Formulae For 3 Dimensional Geometry

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RECAPITULATION, DIRECTION COSINES, DIRECTION RATIOS & EQUATION OF LINES

Important Terms, Definitions & Formulae

01. Distance formula:

The distance between two points $\mathrm{A}(x_1, y_1, z_1)$ and $\mathrm{B}(x_2, y_2, z_2)$ is given by the expression $\mathrm{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ units.

02. Section formula:

The coordinates of a point Q which divides the line joining the points $\mathrm{A}(x_1, y_1, z_1)$ and $\mathrm{B}(x_2, y_2, z_2)$ in the ratio $m : n$

a) internally, are $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}, \dfrac{mz_2 + nz_1}{m+n}\right)$

b) externally *i.e.*, internally in the ratio $(m):(-n)$, are $\left(\dfrac{mx_2 - nx_1}{m-n}, \dfrac{my_2 - ny_1}{m-n}, \dfrac{mz_2 - nz_1}{m-n}\right)$.

03. Direction Cosines of a Line:

If A and B are two points on a given line L then, the direction cosines of vectors $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{BA}}$ are the direction cosines (d.c.'s) of line L. Thus if α, β, γ are the direction-angles which the line L makes with the positive direction of *x, y, z-* axes respectively then, its d.c.'s are $\cos\alpha, \cos\beta, \cos\gamma$ (See Fig.1). If direction of line L is reversed, the direction angles are replaced by their supplements *i.e.*, $\pi - \alpha, \pi - \beta, \pi - \gamma$ and so are the d.c.'s *i.e.,* the direction cosines become $-\cos\alpha, -\cos\beta, -\cos\gamma$. So, a line in space has two sets of d.c.'s *viz.* $\pm\cos\alpha, \pm\cos\beta, \pm\cos\gamma$.

- ❖ The d.c.'s are generally denoted by l, m, n. Also $l^2 + m^2 + n^2 = 1$ and so, we can deduce that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Also $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.
- ❖ The d.c.'s of a line joining the points $\mathrm{A}(x_1, y_1, z_1)$ and $\mathrm{B}(x_2, y_2, z_2)$ are $\pm\dfrac{x_2 - x_1}{\mathrm{AB}}, \pm\dfrac{y_2 - y_1}{\mathrm{AB}}, \pm\dfrac{z_2 - z_1}{\mathrm{AB}}$; where AB is the distance between points A and B *i.e.*, $\mathrm{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- ❖ In order to obtain the d.c.'s of a line which does not pass through the origin, we draw a line through the origin and parallel to the given line. As parallel lines have same set of the d.c.'s, so the d.c.'s of given line can be obtained by taking the d.c.'s of the parallel line through origin.
- ❖ In a unit vector, the coefficient of $\hat{i}, \hat{j}, \hat{k}$ are called d.c.'s. For example in $\hat{a} = l\hat{i} + m\hat{j} + n\hat{k}$ The d.c.'s are l, m, n.

04. Direction Ratios of a Line:

Any three numbers a, b, c (say) which are proportional to d.c.'s i.e., l, m, n of a line are called the **direction ratios** (d.r.'s) of the line. Thus, $a = \lambda l,\ b = \lambda m,\ c = \lambda n$ for any $\lambda \in \mathrm{R} - \{0\}$.

Consider, $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c} = \dfrac{1}{\lambda}$ (say)

$\Rightarrow \quad l = \frac{a}{\lambda},\ m = \frac{b}{\lambda},\ n = \frac{c}{\lambda}$

$\Rightarrow \quad \left(\frac{a}{\lambda}\right)^2 + \left(\frac{b}{\lambda}\right)^2 + \left(\frac{c}{\lambda}\right)^2 = 1$ [Using $l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad \lambda = \pm\sqrt{a^2 + b^2 + c^2}$

Therefore, $l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}},\ m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}},\ n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$.

- The d.c.'s of a line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1$ or $x_1 - x_2,\ y_1 - y_2,\ z_1 - z_2$.
- Direction ratios are sometimes called as **Direction Numbers** as well.
- For a line if a, b, c are its d.r.'s then, ka, kb, kc; $k \neq 0$ is also a set of its d.r.'s. So, for a line there are *infinitely many sets* of the direction ratios.

05. Relation between the direction cosines of a line:

Consider a line L with d.c's l, m, n. Draw a line passing through the origin and $P(x, y, z)$ and parallel to the given line L. From P draw a perpendicular PA on the X-axis. Suppose $OP = r$. (See Fig.2)

Now in ΔOAP we have, $\cos\alpha = \frac{OA}{OP} = \frac{x}{r} \Rightarrow x = lr$.

Similarly we can obtain $y = mr$ and $z = nr$.

Therefore, $x^2 + y^2 + z^2 = r^2\left(l^2 + m^2 + n^2\right)$.

But we know that $x^2 + y^2 + z^2 = r^2$.

Hence, $l^2 + m^2 + n^2 = 1$.

06. Equation of a line in space passing through a given point and parallel to a given vector:

Consider the line L is passing through the given point $A(x_1, y_1, z_1)$ with the position vector $\vec{a}$, $\vec{b}$ is the given vector with d.r.'s a, b, c and $\vec{r}$ is the position vector of any arbitrary point $P(x, y, z)$ on the line. See Fig.3.

Thus $\overrightarrow{OA} = \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k},\ \overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\ \vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$.

a) ***Vector equation of a line:*** As the line L is parallel to given vector $\vec{b}$ and points A and P are lying on the line so, $\overrightarrow{AP}$ is parallel to the $\vec{b}$.

$\Rightarrow \quad \overrightarrow{AP} = \lambda\vec{b}$, where $\lambda \in R$ *i.e.*, set of real nos.

$\Rightarrow \quad \vec{r} - \vec{a} = \lambda\vec{b}$

$\Rightarrow \quad \vec{r} = \vec{a} + \lambda\vec{b}$.

