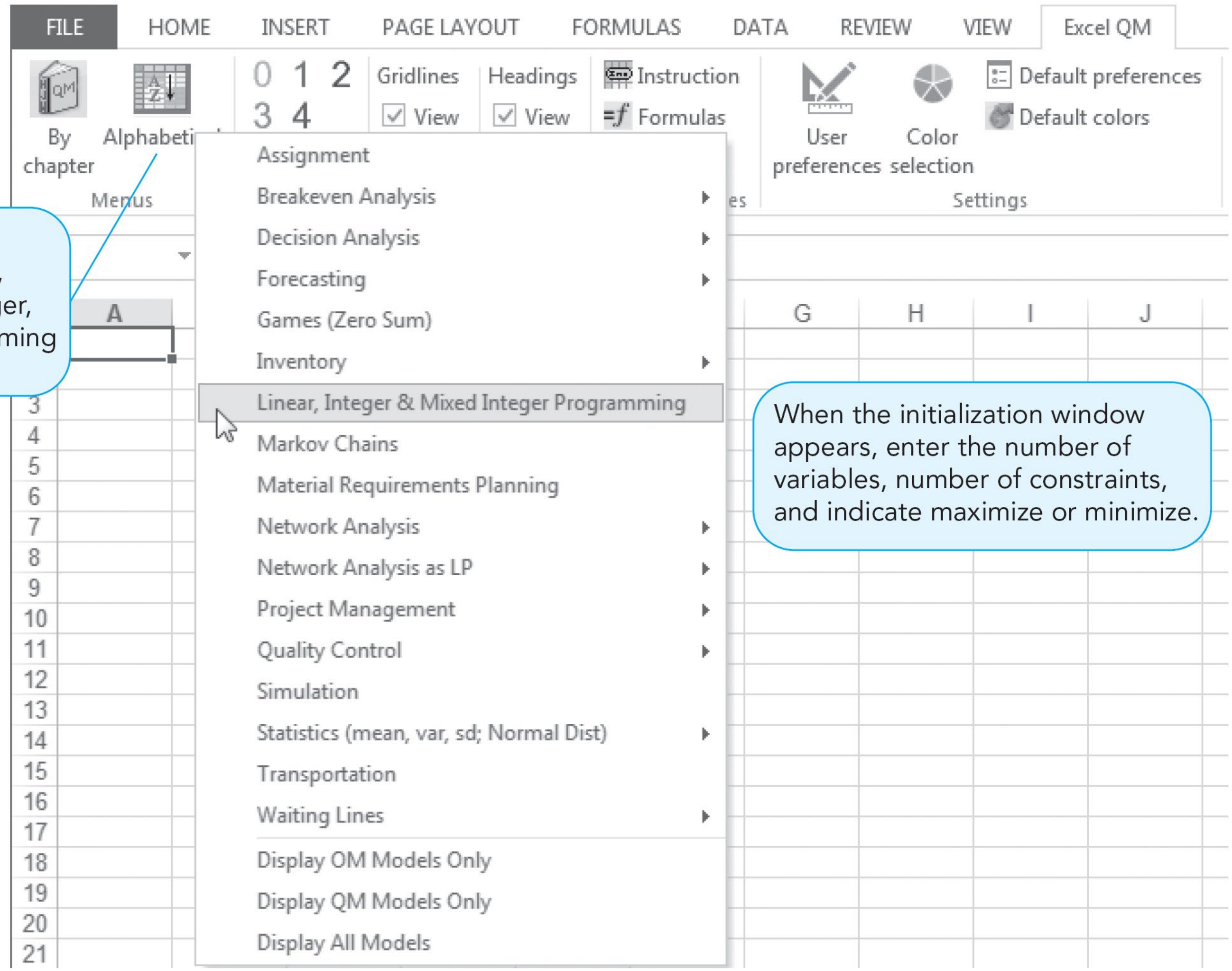
The optimal solution is *T* = 30, *C* = 40, profit = 4,100.

The hours used are given here.

Using Excel QM

PROGRAM 7.3A –
Excel QM in Excel 2013

From the Excel QM ribbon, click the Alphabetical Menu, and then select Linear, Integer, and Mixed Integer Programming from the menu.



When the initialization window appears, enter the number of variables, number of constraints, and indicate maximize or minimize.

Using Excel QM

PROGRAM 7.3B – Excel QM Input Data

After entering the problem, click the Data tab and select Solver from the Data ribbon. When the window for Solver opens, simply click Solve as all the necessary inputs have been entered by Excel QM.

Flair Furniture

Enter the values in the shaded area. Then go to the DATA Tab on the ribbon, click on Solver in the Data Analysis Group and then click SOLVE.
If SOLVER is not on the Data Tab then please see the Help file (Solver) for instructions.

Instructions to access Solver are here.

Signs

<	less than or equal to
=	equals (You need to enter an apostrophe first.)
>	greater than or equal to

Data

	X 1	X 2	sign	RHS
Objective	70	50		
Constraint 1	4	3	<	240
Constraint 2	2	1	<	100

Results

LHS	Slack/Surplus
0	
0	240
0	100

Results

Variables	
Objective	

Enter the data in the appropriate cells. Do not change any other cells in the spreadsheet.

Using Excel QM

PROGRAM 7.3C – Excel QM Output

	A	B	C	D	E	F	G	H
1	**Flair Furniture**							
2–4	Enter the values in the shaded area. Then go to the DATA Tab on the ribbon, click on Solver in the Data Analysis Group and then click SOLVE. If SOLVER is not on the Data Tab then please see the Help file (Solver) for instructions.							
5	Signs							
6		<	less than or equal to					
7		=	equals (You need to enter an apostrophe first.)					
8		>	greater than or equal to					
9								
10	Data						Results	
11		X 1	X 2				LHS	Slack/Surplus
12	Objective	70	50	sign	RHS		4100	
13	Constraint 1	4	3	<	240		240	0
14	Constraint 2	2	1	<	100		100	0
15								
16	Results							
17	Variables	30	40					
18	Objective				4100			

Solution is shown here.

Solving Minimization Problems

- Many LP problems involve minimizing an objective such as cost
- Minimization problems can be solved graphically
  - Set up the feasible solution region
  - Use either the corner point method or an isocost line approach
  - Find the values of the decision variables (e.g., X_1 and X_2) that yield the minimum cost

Holiday Meal Turkey Ranch

- The Holiday Meal Turkey Ranch is considering buying two different brands of turkey feed and blending them to provide a good, low-cost diet for its turkeys

TABLE 7.5 – Holiday Meal Turkey Ranch data

	COMPOSITION OF EACH POUND OF FEED (OZ.)		
INGREDIENT	BRAND 1 FEED	BRAND 2 FEED	MINIMUM MONTHLY REQUIREMENT PER TURKEY (OZ.)
A	5	10	90
B	4	3	48
C	0.5	0	1.5
Cost per pound	2 cents	3 cents	

