Formulae For Probability

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RECAPITULATION & CONDITIONAL PROBABILITY

Important Terms, Definitions & Formulae

01. Basics Of Probability:

Let S and E be the sample space and an event in an experiment respectively.

Then, $\text{Probability} = \dfrac{\text{Number of favourable events}}{\text{Total number of elementary events}} = \dfrac{n(E)}{n(S)}$.

- Also as $\phi \subseteq E \subseteq S$

$\therefore \quad n(\phi) \le n(E) \le n(S)$

or $\quad 0 \le n(E) \le n(S)$

$\Rightarrow \quad \dfrac{0}{n(S)} \le \dfrac{n(E)}{n(S)} \le \dfrac{n(S)}{n(S)}$

$\Rightarrow \quad 0 \le P(E) \le 1$.

Hence, if $P(E)$ denotes the probability of occurrence of an event E then, $0 \le P(E) \le 1$ and $P(\overline{E}) = 1 - P(E)$ such that $P(\overline{E})$ denotes the probability of *non-occurrence* of the event E.

❖ **Note that $P(\overline{E})$ can also be represented as $P(E')$.**

02. Mutually Exclusive or Disjoint Events: Two events A and B are said to be mutually exclusive if occurrence of one *prevents* the occurrence of the other.

Consider an example of throwing a die. We have the sample spaces as, $S = \{1,2,3,4,5,6\}$.

Suppose $A =$ *the event of occurence of a number greater than 4* $= \{5,6\}$,

$B =$ *the event of occurence of an odd number* $= \{1,3,5\}$ and

$C =$ *the event of occurence of an even number* $= \{2,4,6\}$.

In these events, the events B and C are mutually exclusive events but A and B are not mutually exclusive events because they can occur together (when the number 5 comes up). Similarly A and C are not mutually exclusive events as they can also occur together (when the number 6 comes up).

❖ If A and B are mutually exhaustive events then we always have,

$P(A \cap B) = 0 \qquad [\text{As } n(A \cap B) = \phi$

$\therefore \quad P(A \cup B) = P(A) + P(B)$.

❖ If A, B and C are mutually exhaustive events then we always have,

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$.

03. Independent Events: Two events are independent if the occurrence of one *does not affect* the occurrence of the other.

Consider an example of drawing two balls *one by one with replacement* from a bag containing 3 red and 2 black balls.

Suppose $A =$ *the event of occurence of a red ball in first draw*,

$B =$ *the event of occurence of a black ball in the second draw*.

Then $P(A) = \dfrac{3}{5}$, $P(B) = \dfrac{2}{5}$.

Here probability of occurrence of event B is not affected by occurrence or non-occurrence of the event A. Hence event A and B are independent events.

But if the two balls would have been drawn *one by one without replacement,* then the probability of occurrence of a black ball in second draw when a red ball has been drawn in first draw $= P(B) = \frac{2}{4}$.
Also if a red ball is not drawn in the first draw, then the probability of occurrence of a black ball in the second draw $= P(B) = \frac{1}{4}$ (After a black ball is drawn there are only 4 balls left in the bag). In this case the event of drawing a red ball in the first draw and the event of drawing a black ball in the second draw are not independent.

- ❖ For independent events A and B, we always have $\mathbf{P(A \cap B) = P(A).P(B)}$.
- ❖ For independent events A, B and C, we always have $\mathbf{P(A \cap B \cap C) = P(A).P(B).P(C)}$.
- ❖ Also we have for independent events A and B, $\mathbf{P(A \cup B) = 1 - P(\bar{A}).P(\bar{B})}$.
- ❖ Since for independent events A and B we have $\mathbf{P(A \cap B) = P(A).P(B)}$, so the *conditional probability* (discussed later in this chapter) of event A when B has already occurred is given as,

$$\mathbf{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

$$\textbf{i.e.,} \quad \mathbf{P(A|B) = \frac{P(A).P(B)}{P(B)}} \quad \Rightarrow \quad \mathbf{P(A|B) = P(A)}.$$

04. Exhaustive Events: Two or more events say A, B and C of an experiment are said to be exhaustive events if,

a) their *union is the total sample space* i.e. $A \cup B \cup C = S$
b) the events A, B and C are disjoint in pairs i.e. $A \cap B = \phi$, $B \cap C = \phi$ and $C \cap A = \phi$.
c) $P(A) + P(B) + P(C) = 1$.

Consider an example of throwing a die. We have $S = \{1, 2, 3, 4, 5, 6\}$.

Suppose $A =$ *the event of occurence of an even number* $= \{2, 4, 6\}$,

$B =$ *the event of occurence of an odd number* $= \{1, 3, 5\}$ and

$C =$ *the event of getting a number multiple of 3* $= \{3, 6\}$.

In these events, the events A and B are exhaustive events as $A \cup B = S$ but the events A and C or the events B and C are not exhaustive events as $A \cup C \neq S$ and similarly $B \cup C \neq S$.

