Dot product

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"Scalar product" redirects here. For the abstract scalar product, see Inner product space. For the product of a vector and a scalar, see Scalar multiplication.

In mathematics, the **dot product**, or **scalar product** (or sometimes **inner product** in the context of Euclidean space), is an algebraic operation that takes two equal-length sequences of numbers (usually coordinate vectors) and returns a single number. This operation can be defined either algebraically or geometrically. Algebraically, it is the sum of the products of the corresponding entries of the two sequences of numbers. Geometrically, it is the product of the magnitudes of the two vectors and the cosine of the angle between them. The name "dot product" is derived from the centered dot " · " that is often used to designate this operation; the alternative name "scalar product" emphasizes the scalar (rather than vectorial) nature of the result.

In three-dimensional space, the dot product contrasts with the cross product of two vectors, which produces a pseudovector as the result. The dot product is directly related to the cosine of the angle between two vectors in Euclidean space of any number of dimensions.

Contents [hide]

Definition [edit source]

The dot product is often defined in one of two ways: algebraically or geometrically. The equivalence of these definitions is proven later.

The geometric definition is based on the notion of angle. It should be noted that, in the modern presentation of Euclidean geometry, the points are defined as coordinates vectors. In such a presentation of the geometry, the notions of length and angles are not primitive and need to be defined. Therefore, in this case, the length of a vector is defined as the square root of the dot product of the vector by itself, and the geometric definition of the dot product is inverted to define the notion of (non oriented) angle.

Algebraic definition [edit source]

The dot product of two vectors **a** = [a_1, a_2, ..., a_n] and **b** = [b_1, b_2, ..., b_n] is defined as:[1]

$$\mathbf{a}\cdot\mathbf{b} = \sum_{i=1}^n a_ib_i = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

where Σ denotes summation notation and *n* is the dimension of the vector space. For instance, in three-dimensional space, the dot product of vectors [1, 3, −5] and [4, −2, −1] is:

$$[1, 3, -5]\cdot[4, -2, -1] = (1)(4) + (3)(-2) + (-5)(-1) = 4 - 6 + 5 = 3.$$

Geometric definition [edit source]

In Euclidean space, a Euclidean vector is a geometrical object that possesses both a magnitude and a direction. A vector can be pictured as an arrow. Its magnitude is its length, and its direction is the direction the arrow points. The magnitude of a vector **A** is denoted by $\|\mathbf{A}\|$. The dot product of two Euclidean vectors **A** and **B** is defined by[2]

$$\mathbf{A}\cdot\mathbf{B} = \|\mathbf{A}\|\,\|\mathbf{B}\|\cos\theta,$$

where θ is the angle between **A** and **B**.

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In particular, if **A** and **B** are orthogonal, then the angle between them is 90° and

$$\mathbf{A}\cdot\mathbf{B}=0.$$

At the other extreme, if they are codirectional, then the angle between them is 0° and

$$\mathbf{A}\cdot\mathbf{B}=\|\mathbf{A}\|\,\|\mathbf{B}\|$$

This implies that the dot product of a vector **A** by itself is

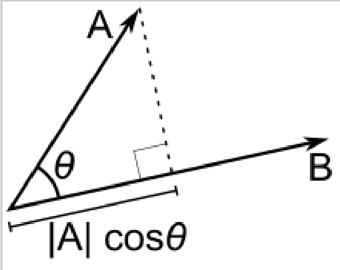
$$\mathbf{A}\cdot\mathbf{A}=\|\mathbf{A}\|^2,$$

which gives

$$\|\mathbf{A}\|=\sqrt{\mathbf{A}\cdot\mathbf{A}},$$

the formula for the Euclidean length of the vector.

Scalar projection and the equivalence of the definitions [edit source]



Scalar projection

The scalar projection (or scalar component) of a Euclidean vector **A** in the direction of a Euclidean vector **B** is given by

$$A_B=\|\mathbf{A}\|\cos\theta$$

where θ is the angle between **A** and **B**.

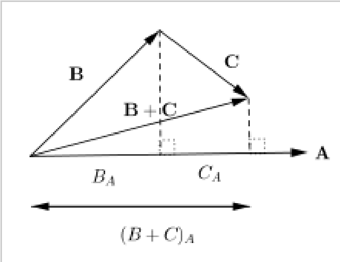
In terms of the geometric definition of the dot product, this can be rewritten

$$A_B=\mathbf{A}\cdot\widehat{\mathbf{B}}$$

where $\widehat{\mathbf{B}}=\mathbf{B}/\|\mathbf{B}\|$ is the unit vector in the direction of **B**.