Holiday Meal Turkey Ranch

Let

X_1 = number of pounds of brand 1 feed purchased

X_2 = number of pounds of brand 2 feed purchased

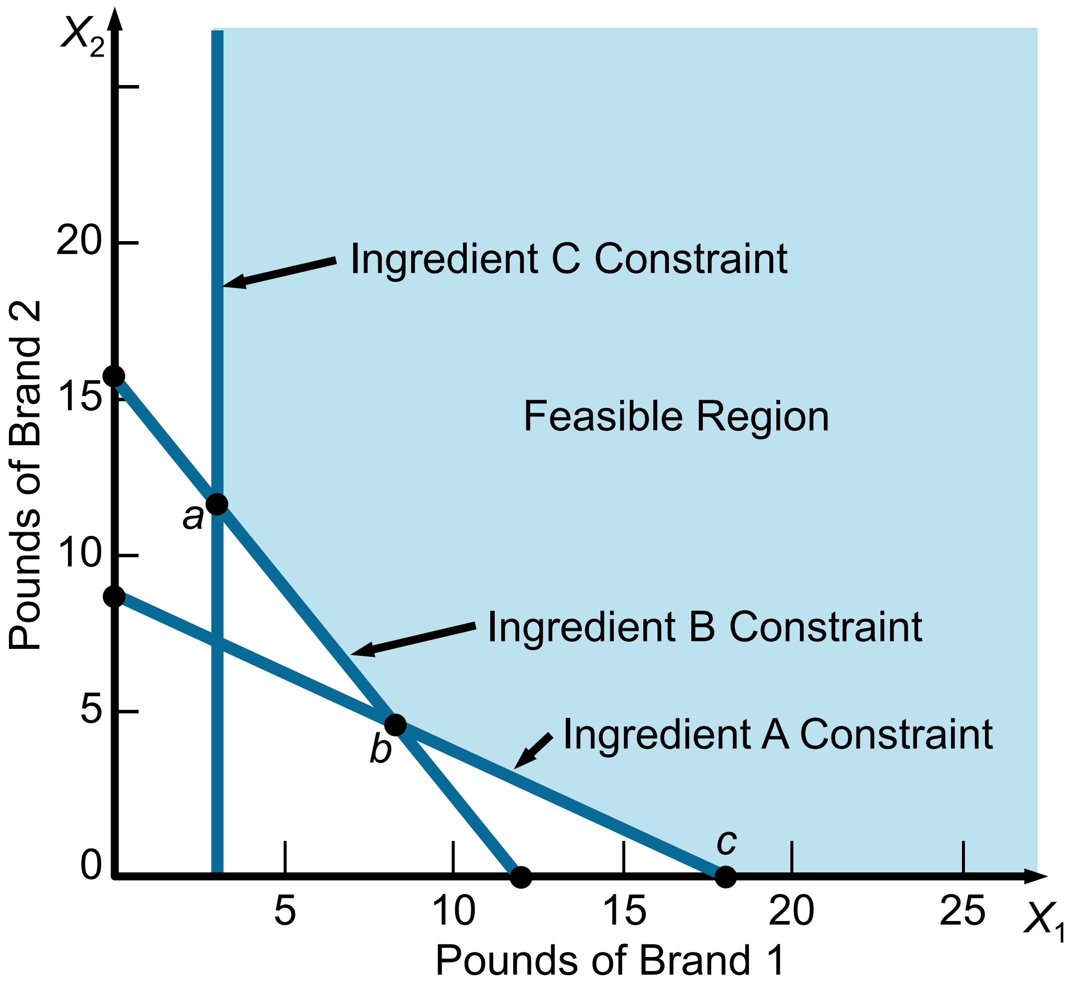
Minimize cost (in cents) = $2X_1 + 3X_2$

subject to:

$5X_1 + 10X_2 \geq$	90 ounces	(ingredient A constraint)
$4X_1 + 3X_2 \geq$	48 ounces	(ingredient B constraint)
$0.5X_1 \geq$	1.5 ounces	(ingredient C constraint)
$X_1 \geq$	0	(nonnegativity constraint)
$X_2 \geq$	0	(nonnegativity constraint)

Holiday Meal Turkey Ranch

FIGURE 7.10 – Feasible Region



Holiday Meal Turkey Ranch

- Solve for the values of the three corner points
  - Point a is the intersection of ingredient constraints C and B

$$4X_1 + 3X_2 = 48$$
$$X_1 = 3$$

  - Substituting 3 in the first equation, we find $X_2 = 12$
  - Solving for point b we find $X_1 = 8.4$ and $X_2 = 4.8$
  - Solving for point c we find $X_1 = 18$ and $X_2 = 0$

Holiday Meal Turkey Ranch

- Substituting these values back into the objective function we find

$$\text{Cost} = 2X_1 + 3X_2$$

$$\text{Cost at point } a = 2(3) + 3(12) = 42$$

$$\boxed{\text{Cost at point } b = 2(8.4) + 3(4.8) = 31.2}$$

$$\text{Cost at point } c = 2(18) + 3(0) = 36$$

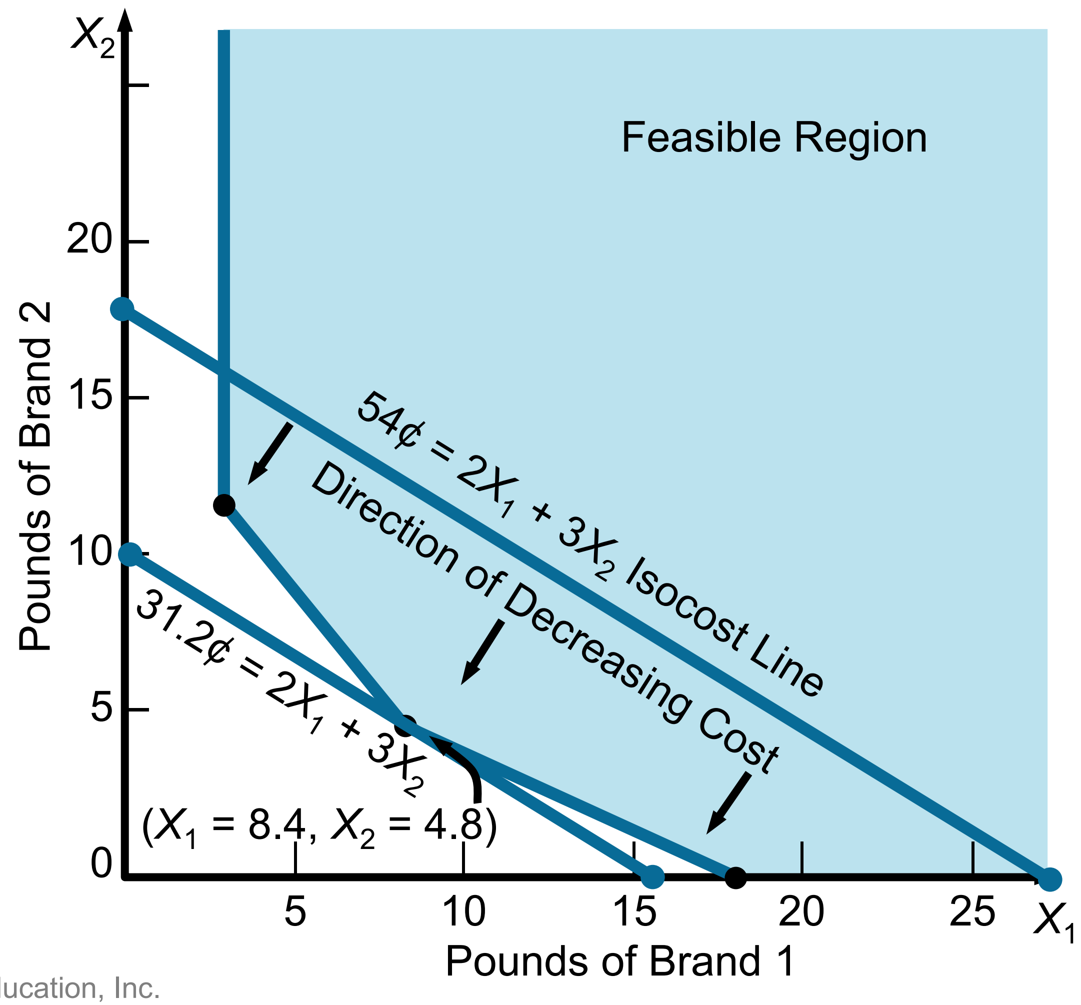
  – The lowest cost solution is to purchase 8.4 pounds of brand 1 feed and 4.8 pounds of brand 2 feed for a total cost of 31.2 cents per turkey

Holiday Meal Turkey Ranch

- Solving using an **isocost line**
- Move the isocost line toward the lower left
- The last point touched in the feasible region will be the optimal solution

Holiday Meal Turkey Ranch

FIGURE 7.11 – Graphical Solution Using the Isocost Approach



Holiday Meal Turkey Ranch

PROGRAM 7.4 – Solution in QM for Windows

Linear Programming Results

Holiday Meal Solution					
	X1	X2		RHS	Dual
Minimize	2	3			
Ingredient A	5	10	>=	90	-0.24
Ingredient B	4	3	>=	48	-0.2
Ingredient C	0.5	0	>=	1.5	0
Solution->	8.4	4.8		31.2	

Holiday Meal Turkey Ranch

PROGRAM 7.5A – Excel 2013 Solution

Formulas are written to find the values in column D.