05. Conditional Probability: By the conditional probability we mean the *probability of occurrence of event A when B has already occurred.*
You can note that in case of occurrence of event A when B has already occurred, the ***event B acts as the sample space and $A \cap B$ acts as the favourable event.***
The 'conditional probability of occurrence of event A when B has already occurred' is sometimes also called as **probability of occurrence of event A w.r.t. B**.

- ❖ $P(A|B) = \frac{P(A \cap B)}{P(B)}$, $B \neq \phi$ *i.e.* $P(B) \neq 0$
- ❖ $P(B|A) = \frac{P(A \cap B)}{P(A)}$, $A \neq \phi$ *i.e.* $P(A) \neq 0$
- ❖ $P(\bar{A}|B) = \frac{P(\bar{A} \cap B)}{P(B)}$, $P(B) \neq 0$
- ❖ $P(A|\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})}$, $P(\bar{B}) \neq 0$
- ❖ $P(\bar{A}|\bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$, $P(\bar{B}) \neq 0$
- ❖ $P(A|B) + P(\bar{A}|B) = 1$, $B \neq \phi$.

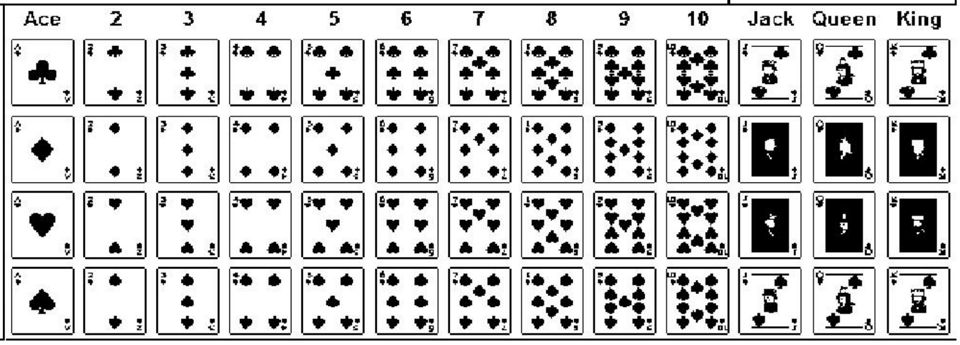
06. Useful formulae:

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ *i.e.* $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
b) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$
c) $P(\bar{A} \cap B) = P(\text{only } B) = P(B - A) = P(B \text{ but not } A) = P(B) - P(A \cap B)$

d) $P\left(A\cap\overline{B}\right)=P\left(\text{only }A\right)=P\left(A-B\right)=P\left(A\text{ but not }B\right)=P\left(A\right)-P\left(A\cap B\right)$

e) $P\left(\overline{A}\cap\overline{B}\right)=P\left(\text{neither }A\text{ nor }B\right)=1-P\left(A\cup B\right)$

Pictorial Description Of The Playing Cards:

SET OF 52 PLAYING CARDS	Face Cards
Suit Clubs (Black card) Diamonds (Red card) Hearts (Red card) Spades (Black card)	

❖ $^{n}C_{r}=\dfrac{n!}{r!\left(n-r\right)!}$, $^{n}C_{r}={}^{n}C_{n-r}$ such that $n\ge r$

❖ $n!=n.(n-1).(n-2)...5.4.3.2.1$. Also, $0!=1$.

07. Events and Symbolic representations:

Verbal description of the event	Equivalent set notation
Event A	A
Not A	$\overline{A}$ or A'
A or B (occurrence of atleast one of A and B)	$A\cup B$ or $A+B$
A and B (simultaneous occurrence of both A and B)	$A\cap B$ or AB
A but not B (A occurs but B does not)	$A\cap\overline{B}$ or $A-B$
Neither A nor B	$\overline{A}\cap\overline{B}$
At least one of A, B or C	$A\cup B\cup C$
All the three of A, B and C	$A\cap B\cap C$

TOTAL PROBABILITY & BAYES' THEOREM

Important Terms, Definitions & Formulae

01. Bayes' Theorem: If $E_1, E_2, E_3,..., E_n$ are n non- empty events constituting a partition of sample spaces S *i.e.,* $E_1, E_2, E_3,..., E_n$ are pair wise disjoint and $E_1\cup E_2\cup E_3\cup...\cup E_n=S$ and A is any event of non-zero probability then,

$$P\left(E_i\,|\,A\right)=\frac{P\left(E_i\right).P\left(A\,|\,E_i\right)}{\sum_{j=1}^{n}P\left(E_j\right).P\left(A\,|\,E_j\right)},\ i=1,2,3,...,n$$

For example, $P\left(E_1|A\right)=\dfrac{P\left(E_1\right).P\left(A\,|\,E_1\right)}{P\left(E_1\right).P\left(A\,|\,E_1\right)+P\left(E_2\right).P\left(A\,|\,E_2\right)+P\left(E_3\right).P\left(A\,|\,E_3\right)}$.

