**Integration..**

The principles of integration were formulated independently by [Isaac Newton](http://en.wikipedia.org/wiki/Isaac_Newton) and [Gottfried Leibniz](http://en.wikipedia.org/wiki/Gottfried_Leibniz) in the late 17th century. Through the [fundamental theorem of calculus](http://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus), which they independently developed, integration is connected with differentiation: if *f* is a continuous real-valued function defined on a [closed interval](http://en.wikipedia.org/wiki/Closed_interval) [*a*, *b*], then, once an antiderivative *F* of *f* is known, the definite integral of *f* over that interval is given by

\int_a^b \! f(x)\,dx = F(b) - F(a).

Integrals and derivatives became the basic tools of calculus, with numerous applications in science and [engineering](http://en.wikipedia.org/wiki/Engineering). The founders of the calculus thought of the integral as an infinite sum of rectangles of [infinitesimal](http://en.wikipedia.org/wiki/Infinitesimal) width. A rigorous mathematical definition of the integral was given by [Bernhard Riemann](http://en.wikipedia.org/wiki/Bernhard_Riemann). It is based on a limiting procedure which approximates the area of a [curvilinear](http://en.wikipedia.org/wiki/Curvilinear) region by breaking the region into thin vertical slabs. Beginning in the nineteenth century, more sophisticated notions of integrals began to appear, where the type of the function as well as the domain over which the integration is performed has been generalised. A [line integral](http://en.wikipedia.org/wiki/Line_integral) is defined for functions of two or three variables, and the interval of integration [*a*, *b*] is replaced by a certain [curve](http://en.wikipedia.org/wiki/Curve) connecting two points on the plane or in the space. In a [surface integral](http://en.wikipedia.org/wiki/Surface_integral), the curve is replaced by a piece of a [surface](http://en.wikipedia.org/wiki/Surface) in the three-dimensional space. Integrals of [differential forms](http://en.wikipedia.org/wiki/Differential_form) play a fundamental role in modern [differential geometry](http://en.wikipedia.org/wiki/Differential_geometry). These generalizations of integrals first arose from the needs of [physics](http://en.wikipedia.org/wiki/Physics), and they play an important role in the formulation of many physical laws, notably those of [electrodynamics](http://en.wikipedia.org/wiki/Classical_electromagnetism). There are many modern concepts of integration, among these, the most common is based on the abstract mathematical theory known as [Lebesgue integration](http://en.wikipedia.org/wiki/Lebesgue_integration), developed by [Henri Lebesgue](http://en.wikipedia.org/wiki/Henri_Lebesgue).

**History[[edit](http://en.wikipedia.org/w/index.php?title=Integral&action=edit&section=1" \o "Edit section: History)]**

*See also:* [*History of calculus*](http://en.wikipedia.org/wiki/History_of_calculus)

**Pre-calculus integration[[edit](http://en.wikipedia.org/w/index.php?title=Integral&action=edit&section=2" \o "Edit section: Pre-calculus integration)]**

The first documented systematic technique capable of determining integrals is the [method of exhaustion](http://en.wikipedia.org/wiki/Method_of_exhaustion) of the [ancient Greek](http://en.wikipedia.org/wiki/Ancient_Greek) astronomer [Eudoxus](http://en.wikipedia.org/wiki/Eudoxus_of_Cnidus) (*ca.* 370 BC), which sought to find areas and volumes by breaking them up into an infinite number of shapes for which the area or volume was known. This method was further developed and employed by [Archimedes](http://en.wikipedia.org/wiki/Archimedes) in the 3rd century BC and used to calculate areas for [parabolas](http://en.wikipedia.org/wiki/Parabola) and an approximation to the area of a circle. Similar methods were independently developed in China around the 3rd century AD by [Liu Hui](http://en.wikipedia.org/wiki/Liu_Hui), who used it to find the area of the circle. This method was later used in the 5th century by Chinese father-and-son mathematicians [Zu Chongzhi](http://en.wikipedia.org/wiki/Zu_Chongzhi) and [Zu Geng](http://en.wikipedia.org/wiki/Zu_Geng_(mathematician)) to find the volume of a sphere ([Shea 2007](http://en.wikipedia.org/wiki/Integral#CITEREFShea2007); [Katz 2004](http://en.wikipedia.org/wiki/Integral#CITEREFKatz2004), pp. 125–126).

