JS module PYU33C01

# Assignment 4: Numerical solution of an ordinary diffenential equation

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## 1 Introduction

The aim of this assignment is to sove the equation

$$f(x,t) = \frac{dx}{dt} = (1+t)x + 1 - 3t + t^2, \tag{1.1}$$

numerically using a variety of methods. I have used mathmathica to solve for $x(t)$ analytically below,

$$x(t) = 3e^{-\frac{1}{2}t(t+2)}\left(\sqrt{2\pi}e^{\frac{t^2}{2}+\frac{1}{2}(t+2)t+t+\frac{1}{2}}\text{erf}\left(\frac{t+1}{\sqrt{2}}\right) - 0.33e^{\frac{t^2}{2}+t}t + 1.33e^{\frac{t^2}{2}+t} - 4.13e^{\frac{t^2}{2}+\frac{1}{2}(t+2)t+t}\right).$$

The sight of error functions in the result, means this integral must be evaluated numerically to determine a result.

### 1.1 Slope Fields

Slope fields are a method of visualising solutions to ordinary differential equations like equation 1.1. They are a grid of points with arrows at each grid point along the plane. The slope of these arrows is $f(x,t)$, and the angle $\theta$ with respect to the $t-x$ plane is

$$\theta = \arctan f(x,t) \tag{1.2}$$

and the $t - x$ components are

$$x_{arrow} = cos(\theta), \quad y_{arrow} = sin(\theta).$$

This is a powerful method of visualising solutions because when given initial conditions on the plane, as you can estimate the rest of the function by following the path of the arrows.

## 1.2 Euler Methods

These methods emerge from the Taylor expansion of the function in question about $t$,

$$x(t + \Delta t) = x(t) + \left.\frac{dx}{dt}\right|_t \Delta t + \frac{1}{2}\left.\frac{d^2x}{dt^2}\right|_t (\Delta t)^2 + \cdots . \tag{1.3}$$

### 1.2.1 Simple Euler method

This method is the crudest of the ones employed in this assignment. Its obtained from by dropping the terms of, and higher order than $(\Delta t)^2$. It can easily be adapted by code by specifying an initial condition and using the repeatedly using the relation

$$x_{i+1} = x_i + f(x_i, t_i)\Delta t; \quad t_i = i\Delta t, xi = x(ti)$$

### 1.2.2 Improved Euler method

The accuracy of this method is improved from the simple version (hence the name). Its based on the trapezoidal rule for integration, and defined as

$$x_{i+1} = x_i + \left(f\left(x_i, t_i\right) + f\left(x_i + f\left(x_i, t_i\right)\Delta t, t_{i+1}\right)\right)\frac{\Delta t}{2}$$

### 1.2.3 Fourth-order Runge Kutta method

This is the most accurate method of evaluation we explore today, and defined as

$$x_{i+1} = x_i + \tfrac{\Delta t}{6}\left(f\left(x'_1, t'_1\right) + 2f\left(x'_2, t'_2\right) + 2f\left(x'_3, t'_3\right) + f\left(x'_4, t'_4\right)\right)$$

where

$$\begin{array}{ll} x'_2 = x_i + \frac{1}{2}f\left(x'_1, t'_1\right)\Delta t & t'_2 = t_i + \frac{\Delta t}{2} \\ x'_3 = x_i + \frac{1}{2}f\left(x'_2, t'_2\right)\Delta t & t'_3 = t_i + \frac{\Delta t}{2} \\ x'_4 = x_i + f\left(x'_3, t'_3\right)\Delta t & t'_4 = t_i + \Delta t \end{array}$$

# 2 Investigation

## 2.1 Producing a direction field for $0 \leq t \leq 5$ and $x$ from $-3 \leq x \leq 3$

In this part I used the numpy 'meshgrid' command to plot a $25 \times 25$ grid of points. I subsequently found the slopes of the arrows using Eq.1.2, and plotted the resulting slope field using the 'quiver' command. In the process I found an alternative way to plot the field through the 'streamplot' command, which in my opinion yielded a prettier plot. I include both figures below. From these plots, I can see that the initial value of $x \approx 0.05$ is the most interesting initial point of $x$. Above this value the solutions diverge to positive infinity, and below to negative infinity.

