# Formulae For Integral Calculus

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❖ ***Standard Formulae Of Integrals:***

**A.** $\int x^n dx = \frac{x^{n+1}}{n+1} + k,\ n \neq -1$

**B.** $\int \frac{1}{x} dx = \log|x| + k$

**C.** $\int a^x dx = \frac{1}{\log a} a^x + k$

**D.** $\int e^{ax} dx = \frac{1}{a} e^{ax} + k$

**E.** $\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + k$

**F.** $\int \cos(ax) dx = \frac{1}{a}\sin(ax) + k$

**G.** $\int \tan x dx = \log|\sec x| + k$

**OR** $= -\log|\cos x| + k$

**H.** $\int \cot x dx = \log|\sin x| + k$

**OR** $= -\log|\text{cosec } x| + k$

**I.** $\int \sec x dx = \log|\sec x + \tan x| + k$

**OR** $= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + k$

**J.** $\int \text{cosec } x dx = \log|\text{cosec } x - \cot x| + k$

**OR** $= \log\left|\tan\frac{x}{2}\right| + k$

**K.** $\int \sec^2 x dx = \tan x + k$

**L.** $\int \text{cosec}^2 x dx = -\cot x + k$

**M.** $\int \sec x.\tan x dx = \sec x + c$

**N.** $\int \text{cosec } x.\cot x dx = -\text{cosec } x + k$

**O.** $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + k$

**P.** $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + k$

**Q.** $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + k$

**R.** $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + k$

**S.** $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x + \sqrt{x^2-a^2}\right| + k$

**T.** $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x + \sqrt{x^2+a^2}\right| + k$

**U.** $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + k$

**V.** $\int \sqrt{x^2-a^2} dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2-a^2}\right| + k$

**W.** $\int \sqrt{x^2+a^2} dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2+a^2}\right| + k$

**X.** $\int \sqrt{a^2-x^2} dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + k$

**Y.** $\int \frac{1}{ax+b} dx = \frac{1}{a}\log|ax+b| + k$, *where 'a' is any constant (and obviously, $k$ is the integral constant)*

**Z.** $\int \lambda\, dx = \lambda x + k$, *where $\lambda$ is a constant (and, $k$ is the integral constant).*

### ❖ *Methods of Integration:*

Though there is no general method for finding the integral of a function, yet here we have considered the following methods based on observations for evaluating the integral of a function:

***a) Integration by Substitution Method–***

In this method we change the integral $\int f(x)dx$, where independent variable is *x*, to another integral in which independent variable is *t* (say) different from *x* such that *x* and *t* are related by $x = g(t)$.

Let $u = \int f(x)dx$ then, $\frac{du}{dx} = f(x)$.

Again as $x = g(t)$ so, we have $\frac{dx}{dt} = g'(t)$.

Now $\frac{du}{dt} = \frac{du}{dx}.\frac{dx}{dt} = f(x).g'(t)$

On integrating both sides w.r.t. *t,* we get

$$\int\left(\frac{du}{dt}\right)dt = \int f(x)g'(t)dt$$

$$\Rightarrow u = \int f\left[g(t)\right]g'(t)dt$$

*i.e.,* $\int f(x)dx = \int f\left[g(t)\right]g'(t)dt$ where $x = g(t)$.

So, it is clear that substituting $x = g(t)$ in $\int f(x)dx$ will give us the same result as obtained by putting $g(t)$ in place of *x* and $g'(t)dt$ in place of $dx$.

***b) Integration by Partial Fractions–***

Consider $\frac{f(x)}{g(x)}$ defines a rational polynomial function.

➲ If the degree of numerator *i.e.* $f(x)$ is **greater than or equal to** the degree of denominator *i.e.* $g(x)$ then, this type of rational function is called an **improper rational function**. And if degree of $f(x)$ is **smaller than** the degree of denominator *i.e.* $g(x)$ then, this type of rational function is called a **proper rational function**.

