MATHEMATICIA GENERALLI

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➲ *Logarithmic Relations*

a) $log_a p = \dfrac{log_b p}{log_b a}$ **b)** $log_a p = \dfrac{1}{log_p a}$

c) $log_a\left(m^n\right) = n log_a\left(m\right)$ **d)** $log_a\left(m.n\right) = log_a\left(m\right) + log_a\left(n\right)$

e) $log_a\left(\dfrac{m}{n}\right) = log_a\left(m\right) - log_a\left(n\right)$ **f)** $log_a a = 1$

g) $log_a a^p = p log_a a = p$ **h)** $a^{log_a f(x)} = f(x)$.

➲ *Exponential Relations*

a) $a^x = 1 + x.\left(log_e a\right) + \dfrac{x^2}{2!}.\left(log_e a\right)^2 + ... + \dfrac{x^n}{n!}.\left(log_e a\right)^n + ...\infty$

b) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ... + \dfrac{x^n}{n!} + ...\infty$ for all x.

➲ *Logarithmic Series*

a) $log_e\left(1+x\right) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + ...\infty, \ -1 < x \le 1$

b) $log_e\left(1-x\right) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - ...\infty, \ -1 \le x < 1$.

➲ *Binomial Expansions*

a) $\left(a+b\right)^n = {}^nC_0 a^{n-0} b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + ... + {}^nC_r a^{n-r} b^r + ... + {}^nC_n a^{n-n} b^n$ if $n \in Z^+$.

b) $\left(1+x\right)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_r x^r + ... + {}^nC_n x^n$ if n is a *positive integer*.

c) $\left(1+x\right)^n = 1 + nx + \dfrac{n\left(n-1\right)}{2!}x^2 + \dfrac{n\left(n-1\right)\left(n-2\right)}{3!}x^3 + ...\infty$ such that $-1 < x < 1$ and $n \in Z^-$ or $n \in Q$.

d) $\dfrac{x^n - a^n}{x - a} = x^{n-1} + a.x^{n-2} + a^2.x^{n-3} + ... + a^{n-1}$.

➲ *Trigonometric Series*

a) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + ...\infty$

b) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + ...\infty$

c) $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + ...\infty$.

➲ *Sum of Special Sequences*

a) Sum of first n natural numbers: $1 + 2 + 3 + ... + n = \sum_{r=1}^{n} r = \dfrac{n(n+1)}{2}$

b) Sum of squares of first n natural numbers: $1^2 + 2^2 + 3^2 + ... + n^2 = \sum_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$

c) Sum of cubes of first n natural numbers: $1^3 + 2^3 + 3^3 + ... + n^3 = \sum_{r=1}^{n} r^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

d) Sum of any constant k to n times: $k + k + k + ... + k\left(\text{to } n \text{ times}\right) = \sum_{r=1}^{n} k = nk$.

➲ ***List of formulae of Mensuration for using in the problems of Maxima and Minima***

01. Circle:

Perimeter *i.e.* circumference $= 2\pi r$

Area $= \pi r^2$

02. Equilateral triangle:

Perimeter $= 3a$

Area $= \dfrac{\sqrt{3}}{4}a^2$

03. Rectangle:

Perimeter $= 2(l+b)$

Area $= lb$

04. Square:

Perimeter $= 4a$

Area $= a^2$

05. Cuboid:

Lateral surface area $= 2(l+b).h$

Total surface area $= 2(lb+bh+hl)$

Volume $= lbh$

06. Cube:

Lateral surface area $= 4a^2$

Total surface area $= 6a^2$

Volume $= a^3$

07. Sphere:

Surface area $= 4\pi r^2$

Volume $= \dfrac{4}{3}\pi r^3$

08. Hemisphere:

Curved surface area $= 2\pi r^2$

Total surface area $= 3\pi r^2$

Volume $= \dfrac{2}{3}\pi r^3$

09. Cylinder:

Curved surface area $= 2\pi rh$

Total surface area $= 2\pi r^2 + 2\pi rh$

Volume $= \pi r^2 h$

10. Cone:

Curved surface area $= \pi rl$, *where* $l^2 = r^2 + h^2$

Total surface area $= \pi r^2 + \pi rl$

Volume $= \dfrac{1}{3}\pi r^2 h$

☞ *Followings are also of **importance**, though questions on them are **rarely** found in maxima and minima:*

11. Frustum of a cone:

Curved surface area $= \pi l(R+r)$, *where* $l = \sqrt{h^2 + (R-r)^2}$

Total surface area $= \pi l(R+r) + \pi(R^2 + r^2)$, *where* $l = \sqrt{h^2 + (R-r)^2}$

Volume $= \dfrac{1}{3}\pi h(R^2 + r^2 + rR)$

12. Sector and segment of Circle:

Area of the sector of angle $\theta = \dfrac{\theta}{360} \times \pi r^2$

Length of the arc of a sector of angle $\theta = \dfrac{\theta}{360} \times 2\pi r$

