Control

M.Sc. in Electrical and Computer Engineering

2018/2019 - First Semester

Control of the Water Level in a Tank

Laboratory guide

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1 Introduction

The problem of controlling the water level in a tank is addressed in this laboratory. To this end, the system will be modeled and different controllers will be proposed and analyzed. Finally, the effects of unmodeled dynamics will also be studied. To test the system, numerical simulations are carried out resorting to Matlab/Simulink.

1.1 Objectives

The following goals are addressed in this laboratory:

1. Modeling of a simple nonlinear dynamic system;
2. Linearization of a nonlinear system about an equilibrium operating point;
3. Comparison of the nonlinear and linearized open-loop responses of a system;
4. Design and analysis of a proportional controller (P);
5. Design and analysis of an integral controller (I);
6. Analysis of the effects of unmodeled dynamics; and
7. Design and analysis of a proportional-integral controller (PI).

At the end of the laboratory, the student should have a thorough understanding of the linearization process, the properties of the different controllers studied, and the potential problems that unmodeled dynamics pose.

1.2 Organization and timeline

This guide consists of a set of questions that explore different aspects related to the control of water level in a tank. There are two kinds of questions: theoretical question, marked as (T), and laboratory questions, marked as (L). All theoretical questions must be solved before the first laboratory, in a first report. The laboratory questions should be addressed in the laboratory sessions and lead to a second report. Nevertheless, it is strongly suggested that the students read the entire guide, including also the questions marked as (L), in order to anticipate the laboratory work and relate it to the theoretical questions.

The laboratory component will take place in two sessions. At the end of the first session, the students must deliver the first report, in person. The second report must be delivered within one week after the second session. This second report is to be dropped in the mailbox of the *Área Científica de Sistemas, Decisão e Controlo, Piso 5 da Torre Norte*.

As a guideline for the laboratory component, in the first laboratory session Sections 3 to 5 should be addressed. Sections 6 and 7 should be addressed in the second laboratory session.

The first report consists simply of answers to all theoretical questions (T). However, in the second report the simulation results should always be commented and, whenever appropriate, these conclusions should be supported by appropriate analytical computations.

Both reports should use the cover available in the course's webpage as a frontpage. The first report may be handwritten.

1.3 Academic ethics code

All members of the academic community of the University of Lisbon (faculty, researchers, staff members, students, and visitors) are required to uphold high ethical standards. Hence, the report submitted by each group of students must be original and correspond to their actual work.

2 Modeling

Consider a water tank with free outflow, as depicted in Fig. 1.

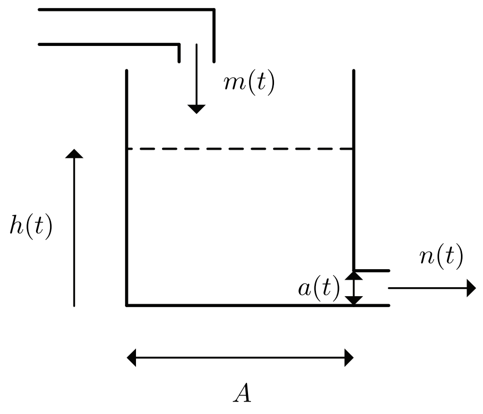

Figure 1: Free flow tank

Let $A > 0$ denote the area of the base and $h(t) \geq 0$ the level of the liquid in the tank. The inflow of the tank is denoted by $m(t)$, whereas the outflow is denoted by $n(t)$. The outflow valve has variable effective area, given by

$$a(t) = Cp^2(t), \tag{1}$$

where $p(t) \geq 0$ is a dimensionless control input that actuates the valve, and C is some known constant.

