# OPG'S

# MATHEMATICS

## LIST OF FORMULAE

## [ For Class XII ]

**Covering all the topics of NCERT Mathematics Text Book Part – I**

***For the session 2013-14***

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# Relations, Functions & Binary Operations

**Important Terms, Definitions & Formulae**

***01. TYPES OF INTERVALS***

**a) Open interval:** If $a$ and $b$ be two real numbers such that $a < b$ then, the set of all the real numbers lying strictly between $a$ and $b$ is called an *open interval*. It is denoted by $]a, b[$ or $(a, b)$ i.e., $\{x \in \mathrm{R} : a < x < b\}$.

**b) Closed interval:** If $a$ and $b$ be two real numbers such that $a < b$ then, the set of all the real numbers lying between $a$ and $b$ such that it includes both $a$ and $b$ as well is known as a *closed interval*. It is denoted by $[a, b]$ i.e., $\{x \in \mathrm{R} : a \le x \le b\}$.

**c) Open Closed interval:** If $a$ and $b$ be two real numbers such that $a < b$ then, the set of all the real numbers lying between $a$ and $b$ such that it excludes $a$ and includes only $b$ is known as an *open closed interval*. It is denoted by $]a, b]$ or $(a, b]$ i.e., $\{x \in \mathrm{R} : a < x \le b\}$.

**d) Closed Open interval:** If $a$ and $b$ be two real numbers such that $a < b$ then, the set of all the real numbers lying between $a$ and $b$ such that it includes only $a$ and excludes $b$ is known as a *closed open interval*. It is denoted by $[a, b[$ or $[a, b)$ i.e., $\{x \in \mathrm{R} : a \le x < b\}$.

## *A. RELATIONS*

***02. Defining the Relation:*** A relation R, from a non-empty set A to another non-empty set B is mathematically defined as an arbitrary subset of $\mathrm{A} \times \mathrm{B}$. Equivalently, any subset of $\mathrm{A} \times \mathrm{B}$ is a relation from A to B.

Thus, R is a relation from A to B $\Leftrightarrow \mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$

$$\Leftrightarrow \mathrm{R} \subseteq \{(a, b) : a \in \mathrm{A},\ b \in \mathrm{B}\}.$$

***Illustrations:***

**a)** Let $\mathrm{A} = \{1, 2, 4\}, \mathrm{B} = \{4, 6\}$. Let $\mathrm{R} = \{(1,4), (1,6), (2,4), (2,6), (4,6)\}$. Here $\mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$ and therefore R is a relation from A to B.

**b)** Let $\mathrm{A} = \{1, 2, 3\}, \mathrm{B} = \{2, 3, 5, 7\}$. Let $\mathrm{R} = \{(2,3), (3,5), (5,7)\}$. Here $\mathrm{R} \not\subset \mathrm{A} \times \mathrm{B}$ and therefore R is not a relation from A to B. Since $(5,7) \in \mathrm{R}$ but $(5,7) \notin \mathrm{A} \times \mathrm{B}$.

**c)** Let $\mathrm{A} = \{-1, 1, 2\}, \mathrm{B} = \{1, 4, 9, 10\}$. Let $a \mathrm{R} b$ means $a^2 = b$ then, $\mathrm{R} = \{(-1,1), (1,1), (2,4)\}$.

➲ ***Note the followings:***

- *A relation from A to B is also called a relation from A into B.*
- $(a, b) \in R$ *is also written as* $aRb$ *(read as* ***a is R related to b****).*
- *Let A and B be two non-empty finite sets having p and q elements respectively. Then* $n(A \times B) = n(A).n(B) = pq$. *Then total number of subsets of* $A \times B = 2^{pq}$. *Since each subset of* $A \times B$ *is a relation from A to B, therefore* ***total number of relations from A to B is given as*** $\boldsymbol{2^{pq}}$.

***03. DOMAIN & RANGE OF A RELATION***

**a) Domain of a relation:** Let R be a relation from A to B. The domain of relation R is the set of all those elements $a \in \mathrm{A}$ such that $(a, b) \in \mathrm{R}$ for some $b \in \mathrm{B}$. Domain of R is precisely written as $Dom.(R)$ symbolically.

Thus, $\mathrm{Dom.(R)} = \{a \in \mathrm{A} : (a, b) \in \mathrm{R} \text{ for some } b \in \mathrm{B}\}$.

That is, the domain of R is **the set of first component of all the ordered pairs which belong to R.**

**b) Range of a relation:** Let R be a relation from A to B. The range of relation R is the set of all those elements $b \in \mathrm{B}$ such that $(a, b) \in \mathrm{R}$ for some $a \in \mathrm{A}$.

Thus, Range of $\mathrm{R} = \{b \in \mathrm{B} : (a, b) \in \mathrm{R} \text{ for some } a \in \mathrm{A}\}$.

That is, the range of R is the set of second components of all the ordered pairs which belong to R.

**c) Codomain of a relation:** Let R be a relation from A to B. Then B is called the codomain of the relation R. So we can observe that codomain of a relation R from A into B is the set B as a whole.

***Illustrations:***

***a)*** *Let* $A=\{1,2,3,7\}, B=\{3,6\}$. *Let aRb means* $a<b$. *Then* $R=\{(1,3),(1,6),(2,3),(2,6),(3,6)\}$.

*Here* $Dom.(R)=\{1,2,3\}$, *Range of* $R=\{3,6\}$, *Codomain of* $R=B=\{3,6\}$.

***b)*** *Let* $A=\{1,2,3\}, B=\{2,4,6,8\}$. *Let* $R_1=\{(1,2),(2,4),(3,6)\}$, $R_2=\{(2,4),(2,6),(3,8),(1,6)\}$.

*Then both* $R_1$ *and* $R_2$ *are relations from A to B because* $R_1 \subseteq A\times B$, $R_2 \subseteq A\times B$.

*Here* $Dom.(R_1)=\{1,2,3\}$, *Range of* $R_1=\{2,4,6\}$; $Dom.(R_2)=\{2,3,1\}$, *Range of* $R_2=\{4,6,8\}$.

***04. TYPES OF RELATIONS FROM ONE SET TO ANOTHER SET***

**a) Empty relation:** A relation R from A to B is called an empty relation or a void relation from A to B if $R=\varphi$.

***For example,*** *let* $A=\{2,4,6\}, B=\{7,11\}$. *Let* $R=\{(a,b): a\in A, b\in B \text{ and } a-b \text{ is even}\}$. *Here R is an empty relation.*

**b) Universal relation:** A relation R from A to B is said to be the universal relation if $R=A\times B$.

***For example,*** *let* $A=\{1,2\}, B=\{1,3\}$. *Let* $R=\{(1,1),(1,3),(2,1),(2,3)\}$. *Here* $R=A\times B$, *so relation R is a universal relation.*

***05. RELATION ON A SET & ITS VARIOUS TYPES***

A relation R from a non-empty set A into itself is called a relation on A. In other words if A is a non-empty set, then a subset of $A\times A=A^2$ is called a relation on A.

***Illustrations:***

*Let* $A=\{1,2,3\}$ *and* $R=\{(3,1),(3,2),(2,1)\}$. *Here R is relation on set A.*

**<u>NOTE</u> If A be a finite set having *n* elements then, number of relations on set A is $2^{n\times n}$ *i.e.*, $2^{n^2}$.**

a) **Empty relation:** A relation R on a set A is said to be empty relation or a void relation if $R=\varphi$. In other words, a relation R in a set A is empty relation, if no element of A is related to any element of A, i.e., $R=\varphi\subset A\times A$.

   ***For example,*** *let* $A=\{1,3\}$, $R=\{(a,b): a\in A, b\in A \text{ and } a+b \text{ is odd}\}$. *Here R contains no element, therefore it is an empty relation on set A.*

b) **Universal relation:** A relation R on a set A is said to be the universal relation on A if $R=A\times A$ i.e., $R=A^2$. In other words, a relation R in a set A is universal relation, if each element of A is related to every element of A, i.e., $R=A\times A$.

   ***For example,*** *let* $A=\{1,2\}$. *Let* $R=\{(1,1),(1,2),(2,1),(2,2)\}$. *Here* $R=A\times A$, *so relation R is universal relation on A.*

   **<u>NOTE</u> The void relation *i.e.*, $\varphi$ and universal relation *i.e.*, $A\times A$ on A are respectively the *smallest* and *largest* relations defined on the set A. Also these are sometimes called *trivial relations*. And, any other relation is called a *non-trivial relation*.**

   ☞ **The relations $R=\varphi$ and $R=A\times A$ are two *extreme relations*.**

c) **Identity relation:** A relation R on a set A is said to be the identity relation on A if $R=\{(a,b): a\in A, b\in A \text{ and } a=b\}$.

   Thus identity relation $R=\{(a,a): \forall\ a\in A\}$.

   The identity relation on set A is also denoted by $I_A$.

***For example,*** *let* $A=\{1,2,3,4\}$. *Then* $I_A=\{(1,1),(2,2),(3,3),(4,4)\}$. *But the relation given by* $R=\{(1,1),(2,2),(1,3),(4,4)\}$ *is not an identity relation because element 1 is related to elements 1 and 3.*

**NOTE In an identity relation on A every element of A should be related to itself only.**

**d)** **Reflexive relation:** A relation R on a set A is said to be reflexive if $a\,\mathrm{R}\,a\ \forall a\in \mathrm{A}$ i.e., $(a,a)\in \mathrm{R}\ \forall a\in \mathrm{A}$.

***For example,*** *let* $A=\{1,2,3\}$, *and* $R_1$, $R_2$, $R_3$ *be the relations given as* $R_1=\{(1,1),(2,2),(3,3)\}$, $R_2=\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3)\}$ *and* $R_3=\{(2,2),(2,3),(3,2),(1,1)\}$. *Here* $R_1$ *and* $R_2$ *are reflexive relations on A but* $R_3$ *is not reflexive as* $3\in A$ *but* $(3,3)\notin R_3$.

**NOTE The identity relation is always a reflexive relation but the opposite may or may not be true. As shown in the example above, $R_1$ is both identity as well as reflexive relation on A but $R_2$ is only reflexive relation on A.**

**e)** **Symmetric relation:** A relation R on a set A is symmetric if $(a,b)\in \mathrm{R}\Rightarrow (b,a)\in \mathrm{R}\ \forall a,b\in \mathrm{A}$ i.e., $a\,\mathrm{R}\,b\Rightarrow b\,\mathrm{R}\,a$ (i.e., whenever $a\mathrm{R}b$ then, $b\mathrm{R}a$).

***For example,*** *let* $A=\{1,2,3\}$, $R_1=\{(1,2),(2,1)\}$, $R_2=\{(1,2),(2,1),(1,3),(3,1)\}$, $R_3=\{(2,3),(3,2),(2,2),(2,2)\}$ *i.e.* $R_3=\{(2,3),(3,2),(2,2)\}$ *and* $R_4=\{(2,3),(3,1),(1,3)\}$. *Here* $R_1$, $R_2$ *and* $R_3$ *are symmetric relations on A. But* $R_4$ *is not symmetric because* $(2,3)\in R_4$ *but* $(3,2)\notin R_4$.