This is the *vector equation of line*.

b) ***Parametric equations:*** If d.r.'s of the line are a, b, c then by using $\vec{r} = \vec{a} + \lambda\vec{b}$ we get,

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda\left(a\hat{i} + b\hat{j} + c\hat{k}\right).$$

Now, as we equate the coefficients of $\hat{i}, \hat{j}, \hat{k}$ we get the *Parametric equations of line* given as,

$$x = x_1 + \lambda a,\ y = y_1 + \lambda b,\ z = z_1 + \lambda c.$$

- *Coordinates of any point on the line considered here are* $\left(x_1 + \lambda a,\ y_1 + \lambda b,\ z_1 + \lambda c\right)$.

c) ***Cartesian equation of a line:*** If we eliminate the parameter λ from the Parametric equations of a line, we get the *Cartesian equation of line* as

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}.$$

❖ If l,m,n are the d.c.'s of the line then, *Cartesian equation of line* becomes

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}.$$

❖ The Cartesian equation of line is also called the ***symmetrical equation*** *or* ***one point form of line***. In the symmetrical form the coefficient of *x, y, z* are unity i.e., 1.

◆ Note that $\vec{b}$ is parallel to the line L. So they both have the ***same*** d.r.'s.

07. Equation of a Line passing through two given points:

Consider the two given points as $\text{A}(x_1,y_1,z_1)$ and $\text{B}(x_2,y_2,z_2)$ with position vectors $\vec{a}$ and $\vec{b}$ respectively. Also assume $\vec{r}$ as the position vector of any arbitrary point $\text{P}(x,y,z)$ on the line L passing through A and B. See Fig.4.

Thus $\overrightarrow{\text{OA}}=\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k},\ \overrightarrow{\text{OB}}=\vec{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k},\ \overrightarrow{\text{OP}}=\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

a) Vector equation of a line: Since the points A, B and P all lie on the same line which means they are all ***collinear points***.

Further it means, $\overrightarrow{\text{AP}}=\vec{r}-\vec{a}$ and $\overrightarrow{\text{AB}}=\vec{b}-\vec{a}$ are collinear vectors, i.e.,

$$\overrightarrow{\text{AP}}=\lambda\overrightarrow{\text{AB}}$$

$$\Rightarrow \quad \vec{r}-\vec{a}=\lambda\left(\vec{b}-\vec{a}\right)$$

$$\Rightarrow \quad \vec{r}=\vec{a}+\lambda\left(\vec{b}-\vec{a}\right),\ \text{where } \lambda\in\text{R}.$$

This is the *vector equation of line*.

b) Cartesian equation of a line: By using the vector equation of the line $\vec{r}=\vec{a}+\lambda\left(\vec{b}-\vec{a}\right)$ we get,

$$x\hat{i}+y\hat{j}+z\hat{k}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}+\lambda\left[(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}\right]$$

On equating the coefficients of $\hat{i},\hat{j},\hat{k}$ we get,

$$x=x_1+\lambda(x_2-x_1),\ y=y_1+\lambda(y_2-y_1),\ z=z_1+\lambda(z_2-z_1) \qquad ....(i)$$

On eliminating λ we have,

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}.$$

This is the *Cartesian equation of line*.

c) Parametric equations: By using *(i)*, we get

$$x=x_1+\lambda(x_2-x_1),\ y=y_1+\lambda(y_2-y_1),\ z=z_1+\lambda(z_2-z_1).$$

These are called the *Parametric equations of line*.

08. Angle between two Lines:

a) When d.r.'s or d.c.'s of the two lines are given:

Consider two lines L_1 and L_2 with d.r.'s as a_1,b_1,c_1 and a_2,b_2,c_2; d.c.'s as l_1,m_1,n_1 and l_2,m_2,n_2. Consider $\vec{b}_1=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and $\vec{b}_2=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$. These vectors $\vec{b}_1$ and $\vec{b}_2$ are parallel to the given lines L_1 and L_2. So in order to find the angle between the Lines L_1 and L_2, we need to get the angle between the vectors $\vec{b}_1$ and $\vec{b}_2$. Consider the Fig.5.

So the acute angle θ between the vectors $\vec{b}_1$ and $\vec{b}_2$ (and hence lines L_1 and L_2) can be obtained as,

$$\vec{b}_1.\vec{b}_2=\left|\vec{b}_1\right|\left|\vec{b}_2\right|\cos\theta$$

Thus, $$\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|.$$

- Also, in terms of d.c.'s : $\cos\theta = \left| l_1l_2 + m_1m_2 + n_1n_2 \right|$.
- Sine of angle is given as: $\sin\theta = \left| \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$.

b) *When vector equations of two lines are given:*

Consider vector equations of lines L_1 and L_2 as $\vec{r_1} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r_2} = \vec{a_2} + \mu\vec{b_2}$ respectively. Then, the acute angle θ between the two lines is given by the relation

$$\cos\theta = \left| \frac{\vec{b_1}.\vec{b_2}}{\left|\vec{b_1}\right|\left|\vec{b_2}\right|} \right|.$$

c) *When Cartesian equations of two lines are given:*

Consider the lines L_1 and L_2 in Cartesian form as,

$$L_1 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}, \quad L_2 : \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}.$$

Then the acute angle θ between the lines L_1 and L_2 can be obtained by,

$$\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|.$$

- ***For two perpendicular lines:*** $\mathbf{a_1a_2 + b_1b_2 + c_1c_2 = 0}$; $\mathbf{l_1l_2 + m_1m_2 + n_1n_2 = 0}$.
- ***For two parallel lines:*** $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$; $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

09. Shortest Distance between two Lines:

If two lines are in the same plane i.e. they are coplanar, they will intersect each other if they are non-parallel. Hence shortest distance between them is *zero*. If the lines are parallel then the shortest distance between them will be the perpendicular distance between the lines i.e. the length of the perpendicular drawn from a point on one line onto the other line. Adding to this discussion, in space, there are lines which are *neither intersecting nor parallel*. In fact, such pair of lines are *non-coplanar* and are called the *skew lines*.

