Part 1_Ch 02_미적분학

date	@2023/04/19
time	19:00
class	Monthly Task
category	Study
상태	☐
note	
day	수요일
goal	
period	

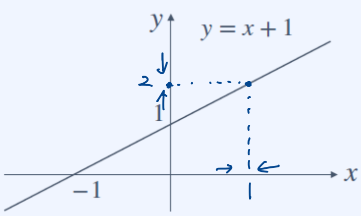
Chapter 02. 미적분학

2.1. Limits

Limits : 극한

$f(x)$ approaches L as x approaches a

$$\lim_{x \to a} f(x) = L$$: x가 a로 접근할 때, y가 L에 수렴한다.



$$x \to 1 \longrightarrow y \to 2 \longrightarrow \lim_{x \to 1} y = \lim_{x \to 1} f(x) = 2$$

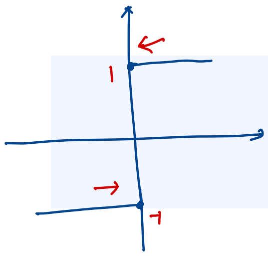
$$x \to 0 \longrightarrow y \to 1 \longrightarrow \lim_{x \to 0} f(x) = 1$$

$$x \to -1 \longrightarrow y \to 0 \longrightarrow \lim_{x \to -1} f(x) = 0$$

Left / Right-hand Limits : 좌극한, 우극한

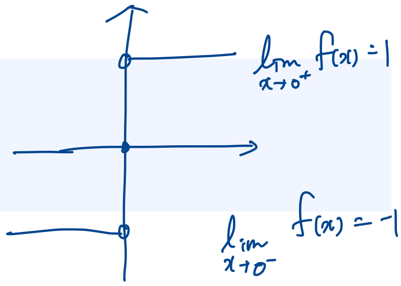
: 극한값의 정의를 위해!

$y = \begin{cases} 1, & x \ge 0 \\ -1, & x < 0 \end{cases}$



좌극한: left-hand limits $\lim_{x \to a^-} f(x)$

우극한: right-hand limits $\lim_{x \to a^+} f(x)$

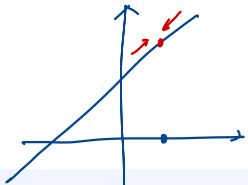


Existence of Limits : 극한값의 존재 조건

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Continuity

연속이기 위한 조건
① 극한값 = 함수값



$f(x)$ is continuous at $x = a$, if $\lim_{x \to a} f(x) = f(a)$

└ 극한값과 함수값이 같을 때, 연속이다

Continuity on Intervals

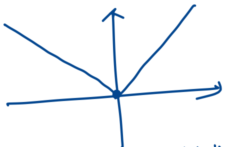
$x_1 \le x \le x_2$

$\forall a \in [x_1, x_2],\ \lim_{x \to a} f(x) = f(a) \quad \longrightarrow \quad f(x)$ is continuous on $[x_1, x_2]$

$\forall a \in D,\ \lim_{x \to a} f(x) = f(a) \quad \longrightarrow \quad f(x)$ is continuous

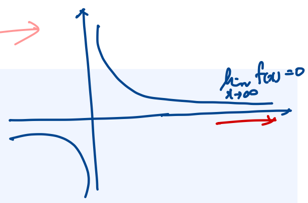
연속함수

$y = |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

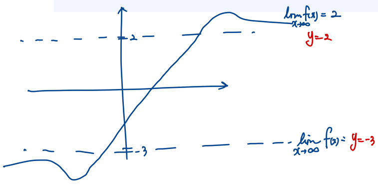


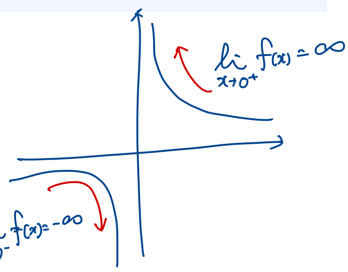
Limits and Asymptotes : 점근선을 따져볼 때, 수렴값과 발산값을 알아볼 수 있다.

horizontal asymptotes $\lim_{x \to \pm\infty} f(x) = L$



vertical asymptotes $\lim_{x \to a} f(x) = \pm\infty$





2.2. Differentiation and Derivatives

Rates of Change

평균 변화율

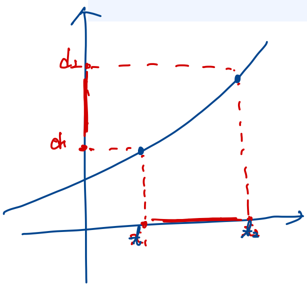
Average Rates of Change

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

순간변화율 : 접선에서의 기울기

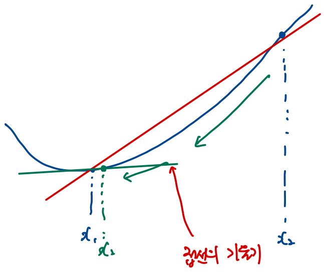
Instantaneous Rates of Change

$$\frac{\delta y}{\delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



$\frac{d_2 - d_1}{t_2 - t_1}$: 평균의 변화량

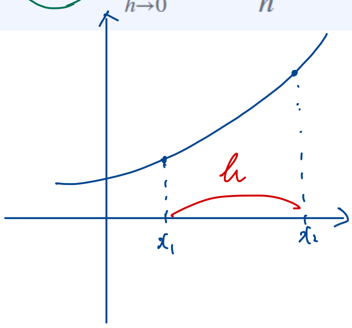
$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$



