A Continuous Petri Net Model for the Management and Design of Emergency Cardiology Departments

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Abstract: The efficient management of the Emergency Department (ED) has become an important issue in the past decade. Indeed, the increased demand for emergency services has saturated the capacity of EDs that require suitable tools for the efficient flow of work and people. The paper proposes a model to describe in a concise and effective way the structure and dynamics of a critical ED of the general hospital of Bari (Italy): the Emergency Cardiology Department (ECD). The model describes in a continuous Petri net framework the complete workflow and management of patients starting from their arrival to the ED until either their discharge from the hospital or their admission in a suitable hospital department. The fluid approximation allows defining suitable optimization problems to optimize the system performances and determine the optimal resource dimension guaranteeing the efficient management of the ECD. A simulation study shows that the optimized parameters lead to an effective workflow organization while maximizing the patient flow.

Keywords: Modeling, Continuous Petri nets, Simulation, Performance evaluation, Healthcare systems.

1. INTRODUCTION

Providing high quality healthcare calls for improved organization and management in hospital departments. In particular, the management of Emergency Departments (EDs) has become an important issue in the past decade. In such systems the main problems may be classified as follows (Xiong *et al.* 1994): i) dimensioning the system, i.e., determining the type and number of resources to provide (staff, rooms, beds, etc.); ii) understanding the workflow and detecting anomalies such as bottlenecks, waiting times, etc.; iii) improving efficiency, i.e., using resources in a better way, by decreasing patients length of stay, reacting to problems such as staff absence, etc.; iv) studying the system reactivity with respect to an increased workload.

Simulation and performance evaluation provides a useful tool for capacity planning and efficiency improvement. The ED may be effectively described as a Discrete Event System (DES) whose dynamics depends on the interaction of discrete events with a high degree of concurrency and parallelism (Gunal and Pidd 2007, Kumar and Shim 2007). Among the DES models, Petri Nets (PNs) may be employed to model emergency medical services and hospitals (Combes *et al.* 1993, Criswell *et al.* 2007, Hughes, *et al.* 1998, Xiong *et al.* 1994). Indeed, PNs are analytical and graphical tools suitable to model asynchronous and concurrent processes in numerous man-made systems, e.g. communication, computer and manufacturing systems. However, PN models typically suffer from the so called state explosion problem. One way to deal with such an issue is to employ some kind of relaxation technique, in particular applicable to some discrete event models. Since hospitals can be considered DESs whose number of reachable states is very large, PN formalisms using fluid approximations provide an aggregate formulation to deal effectively with such complex systems, while determining the optimal value of the performances (Silva and Recalde 2004).

This paper proposes a model to describe in a concise way the structure and dynamics of the ED of the general hospital of Bari (Italy), namely the Emergency Cardiology Department (ECD). The paper starting point is a PN model previously introduced in (Amodio *et al.* 2009) to describe and simulate the ECD. This former model describes in a timed PNs framework the workflow and management of patients starting from their arrival to the ED until their discharge from the hospital or admission in the chest pain unit of the ECD. In particular, places with finite capacities model the medical and nursing staff as well as resources such as the available beds and medical devices, while transitions describe the flow of patients and the examination/treatment actions. This PN model is validated in (Amodio *et al.* 2009) by the evaluation of some suitable performance indices. In order to optimize the system performance, in this paper we propose an alternative model of the ED based on the timed continuous PN framework (Silva and Recalde 2004) instead of discrete Petri nets. In contrast to discrete nets, in continuous Petri nets the state is a vector of nonnegative real numbers and the firings of transitions correspond to real valued flows from input places to output places. This feature enables the model to accomplish the following two main goals: 1) obtaining a macroscopic model of ECD; 2) optimizing the system performances determining the optimal resource dimension using such a model.