- ***Skew lines:*** Two straight lines in space which are neither parallel nor intersecting are known as the *skew lines*. They lie in different planes and are non-coplanar.
- ***Shortest Distance:*** There exists unique line perpendicular to each of the skew lines L_1 and L_2, this line is known as line of *shortest distance* (S.D.).

a) *Shortest distance between two Skew lines:*

➲ ***When lines are in vector form:***

Consider the two skew lines $\vec{r_1} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r_2} = \vec{a_2} + \mu\vec{b_2}$. Assume that A and B are two points on the lines L_1 and L_2 with position vectors $\vec{a_1}$ and $\vec{a_2}$ respectively. Let us assume that the Shortest Distance between the two lines is $PQ = d$. See Fig.6.

Now PQ is perpendicular to both the lines L_1 and L_2. That means $\overrightarrow{PQ}$ is perpendicular to both $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$. But we know that $\overrightarrow{b_1}\times\overrightarrow{b_2}$ is perpendicular to both $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$. So we can deduce that $\overrightarrow{PQ}$ is in the direction of $\overrightarrow{b_1}\times\overrightarrow{b_2}$.

Let the unit vector in the direction of $\overrightarrow{b_1}\times\overrightarrow{b_2}$ is $\hat{n}$.

So, $$\hat{n}=\pm\frac{\overrightarrow{b_1}\times\overrightarrow{b_2}}{\left|\overrightarrow{b_1}\times\overrightarrow{b_2}\right|}$$

$$\Rightarrow \quad \overrightarrow{PQ}=PQ.\,\hat{n}$$

$$\Rightarrow \quad \overrightarrow{PQ}=\pm PQ\left(\frac{\overrightarrow{b_1}\times\overrightarrow{b_2}}{\left|\overrightarrow{b_1}\times\overrightarrow{b_2}\right|}\right)$$

The shortest distance PQ is basically the *projection of* $\overrightarrow{AB}$ *on* $\overrightarrow{PQ}$.

i.e., $PQ=\overrightarrow{AB}.\,\hat{n}$

Then the S.D. between them is given as follow,

$$PQ=d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right).\left(\vec{a}_2-\vec{a}_1\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|.$$

➲ ***When the lines are in Cartesian form:***

Consider the two skew lines as,

$$L_1:\ \frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1},\ \ L_2:\ \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}.$$

Then the S.D. between them is given as follow

$$d=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}}{\sqrt{\left(a_1b_2-a_2b_1\right)^2+\left(b_1c_2-b_2c_1\right)^2+\left(c_1a_2-c_2a_1\right)^2}}\right|.$$

b) *Shortest distance between two parallel lines:*

If the lines are parallel then they are coplanar i.e. they lie in the same plane.

Consider the two parallel lines as $L_1:\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $L_2:\vec{r}=\vec{a}_2+\mu\vec{b}$. Assume that A and B are two points on the lines L_1 and L_2 with position vectors $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$ respectively. Also assume that the lines are parallel to $\vec{b}$. Let $\overrightarrow{AB}$ makes angle θ with the line L_1. So the angle between the $\overrightarrow{AB}$ and $\vec{b}$ will be $\pi-\theta$. See Fig.7.

Draw $BP\perp L_1$. Now BP represents the perpendicular distance between L_1 and L_2.

In $\Delta\,APB$, we have $BP=AB\sin\theta$ ...(*i*)

Now consider $\overrightarrow{AB}\times\vec{b}=\left|\overrightarrow{AB}\right|\left|\vec{b}\right|\sin(\pi-\theta)\,\hat{n}$

$$\Rightarrow \quad \left|\overrightarrow{AB}\times\vec{b}\right|=\left|\left|\overrightarrow{AB}\right|\left|\vec{b}\right|\sin\theta\right| \qquad [\because\ \hat{n}=1$$

$$\Rightarrow \quad \left|\overrightarrow{AB}\times\vec{b}\right|=\left|\left(\left|\overrightarrow{AB}\right|\sin\theta\right)\ \left|\vec{b}\right|\right|$$

$$\Rightarrow \quad \left|\overrightarrow{AB}\times\vec{b}\right|=BP\left|\vec{b}\right| \qquad [\textit{By using } (i)$$

$$\Rightarrow \qquad BP = \frac{\left|\overrightarrow{AB} \times \vec{b}\right|}{\left|\vec{b}\right|}.$$

Assume that the Shortest Distance between the two lines is $BP = d$.

Then the S.D. between them is given as follow,

$$d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{\left|\vec{b}\right|}\right|.$$

❖ *Note that the S.D. between two parallel lines in the Cartesian form can be obtained by simply replacing* $\vec{a}_2 - \vec{a}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ and $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$ *in the expression obtained above for the Vector form.*

PLANE & ITS EQUATION IN VARIOUS FORMS

Important Terms, Definitions & Formulae

01. Plane and its equation:

A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface. Plane is symbolized by the Greek letter π.

a) ***Equation of plane in Normal unit vector form:***

Consider a plane at distance d from the origin such that $\overrightarrow{ON}$ is the normal from the origin to plane and $\hat{n}$ is a unit vector along $\overrightarrow{ON}$. Then $\overrightarrow{ON} = d\hat{n}$ if $ON = d$. Consider $\vec{r}$ be the position vector of any arbitrary point $P(x, y, z)$ on the plane. See Fig.8.

