Analytical and Numerical study of Delay Differential Equations and its Applications to Real Life Problems - Economics Model

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Abstract

In this paper, Delay Differential Equations is introduced and applications of DDE in economics is reviewed by analysing the Kaldor–Kalecki model of business cycle with two discrete time delays and hopf bifurication of the positive equilibrium.
Later, the Methods of solving DDEs and finally using the dde23 solver is discussed.
Keywords:
business model , time lag ,Kaldor–Kalecki Model , investment function, capital stock, Hopf Bifurication

1 Introduction

1.1 What is a Delay Differential Equation?

A delay differential equation (DDE) is a differential equation in which the time derivatives at the current time depend on the solution and possibly its derivatives at previous time.
A general form of the time-delay differential equation :

$$y'(t) = F(t, y(t), y(t-\theta_1), ..., y(t-\theta_n), t \geq t_0$$

$$y'(t-\sigma_1), .., y'(t-\sigma_m))$$

given the initial history function

$$y(t) = \phi(t), t \leq t_0$$

1.2 Different Forms of DDEs:

- Continuous delay:

$$x'(t) = f(t, x(t), \int_{-\infty}^{0} x(t+\tau) d\mu(\tau))$$

- Discrete Delay:

$$x'(t) = f(t, x(t), x(t-\tau_1), ..., x(t-\tau_m)); \tau_m \geq 0$$

- Pantograph Delay:

$$x'(t) = ax(t) + bx(\lambda t); 0 \leq \lambda \leq 1$$

  where a and b are constants

2 Applications of DDE in Economics

Business cycle (or named economic cycle) is a hot topic in the study of the macroeconomic theory.The study of factors that cause fluctuations in the economic cycle and the duration of the economic cycle have important theoretical and practical significance and will help us to better understand the law of economic operation and to gain a reasonable understanding of the leading role of investment in economic development.

Business cycle is an economic phenomenon also called economic cycle which is defined as the fluctuation of the macroeconomic variables caused by the instability of business systems. In the 19th century, several cyclical fluctuations were identified in economy. Therefore, many mathematical models have been developed in order to understand the dynamics of business cycle.

2.1 Kaldor–Kalecki Model of Business Cycle:

The Kalecki model of business cycle assumes that the saved part of profit is invested and the capital growth is due to past investment decisions. There is a time lag after which capital equipment is available for production. Recently, business cycle, as one of the important economic phenomena, has received attractive attentions due its widely application in many fields such as economic decisions, macroeconomic regulation, and market regulation . In order to understand the mechanisms of business cycle, many models are proposed. One of the most famous business cycle models is the Kaldor–Kalecki business cycle , which is described as:

$$Y'(t) = \alpha[I(Y(t), K(t)) - \gamma Y(t)]$$
$$K'(t) = I(Y(t), K(t)) - \delta K(t)$$

where Y(t)is the gross product,K(t) is the capital stock at time, α is the adjustment coefficient in the goods market, δ is the depreciation rate of the capital stock, γ represents the propensity to save, and I(Y(t),K(t)) is the investment.

On the other hand, in 1935 Kalecki released a business cycle model where he pointed out the existence of a time lag between a decision of investment and its effect on the capital stock. He assumed that the saved part of profit is invested and the capital growth is due to past investment decisions. There is a gestation period or a time lag, after which capital equipment is available for production.

Based on Kaldor's idea of introducing nonlinear functional forms and Kalecki's idea of introducing time lags, a Kaldor–Kalecki type model was proposed:

$$Y'(t) = \alpha[I(Y(t), K(t)) - \gamma Y(t)]$$
$$K'(t) = I(Y(t-\tau), K(t)) - \delta K(t)$$

where τ is the time delay between the decision of investment and implementation , δ is the depreciation rate of capital stock.
Taking into account the impact of capital stock in the past also, in 2008, Kaddar proposed a new Kaldor–Kalecki model of business cycle with time delay in the following form:

$$Y'(t) = \alpha[I(Y(t), K(t)) - \gamma Y(t)]$$

$$K'(t) = I(Y(t-\tau_1), K(t-\tau_2) - \delta K(t)$$

2.2 Local stability and Hopf bifurcation

In this section, the stability and Hopf bifurcation of the positive equilibrium point will be investigated.