The fluid approximation allows us to analyze and simulate concisely the workflow in the considered hospital department while solving a Linear Programming (LP) problem for the determination of the optimal resources, rather than a more complex integer LP problem, necessary with a discrete PN model. Hence, the proposed model can be used as a tool for the initial design of the key element in the healthcare chain, i.e., the hospital. Indeed, the optimal resources determined by the continuous model can be considered as the initial marking of the discrete PN model, in the sense that the operational functioning of such a department can subsequently be estimated in a more comprehensive way by a discrete event simulation based on a detailed timed PN model such as the one presented in (Amodio *et al.* 2009).

The paper is organized as follows. Section 2 recalls the basics of PNs, particularly of timed continuous PNs. Section 3 describes the ED and presents its timed continuous PNs model. Section 4 describes and solves the ED design problem by means of the proposed PNs model to optimize the system performance; moreover, a discrete event simulation is performed in the MATLAB environment to verify the system behavior. A conclusion section closes the paper.

2. BASICS OF PETRI NETS

2.1 Discrete Petri Nets

A discrete PN (Peterson 1981) is bipartite graph described by the four-tuple $PN=(P, T, \mathbf{Pre}, \mathbf{Post})$, where P is a set of places with cardinality m, T is a set of transitions with cardinality n, $\boldsymbol{Pre}$: $P\times T\rightarrow\mathbb{N}^{m\times n}$ and $\boldsymbol{Post}$: $P\times T\rightarrow\mathbb{N}^{m\times n}$ are the *pre-* and *post-incidence matrices*, respectively, which specify the arcs connecting places and transitions. More precisely, for each $p\in P$ and $t\in T$ element $\mathbf{Pre}(p,t)$ ($\mathbf{Post}(p,t)$) is equal to a natural number indicating the arc multiplicity if an arc going from p to t (from t to p) exists, and it equals 0 otherwise. Note that $\mathbb{N}$ is the set of non-negative integers. The m×n incidence matrix of the net is $\mathbf{C}=\mathbf{Post}-\mathbf{Pre}$.

For the pre- and post-set we use the dot notation, e.g., $\bullet t=\{p\in P\text{: } \boldsymbol{Pre}(p,t)>0\}$.

The state of a PN is given by its current marking, which is a mapping $\mathbf{m}$: $P\rightarrow\mathbb{N}^m$, assigning to each place of the net a nonnegative number of tokens. A PN system $\langle PN,\mathbf{m}_0\rangle$ is a net PN with an initial marking $\mathbf{m}_0$.

A transition $t\in T$ is enabled at a marking $\mathbf{m}$ if and only if (iff) for each $p\in{}^{\bullet}t$, it holds $\mathbf{m}(p)\geq\mathbf{Pre}(p,t)$ and we write $\mathbf{m}[t\rangle$ to denote that $t\in T$ is enabled at marking $\mathbf{m}$. When fired, t produces a new marking $\mathbf{m'}$, denoted by $\mathbf{m}[t\rangle\mathbf{m'}$ that is computed by the PN state equation:

$$\mathbf{m'}=\mathbf{m}+\mathbf{C}\ \vec{\boldsymbol{t}}\ , \tag{1}$$

where $\vec{\boldsymbol{t}}$ is the firing vector.

Let σ be a sequence of transitions (or firing sequence). The notation $\mathbf{m}[\sigma\rangle\mathbf{m'}$ indicates that the sequence of enabled transitions σ may fire at $\mathbf{m}$ yielding $\mathbf{m'}$. We also denote $\boldsymbol{\sigma}$: $T\rightarrow\mathbb{N}^n$ the firing vector associated with a sequence σ, i.e., $\boldsymbol{\sigma}(t)=v$ if transition t is contained v times in σ. A marking $\mathbf{m}$ is said reachable from $\langle PN,\mathbf{m}_0\rangle$ iff there exists a firing sequence σ such that $\mathbf{m}_0[\sigma\rangle\mathbf{m}$.