**f)** **Transitive relation:** A relation R on a set A is transitive if $(a,b)\in \mathrm{R}$ and $(b,c)\in \mathrm{R}\Rightarrow (a,c)\in \mathrm{R}$ i.e., $a\,\mathrm{R}\,b$ and $b\,\mathrm{R}\,c\Rightarrow a\,\mathrm{R}\,c$.

***For example,*** *let* $A=\{1,2,3\}$, $R_1=\{(1,2),(2,3),(1,3),(3,2)\}$ *and* $R_2=\{(1,3),(3,2),(1,2)\}$. *Here* $R_2$ *is transitive relation whereas* $R_1$ *is not transitive because* $(2,3)\in R_1$ *and* $(3,2)\in R_1$ *but* $(2,2)\notin R_1$.

**g)** **Equivalence relation:** Let A be a non-empty set, then a relation R on A is said to be an equivalence relation if
(i) R is reflexive i.e. $(a,a)\in \mathrm{R}\ \forall a\in \mathrm{A}$ i.e., $a\mathrm{R}a$.
(ii) R is symmetric i.e. $(a,b)\in \mathrm{R}\Rightarrow (b,a)\in \mathrm{R}\ \forall a,b\in \mathrm{A}$ i.e., $a\mathrm{R}b\Rightarrow b\mathrm{R}a$.
(iii) R is transitive i.e. $(a,b)\in \mathrm{R}$ and $(b,c)\in \mathrm{R}\Rightarrow (a,c)\in \mathrm{R}\ \forall a,b,c\in \mathrm{A}$ i.e., $a\mathrm{R}b$ and $b\mathrm{R}c\Rightarrow a\mathrm{R}c$.

***For example,*** *let* $A=\{1,2,3\}$, $R=\{(1,2),(1,1),(2,1),(2,2),(3,3)\}$. *Here R is reflexive, symmetric and transitive. So R is an equivalence relation on A.*

- ***Equivalence classes****:* Let A be an equivalence relation in a set A and let $a\in \mathrm{A}$. Then, the set of all those elements of A which are related to $a$, is called equivalence class determined by $a$ and it is denoted by $[a]$. Thus, $[a]=\{b\in \mathrm{A}:(a,\,b)\in \mathrm{A}\}$.

  **NOTE (i) Two equivalence classes are either disjoint or identical.**
  **(ii) An equivalence relation R on a set A partitions the set into mutually disjoint equivalence classes.**

### *06. TABULAR REPRESENTATION OF A RELATION*

In this form of representation of a relation R from set A to set B, elements of A and B are written in the first column and first row respectively. If $(a,b)\in \mathrm{R}$ then we write ‘1’ in the row containing $a$ and column containing $b$ and if $(a,b)\notin \mathrm{R}$ then we write ‘0’ in the same manner.

***For example,*** *let* $A=\{1,2,3\}, B=\{2,5\}$ *and* $R=\{(1,2),(2,5),(3,2)\}$ *then,*

| ***R*** | ***2*** | ***5*** |
|---|---|---|
| ***1*** | *1* | *0* |
| ***2*** | *0* | *1* |
| ***3*** | *1* | *0* |

### *07. INVERSE RELATION*

Let $\mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$ be a relation from A to B. Then, the inverse relation of R, to be denoted by $\mathrm{R}^{-1}$, is a relation from B to A defined by $\mathrm{R}^{-1}=\{(b,a):(a,b)\in\mathrm{R}\}$.

Thus $(a,b)\in\mathrm{R} \Leftrightarrow (b,a)\in\mathrm{R}^{-1}\ \forall a\in\mathrm{A}, b\in\mathrm{B}$.

Clearly, $\mathrm{Dom.}\left(\mathrm{R}^{-1}\right)=\text{Range of R}$, $\text{Range of } \mathrm{R}^{-1}=\mathrm{Dom.}(\mathrm{R})$.

Also, $\left(\mathrm{R}^{-1}\right)^{-1}=\mathrm{R}$.

***For example,*** *let* $A=\{1,2,4\}, B=\{3,0\}$ *and let* $R=\{(1,3),(4,0),(2,3)\}$ *be a relation from A to B then,* $R^{-1}=\{(3,1),(0,4),(3,2)\}$.

**Summing up all the discussion given above, here is a recap of all these for quick grasp:**

| | |
|---|---|
| **01.** | **a)** A relation R from A to B is an empty relation or void relation iff $\mathrm{R}=\varphi$. |
| | **b)** A relation R on a set A is an empty relation or void relation iff $\mathrm{R}=\varphi$. |
| **02.** | **a)** A relation R from A to B is a universal relation iff $\mathrm{R}=\mathrm{A}\times\mathrm{B}$. |
| | **b)** A relation R on a set A is a universal relation iff $\mathrm{R}=\mathrm{A}\times\mathrm{A}$. |
| **03.** | A relation R on a set A is reflexive iff $a\mathrm{R}a, \forall a\in\mathrm{A}$. |
| **04.** | A relation R on set A is symmetric iff whenever $a$R$b$, then $b$R$a$ for all $a,b\in\mathrm{A}$. |
| **05.** | A relation R on a set A is transitive iff whenever $a$R$b$ and $b$R$c$, then $a$R$c$. |
| **06.** | A relation R on A is identity relation iff $\mathrm{R}=\{(a,a),\forall a\in\mathrm{A}\}$ i.e., R contains only elements of the type $(a,a)\forall a\in\mathrm{A}$ and it contains no other element. |
| **07.** | A relation R on a non-empty set A is an equivalence relation iff the following conditions are satisfied:<br>i) **R is reflexive** i.e., for every $a\in\mathrm{A}$, $(a,a)\in\mathrm{R}$ i.e., $a\mathrm{R}a$.<br>ii) **R is symmetric** i.e., for $a,b\in\mathrm{A}$, $a\mathrm{R}b\Rightarrow b\mathrm{R}a$ i.e., $(a,b)\in\mathrm{R}\Rightarrow(b,a)\in\mathrm{R}$.<br>iii) **R is transitive** i.e., for all $a,b,c\in\mathrm{A}$ we have, $a\mathrm{R}b$ and $b\mathrm{R}c\Rightarrow a\mathrm{R}c$ i.e., $(a,b)\in\mathrm{R}$ and $(b,c)\in\mathrm{R}\Rightarrow(a,c)\in\mathrm{R}$. |

## *B. FUNCTIONS*

### *01. CONSTANT & TYPES OF VARIABLES*

**a) Constant:** A constant is a symbol which retains the same value throughout a set of operations. So, a symbol which denotes a particular number is a constant. Constants are usually denoted by the symbols ***a, b, c, k, l, m,…*** etc.

**b) Variable:** It is a symbol which takes a number of values i.e., it can take any arbitrary values over the interval on which it has been defined. For example if $x$ is a variable over R (set of real numbers) then we mean that $x$ can denote any arbitrary real number. Variables are usually denoted by the symbols ***x, y, z, u, v,…*** etc.

**c) Independent variable:** That variable which can take an arbitrary value from a given set is termed as an independent variable.

**d) Dependent variable:** That variable whose value depends on the independent variable is called a dependent variable.

***02. Defining A Function:*** Consider A and B be two non- empty sets then, a rule $f$ which associates **each element of A with a unique element of B** is called a *function* or the *mapping from A to B* or *f maps A to B*. If $f$ is a mapping from $A$ to $B$ then, we write $f:\mathrm{A}\to\mathrm{B}$ which is read as *'f is a mapping from A to B'* or *'f is a function from A to B'*.

If $f$ associates $a\in\mathrm{A}$ to $b\in\mathrm{B}$, then we say that '***b* is the image of the element *a* under the function *f*'** or '***b* is the *f* - image of *a*'** or '**the value of *f* at *a***' and denote it by $f(a)$ and we write $b=f(a)$. The element $a$ is called the **pre-image** or **inverse-image of *b***.

Thus for a function from A to B,

*(i)* A and B should be non-empty.

*(ii)* Each element of A should have image in B.

*(iii) No element of A should have more than one image in B.*

*(iv)* If A and B have respectively $m$ and $n$ number of elements then the **number of functions defined from A to B is $n^m$**.

***03. Domain, Co-domain & Range of a function***

The **set A is called the domain** of the function $f$ and the **set B is called the co- domain**. The set of the images of all the elements of A under the function $f$ is called the **range of the function *f*** and is denoted as $f(\mathrm{A})$.

Thus range of the function $f$ is $f(\mathrm{A})=\{f(x):x\in\mathrm{A}\}$.

Clearly $f(\mathrm{A})\subseteq\mathrm{B}$.

➲ ***Note the followings:***

i) It is necessary that every $f$ -image is in B; but there may be some elements in B which are not the $f$-images of any element of A i.e., whose pre-image under $f$ is not in A.

ii) Two or more elements of A may have same image in B.

iii) $f:x\to y$ means that under the function $f$ from A to B, an element $x$ of A has image $y$ in B.

iv) Usually we denote the function $f$ by writing $y=f(x)$ and read it as '***y* is a function of *x***'.

**<u>POINTS TO REMEMBER FOR FINDING THE DOMAIN & RANGE</u>**

**Domain:** If a function is expressed in the form $y=f(x)$, then domain of $f$ means **set of all those real values of *x* for which *y* is real (i.e., *y* is well - defined)**.

Remember the following points:

*i) Negative number should not occur under the square root (even root) i.e., expression under the square root sign must be always $\geq 0$.*

*ii) Denominator should never be zero.*

*iii) For $\log_b a$ to be defined $a>0$, $b>0$ and $b\neq 1$. Also note that $\log_b 1$ is equal to zero i.e. $0$.*

**Range:** If a function is expressed in the form $y=f(x)$, then range of $f$ means **set of all possible real values of *y* corresponding to every value of *x* in its domain**.

Remember the following points:

*i) Firstly find the domain of the given function.*

*ii) If the domain does not contain an interval, then find the values of y putting these values of $x$ from the domain. The set of all these values of y obtained will be the range.*

*iii) If domain is the set of all real numbers R or set of all real numbers except a few points, then express x in terms of y and from this find the real values of y for which x is real and belongs to the domain.*

***04. Function as a special type of relation:*** A relation $f$ from a set A to another set B is said be a function (or mapping) from A to B if with every element (say $x$) of A, the relation $f$ relates a unique element (say $y$) of B. This $y$ is called $f$-image of $x$. Also $x$ is called pre-image of $y$ under $f$.