	A	B				
1	Holiday Meal Turkey Ran					
2						
3	Variables	Brand 1	Brand 2			
4	Units Produced	8.4	4.8	Cost		
5	Objective function	2	3	31.2		
6						
7	Constraints			LHS (Amt. of Ing.)		RHS
8	Ingredient A	5	10	90	≥	90
9	Ingredient B	4	3	48	≥	48
10	Ingredient C	0.5	0	4.2	≥	1.5

Holiday Meal Turkey Ranch

PROGRAM 7.5B – Excel 2013 Formulas

	D
4	**Cost**
5	=SUMPRODUCT($B$4:$C$4,B5:C5)
6	
7	**LHS (Amt. of Ing.)**
8	=SUMPRODUCT($B$4:$C$4,B8:C8)
9	=SUMPRODUCT($B$4:$C$4,B9:C9)
10	=SUMPRODUCT($B$4:$C$4,B10:C10)

Four Special Cases in LP

- Four special cases and difficulties arise at times when using the graphical approach
  1. No feasible solution
  2. Unboundedness
  3. Redundancy
  4. Alternate Optimal Solutions

No Feasible Solution

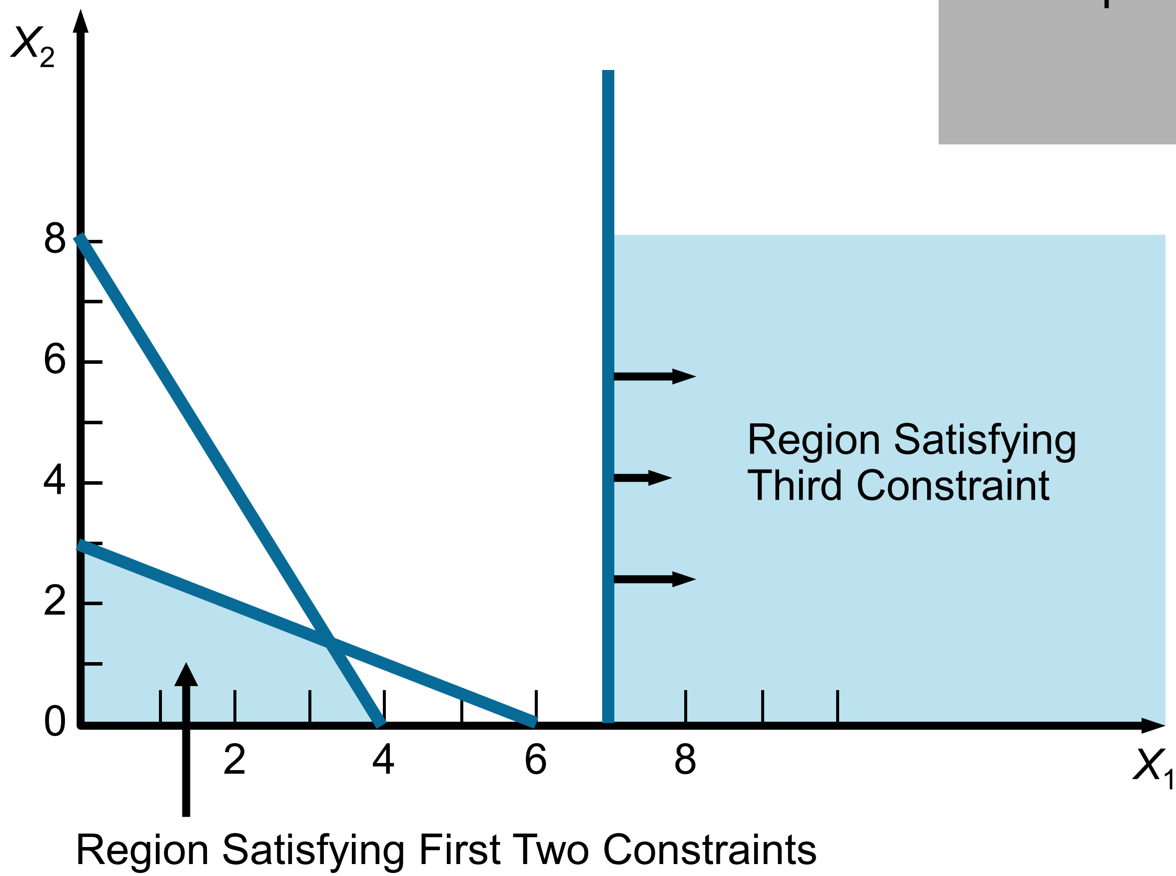
- No solution to the problem that satisfies all the constraint equations
- No feasible solution region exists
- A common occurrence in the real world
- Generally one or more constraints are relaxed until a solution is found
- Consider the following three constraints

$$X_1 + 2X_2 \leq 6$$
$$2X_1 + X_2 \leq 8$$
$$X_1 \geq 7$$

No Feasible Solution

FIGURE 7.12 – A problem with no feasible solution

$$X_1 + 2X_2 \le 6$$
$$2X_1 + X_2 \le 8$$
$$X_1 \ge 7$$

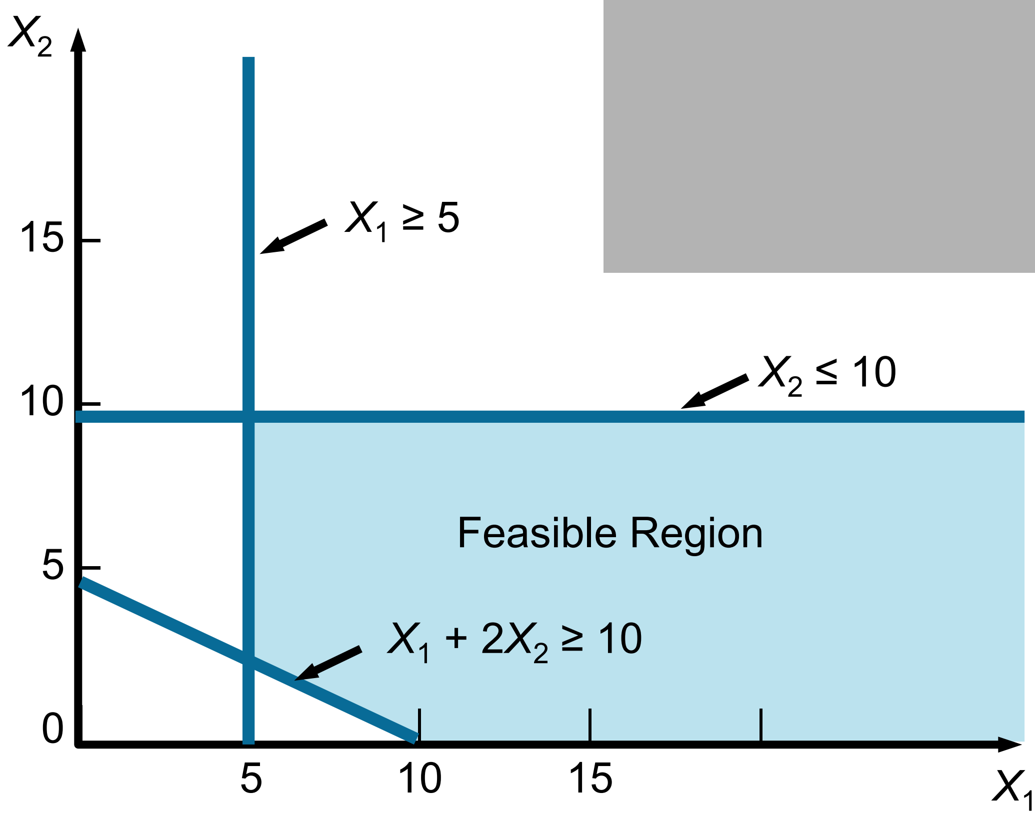


Unboundedness

- Sometimes a linear program will not have a finite solution
- In a maximization problem
  - One or more solution variables, and the profit, can be made infinitely large without violating any constraints
- In a graphical solution, the feasible region will be open ended
- Usually means the problem has been formulated improperly

Unboundedness

FIGURE 7.13 – A Feasible Region That Is Unbounded to the Right



$$
\begin{aligned}
\text{Maximize profit} = \quad & \$3X_1 + \$5X_2 \\
\text{subject to} \quad & X_1 \geq 5 \\
& X_2 \leq 10 \\
& X_1 + 2X_2 \geq 10 \\
& X_1, X_2 \geq 0
\end{aligned}
$$