2.2 Continuous Petri Nets

Continuous PNs (ContPNs) are a straightforward relaxation of discrete PNs (Silva and Recalde 2004). Unlike discrete PNs, the amount in which a transition can be fired in ContPNs is not restricted to a natural number. The structure of a ContPN is identical to that of a discrete PN. However, the initial marking $\mathbf{m}_0$ is a vector of non negative *real* numbers. A transition t is enabled at $\mathbf{m}$ iff $\forall p\in{}^{\bullet}t$, $\mathbf{m}(p)>0$. The enabling degree of t is:

$$\text{enab}(t,\mathbf{m})=\min_{p\in{}^{\bullet}t}\left\{\frac{\mathbf{m}(p)}{\mathbf{Pre}(p,t)}\right\}=$$

$$=\max\left\{k\in\mathbb{R}_0^+ \mid k\cdot\mathbf{Pre}(\cdot,t)\leq\mathbf{m}\right\}, \tag{2}$$

with $\mathbb{R}_0^+=\mathbb{R}^+\cup\{0\}$ and t can fire in a certain amount $\alpha\in\mathbb{R}$, with $0\leq\alpha\leq\text{enab}(t,\mathbf{m})$ leading to a new marking $\mathbf{m'}=\mathbf{m}+\alpha\cdot\mathbf{C}(\cdot,t)$, where the incidence matrix $\mathbf{C}$ is also called the *token flow matrix*. If $\mathbf{m}$ is reachable from $\mathbf{m}_0$ by the firing of a sequence σ, the fundamental equation $\mathbf{m}=\mathbf{m}_0+\mathbf{C}\cdot\boldsymbol{\sigma}$ can be written, where $\boldsymbol{\sigma}\in{\mathbb{R}_0^+}^n$ is the firing count vector associated with σ.

2.3 Timed Continuous Petri Nets

Time can be associated with places, transitions or arcs of a ContPN. In this paper we assume that time is associated with transitions and the following definitions specify and characterize timed ContPNs.

Definition 1: A timed ContPN $\langle PN,\boldsymbol{\lambda}\rangle$ is the untimed ContPN PN together with a vector $\boldsymbol{\lambda}\in{\mathbb{R}^+}^n$, where $\boldsymbol{\lambda}[t_i]=\lambda_i$ is the firing rate of transition t_i.

Definition 2: A timed ContPN system is a tuple $\Sigma = \langle PN, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$, where $\langle PN, \boldsymbol{\lambda} \rangle$ is a timed ContPN and $\mathbf{m}_0$ is the initial marking of the net.

The fundamental equation describing the timed ContPN system evolution explicitly depends on time τ and is as follows: $\mathbf{m}(\tau) = \mathbf{m}_0 + \mathbf{C} \cdot \boldsymbol{\sigma}(\tau)$. Taking the derivative of this equation with respect to time, we obtain $\dot{\mathbf{m}}(\tau) = \mathbf{C} \cdot \dot{\boldsymbol{\sigma}}(\tau)$. Using the notation $\boldsymbol{f}(\tau) = \dot{\boldsymbol{\sigma}}(\tau)$ to represent the flow of transitions as a function of time, the state equation becomes:

$$\dot{\mathbf{m}}(\tau) = \mathbf{C} \cdot \boldsymbol{f}(\tau) . \tag{3}$$

Depending on the flow definition, in the literature different semantics, i.e. different approximations of the discrete net system that the ContPN relaxes, have been defined for continuous timed transitions. In this paper we consider the so-called infinite server (or variable speed) semantics (Silva and Recalde 2004). Under this semantics, the flow f_i through a timed transition t_i is the product of its speed $\boldsymbol{\lambda}[t_i]$ and its instantaneous enabling degree, as follows:

$$f_i = \boldsymbol{f}\left[t_i\right] = \lambda_i \cdot \mathrm{enab}(t_i, \mathbf{m}) = \lambda_i \cdot \min_{p_j \in {}^\bullet t_i} \left\{ \frac{m_j}{\mathbf{Pre}(p_j, t_i)} \right\}. \tag{4}$$

where $m_j = \mathbf{m}(p_j)$ is the marking of place p_j.