The dot product is thus characterized geometrically by[3]

$$\mathbf{A}\cdot\mathbf{B}=A_B\|\mathbf{B}\|=B_A\|\mathbf{A}\|.$$



Distributive law for the dot product

The dot product, defined in this manner, is homogeneous under scaling in each variable, meaning that for any scalar α,

$$(\alpha\mathbf{A})\cdot\mathbf{B}=\alpha(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\cdot(\alpha\mathbf{B}).$$

It also satisfies a distributive law, meaning that

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}.$$

As a consequence, if $\mathbf{e}_1,\ldots,\mathbf{e}_n$ are the standard basis vectors in $\mathbb{R}^n$, then writing

$$\mathbf{A}=[A_1,\ldots,A_n]=\sum_i A_i\mathbf{e}_i$$

$$\mathbf{B}=[B_1,\ldots,B_n]=\sum_i B_i\mathbf{e}_i$$

we have

$$\mathbf{A}\cdot\mathbf{B}=\sum_i B_i(\mathbf{A}\cdot\mathbf{e}_i)=\sum_i B_iA_i$$

which is precisely the algebraic definition of the dot product. More generally, the same identity holds with the $\mathbf{e}_i$ replaced by any orthonormal basis.

Properties [edit source]

The dot product fulfils the following properties if **a**, **b**, and **c** are real vectors and *r* is a scalar.[1][2]

1. **Commutative:**

   $$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}.$$

   which follows from the definition (*θ* is the angle between **a** and **b**):

   $$\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta=\|\mathbf{b}\|\|\mathbf{a}\|\cos\theta=\mathbf{b}\cdot\mathbf{a}$$

2. **Distributive over vector addition:**

   $$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}.$$

3. **Bilinear**:

   $$\mathbf{a}\cdot(r\mathbf{b}+\mathbf{c})=r(\mathbf{a}\cdot\mathbf{b})+(\mathbf{a}\cdot\mathbf{c}).$$

4. **Scalar multiplication:**

   $$(c_1\mathbf{a})\cdot(c_2\mathbf{b})=c_1c_2(\mathbf{a}\cdot\mathbf{b})$$

5. **Orthogonal:**

   Two non-zero vectors **a** and **b** are *orthogonal* if and only if **a** · **b** = 0.

6. **No cancellation:**

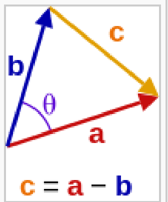
   Unlike multiplication of ordinary numbers, where if *ab* = *ac*, then *b* always equals *c* unless *a* is zero, the dot product does not obey the cancellation law:

   If **a** · **b** = **a** · **c** and **a** ≠ **0**, then we can write: **a** · (**b** − **c**) = 0 by the distributive law; the result above says this just means that **a** is perpendicular to (**b** − **c**), which still allows (**b** − **c**) ≠ **0**, and therefore **b** ≠ **c**.

7. **Derivative:** If **a** and **b** are functions, then the derivative (denoted by a prime ′) of **a** · **b** is **a′** · **b** + **a** · **b′**.

Application to the cosine law [edit source]

Main article: law of cosines



Triangle with vector edges **a** and **b**, separated by angle *θ*.

Given two vectors **a** and **b** separated by angle *θ* (see image right), they form a triangle with a third side **c** = **a** − **b**. The dot product of this with itself is:

$$\begin{aligned}\mathbf{c}\cdot\mathbf{c} &= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}\\ &= a^2-\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{b}+b^2\\ &= a^2-2\mathbf{a}\cdot\mathbf{b}+b^2\\ c^2 &= a^2+b^2-2ab\cos\theta\end{aligned}$$

which is the law of cosines.

Triple product expansion [edit source]

Main article: Triple product

This is a very useful identity (also known as **Lagrange's formula**) involving the dot- and cross-products. It is written as:[1][2]

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$

which is easier to remember as "BAC minus CAB", keeping in mind which vectors are dotted together. This formula is commonly used to simplify vector calculations in physics.