Redundancy

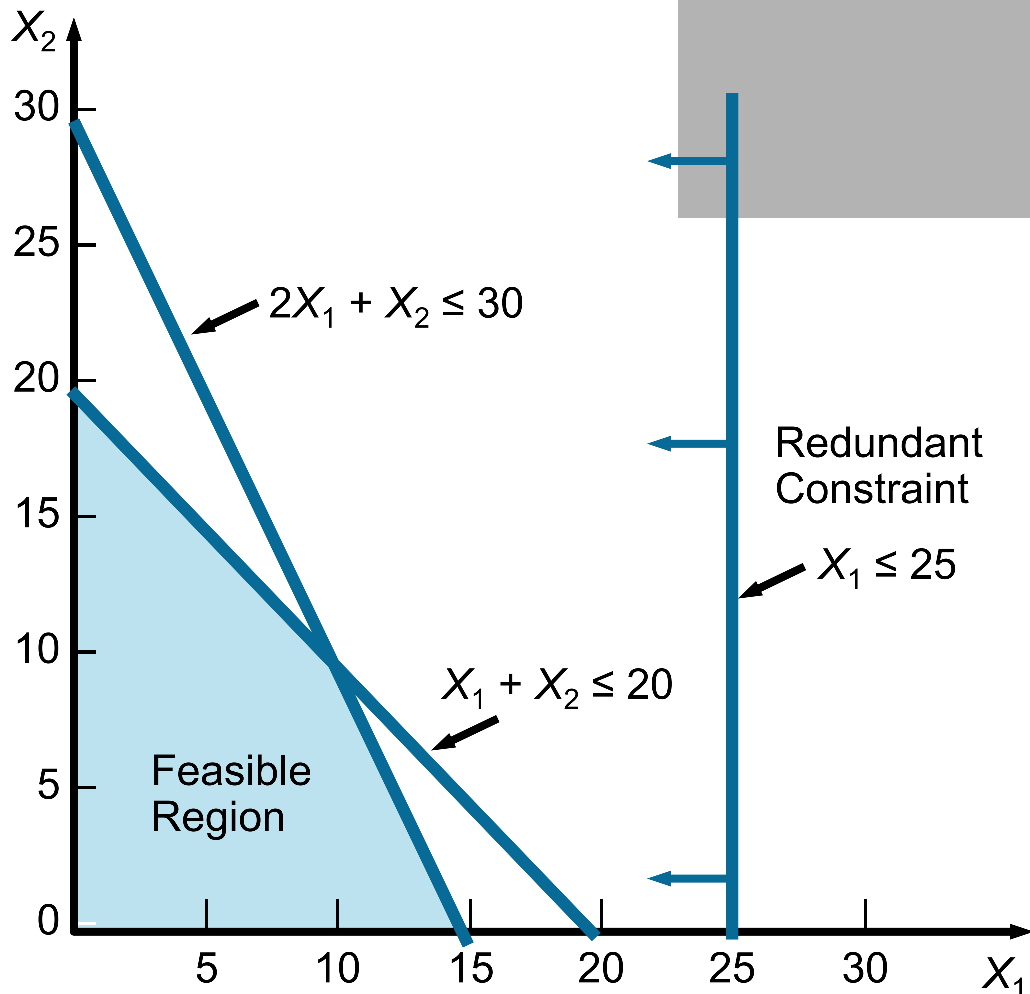
- A redundant constraint is one that does not affect the feasible solution region
- One or more constraints may be binding
- This is a very common occurrence in the real world
- Causes no particular problems, but eliminating redundant constraints simplifies the model

$$
\begin{aligned}
\text{Maximize profit} = \quad & \$1X_1 + \$2X_2 \\
\text{subject to} \quad & X_1 + X_2 \le 20 \\
& 2X_1 + X_2 \le 30 \\
& X_1 \le 25 \\
& X_1, X_2 \ge 0
\end{aligned}
$$

Redundancy

FIGURE 7.14 – Problem with a Redundant Constraint

$$\begin{aligned}
\text{Maximize profit} = \quad & \$1X_1 + \$2X_2 \\
\text{subject to} \quad & X_1 + X_2 \le 20 \\
& 2X_1 + X_2 \le 30 \\
& X_1 \le 25 \\
& X_1, X_2 \ge 0
\end{aligned}$$



Alternate Optimal Solutions

- Occasionally two or more optimal solutions may exist
- Graphically this occurs when the objective function's isoprofit or isocost line runs perfectly parallel to one of the constraints
- Allows management great flexibility in deciding which combination to select as the profit is the same at each alternate solution

$$
\begin{array}{lrrl}
\text{Maximize profit} = & \$3X_1 & + \$2X_2 & \\
\text{subject to} & 6X_1 & + 4X_2 & \leq 24 \\
 & X_1 & & \leq 3 \\
 & & X_1, X_2 & \geq 0
\end{array}
$$

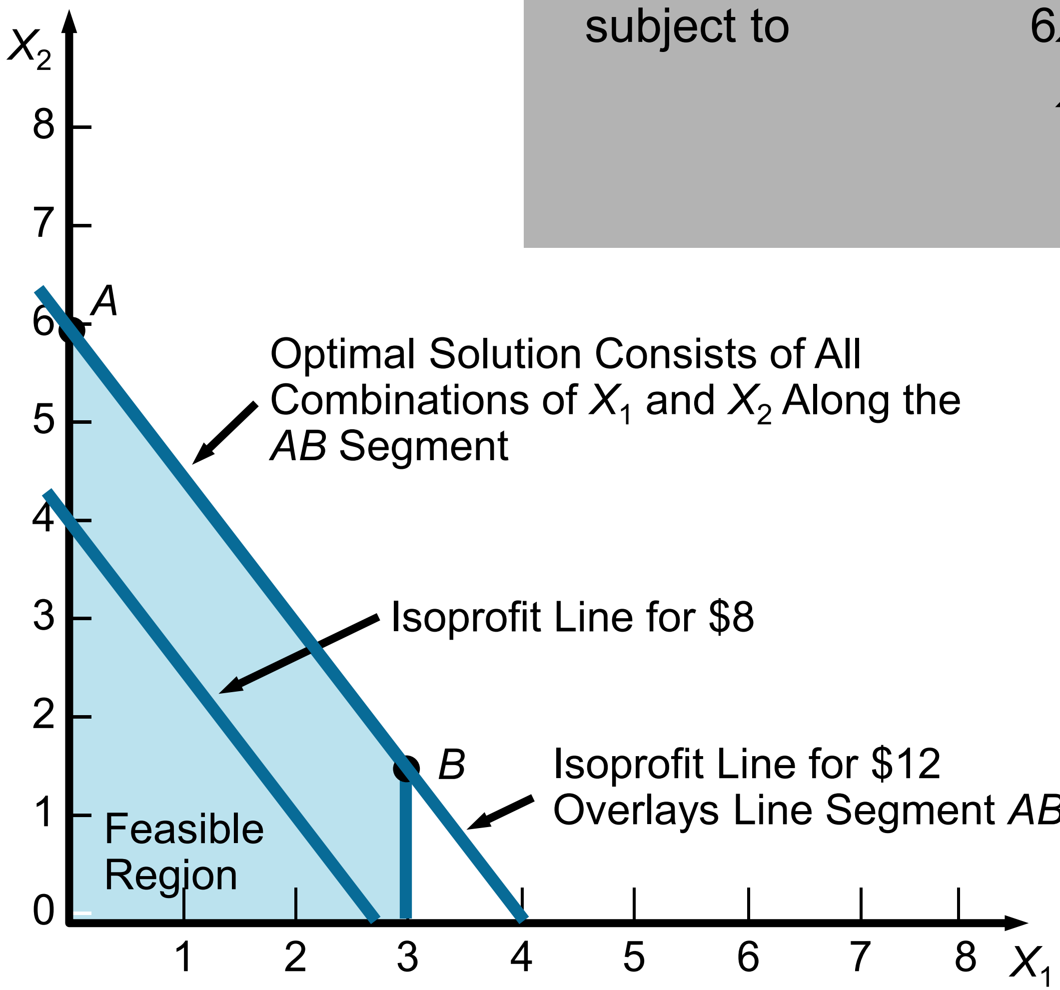
Alternate Optimal Solutions

FIGURE 7.15 – Example of Alternate Optimal Solutions

Maximize profit = $\$3X_1 + \$2X_2$

subject to

$$6X_1 + 4X_2 \le 24$$
$$X_1 \le 3$$
$$X_1, X_2 \ge 0$$



Sensitivity Analysis

- Optimal solutions to LP problems thus far have been found under *deterministic assumptions*
  - We assume complete certainty in the data and relationships of a problem
- Real world conditions are dynamic
- Analyze how sensitive a deterministic solution is to changes in the assumptions of the model
- This is called **sensitivity analysis**, *postoptimality analysis*, *parametric programming*, or *optimality analysis*