The next significant advances in integral calculus did not begin to appear until the 16th century. At this time the work of [Cavalieri](http://en.wikipedia.org/wiki/Bonaventura_Cavalieri) with his [*method of indivisibles*](http://en.wikipedia.org/wiki/Cavalieri%27s_principle), and work by [Fermat](http://en.wikipedia.org/wiki/Pierre_de_Fermat), began to lay the foundations of modern calculus, with Cavalieri computing the integrals of *xn* up to degree *n* = 9 in [Cavalieri's quadrature formula](http://en.wikipedia.org/wiki/Cavalieri%27s_quadrature_formula). Further steps were made in the early 17th century by [Barrow](http://en.wikipedia.org/wiki/Isaac_Barrow) and [Torricelli](http://en.wikipedia.org/wiki/Evangelista_Torricelli), who provided the first hints of a connection between integration and [differentiation](http://en.wikipedia.org/wiki/Differential_calculus). Barrow provided the first proof of the [fundamental theorem of calculus](http://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus). [Wallis](http://en.wikipedia.org/wiki/John_Wallis) generalized Cavalieri's method, computing integrals of *x* to a general power, including negative powers and fractional powers.

**Newton and Leibniz[[edit](http://en.wikipedia.org/w/index.php?title=Integral&action=edit&section=3" \o "Edit section: Newton and Leibniz)]**

The major advance in integration came in the 17th century with the independent discovery of the [fundamental theorem of calculus](http://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus) by [Newton](http://en.wikipedia.org/wiki/Isaac_Newton) and [Leibniz](http://en.wikipedia.org/wiki/Gottfried_Leibniz). The theorem demonstrates a connection between integration and differentiation. This connection, combined with the comparative ease of differentiation, can be exploited to calculate integrals. In particular, the fundamental theorem of calculus allows one to solve a much broader class of problems. Equal in importance is the comprehensive mathematical framework that both Newton and Leibniz developed. Given the name infinitesimal calculus, it allowed for precise analysis of functions within continuous domains. This framework eventually became modern [calculus](http://en.wikipedia.org/wiki/Calculus), whose notation for integrals is drawn directly from the work of Leibniz.

**Formalizing integrals[[edit](http://en.wikipedia.org/w/index.php?title=Integral&action=edit&section=4" \o "Edit section: Formalizing integrals)]**

While Newton and Leibniz provided a systematic approach to integration, their work lacked a degree of [rigour](http://en.wikipedia.org/wiki/Rigor#Mathematical_rigour). [Bishop Berkeley](http://en.wikipedia.org/wiki/George_Berkeley) memorably attacked the vanishing increments used by Newton, calling them "[ghosts of departed quantities](http://en.wikipedia.org/wiki/The_Analyst#Content)". Calculus acquired a firmer footing with the development of [limits](http://en.wikipedia.org/wiki/Limit_(mathematics)). Integration was first rigorously formalized, using limits, by [Riemann](http://en.wikipedia.org/wiki/Bernhard_Riemann). Although all bounded piecewise continuous functions are Riemann integrable on a bounded interval, subsequently more general functions were considered—particularly in the context of [Fourier analysis](http://en.wikipedia.org/wiki/Fourier_analysis)—to which Riemann's definition does not apply, and [Lebesgue](http://en.wikipedia.org/wiki/Henri_Lebesgue) formulated a different definition of integral, founded in [measure theory](http://en.wikipedia.org/wiki/Measure_(mathematics)) (a subfield of [real analysis](http://en.wikipedia.org/wiki/Real_analysis)). Other definitions of integral, extending Riemann's and Lebesgue's approaches, were proposed. These approaches based on the real number system are the ones most common today, but alternative approaches exist, such as a definition of integral as the [standard part](http://en.wikipedia.org/wiki/Standard_part) of an infinite Riemann sum, based on the [hyperreal number](http://en.wikipedia.org/wiki/Hyperreal_number) system.

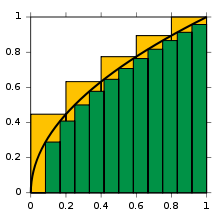
**Historical notation[[edit](http://en.wikipedia.org/w/index.php?title=Integral&action=edit&section=5" \o "Edit section: Historical notation)]**

[Isaac Newton](http://en.wikipedia.org/wiki/Isaac_Newton) used a small vertical bar above a variable to indicate integration, or placed the variable inside a box. The vertical bar was easily confused with \dot{x}or x'\,\!, which Newton used to indicate differentiation, and the box notation was difficult for printers to reproduce, so these notations were not widely adopted.