Area of the segment of a circle = Area of the corresponding sector – Area of corresponding triangle

$$= \frac{\theta}{360} \times \pi r^2 - r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{\theta}{360} \times \pi r^2 - \frac{1}{2}r^2 \sin\theta$$

Circumference i.e. arc length of semicircle of radius $r = \pi r$

Perimeter of semicircle of radius $r = \pi r + 2r$

13. Area of a trapezium $= \dfrac{1}{2}$ (Sum of *parallel* sides) × (Distance between *parallel* sides)

14. Area of a rhombus $= \dfrac{1}{2}$ (Product of *diagonals*)

15. Area of a triangle ABC $= \dfrac{1}{2}ab\sin C = \dfrac{1}{2}bc\sin A = \dfrac{1}{2}ca\sin B$

16. Area of a triangle by Heron's formula $= \sqrt{s(s-a)(s-b)(s-c)}$ *where,* $s = \dfrac{a+b+c}{2}$.

➲ *Algebraic Identities*

a) $(a \pm b)^2 = a^2 \pm 2ab + b^2$

b) $a^2 - b^2 = (a+b)(a-b)$

c) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

d) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

e) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

f) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

g) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

h) $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$.

➲ *Concept of Infinity*

We consider the existence of **two symbols** $-\infty$ and ∞ **outside** the set of real numbers R and would call them ***minus infinity*** and ***plus infinity*** respectively with the fact $-\infty < x < \infty$ for every $x \in \mathrm{R}$.
Thus $-\infty$ and ∞ are not real numbers but just the symbols (like we use *x, y* etc.).

When we write $x = \infty$, we mean that:

a) x is larger than any real number however large.

b) x is not a fixed number.

Also $c^{\infty} = \begin{cases} \infty, \text{ if } \; c > 1 \\ 0, \text{ if } \; 0 \le c < 1. \\ 1, \text{ if } \; c = 1 \end{cases}$

➲ *Symbols and their meanings*

S. No.	Symbol	Meaning
01.	N	Set of natural numbers
02.	I *or* Z	Set of integers
03.	Q	Set of rational numbers
04.	T	Set of irrational numbers
05.	R	Set of real numbers
06.	C	Set of complex numbers
07.	$\in$	is an element of (or belongs to)
08.	$\notin$	is not an element of (or does not belong to)
09.	S *or* ξ *or* U	Universal set
10.	: *or* /	Such that
11.	ϕ	Empty set *or* Null set
12.	$\subseteq$	is subset of
13.	$\supseteq$	is superset of
14.	$\subset$	is proper subset of
15.	$\supset$	is proper superset of
16.	$\cup$	Union
17.	$\cap$	Intersection
18.	$\forall$	For all
19.	$\Rightarrow$	Implies
20.	$\Leftrightarrow$	if and only if

TRIGONOMETRIC FORMULAE

Relation between trigonometric ratios

a) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ ***b)*** $\tan\theta = \dfrac{1}{\cot\theta}$ ***c)*** $\tan\theta\,\cot\theta = 1$

d) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ ***e)*** $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ ***f)*** $\sec\theta = \dfrac{1}{\cos\theta}$

Trigonometric identities

a) $\sin^2\theta + \cos^2\theta = 1$

b) $1 + \tan^2\theta = \sec^2\theta$

c) $1 + \cot^2\theta = \text{cosec}^2\theta$

Addition / subtraction formulae & some related results

a) $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

b) $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

c) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

d) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

e) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

f) $\cot(A \pm B) = \dfrac{\cot B\cot A \mp 1}{\cot B \pm \cot A}$

Transformation of sums / differences into products & vice-versa

a) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

b) $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

c) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

d) $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

e) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

f) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

g) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

h) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Multiple angle formulae involving 2A and 3A

a) $\sin 2A = 2\sin A\cos A$

b) $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2}$

c) $\cos 2A = \cos^2 A - \sin^2 A$

d) $\cos A = \cos^2\dfrac{A}{2} - \sin^2\dfrac{A}{2}$

e) $\cos 2A = 2\cos^2 A - 1$

f) $2\cos^2 A = 1 + \cos 2A$

g) $\cos 2A = 1 - 2\sin^2 A$

h) $2\sin^2 A = 1 - \cos 2A$

i) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$

j) $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

k) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

l) $\sin 3A = 3\sin A - 4\sin^3 A$

m) $\cos 3A = 4\cos^3 A - 3\cos A$

n) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Relations in Different Measures of Angle

➲ $\textit{Angle in Radian Measure} = (\textit{Angle in Degree Measure}) \times \dfrac{\pi}{180}$

➲ $\textit{Angle in Degree Measure} = (\textit{Angle in Radian Measure}) \times \dfrac{180}{\pi}$

➲ $\theta\,(\textit{in radian measure}) = \dfrac{l}{r}$

Also followings are of importance as well:

➲ $1\,\textit{Right angle} = 90^o$ ➲ $1^o = 60',\ 1' = 60''$

➲ $1^o = \dfrac{\pi}{180} = 0.01745\ \textit{radians}\,(\textit{approximately})$ ➲ $1\ \textit{radian} = 57^o 17' 45''$ *or 206265 seconds* .