In order to derive a model of the tank, consider Bernoulli's law, which describes the relationship between the outflow speed $v(t)$ and the liquid level $h(t)$, given by

$$v(t) = \sqrt{2Gh(t)}, \tag{2}$$

where G is the acceleration of gravity. The outflow of the tank is given by

$$n(t) = a(t)v(t). \tag{3}$$

The volume of the liquid in the tank is

$$q(t) = Ah(t). \tag{4}$$

Assuming that the density of the liquid in the tank remains constant, the variation of volume satisfies

$$\frac{dq(t)}{dt} = m(t) - n(t). \tag{5}$$

2.1. **(T)** Compute the outflow of the tank as a function of its level $h(t)$ and the control input $p(t)$.

2.2. **(T)** Determine the nonlinear differential equation that models the evolution of the liquid level $h(t)$ as a function of the control input $p(t)$ and the inflow $m(t)$. Write it also in the form

$$\frac{dh(t)}{dt} = f\left(h(t), p(t), m(t)\right).$$

2.3. **(T)** Suppose that the tank is operated around an equilibrium point determined by a constant inflow M_{eq} and a constant level H_{eq}. Determine the corresponding control input, at equilibrium, P_{eq}.

2.4. **(T)** The equilibrium input P_{eq} does not depend on the area of the base of the tank. Give an intuitive explanation for that.

2.5. **(T)** Consider now incremental variables around the equilibrium point, i.e., let $h(t) := H_{eq} + x(t)$, $p(t) = P_{eq} + r(t)$, and $m(t) = M_{eq} + d(t)$, where $x(t)$ corresponds to small deviations of the liquid level around the equilibrium level H_{eq}, $r(t)$ corresponds to small deviations of the control input around the equilibrium input P_{eq}, and $d(t)$ corresponds to small deviations of the inflow around the equilibrium inflow M_{eq}. Derive the linear differential equation

$$\frac{dx(t)}{dt} = g\left(x(t), r(t), d(t)\right)$$

that approximately describes the system operating close to the equilibrium point.

Suggestion: Note that the Taylor series of a nonlinear function $w\left(x_1, x_2\right)$ about the point $\left(x_{10}, x_{20}\right)$ is given by

$$w\left(x_1, x_2\right) = w\left(x_{10}, x_{20}\right) + \frac{\partial w\left(x_{10}, x_{20}\right)}{\partial x_1}\left(x_1 - x_{10}\right) + \frac{\partial w\left(x_{10}, x_{20}\right)}{\partial x_2}\left(x_2 - x_{20}\right) + \ldots$$

2.6. **(T)** Show that the transfer function that describes the linearized system with input $r(t)$ and output $x(t)$, for $d(t) = 0$, can be written as

$$G_1(s) = K_1 \frac{p}{s+p}.$$

Determine the constants K_1 and p.

2.7. **(T)** Show that the transfer function that describes the linearized system with input $d(t)$ and output $x(t)$, for $r(t) = 0$, can be written as

$$G_2(s) = K_2 \frac{p}{s+p}.$$

Determine K_2.

3 Open-loop simulations

Consider, from now on, the parameters described in Table 1.

A	10 m^2
C	10^{-3} m^2

Table 1: Parameters of the tank

For the sake of simplicity, suppose that the acceleration of gravity is $G = 10$ m/s^2. Further suppose that the tank is operated around the equilibrium level $H_{eq} = 5$ m and that the equilibrium inflow is $M_{eq} = 1$ m^3/s.

3.1. **(T)** Determine the equilibrium input P_{eq}.

3.2. **(T)** Determine the transfer functions $G_1(s)$ and $G_2(s)$. In particular, compute p, K_1, and K_2. Determine also the time constant τ of both systems.

3.3. **(L)** Simulate and plot the response of the linear system $G_1(s)$ with inputs

$$\text{(i)} \quad r(t) = 0.1u(t) \qquad \text{and then} \qquad \text{(ii)} \quad r(t) = u(t)$$

where $u(t)$ is the unit step. Plot the evolution of $x(t)$, and discuss the results.

Suggestions:

(a) The simulation time should be adjusted to highlight both the initial transients and the steady-state behavior.