Differential Coefficients

미분계수 : 그 점에서 순간 변화율

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1+h) - f(x_1)}{(x_1+h) - x_1} \Rightarrow \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Derivatives : 도함수

도함수에 x 값을 넣으면 미분계수 값이 나온다.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

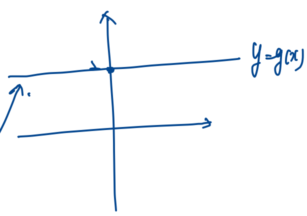
ex)

$y = 2x$

$f(x+h) = 2(x+h)$

$\lim_{h \to 0} \frac{2(x+h) - 2x}{h}$

$= \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2$



$\Rightarrow y = 2x$

$y' = 2$

Notations

$$f'(x)\ /\ y'$$

$$\frac{df(x)}{dx} \Big/ \frac{dy}{dx} \Big/ \frac{d}{dx}[f(x)]$$

Derivative Formulas

$$f(x) = c \longrightarrow f'(x) = 0$$

$$f(x) = x^n \longrightarrow f'(x) = nx^{n-1}$$

$$f(x) = a^x \longrightarrow f'(x) = a^x \cdot ln(a)$$

$$f(x) = log_a x \longrightarrow f'(x) = \frac{1}{x \cdot ln(a)}$$

자연상수는 simple한 결과를 만들 수 있다

$$f(x) = e^x \longrightarrow f'(x) = e^x \cdot \cancel{ln(e)}^{\,1} = e^x$$

$$g(x) = ln(x) \rightarrow g'(x) = \frac{1}{x \cdot \cancel{ln\,e}^{\,1}} = \frac{1}{x}$$

$$f(x) = sin(x) \longrightarrow f'(x) = cos(x)$$

$$f(x) = cos(x) \longrightarrow f'(x) = -sin(x)$$

$$f(x) = tan(x) \longrightarrow f'(x) = sec^2(x) = \frac{1}{cos^2 x}$$

Differentiation Rules

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$$: 도함수에 그대로 상수배.

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{\{g(x)\}^2}$$

Note! linearity of differentiation

Chain Rule

합성함수의 미분

$$\frac{d}{dx}\left[f(g(x))\right] = \underline{f'(g(x))} \cdot \underline{g'(x)}$$

$f(g(x))$: 합성함수 ⇒ $x \to \boxed{g} \to \boxed{f} \to y$

$$y = f(g(h(x))) \Rightarrow \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

$$x \longrightarrow \boxed{u = h(x)} \longrightarrow \boxed{v = g(u)} \longrightarrow \boxed{y = f(v)} \longrightarrow y$$

$\frac{du(x)}{dx} = \frac{du}{dx}$ $\quad \frac{dg(u)}{du} = \frac{dv}{du}$ $\quad \frac{df(v)}{dv} = \frac{dy}{dv}$

ex) $\overbrace{(\underbrace{e^x + x^2}_{g(x)})^3}^{f(x)} = f(g(x)) \to f'(g(x)) \cdot g'(x)$

$e^x + x^2 = g(x) \longrightarrow u = e^x + x^2 \to \frac{du}{dx} = e^x + 2x$

$x^3 = f(x) \longrightarrow y = u^3 \to \frac{dy}{du} = 3u^2 = 3(e^x + x^2)^2$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3(e^x + x^2)^2 \cdot (e^x + 2x)$

2.3. Gradient-based Optimization

Case Study: Quadratic Functions

$$x := x - \alpha \frac{dy}{dx}$$

2.4. Matrix Calculus

Multivariate Functions

$$\begin{tikzcd}
& f(x_1, x_2) = y \arrow[dl] \arrow[dr] & \\
\dfrac{\partial f}{\partial x_1} & & \dfrac{\partial f}{\partial x_2}
\end{tikzcd}$$

$$\begin{tikzcd}
& & \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, f(\mathbf{x}) = y \arrow[dll] \arrow[dl] \arrow[dr] & \\
\dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots & \dfrac{\partial f}{\partial x_n}
\end{tikzcd}$$

Gradients

$$f(x_1, x_2) = y \quad \longrightarrow \quad \nabla_{(x_1, x_2)} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix}$$

$$f(\mathbf{x}) = y \quad \longrightarrow \quad \nabla_{\mathbf{x}} f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} & \frac{\partial f(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f(\mathbf{x})}{\partial x_n} \end{pmatrix}$$

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = \frac{\partial f}{\partial \mathbf{x}}$$