The modern notation for the indefinite integral was introduced by [Gottfried Leibniz](http://en.wikipedia.org/wiki/Gottfried_Leibniz) in 1675 ([Burton 1988](http://en.wikipedia.org/wiki/Integral#CITEREFBurton1988), p. 359; [Leibniz 1899](http://en.wikipedia.org/wiki/Integral#CITEREFLeibniz1899), p. 154). He adapted the [integral symbol](http://en.wikipedia.org/wiki/Integral_symbol), **∫**, from the letter *ſ* ([long s](http://en.wikipedia.org/wiki/Long_s)), standing for *summa* (written as *ſumma*; Latin for "sum" or "total"). The modern notation for the definite integral, with limits above and below the integral sign, was first used by [Joseph Fourier](http://en.wikipedia.org/wiki/Joseph_Fourier) in *Mémoires* of the French Academy around 1819–20, reprinted in his book of 1822 ([Cajori 1929](http://en.wikipedia.org/wiki/Integral" \l "CITEREFCajori1929), pp. 249–250; [Fourier 1822](http://en.wikipedia.org/wiki/Integral#CITEREFFourier1822), §231).

**Introduction[[edit](http://en.wikipedia.org/w/index.php?title=Integral&action=edit&section=7" \o "Edit section: Introduction)]**

Integrals appear in many practical situations. If a swimming pool is rectangular with a flat bottom, then from its length, width, and depth we can easily determine the volume of water it can contain (to fill it), the area of its surface (to cover it), and the length of its edge (to rope it). But if it is oval with a rounded bottom, all of these quantities call for integrals. Practical approximations may suffice for such trivial examples, but [precision engineering](http://en.wikipedia.org/wiki/Precision_engineering) (of any discipline) requires exact and rigorous values for these elements.

[](http://en.wikipedia.org/wiki/File:Integral_approximations.svg)

[http://bits.wikimedia.org/static-1.23wmf9/skins/common/images/magnify-clip.png](http://en.wikipedia.org/wiki/File:Integral_approximations.svg)

Approximations to integral of √*x* from 0 to 1, with 5 ■ (yellow) right endpoint partitions and 12 ■ (green) left endpoint partitions

To start off, consider the curve *y* = *f*(*x*) between *x* = 0 and *x* = 1 with *f*(*x*) = √*x*. We ask:

What is the area under the function *f*, in the interval from 0 to 1?

and call this (yet unknown) area the **integral** of *f*. The notation for this integral will be

 \int_0^1 \sqrt x \, dx \,\!.

As a first approximation, look at the unit square given by the sides *x* = 0 to *x* = 1 and *y* = *f*(0) = 0 and *y* = *f*(1) = 1. Its area is exactly 1. As it is, the true value of the integral must be somewhat less. Decreasing the width of the approximation rectangles shall give a better result; so cross the interval in five steps, using the approximation points 0, 1/5, 2/5, and so on to 1. Fit a box for each step using the right end height of each curve piece, thus √(1⁄5), √(2⁄5), and so on to √1 = 1. Summing the areas of these rectangles, we get a better approximation for the sought integral, namely

\textstyle \sqrt {\frac {1} {5}} \left ( \frac {1} {5} - 0 \right ) + \sqrt {\frac {2} {5}} \left ( \frac {2} {5} - \frac {1} {5} \right ) + \cdots + \sqrt {\frac {5} {5}} \left ( \frac {5} {5} - \frac {4} {5} \right ) \approx 0.7497.\,\!

**Trigonometry**

**History[[edit](http://en.wikipedia.org/w/index.php?title=Trigonometry&action=edit&section=1" \o "Edit section: History)]**

*Main article:* [*History of trigonometry*](http://en.wikipedia.org/wiki/History_of_trigonometry)

[](http://en.wikipedia.org/wiki/File:Hipparchos_1.jpeg)

[http://bits.wikimedia.org/static-1.23wmf10/skins/common/images/magnify-clip.png](http://en.wikipedia.org/wiki/File:Hipparchos_1.jpeg)

The first [trigonometric table](http://en.wikipedia.org/wiki/Generating_trigonometric_tables) was apparently compiled by [Hipparchus](http://en.wikipedia.org/wiki/Hipparchus), who is now consequently known as "the father of trigonometry."[[3]](http://en.wikipedia.org/wiki/Trigonometry#cite_note-3)