2.2.1 Hopf Bifurication

The term Hopf bifurcation (also sometimes called Poincar´e-Andronov-Hopf bifurcation) refers to the local birth or death of a periodic solution (self-excited oscillation) from an equilibrium as a parameter crosses a critical value.

2.2.2 Assumptions

In a Keynesian framework, savings are assumed to be proportional to the current level of income, $S(Y, K) = \gamma Y$, where the coefficient γ, $0 < \gamma < 1$, represents the propensity to save.
In the Kaldor model the saving function is assumed to be nonlinear, we prefer a linear specification, both for its analytical simplicity and for its sounder microfoundation.
The investment demand is assumed to be an increasing and sigmoid-shaped function of income.

$$I(Y, K) = \sigma\mu + \gamma(\frac{\sigma\mu}{\delta} - K + arctan(Y - \mu))$$

[all notations are same as mentioned in the previous section]
Now, the model becomes,

$$Y'(t) = \alpha[I(Y(t) - \beta K(t)) - \gamma Y(t)]$$

$$K'(t) = I(Y(t - \tau_1)) - \beta K(t - \tau_2) - \delta K(t)$$

2.3 Conclusion

Hopf bifurcation occurs due to the nonlinearity of the investment function or time delay in output. Hopf bifurcation is just a case where the Kaldor–Kalecki model produces periodic solutions.

3 Methods to solve DDE

There are mainly two analytical methods to solve a single / system of delay differential equations.

1. Method of Steps
2. Laplace Transform

In this paper, we will discuss on how to solve DDE using Method of Steps.

3.1 Method of Steps

The method of steps is used to write a differential delay equation as a system of ordinary differential equations.

3.1.1 Example 1

Consider the DDE of the form

$$x'(t) = y(t-1), t \geq 0$$

with history function

$$x(t) = t; -1 \leq t \leq 0$$

On interval [0,1]:The above equation can be written as ODE as:

$$x'(t) = f(x, x(t))$$

with

$$f(t, x(t)) = \phi_0(t-1)$$

where

$$\phi_0 = 1, t \in [-1, 0]$$

We have,

$$x(t) = x(0) + \int_0^t f(s, x(s))\, ds = x(0) + \int_0^t \phi_0(s-1)\, ds$$

$$= x(0) + t = 1 + t$$

as x(t)=1 so,

$$x(0) = 1$$

On interval [1,2]:

$$x'(t) = f(t, x(t))$$

with,

$$f(t, x(t)) = \phi_1(t-1)$$

where,

$$\phi_1(t) = 1, t \in [0, 1]$$

We have,

$$x(t) = x(1) + \int_1^t \phi_1(s-1)\, ds = x(1) + \int_1^t \phi_1(s)\, ds$$

From solution we found earlier ,

$$x(1) = 2 \qquad as \;\; x(t) = t + 1$$

Therefore for t $\in [1, 2]$

$$x(t) = 2 + t + \frac{t^2}{2}$$

as previously , for t $\in [2, 3]$,

$$x(t) = \frac{10}{3} + 2t + \frac{t^2}{2} + \frac{t^3}{3}$$

We can continue further , this will give the general form of the solution.

4 MATLAB Code

In this section , we will use the dde23 solver to solve two kinds of Delay Differential equations .