Physics [edit source]

In physics, vector magnitude is a scalar in the physical sense, i.e. a physical quantity independent of the coordinate system, expressed as the product of a numerical value and a physical unit, not just a number. The dot product is also a scalar in this sense, given by the formula, independent of the coordinate system. Examples include:[4][5]

- Mechanical work is the dot product of force and displacement vectors.
- Magnetic flux is the dot product of the magnetic field and the area vectors.

Generalizations [edit source]

Complex vectors [edit source]

For vectors with complex entries, using the given definition of the dot product would lead to quite different properties. For instance the dot product of a vector with itself would be an arbitrary complex number, and could be zero without the vector being the zero vector (such vectors are called isotropic); this in turn would have consequences for notions like length and angle. Properties such as the positive-definite norm can be salvaged at the cost of giving up the symmetric and bilinear properties of the scalar product, through the alternative definition[1]

$$\mathbf{a}\cdot\mathbf{b} = \sum a_i\overline{b_i}$$

where $\overline{b_i}$ is the complex conjugate of b_i. Then the scalar product of any vector with itself is a non-negative real number, and it is nonzero except for the zero vector. However this scalar product is thus sesquilinear rather than bilinear: it is conjugate linear and not linear in **b**, and the scalar product is not symmetric, since

$$\mathbf{a}\cdot\mathbf{b} = \overline{\mathbf{b}\cdot\mathbf{a}}.$$

The angle between two complex vectors is then given by

$$\cos\theta = \frac{\operatorname{Re}(\mathbf{a}\cdot\mathbf{b})}{\|\mathbf{a}\|\,\|\mathbf{b}\|}.$$

This type of scalar product is nevertheless useful, and leads to the notions of Hermitian form and of general inner product spaces.

Inner product [edit source]

Main article: Inner product space

The inner product generalizes the dot product to abstract vector spaces over a field of scalars, being either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. It is usually denoted by $\langle\mathbf{a}\,,\mathbf{b}\rangle$.

The inner product of two vectors over the field of complex numbers is, in general, a complex number, and is sesquilinear instead of bilinear. An inner product space is a normed vector space, and the inner product of a vector with itself is real and positive-definite.

Functions [edit source]

Vectors have a discrete number of entries, that is, an integer correspondence between natural number indices and the entries.

A function *f*(*x*) is the continuous analogue: an uncountably infinite number of entries where the correspondence is between the variable *x* and value *f*(*x*) (see domain of a function for details).

Just as the inner product on vectors uses a sum over corresponding components, the inner product on functions is defined as an integral over some interval. For example, a the inner product of two real continuous functions *u*(*x*), *v*(*x*) may be defined on the interval $a \le x \le b$ (also denoted [*a*, *b*]):[1]

$$(u, v) \equiv \langle u, v \rangle = \int_a^b u(x)v(x)dx$$

This can be generalized to complex functions ψ(x) and χ(x), by analogy with the complex inner product above:[1]

$$(\psi, \chi) \equiv \langle \psi, \chi \rangle = \int_a^b \psi(x)\overline{\chi(x)}dx.$$

Weight function [edit source]

Inner products can have a weight function, i.e. a function which weight each term of the inner product with a value.

Dyadics and matrices [edit source]

Matrices have the Frobenius inner product, which is analogous to the vector inner product. It is defined as the sum of the products of the corresponding components of two matrices **A** and **B** having the same size:

$$\mathbf{A} : \mathbf{B} = \sum_i \sum_j A_{ij}\overline{B_{ij}} = \mathrm{tr}(\mathbf{A}^*\mathbf{B}) = \mathrm{tr}(\mathbf{A}\mathbf{B}^*).$$

$$\mathbf{A} : \mathbf{B} = \sum_i \sum_j A_{ij}B_{ij} = \mathrm{tr}(\mathbf{A}^\mathrm{T}\mathbf{B}) = \mathrm{tr}(\mathbf{A}\mathbf{B}^\mathrm{T}).$$ (For real matrices)

Dyadics have a dot product and "double" dot product defined on them, see Dyadics (Product of dyadic and dyadic) for their definitions.

Tensors [edit source]

The inner product between a tensor of order n and a tensor of order m is a tensor of order $n + m - 2$, see tensor contraction for details.

See also [edit source]

- Cauchy–Schwarz inequality
- Cross product
- Matrix multiplication

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