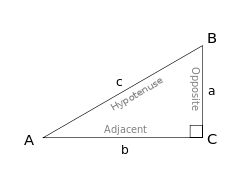
[Sumerian](http://en.wikipedia.org/wiki/Sumer) astronomers studied angle measure, using a division of circles into 360 degrees.[[4]](http://en.wikipedia.org/wiki/Trigonometry#cite_note-4) They, and later the [Babylonians](http://en.wikipedia.org/wiki/Babylonians), studied the ratios of the sides of [similar](http://en.wikipedia.org/wiki/Similarity_(geometry)) triangles and discovered some properties of these ratios, but did not turn that into a systematic method for finding sides and angles of triangles. The [ancient Nubians](http://en.wikipedia.org/wiki/Nubia) used a similar method.[[5]](http://en.wikipedia.org/wiki/Trigonometry#cite_note-5) The [ancient Greeks](http://en.wikipedia.org/wiki/Ancient_Greeks) transformed trigonometry into an ordered science.[[6]](http://en.wikipedia.org/wiki/Trigonometry#cite_note-6)

In the 3rd century BCE, classical [Greek mathematicians](http://en.wikipedia.org/wiki/Greek_mathematics) (such as [Euclid](http://en.wikipedia.org/wiki/Euclid) and [Archimedes](http://en.wikipedia.org/wiki/Archimedes)) studied the properties of [chords](http://en.wikipedia.org/wiki/Chord_(geometry)) and [inscribed angles](http://en.wikipedia.org/wiki/Inscribed_angle) in circles, and proved theorems that are equivalent to modern trigonometric formulae, although they presented them geometrically rather than algebraically. [Claudius Ptolemy](http://en.wikipedia.org/wiki/Ptolemy) expanded upon [Hipparchus](http://en.wikipedia.org/wiki/Hipparchus)' *Chords in a Circle* in his [*Almagest*](http://en.wikipedia.org/wiki/Almagest).[[7]](http://en.wikipedia.org/wiki/Trigonometry#cite_note-7)

The modern [sine function](http://en.wikipedia.org/wiki/Trigonometric_functions) was first defined in the [*Surya Siddhanta*](http://en.wikipedia.org/wiki/Surya_Siddhanta), and its properties were further documented by the 5th century (CE) [Indian mathematician](http://en.wikipedia.org/wiki/Indian_mathematics) and astronomer [Aryabhata](http://en.wikipedia.org/wiki/Aryabhata).[[8]](http://en.wikipedia.org/wiki/Trigonometry#cite_note-8) These Greek and Indian works were translated and expanded by [medieval Islamic mathematicians](http://en.wikipedia.org/wiki/Mathematics_in_medieval_Islam). By the 10th century, Islamic mathematicians were using all six trigonometric functions, had tabulated their values, and were applying them to problems in [spherical geometry](http://en.wikipedia.org/wiki/Spherical_geometry).[[*citation needed*](http://en.wikipedia.org/wiki/Wikipedia:Citation_needed)] At about the same time, [Chinese](http://en.wikipedia.org/wiki/Chinese_mathematics) mathematicians developed trigonometry independently, although it was not a major field of study for them. Knowledge of trigonometric functions and methods reached [Europe](http://en.wikipedia.org/wiki/Europe) via [Latin translations](http://en.wikipedia.org/wiki/Latin_translations_of_the_12th_century) of the works of [Persian and Arabic astronomers](http://en.wikipedia.org/wiki/Astronomy_in_medieval_Islam) such as [Al Battani](http://en.wikipedia.org/wiki/Muhammad_ibn_J%C4%81bir_al-Harr%C4%81n%C4%AB_al-Batt%C4%81n%C4%AB) and [Nasir al-Din al-Tusi](http://en.wikipedia.org/wiki/Nasir_al-Din_al-Tusi).[[9]](http://en.wikipedia.org/wiki/Trigonometry#cite_note-9) One of the earliest works on trigonometry by a European mathematician is *De Triangulis* by the 15th century [German](http://en.wikipedia.org/wiki/Germany) mathematician [Regiomontanus](http://en.wikipedia.org/wiki/Regiomontanus). Trigonometry was still so little known in 16th-century Europe that [Nicolaus Copernicus](http://en.wikipedia.org/wiki/Nicolaus_Copernicus) devoted two chapters of [*De revolutionibus orbium coelestium*](http://en.wikipedia.org/wiki/De_revolutionibus_orbium_coelestium) to explain its basic concepts.

