CS480 Computational Photography:
Notes on Binary Image Processing

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In Figure 1

$$C = (-\rho \sin\theta, \rho\cos\theta)\ . \tag{1}$$

Thus, we can write the orientation axis that we will be estimating as a line passing through C with slope $\tan\theta$. Then an arbitrary point, (x_0, y_0), on the line will be expressed as

$$(x_0, y_0) = (-\rho\sin\theta + s\cos\theta, \rho\cos\theta + s\sin\theta)\ , \tag{2}$$

where s is the 1D parameter which encodes the signed distance from point C.

Then, for a point, (x, y), lying in the region whose orientation we are seeking for, its distance to an arbitrary point on the line becomes

$$\begin{aligned} r^2 &= (x - x_0)^2 + (y - y_0)^2 \\ &= (x^2 + y^2) - 2(x_0 x + y_0 y) + (x_0^2 + y_0^2) \\ &= (x^2 + y^2) + 2\rho(x\sin\theta - y\cos\theta) - 2s(x\cos\theta + y\sin\theta) + s^2 + \rho^2\ (\because (2)) \end{aligned} \tag{3}$$

The distance of the point, (x, y), to the line should be the shortest distance. So let us first find the point on the line, (x_0, y_0), that minimizes the distance (Eq. (3)). For this, we first take the derivative of Eq. (3) w.r.t. s (since that is our parameter of the line) and equate it to zero:

$$\frac{\partial r^2}{\partial s} = -2(x\cos\theta + y\sin\theta) + 2s = 0\ . \tag{4}$$

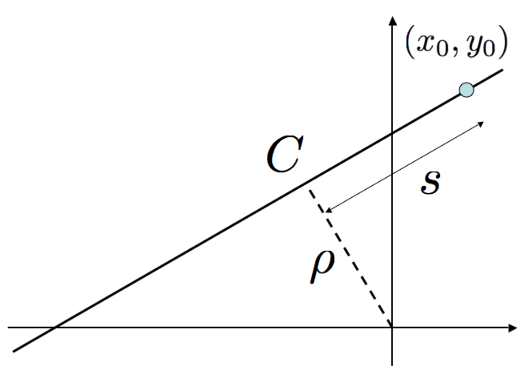

Figure 1: The orientation axis we want to estimate.

Thus, we obtain

$$s = x\cos\theta + y\sin\theta \tag{5}$$

Plugging Eq. (5) back to Eq. (2) gives us the point on the line, (x_0, y_0), that is closest to the point, (x, y):

$$\begin{aligned} x_0 &= -\rho\sin\theta + x\cos^2\theta + y\cos\theta\sin\theta & (6) \\ y_0 &= \rho\cos\theta + x\cos\theta\sin\theta + y\sin^2\theta\ . & (7) \end{aligned}$$

From this we can compute the shortest distance, which is the distance of the point, (x, y), to that closest point, (x_0, y_0), on the line, as

$$r^2 = (x - x_0)^2 + (y - y_0)^2\ ,$$

where

$$\begin{aligned} x - x_0 &= x + \rho\sin\theta - x\cos^2\theta - y\cos\theta\sin\theta \\ &= x(1 - \cos^2\theta) + \rho\sin\theta - y\cos\theta\sin\theta \\ &= x\sin^2\theta + \rho\sin\theta - y\cos\theta\sin\theta \\ &= sin\theta(x\sin\theta - y\cos\theta + \rho)\ , \end{aligned}$$

and similary

$$y - y_0 = -\cos(x\sin\theta - y\cos\theta + \rho)\ .$$

These result in

$$r^2 = (x\sin\theta - y\cos\theta + \rho)^2\ . \tag{8}$$

Recall that the line can be written as $x\sin\theta - y\cos\theta + \rho = 0$. Eq. (8) tells us that the residue after plugging the coordinate values into the line equation is the distance of that point to the line!

