Control Systems

Lab 5

State Feedback Controller Design

Dhyey Findoriya (B22EE024)

1. Objective

The primary objective of this design is to find the feedback matrix k that provides closed loop system stability. We want to do this through analysing the system's controllability and constructing a suitable state feedback controller.

2. Theory

Design of State Feedback Controller:

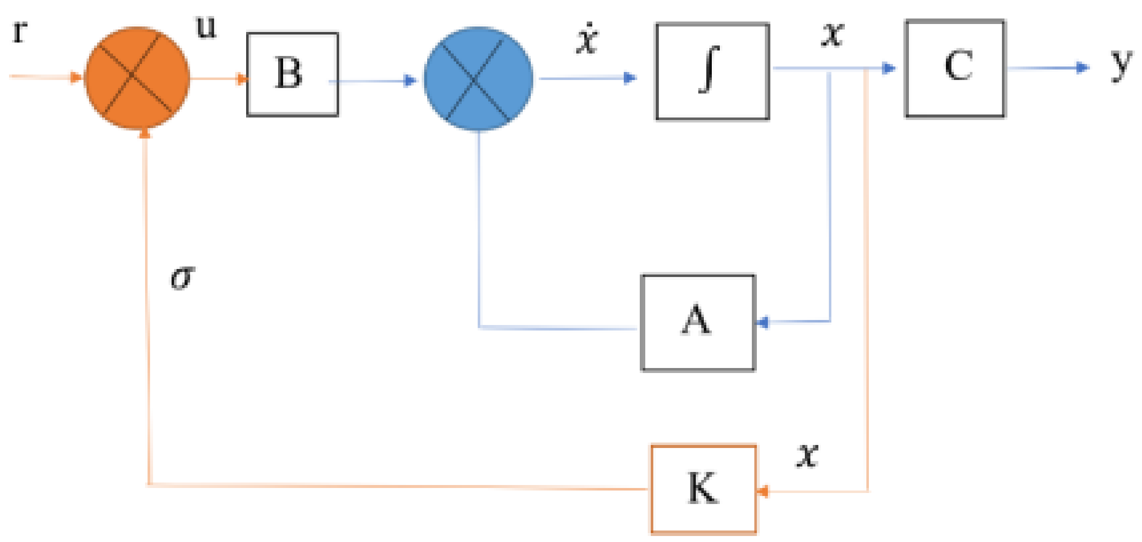
a) Check Controllability of the system

b) Determine the Characteristic equation of the original system

c) Determine the transformation matrix which converts given state model to Controllable Canonical form (CCF) (If the state model is already in CCF P = I, Identity matrix with order same as that of the system matrix A)

d) Determine the state feedback gain matrix

3. Block Diagram



Block Diagram of the system with state feedback

4. Calculation

$$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 16.3106 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1.0637 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -1.4458 \\ 0 \\ 0.9639 \end{bmatrix}$$

$$\text{Poles} = [-1.3 \quad -1 \quad -2 \quad -4]$$

Step-1) Check Controllability of the Given System.

Controllability matrix

$$Q_c = [B \quad AB \quad A^2B \quad A^3B]$$

$$= \begin{bmatrix} 0 & -1.4458 & 0 & -23.5819 \\ -1.4458 & 0 & -23.5819 & 0 \\ 0 & 0.9639 & 0 & 1.5379 \\ 0.9639 & 0 & 1.5379 & 0 \end{bmatrix}$$

$$Q_c \neq 0$$

$\Rightarrow$ System is completely controllable.

Step-2) Characteristic equation

$$|\lambda I - A| = 0$$

$$\Rightarrow \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 16.3106 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1.0637 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \lambda^4 - 16.3106\lambda^2 = 0$$

Here, $a_1 = 0$ (coefficient of λ^3)

$a_2 = -16.3106$ (coefficient of λ^2)

$a_3 = 0$ (coefficient of λ)

$a_4 = 0$

Step-3) Poles $= [-1.3 \quad -1 \quad -2 \quad -4]$

$$(\lambda+1.3)(\lambda+1)(\lambda+2)(\lambda+4)$$

$$\Rightarrow \lambda^4 + 8.3\lambda^3 + 23.1\lambda^2 + 26.2\lambda + 10.4 = 0$$