(b) For the linear system, consider zero initial conditions, as the system is assumed to be initially at equilibrium.

3.4. **(L)** Repeat the previous question for $G_2(s)$, as well as the corresponding original nonlinear system. In particular, consider the linearized and the original nonlinear system with $r(t) = 0$ and two different deviations around the equilibrium inflow, here interpreted as disturbances, as follows:

$$\text{(i)} \quad d(t) = 0.1u(t) \qquad \text{and then} \qquad \text{(ii)} \quad d(t) = u(t) \quad .$$

3.5. **(L)** Nonlinear vs Linearized system. In this question are considered total variables, $p(t)$, $m(t)$ and $h(t)$ instead of just using incremental variables as in previous and next questions (see Fig. 2).

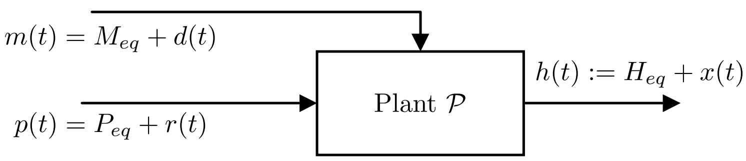

Figure 2: System plant $\mathcal{P}$, inputs and outputs, total and incremental variables.

Simulate and plot the responses of (i) the nonlinear system $\mathcal{P} = \mathcal{P}_{NL}$ implemented as detailed by equations 1 to 5 and (ii) the linearized system $\mathcal{P} = \mathcal{P}_{lin}$ based on the transfer functions $G_1(s)$ and $G_2(s)$ whose incremental inputs/outputs are adapted from/to total variables. In both cases consider that $\mathcal{P}$ is initially at equilibrium, i.e. the inputs imply an equilibrium operation and in the nonlinear case initial conditions of pure integrators are set accordingly. Consider small and large incremental steps in the inputs as suggested in the previous questions. In your report please include two block diagrams representing the two implementations of $\mathcal{P}$. Suggestion: consider blocks such as `Transfer Fcn`, `Sum`, `Constant`, `Integrator`, `Product`, `Sqrt`, ... as found in the Simulink library.

4 Closed-loop control: Proportional controller

In order to control the level of the water in the tank, consider the linearized system with a proportional controller, depicted in Fig. 3, where

$$K(s) = -K_P, \quad K_P \in \mathbb{R}, \tag{6}$$

is the proportional controller, which in this case is a simple gain. Here, $r(t)$ can be interpreted as a reference signal and $d(t)$ as a disturbance. Notice also the negative signal, which is included to simplify the design and analysis.

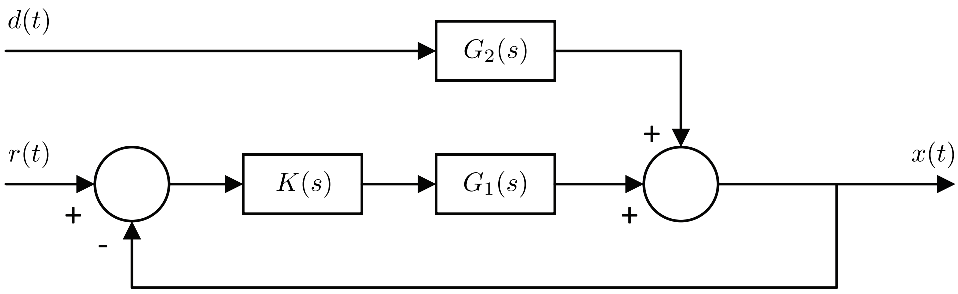

Figure 3: Closed-loop control scheme

4.1. **(T)** Provide an intuitive explanation for the presence of the *"minus"* sign in (6) assuming a positive gain K_P for the controller.