➲ In rational polynomial functions if the degree (*i.e.* highest power of the variable) of numerator (Nr) is **greater than or equal to** the degree of denominator (Dr), then *(without any doubt)* **always perform the division** *i.e.*, divide the Nr by Dr **before doing anything** and thereafter use the following:

$$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}.$$

On doing this, the rational function is resolved into partial fractions. The table *shown below* lists the types of simpler partial fractions that are to be associated with various kinds of rational functions which will be dealt in our current study:

(**TABLE DEMONSTRATING PARTIAL FRACTIONS OF VARIOUS FORMS**)

| Form of the Rational Function | Form of the Partial Fraction |
|---|---|
| $\frac{px+q}{(x-a)(x-b)}$, $a \ne b$ | $\frac{A}{x-a} + \frac{B}{x-b}$ |
| $\frac{px+q}{(x-a)^2}$ | $\frac{A}{x-a} + \frac{B}{(x-a)^2}$ |
| $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$ | $\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$ |
| $\frac{px^2+qx+r}{(x-a)^2(x-b)}$ | $\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$ |
| $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where $x^2+bx+c$ can't be factorized further. | $\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$ |

***c) Integral By Parts–***

If $\upsilon$ and $V$ be two functions of $x$ then,

$$\int \underset{(\mathrm{I})}{\upsilon} \,.\, \underset{(\mathrm{II})}{V}\, dx = \upsilon\left(\int V dx\right) - \int \left\{\frac{d\upsilon}{dx}\int V dx\right\} dx$$

In finding integrals by this method, proper choice of functions $\upsilon$ and $V$ is crucial. Though there is *no fixed rule* for taking $\upsilon$ and $V$ (their choice is possible by ***practice***) yet, following rule is found to be quite helpful in deciding the functions $\upsilon$ and $V$ :

➲ If $\upsilon$ and $V$ are of different types, take that function as $\upsilon$ which comes first in the word **ILATE**.

*Here **I** stands for Inverse trigonometrical function, **L** stands for Logarithmic function, **A** stands for Algebraic function, **T** stands for Trigonometrical function and **E** stands for the Exponential function.*

➲ If both the functions are trigonometrical, take that function as $V$ whose integral is easier.

➲ If both the functions are algebraic, take that function as $\upsilon$ whose differentiation is easier.

➲ Some integrands are such that they are not product of two functions. Their integrals may be found by integrals by parts taking **1** as the second function. Logarithmic and inverse trigonometric functions are examples of such functions.

➲ The result of integral $\int e^x\left[f(x)+f'(x)\right]dx = e^x f(x)+k$ can be directly applied in case of the objective type questions.

❖ ***Making the Perfect Square:***

**STEP1–** Consider the expression $ax^2+bx+c$.

**STEP2–** Make the coefficient of $x^2$ as unity ( *i.e.,* $-1$ ) by taking ***a*** common, after doing so the original expression will look like, $a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$.

**STEP3–** Add and subtract $\left(\frac{b}{2a}\right)^2$ to the expression obtained in STEP2 as depicted here *i.e.*, $a\left[x^2+\frac{b}{a}x+\frac{c}{a}+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right]$.

**STEP4–** The perfect square of $ax^2+bx+c$ will be $a\left\{\left(x+\frac{b}{2a}\right)^2+\left[\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right]\right\}$.

❖ ***Various Integral forms:***

➲ Integrals of the form $\int\frac{px+q}{ax^2+bx+c}dx$, $\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$, $\int(px+q)\sqrt{ax^2+bx+c}\,dx$: Express the numerator $px+q$ as shown here, *i.e.,* $px+q=A\frac{d}{dx}\left(ax^2+bx+c\right)+B$. Then on, obtain the values of $A$ and $B$ by equating the coefficients of like powers of $x$ and constants terms on both the sides. Then, integrate it after replacing $px+q$ by $A\frac{d}{dx}\left(ax^2+bx+c\right)+B$ using the values of $A$ and $B$.

➲ Integrals of the form $\int\frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx$: Express $\text{Numerator}=A\frac{d}{dx}(\text{Denominator})+B(\text{Denominator})$. Then obtain the values of $A$ and $B$ by equating the coefficients of $\sin x$ and $\cos x$ on both the sides and proceed.

➲ Integrals of the form $\int\frac{a\sin x+b\cos x+c}{p\sin x+q\cos x+r}dx$ : Note that the previous integral form can be considered as a special case of this form. Express $\text{Numerator}=A\frac{d}{dx}(\text{Denominator})+B(\text{Denominator})+C$. Then obtain the values of unknowns *i.e.*, $A$, $B$ and $C$ by equating the coefficients of $\sin x$, $\cos x$ and the constant terms on both the sides and hence proceed.