***05. Difference between relation and function:*** A relation from a set A to another set B is any subset of $A \times B$; while a function $f$ from A to B is a subset of $A \times B$ satisfying following conditions:

*i) For every* $x \in A$, there exists $y \in B$ such that $(x, y) \in f$

*ii) If* $(x, y) \in f$ and $(x, z) \in f$ then, $y = z$.

| Sl. No. | Function | Relation |
|---|---|---|
| 01. | Each element of A must be related to some element of B. | There may be some element of A which are not related to any element of B. |
| 02. | An element of A should not be related to more than one element of B. | An element of A may be related to more than one elements of B. |

***06. Real valued function of a real variable:*** If the domain and range of a function $f$ are subsets of R (the set of real numbers), then $f$ is said to be **a real valued function of a real variable** or **a real function**.

***07. Some important real functions and their domain & range***

| FUNCTION | REPRESENTATION | DOMAIN | RANGE |
|---|---|---|---|
| **a)** Identity function | $I(x) = x \ \forall x \in R$ | R | R |
| **b)** Modulus function or Absolute value function | $f(x) = \lvert x\rvert = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$ | R | $[0, \infty)$ |
| **c)** Greatest integer function or Integral function or Step function | $f(x) = [x]$ or $f(x) = \lfloor x \rfloor \ \forall x \in R$ | R | Z |
| **d)** Smallest integer function | $f(x) = \lceil x \rceil \ \forall x \in R$ | R | Z |
| **e)** Signum function | $f(x) = \begin{cases} \dfrac{\lvert x\rvert}{x} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$ i.e., $f(x) = \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$ | R | $\{-1, 0, 1\}$ |
| **f)** Exponential function | $f(x) = a^x \ \forall a \neq 1, a > 0$ | R | $(0, \infty)$ |
| **g)** Logarithmic function | $f(x) = \log_a x \ \ \forall a \neq 1, a > 0$ and $x > 0$ | $(0, \infty)$ | R |

***08. TYPES OF FUNCTIONS***

**a) One-one function (Injective function or Injection):** A function $f : A \to B$ is one- one function or injective function if distinct elements of A have distinct images in B.

Thus, $f : A \to B$ is one-one $\Leftrightarrow f(a) = f(b) \Rightarrow a = b \ \forall a, b \in A$

$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b) \ \forall a, b \in A$.

➲ If A and B are two sets having $m$ and $n$ elements respectively such that $m \leq n$, then **total number of one-one functions** from set A to set B is ${}^nC_m \times m!$ *i.e.,* ${}^nP_m$.

<u>**ALGORITHM TO CHECK THE INJECTIVITY OF A FUNCTION**</u>

**STEP1-** Take any two arbitrary elements *a, b* in the domain of $f$.

**STEP2-** Put $f(a) = f(b)$.

**STEP3-** Solve $f(a) = f(b)$. If it gives $a = b$ only, then $f$ is a one-one function.

**b) Onto function (Surjective function or Surjection):** A function $f : \mathrm{A} \to \mathrm{B}$ is onto function or a surjective function if every element of B is the *f*-image of some element of A. That implies $f(\mathrm{A}) = \mathrm{B}$ or range of *f* is the co-domain of *f*.
Thus, $f : \mathrm{A} \to \mathrm{B}$ is onto $\Leftrightarrow f(\mathrm{A}) = \mathrm{B}$ *i.e.*, range of $f =$ co-domain of $f$.

**ALGORITHM TO CHECK THE SURJECTIVITY OF A FUNCTION**

**STEP1-** Take an element $b \in \mathrm{B}$.
**STEP2-** Put $f(x) = b$.
**STEP3-** Solve the equation $f(x) = b$ for *x* and obtain *x* in terms of *b*. Let $x = g(b)$.
**STEP4-** If for all values of $b \in \mathrm{B}$, the values of *x* obtained from $x = g(b)$ are in A, then *f* is onto. If there are some $b \in \mathrm{B}$ for which values of *x,* given by $x = g(b)$, is not in A. Then *f* is not onto.

**c) One-one onto function (Bijective function or Bijection):** A function $f : \mathrm{A} \to \mathrm{B}$ is said to be one-one onto or bijective if it is both one-one and onto i.e., if the distinct elements of A have distinct images in B and each element of B is the image of some element of A.

- Also note that a **bijective function is also called a one-to-one function or one-to-one correspondence.**
- If $f : \mathrm{A} \to \mathrm{B}$ is a function such that,
  *i) f is one-one* $\Rightarrow n(\mathrm{A}) \le n(\mathrm{B})$.
  *ii) f is onto* $\Rightarrow n(B) \le n(\mathrm{A})$.
  *iii) f is one-one onto* $\Rightarrow n(\mathrm{A}) = n(\mathrm{B})$.
- For an ordinary finite set A, a one-one function $f : \mathrm{A} \to \mathrm{A}$ is necessarily onto and an onto function $f : \mathrm{A} \to \mathrm{A}$ is necessarily one-one for every finite set A.

**d) Identity Function:** The function $I_A : \mathrm{A} \to \mathrm{A}$; $I_{\mathrm{A}}(x) = x \,\forall\, x \in \mathrm{A}$ is called an identity function on A.
**NOTE** Domain$\left(I_{\mathrm{A}}\right) = \mathrm{A}$ and Range$\left(I_{\mathrm{A}}\right) = \mathrm{A}$.

**e) Equal Functions:** Two function *f* and *g* having the same domain *D* are said to be equal if $f(x) = g(x)$ for all $x \in D$.

***09. COMPOSITION OF FUNCTIONS***

Let $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ be any two functions. Then *f* maps an element $x \in \mathrm{A}$ to an element $f(x) = y \in \mathrm{B}$; and this *y* is mapped by *g* to an element $z \in \mathrm{C}$. Thus, $z = g(y) = g\left(f(x)\right)$. Therefore, we have a rule which associates with each $x \in \mathrm{A}$, a unique element $z = g\left(f(x)\right)$ of $\mathrm{C}$. This rule is therefore a mapping from A to C. We denote this mapping by *gof* (read as ***g composition f***) and call it the '**composite mapping of *f* and *g***'.

**Definition:** *If $f : A \to B$ and $g : B \to C$ be any two mappings (functions), then the composite mapping gof of f and g is defined by $gof : A \to C$ such that $\left(gof\right)(x) = g\left(f(x)\right)$ for all $x \in A$.*

- The composite of two functions is also called the **resultant of two functions** or **the function of a function**.
- Note that for the composite function *gof* to exist, it is essential that range of *f* must be a subset of the domain of *g*.
- Observe that the order of events occur from the right to left i.e., *gof* reads composite of *f* and *g* and it means that we have to first apply *f* and then, follow it up with *g*.
- *Dom.*(*gof*) = *Dom.*(*f*) and *Co-dom.*(*gof*) = *Co-dom.*(*g*).
- Remember that *gof* is well defined only if *Co-dom.*(*f*) = *Dom.*(*g*).
- If *gof* is defined then, it is not necessary that *fog* is also defined.

- If $gof$ is one-one then $f$ is also one-one function. Similarly, if $gof$ is onto then $g$ is onto function.

***10. INVERSE OF A FUNCTION***

Let $f: \mathrm{A} \to \mathrm{B}$ be a bijection. Then a function $g: \mathrm{B} \to \mathrm{A}$ which associates each element $y \in \mathrm{B}$ to a unique element $x \in \mathrm{A}$ such that $f(x) = y$ is called the inverse of $f$ i.e., $f(x) = y \Leftrightarrow g(y) = x$.

The inverse of $f$ is generally denoted by $f^{-1}$.

Thus, if $f: \mathrm{A} \to \mathrm{B}$ is a bijection, then a function $f^{-1}: \mathrm{B} \to \mathrm{A}$ is such that $f(x) = y \Leftrightarrow f^{-1}(y) = x$.

**ALGORITHM TO FIND THE INVERSE OF A FUNCTION**

**STEP1-** Put $f(x) = y$ where $y \in \mathrm{B}$ and $x \in \mathrm{A}$.

**STEP2-** Solve $f(x) = y$ to obtain $x$ in terms of $y$.

**STEP3-** Replace $x$ by $f^{-1}(y)$ in the relation obtained in STEP2.

**STEP4-** In order to get the required inverse of $f$ i.e. $f^{-1}(x)$, replace $y$ by $x$ in the expression obtained in STEP3 i.e. in the expression $f^{-1}(y)$.

## *C. BINARY OPERATIONS*

***01. Definition & Basic Understanding***

An operation is a process which produces a new element from two given elements; e.g., addition, subtraction, multiplication and division of numbers. If the new element belongs to the same set to which the two given elements belong, the operation is called a binary operation.

**DEFINITION:** A binary operation (or binary composition) $*$ on a non-empty set A is a mapping which associates with each ordered pair $(a,b) \in \mathrm{A} \times \mathrm{A}$ a unique element of A. This unique element is denoted by $a * b$.

Thus, a binary operation $*$ on A is a mapping $*: \mathrm{A} \times \mathrm{A} \to \mathrm{A}$ defined by $*(a,b) = a * b$.

Clearly $a \in \mathrm{A},\ b \in \mathrm{A} \Rightarrow a * b \in \mathrm{A}$.

__NOTE__ **i)** If set A has *m* elements, then number of binary operations on A is $m^{m \times m}$ i.e., $m^{m^2}$.

**ii)** If set A has *m* elements, then number of Commutative binary operations on A is $m^{m(m-1)/2}$.

***02. Terms related to binary operations***

- A ***binary operation*** $*$ on a set A is a function $*$ from $\mathrm{A} \times \mathrm{A}$ to A.
- An operation $*$ on A is ***commutative*** if $a * b = b * a \quad \forall\, a,b \in \mathrm{A}$.
- An operation $*$ on A is ***associative*** if $(a * b) * c = a * (b * c) \ \forall\, a,b,c \in \mathrm{A}$.
- An operation $*$ on A is ***distributive*** if $a * (b \otimes c) = (a * b) \otimes (a * c) \ \forall\, a,b,c \in \mathrm{A}$. Also an operation $*$ on A is ***distributive*** if $(b \otimes c) * a = (b * a) \otimes (c * a) \ \forall\, a,b,c \in \mathrm{A}$.
- An ***identity element*** of a binary operation $*$ on A is an element $e \in \mathrm{A}$ such that $e * a = a * e = a \ \ \forall\, a \in \mathrm{A}$. Identity element for a binary operation, if it exists is **unique**.

***03. Inverse of an element***

Let $*$ be a binary operation on a non-empty set A and $e$ be the identity element for the binary operation $*$. An element $b \in \mathrm{A}$ is called the inverse element or simply inverse of *a* for binary operation $*$ if $a * b = b * a = e$.

Thus, an element $a \in \mathrm{A}$ is invertible if and only if its inverse exists.