$b_1 = 8.3$

$b_2 = 23.1$

$b_3 = 26.2$

$b_4 = 10.4$

Transformation matrix $P_c = \begin{bmatrix} P_1 \\ P_1 A \\ P_1 A^2 \\ P_1 A^3 \end{bmatrix}$

where $P_1 = [0 \quad 0 \quad 0 \quad 1] Q_c^{-1}$

$$Q_c^{-1} = \begin{bmatrix} 0 & 0.0750 & 0 & 1.1499 \\ 0.0750 & 0 & 1.1499 & 0 \\ 0 & -0.0470 & 0 & -0.0705 \\ -0.0470 & 0 & -0.0705 & 0 \end{bmatrix}$$

$$\therefore P_c = \begin{bmatrix} -0.0470 & 0 & -0.0705 & 0 \\ 0 & -0.0470 & 0 & -0.0705 \\ -0.6917 & 0 & 0 & 0 \\ 0 & -0.6917 & 0 & 0 \end{bmatrix}$$

Step-4) State feedback gain matrix

$$K = [b_4 - a_4 \quad b_3 - a_3 \quad b_2 - a_2 \quad b_1 - a_1] P_c$$

$$= [10.4 \quad 26.2 \quad 39.4106 \quad 8.3] \begin{bmatrix} -0.0470 & 0 & -0.0705 & 0 \\ 0 & -0.0470 & 0 & -0.0705 \\ -0.6917 & 0 & 0 & 0 \\ 0 & -0.6917 & 0 & 0 \end{bmatrix}$$

$$= [-27.7475 \quad -6.9723 \quad -0.7332 \quad -1.8472]$$

5. MATLAB Code

```matlab
    % Get the dimensions of matrices A and B
    dim_A = size(A, 1);
    dim_B = size(B, 1);

    % Initialize the matrix Qc to store intermediate values
    Qc = zeros(dim_A, dim_A);

    % Calculate Qc matrix using matrix exponentiation
    for element = 1:dim_A
        a_temp = A^(element-1);
        A_val = a_temp * B;
        Qc(:, element) = A_val;
    end

    % Compute the inverse of the Qc matrix
    Qc_inv = inv(Qc);

    % Initialize the first row of the desired closed-loop system matrix P
    P_row1 = zeros(1, dim_A);
    P_row1(end) = 1; % Set the last element to 1
    P1 = P_row1 * Qc_inv;

    % Initialize the Pc matrix
    Pc = zeros(dim_A, dim_A);

    % Calculate the entire Pc matrix using matrix exponentiation
    for i = 1:dim_A
        p_temp = A^(i-1);
        p_val = P1 * p_temp;
        Pc(i, :) = p_val;
    end

    % Create a diagonal matrix from the desired closed-loop poles P0
    P0_mtx = diag(P0);

    % Calculate the coefficients of the characteristic polynomial of A
    A_coeff = poly(eig(A));

    % Calculate the coefficients of the characteristic polynomial of P0
    B_coeff = poly(P0);

    % Compute the difference between the two sets of coefficients
    Diff_Matrix = B_coeff - A_coeff;

    % Flip the difference matrix for compatibility
    K_mod = flip(Diff_Matrix);

    % Remove the last element (constant term) from K_mod
    K = K_mod(1: end-1) * Pc;

    % Display various matrices and results for debugging
    disp("A:"); disp(A);
    disp("B:"); disp(B);
    disp("Qc:"); disp(Qc);
    disp("A_coeff:"); disp(A_coeff);
    disp("B_coeff:"); disp(B_coeff);
    disp("Pc:"); disp(Pc);
    disp("K:"); disp(K);
end
```

6. Output in Terminal

```
A:
         0    1.0000         0         0
   16.3106         0         0         0
         0         0         0    1.0000
   -1.0637         0         0         0

B:
         0
   -1.4458
         0
    0.9639

Qc:
         0   -1.4458         0  -23.5819
   -1.4458         0  -23.5819         0
         0    0.9639         0    1.5379
    0.9639         0    1.5379         0

A_coeff:
    1.0000   -0.0000  -16.3106         0         0

B_coeff:
    1.0000    8.3000   23.1000   26.2000   10.4000

Pc:
   -0.0470         0   -0.0705         0
         0   -0.0470         0   -0.0705
   -0.6917         0         0         0
         0   -0.6917         0         0

K:
  -27.7475   -6.9723   -0.7332   -1.8472

>>
```

7. Conclusion:

- **Successful Design and Implementation:** The experiment aimed to design a state feedback controller to stabilize a control system. The fact that the manually calculated and code-calculated feedback matrix values match indicates that the design and implementation of the state feedback controller were successful. This suggests that the control theory and methods applied in the experiment are consistent and accurate.