2.4 Optimization of Timed Continuous Petri Nets

The use of timed ContPNs to model a DES allows us to consider off-line problems in which, given the system configuration, the objective is to optimally parameterize it. Among the problems belonging to this class are those devoted to the minimization of a cost function that may be formulated in linear terms with respect to the initial marking elements, i.e., as a weighting of the initial marking $\mathbf{b} \cdot \mathbf{m}_0$, where **b** represents a gain vector (e.g., if $\mathbf{m_0}(p_1)$ is to be minimized, $\mathbf{b}(p_1)$=1, while the rest of the weights of the gain vector should be zero). This kind of optimization problems, under some conditions depending by the structure of the ContPN described in (Silva and Recalde 2004), admits a particularly elegant and efficient solution by solving a LP problem. Indeed, it is either possible to determine the exact value of the optimal initial marking or in the worst case to obtain an upper bound of $\mathbf{m}_0$.

3. THE EMERGENCY DEPARTMENT MODEL

3.1. The System Description

Figure 1 shows the workflow organization of the case study, i.e. the ECD of the Bari general hospital (Italy). The workflow comprises two phases and the management of patients is performed according to the guidelines of the European Society of Cardiology/American College of Cardiology. In the first phase patients arrive at the ED at random time instants. In general, the ED serves various categories of patients presenting chest pain (dyspnoea, palpitations and syncope). An early anamnesis based on a physical examination and the patient ElectroCardioGram (ECG) are urgently completed in the ED. The ECG is transmitted to the ECD where it is evaluated in real time and the evaluation results are sent to the ED. If the patient is considered urgent, for instance a myocardial infarction is diagnosed, then he is hospitalized in a suitable department before entering the ECD. If this is not the case, the patient continues his visit in the ECD where he successively undergoes a complete cardiac examination. In particular, after the ECG and the blood sample evaluation, the patient undergoes a Doppler echocardiography so that the doctor can have a complete clinical picture. At this point the second phase of the workflow begins and the doctor decides whether the patient can be discharged or a follow-up of 24 hours in the Chest Pain Unit (CPU) of the ECD is necessary. During the short hospitalization of at most 24 hours the patient conditions can get worse so that he is admitted in an hospital department. Otherwise, the patient undergoes an echo-stress examination: based on the outcome, he is either admitted in a hospital department or discharged from the ECD and he can be eventually revisited in the ED.

3.2 The System Model by Timed Continuous Petri Nets

We model the ECD in the timed ContPN framework: places with finite capacities model the staff of doctors, the available analysis devices and beds. In addition, transitions describe the flow of patients and the staffing actions. The ContPN in Fig. 2 models the patient flow process steps of Fig. 1. Marking m_1 represents the number of patients entering the ED. Some patients (5%) are in a life threatening situation and are transferred to a hospital department, the others (m_5) have to wait for the visit in the ECD, represented by the timed transition t_6. The visit of each patient is performed by only one doctor: marking m_7 represents the available doctors in the ECD. Successively the patients wait for a series of tests in the ECD (marking m_6). In particular, patients undergo a Doppler echocardiography and blood examination: markings m_{15} and m_{13} describe the available echo-Doppler and blood analysis devices, respectively. Transition t_{16} models the doctors decision: the patient is discharged (94%), or admitted to a 24 hours follow-up in the ECD CPU (transition t_{18}). The failure and repair of the blood analysis and the echo-Doppler devices correspond to the firing of t_{12}, t_{13}, t_{15} and t_{14}, respectively. After a hospitalization of at most 24 hours and an eventual echo-stress examination, the patient may be admitted in a hospital department or may be discharged (transition t_{21}). During the hospitalization, the patient conditions can become serious (20%). In this case such patients are transferred to a hospital department (transition t_{22}).

4. THE SYSTEM OPTIMIZATION AND SIMULATION BY TIMED CONTINUOUS PETRI NETS

This section presents a parameterization problem aiming to optimize suitable performance indices on the basis of the defined ContPN model of the ECD. A subsequent analysis applies and verifies the obtained policy.