# Inverse Trigonometric Functions

## Important Terms, Definitions & Formulae

### *01. Trigonometric Formulae:*

***Relation between trigonometric ratios***

***a)*** $\tan\theta = \frac{\sin\theta}{\cos\theta}$ ***b)*** $\tan\theta = \frac{1}{\cot\theta}$ ***c)*** $\tan\theta.\cot\theta = 1$

***d)*** $\cot\theta = \frac{\cos\theta}{\sin\theta}$ ***e)*** $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ ***f)*** $\sec\theta = \frac{1}{\cos\theta}$

***Trigonometric identities***

***a)*** $\sin^2\theta + \cos^2\theta = 1$

***b)*** $1 + \tan^2\theta = \sec^2\theta$

***c)*** $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

***Addition / subtraction formulae & some related results***

***a)*** $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

***b)*** $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

***c)*** $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

***d)*** $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

***e)*** $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

***f)*** $\cot(A \pm B) = \frac{\cot B\cot A \mp 1}{\cot B \pm \cot A}$

***Transformation of sums / differences into products & vice-versa***

***a)*** $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

***b)*** $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

***c)*** $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

***d)*** $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

***e)*** $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

***f)*** $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

***g)*** $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

***h)*** $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

***Multiple angle formulae involving 2A and 3A***

***a)*** $\sin 2A = 2\sin A\cos A$

***b)*** $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

***c)*** $\cos 2A = \cos^2 A - \sin^2 A$

***d)*** $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

***e)*** $\cos 2A = 2\cos^2 A - 1$

***f)*** $2\cos^2 A = 1 + \cos 2A$

***g)*** $\cos 2A = 1 - 2\sin^2 A$

***h)*** $2\sin^2 A = 1 - \cos 2A$

***i)*** $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$

***j)*** $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

***k)*** $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

***l)*** $\sin 3A = 3\sin A - 4\sin^3 A$

***m)*** $\cos 3A = 4\cos^3 A - 3\cos A$

***n)*** $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

***Relations in Different Measures of Angle***

➲ $\text{Angle in Radian Measure} = (\text{Angle in Degree Measure}) \times \frac{\pi}{180}$

➲ $\text{Angle in Degree Measure} = (\text{Angle in Radian Measure}) \times \frac{180}{\pi}$

➲ $\theta\,(\text{in radian measure}) = \frac{l}{r}$

*Also followings are of importance as well:*

➲ $1\,\text{Right angle} = 90^o$ ➲ $1^o = 60'$, $1' = 60''$

➲ $1^o = \dfrac{\pi}{180} = 0.01745 \text{ radians} (\text{approximately})$ ➲ $1 \text{ radian} = 57^o 17' 45''$ *or* $206265$ *seconds* .

***General Solutions***

***a)*** $\sin x = \sin y \;\Rightarrow\; x = n\pi + (-1)^n y$, *where* $n \in Z$ .

***b)*** $\cos x = \cos y \;\Rightarrow\; x = 2n\pi \pm y$, *where* $n \in Z$ .

***c)*** $\tan x = \tan y \;\Rightarrow\; x = n\pi + y$, *where* $n \in Z$ .

***Relation in Degree & Radian Measures***

| **Angles in Degree** | $0^\circ$ | $30^\circ$ | $45^\circ$ | $60^\circ$ | $90^\circ$ | $180^\circ$ | $270^\circ$ | $360^\circ$ |
|---|---|---|---|---|---|---|---|---|
| **Angles in Radian** | $0^c$ | $\left(\dfrac{\pi}{6}\right)^c$ | $\left(\dfrac{\pi}{4}\right)^c$ | $\left(\dfrac{\pi}{3}\right)^c$ | $\left(\dfrac{\pi}{2}\right)^c$ | $(\pi)^c$ | $\left(\dfrac{3\pi}{2}\right)^c$ | $(2\pi)^c$ |

➯ *In actual practice, we omit the exponent '**c**' and instead of writing* $\pi^c$ *we simply write* $\pi$ *and similarly for others.*

***Trigonometric Ratio of Standard Angles***

| **Degree /Radian** $(\rightarrow)$ | $0^\circ$ | $30^\circ$ | $45^\circ$ | $60^\circ$ | $90^\circ$ |
|---|---|---|---|---|---|
| **T – Ratios** $(\downarrow)$ | $0$ | $\dfrac{\pi}{6}$ | $\dfrac{\pi}{4}$ | $\dfrac{\pi}{3}$ | $\dfrac{\pi}{2}$ |
| sin | $0$ | $\dfrac{1}{2}$ | $\dfrac{1}{\sqrt{2}}$ | $\dfrac{\sqrt{3}}{2}$ | $1$ |
| cos | $1$ | $\dfrac{\sqrt{3}}{2}$ | $\dfrac{1}{\sqrt{2}}$ | $\dfrac{1}{2}$ | $0$ |
| tan | $0$ | $\dfrac{1}{\sqrt{3}}$ | $1$ | $\sqrt{3}$ | $\infty$ |
| cosec | $\infty$ | $2$ | $\sqrt{2}$ | $\dfrac{2}{\sqrt{3}}$ | $1$ |
| sec | $1$ | $\dfrac{2}{\sqrt{3}}$ | $\sqrt{2}$ | $2$ | $\infty$ |
| cot | $\infty$ | $\sqrt{3}$ | $1$ | $\dfrac{1}{\sqrt{3}}$ | $0$ |

***Trigonometric Ratios of Allied Angles***

| **Angles** $(\rightarrow)$ / **T- Ratios** $(\downarrow)$ | $\dfrac{\pi}{2}-\theta$ | $\dfrac{\pi}{2}+\theta$ | $\pi-\theta$ | $\pi+\theta$ | $\dfrac{3\pi}{2}-\theta$ | $\dfrac{3\pi}{2}+\theta$ | $2\pi-\theta$ **OR** $-\theta$ | $2\pi+\theta$ |
|---|---|---|---|---|---|---|---|---|
| sin | $\cos\theta$ | $\cos\theta$ | $\sin\theta$ | $-\sin\theta$ | $-\cos\theta$ | $-\cos\theta$ | $-\sin\theta$ | $\sin\theta$ |
| cos | $\sin\theta$ | $-\sin\theta$ | $-\cos\theta$ | $-\cos\theta$ | $-\sin\theta$ | $\sin\theta$ | $\cos\theta$ | $\cos\theta$ |
| tan | $\cot\theta$ | $-\cot\theta$ | $-\tan\theta$ | $\tan\theta$ | $\cot\theta$ | $-\cot\theta$ | $-\tan\theta$ | $\tan\theta$ |
| cot | $\tan\theta$ | $-\tan\theta$ | $-\cot\theta$ | $\cot\theta$ | $\tan\theta$ | $-\tan\theta$ | $-\cot\theta$ | $\cot\theta$ |
| sec | $\text{cosec}\,\theta$ | $-\text{cosec}\,\theta$ | $-\sec\theta$ | $-\sec\theta$ | $-\text{cosec}\,\theta$ | $\text{cosec}\,\theta$ | $\sec\theta$ | $\sec\theta$ |
| cosec | $\sec\theta$ | $\sec\theta$ | $\text{cosec}\,\theta$ | $-\text{cosec}\,\theta$ | $-\sec\theta$ | $-\sec\theta$ | $-\text{cosec}\,\theta$ | $\text{cosec}\,\theta$ |

***02.*** **a)** $\sin^{-1}(x)=\operatorname{cosec}^{-1}\left(\frac{1}{x}\right),\ x\in[-1,1]$ **b)** $\operatorname{cosec}^{-1}(x)=\sin^{-1}\left(\frac{1}{x}\right),\ x\in(-\infty,-1]\cup[1,\infty)$

**c)** $\cos^{-1}(x)=\sec^{-1}\left(\frac{1}{x}\right),\ x\in[-1,1]$ **d)** $\sec^{-1}(x)=\cos^{-1}\left(\frac{1}{x}\right),\ x\in(-\infty,-1]\cup[1,\infty)$

**e)** $\tan^{-1}(x)=\begin{cases}\cot^{-1}\left(\frac{1}{x}\right), & x>0\\ -\pi+\cot^{-1}\left(\frac{1}{x}\right), & x<0\end{cases}$ **f)** $\cot^{-1}(x)=\begin{cases}\tan^{-1}\left(\frac{1}{x}\right), & x>0\\ \pi+\tan^{-1}\left(\frac{1}{x}\right), & x<0\end{cases}$

***03.*** **a)** $\sin^{-1}(-x)=-\sin^{-1}x,\ x\in[-1,1]$ **b)** $\cos^{-1}(-x)=\pi-\cos^{-1}x,\ x\in[-1,1]$

**c)** $\tan^{-1}(-x)=-\tan^{-1}x,\ x\in\mathrm{R}$ **d)** $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x,\ |x|\geq 1$

**e)** $\sec^{-1}(-x)=\pi-\sec^{-1}x,\ |x|\geq 1$ **f)** $\cot^{-1}(-x)=\pi-\cot^{-1}x,\ x\in\mathrm{R}$

***04.*** **a)** $\sin^{-1}(\sin x)=x,\ -\frac{\pi}{2}\leq x\leq\frac{\pi}{2}$ **b)** $\cos^{-1}(\cos x)=x,\ 0\leq x\leq\pi$

**c)** $\tan^{-1}(\tan x)=x,\ -\frac{\pi}{2}<x<\frac{\pi}{2}$ **d)** $\operatorname{cosec}^{-1}(\operatorname{cosec} x)=x,\ -\frac{\pi}{2}\leq x\leq\frac{\pi}{2}, x\neq 0$

**e)** $\sec^{-1}(\sec x)=x,\ 0\leq x\leq\pi,\ x\neq\frac{\pi}{2}$ **f)** $\cot^{-1}(\cot x)=x,\ 0<x<\pi$

***05.*** **a)** $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2},\ x\in[-1,1]$

**b)** $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2},\ x\in\mathrm{R}$

**c)** $\operatorname{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2}, |x|\geq 1$ *i.e.*, $x\leq -1$ or $x\geq 1$

***06.*** **a)** $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\left[x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right]$

**b)** $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\left[xy\mp\sqrt{1-x^2}\sqrt{1-y^2}\right]$

**c)** $\tan^{-1}x+\tan^{-1}y=\begin{cases}\tan^{-1}\left(\frac{x+y}{1-xy}\right), & xy<1\\ \pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), & x>0,\ y>0,\ xy>1\\ -\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), & x<0,\ y<0,\ xy>1\end{cases}$

**d)** $\tan^{-1}x-\tan^{-1}y=\begin{cases}\tan^{-1}\left(\frac{x-y}{1+xy}\right), & xy>-1\\ \pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right), & x>0,\ y<0,\ xy<-1\\ -\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right), & x<0,\ y>0,\ xy<-1\end{cases}$

**e)** $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

***07.*** **a)** $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right),\ |x|\leq 1$

**b)** $2\tan^{-1}x = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right),\ x \geq 0$

**c)** $2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right),\ -1 < x < 1$

***08. Principal Value:*** Numerically *smallest angle* is known as the principal value.

**<u>Finding the principal value:</u>** For finding the principal value, following algorithm can be followed–

**STEP1–** Firstly, draw a trigonometric circle and mark the quadrant in which the angle may lie.

**STEP2–** Select anticlockwise direction for 1st and 2nd quadrants and clockwise direction for 3rd and 4th quadrants.

**STEP3–** Find the angles in the first rotation.

**STEP4–** Select the numerically least (magnitude wise) angle among these two values. The angle thus found will be the principal value.