- **Validation of Control Theory:** The consistency between the manual and code calculated feedback matrix values validates the control theory and mathematical methods used in the experiment. This demonstrates that the theoretical principles of state feedback control, such as pole placement or eigenvalue assignment, have been effectively applied and verified in practice.

- **Stability Achievement:** Since the objective of the experiment was to make the closed loop system stable, the matching feedback matrix values affirm that the chosen control strategy has indeed achieved system stability. This is a fundamental goal in control system design, as it ensures that the system responds predictably and without oscillations to control inputs.

In summary, this lab experiment emphasizes the significance of feedback control in stabilizing dynamic systems, demonstrates the consistency between manual and automated calculations, underscores the importance of accurate mathematical modelling, and highlights the computational aspects involved in control design. It also reinforces the practical relevance of feedback control in engineering applications.

Lab Calculation:

$Q = C$

$S_{12} = -1 \pm i$

$A = \begin{bmatrix} 0 & 10 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Q = \begin{bmatrix} 0 & 10 \\ 1 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 10 \\ 1 & 2 \end{bmatrix}_{2\times 2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2\times 1}$

$\boxed{|Q_c| \neq 0}$ $= \begin{bmatrix} 10 \\ 2 \end{bmatrix}$

controlable

$\begin{bmatrix} \lambda - 0 & -10 \\ 1 & \lambda - 2 \end{bmatrix} = 0$

$\boxed{\phi(s) = \{s - (-1+i)\}\{s - (-1-i)\}}$

$= \{s + 1 - i\}\{s + 1 + i\}$

$= (s+1)^2 - i^2$

$= (s+1)^2 + 1$

$\boxed{\phi(s) = s^2 + 2s + 2}$

$$Q_c^{-1} = \frac{\begin{bmatrix} 2 & -10 \\ -1 & 0 \end{bmatrix}}{10}$$

$$\boxed{Q_c^{-1} = \begin{bmatrix} 0.2 & -1 \\ -0.1 & 0 \end{bmatrix}}$$

$$P_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}_{1\times 2} \begin{bmatrix} 0.2 & -1 \\ -0.1 & 0 \end{bmatrix}_{2\times 2}$$

$$P_1 =$$

$$\boxed{\phi(A) = [A]^2 + 2A + 2I}$$

$$K_p = [0 \ 1] \, Q_c^{-1} \phi(A)$$

$$\phi(A) = \begin{bmatrix} 0 & 10 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 10 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 20 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 20 \\ 2 & 14 \end{bmatrix} + \begin{bmatrix} 2 & 20 \\ 2 & 6 \end{bmatrix}$$

$$\boxed{\phi(A) = \begin{bmatrix} 12 & 40 \\ 4 & 20 \end{bmatrix}}$$

$$K_p = \begin{bmatrix} 0 & 1 \end{bmatrix}_{1\times 2} \begin{bmatrix} 0.2 & -1 \\ -0.1 & 0 \end{bmatrix}_{2\times 2} \begin{bmatrix} 12 & 40 \\ 4 & 20 \end{bmatrix}$$

$|Q_c| \neq 0 \Rightarrow |Q_c| = -1$

$$\phi(s) = \{s+1-j\sqrt{3}\}\{s+1+j\sqrt{3}\}\{s+10\}$$

$$= \left((s+1)^2+3\right)(s+10)$$

$$= \left(s^2+2s+4\right)(s+10)$$

$$= s^3+10s^2+2s^2+20s+4s+40$$

$$= s^3+12s^2+24s+40$$

$$\phi(A) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} +$$

$$12 \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & -3 & 3 \end{bmatrix}$$

12

$$A^3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -2 & -5 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & -2 & -5 \end{bmatrix} + \begin{bmatrix} 0 & 12 & 12 \\ 0 & 0 & -12 \\ 0 & 12 & 36 \end{bmatrix} + \begin{bmatrix} 0 & 24 & 0 \\ 0 & 24 & 24 \\ 0 & -24 & -48 \end{bmatrix}$$

$$+ \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & -2 & -5 \end{bmatrix} + \begin{bmatrix} 0 & 36 & 12 \\ 0 & 24 & 12 \\ 0 & -12 & -12 \end{bmatrix} + \begin{pmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{pmatrix}$$