➲ ***Vector form of the equation of plane:*** Since P lies on the plane so $\overrightarrow{NP}$ is perpendicular to the vector $\overrightarrow{ON}$. That implies $\overrightarrow{NP}.\overrightarrow{ON} = 0$

$$\Rightarrow \quad (\vec{r} - d\hat{n}).d\hat{n} = 0$$

$$\Rightarrow \quad (\vec{r} - d\hat{n}).\hat{n} = 0 \qquad [\because d \neq 0$$

$$\Rightarrow \quad \vec{r}.\hat{n} - d\hat{n}.\hat{n} = 0$$

$$\Rightarrow \quad \vec{r}.\hat{n} = d \qquad [\because \hat{n}.\hat{n} = 1$$

This is the *vector equation of the plane.*

➲ ***Cartesian form of the equation of plane:*** If l, m, n are d.c.'s of the normal $\hat{n}$ to the given plane. Then by using $\vec{r}.\hat{n} = d$ we get,

$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(l\hat{i} + m\hat{j} + n\hat{k}\right) = d$$

$$\Rightarrow \qquad lx + my + nz = d.$$

This is the *Cartesian equation of the plane.*

❖ Also if a, b, c are the d.r.'s of the normal $\hat{n}$ to the plane then, the *Cartesian equation of plane* becomes $ax + by + cz = d$.

b) ***Equation of plane Perpendicular to a given vector and passing through a given point:***

Assume that the plane passes through a point $A(x_1, y_1, z_1)$ with the position vector $\vec{a}$ and is perpendicular to the vector $\vec{m}$ with d.r.'s as A, B, C $\left(\therefore\ \vec{m} = A\hat{i} + B\hat{j} + C\hat{k}\right)$.

Also consider $P(x, y, z)$ as any arbitrary point on the plane with position vector as $\vec{r}$. Consider the Fig.9.

➲ ***Vector form of the equation of plane:*** As $\overrightarrow{AP}$ lies in the plane and $\vec{m}$ is perpendicular to the plane. So $\overrightarrow{AP}$ is perpendicular to $\vec{m}$.

$$\Rightarrow \quad \overrightarrow{AP}.\vec{m} = 0$$
$$\Rightarrow \quad (\vec{r} - \vec{a}).\vec{m} = 0.$$

This is the *Vector equation of the plane*.

❖ The above obtained equation of plane can also be expressed as $\vec{r}.\vec{m} = \vec{a}.\vec{m}$.

➲ ***Cartesian form of the equation of plane:*** As $\overrightarrow{AP} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$, so by using $(\vec{r} - \vec{a}).\vec{m} = 0$ we get,

$$\left[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\right].\left(A\hat{i} + B\hat{j} + C\hat{k}\right) = 0$$
$$\Rightarrow \quad A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.$$

This is the *Cartesian equation of the plane*.

c) Equation of plane passing through three non- collinear points:

Assume that the plane contains three non- collinear points $R(x_1, y_1, z_1)$, $S(x_2, y_2, z_2)$ and $T(x_3, y_3, z_3)$ with the position vectors as $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Let $P(x, y, z)$ be any arbitrary point in the plane whose position vector is $\vec{r}$.

➲ ***Vector form of the equation of plane:*** As $\overrightarrow{RS}$ and $\overrightarrow{RT}$ are in the plane, so $\overrightarrow{RS} \times \overrightarrow{RT} = \vec{m}$ (*say*) will be perpendicular to the plane containing the points R, S and T. Also since $\vec{r}$ is position vector of P which lies in the plane, therefore $\overrightarrow{RP} \perp \vec{m}$. See Fig.10.

$$\Rightarrow \quad \overrightarrow{RP}.\vec{m} = 0$$
$$\Rightarrow \quad (\vec{r} - \vec{a}).\left(\overrightarrow{RS} \times \overrightarrow{RT}\right) = 0 \qquad [\because \vec{m} = \overrightarrow{RS} \times \overrightarrow{RT}$$
$$\Rightarrow \quad (\vec{r} - \vec{a}).\left[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right] = 0.$$

This is the *Vector equation of the plane.*

➲ ***Cartesian form of the equation of plane:***

The position vector of $\overrightarrow{RP}$, $\overrightarrow{RS}$ and $\overrightarrow{RT}$ is given as,

$$\overrightarrow{RP} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$$
$$\overrightarrow{RS} = \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$
$$\overrightarrow{RT} = \left(x_3\hat{i} + y_3\hat{j} + z_3\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}$$

Substituting these in the above obtained *vector equation of plane*, we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

This is the *Cartesian equation of the plane*.

d) Intercept form of the equation of plane:

Consider the equation of plane $Ax + By + Cz + D = 0,\ D \neq 0$ and the plane makes intercepts a, b, c on x, y, z*-axes* respectively. This implies that the plane meets x, y, z*-axes* at $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ respectively. See Fig.11.

Therefore, $\quad A.a + B.0 + C.0 + D = 0 \Rightarrow A = -\dfrac{D}{a}$,

$$\text{A}.0+\text{B}.b+\text{C}.0+\text{D}=0 \Rightarrow \text{B}=-\frac{\text{D}}{b} \text{ and}$$

$$\text{A}.0+\text{B}.0+\text{C}.c+\text{D}=0 \Rightarrow \text{C}=-\frac{\text{D}}{c}.$$

Substituting these values in $\text{A}x+\text{B}y+\text{C}z+\text{D}=0$,, we get $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

This is the *equation of plane in intercept form.*

NOTE
Equation of XY-plane : $z=0$,
Equation of YZ-plane : $x=0$,
Equation of ZX-plane : $y=0$.