4.2. **(T)** Considering $d(t) = 0$, determine the transfer function of the closed-loop system with input $r(t)$ and output $x(t)$, i.e., compute

$$G_{clr}(s) = \left.\frac{X(s)}{R(s)}\right|_{D(s)=0}.$$

4.3. **(T)** Discuss the stability of $G_{clr}(s)$ as a function of $K_P \in \mathbb{R}$ and compute the corresponding static gain.

4.4. **(T)** Considering $r(t) = 0$, determine the transfer function of the closed-loop system with input $d(t)$ and output $x(t)$, i.e., compute

$$G_{cld}(s) = \left.\frac{X(s)}{D(s)}\right|_{R(s)=0}.$$

4.5. **(T)** Discuss the stability of $G_{cld}(s)$ as a function of K_P and compute the corresponding static gain.

4.6. **(L)** Simulate and plot the response of the closed-loop system when $r(t)$ is a unit step and $d(t)$ is zero, for three different gains: i) $K_P = 0.5$; ii) $K_P = 5$; and iii) $K_P = 50$. Discuss the reference following properties of the closed-loop system with proportional control.

Suggestion: Recall question 4.3.

4.7. **(L)** Simulate and plot the response of the closed-loop system when $d(t)$ is a unit step and $r(t)$ is zero for three different gains: i) $K_P = 0.5$; ii) $K_P = 5$; and iii) $K_P = 50$. Discuss the disturbance rejection properties of the closed-loop system with proportional control.

Suggestion: Recall question 4.5.

5 Closed-loop control: Integral Controller

An integral controller is analyzed in this section. In particular, the closed-loop scheme of Fig. 3 is considered, with the control law now given by

$$K(s) = -\frac{K_I}{s}, \tag{7}$$

where $K_I \in \mathbb{R}$ is the integral gain.

5.1. **(T)** Considering $d(t) = 0$, determine the transfer function of the closed-loop system with input $r(t)$ and output $x(t)$, i.e., compute

$$G_{clr}(s) = \left.\frac{X(s)}{R(s)}\right|_{D(s)=0}.$$

5.2. **(T)** Discuss the stability of $G_{clr}(s)$ as a function of $K_I \in \mathbb{R}$ and compute the corresponding static gain.

5.3. **(T)** Considering $r(t) = 0$, determine the transfer function of the closed-loop system with input $d(t)$ and output $x(t)$, i.e., compute

$$G_{cld}(s) = \left.\frac{X(s)}{D(s)}\right|_{R(s)=0}.$$

5.4. **(T)** Discuss the stability of $G_{cld}(s)$ as a function of K_I and compute the corresponding static gain.

5.5. **(L)** Simulate and plot the response of the closed-loop system when $r(t)$ is a unit step and $d(t)$ is zero for different gains, to illustrate the different possible types of responses. Discuss the reference following properties of the closed-loop system with integral control.

Suggestions:

(a) Recall that the closed-loop system is a second order LTI system.

(b) Recall question 5.2.

5.6. **(L)** Using the command `rlocus`, plot the root-locus of the closed-loop system $G_{clr}(s)$. Relate the closed-loop responses that were obtained in the previous question to the position of the closed-loop poles.

Suggestions:

(a) Use `doc rlocus` to see the documentation of the `rlocus` command.

(b) The `rlocus` command plots the loci of the closed-loop poles of a system for positive gains <u>based on the open-loop transfer function</u>.

(c) The command `grid on` applied to the figure produced by the `rlocus` command yields useful results.

5.7. **(L)** Simulate and plot the response of the closed-loop system when $d(t)$ is a unit step and $r(t)$ is zero for the gains selected in question 5.5. Discuss the disturbance rejection properties of the closed-loop system with integral control.

Suggestion: Recall question 5.4.

6 Unmodeled dynamics

The effect of unmodeled dynamics is studied in this section. To this purpose, suppose that the valve actuator is not perfect, and hence the actual (linearized) system is better modeled as depicted in Fig. 4, where

$$V(s) = \frac{0.1}{s + 0.1}$$

corresponds to the unmodeled dynamics of the outflow valve.