Driven by the demands of [navigation](http://en.wikipedia.org/wiki/Navigation) and the growing need for accurate maps of large geographic areas, trigonometry grew into a major branch of mathematics.[[10]](http://en.wikipedia.org/wiki/Trigonometry#cite_note-10) [Bartholomaeus Pitiscus](http://en.wikipedia.org/wiki/Bartholomaeus_Pitiscus) was the first to use the word, publishing his *Trigonometria* in 1595.[[11]](http://en.wikipedia.org/wiki/Trigonometry#cite_note-11) [Gemma Frisius](http://en.wikipedia.org/wiki/Gemma_Frisius) described for the first time the method of [triangulation](http://en.wikipedia.org/wiki/Triangulation) still used today in surveying. It was [Leonhard Euler](http://en.wikipedia.org/wiki/Leonhard_Euler) who fully incorporated [complex numbers](http://en.wikipedia.org/wiki/Complex_number) into trigonometry. The works of [James Gregory](http://en.wikipedia.org/wiki/James_Gregory_(astronomer_and_mathematician)) in the 17th century and [Colin Maclaurin](http://en.wikipedia.org/wiki/Colin_Maclaurin) in the 18th century were influential in the development of [trigonometric series](http://en.wikipedia.org/wiki/Trigonometric_series).[[12]](http://en.wikipedia.org/wiki/Trigonometry#cite_note-12) Also in the 18th century, [Brook Taylor](http://en.wikipedia.org/wiki/Brook_Taylor) defined the general [Taylor series](http://en.wikipedia.org/wiki/Taylor_series).[[13]](http://en.wikipedia.org/wiki/Trigonometry#cite_note-13)

## Overview[[edit](http://en.wikipedia.org/w/index.php?title=Trigonometry&action=edit&section=2" \o "Edit section: Overview)]

[](http://en.wikipedia.org/wiki/File:TrigonometryTriangle.svg)

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In this right triangle: sin *A* = *a*/*c*; cos *A* = *b*/*c*; tan *A* = *a*/*b*.

If one [angle](http://en.wikipedia.org/wiki/Angle) of a triangle is 90 degrees and one of the other angles is known, the third is thereby fixed, because the three angles of any triangle add up to 180 degrees. The two acute angles therefore add up to 90 degrees: they are [complementary angles](http://en.wikipedia.org/wiki/Complementary_angles). The [shape](http://en.wikipedia.org/wiki/Shape) of a triangle is completely determined, except for [similarity](http://en.wikipedia.org/wiki/Similarity_(geometry)), by the angles. Once the angles are known, the [ratios](http://en.wikipedia.org/wiki/Ratio) of the sides are determined, regardless of the overall size of the triangle. If the length of one of the sides is known, the other two are determined. These ratios are given by the following [trigonometric functions](http://en.wikipedia.org/wiki/Trigonometric_function) of the known angle *A*, where *a*, *b* and *c* refer to the lengths of the sides in the accompanying figure:

* [**Sine**](http://en.wikipedia.org/wiki/Sine) function (sin), defined as the ratio of the side opposite the angle to the [hypotenuse](http://en.wikipedia.org/wiki/Hypotenuse).

\sin A=\frac{\textrm{opposite}}{\textrm{hypotenuse}}=\frac{a}{\,c\,}\,.

* [**Cosine**](http://en.wikipedia.org/wiki/Cosine) function (cos), defined as the ratio of the [adjacent](http://en.wikipedia.org/wiki/Adjacent_side_(right_triangle)) leg to the hypotenuse.

\cos A=\frac{\textrm{adjacent}}{\textrm{hypotenuse}}=\frac{b}{\,c\,}\,.

* [**Tangent**](http://en.wikipedia.org/wiki/Tangent) function (tan), defined as the ratio of the opposite leg to the adjacent leg.

\tan A=\frac{\textrm{opposite}}{\textrm{adjacent}}=\frac{a}{\,b\,}=\frac{\sin A}{\cos A}\,.

The **hypotenuse** is the side opposite to the 90 degree angle in a right triangle; it is the longest side of the triangle, and one of the two sides adjacent to angle *A*. The **adjacent leg** is the other side that is adjacent to angle *A*. The **opposite side** is the side that is opposite to angle *A*. The terms **perpendicular** and **base** are sometimes used for the opposite and adjacent sides respectively. Many people find it easy to remember what sides of the right triangle are equal to sine, cosine, or tangent, by memorizing the word SOH-CAH-TOA (see below under [Mnemonics](http://en.wikipedia.org/wiki/Trigonometry#Mnemonics)).

