CS 610: Midterm
Winter 2013

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February 27, 2013

0. (a) We consider three probabilities:

    - The probability of being in the initial start state, S_0, where:

    $$S_0 = (\texttt{LanguageA}/\text{Anti-Polka})$$
    $$P_0 = 0.7 \cdot 0.6 = 0.42$$

    - The probability of a transition from S_0 to S_1, where:

    $$S_1 = (\texttt{LanguageB}/\text{Anti-Polka})$$
    $$P_1 = 0.15 \cdot 0.7 = 0.105$$

    - The transition from S_1 to S_2, where:

    $$S_1 = (\texttt{LanguageC}/\text{Dummy})$$
    $$P_1 = 0.7 \cdot 0.3 = 0.21$$

    Thus, the probability that *at the beginning of the day* the machine will produce chips in the order $S_0 \to S_1 \to S_2$ is:

    $$\begin{aligned} P_0 \cdot P_1 \cdot P_2 &= 0.42 \cdot 0.105 \cdot 0.21 \\ &= 0.009261 \\ &\approx 1\% \end{aligned}$$

   (b) We consider the first, heavy TV. For simplicity, we use the following notation:

    $$\begin{aligned} P(C|W_h) &= P(\text{chip}_1, \text{chip}_2|\text{weight} = \text{heavy}) \\ P(W_h|C) &= P(\text{weight} = \text{heavy}|\text{chip}_1, \text{chip}_2) \\ P(W_h) &= P(\text{weight} = \text{heavy}) \\ P(C_s) &= P_{start}(\text{chip}_1, \text{chip}_2) \end{aligned}$$

    The probability for the chips at the start of the run, $P(C_s)$ is given in part (a) and is as follows:

	LanguageA	LanguageB	LanguageC
Anti-Polka	0.42	0.14	0.14
Dummy	0.18	0.06	0.06

We are given $P(W_h|C)$ by the table in the question. $P(W_h)$ is given as follows:

$$P(W_h) = \sum (P(W_h|C) \cdot P(C_s)) = \mathbf{0.486}$$

Using Bayes' theorem:

$$P(C|W_h) = \frac{P(W_h|C) \cdot P(C_s)}{P(W_h)}$$

Thus, the resulting values for $P(C|W_h)$ for the first, heavy TV are as follows:

$P(\text{Anti-Polka}, \texttt{LanguageA}\|\text{weight} = \text{heavy})$	**0.60**
$P(\text{Anti-Polka}, \texttt{LanguageB}\|\text{weight} = \text{heavy})$	**0.14**
$P(\text{Anti-Polka}, \texttt{LanguageC}\|\text{weight} = \text{heavy})$	**0.12**
$P(\text{Dummy}, \texttt{LanguageA}\|\text{weight} = \text{heavy})$	**0.11**
$P(\text{Dummy}, \texttt{LanguageB}\|\text{weight} = \text{heavy})$	**0.01**
$P(\text{Dummy}, \texttt{LanguageC}\|\text{weight} = \text{heavy})$	**0.01**

For the second, light TV, we use the following

$$\begin{aligned}
P(C|W_l) &= P(\text{chip}_1, \text{chip}_2|\text{weight} = \text{light}) \\
P(W_l|C) &= P(\text{weight} = \text{light}|\text{chip}_1, \text{chip}_2) \\
P(W_l) &= P(\text{weight} = \text{light}) \\
P(C_t) &= P_{transition}(\text{chip}_1, \text{chip}_2)
\end{aligned}$$

We no longer need to consider the probabilities of the starting state, but rather the transition between any two states:

	AP, `LA`	AP, `LB`	AP, `LC`	D, `LA`	D, `LB`	D, `LC`
AP, `LA`	0.490	0.105	0.105	0.210	0.045	0.045
AP, `LB`	0.105	0.490	0.105	0.045	0.210	0.045
AP, `LC`	0.105	0.105	0.490	0.045	0.045	0.210
D, `LA`	0.210	0.045	0.045	0.490	0.105	0.105
D, `LB`	0.045	0.210	0.045	0.105	0.490	0.105
D, `LC`	0.045	0.045	0.210	0.105	0.105	0.490

However, this probability of transition is assuming that we are equally likely to be in any given preceding state. As we have shown in the first half of the question where the first TV is heavy, in this case there is a known probability for being in a certain state before the transition. We must multiply this across the transition table to get the probability of transition if the first TV was heavy:

	AP, `LA`	AP, `LB`	AP, `LC`	D, `LA`	D, `LB`	D, `LC`
AP, `LA`	0.296	0.064	0.064	0.127	0.027	0.027
AP, `LB`	0.015	0.071	0.015	0.006	0.030	0.006
AP, `LC`	0.012	0.012	0.056	0.005	0.005	0.024
D, `LA`	0.023	0.005	0.005	0.054	0.012	0.012
D, `LB`	0.001	0.003	0.001	0.001	0.006	0.001
D, `LC`	0.001	0.001	0.003	0.001	0.001	0.006

And then sum the probabilities that result in the same state to get $P(C_t)$ for a TV following a heavy TV:

	`LanguageA`	`LanguageB`	`LanguageC`
Anti-Polka	0.348	0.154	0.143
Dummy	0.196	0.082	0.077

Again, using Bayes' theorem:

$$P(C|W_l) = \frac{P(W_l|C) \cdot P(C_t)}{P(W_l)}$$

and:

$$P(W_l) = \sum (P(W_l|C) \cdot P(C_t)) = \mathbf{0.211}$$

Gives us the following probabilities for $P(C|W_h)$ for the second, light TV:

$P(\text{Anti-Polka}, \texttt{LanguageA}\vert\text{weight} = \text{light})$	**0.165**
$P(\text{Anti-Polka}, \texttt{LanguageB}\vert\text{weight} = \text{light})$	**0.146**
$P(\text{Anti-Polka}, \texttt{LanguageC}\vert\text{weight} = \text{light})$	**0.204**
$P(\text{Dummy}, \texttt{LanguageA}\vert\text{weight} = \text{light})$	**0.186**
$P(\text{Dummy}, \texttt{LanguageB}\vert\text{weight} = \text{light})$	**0.116**
$P(\text{Dummy}, \texttt{LanguageC}\vert\text{weight} = \text{light})$	**0.182**

(c) We would create a Markov Decision Process and a stationary distribution to calculate the expected value for the percentage of TVs that have anti-polka chips at the end of a significantly large run. The probability of any given TV being heavy or light (i.e., not heavy) is simply:

$$P(W_h|\text{chip}_1 = AP) = \frac{\sum P(W_h|\text{chip}_2 = \{LA, LB, LC\})}{3}$$

$$P(W_{l,m}|\text{chip}_1 = AP) = \frac{\sum P(W_{l,m}|\text{chip}_2 = \{LA, LB, LC\})}{6}$$

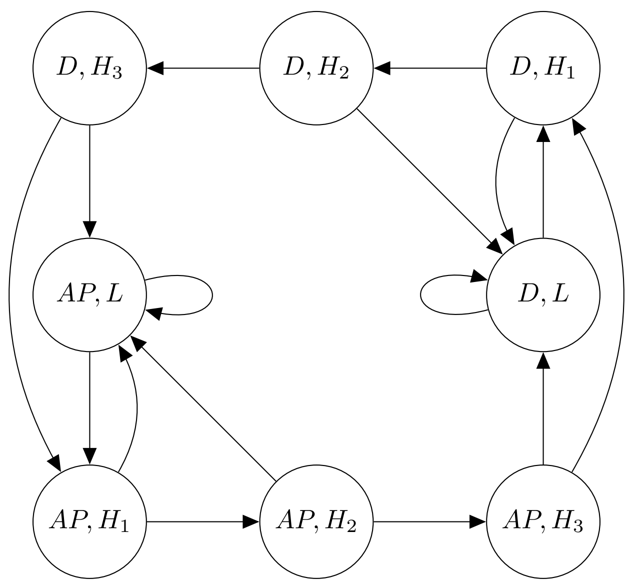
(and similarly for the case where the dummy chip is installed) This can be reduced from the given table as follows:

	H	L
AP	0.533	0.467
D	0.167	0.833

We also know with 100% probability when the chimp will flip the switch.

Here, we make the assumption that every time three heavy TVs drop off the conveyor belt, the chimp not only flips the switch, but resets his count of heavy TVs, so that if a *fourth* heavy TV drops off, he does not flip the switch again.

The following is a MDP diagram for the system, where AP is a TV with an anti-polka chip, D is a TV with a dummy chip, L is a TV that has been observed to be light (or medium), and $\{H_1, H_2, H_3\}$ correspond to the first, second and third heavy TVs that the chip has seen:



Using the MDP above, we can produce a stationary distribution from it and the probabilities for a given TV's weight:

	AP, L	AP, H_1	AP, H_2	AP, H_3	D, L	D, H_1	D, H_2	D, H_3
AP, L	0.467	0.533	0	0	0	0	0	0
AP, H_1	0.467	0	0.533	0	0	0	0	0
AP, H_2	0.467	0	0	0.533	0	0	0	0
AP, H_3	0	0	0	0	0.833	0.167	0	0
D, L	0	0	0	0	0.833	0.167	0	0
D, H_1	0	0	0	0	0.833	0	0.167	0
D, H_2	0	0	0	0	0.833	0	0	0.167
D, H_3	0.467	0.533	0	0	0	0	0	0

Then, using the following formula for a stationary distribution, where T is the transition matrix from the previous part:

$$\pi = [1, 0, \dots, 0] \cdot ((T - I)_1)^{-1}$$

We would sum the probabilities for the TVs with anti-polka chips installed to get the expected value for the percentage of TVs that have anti-polka chips at the end of the day.

(d) We find π to be:

AP, L	AP, H_1	AP, H_2	AP, H_3	D, L	D, H_1	D, H_2	D, H_3
0.021	0.013	0.007	0.004	0.796	0.134	0.022	0.004

Thus the expected value for the percentage of TVs that have anti-polka chips at the end of the day would be:

$$0.021 + 0.013 + 0.007 + 0.004 = 0.045 \approx \mathbf{4.5\%}$$