02. Equation of plane passing through the intersection of two given planes:

The intersection of two planes say π_1 and π_2 is always a straight line. For instance, we can visualize the intersection of *xy - plane* and *xz - plane* to form *x- axis*.

a) Vector equation of the plane: Consider two planes $\pi_1 : \vec{r}.\vec{m}_1 = d_1$ and $\pi_2 : \vec{r}.\vec{m}_2 = d_2$. So if $\vec{h}$ is the position vector of any arbitrary point on the line of intersection of π_1 and π_2 then, it must satisfy both the equations of planes i.e.,

$$\vec{h}.\vec{m}_1 = d_1 \text{ and } \vec{h}.\vec{m}_2 = d_2$$

$$\Rightarrow \quad \vec{h}.\vec{m}_1 - d_1 = 0 \text{ and } \vec{h}.\vec{m}_2 - d_2 = 0.$$

Therefore for all $\lambda \in \text{R}$ (set of all real nos.), we get

$$\left(\vec{h}.\vec{m}_1 - d_1\right) + \lambda\left(\vec{h}.\vec{m}_2 - d_2\right) = 0$$

$$\Rightarrow \quad \vec{h}.\left(\vec{m}_1 + \lambda\vec{m}_2\right) = d_1 + \lambda d_2$$

As the obtained equation is of the form $\vec{r}.\vec{m} = d$ (Note that in $\vec{r}.\vec{m} = d$, d is not the perpendicular distance of plane from the origin. Rather d is perpendicular distance from the origin in $\vec{r}.\hat{n} = d$.)

So it represents a plane π_3 (say).

Hence, the required plane is: $\vec{r}.\left(\vec{m}_1 + \lambda\vec{m}_2\right) = d_1 + \lambda d_2$.

This is the *Vector equation of plane.*

b) Cartesian equation of the plane:

Assume $\vec{m}_1 = \text{A}_1\hat{i} + \text{B}_1\hat{j} + \text{C}_1\hat{k}$, $\vec{m}_2 = \text{A}_2\hat{i} + \text{B}_2\hat{j} + \text{C}_2\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Then by using $\vec{r}.\left(\vec{m}_1 + \lambda\vec{m}_2\right) = d_1 + \lambda d_2$, we get

$$x\left(\text{A}_1 + \lambda\,\text{A}_2\right) + y\left(\text{B}_1 + \lambda\,\text{B}_2\right) + z\left(\text{C}_1 + \lambda\,\text{C}_2\right) = d_1 + \lambda d_2$$

i.e., $$\left(\text{A}_1x + \text{B}_1y + \text{C}_1z - d_1\right) + \lambda\left(\text{A}_2x + \text{B}_2y + \text{C}_2z - d_2\right) = 0.$$

This is the *Cartesian equation of plane.*

(You can visualize the situation discussed here in the Fig.12)

03. Co- planarity of two Lines:

Assume the given lines are $\text{L}_1 : \vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\text{L}_2 : \vec{r} = \vec{a}_2 + \mu\vec{b}_2$ such that L_1 passes through $\text{A}\left(x_1, y_1, z_1\right)$ with position vector $\vec{a}_1$ and is parallel to $\vec{b}_1$ with d.r.'s a_1, b_1, c_1. Also L_2 passes through $\text{B}\left(x_2, y_2, z_2\right)$ with position vector $\vec{a}_2$ and is parallel to $\vec{b}_2$ with the d.r.'s a_2, b_2, c_2.

a) Vector form of co- planarity of lines:

We know $\overrightarrow{AB} = \vec{a}_2 - \vec{a}_1$. Now the lines L_1 and L_2 are coplanar iff $\overrightarrow{AB}$ is perpendicular to $\vec{b}_1 \times \vec{b}_2$. That implies, $\overrightarrow{AB}.\left(\vec{b}_1 \times \vec{b}_2\right) = 0$

$$\Rightarrow \quad \left(\vec{a}_2 - \vec{a}_1\right).\left(\vec{b}_1 \times \vec{b}_2\right) = 0.$$

b) *Cartesian form of co- planarity of lines:*

We know that $\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$, $\vec{b}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$. So by using $(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = 0$, we get

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

❖ *Note that only coplanar lines can intersect each other in the plane they exist.*

04. Angle between two planes:

The angle between two planes is the angle between their normals $\vec{m}_1$ and $\vec{m}_2$ (say). Therefore if θ is the angle between the planes π_1 and π_2 then $180^\circ - \theta$ is also the angle between the two planes. Though we shall be taking **acute angle θ only** as the angle between two planes.
Observe the Fig.13.

a) *Vector form for the angle between two planes:*

Consider the planes $\pi_1 : \vec{r}.\vec{m}_1 = d_1$ and $\pi_2 : \vec{r}.\vec{m}_2 = d_2$. If θ is the angle between the normals to the plane drawn from some common point. Then,

$$\cos\theta = \left|\frac{\vec{m}_1.\vec{m}_2}{|\vec{m}_1||\vec{m}_2|}\right| \qquad \text{[Using dot product of vectors}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\left|\frac{\vec{m}_1.\vec{m}_2}{|\vec{m}_1||\vec{m}_2|}\right|\right).$$

b) *Cartesian form for the angle between two planes:*

Assume the planes, $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ where A_1, B_1, C_1 and A_2, B_2, C_2 are the d.r.'s of normals (to the planes) $\vec{m}_1$ and $\vec{m}_2$ respectively.

Then, $$\cos\theta = \left|\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$$

$$\Rightarrow \quad \theta = \cos^{-1}\left|\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|.$$

❖ ***For the parallel planes, we have:*** $\boldsymbol{\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}}$.

❖ ***For the perpendicular planes, we have:*** $\boldsymbol{A_1A_2 + B_1B_2 + C_1C_2 = 0}$.