❖ Bayes' Theorem is also known as the ***formula for the probability of causes***.

❖ If $E_1, E_2, E_3,..., E_n$ form a partition of S and A be any event then,

$$P\left(A\right)=P\left(E_1\right).P\left(A\,|\,E_1\right)+P\left(E_2\right).P\left(A\,|\,E_2\right)+...+P\left(E_n\right).P\left(A\,|\,E_n\right)$$

$[\because P(E_i\cap A)=P(E_i).P(A|E_i))$

- The probabilities $\mathrm{P}(\mathrm{E}_1), \mathrm{P}(\mathrm{E}_2), \ldots, \mathrm{P}(\mathrm{E}_n)$ which are known before the experiment takes place are called **priori probabilities** and $\mathrm{P}(\mathrm{A} \mid \mathrm{E}_n)$ are called **posteriori probabilities**.

BERNOULLI TRIALS & BINOMIAL DISTRIBUTION

Important Terms, Definitions & Formulae

01. Bernoulli Trials: Trials of a random experiment are called Bernoulli trials, if they satisfy the following four conditions:

a) *The trials should be finite in numbers.*
b) *The trials should be independent of each other.*
c) *Each of the trial yields exactly two outcomes i.e. success or failure.*
d) *The probability of success or the failure remains the same in each of the trial.*

If an experiment is repeated *n* times under the similar conditions, we say that *n* trials of the experiment have been made.

02. Binomial Distribution: Let E be an event. Let $p =$ *probability of success in one trial (i.e., occurrence of event* E *in one trial)* and, $q = 1 - p =$ *probability of failure in one trial (i.e.,non-occurrence of event* E *in one trial).*
Let X = number of successes (*i.e.*, number of times event E occurs in *n* trials).
Then, Probability of X successes in *n* trials is given by the relation,

$$\mathrm{P}(\mathrm{X}=r) = \mathrm{P}(r) = {}^n\mathrm{C}_r p^r q^{n-r}$$

where $r = 0,1,2,3,\ldots,n$; $p =$ *probability of success in one trial* and $q = 1 - p =$ *probability of failure in one trial.*

The result $\mathrm{P}(\mathrm{X}=r) = \mathrm{P}(r) = {}^n\mathrm{C}_r p^r q^{n-r}$ can be used only when:
(i) the probability of success in each trial is the same.
(ii) each trial must surely result in either a success or a failure.

- $\mathrm{P}(\mathrm{X}=r)$ or $\mathrm{P}(r)$ is also called **probability of occurrence of event E exactly *r* times in *n* trials.**
- Here ${}^n\mathrm{C}_r = \dfrac{n!}{r!\,(n-r)!}$.
- Note that ${}^n\mathrm{C}_r p^r q^{n-r}$ is the $(r+1)^{th}$ term in the binomial expansion of $(q+p)^n$.
- $Mean = \sum_{r=0}^{n} r.\mathrm{P}(r) = np$
- $Variance = \sum_{r=0}^{n} r^2.\mathrm{P}(r) - (\mathrm{Mean})^2 = npq$
- $Standard\ Deviation = \sqrt{npq}$
- $Recurrence\ formula$, $\mathrm{P}(x = r+1) = \left(\dfrac{n-r}{r+1}\right)\left(\dfrac{p}{q}\right)\mathrm{P}(r)$.
- A Binomial Distribution with n Bernoulli trials and probability of success in each trial as p is denoted by $\mathrm{B}(n,p)$. Here n and p are known as **the parameters of binomial distribution**.
- The expression $\mathrm{P}(x=r)$ or $\mathrm{P}(r)$ is called the **probability function of the binomial distribution.**

PROBABILITY DISTRIBUTION

Important Terms, Definitions & Formulae

01. Random Variable: A random variable is a real valued function defined over the sample space of an experiment. In other words, a random variable is a real-valued function whose domain is the sample space of a random experiment. A random variable is usually denoted by upper case letters X, Y, Z etc.

- **Discrete random variable:** It is a random variable which can take only finite or countably infinite number of values.
- **Continuous random variable:** It is a random variable which can take any value between two given limits is called a continuous random variable.

02. Probability Distribution Of A Random Variable: If the values of a random variable together with the corresponding probabilities are given, then this description is called a probability distribution of the random variable.

- *Mean or Expectation of a random variable* $X = \mu = \sum_{i=1}^{n} x_i \, \mathrm{P}_i$
- *Variance* $= (\sigma^2) = \sum_{i=1}^{n} \mathrm{P}_i \, x_i^2 - \mu^2$
- *Standard Deviation* $= \sigma = \sqrt{Variance}$.

Hii, All!

I hope this texture may have proved beneficial for you.

While going through this material, if you noticed any error(s) or, something which doesn't make sense to you, **please** bring it in my notice through SMS or Call at **+91-9650 350 480** or Email at **theopgupta@gmail.com**.

With lots of Love & Blessings!

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