The [reciprocals](http://en.wikipedia.org/wiki/Multiplicative_inverse) of these functions are named the **cosecant** (csc or cosec), **secant** (sec), and **cotangent** (cot), respectively:

\csc A=\frac{1}{\sin A}=\frac{c}{a} ,

\sec A=\frac{1}{\cos A}=\frac{c}{b} ,

\cot A=\frac{1}{\tan A}=\frac{\cos A}{\sin A}=\frac{b}{a} .

The [inverse functions](http://en.wikipedia.org/wiki/Inverse_trigonometric_function) are called the **arcsine**, **arccosine**, and **arctangent**, respectively. There are arithmetic relations between these functions, which are known as [trigonometric identities](http://en.wikipedia.org/wiki/Trigonometric_identities). The cosine, cotangent, and cosecant are so named because they are respectively the sine, tangent, and secant of the complementary angle abbreviated to "co-".

With these functions one can answer virtually all questions about arbitrary triangles by using the [law of sines](http://en.wikipedia.org/wiki/Law_of_sines) and the [law of cosines](http://en.wikipedia.org/wiki/Law_of_cosines). These laws can be used to compute the remaining angles and sides of any triangle as soon as two sides and their included angle or two angles and a side or three sides are known. These laws are useful in all branches of geometry, since every [polygon](http://en.wikipedia.org/wiki/Polygon) may be described as a finite combination of triangles.

**Probability**

## History[[edit](http://en.wikipedia.org/w/index.php?title=Probability&action=edit&section=3" \o "Edit section: History)]

*Main article:* [*History of probability*](http://en.wikipedia.org/wiki/History_of_probability)

The scientific study of probability is a modern development. [Gambling](http://en.wikipedia.org/wiki/Gambling) shows that there has been an interest in quantifying the ideas of probability for millennia, but exact mathematical descriptions arose much later. There are reasons of course, for the slow development of the mathematics of probability. Whereas games of chance provided the impetus for the mathematical study of probability, fundamental issues are still obscured by the superstitions of gamblers.[[9]](http://en.wikipedia.org/wiki/Probability#cite_note-9)

[](http://en.wikipedia.org/wiki/File:Christiaan_Huygens-painting.jpeg)

[http://bits.wikimedia.org/static-1.23wmf10/skins/common/images/magnify-clip.png](http://en.wikipedia.org/wiki/File:Christiaan_Huygens-painting.jpeg)

Christiaan Huygens probably published the first book on probability

According to Richard Jeffrey, "Before the middle of the seventeenth century, the term 'probable' (Latin *probabilis*) meant *approvable*, and was applied in that sense, univocally, to opinion and to action. A probable action or opinion was one such as sensible people would undertake or hold, in the circumstances."[[10]](http://en.wikipedia.org/wiki/Probability#cite_note-Jeffrey-10) However, in legal contexts especially, 'probable' could also apply to propositions for which there was good evidence.[[11]](http://en.wikipedia.org/wiki/Probability#cite_note-Franklin-11)

Aside from elementary work by [Gerolamo Cardano](http://en.wikipedia.org/wiki/Gerolamo_Cardano) in the 16th century, the doctrine of probabilities dates to the correspondence of [Pierre de Fermat](http://en.wikipedia.org/wiki/Pierre_de_Fermat) and [Blaise Pascal](http://en.wikipedia.org/wiki/Blaise_Pascal) (1654). [Christiaan Huygens](http://en.wikipedia.org/wiki/Christiaan_Huygens) (1657) gave the earliest known scientific treatment of the subject.[[12]](http://en.wikipedia.org/wiki/Probability#cite_note-12) [Jakob Bernoulli's](http://en.wikipedia.org/wiki/Jakob_Bernoulli) [*Ars Conjectandi*](http://en.wikipedia.org/wiki/Ars_Conjectandi) (posthumous, 1713) and [Abraham de Moivre's](http://en.wikipedia.org/wiki/Abraham_de_Moivre) [*Doctrine of Chances*](http://en.wikipedia.org/wiki/Doctrine_of_Chances) (1718) treated the subject as a branch of mathematics.[[13]](http://en.wikipedia.org/wiki/Probability#cite_note-13) See [Ian Hacking's](http://en.wikipedia.org/wiki/Ian_Hacking) *The Emergence of Probability*[[8]](http://en.wikipedia.org/wiki/Probability#cite_note-Emergence-8) and [James Franklin's](http://en.wikipedia.org/wiki/James_Franklin_(philosopher)) *The Science of Conjecture*[[*full citation needed*](http://en.wikipedia.org/wiki/Wikipedia:Citing_sources#What_information_to_include)] for histories of the early development of the very concept of mathematical probability.