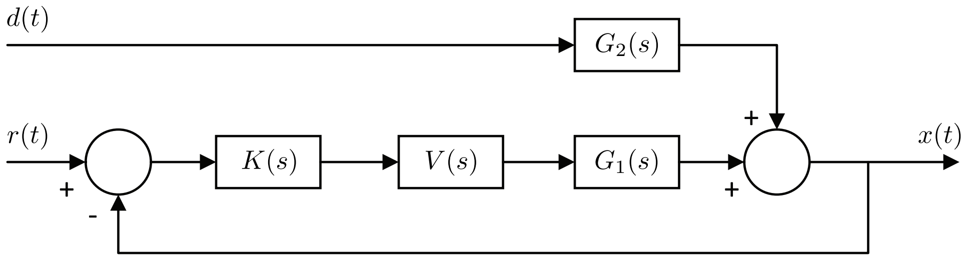

Figure 4: Closed-loop control scheme considering unmodeled dynamics

6.1. **(L)** Using the `rlocus` command, plot the loci of the closed-loop poles of $G_{clr}(s)$ considering the control-scheme shown in Fig. 4 with the integral control law (7) for positive gains $K_I > 0$. Discuss the impact of the unmodeled dynamics on the stability of the system.

6.2. **(T)** Without using the Routh-Hurwitz method, compute the critical gain for which the closed-loop system is marginally stable.

Suggestions:

(a) Start by computing the closed-loop transfer function $G_{clr}(s)$ corresponding to the control scheme depicted in Fig. 4 with integral controller (7).

(b) Notice that when the closed-loop system is marginally stable there are three poles: a real pole and a pair of pure conjugate imaginary poles.

6.3. **(L)** Illustrate the results of this section with simulations for $r(t) = u(t)$ and $d(t)$ identical to zero for all time, i.e., simulate and plot the response of the system for three different gains such that the system is stable, marginally stable, and unstable, respectively.

7 Closed-loop control: Proportional-Integral controller

A proportional-integral controller is considered in this section, given by

$$K(s) = -K_{PI}\frac{s+z}{s}, \tag{8}$$

where $K_{PI} > 0$ is a gain and $-z$, $z > 0$, corresponds to the zero introduced by the PI controller. The closed-loop control scheme depicted in Fig. 4 is considered, assuming $d(t) = 0$.

7.1. **(T)** Determine the parameters of the PI controller such that two of the closed-loop poles of the transfer function $G_{clr}(s)$ correspond to the poles of a second order LTI system with the following specifications: i) overshoot $S\% = 25\%$; and ii) settling time $t_s(5\%) = 120$ s.

Suggestions:

(a) Compute the poles and the denominator of the transfer function of a second order system that meets the required specifications.

(b) Compute the closed-loop transfer function $G_{clr}(s)$ corresponding to the control scheme depicted in Fig. 4 and the PI controller (8).

(c) Compare the denominators of both transfer functions, with the appropriate modifications to the second order polynomial.

7.2. **(L)** Simulate and plot the response of the closed-loop system when $r(t)$ is a unit step and $d(t)$ is zero with the parameters determined in the previous question. Discuss the results.

7.3. **(L)** Based on root-locus procedures, tune the parameters of the PI controller such that the overshoot is below 25% and the settling time is below 120 s. Show the original root-locus, the root-locus with the proposed zero, and the final response of the system. Describe the procedure in detail.

Suggestions:

(a) The *"branches"* of the root-locus are attracted by the zeros, whereas excess branches approach infinity as the gain increases.

(b) Establish the correspondence between the specifications and the admissible regions for the closed-loop poles.

(c) Try changing the position of the zero to gain insight to its influence on the shape of the root-locus.

(d) For each new position of the zero, use the root-locus tool to select a new gain.

(e) For each new pair of parameters (z, K_{PI}), test the closed-loop system.