Now we know how to compute the distance of a point to a line. Recall that the orientation axis is the line that minimizes the second moment,

$$E = \int\int r^2 b(x, y)dxdy\ , \tag{9}$$

where $b(x, y)$ is the characteristic function of the binary image (1 inside the region of interest, 0 elsewhere). We want to find the line parameters (ρ, θ) of the orientation axis $x\sin\theta - y\cos\theta + \rho = 0$ which should actually minimize Eq. (9). So, let's plug Eq. (8) which tells us how the distance is expressed in terms of (ρ, θ) into Eq. (9) and minimize it w.r.t. (ρ, θ). We have

$$E = \int\int (x\sin\theta - y\cos\theta + \rho)^2 b(x, y)dxdy\ (\because (8))\ ,$$

and thus

$$\begin{aligned} \frac{\partial E}{\partial\rho} &= \int\int 2(x\sin\theta - y\cos\theta + \rho)b(x, y)dxdy \\ &= 2\left[\sin\theta\int\int xb(x, y)dxdy - \cos\theta\int\int yb(x, y)dxdy + \rho\int\int b(x, y)dxdy\right]\ . & (10) \end{aligned}$$

Now recall that

$$\begin{aligned} A &= \int\int b(x,y)dxdy \ (\textbf{Area}) , \\ (\bar{x},\bar{y}) &= \left(\frac{1}{A}\int\int xb(x,y)dxdy, \frac{1}{A}\int\int yb(x,y)dxdy\right) \ (\textbf{Position}) . \end{aligned}$$

Thus, Eq. (10) which should be equated to zero becomes

$$\begin{aligned} 2(\sin\theta A\bar{x} - \cos\theta A\bar{y} + \rho A) &= \\ A(\bar{x}\sin\theta - \bar{y}\cos\theta + \rho) &= 0 . \end{aligned} \tag{11}$$

Since the area, A, is non-zero,

$$\bar{x}\sin\theta - \bar{y}\cos\theta + \rho = 0 . \tag{12}$$

This tells us that the orientation axis, which is the line represented by (ρ,θ), passes through the position point, $(\bar{x},\bar{y})$. Thus, let us reparameterize the orientation axis as a line relative to the position point (translate the origin there):

$$(x',y') = (x-\bar{x}, y-\bar{y}) . \tag{13}$$

Then, from Eq. (8) and Eq. (13), we get

$$\begin{aligned} r^2 &= (x\sin\theta - y\cos\theta + \rho)^2 \\ &= [(x'+\bar{x})\sin\theta - (y'+\bar{y})\cos\theta + \rho]^2 \\ &= (x'\sin\theta - y'\cos\theta)^2 \ (\because (12)) . \end{aligned} \tag{14}$$

Plugging this into Eq. (9), we now have

$$\begin{aligned} E &= \int\int (x'\sin\theta - y'\cos\theta)^2 b(x,y)dxdy \\ &= \sin^2\theta\int\int (x')^2 b(x,y)dxdy - 2\sin\theta\cos\theta\int\int (x'y')b(x,y)dxdy + \cos^2\theta\int\int (y')^2 b(x,y)dxdy \\ &= a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta , \end{aligned} \tag{15}$$

where

$$\begin{aligned} a &= \int\int (x')^2 b(x,y)dxdy , \\ b &= 2\int\int (x'y')b(x,y)dxdy , \\ c &= \int\int (y')^2 b(x,y)dxdy . \end{aligned}$$

Using $\sin 2\theta = 2\sin\theta\cos\theta, \cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$,

$$E = \frac{1}{2}(a+c) - \frac{1}{2}(a-c)\cos 2\theta - \frac{1}{2}b\sin 2\theta .$$

If we take the derivative of E w.r.t. θ this time and equate it to zero, we get

$$\begin{aligned}\frac{\partial E}{\partial \theta} &= -(a-c)\sin 2\theta + b\cos 2\theta \\ &= 0\ .\end{aligned}$$

Thus,

$$\tan 2\theta = \frac{b}{a-c}\ . \tag{16}$$

This gives us

$$\sin 2\theta = \pm\frac{b}{\sqrt{b^2+(a-c)^2}}\ , \tag{17}$$

$$\cos 2\theta = \pm\frac{a-c}{\sqrt{b^2+(a-c)^2}}\ . \tag{18}$$

By checking the second order derivative of E w.r.t. θ, it can easily be shown that the positive values in Eq. (17) and Eq. (18) gives the θ that minimizes E (our solution for the orientation axis) and the negative values gives the θ that maximizes E.