$$\begin{bmatrix} 0 & 36 & -11 \\ 0 & 25 & 10 \\ 0 & -14 & -17 \end{bmatrix} + \begin{pmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{pmatrix}$$

$$\phi(A) = \begin{bmatrix} 40 & 36 & -11 \\ 0 & 65 & 10 \\ 0 & -14 & 23 \end{bmatrix}$$

$$K_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}_{1\times3} \begin{bmatrix} -9 & 2 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 40 & 36 & -1 \\ 0 & 65 & 10 \\ 0 & -14 & 23 \end{bmatrix}$$

$$K_p = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 40 & 36 & -1 \\ 0 & 65 & 10 \\ 0 & -14 & 23 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 36 & +11 \end{bmatrix}$$ { Q_c calc once again }

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$S_1 = -2 \quad S_2 = -1+j \quad S_3 = -1-j$

$$Q_c = [B \quad AB \quad A^2B]$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -3 & -4 \\ 0 & 12 & 13 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 1 \\ -4 \\ 13 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix} \qquad \underline{\underline{Q_c = -1}}$$

$$Q_c^{-1} = \begin{bmatrix} 3 & 4 & 1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$(s+2)\left[(s+1)^2+1\right]$$

$$(s+2)(s^2+2s+2)$$

$$s^3+2s^2+2s+2s^2+4s+4$$

$$s^3+4s^2+6s+4$$

$$A^3 = \begin{bmatrix} 0 & -3 & -4 \\ 0 & 12 & 13 \\ 0 & -39 & -40 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & -4 \\ 0 & 12 & 13 \\ 0 & -39 & -40 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 4 \\ 0 & -12 & -16 \\ 0 & 48 & 52 \end{bmatrix} + \begin{bmatrix} 0 & 6 & 0 \\ 0 & 0 & 6 \\ 0 & -18 & -24 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & -3 \\ 0 & 9 & 12 \end{bmatrix} + \begin{bmatrix} 4 & 6 & 0 \\ 0 & 4 & 6 \\ 0 & -18 & -20 \end{bmatrix}$$

$$\phi(A) = \begin{bmatrix} 4 & 3 & 0 \\ 0 & 4 & 3 \\ 0 & -9 & -8 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ M & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 \\ 0 & 4 & 3 \\ 0 & -9 & -8 \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \cancel{\begin{bmatrix} 4 & 3 & 0 \\ 0 & 4 & 3 \\ 0 & -9 & -8 \end{bmatrix}}$$

$$\boxed{\cancel{K = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}}} \qquad = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ \end{bmatrix}$$

$$\boxed{K = \begin{bmatrix} 4 & 3 & 0 \end{bmatrix}}$$

12

$$H(s) = \frac{8s^3 - 6s^2 - 12s - 4}{4s^3 + 3s^2 - 6s + 2}$$

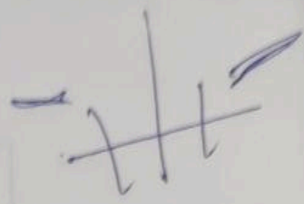
$$s_{1,2} = -1 \pm j\sqrt{2}$$
$$s_3 = -3$$

$$Q_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 6 & -3 \end{bmatrix}$$

$$\boxed{Q_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 15 \end{bmatrix}}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 6 & -3 \\ 6 & -20 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 15 \end{bmatrix}$$

$$Q_c^{-1} = \begin{bmatrix} -6 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



$$(s+3)\left((s+1)^2+2\right)$$

$$(s+3)\,(s^2+2s+3)$$

$$s^3+2s^2+3s+3s^2+6s+9$$

$$\boxed{s^3+5s^2+9s+9}$$

$$A^3 = \begin{bmatrix} -2 & 6 & -3 \\ 6 & -20 & 15 \\ -30 & 96 & -65 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ -10 & 30 & -15 \\ 30 & -100 & 75 \end{bmatrix} + \begin{bmatrix} 0 & 9 & 0 \\ 0 & 0 & 9 \\ -18 & 54 & -27 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 6 & 2 \\ -4 & 10 & 0 \\ 0 & -4 & 10 \end{bmatrix} + \begin{bmatrix} 9 & 9 & 0 \\ 0 & 9 & 9 \\ -18 & 54 & -18 \end{bmatrix}$$

$$\boxed{\phi(A) = \begin{bmatrix} 7 & 15 & 2 \\ -4 & 19 & 9 \\ -18 & 50 & -8 \end{bmatrix}}$$