The [theory of errors](http://en.wikipedia.org/wiki/Theory_of_errors) may be traced back to [Roger Cotes's](http://en.wikipedia.org/wiki/Roger_Cotes) *Opera Miscellanea* (posthumous, 1722), but a memoir prepared by [Thomas Simpson](http://en.wikipedia.org/wiki/Thomas_Simpson) in 1755 (printed 1756) first applied the theory to the discussion of errors of observation.[[*citation needed*](http://en.wikipedia.org/wiki/Wikipedia:Citation_needed)] The reprint (1757) of this memoir lays down the axioms that positive and negative errors are equally probable, and that certain assignable limits define the range of all errors. Simpson also discusses continuous errors and describes a probability curve.

The first two laws of error that were proposed both originated with [Pierre-Simon Laplace](http://en.wikipedia.org/wiki/Pierre-Simon_Laplace). The first law was published in 1774 and stated that the frequency of an error could be expressed as an exponential function of the numerical magnitude of the error, disregarding sign. The second law of error was proposed in 1778 by Laplace and stated that the frequency of the error is an exponential function of the square of the error.[[14]](http://en.wikipedia.org/wiki/Probability#cite_note-Wilson1923-14) The second law of error is called the normal distribution or the Gauss law. "It is difficult historically to attribute that law to Gauss, who in spite of his well-known precocity had probably not made this discovery before he was two years old."[[14]](http://en.wikipedia.org/wiki/Probability#cite_note-Wilson1923-14)

[Daniel Bernoulli](http://en.wikipedia.org/wiki/Daniel_Bernoulli) (1778) introduced the principle of the maximum product of the probabilities of a system of concurrent errors.

[](http://en.wikipedia.org/wiki/File:Bendixen_-_Carl_Friedrich_Gau%C3%9F,_1828.jpg)

[http://bits.wikimedia.org/static-1.23wmf10/skins/common/images/magnify-clip.png](http://en.wikipedia.org/wiki/File:Bendixen_-_Carl_Friedrich_Gau%C3%9F,_1828.jpg)

Carl Friedrich Gauss

[Adrien-Marie Legendre](http://en.wikipedia.org/wiki/Adrien-Marie_Legendre) (1805) developed the [method of least squares](http://en.wikipedia.org/wiki/Method_of_least_squares), and introduced it in his *Nouvelles méthodes pour la détermination des orbites des comètes* (*New Methods for Determining the Orbits of Comets*).[[*citation needed*](http://en.wikipedia.org/wiki/Wikipedia:Citation_needed)] In ignorance of Legendre's contribution, an Irish-American writer, [Robert Adrain](http://en.wikipedia.org/wiki/Robert_Adrain), editor of "The Analyst" (1808), first deduced the law of facility of error,

\phi(x) = ce^{-h^2 x^2},

where his a constant depending on precision of observation, and cis a scale factor ensuring that the area under the curve equals 1. He gave two proofs, the second being essentially the same as [John Herschel's](http://en.wikipedia.org/wiki/John_Herschel) (1850).[*[citation needed](http://en.wikipedia.org/wiki/Wikipedia:Citation_needed" \o "Wikipedia:Citation needed)*] [Gauss](http://en.wikipedia.org/wiki/Carl_Friedrich_Gauss) gave the first proof that seems to have been known in Europe (the third after Adrain's) in 1809. Further proofs were given by Laplace (1810, 1812), Gauss (1823), [James Ivory](http://en.wikipedia.org/wiki/James_Ivory_(mathematician)) (1825, 1826), Hagen (1837), [Friedrich Bessel](http://en.wikipedia.org/wiki/Friedrich_Bessel) (1838), [W. F. Donkin](http://en.wikipedia.org/w/index.php?title=W._F._Donkin&action=edit&redlink=1) (1844, 1856), and [Morgan Crofton](http://en.wikipedia.org/wiki/Morgan_Crofton) (1870). Other contributors were Ellis (1844), [De Morgan](http://en.wikipedia.org/wiki/Augustus_De_Morgan) (1864), [Glaisher](http://en.wikipedia.org/wiki/James_Whitbread_Lee_Glaisher) (1872), and [Giovanni Schiaparelli](http://en.wikipedia.org/wiki/Giovanni_Schiaparelli) (1875). [Peters](http://en.wikipedia.org/wiki/Christian_August_Friedrich_Peters)'s (1856) formula[*[clarification needed](http://en.wikipedia.org/wiki/Wikipedia:Please_clarify" \o "Wikipedia:Please clarify)*] for *r*, the [probable error](http://en.wikipedia.org/wiki/Probable_error) of a single observation, is well known.[[*to whom?*](http://en.wikipedia.org/wiki/Wikipedia:Avoid_weasel_words)]