1

$$y'(t) = ay(t-1) - by(t-3)$$

where a,b are constants and y(t) =1

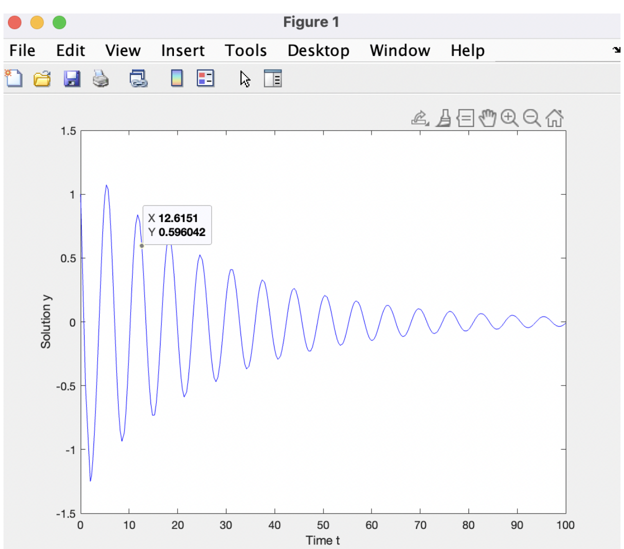
Listing 1: First Order DDE with multiple delays

```
% solving first order DDE with multiple delays using dde23
% y'(t)=ay(t-1)-by(t-3) , where a and b are constants and y(t)=1
% time = 1000

tspan=linspace(0,100,1000);
lags= [1 3];
sol = dde23(@funcl,lags,@history,tspan);
figure(1)
y=deval(sol,t);
plot(sol.x,sol.y(1,:),'b')
xlabel ( ' Time t ') ;
ylabel ( ' Solution y ') ;


function yf= funcl(t,y,z)
a=-1,b=0.5;
yf=a*z(1)-b*z(2);
end

function yh=history(t)
yh=1;
end
```



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2

$$y_1'(t) = y_1(t-1)$$
$$y_2'(t) = y_1(t) - y_1(t-1) + y_2(t-0.5)$$
$$y_3'(t) = y_2(t) - y_3(t)$$

with

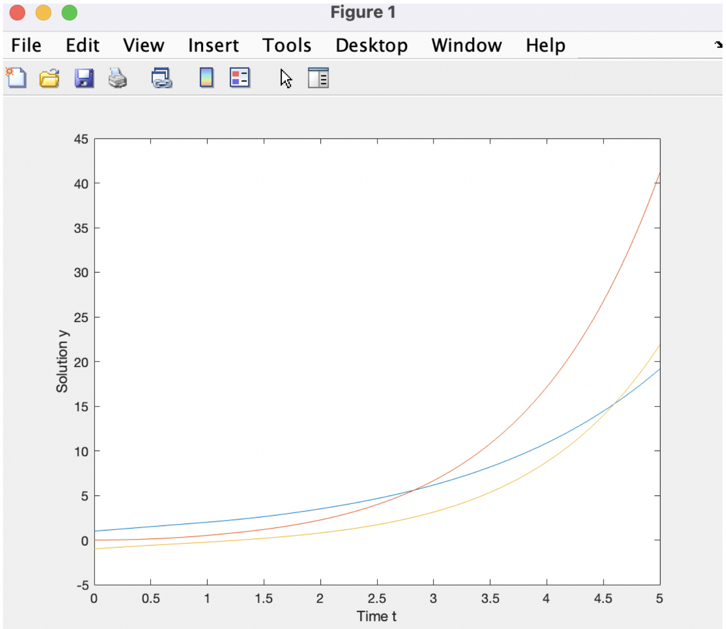
$$y_1(t) = 1; y_2(t) = 0; y_3 = -1; t \leq 0$$

Listing 2: Second Order DDE with multiple delays

```
%solution of dde with system of equation
%y_1'(t)=y_1(t-1)
%y_2'(t)=-y_1(t)-y_1(t-1)+ y_2(t-0.5)
%y_3'(t)=y_2(t)-y_3(t)
%with y_1(t)=1 ; y_2(t)=0; y_3=-1 ; t<=0

lags=[1 0.5];
tspan=[0 5];
sol=dde23(@ddefunc,lags,@history,tspan);
t=linspace(0,5,100);
y=deval(sol,t);
figure;
plot(t,y)
xlabel ( ' Time t ') ;
ylabel ( ' Solution y ') ;

function yp=ddefunc(t,y,yl)
ylag1=yl(:,1);%denote the first delay 1
ylag2=yl(:,2);%denote the second delay 0.2
yp=[ylag1(1);
    y(1)-ylag1(1)+ylag2(2);
    y(2)-y(3)];
end
function yh=history(t)
yh=[1 0 -1]';
end
```



4.1 Observation

The delay parameter has a profound effect on each of the equations.

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