General Solutions

a) $\sin x = \sin y \;\Rightarrow\; x = n\pi + (-1)^n y$, *where* $n \in Z$.

b) $\cos x = \cos y \;\Rightarrow\; x = 2n\pi \pm y$, *where* $n \in Z$.

c) $\tan x = \tan y \;\Rightarrow\; x = n\pi + y$, *where* $n \in Z$.

Relation in Degree & Radian Measures

Angles in Degree	0°	30°	45°	60°	90°	180°	270°	360°
Angles in Radian	0^c	$\left(\frac{\pi}{6}\right)^c$	$\left(\frac{\pi}{4}\right)^c$	$\left(\frac{\pi}{3}\right)^c$	$\left(\frac{\pi}{2}\right)^c$	$(\pi)^c$	$\left(\frac{3\pi}{2}\right)^c$	$(2\pi)^c$

↳ *In actual practice, we omit the exponent '**c**' and instead of writing* π^c *we simply write* π *and similarly for others.*

Trigonometric Ratio of Standard Angles

Degree /Radian $(\rightarrow)$	0°	30°	45°	60°	90°
T – Ratios $(\downarrow)$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Trigonometric Ratios of Allied Angles

Angles $(\rightarrow)$ / **T- Ratios** $(\downarrow)$	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$ **OR** $-\theta$	$2\pi+\theta$
sin	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
cos	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tan	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$
cot	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$
sec	$\mathrm{cosec}\,\theta$	$-\mathrm{cosec}\,\theta$	$-\sec\theta$	$-\sec\theta$	$-\mathrm{cosec}\,\theta$	$\mathrm{cosec}\,\theta$	$\sec\theta$	$\sec\theta$
cosec	$\sec\theta$	$\sec\theta$	$\mathrm{cosec}\,\theta$	$-\mathrm{cosec}\,\theta$	$-\sec\theta$	$-\sec\theta$	$-\mathrm{cosec}\,\theta$	$\mathrm{cosec}\,\theta$

NUMBER SYSTEM

01. Natural numbers: The numbers used in ordinary counting *i.e.* $1, 2, 3, ...,$ are called natural numbers (and *positive integers* as well). The set of natural nos. is denoted by N. Also if we include 0 to the set of natural numbers, we get set of the *whole numbers* which is denoted by the symbol W.

Therefore $\mathrm{N}=\{1, 2, 3, ...\}$ and, $\mathrm{W}=\{0, 1, 2, 3, ...\}$.

02. Integers: The numbers $... -3, -2, -1, 0, 1, 2, 3, ...$ are called integers. The set of integers is denoted by I *or* Z. Though now we use Z instead of I to symbolize the set of integers.

Therefore, I *or* $\mathrm{Z}=\{... -3, -2, -1, 0, 1, 2, 3, ...\}$.

Clearly $\mathrm{N} \subset \mathrm{Z}$.

- Also from the above discussion, it is evident that integers are of three types viz.:

a) *Positive integers i.e.* $\mathrm{Z}^{+}=\{1, 2, 3, ...\}$

b) *Negative integers i.e.* $\mathrm{Z}^{-}=\{-1, -2, -3, ...\}$

c) *Zero integer i.e. non-positive and non-negative integer.*

03. Rational numbers: A number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, is called a rational no. The set of rational nos. is denoted by Q.

Therefore $\mathrm{Q}=\left\{\frac{p}{q}; p, q \in Z \text{ and } q \neq 0\right\}$

Clearly $\mathrm{N} \subset \mathrm{Z} \subset \mathrm{Q}$.

- Zero being an integer, is also a rational number.

04. Irrational numbers: An irrational number has a non-terminating and non-repeating decimal representation *i.e.* it can't be expressed in the form of $\frac{p}{q}$. The set of irrational nos. is denoted by T.

Few examples of irrational numbers are $\sqrt{2}, 5\sqrt{7}, 8+\sqrt{3}, \sqrt[3]{5}, e, \pi, ...$ *etc.*

- Note that π is irrational while $\frac{22}{7}$ is rational.

05. Real numbers: The set of all numbers either rational or irrational, is called real number. The set of real nos. is denoted by R.

Clearly $\mathrm{N} \subset \mathrm{Z} \subset \mathrm{Q} \subset \mathrm{R}$.

➲ *Solving of a Quadratic Equation:*

Consider a quadratic equation of the form, $ax^2+bx+c=0$ then, its roots are given by

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and, } x=\frac{-b-\sqrt{b^2-4ac}}{2a} \quad \textit{where } D=b^2-4ac \ .$$