**STEP5–** In case, two angles one with positive sign and the other with the negative sign qualify for the numerically least angle then, it is the convention to select the angle with positive sign as principal value.

*The principal value is **never** numerically greater than* $\pi$ .

***09. Table demonstrating domains and ranges of Inverse Trigonometric functions:***

| Inverse Trigonometric Functions *i.e.,* $f(x)$ | Domain/ Values of $x$ | Range/ Values of $f(x)$ |
|---|---|---|
| $\sin^{-1}x$ | $[-1, 1]$ | $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ |
| $\cos^{-1}x$ | $[-1, 1]$ | $[0, \pi]$ |
| $\text{cosec}^{-1}x$ | $R-(-1, 1)$ | $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]-\{0\}$ |
| $\sec^{-1}x$ | $R-(-1, 1)$ | $[0, \pi]-\left\{\dfrac{\pi}{2}\right\}$ |
| $\tan^{-1}x$ | R | $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ |
| $\cot^{-1}x$ | R | $(0, \pi)$ |

**Discussion about the range of inverse circular functions other than their respective principal value branch**

We know that the domain of sine function is the set of real numbers and range is the closed interval [–1, 1]. If we restrict its domain to $\left[-\dfrac{3\pi}{2}, -\dfrac{\pi}{2}\right]$, $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$ etc. then, it becomes bijective with the range [–1, 1]. So, we can define the inverse of sine function in each of these intervals. Hence, all the intervals of $\sin^{-1}$ function, **except principal value branch** (here except of $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ for $\sin^{-1}$ function) are known as the **range of $\sin^{-1}$ other than its principal value branch**. The same discussion can be extended for other inverse circular functions.

***10. To simplify inverse trigonometrical expressions, following substitutions can be considered:***

| Expression | Substitution |
|---|---|
| $a^2+x^2$ or $\sqrt{a^2+x^2}$ | $x=a\tan\theta$ or $x=a\cot\theta$ |
| $a^2-x^2$ or $\sqrt{a^2-x^2}$ | $x=a\sin\theta$ or $x=a\cos\theta$ |
| $x^2-a^2$ or $\sqrt{x^2-a^2}$ | $x=a\sec\theta$ or $x=a\ \text{cosec}\,\theta$ |
| $\sqrt{\dfrac{a-x}{a+x}}$ or $\sqrt{\dfrac{a+x}{a-x}}$ | $x=a\cos 2\theta$ |
| $\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\dfrac{a^2+x^2}{a^2-x^2}}$ | $x^2=a^2\cos 2\theta$ |
| $\sqrt{\dfrac{x}{a-x}}$ or $\sqrt{\dfrac{a-x}{x}}$ | $x=a\sin^2\theta$ or $x=a\cos^2\theta$ |
| $\sqrt{\dfrac{x}{a+x}}$ or $\sqrt{\dfrac{a+x}{x}}$ | $x=a\tan^2\theta$ or $x=a\cot^2\theta$ |

*Note the followings and keep them in mind:*

➲ *The symbol $sin^{-1}x$ is used to denote the* ***smallest angle*** *whether positive or negative, the sine of this angle will give us x. Similarly $cos^{-1}x$, $tan^{-1}x$, $cosec^{-1}x$, $sec^{-1}x$, and $cot^{-1}x$ are defined.*

➲ *You should note that $sin^{-1}x$ can be written as* ***arcsinx***. *Similarly other Inverse Trigonometric Functions can also be written as arccosx, arctanx, arcsecx etc.*

➲ *Also* ***note that*** *$sin^{-1}x$ (and similarly other Inverse Trigonometric Functions)* ***is entirely different from*** *$(sin\,x)^{-1}$. In fact, $sin^{-1}x$ is the measure of an angle in Radians whose sine is x whereas $(sin\,x)^{-1}$ is $\dfrac{1}{sin\,x}$ (which is obvious as per the* ***laws of exponents****).*

➲ *Keep in mind that these inverse trigonometric relations are* ***true only in their domains*** *i.e., they are valid only for some values of 'x' for which inverse trigonometric functions are well defined!*

# Algebra Of Matrices & Determinants

## Important Terms, Definitions & Formulae

***01. Matrix - a basic introduction:*** A matrix having m rows and n columns is called a matrix of order $m \times n$ (read as '$m$ *by* $n$' matrix). And a matrix A of order $m \times n$ is depicted as $A = \left[a_{ij}\right]_{m\times n}$; $i, j \in \mathrm{N}$.

➢ Also in general, $a_{ij}$ means an element lying in the $i^{th}$ row and $j^{th}$ column.

➢ No. of elements in the matrix $A = \left[a_{ij}\right]_{m\times n}$ is given as $(m)(n)$.

***02. Types of Matrices:***

***a) Column matrix:*** A matrix having only one column is called a *column matrix* or *column vector*.

e.g. $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times1}$, $\begin{bmatrix} 8 \\ 5 \end{bmatrix}_{2\times1}$.

➲ **General notation:** $A = \left[a_{ij}\right]_{m\times 1}$.

***b) Row matrix:*** A matrix having only one row is called a *row matrix* or *row vector*.

e.g. $\begin{bmatrix} -1 & 2 & \sqrt{3} & 4 \end{bmatrix}_{1\times4}$, $\begin{bmatrix} 2 & 5 & 0 \end{bmatrix}_{1\times3}$

➲ **General notation:** $A = \left[a_{ij}\right]_{1\times n}$.

***c) Square matrix:*** It is a matrix in which the number of rows is equal to the number of columns *i.e.*, an $m \times n$ matrix is said to constitute a square matrix if $m = n$ and is known as a ***square matrix of order 'n'***.

e.g. $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times3}$ is a square matrix of order 3.

➲ **General notation:** $A = \left[a_{ij}\right]_{n\times n}$.

***d) Diagonal matrix:*** A square matrix $A = \left[a_{ij}\right]_{m\times m}$ is said to be a *diagonal matrix* if $a_{ij} = 0$, when $i \neq j$ *i.e.,* all its non- diagonal elements are *zero*.

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times3}$ is a diagonal matrix of order 3.

➢ *Also there is **one more notation** specifically used for the diagonal matrices. For instance, consider the matrix depicted above, it can be also written as* $diag\begin{pmatrix} 2 & 5 & 4 \end{pmatrix}$.

➢ *Note that the elements* $a_{11}, a_{22}, a_{33}, \ldots, a_{mm}$ *of a square matrix* $A = \left[a_{ij}\right]_{m\times n}$ *of **order m** are said to constitute the **principal diagonal** or simply **the diagonal of the square matrix A**. And these elements are known as **diagonal elements of matrix A**.*

***e) Scalar matrix:*** A *diagonal matrix* $A = \left[a_{ij}\right]_{m\times m}$ is said to be a *scalar matrix* if its *diagonal elements* are **equal** *i.e.*, $a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ k, & \text{when } i = j \text{ for some constant } k \end{cases}$.

e.g. $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3\times3}$ is a scalar matrix of order 3.

***f) Unit or Identity matrix:*** A square matrix $A = \left[a_{ij}\right]_{m\times n}$ is said to be an *identity matrix* if $a_{ij} = \begin{cases} 1, & if \;\; i = j \\ 0, & if \;\; i \neq j \end{cases}$.

A *unit matrix* can also be defined as the *scalar matrix* each of whose diagonal elements is *unity*. We denote the identity matrix of order $m$ by $\mathrm{I}_m$ or $\mathrm{I}$.

e.g. $\mathrm{I}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $\mathrm{I}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

***g) Zero matrix or Null matrix:*** A matrix is said to be a *zero matrix* or *null matrix* if each of its elements is *zero*.

e.g. $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$, $\begin{bmatrix}0&0\\0&0\end{bmatrix}$, $\begin{bmatrix}0&0\end{bmatrix}$.

***h) Horizontal matrix:*** A $m\times n$ matrix is said to be a *horizontal matrix* if $m < n$.

e.g. $\begin{bmatrix}1&2&0\\5&4&7\end{bmatrix}_{2\times3}$.

***i) Vertical matrix:*** A $m\times n$ matrix is said to be a *vector matrix* if $m > n$.

e.g. $\begin{bmatrix}2&5\\0&7\\3&1\end{bmatrix}_{3\times2}$.

***j) Triangular matrix:***

***<u>Lower triangular matrix:</u>*** A square matrix is called a lower triangular matrix if $a_{ij}=0$ *when* $i<j$.

e.g. $\begin{bmatrix}1&0&0\\2&2&0\\0&5&3\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&0&0\\0&5&0\end{bmatrix}$, $\begin{bmatrix}2&0&0\\3&2&0\\4&5&7\end{bmatrix}$.

***<u>Upper triangular matrix:</u>*** A square matrix is called an upper triangular matrix if $a_{ij}=0$ *when* $i>j$.

e.g. $\begin{bmatrix}1&2&4\\0&5&8\\0&0&3\end{bmatrix}$, $\begin{bmatrix}1&3&9\\0&0&0\\0&0&5\end{bmatrix}$.

***03. Properties of matrix addition:***

- *Commutative property:* $A+B=B+A$
- *Associative property:* $A+(B+C)=(A+B)+C$
- *Cancellation laws:* ***i)*** *Left cancellation-* $A+B=A+C\Rightarrow B=C$
  ***ii)*** *Right cancellation-* $B+A=C+A\Rightarrow B=C$.

***04. Properties of matrix multiplication:***

- Note that the **product *AB* is defined *only* when** the number of columns in matrix $A$ is equal to the number of rows in matrix $B$.
- If $A$ and $B$ are $m\times n$ and $n\times p$ matrices respectively then matrix $AB$ will be an $m\times p$ matrix *i.e.,* order of matrix $AB$ will be $m\times p$.
- In the product $AB$, $A$ is called the ***pre-factor*** and $B$ is called the ***post-factor***.
- If the product $AB$ is possible then it is ***not necessary*** that the product $BA$ is also possible.
- If $A$ is a $m\times n$ matrix and both $AB$ and $BA$ are defined then $B$ will be a $n\times m$ matrix.
- If $A$ is a $n\times n$ matrix and $\mathrm{I}_n$ be the unit matrix of order $n$ then, $A\,\mathrm{I}_n=\mathrm{I}_n\,A=A$.
- Matrix multiplication is ***associative*** *i.e.,* $A(BC)=(AB)C$.
- Matrix multiplication is ***distributive over the addition*** *i.e.,* $A(B+C)=AB+AC$.

➲ **<u>Idempotent matrix:</u>** A square matrix $A$ is said to be an *idempotent matrix* if $\boldsymbol{A^2=A}$.

For example, $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}0&1\\0&1\end{bmatrix}$.