➲ Integrals of the form $\int \frac{dx}{a\sin^2 x + b\cos^2 x}$, $\int \frac{dx}{a + b\sin^2 x}$, $\int \frac{dx}{a + b\cos^2 x}$, $\int \frac{dx}{(a\sin x + b\cos x)^2}$ and $\int \frac{dx}{a + b\sin^2 x + c\cos^2 x}$: Divide the Nr and Dr both by $\cos^2 x$. Replace $\sec^2 x$, if any, in Dr by $1+\tan^2 x$ and then put $\tan x = \mathrm{t}$ and proceed.

➲ Integrals of the form $\int \frac{dx}{a\sin x + b\cos x}$, $\int \frac{dx}{a + b\sin x}$, $\int \frac{dx}{a + b\cos x}$ and $\int \frac{dx}{a\sin x + b\cos x}$:

Use $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and/ or $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$. Replace $1+\tan^2\frac{x}{2}$ in the Nr by $\sec^2\frac{x}{2}$ and then put $\tan\frac{x}{2} = \mathrm{t}$ and then after proceed.

➲ Integrals of the form $\int \frac{1}{\mathrm{M}\sqrt{\mathrm{N}}}dx$ where M and N are linear or quadratic expressions in $x$:

| M | N | Substitutions |
|---|---|---|
| Linear | Linear | $\mathrm{t}^2 = \mathrm{N}$ |
| Quadratic | Linear | $\mathrm{t}^2 = \mathrm{N}$ |
| Linear | Quadratic | $\mathrm{t} = \frac{1}{\mathrm{M}}$ |
| Quadratic | Quadratic | $\mathrm{t}^2 = \frac{\mathrm{N}}{\mathrm{M}}$ or $\mathrm{t} = \frac{1}{x}$ |

❖ ***A Few Useful Quickies***:

***a)*** $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + k,\ n \neq -1$

***b)*** $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + k$

***c)*** $\int \frac{f'(x)}{[f(x)]^n}dx = \frac{[f(x)]^{-n+1}}{-n+1} + k$

***d)*** $\int (ax+b)^n\,dx = \frac{1}{a}\left[\frac{(ax+b)^{n+1}}{n+1}\right] + k$.

❖ ***Formulae & Properties Of Definite Integrals:***

**P.01** $\int_a^b f(x)\,dx = [\mathrm{F}(x)]_a^b = \mathrm{F}(b) - \mathrm{F}(a)$

**P.02** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

**P.03** $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

**P.04** $\int_a^b f(x)\,dx = \int_a^m f(x)\,dx + \int_m^b f(x)\,dx,\ a < m < b$

**P.05** $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

**P.06** $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is even function } i.e., f(-x)=f(x) \\ 0, \text{ if } f(x) \text{ is odd function } i.e., f(-x)=-f(x) \end{cases}$

**P.07** $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

$$= \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

**P.08** $\int_0^m f(x)\,dx = \begin{cases} 2\int_0^{m/2} f(x)\,dx, \text{ if } f(m-x)=f(x) \\ 0, \text{ if } f(m-x)=-f(x) \end{cases}$

❖ ***Proof Of A Few Important Properties:***

**P.04** $\int_a^b f(x)dx = \int_a^m f(x)dx + \int_m^b f(x)dx,\ a<m<b$.

**PROOF** We know, $\int_a^b f(x)dx = \left[\mathrm{F}(x)\right]_a^b = \mathrm{F}(b)-\mathrm{F}(a)$ …(i)

Consider $\int_a^m f(x)dx = \left[\mathrm{F}(x)\right]_a^m = \mathrm{F}(m)-\mathrm{F}(a)$ …(ii)

And $\int_m^b f(x)dx = \left[\mathrm{F}(x)\right]_m^b = \mathrm{F}(b)-\mathrm{F}(m)$ …(iii)

Adding the equations (ii) and (iii), we get

$$\int_a^m f(x)dx + \int_m^b f(x)dx = \mathrm{F}(b)-\mathrm{F}(a)$$

$$= \int_a^b f(x)\,dx.$$ [ By (i)

Hence, $\int_a^b f(x)dx = \int_a^m f(x)dx + \int_m^b f(x)dx,\ a<m<b$. **[H.P.]**

**P.05** $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$.

**PROOF** Consider $\int_a^b f(x)\,dx$.

Let $x=a+b-t \Rightarrow dx=-dt$.
Also when $x=a \Rightarrow t=b$ and, when $x=b \Rightarrow t=a$.