Sensitivity Analysis

- Involves a series of what-if? questions concerning constraints, variable coefficients, and the objective function
- Trial-and-error method
  - Values are changed and the entire model is resolved
- Preferred way is to use an analytic postoptimality analysis
  - After a problem has been solved, we determine a range of changes in problem parameters that will not affect the optimal solution or change the variables in the solution

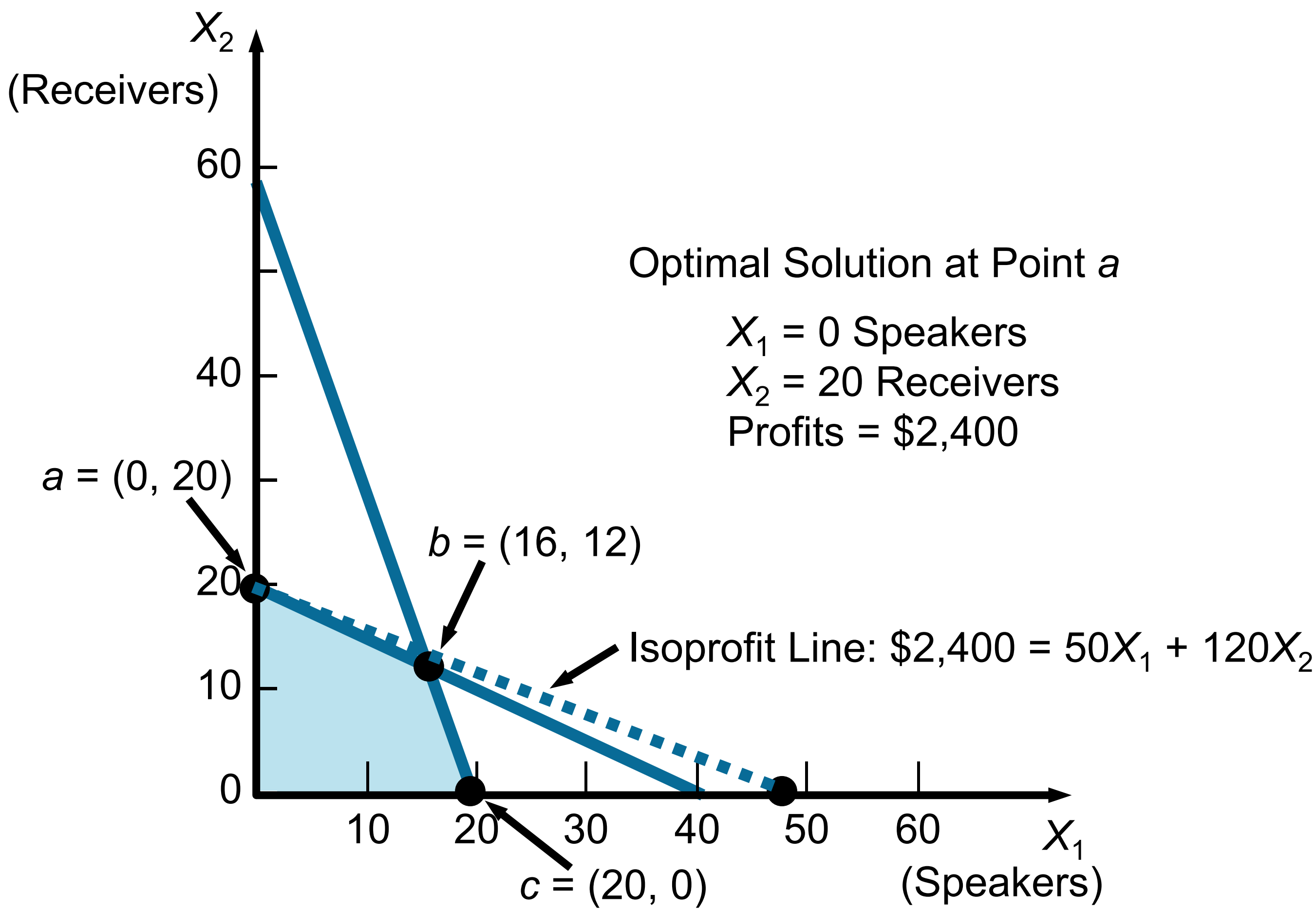
High Note Sound Company

- The company manufactures quality speakers and stereo receivers
- Products require a certain amount of skilled artisanship which is in limited supply
- Product mix LP model

Maximize profit =	$\$50X_1$	$+ \$120X_2$		
subject to	$2X_1$	$+ 4X_2$	≤ 80	(hours of electricians' time available)
	$3X_1$	$+ 1X_2$	≤ 60	(hours of audio technicians' time available)
		X_1, X_2	≥ 0	

High Note Sound Company

FIGURE 7.16 – The High Note Sound Company Graphical Solution



High Note Sound Company

- Electrician hours used are

$$2X_1 + 4X_2 = 2(0) + 4(20) = 80$$

  – All hours are utilized so slack = 0
  – Additional units of a binding constraint will generally increase profits
- Technician hours used are

$$3X_1 + 1X_2 = 3(0) + 1(20) = 20$$

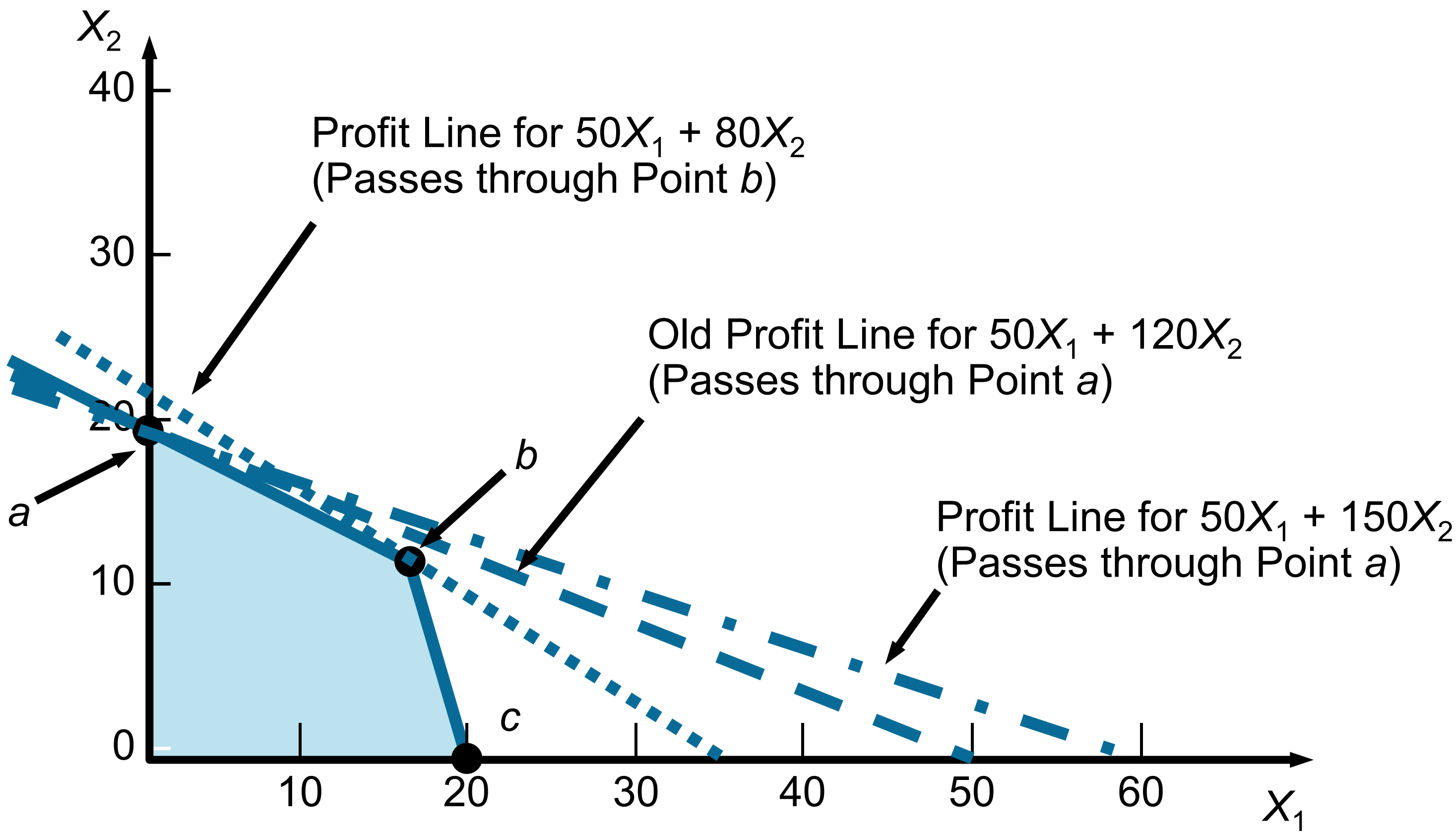
  – Available hours = 60 so slack = 60 – 20 = 40
  – Additional units of a nonbinding constraint will only increase slack