05. Distance of a point from a plane:

a) *Vector form for the distance of a point from a plane:*

Let $\pi : \vec{r}.\vec{m} = d$ be the plane and $P(x_1, y_1, z_1)$ be the point with position vector $\vec{a}$. Let PA be the length of perpendicular on the plane. See Fig.14. Since line PA passes through $P(\vec{a})$ and is parallel to the $\vec{m}$ which is normal to the plane.
So the vector equation of the line PA is $\vec{r} = \vec{a} + \lambda\vec{m}$ ...(*i*)

Since A is point of intersection of line *(i)* and the given plane. So we have,

$$\left(\vec{a}+\lambda\vec{m}\right).\vec{m}=d$$

$$\Rightarrow \quad \lambda=\frac{d-\vec{a}.\vec{m}}{|\vec{m}|^2}$$

Putting the value of λ in *(i)*, we get the position vector of A given as follow,

$$\vec{r}=\vec{a}+\left(\frac{d-\vec{a}.\vec{m}}{|\vec{m}|^2}\right)\vec{m}$$

Since $\overrightarrow{\mathrm{PA}}=\overrightarrow{\mathrm{OA}}-\overrightarrow{\mathrm{OP}}$

$$\Rightarrow \quad \overrightarrow{\mathrm{PA}}=\vec{r}-\vec{a}$$

$$\therefore \quad \overrightarrow{\mathrm{PA}}=\vec{a}+\left(\frac{d-\vec{a}.\vec{m}}{|\vec{m}|^2}\right)\vec{m}-\vec{a}$$

$$\Rightarrow \quad \overrightarrow{\mathrm{PA}}=\left(\frac{d-\vec{a}.\vec{m}}{|\vec{m}|^2}\right)\vec{m}$$

$$\Rightarrow \quad \mathrm{PA}=\left|\overrightarrow{\mathrm{PA}}\right|=\left|\left(\frac{d-\vec{a}.\vec{m}}{|\vec{m}|^2}\right)\vec{m}\right|$$

$$\Rightarrow \quad \mathrm{PA}=\frac{|d-\vec{a}.\vec{m}|}{|\vec{m}|^2}|\vec{m}|$$

$$i.e., \quad \mathrm{PA}=\frac{|d-\vec{a}.\vec{m}|}{|\vec{m}|}$$

Hence, length of the perpendicular $\mathrm{PA}=p$ (*say*) from a point having position vector $\vec{a}$ to the plane $\vec{r}.\vec{m}=d$ is given by

$$p=\frac{|d-\vec{a}.\vec{m}|}{|\vec{m}|}.$$

b) ***Cartesian form for the distance of a point from a plane:*** Let A, B, C be the d.r.'s of the normal $\vec{m}$ to the given plane. So by using the relation $p=\frac{|d-\vec{a}.\vec{m}|}{|\vec{m}|}$ we can obtain,

$$p=\left|\frac{\mathrm{A}x_1+\mathrm{B}y_1+\mathrm{C}z_1-d}{\sqrt{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2}}\right|.$$

- ❖ If *d* is the distance from the origin and *l, m, n* are the d.c.'s of the normal vector to the plane through origin, then the coordinates of the ***foot of perpendicular*** is (ld, md, nd).

c) ***Distance between two parallel palnes:*** Assume the two planes as,
$\vec{r}.\vec{m}=d_1$ i.e., $\mathrm{A}x+\mathrm{B}y+\mathrm{C}z+\mathrm{D}_1=0$ and $\vec{r}.\vec{m}=d_2$ i.e., $\mathrm{A}x+\mathrm{B}y+\mathrm{C}z+\mathrm{D}_2=0$.
Then the distance *p (say)* between them is given as

(i) Vector form: $p=\frac{|d_1-d_2|}{|\vec{m}|}$.

(ii) Cartesian form: $p=\frac{|\mathrm{D}_1-\mathrm{D}_2|}{\sqrt{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2}}$.

06. Angle between a line and a plane:

The angle between a line and a plane is complementary to the angle between the line and normal to the plane. Let θ is the angle between $\vec{b}$ (which is parallel to the line) and normal $\vec{m}$ of the plane. This implies that $90^\circ - \theta$ is the angle between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and plane $\vec{r}.\vec{m} = d$.
Consider Fig.15.

Now the angle between $\vec{b}$ and $\vec{m}$, $\cos\theta = \left|\dfrac{\vec{b}.\vec{m}}{|\vec{b}||\vec{m}|}\right|$ [By using dot product of vectors

So the angle φ (*say*) between the line and plane is given as $\varphi = 90^\circ - \theta$ i.e.,

$$\sin\varphi = \sin\left(90^\circ - \theta\right) = \cos\theta$$

i.e., $$\sin\varphi = \left|\frac{\vec{b}.\vec{m}}{|\vec{b}||\vec{m}|}\right|$$

$\Rightarrow$ $$\varphi = \sin^{-1}\left(\left|\frac{\vec{b}.\vec{m}}{|\vec{b}||\vec{m}|}\right|\right).$$

This is the angle between line and a plane.

Click on the following link to go for a pleasant surprise:
http://theopgupta.wordpress.com/maths-rockers/

Hii, All!
I hope this texture may have proved beneficial for you.
While going through this material, if you noticed any error(s) or, something which doesn't make sense to you, **please** bring it in my notice through SMS or Call at **+91-9650 350 480** or Email at **theopgupta@gmail.com**.

With lots of Love & Blessings!

- OP Gupta
Electronics & Communications Engineering, Indira Award Winner
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VARIOUS FIGURES RELATED TO THREE DIMENSIONAL GEOMETRY

