21-127-T: Concepts of Mathematics

Homework 5

Due date: 6/19/2015, 11:59 PM

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Collaborators:

Question 1

Compute the following quantities:

Part a

$(a+b)^5 = (\text{mod } 5)$

Answer: $(a+b)^5 = (\text{mod } 5) =$
$= a^5 + b^5 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 =$
$= a^5 + b^5 + 5(a^4b^1 + 2a^3b^2 + 2a^2b^3 + a^1b^4)(\text{mod } 5) =$
$= a^5 + b^5$

Part b

$(a+b)^3 = ?(\text{mod } 3)$

Answer: $(a+b)^3 = (\text{mod } 3) =$
$= a^3 + b^3 + 3a^2b + 3ab^2 =$
$= a^3 + b^3 + 3(a^2b + ab^2)(\text{mod } 3) =$
$= a^3 + b^3$

Part c

$(a+b)^2 = ?\ (\text{mod } 2)$

Answer:
$(a+b)^2 = (\text{mod } 2) =$
$= a^2 + b^2 + 2ab =$
$= a^2 + b^2 + 2ab(\text{mod } 2) =$
$= a^2 + b^2$

Part d

Let p be a positive prime. $(a+b)^p = ?\ (\text{mod } p)$

Answer:

$(a+b)^p = (\text{mod } p) =$

Using binomial expansion:

$(a+b)^p = \binom{p}{0}a^p b^0 + \binom{p}{1}a^{p-1}b^1 + ... + \binom{p}{1}a^1 b^{p-1} + \binom{p}{p}a^0 b^p$

By computing binomial coefficients:

$= a^p + b^p + p(a^{p-1}b^1 + 2a^{p-2}b^2 + ...2a^2 b^{p-2} + a^1 b^{p-1}) =$

$a^p + b^p + p(a^{p-1}b^1 + 2a^{p-2}b^2 + ...2a^2 b^{p-2} + a^1 b^{p-1})(\text{mod } p) = = a^p + b^p$ So, $(a+b)^p(\text{mod } p) = a^p + b^p$

Question 2

Let $p > 1$ be an integer. Suppose the following property holds

$$\forall a, b \in \{0, ..., p-1\}, \langle ab = 0 \text{ (mod p)}\rangle \Rightarrow \langle\langle a = 0 \text{ (mod p)}\rangle \vee \langle b = 0 \text{ (mod p)}\rangle\rangle$$

Prove that p is prime.

Answer: Proof:
By contradiction,assume $\exists a, b \in \{0, ..., p-1\}$ s.t $ab(modp) = 0$ and $a(modp) \neq 0^b(modp) \neq 0$
So, if $abmodp$ then $\exists k \in Z$ s.t $ab = pk$
Then $amodp \neq 0$ and $bmodp \neq 0$
Observe that this is absurd
So there exists prime number p that satisfies the above property

Question 3

Part a

Let $\{S_i\}_{i\in\mathbb{N}}$ be a collection of subsets of the real line satisfying the following property

1. $i < j \Rightarrow S_j \subseteq S_i$
2. $\forall\epsilon > 0, \exists n(\epsilon) \in \mathbb{N}$ s.t. $S_{n(\epsilon)}$ has measure at most ϵ

Prove that if $S \subseteq \bigcap_{i\in\mathbb{N}} S_i$ then S has measure zero.

Answer: Prrof:
By definition of measure, if $\forall\epsilon > 0, \exists n(\epsilon) \in \mathbb{N}$ s.t. $S_{n(\epsilon)}$ has measure at mostϵ then $\bigcap_{i\in\mathbb{N}} S_i$ has measure zero.
Choose $S_i \subseteq \bigcap_{i\in\mathbb{N}} S_i$
Then $\forall\epsilon > 0, \exists n(\epsilon) \in \mathbb{N}$ S_i has $\epsilon > 0$
Observe that subset has the same sequence of intervals

Part b

Prove the Cantor Set has measure zero.

Answer: Proof:Let C_k denote the kth element of Cantor set C
According to Cantor set definition, $C_0 = [0, 1]$,$[0, 1/3] \cup [2/3, 1]$ and so on, and $bigcap_i C_i$
Adding up size of intervals of Cantor set give us:
$1/3 + 2(1/3)^2 + 2^2(1/3)^2 + 2^3(1/3)^3 + ... + 2^{i-1}(1/3)^i$
So C is the complement of a set size $1/3 + 2(1/3)^2 + 2^2(1/3)^2 + 2^3(1/3)^3 + ... = 1$,observe that it has measure 0 as it has 1:1 correspondence with rational numbers

Question 4

Do problem 16 from page 105.

Answer: Proof:
By definition of upper bound, $\exists x \in R$ s.t $x \geq s, \forall s \in S$
Observe that the complement of upper bound is $x \in R$ s.t $x < s, \forall s \in S$
We see that the complement of $U(S)$ is an open set
The set is closed if its complement is open
Then by definition, $U(S)$ is closed set

Question 5

Formulate modular equality as a relation and show it's reflexive, symmetric, and transitive.

Answer: Reflexive.Equality is reflexive since for each $x \in R$, $x = x$

Symmetric. Equality is symmetric since for each $x, y \in R$ if $x = y$ then $y = x$

Transitive. Equqlity is transitive since for each $x, y, z \in R$ if $x = y$ and $y = z$ then $x = z$