In the nineteenth century authors on the general theory included [Laplace](http://en.wikipedia.org/wiki/Laplace), [Sylvestre Lacroix](http://en.wikipedia.org/wiki/Sylvestre_Lacroix) (1816), Littrow (1833), [Adolphe Quetelet](http://en.wikipedia.org/wiki/Adolphe_Quetelet) (1853), [Richard Dedekind](http://en.wikipedia.org/wiki/Richard_Dedekind) (1860), Helmert (1872), [Hermann Laurent](http://en.wikipedia.org/wiki/Hermann_Laurent) (1873), Liagre, Didion, and [Karl Pearson](http://en.wikipedia.org/wiki/Karl_Pearson). [Augustus De Morgan](http://en.wikipedia.org/wiki/Augustus_De_Morgan) and [George Boole](http://en.wikipedia.org/wiki/George_Boole) improved the exposition of the theory.

[Andrey Markov](http://en.wikipedia.org/wiki/Andrey_Markov) introduced[*[citation needed](http://en.wikipedia.org/wiki/Wikipedia:Citation_needed" \o "Wikipedia:Citation needed)*] the notion of [Markov chains](http://en.wikipedia.org/wiki/Markov_chains) (1906), which played an important role in [stochastic processes](http://en.wikipedia.org/wiki/Stochastic_process) theory and its applications. The modern theory of probability based on the [measure theory](http://en.wikipedia.org/wiki/Measure_(mathematics)) was developed by [Andrey Kolmogorov](http://en.wikipedia.org/wiki/Andrey_Kolmogorov) (1931).[[*citation needed*](http://en.wikipedia.org/wiki/Wikipedia:Citation_needed)]

On the geometric side (see [integral geometry](http://en.wikipedia.org/wiki/Integral_geometry)) contributors to [*The Educational Times*](http://en.wikipedia.org/w/index.php?title=The_Educational_Times&action=edit&redlink=1) were influential (Miller, Crofton, McColl, Wolstenholme, Watson, and [Artemas Martin](http://en.wikipedia.org/wiki/Artemas_Martin)).[[*citation needed*](http://en.wikipedia.org/wiki/Wikipedia:Citation_needed)]

*Further information:* [*History of statistics*](http://en.wikipedia.org/wiki/History_of_statistics)

Like other [theories](http://en.wikipedia.org/wiki/Theory), the [theory of probability](http://en.wikipedia.org/wiki/Probability_theory) is a representation of probabilistic concepts in formal terms—that is, in terms that can be considered separately from their meaning. These formal terms are manipulated by the rules of mathematics and logic, and any results are interpreted or translated back into the problem domain.

There have been at least two successful attempts to formalize probability, namely the [Kolmogorov](http://en.wikipedia.org/wiki/Kolmogorov) formulation and the [Cox](http://en.wikipedia.org/wiki/Richard_Threlkeld_Cox) formulation. In Kolmogorov's formulation (see [probability space](http://en.wikipedia.org/wiki/Probability_space)), [sets](http://en.wikipedia.org/wiki/Set_(mathematics)) are interpreted as [events](http://en.wikipedia.org/wiki/Event_(probability_theory)) and probability itself as a [measure](http://en.wikipedia.org/wiki/Measure_(mathematics)) on a class of sets. In [Cox's theorem](http://en.wikipedia.org/wiki/Cox%27s_theorem), probability is taken as a primitive (that is, not further analyzed) and the emphasis is on constructing a consistent assignment of probability values to propositions. In both cases, the [laws of probability](http://en.wikipedia.org/wiki/Probability_axioms) are the same, except for technical details.

There are other methods for quantifying uncertainty, such as the [Dempster–Shafer theory](http://en.wikipedia.org/wiki/Dempster%E2%80%93Shafer_theory) or [possibility theory](http://en.wikipedia.org/wiki/Possibility_theory), but those are essentially different and not compatible with the laws of probability as usually understood.