Changes in the Objective Function Coefficient

- Contribution rates in the objective functions fluctuate
  - The feasible solution region remains exactly the same
  - The slope of the isoprofit or isocost line changes
- Modest increases or decreases in objective function coefficients may not change the current optimal corner point
- Know how much an objective function coefficient can change before the optimal solution would be at a different corner point

Changes in the Objective Function Coefficient

FIGURE 7.17 – Changes in the Receiver Contribution Coefficients



QM for Windows

PROGRAM 7.6A – Input to QM for Windows High Note Sound

Objective
- (•) Maximize
- ( ) Minimize

Instruction
Use these option buttons to set the objective.

High Note Sound Company

	X1	X2		RHS	Equation form
Maximize	50	120			Max 50X1 + 120X2
Electrician hours	2	4	<=	80	2X1 + 4X2 <= 80
Audio technician hours	3	1	<=	60	3X1 + X2 <= 60

QM for Windows

PROGRAM 7.6B – High Note Sound Sensitivity Analysis

Ranging

High Note Sound Company Solution

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	0	10	50	-Infinity	60
X2	20	0	120	100	Infinity
Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Electrician hours	30	0	80	0	240
Audio technician hours	0	40	60	20	Infinity

Excel Solver

PROGRAM 7.7A – Excel Spreadsheet for High Note Sound

The By Changing Variable Cells in the Solver Dialog Box are B4:C4.

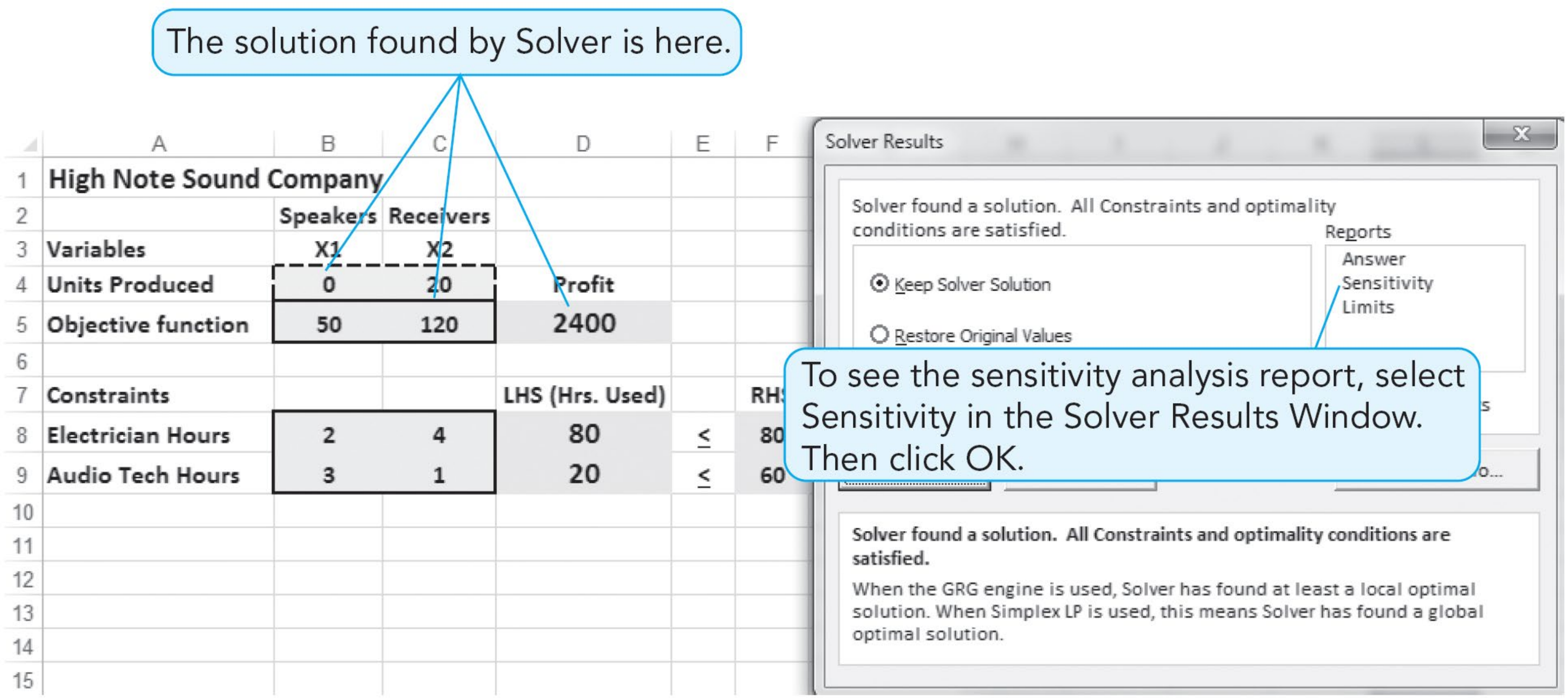
The Set Objective cell in the Solver Dialog Box is D5.

	A	B	C	D	E	F
1	**High Note Sound**					
2		**Speakers**				
3	**Variables**	**X1**	**X2**			
4	**Units Produced**	**0**	**20**	**Profit**		
5	**Objective function**	**50**	**120**	**=SUMPRODUCT($B$4:$C$4,B5:C5)**		
6						
7	**Constraints**			**LHS (Hrs. Used)**		**RHS**
8	**Electrician Hours**	**2**	**4**	**=SUMPRODUCT($B$4:$C$4,B8:C8)**	**≤**	**80**
9	**Audio Tech Hours**	**3**	**1**	**=SUMPRODUCT($B$4:$C$4,B9:C9)**	**≤**	**60**

The constraints added into Solver will be D8:D9 <=F8:F9.

Excel Solver

PROGRAM 7.7B – Excel 2013 Solution and Solver Results



Excel Solver

PROGRAM 7.7C – Excel 2013 Sensitivity Report

The names presented in the Sensitivity Report combine the text in column A and the text above the data, unless the cells have been named using the Name Manager from the Formulas tab.