***05. Transpose of a Matrix:*** If $A=\left[a_{ij}\right]_{m\times n}$ be an $m\times n$ matrix, then the matrix obtained by interchanging the rows and columns of matrix $A$ is said to be a ***transpose of matrix A***. The transpose of $A$ is denoted by $A'$ or $A^T$ or $A^c$ *i.e.,* if $A=\left[a_{ij}\right]_{m\times n}$ then, $A^T=\left[a_{ji}\right]_{n\times m}$.

For example, $\begin{bmatrix}3&2&0\\1&-2&6\end{bmatrix}^T=\begin{bmatrix}3&1\\2&-2\\0&6\end{bmatrix}$.

**➲ Properties of Transpose of matrices:**

- $(A+B)^T = A^T + B^T$
- $(A-B)^T = A^T - B^T$
- $(A^T)^T = A$
- $(kA)^T = kA^T$ where, $k$ is any constant
- $(AB)^T = B^T A^T$
- $(ABC)^T = C^T B^T A^T$

***06. Symmetric matrix:*** A square matrix $A=\left[a_{ij}\right]$ is said to be a *symmetric matrix* if $A^T = A$.

That is, if $A=\left[a_{ij}\right]$ then, $A^T=\left[a_{ji}\right]=\left[a_{ij}\right] \Rightarrow A^T = A$.

For example: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \begin{bmatrix} 2+i & 1 & 3 \\ 1 & 2 & 3+2i \\ 3 & 3+2i & 4 \end{bmatrix}$.

***07. Skew-symmetric matrix:*** A square matrix $A=\left[a_{ij}\right]$ is said to be a *skew-symmetric matrix* if $A^T = -A$

*i.e.*, if $A=\left[a_{ij}\right]$ then, $A^T=\left[a_{ji}\right]=-\left[a_{ij}\right] \Rightarrow A^T = -A$.

For example: $\begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

***Facts you should know:***

- Note that $\left[a_{ji}\right]=-\left[a_{ij}\right] \Rightarrow \left[a_{ii}\right]=-\left[a_{ii}\right] \Rightarrow 2\left[a_{ii}\right]=0$ {Replacing $j$ by $i$ }
  That is, ***all the diagonal elements in a skew-symmetric matrix are zero***.
- The matrices $AA^T$ and $A^T A$ are symmetric matrices.
- For any square matrix $A$, $A+A^T$ is a symmetric matrix and $A-A^T$ is a skew- symmetric matrix *always*.
- Also ***any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix*** *i.e.,* $A=\frac{1}{2}(P)+\frac{1}{2}(Q)$, where $P=A+A^T$ is a symmetric matrix and $Q=A-A^T$ is a skew- symmetric matrix.

***08. Orthogonal matrix:*** A matrix $A$ is said to be orthogonal if $A.A^T = \mathrm{I}$ where $A^T$ is the transpose of $A$.

***09. Invertible Matrix:*** If $A$ is a square matrix of order $m$ and if there exists another square matrix $B$ of the same order $m$, such that $AB = BA = I$, then $B$ is called the *inverse* matrix of $A$ and it is denoted by $A^{-1}$. A matrix having an inverse is said to be ***invertible***.

➢ *It is to note that if B is inverse of A, then A is also the inverse of B. In other words, if it is known that* $AB = BA = I$ *then,* $A^{-1} = B \Leftrightarrow B^{-1} = A$.

***10. Determinants, Minors & Cofactors:***

***a) Determinant:*** A unique number (real or complex) can be associated to every square matrix $A=\left[a_{ij}\right]$ of order $m$. This number is called the determinant of the square matrix $A$, where $a_{ij}=(i,\ j)^{th}$ element of $A$.

For instance, if $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then, determinant of matrix $A$ is written as $|A|=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=\det(A)$ and its value is given by $ad-bc$.

***b) Minors:*** Minors of an element $a_{ij}$ of a determinant *(or a determinant corresponding to matrix A)* is the determinant obtained by deleting its $i^{th}$ row and $j^{th}$ column in which $a_{ij}$ lies. Minor of $a_{ij}$ is denoted by $M_{ij}$. Hence we can get 9 minors corresponding to the 9 elements of a third order (*i.e.*, $3\times3$) determinant.

***c) Cofactors:*** Cofactor of an element $a_{ij}$, denoted by $A_{ij}$, is defined by, $A_{ij} = (-1)^{i+j} M_{ij}$, where $M_{ij}$ is minor of $a_{ij}$. Sometimes $C_{ij}$ is used in place of $A_{ij}$ to denote the cofactor of element $a_{ij}$.

***11. Adjoint of a square matrix:*** Let $A = \left[a_{ij}\right]$ be a square matrix. Also assume $B = \left[A_{ij}\right]$ where $A_{ij}$ is the cofactor of the elements $a_{ij}$ in matrix $A$. Then the transpose $B^T$ of matrix $B$ is called the ***adjoint of matrix*** *A* and it is denoted by ' ***adjA***'.

➢ ***To find adjoint of a*** $2\times 2$ ***matrix:*** Follow $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For example, consider a square matrix of order 3 as $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$ then, in order to find the adjoint of matrix *A*, we find a matrix *B* (formed by the cofactors of elements of matrix *A* as mentioned above in the definition) *i.e.,* $B=\begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$. Hence, $adjA = B^T = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$.

***12. Singular matrix & Non-singular matrix:***

***a) Singular matrix:*** A square matrix *A* is said to be *singular if* $|A| = 0$ *i.e.*, its **determinant is *zero***.

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 12 \\ 1 & 1 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}$.

***b) Non-singular matrix:*** A square matrix *A* is said to be *non-singular if* $|A| \neq 0$.

e.g. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$.

➢ *A square matrix A is* ***invertible*** *if and only if A is* ***non-singular***.

***13. Elementary Operations or Transformations of a Matrix:*** The following three operations applied on the row (or column) of a matrix are called elementary row (or column) transformations-

*a) Interchange of any two rows (or columns)***:** When $i^{th}$ row (or column) of a matrix is interchanged with the $j^{th}$ row (or column), it is denoted as $R_i \leftrightarrow R_j$ (or $C_i \leftrightarrow C_j$ ).

***b)*** *Multiplying all elements of a row (or column) of a matrix by a non-zero scalar***:** When the $i^{th}$ row (or column) of a matrix is multiplied by a scalar *k*, it is denoted as $R_i \to kR_i$ (or $C_i \to kC_i$ ).

***c)*** *Adding to the elements of a row (or column), the corresponding elements of any other row (or column) multiplied by any scalar k***:** When *k* times the elements of $j^{th}$ row (or column) is added to the corresponding elements of the $i^{th}$ row (or column), it is denoted as $R_i \to R_i + kR_j$ (or $C_i \to C_i + kC_j$ ).

**<u>NOTE</u>: In case, after applying one or more elementary row (or column) operations on A = IA (or A = AI), if we obtain all zeros in one or more rows of the matrix A on LHS, then $A^{-1}$ does not exist.**

***14. Inverse or reciprocal of a square matrix:*** If *A* is a square matrix of order *n*, then a matrix *B* (if such a matrix exists) is called the inverse of $A$ if $AB = BA = \mathrm{I}_n$. Also note that the inverse of a square matrix $A$ is denoted by $A^{-1}$ and we write, $A^{-1} = B$.

➢ *Inverse of a square matrix* $A$ *exists if and only if* $A$ *is non-singular matrix i.e.,* $|A| \neq 0$.

➢ *If B is inverse of* $A$*, then* $A$ *is also the inverse of* $B$.

***15. Algorithm to find Inverse of a matrix by Elementary Operations or Transformations:***

➲ **By Row Transformations:**

**STEP1-** Write the given square matrix as $A = \mathrm{I}_n A$.

**STEP2-** Perform a sequence of elementary row operations successively on $A$ on the LHS and pre-factor $\mathrm{I}_n$ on the RHS till we obtain the result $\mathrm{I}_n = BA$ (or $\mathrm{I}_n = AB$).

**STEP3-** Write $A^{-1} = B$.

➲ **By Column Transformations:**

**STEP1-** Write the given square matrix as $A = A\,\mathrm{I}_n$.

**STEP2-** Perform a sequence of elementary column operations successively on $A$ on the LHS and post-factor $\mathrm{I}_n$ on the RHS till we obtain the result $\mathrm{I}_n = AB$.

**STEP3-** Write $A^{-1} = B$.

***16. Algorithm to find $A^{-1}$ by Determinant method:***

**STEP1-** Find $|A|$.

**STEP2-** If $|A| = 0$ then, write **"*A* is a singular matrix and hence not invertible"**. Else write **"*A* is a non-singular matrix and hence invertible"**.

**STEP3-** Calculate the cofactors of elements of matrix $A$.

**STEP4-** Write the matrix of cofactors of elements of $A$ and then obtain its transpose to get $adjA$.

**STEP5-** Find the inverse of $A$ by using the relation $A^{-1} = \frac{1}{|A|} adjA$.

***17. Properties associated with the Inverse of Matrix & the Determinants:***

**a)** $AB = \mathrm{I} = BA$

**b)** $A\,A^{-1} = \mathrm{I}$ or $A^{-1}\,\mathrm{I} = A^{-1}$

**c)** $(AB)^{-1} = B^{-1}A^{-1}$

**d)** $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

**e)** $\left(A^{-1}\right)^{-1} = A$

**f)** $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$

**g)** $A(adjA) = (adjA)A = |A|\,\mathrm{I}$

**h)** $adj(AB) = (adjB)(adj\ A)$

**i)** $adj\left(A^{T}\right) = (adjA)^{T}$

**j)** $(adjA)^{-1} = \left(adjA^{-1}\right)$

**k)** $|adjA| = |A|^{n-1}$ *if* $|A| \neq 0$, where $n$ is order of $A$

**l)** $|AB| = |A|\,|B|$

**m)** $|A.adjA| = |A|^{n}$, where $n$ is order of $A$

**n)** $\left|A^{-1}\right| = \frac{1}{|A|}$ provided matrix $A$ is invertible

**o)** $|A| = \left|A^{T}\right|$

- $|kA| = k^{n}|A|$ where $n$ is order of square matrix $A$ and $k$ is any scalar.
- If $A$ is a non-singular matrix of order $n$, then $|adjA| = |A|^{n-1}$ [ point ( k) given above].
- If $A$ is a non-singular matrix of order $n$, then $adj(adjA) = |A|^{n-2}A$.

***18. Properties of Determinants:***

**a)** If any two rows or columns of a determinant are *proportional* or *identical*, then its value is *zero*.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$ [As $R_1$ and $R_3$ are the same.

**b)** The value of a determinant remains *unchanged* if its rows and columns are *interchanged*.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Here rows and columns have been interchanged, but there is *no effect* on the value of determinant.

**c)** If each element of a row or a column of a determinant is multiplied by a constant $k$, then the value of new determinant is $k$ times the value of the original determinant.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow \Delta_1 = k\Delta$ .

**d)** If *any two* rows or columns are *interchanged*, then the determinant retains its *absolute* value, but its *sign* is changed.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} \Rightarrow \Delta_1 = -\Delta$ [Here $R_1 \leftrightarrow R_3$ .