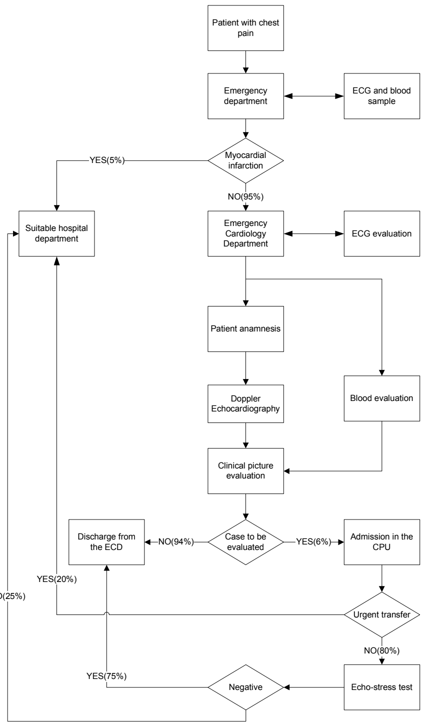

Fig. 1. The ECD workflow organization.

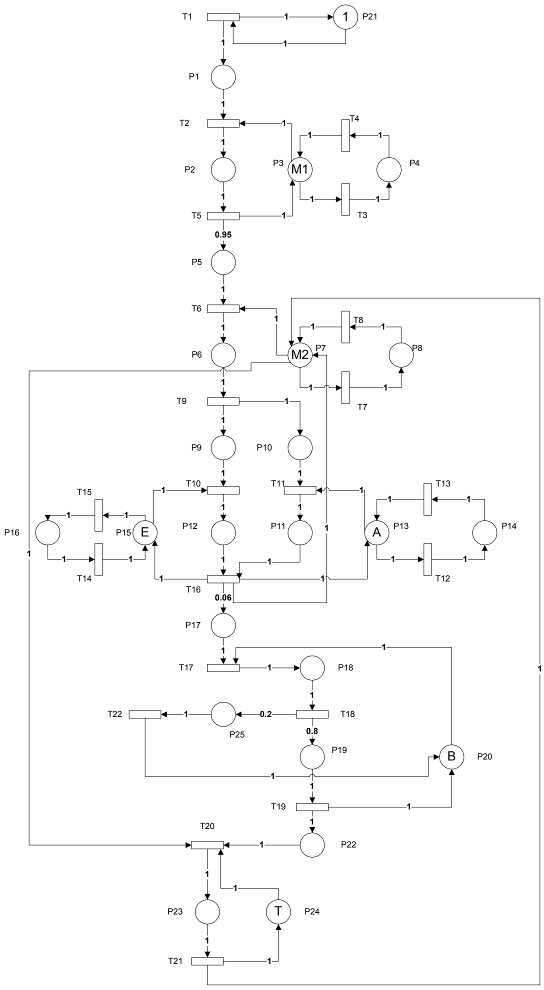

Fig. 2. The timed ContPN model of the ECD workflow.

4.1 The System Optimization

The objective of the considered design problem is establishing the minimum number of ED doctors, ECD doctors, instruments for the blood evaluation, echo-Doppler devices, beds and physical effort test instruments to obtain the maximum patient output flow from the ECD. Hence, we choose as performance index the number of discharged patients per time unit (t.u.). In the ContPN model maximizing this performance index means maximizing the flow of transition t_{21} that represents the flow of patients discharged at the last step of the ECD workflow protocol. Moreover, in order to establish the minimum number of ED doctors, ECD doctors, instruments for the blood evaluation, echo-Doppler devices, beds and physical effort test, we minimize the initial markings of the capacity places p_3, p_7, p_{13}, p_{15}, p_{20} and p_{24} respectively. The considered design problem can be solved by the following LP problem (Silva and Recalde 2002):