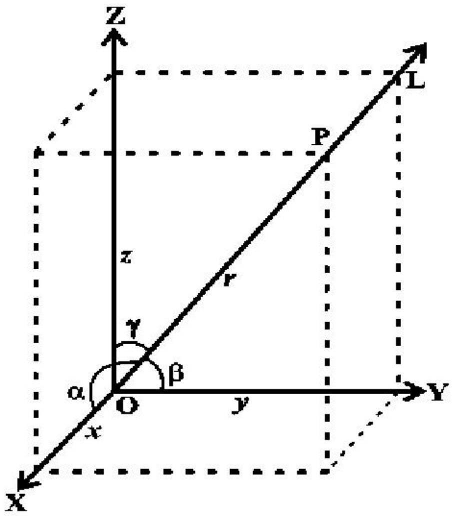


Fig. 1

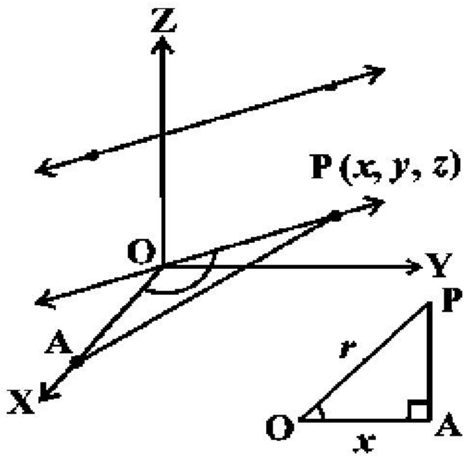


Fig. 2

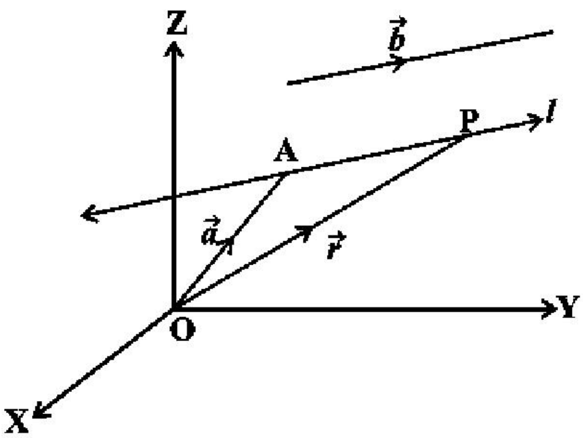


Fig. 3

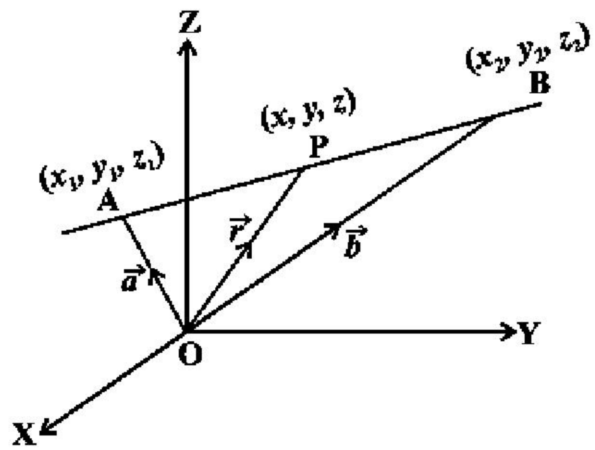


Fig. 4

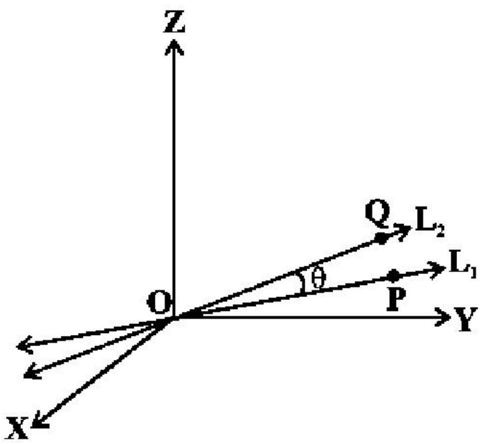


Fig. 5

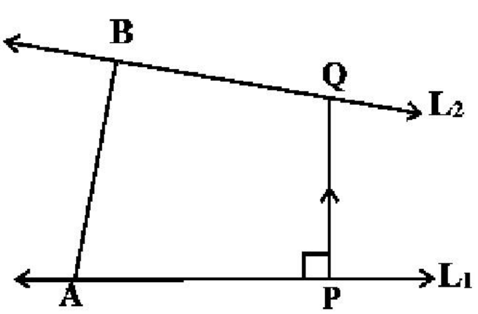


Fig. 6

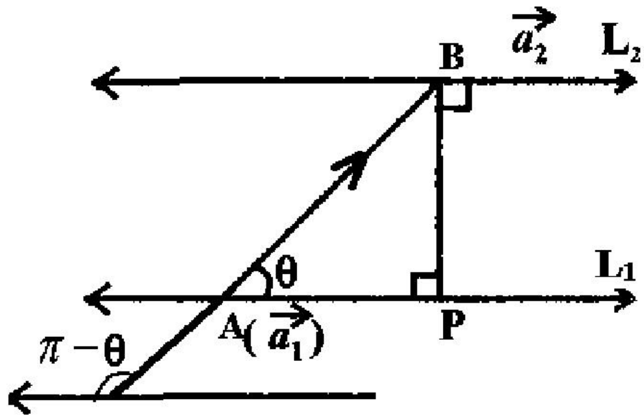


Fig. 7

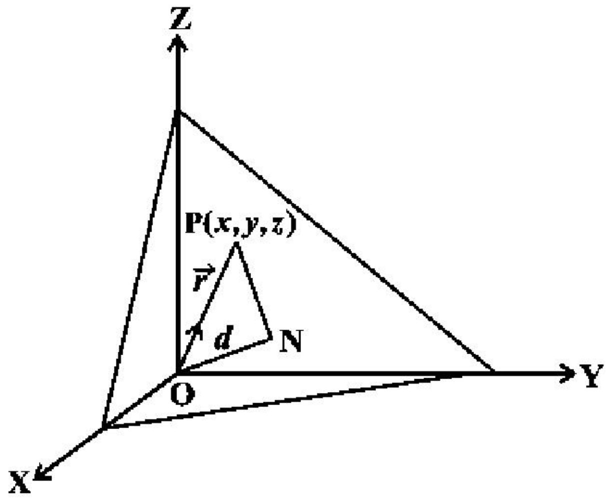