The profit on speakers may change by these amounts and the current corner point will remain optimal.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$4	Units Produced X1	0	-10	50	10	1E+30
$C$4	Units Produced X2	20	0	120	1E+30	20

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$8	Electrician Hours LHS (Hrs. Used)	80	30	80	160	80
$D$9	Audio Tech Hours LHS (Hrs. Used)	20	0	60	1E+30	40

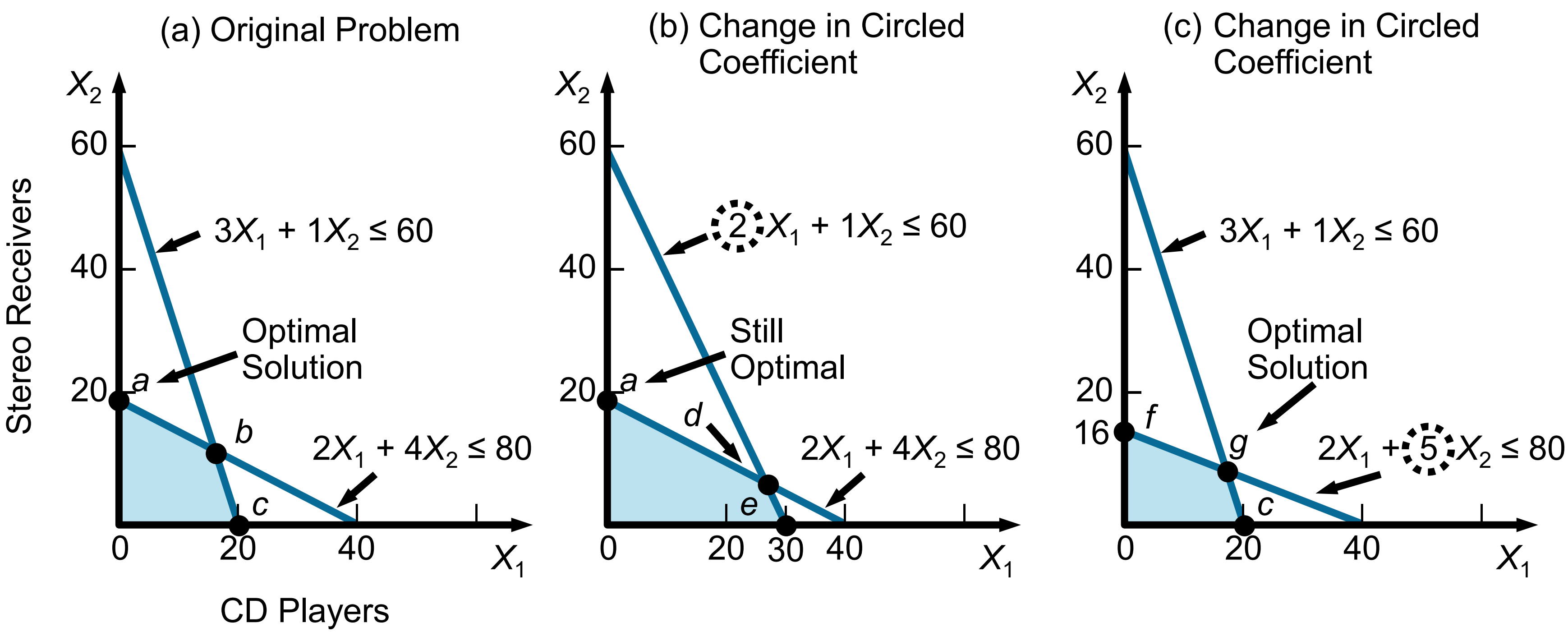
The resources used are here. The RHS can change by these amounts, and the shadow price will still be relevant.

Changes in the Technological Coefficients

- Changes in the **technological coefficients** often reflect changes in the state of technology
- If the amount of resources needed to produce a product changes, coefficients in the constraint equations will change
- Objective function does not change
- May produce significant change in the shape of the feasible region
- May cause a change in the optimal solution

Changes in the Technological Coefficients

FIGURE 7.18 – Change in the Technological Coefficients



Changes in Resources or Right-Hand-Side Values

- Right-hand-side values of the constraints often represent resources available to the firm
- Additional resources may lead to higher total profit
- Sensitivity analysis about resources helps answer questions about
  – How much should be paid for additional resources
  – How much more of a resource would be useful

Changes in Resources or Right-Hand-Side Values

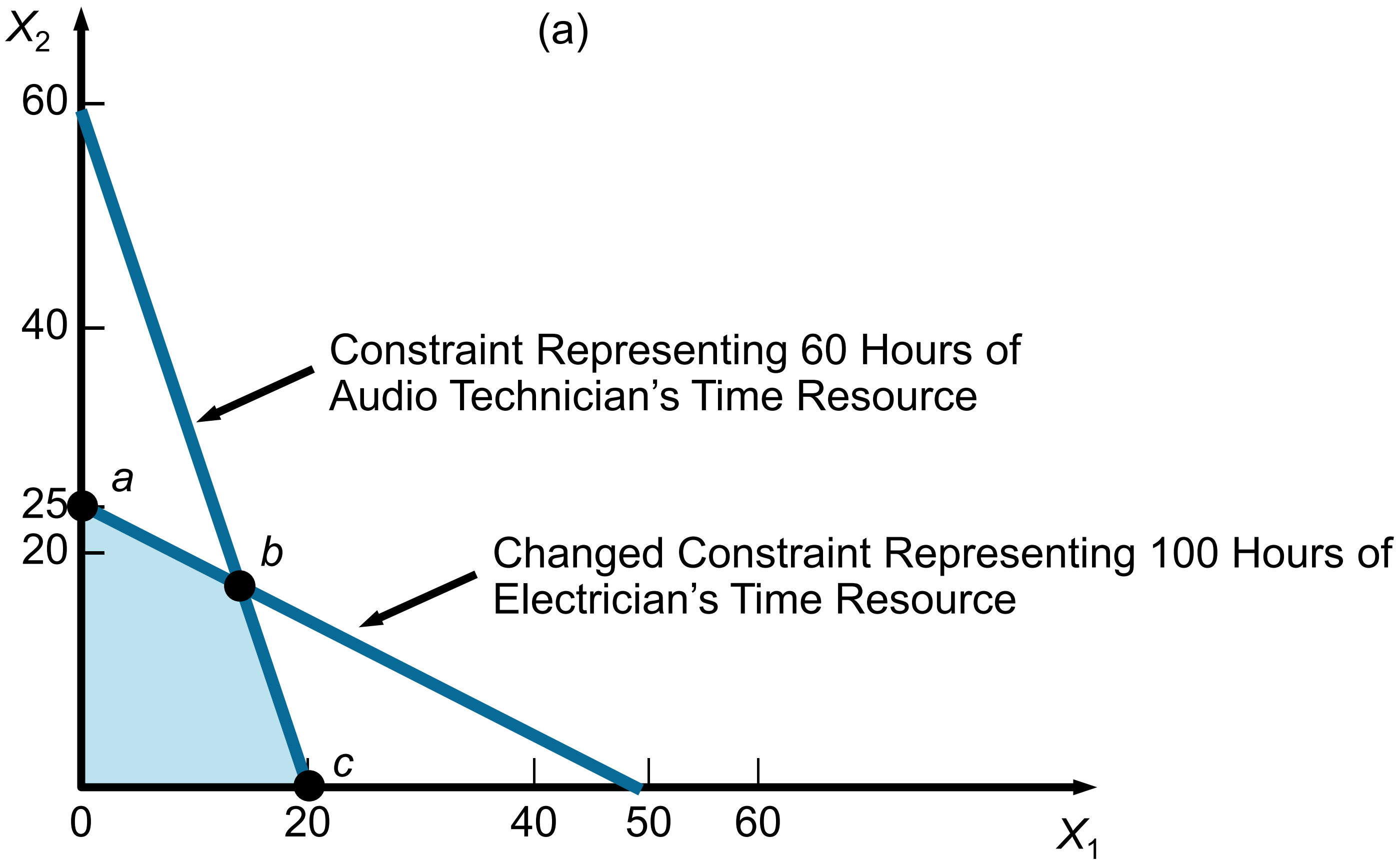
- Changing the RHS will change the feasible region, unless the constraint is redundant
- Often changes the optimal solution
- The **dual price** or dual value
  - The amount of change in the objective function value that results from a unit change in one of the resources
  - The dual price for a constraint is the improvement in the objective function value that results from a one-unit increase in the right-hand side of the constraint

Changes in Resources or Right-Hand-Side Values

- The amount of possible increase in the RHS is limited
- If the RHS is increased beyond the upper bound, then the objective function would no longer increase by the dual price
- There would be excess (slack) resources or the objective function may change by an amount different from the dual price
- The dual price is relevant only within limits