So, $\int_a^b f(x)dx = \int_b^a f(a+b-t)(-dt)$

$$= -\left[-\int_a^b f(a+b-t)\,dt\right]$$ [By using **P.02**

$$= \int_a^b f(a+b-t)\,dt$$

$$\Rightarrow \int_a^b f(x)\,dx = \int_b^a f(a+b-x)\,dx$$ [Replacing *t* by *x*, **P.03**

Hence, $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$. **[H.P.]**

**SPECIAL CASE OF P.05** Take $a=0$ and $b=a$.

Then, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

◆ *The proof for the special case is same as is for the* ***P.05****, so it has been left as an exercise for you!*

**P.06** $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is even function i.e., } f(-x)=f(x) \\ 0, \text{ if } f(x) \text{ is odd function i.e., } f(-x)=-f(x) \end{cases}.$$

**PROOF** We know that $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$ ...(i) [By using **P.04**

Consider $\int_{-a}^{0} f(x)\,dx$.

Let $x=-t \Rightarrow dx=-dt$.

Also when $x=-a \Rightarrow t=a$ and when $x=0 \Rightarrow t=0$.

So, $$\int_{-a}^{0} f(x)\,dx = \int_a^0 f(-t)(-dt)$$

$$= -\left[-\int_0^a f(-t)\,dt\right]$$ [By using **P.02**

$$= \int_0^a f(-t)\,dt$$

$$\Rightarrow \int_{-a}^{0} f(x)\,dx = \int_0^a f(-x)\,dt$$ [Replacing *t* by *x*, **P.03**

Therefore equation (i) becomes,

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_0^a \left[f(-x)+f(x)\right]dx$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{ if } f(-x)=f(x) \\ 0, \text{ if } f(-x)=-f(x) \end{cases}$$

$$\text{i.e.,. } \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is even function} \\ 0, \text{ if } f(x) \text{ is odd function} \end{cases}$$ **[H.P.]**

**P.07** $$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$

$$= \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx.$$

**PROOF** We know $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$ …(i) [By using **P.04**

Consider $\int_a^{2a} f(x)\,dx$.

Let $x = 2a - t \Rightarrow dx = -dt$.
Also when $x = a \Rightarrow t = a$ and when $x = 2a \Rightarrow t = 0$.

So, $\int_a^{2a} f(x)\,dx = \int_a^{0} f(2a-t)(-dt)$

$= -\left[-\int_0^{a} f(2a-t)\,dt\right]$ [By using **P.02**

$= \int_0^{a} f(2a-t)\,dt$

$\Rightarrow \int_a^{2a} f(x)\,dx = \int_0^{a} f(2a-x)\,dx$ [Replacing *t* by *x*, **P.03**

So equation (i) becomes,

$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$. **[H.P]**

**P.08** $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x) \\ 0, \text{ if } f(2a-x) = -f(x) \end{cases}$.

**PROOF** We know $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$ …(i)

Consider $\int_a^{2a} f(x)\,dx$.

Let $x = 2a - t \Rightarrow dx = -dt$.
Also when $x = a \Rightarrow t = a$ and when $x = 2a \Rightarrow t = 0$.

So, $\int_a^{2a} f(x)\,dx = \int_a^{0} f(2a-t)(-dt)$

$= -\left[-\int_0^{a} f(2a-t)\,dt\right]$ [By using **P.02**

$= \int_0^{a} f(2a-t)\,dt$

$\Rightarrow \int_a^{2a} f(x)\,dx = \int_0^{a} f(2a-x)\,dx$ [Replacing *t* by *x*, **P.03**

So equation (i) becomes,

$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$

$= \int_0^{a} \left[f(x) + f(2a-x)\right]dx$

Hence, $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(2a-x)=f(x) \\ 0, \text{ if } f(2a-x)=-f(x) \end{cases}$. **[H.P.]**

❖ ***Definite integral as the Limit Of A Sum (First Principle Of Integrals):***

Take that function whose integral value is to be calculated as $f(x)$ and then use the given relation,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\ldots+f(a+(n-1)h)\right]$$

***or,*** $$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a+h)+f(a+2h)+f(a+3h)+\ldots+f(a+nh)\right]$$

***i.e.,*** $\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh)$, such that as $n\to\infty$, $h\to 0$ and $nh=b-a$

***or,*** $\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=1}^{n} f(a+rh)$, such that as $n\to\infty$, $h\to 0$ and $nh=b-a$.



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Hii, All!

I hope this texture may have proved beneficial for you.

While going through this material, if you noticed any error(s) or, something which doesn't make sense to you, **please** bring it in my notice through SMS or Call at **+91-9650 350 480** or Email at **theopgupta@gmail.com**.

With lots of Love & Blessings!

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