Gradient-based Optimization

$$\mathbf{x} := \mathbf{x} - \alpha \nabla_{\mathbf{x}} f(\mathbf{x})$$

Optimizations of Artificial Neurons

$$z = f(\mathbf{x};\theta) = \theta_n x_n + \theta_{n-1}x_{n-1} + \ldots + \theta_1 x_1 + \theta_0$$
$$a = g(z) = \frac{1}{1+e^{-z}} \longrightarrow \nu(\mathbf{x};\theta) = g(f(\mathbf{x};\theta))$$

$$\theta := \theta - \alpha\frac{\partial \nu(\mathbf{x};\theta)}{\partial \theta}$$

Vector Functions

$$\begin{matrix} f_1(x) = y_1 \\ f_2(x) = y_2 \\ \vdots \\ f_m(x) = y_m \end{matrix} \longrightarrow \mathbf{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

Vector Functions and Gradients

$$\mathbf{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix} \longrightarrow \frac{\partial \mathbf{f}}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial x} \\ \vdots \\ \frac{\partial f_m}{\partial x} \end{pmatrix}$$

Multivariate and Vector Functions

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{matrix} f_1(\mathbf{x}) = y_1 \\ f_2(\mathbf{x}) = y_2 \\ \vdots \\ f_m(\mathbf{x}) = y_m \end{matrix} \quad \longrightarrow \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

General Jacobians

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} \quad \longrightarrow \quad \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial \mathbf{x}} \\ \frac{\partial f_2}{\partial \mathbf{x}} \\ \vdots \\ \frac{\partial f_m}{\partial \mathbf{x}} \end{pmatrix} = \begin{pmatrix} \nabla_{\mathbf{x}} f_1 \\ \nabla_{\mathbf{x}} f_2 \\ \vdots \\ \nabla_{\mathbf{x}} f_m \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

$$\mathbf{f} \in \mathbb{R}^m, \ \mathbf{x} \in \mathbb{R}^n \quad \longrightarrow \quad \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \in \mathbb{R}^{m \times n}$$

Matrix Calculus

$$
\begin{array}{ccccccc}
\mathbf{x} \in \mathbb{R}^{\alpha} & \longrightarrow & \mathbf{f}(\mathbf{x}) = \mathbf{u} \in \mathbb{R}^{\beta} & \longrightarrow & \mathbf{g}(\mathbf{u}) = \mathbf{v} \in \mathbb{R}^{\gamma} & \longrightarrow & h(\mathbf{v}) = y \in \mathbb{R} \\
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_\alpha \end{pmatrix} & & \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_\beta(\mathbf{x}) \end{pmatrix} & & \mathbf{g}(\mathbf{u}) = \begin{pmatrix} g_1(\mathbf{u}) \\ g_2(\mathbf{u}) \\ \vdots \\ g_\gamma(\mathbf{u}) \end{pmatrix} & & \\
& & \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \in \mathbb{R}^{\beta \times \alpha} & & \frac{\partial \mathbf{v}}{\partial \mathbf{u}} \in \mathbb{R}^{\gamma \times \beta} & & \frac{\partial y}{\partial \mathbf{v}} \in \mathbb{R}^{1 \times \gamma}
\end{array}
$$

$$
\frac{\partial y}{\partial \mathbf{x}} = \frac{\partial y}{\partial \mathbf{v}} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{u}} \cdot \frac{\partial \mathbf{u}}{\partial \mathbf{x}}
$$

2.5. Integration

Equally Spaced Intervals

$$
\Delta x = \frac{b - a}{n}
$$

Mensuration by Parts

$$
S' = \sum_{k=0}^{n-1} f(a + \Delta x \cdot k) \cdot \Delta x
$$

Definite Integrals

$$[n \to \infty] \longrightarrow \left[\sum_{k=0}^{n-1} f(a + \Delta x \cdot k) \cdot \Delta x \to S\right]$$

$$S = \lim_{n \to \infty} \sum_{k=0}^{n-1} f\left(a + \frac{b-a}{n} \cdot k\right) \cdot \frac{b-a}{n}$$

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} f\left(a + \frac{b-a}{n} \cdot k\right) \cdot \frac{b-a}{n} = \int_a^b f(x)dx$$

note! definite integrals are not areas

Indefinite Integrals

$$\int f(x)dx = F(x) + C$$

Integral Formulas

$$\int f(x)dx = F(x) + C$$

$$f(x) = a \longrightarrow \int f(x)dx = ax + C$$

$$f(x) = x^n \longrightarrow \int f(x)dx = \frac{1}{n+1}x^{n+1} + C$$

$$f(x) = a^x \longrightarrow \int f(x)dx = \frac{a^x}{ln(a)} + C$$

$$f(x) = \frac{1}{x} \longrightarrow \int f(x)dx = ln\,|\,x\,| + C$$

$$f(x) = sin(x) \longrightarrow \int f(x)dx = -\,cos(x) + C$$

$$f(x) = cos(x) \longrightarrow \int f(x)dx = sin(x) + C$$

Integration Rules

$$\int \left[k \cdot f(x)\right] dx = k \cdot \int f(x) dx$$

$$\int \left[f(x) \pm g(x)\right] dx = \int f(x) dx \pm \int g(x) dx$$

Fundamental Theorem of Calculus

Part.2 $\int f(x) dx = F(x) + C \longrightarrow \int_a^b f(x) dx = F(b) - F(a)$