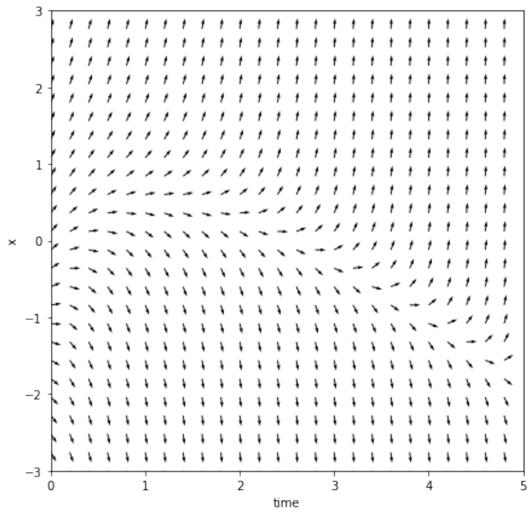


Figure 2.1: Direction field

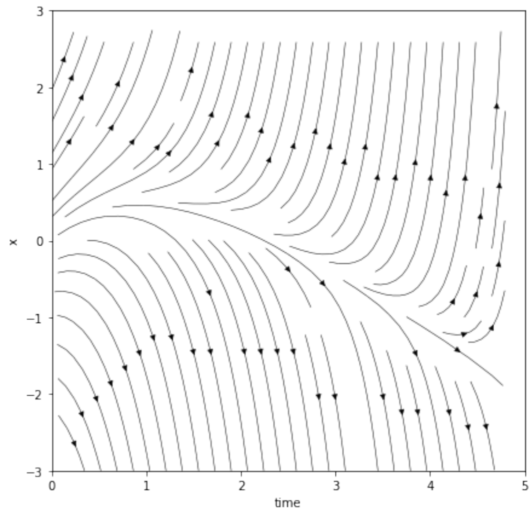


Figure 2.2: Stream plot

## 2.2 Implementation of the simple Euler method

Now with the help of Professor Hutzler's code, I implemented the simple Euler method discussed in 1.2.1. Once the method was up and running, I used step size $\Delta t = 0.04$ and applied the methodp to the point $x(t = 0) = 0.0655$, which is below the critical point $x_c = 0.065923...$ separating the solutions of $f(x, t)$ which tend to $\pm\infty$. Thus we should expect the solution to tend to $-\infty$. However on inspection of the slope field, and simple Euler method reveal in figure 2.3/4, we see that the solution tends to positive infinity. Thus the method is picking up on the wrong solution due to its inaccuracies and error with each step. We need an improved method to reveal a more accurate result.

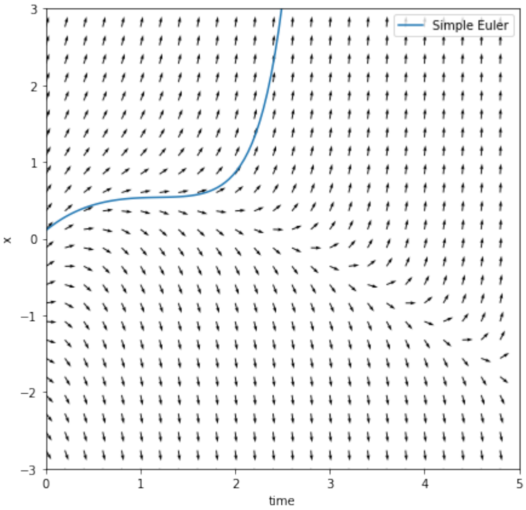


Figure 2.3: Direction field, simple Euler

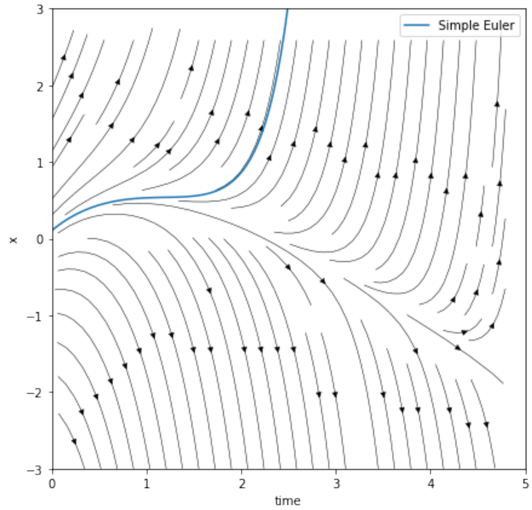


Figure 2.4: Stream plot, simple Euler

## 2.3 Implementation of improved Euler method and Runge-Kutta method

Finally, I have implemented the improved Euler and Runge-Kutta method as described in 1.2.2, and 1.2.3. Initially I have used the step size $\Delta t = 0.04$ as in the previous section and plotted the results (Fig.2.5/6) then I reduced the size to $\Delta t = 0.02$ and plotted the results (Fig.2.7/8). I have also plotted the resulting graph without field lines for a clearer view of the differences in the methods (Fig.2.9/10). Also, I included a plot of the methods at both step sizes overlaid on each other (Fig.2.11).