$$\min \mathbf{b}\cdot\boldsymbol{\mu}_0. \tag{5}$$

$$\text{s.t.} \begin{cases} \mu = \mu_0 + \mathbf{C}\cdot\boldsymbol{\sigma} \\ \phi[t] \le \lambda[t]\cdot\dfrac{\mu[p]}{\mathbf{Pre}(p,t)} \quad \forall t \in T_S,\ \forall p \in {}^{\bullet}t \\ \phi[t] = \lambda[t]\cdot\dfrac{\mu[p]}{\mathbf{Pre}(p,t)} \quad \forall t \in T_U,\ p = {}^{\bullet}t \\ \mathbf{C}\cdot\phi = 0 \\ \boldsymbol{\sigma}, \mu, \mu_0, \phi \ge 0 \\ \mu_0[p_{21}] = 1 \\ \mu_0[p_i] = 0 \text{ for } i=1,2,4,5,6,8,\ldots,12,14,16,\ldots,19,22,23 \\ \phi[t_{21}] \ge 1/\Gamma_{21} \end{cases} \tag{6}$$

where $\mathbf{b}$ is the objective function cost vector, while $\boldsymbol{\sigma}$, $\boldsymbol{\mu}$, $\boldsymbol{\mu}_0$, $\phi[t]$ are respectively the firing count vector and the approximations of marking $\mathbf{m}$, initial marking $\mathbf{m}_0$ and flow $\boldsymbol{f}[t]$. T_U is the set of transitions with one input place, while T_S is the remaining transitions set, in which synchronizations are present. Moreover, Γ_{21} is the cycle time of transition t_{21}.

The first five constraints of (6) follow from the enabling conditions, state equation and the fact ϕ has to be a T-semiflow (Silva and Recalde 2002). We add the following constraints on the initial markings: i) $\mu_0[p_{21}]=1$ to impose at each time instant that flow $\boldsymbol{f}[t_1]$ is constant and equal to λ_1; ii) the initial markings that do not represent department resources are set to zero. The last constraint of (6) imposes a lower bound on the flow of transition t_{21}. In particular, we impose that the flow of t_{21} is maximum ($\phi[t_{21}] \ge 1/\Gamma_{21}$), where $1/\Gamma_{21} = 0.456\lambda(t_1) = 0.1824$, i.e., the flow of t_{21} equals 4.56% of the total number of patients entered into the system since it is evaluated that only 4.56% of patients reaches the last step of the workflow (the physical effort test).

As previously specified, the aim of the optimization problem is choosing the minimum number of available doctors, instruments and beds in order to maximize the patient flow. However, we assume that the cost of a doctor is higher than that of a physical effort test device, which is in turn higher than that of an echo-Doppler instrument, or of a bed or of blood analysis devices. Hence, we set b_3=100, b_7=100, b_{13}=3, b_{15}=5, b_{20}=3, b_{24}=10 in vector **b** and objective function (5) is:

$$\min\begin{pmatrix}100\mu_0[p_3]+100\mu_0[p_7]+3\mu_0[p_{13}]+\\+5\mu_0[p_{15}]+3\mu_0[p_{20}]+10\mu_0[p_{24}]\end{pmatrix}. \tag{7}$$

Table 1 reports the descriptions and firing rates of the transitions in Fig. 2. Solving (7)-(6) provides the markings:

$$\begin{cases}\mu_0[p_3]=0.45,\ \mu_0[p_7]=1.88,\ \mu_0[p_{13}]=1.37\\ \mu_0[p_{15}]=1.68,\ \mu_0[p_{20}]=5.29,\ \mu_0[p_{24}]=0.12.\end{cases} \tag{8}$$

Obviously, since the initial markings m_3, m_7, m_{13}, m_{15}, m_{20} and m_{24} have to be defined by integer quantities, we set:

$$\begin{cases}\mu_0[p_3]=1 & \text{number of ED doctors}\\ \mu_0[p_7]=2 & \text{number of ECD doctors}\\ \mu_0[p_{13}]=2 & \text{number of analysis devices}\\ \mu_0[p_{15}]=2 & \text{number of echo-Doppler devices}\\ \mu_0[p_{20}]=6 & \text{number of beds}\\ \mu_0[p_{24}]=1 & \text{number of physical effort test instruments.}\end{cases} \tag{9}$$