Fig. 8

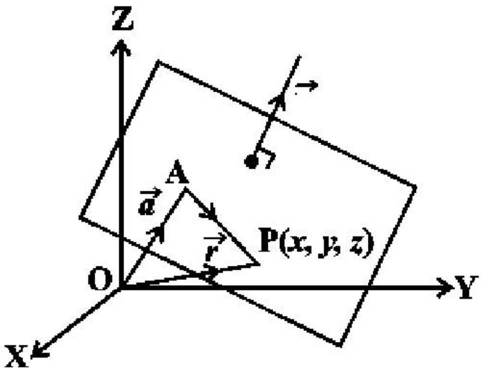


Fig. 9

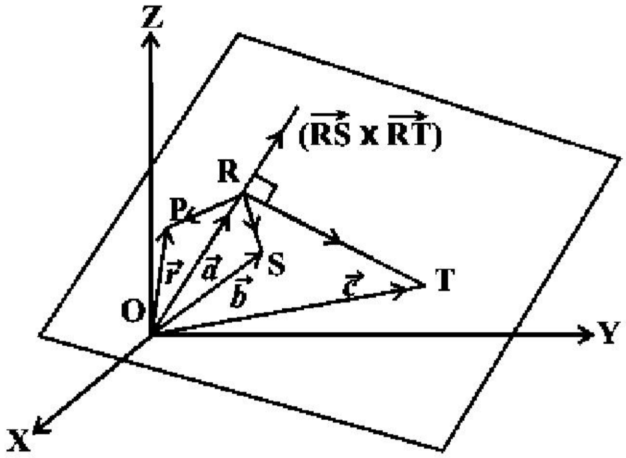


Fig. 10

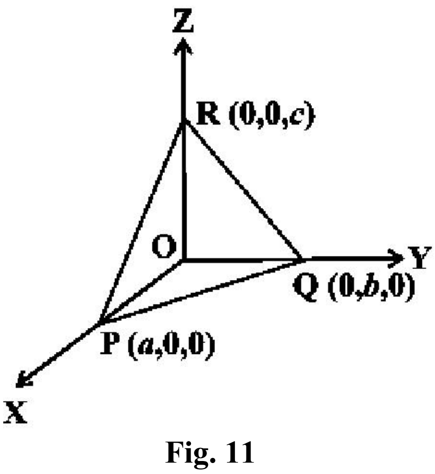


Fig. 11

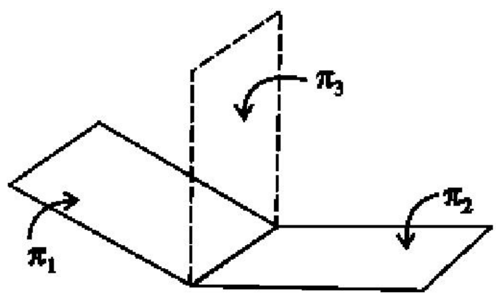
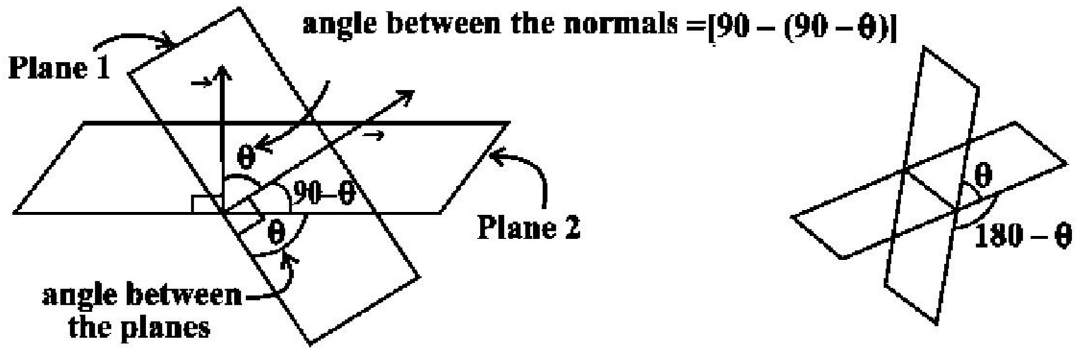


Fig. 12



(a)

(b)

Fig. 13

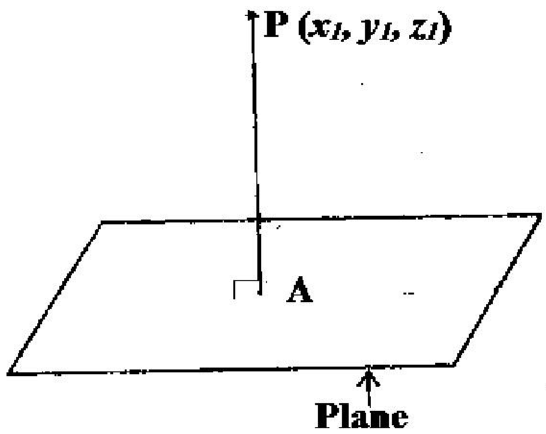


Fig. 14

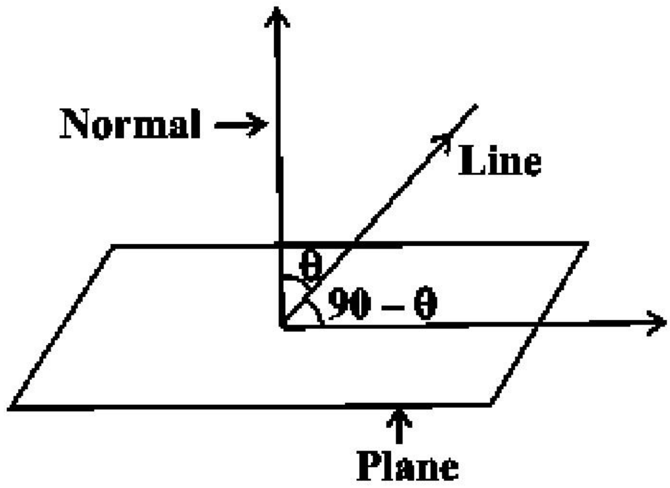


Fig. 15