**e)** If every element of some column or row is the *sum* of two terms, then the determinant is equal to the sum of two determinants; one containing only the first term in place of each sum, the other only the second term. The remaining elements of both determinants are the same as given in the original determinant.

e.g. $\Delta = \begin{vmatrix} a_1+\alpha & b_1 & c_1 \\ a_2+\beta & b_2 & c_2 \\ a_3+\gamma & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & b_1 & c_1 \\ \beta & b_2 & c_2 \\ \gamma & b_3 & c_3 \end{vmatrix}$.

***19. Area of triangle:*** Area of a triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$ is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \; sq.units .$$

*As the area is a positive quantity, we take* ***absolute value of the determinant*** *given above.*

➲ *If the points $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$ are* ***collinear*** *then $\Delta = 0$ .*

➲ *The* ***equation of a line*** *passing through the points $(x_1, y_1)$ and $(x_2, y_2)$ can be obtained by the expression given here:*

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 .$$

***20. Solutions of System of Linear equations:***

***a) Consistent and Inconsistent system:*** A system of equations is consistent if it has *one or more* solutions otherwise it is said to be an inconsistent system. In other words an inconsistent system of equations has *no solution*.

***b) Homogeneous and Non-homogeneous system:*** A system of equations $AX = B$ is said to be a homogeneous system if $B = 0$ . Otherwise it is called a non-homogeneous system of equations.

***21. Solving of system of equations by Matrix method*** $[$ ***Inverse Method*** $]$*:*

Consider the following system of equations,

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3.$$

**STEP1-** Assume $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ .

**STEP2-** Find $|A|$. Now there may be following situations:

**a)** $|A| \neq 0 \Rightarrow A^{-1}$ exists. It implies that the given system of equations is *consistent* and therefore, the system has ***unique solution***. In that case, write

$$AX = B$$

$\Rightarrow \quad X = A^{-1}B$ $\qquad \left[\text{where } A^{-1} = \frac{1}{|A|}(adjA)\right.$

$\Rightarrow$ Then by using the ***definition of equality of matrices***, we can get the values of *x, y* and *z*.

**b)** $|A| = 0 \Rightarrow A^{-1}$ does not exist. It implies that the given system of equations may be *consistent* or *inconsistent*. In order to check proceed as follow:

$\Rightarrow$ Find $(adjA)B$. Now we may have either $(adjA)B \neq 0$ or $(adjA)B = 0$.

- If $(adjA)B = 0$, then the given system may be *consistent* or *inconsistent.* To check, put $z = k$ in the given equations and proceed in the same manner in the new *two variables* system of equations assuming $d_i - c_i k$, $1 \le i \le 3$ as constant.
- And if $(adjA)B \neq 0$, then the given system is *inconsistent* with *no solutions.*

# Continuity & Differential Calculus

**Important Terms, Definitions & Formulae**

***01. Formulae for Limits & Differential Calculus:***

## (LIMITS FOR SOME STANDARD FORMS)

**a)** $\lim_{x\to 0}\cos x = 1$

**b)** $\lim_{x\to 0}\frac{\sin x}{x} = 1$

**c)** $\lim_{x\to 0}\frac{\tan x}{x} = 1$

**d)** $\lim_{x\to 0}\frac{\sin^{-1} x}{x} = 1$

**e)** $\lim_{x\to 0}\frac{\tan^{-1} x}{x} = 1$

**f)** $\lim_{x\to 0}\frac{a^x - 1}{x} = \log_e a$, $a > 0$

**g)** $\lim_{x\to 0}\frac{e^x - 1}{x} = 1$

**h)** $\lim_{x\to 0}\frac{\log_e(1+x)}{x} = 1$

**i)** $\lim_{x\to a}\frac{x^n - a^n}{x - a} = na^{n-1}$

**j)** $\lim_{x\to 0}(1+kx)^{1/x} = e^k$, where $k$ is any constant.

## (DERIVATIVE OF SOME STANDARD FUNCTIONS)

**a)** $\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}$

**b)** $\frac{d}{dx}(k) = 0$, where $k$ is any constant

**c)** $\frac{d}{dx}\left(a^x\right) = a^x \log_e a$, $a > 0$

**d)** $\frac{d}{dx}\left(e^x\right) = e^x$

**e)** $\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\log_e a} = \frac{1}{x}\log_a e$

**f)** $\frac{d}{dx}\left(\log_e x\right) = \frac{1}{x}$

**g)** $\frac{d}{dx}\left(\sin x\right) = \cos x$

**h)** $\frac{d}{dx}\left(\cos x\right) = -\sin x$

**i)** $\frac{d}{dx}\left(\tan x\right) = \sec^2 x$

**j)** $\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$

**k)** $\frac{d}{dx}\left(\cot x\right) = -\text{cosec}^2 x$

**l)** $\frac{d}{dx}\left(\text{cosec}\, x\right) = -\text{cosec}\, x\, \cot x$

**m)** $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}, x \in (-1,1)$

**n)** $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}, x \in (-1,1)$

**o)** $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}, x \in \mathrm{R}$

**p)** $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}, x \in \mathrm{R}$

**q)** $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty,-1)\cup(1,\infty)$

**r)** $\frac{d}{dx}\left(\text{cosec}^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty,-1)\cup(1,\infty)$

*Do you know for trigonometric functions, angle '**x**' is in **Radians**?*

*Following derivatives should also be **memorized** by you for quick use:*

- $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
- $\frac{d}{dx}\left(x^x\right) = x^x\left(1+\log x\right)$

***02. Important terms and facts about Limits and Continuity of a function:***

➲ For a function $f(x)$, $\lim_{x\to m} f(x)$ exists *iff* $\lim_{x\to m^-} f(x) = \lim_{x\to m^+} f(x)$.

- A function $f(x)$ is continuous at a point $x = m$ *iff* $\lim_{x\to m^-} f(x) = \lim_{x\to m^+} f(x) = f(m)$, where $\lim_{x\to m^-} f(x)$ is **Left Hand Limit** of $f(x)$ at $x = m$ and $\lim_{x\to m^+} f(x)$ is **Right Hand Limit** of $f(x)$ at $x = m$. Also $f(m)$ is the value of function $f(x)$ at $x = m$.
- A function $f(x)$ is *continuous* at $x = m$ *(say)* if, $f(m) = \lim_{x\to m} f(x)$ *i.e.,* a function is *continuous* at a point in its **domain** if the **limit value of the function** at that point ***equals*** the value of the function at the same point.
- For a continuous function $f(x)$ at $x = m$, $\lim_{x\to m} f(x)$ can be directly obtained by evaluating $f(m)$.
- **Indeterminate forms or meaningless forms:**

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0.\infty,\ \infty-\infty,\ 1^{\infty},\ 0^{0},\ \infty^{0}.$$

*The first two forms i.e.* $\frac{0}{0}, \frac{\infty}{\infty}$ *are the forms which are suitable for **L'Hospital Rule**.*

***03. Important terms and facts about Derivatives and Differentiability of a function:***

- **Left Hand Derivative of $f(x)$ at $x = m$,**

$$\mathrm{L}f'(m) = \lim_{x\to m^-} \frac{f(x)-f(m)}{x-m} \text{ and,}$$

**Right Hand Derivative of $f(x)$ at $x = m$,**

$$\mathrm{R}f'(m) = \lim_{x\to m^+} \frac{f(x)-f(m)}{x-m}.$$

*For a **function to be differentiable** at a point, the **LHD and RHD at that point should be equal**.*

- **Derivative of $y$ w.r.t. $x$:** $\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x}$.

*Also for **very- very small value h**,* $f'(x) = \frac{f(x+h)-f(x)}{h}$, $(as\ h \to 0)$.

***04. Relation between Continuity and Differentiability:***

a) If a function is *differentiable* at a point, it is *continuous* at that point as well.
b) If a function is *not differentiable* at a point, it *may* or *may not be continuous* at that point.
c) If a function is *continuous* at a point, it *may* or *may not be differentiable* at that point.
d) If a function is *discontinuous* at a point, it is *not differentiable* at that point.

***05. Rules of derivatives:***

- Product or Leibnitz Rule of derivatives: $\frac{d}{dx}(u\,\mathrm{v}) = u\frac{d}{dx}(\mathrm{v}) + \mathrm{v}\frac{d}{dx}(u)$
- Quotient Rule of derivatives: $\frac{d}{dx}\left(\frac{u}{\mathrm{v}}\right) = \frac{\mathrm{v}\frac{d}{dx}(u) - u\frac{d}{dx}(\mathrm{v})}{\mathrm{v}^2}$.

# Applications Of Derivatives

## Important Terms, Definitions & Formulae

### ❖ *Rolle's Theorem*

***01. The statement of Rolle's Theorem:*** If a function $f(x)$ is,

a) continuous in the closed interval $[a,b]$,

b) differentiable in the open interval $(a,b)$, and

c) $f(a)=f(b)$

Then, there will be **at least** one point ***c*** in $(a,b)$ such that $f'(c)=0$.

➲ *Note that the **converse of Rolle's Theorem is not true** i.e., if a function $f(x)$ is such that $f'(c)=0$ for at least one c in the open interval $(a,b)$ then it is not necessary that*

***a)*** $f(x)$ *is continuous in* $[a,b]$,

***b)*** $f(x)$ *is differentiable in* $(a,b)$,

***c)*** $f(a)=f(b)$.

### ❖ *Mean Value Theorem*

***01. The statement of Mean Value Theorem:*** If a function $f(x)$ is,

a) continuous in the closed interval $[a,b]$,

b) differentiable in the open interval $(a,b)$,

Then, there will be at least one point *c*, such that $a<c<b$ *i.e.*

$c\in(a,b)$ for which, $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

➲ *Note that the Mean Value Theorem is **an extension** of Rolle's Theorem.*

**<u>On Rolle's Theorem & Mean Value Theorem, we are generally asked three types of problems:</u>**

***(i)*** *To check the applicability of Rolle's Theorem to a given function on a given interval,*
***(ii)*** *To verify Rolle's theorem for a given function on a given interval, and*
***(iii)*** *Problems based on the geometrical interpretation of Rolle's theorem.*

*In the first two cases, we first check whether the function satisfies conditions of Rolle's theorem or not. The following results may be very helpful in doing so:*

***(a)*** *A polynomial function is everywhere continuous and differentiable.*

***(b)*** *The exponential function, sine and cosine functions are everywhere continuous and differentiable.*

***(c)*** *Logarithmic function is continuous and differentiable in its domain.*

***(d)*** $\tan x$ *is not continuous at* $x=\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2,...$

***(e)*** $|x|$ *is not differentiable at* $x=0$.

***(f)*** *The sum, difference, product and quotient of continuous (differentiable) function is continuous (differentiable).*

### ❖ *Approximate Values*

***01. Approximate change in the value of function*** $y=f(x)$**:**

Given function is $y=f(x)$.