Table 1. Transition descriptions and firing rates

Transition	Description	Firing rate λ [t.u.]$^{-1}$
t_1	Patients arrive to the ED	4.00000
t_2	Visit of the ED doctor	12.5000
t_3	An ED doctor is absent	0.00046
t_4	An ED doctor comes back to work	0.01400
t_5	A non serious patient is visited	33.34000
t_6	Visit of the doctor of ECD	33.34000
t_7	An ECD doctor is absent	0.00046
t_8	An ECD doctor comes back to work	0.01400
t_9	A non serious patient undergoes the Doppler echocardiography	33.34000
t_{10}	Doppler echocardiography	2.50000
t_{11}	Blood analysis evaluation	3.12500
t_{12}	Failure of the analysis device	0.00070
t_{13}	Repair of the analysis device	0.02000
t_{14}	Echo-Doppler facility repair	0.01400
t_{15}	Failure of an Echo-Doppler facility	0.00046
t_{16}	Diagnosis	33.34000
t_{17}	Decision about the patient hospitalization in the CPU	62.50000
t_{18}	Patient is hospitalized in the CPU	6.25000
t_{19}	End of the hospitalization	0.04000
t_{20}	Delay time for Physical effort test	1.72000
t_{21}	The patient is discharged	18.00000
t_{22}	The patient is transferred to a hospital department	0.06600

Table 2. Flows and markings at steady state

Transition	$f[t_i]$	Marking	Patient
t_1	4.0000	m_1	0.3200
t_2	4.0000	m_2	0.1200
t_3	0.0004	m_3	0.8536
t_4	0.0004	m_4	0.0264
t_5	4.0000	m_5	0.1140
t_6	3.8000	m_6	0.1140
t_7	0.0001	m_7	0.2346
t_8	0.0001	m_8	0.0073
t_9	3.8000	m_9	1.5200
t_{10}	3.8000	m_{10}	1.2160
t_{11}	3.8000	m_{11}	0.4180
t_{12}	0.0011	m_{12}	0.1140
t_{13}	0.0011	m_{13}	1.5294
t_{14}	0.0008	m_{14}	0.0527
t_{15}	0.0008	m_{15}	1.8294
t_{16}	3.8000	m_{16}	0.0567
t_{17}	0.2316	m_{17}	0.0032
t_{18}	0.2316	m_{18}	0.0370
t_{19}	0.1824	m_{19}	4.5584
t_{20}	0.1824	m_{20}	0.7137
t_{21}	0.1824	m_{21}	1
t_{22}	0.0456	m_{22}	0.1060
		m_{23}	0.0101
		m_{24}	0.9899
		m_{25}	0.6909

The obtained values are the minimum values to assign to the resources to obtain the optimal performance values.

4.2 The System Evolution

The LP problem (7)-(6) is a relaxation of the following nonlinear programming problem:

$$\min\begin{pmatrix}100\mu_0[p_3]+100\mu_0[p_7]+3\mu_0[p_{13}]+\\+5\mu_0[p_{15}]+3\mu_0[p_{20}]+10\mu_0[p_{24}]\end{pmatrix} \tag{10}$$

$$\begin{cases}\mu=\mu_0+\mathbf{C}\cdot\boldsymbol{\sigma}\\ \phi[t]=\lambda[t]\cdot\min\limits_{t\in p^\bullet}\left\{\dfrac{\mu[p]}{\mathbf{Pre}(p,t)}\right\},\ \forall t\in T\\ \mathbf{C}\cdot\boldsymbol{\phi}=0\\ \sigma,\mu,\mu_0,\phi\geq 0\\ \mu_0[p_{21}]=1\\ \mu_0[p_i]=0\\ \text{for } i=1,2,4,5,6,8,...,12,14,16,...,19,22,23\\ \phi[t_{21}]\geq 1/\Gamma_{21}.\end{cases} \tag{11}$$