On analysis of the methods with step size $\Delta t = 0.04$, we can see that the Runge-Kutta method tends to $-\infty$ as expected from a solution to the function. The simple and improved Euler methods are not accurate enough at this step size to reveal a plausible solution. Now from the plots of step size $\Delta t = 0.02$, the improved Euler method has now become a more reasonable solution as it tends to $-\infty$. The Runge-Kutta remains an acceptable solution as expected. The simple Euler is still picking up on the wrong solution.

On inspecting figure 2.11, we see that as the step size increases, the simple Euler method veers to $+\infty$ at a slightly later point in time $t \approx 2.6$. The improved Euler as stated before now tends to $-\infty$, slightly later then the Runge-Kutta methods which have not changed at all as they are superimposed on each other tending to $-\infty$ at $t \approx 3.25$. This shows that decreasing the step size for this method here doesn't greatly improve the accuracy and is a waste of computational time. However for the other methods improving the step size greatly changes the result and is thus necessary. This is a negative for the Euler methods as it uses more computational time, and is more subject to rounding errors because of the necessity to use lower $\Delta t$.

From this investigation, the benefits of using improved integration schemes crystal clear. The reduction of computation time, the lack of a need for a smaller $\Delta t$, and the solutions which can be drastically more accurate.

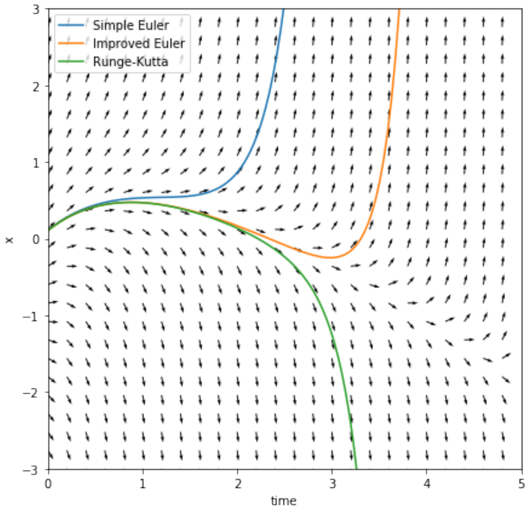


Figure 2.5: Direction field$\Delta t = 0.04$

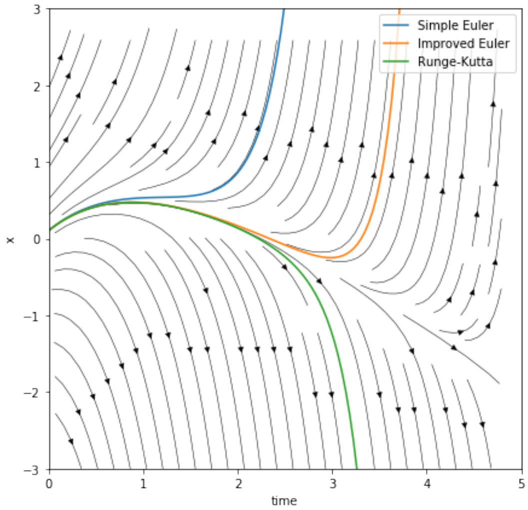


Figure 2.6: Stream plot $\Delta t = 0.04$

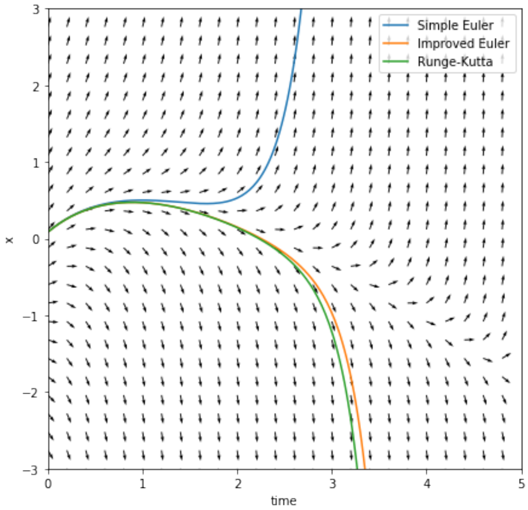


Figure 2.7: Direction field$\Delta t = 0.02$

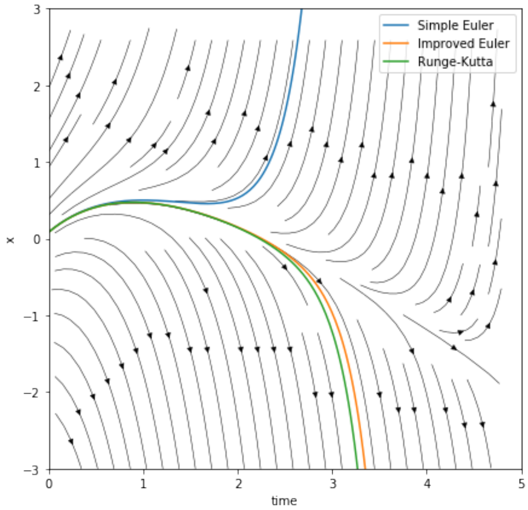


Figure 2.8: Stream plot $\Delta t = 0.02$

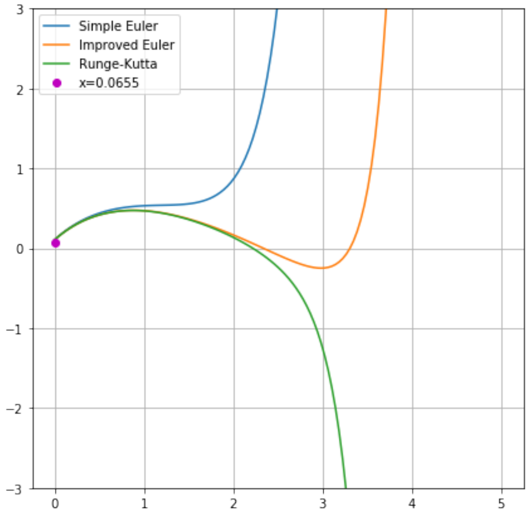


Figure 2.9: Various methods, $\Delta t = 0.04$

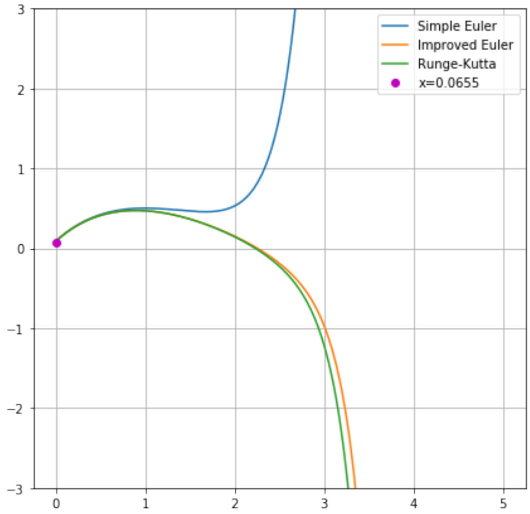


Figure 2.10: Various methods, $\Delta t = 0.02$

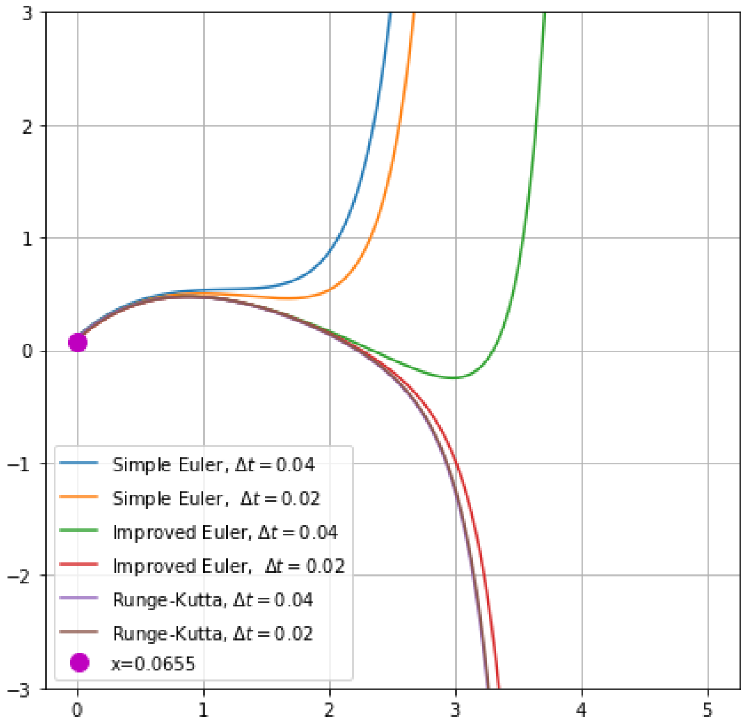


Figure 2.11: Both method step sizes overlaid