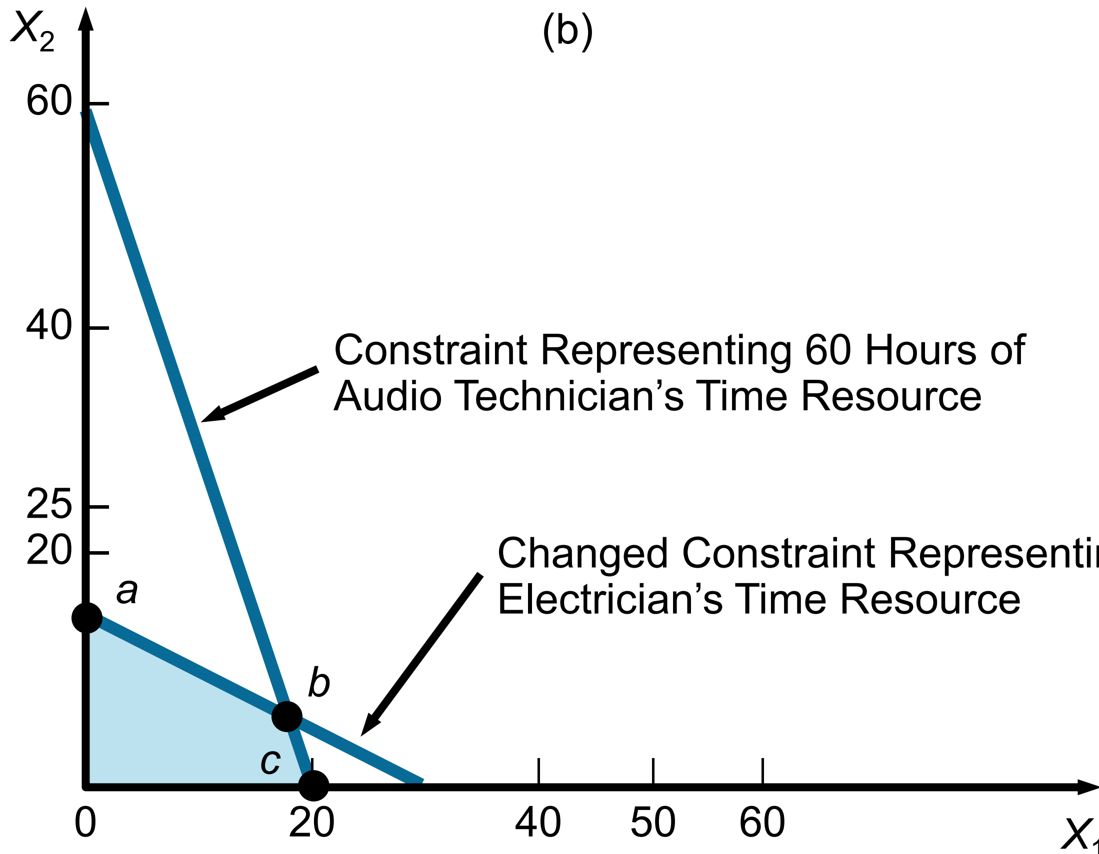
Changes in the Electricians' Time Resource

FIGURE 7.19



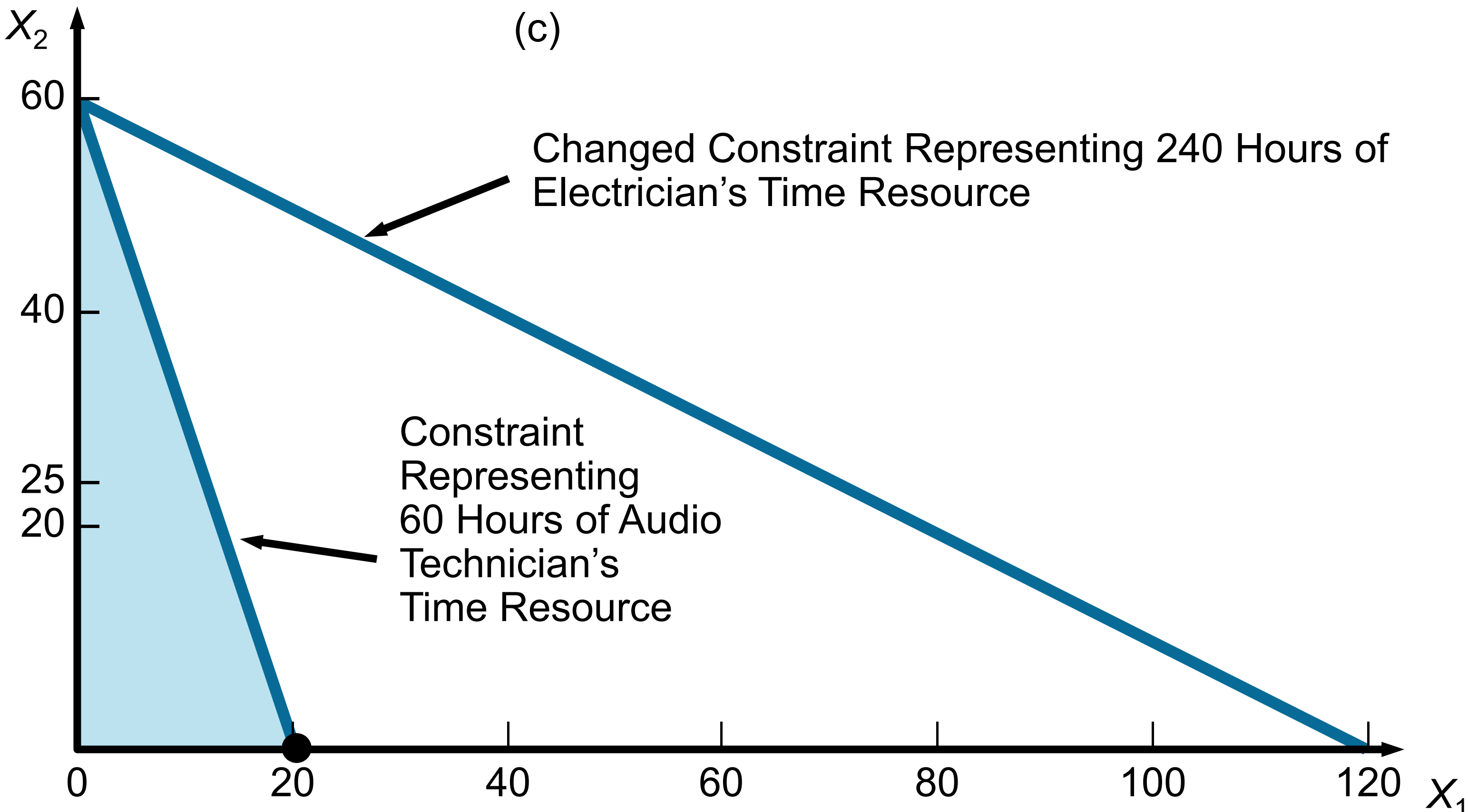
Changes in the Electricians' Time Resource

FIGURE 7.19



Changes in the Electricians' Time Resource

FIGURE 7.19



QM for Windows

PROGRAM 7.6B – High Note Sound Sensitivity Analysis

Ranging

High Note Sound Company Solution

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	0	10	50	-Infinity	60
X2	20	0	120	100	Infinity

Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Electrician hours	30	0	80	0	240
Audio technician hours	0	40	60	20	Infinity

Excel Solver

PROGRAM 7.7C – Excel 2013 Sensitivity Report

The names presented in the Sensitivity Report combine the text in column A and the text above the data, unless the cells have been named using the Name Manager from the Formulas tab.

The profit on speakers may change by these amounts and the current corner point will remain optimal.

	A	B	C	D	E	F	G	H
6	Variable Cells							
7				Final	Reduced	Objective	Allowable	Allowable
8		Cell	Name	Value	Cost	Coefficient	Increase	Decrease
9		$B$4	Units Produced X1	0	-10	50	10	1E+30
10		$C$4	Units Produced X2	20	0	120	1E+30	20
11								
12	Constraints							
13				Final	Shadow	Constraint	Allowable	Allowable
14		Cell	Name	Value	Price	R.H. Side	Increase	Decrease
15		$D$8	Electrician Hours LHS (Hrs. Used)	80	30	80	160	80
16		$D$9	Audio Tech Hours LHS (Hrs. Used)	20	0	60	1E+30	40

The resources used are here. The RHS can change by these amounts, and the shadow price will still be relevant.

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