Indeed, the second constraint of (11), representing the flow equation for each transition, is different from the second and third constraints in (6) that are linear due to the absence of the min operator. Hence, the LP problem (7)-(6) provides a relaxed solution of the actual objective problem (10)-(11). In order to validate the solution (8) and consequently the integer solution (9), the system dynamics is analyzed adopting the fundamental equation describing the timed ContPN system evolution defined in Section 2.3 and assuming as initial marking the solution (9). The other initial markings are set to zero. The flow and marking evolutions are determined in the MATLAB environment (The Mathworks 2006). Table 2 reports all values of transition flows and markings in the steady state that is reached by a simulation run of about 200 hours (if we associate one hour to one t.u.). Note that the flow of t_{21} reaches a value close to the corresponding imposed bound $1/\Gamma_{21}$ (see Fig. 3). Hence, the ContPN dynamics shows that the initial markings provided by the solution of the LP problem are appropriate to reach the objective performance index values. The evolution of the flows $\boldsymbol{f}[t_6]$ and $\boldsymbol{f}[t_{16}]$ (not depicted for lack of space), respectively representing the flow of patients entering the ECD and ending the clinical evaluation, both reach immediately the same steady state value, indicating that the available ECD resources are sufficient to treat all patients. Moreover, such a steady state value is the maximum possible value of 3.8 patients/hour, even if the LP problem does not constrain $\boldsymbol{f}[t_6]$ and $\boldsymbol{f}[t_{16}]$. Indeed, it is sufficient to impose that the system throughput (flow of t_{21}) is maximum to maximize the patient flow in all the system. Note that the patient flow of transitions t_1, t_2, t_5, t_6, t_9, t_{10}, t_{11}, t_{16}, t_{17}, t_{18}, t_{19}, t_{20} reach the maximum value in steady state (see Table 2). Figure 4 shows the bed occupation in the CPU, which does not overcome 6 units at the steady state. Hence, six beds are sufficient to hospitalize all patients admitted in the CPU of the ECD. The next step is to use the optimal resources determined by the presented model as the initial marking of a discrete PN model, e.g., the one introduced in (Amodio *et al.* 2009). Appropriate changes in the initial marking can be performed if the actual steady state differs. Unfortunately, due to the state explosion problem, it is not an easy task to analytically state how well the continuous model approximates the populated discrete model. Nevertheless, David and Alla (2004) showed that timed continuous PNs are a limit case of discrete PNs and in most cases the optimal resources in continuous models remain optimal in discrete PNs.

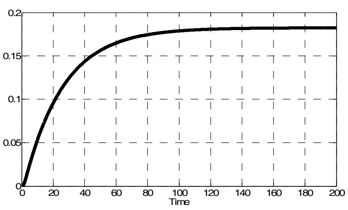

Fig. 3. Evolution of flow $\boldsymbol{f}[t_{21}]$ of discharged patients.

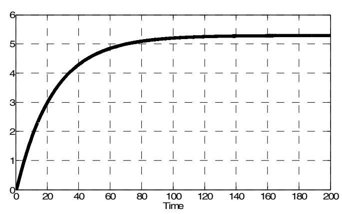

Fig. 4. Bed occupation in the CPU: marking $m_{18}+m_{19}+m_{25}$.

5. CONCLUSIONS

We propose a continuous Petri net model for analyzing and simulating the workflow of a hospital emergency department, starting from the arrival of patients to their discharge. The fluid approximation allows us to define suitable optimization problems in order to determine the optimal value of key design parameters. In particular, we consider the planning of the optimal number of beds, doctors and inspection devices to guarantee efficiency and maximize the patient flows. The proposed model can be used as a tool for the initial design of a key element in the healthcare chain such as the emergency department. The operational functioning of such a system can subsequently be estimated in a more comprehensive way by a discrete event simulation in a timed Petri Net framework.

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