From the definition of derivatives, $\underset{\delta x\to 0}{Lt}\dfrac{\delta y}{\delta x}=\dfrac{dy}{dx}$.

$\therefore$ by definition of limit as $\delta x\to 0$, $\dfrac{\delta y}{\delta x}\to\dfrac{dy}{dx}$.

$\therefore$ if $\delta x$ is very near to *zero,* then we have

$$\frac{\delta y}{\delta x}=\frac{dy}{dx}\ \textit{(approximately).}$$

Therefore, $\delta y = \frac{dy}{dx}.\delta x$, where $\delta y$ respresents the ***approximate change in y***.

➲ *In case $dx = \delta x$ is relatively small when compared with x, dy is a good approximation of $\delta y$ and we denote it by $dy \approx \delta y$.*

***02. Approximate value:***

By the definition of derivatives *(first principle)*,

$$f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}.$$

$\therefore$ by the definition of limit when $h \to 0$, we have $\frac{f(x+h) - f(x)}{h} \to f'(x)$.

$\therefore$ if $h$ is very near to *zero,* then we have

$$\frac{f(x+h) - f(x)}{h} = f'(x) \textit{ (approximately)}.$$

*or* $f(x+h) = f(x) + h\,f'(x)$ ***approximately as*** $h \to 0$.

## ❖ *Rate Of Change*

***01. Interpretation of $\frac{dy}{dx}$ as a rate measurer:***

If two variables *x* and *y* are varying with respect to another variable say *t*, *i.e.*, if $x = f(t)$ and $y = g(t)$, then by the Chain Rule, we have $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, $\frac{dx}{dt} \neq 0$.

Thus, the rate of change of *y* with respect to *x* can be calculated using the rate of change of *y* and that of *x* both with respect to *t*.

*Also, if y is a function of x and they are related as $y = f(x)$ then, $f'(\alpha)$ i.e., $\left.\frac{dy}{dx}\right]_{at\ x=\alpha}$ represents the rate of change of y with respect to x at the **instant** when $x = \alpha$.*

## ❖ *Tangents & Normals*

***01. Slope or gradient of a line:*** If a line makes an angle $\theta$ with the *positive direction* of X-axis in *anticlockwise direction*, then $\tan\theta$ is called slope or gradient of the line. *[Note that $\theta$ is taken as positive or negative according as it is measured in anticlockwise **(i.e., from positive direction of X-axis to the positive direction of Y-axis)** or clockwise direction respectively.]*

***02. Facts about the slope of a line:***

**a)** If a line is parallel to *x*- axis (or perpendicular to *y* - axis) then, its slope is $0$ (zero).

**b)** If a line is parallel to *y*- axis (or perpendicular to *x*- axis) then, its slope is $\frac{1}{0}$ *i.e.* not defined.

**c)** If two lines are perpendicular then, product of their slopes equals $-1$ *i.e.,* $m_1 \times m_2 = -1$. Whereas for two parallel lines, their slopes are equal *i.e.,* $m_1 = m_2$. *(Here in both the cases, $m_1$ and $m_2$ represent the slopes of respective lines).*

***03. Equation of Tangent at $(x_1, y_1)$:***

$(y - y_1) = m_T(x - x_1)$ where, $m_T$ is the slope of tangent such that $m_T = \left.\frac{dy}{dx}\right]_{at\ (x_1, y_1)}$

***04. Equation of Normal at $(x_1, y_1)$:***

$(y - y_1) = m_N(x - x_1)$ where, $m_N$ is the slope of normal such that $m_N = \dfrac{-1}{\left.\frac{dy}{dx}\right]_{at\ (x_1, y_1)}}$

*Note that* $m_T \times m_N = -1$, *which is obvious because tangent and normal are* **perpendicular** *to each other. In other words, the tangent and normal lines are inclined at right angle on each other.*

***05. Acute angle between the two curves whose slopes*** $m_1$ ***and*** $m_2$ ***are known:***

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1.m_2}\right| \Rightarrow \theta = \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_1.m_2}\right|.$$

It is absolutely *sufficient to find* one angle (**generally the acute angle**) between the two curves. Other angle between the two curve is given by $\pi - \theta$.

*Note that if the curves cut* ***orthogonally*** *(i.e., they cut each other at right angles) then, it means* $m_1 \times m_2 = -1$ *where* $m_1$ *and* $m_2$ *represent slopes of the tangents of curves at the intersection point.*

***06. Finding the slope of a line*** $ax + by + c = 0$ ***:***

**STEP1-** Express the given line in the standard *slope-intercept form* $y = mx + c$ *i.e.,* $y = \left(-\frac{a}{b}\right)x - \frac{c}{b}$.

**STEP2-** By comparing to the standard form $y = mx + c$, we can conclude $-\frac{a}{b}$ as the slope of given line $ax + by + c = 0$.

### ❖ *Maxima & Minima*

***01. Understanding maxima and minima:***

Consider $y = f(x)$ be a well defined function on an interval $I$. Then

**a)** $f$ is said to have a ***maximum value*** in *I,* if there exists a point *c* in *I* such that *f(c) > f(x),* for all $x \in I$.

The value corresponding to *f(c)* is called the maximum value of $f$ in *I* and the point *c* called a ***point of maximum value of*** $f$ ***in I****.*

**b)** $f$ is said to have a ***minimum value*** in *I,* if there exists a point *c* in *I* such that *f(c) < f(x),* for all $x \in I$.

The value corresponding to *f(c)* is called the minimum value of $f$ in *I* and the point *c* called a ***point of minimum value of*** $f$ ***in I****.*

**c)** $f$ is said to have an ***extreme value*** in *I,* if there exists a point *c* in *I* such that *f(c)* is either a maximum value or a minimum value of $f$ in *I.*

The value *f(c)* in this case, is called an extreme value of $f$ in *I* and the point *c* called an ***extreme point****.*

***02. Meaning of local maxima and local minima:***

Let $f$ be a real valued function and also take a point *c* from its domain. Then

**a)** *c* is called a point of ***local maxima*** if there exists a number *h>0* such that *f(c) > f(x),* for all *x* in $(c-h, c+h)$. The value *f(c)* is called the ***local maximum value of*** $f$.

**b)** *c* is called a point of ***local minima*** if there exists a number *h>0* such that *f(c) < f(x),* for all *x* in $(c-h, c+h)$. The value *f(c)* is called the ***local minimum value of*** $f$.

***Critical points:*** *It is a point c (say) in the domain of a function f(x) at which either* $f'(x)$ *vanishes i.e,* $f'(c) = 0$ *or f is not differentiable.*

***03. First Derivative Test:***

Consider $y = f(x)$ be a well defined function on an open interval $I$. Now proceed as have been mentioned in the following algorithm:

**STEP1-** Find $\frac{dy}{dx}$.

**STEP2-** Find the *critical* point(s) by putting $\frac{dy}{dx} = 0$. Suppose $c \in I$ (where $I$ is the interval) be any critical point and *f* be continuous at this point $c$. Then we may have following situations:

- $\frac{dy}{dx}$ changes sign from ***positive to negative*** as $x$ increases through $c$, then the function attains a ***local maximum*** at $x = c$.
- $\frac{dy}{dx}$ changes sign from ***negative to positive*** as $x$ increases through $c$, then the function attains a ***local minimum*** at $x = c$.
- $\frac{dy}{dx}$ ***does not change sign*** as $x$ increases through $c$, then $x = c$ is **neither** a point of ***local maximum*** **nor** a point of ***local minimum***. Rather in this case, the point $x = c$ is called the ***point of inflection***.

***04. Second Derivative Test:***

Consider $y = f(x)$ be a well defined function on an open interval $I$ and twice differentiable at a point $c$ in the interval. Then we observe that:

- $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$.

  The value $f(c)$ is called local maximum value of $f$.
- $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$.

  The value $f(c)$ is called local minimum value of $f$.

*This test fails if $f'(c) = 0$ and $f''(c) = 0$. In such a case, we use **first derivative test** as discussed in the **para 03**.*

***05. Absolute maxima and absolute minima:***

If $f$ is a continuous function on a ***closed interval*** $I$ then, $f$ has the absolute maximum value and $f$ attains it at least once in $I$. Also $f$ has the absolute minimum value and the function attains it at least once in $I$.

**ALGORITHM**

**STEP1-** Find all the critical points of $f$ in the given interval, *i.e.,* find all points $x$ where either $f'(x) = 0$ or $f$ is not differentiable.

**STEP2-** Take the end points of the given interval.

**STEP3-** At all these points (*i.e.,* the points found in STEP1 and STEP2), calculate the values of $f$.

**STEP4-** Identify the maximum and minimum values of $f$ out of the values calculated in STEP3. This maximum value will be the ***absolute maximum value*** of $f$ and the minimum value will be the ***absolute minimum value*** of the function $f$.

*Absolute maximum value is also called as **global maximum value** or **greatest value**. Similarly absolute minimum value is called as **global minimum value** or the **least value**.*

## ❖ *Increasing & Decreasing*

***01.*** A function $f(x)$ is said to be an **increasing function** in $[a,b]$ if as $x$ increases, $f(x)$ also increases *i.e.,* if $\alpha,\beta \in [a,b]$ and $\alpha > \beta \Rightarrow f(\alpha) > f(\beta)$.

*If $f'(x) \geq 0$ lies in $(a,b)$ then, $f(x)$ is an increasing function in $[a,b]$ provided $f(x)$ is continuous at $x = a$ and $x = b$.*

***02.*** A function $f(x)$ is said to be a **decreasing function** in $[a,b]$ if as $x$ increases, $f(x)$ decreases *i.e.* if $\alpha,\beta \in [a,b]$ and $\alpha > \beta \Rightarrow f(\alpha) < f(\beta)$.

*If $f'(x) \leq 0$ lies in $(a,b)$ then, $f(x)$ is a decreasing function in $[a,b]$ provided $f(x)$ is continuous at $x = a$ and $x = b$.*

➲ *A function $f(x)$ is a **constant function** in $[a,b]$ if $f'(x) = 0$ for each $x \in (a,b)$.*

➲ *By **monotonic function** $f(x)$ in interval I, we mean that $f$ is either **only increasing** in I or **only decreasing** in I.*

***03. Finding the intervals of increasing and/ or decreasing of a function:***

**ALGORITHM**

**STEP1-** Consider the function $y = f(x)$.

**STEP2-** Find $f'(x)$.

**STEP3-** Put $f'(x) = 0$ and solve to get the critical point(s).

**STEP4-** The value(s) of $x$ for which $f'(x) > 0$, $f(x)$ is increasing; and the value(s) of $x$ for which $f'(x) < 0$, $f(x)$ is decreasing.



Hii, All!

I hope this texture may have proved beneficial for you.

While going through this material, if you noticed any error(s) or, something which doesn't make sense to you, please bring it in my notice through SMS or Call at +91-9650 350 480 or Email at **theopgupta@gmail.com**.

With lots of Love & Blessings!

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