# On the Liouville Approach to Correlation Functions for 2-D Quantum Gravity\*

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### ABSTRACT

We evaluate the three point function for arbitrary states in bosonic minimal models on the sphere coupled to quantum gravity in two dimensions. The validity of the formal continuation in the number of Liouville screening charge insertions is shown directly from the Liouville functional integral using semi-classical methods.

<sup>★</sup> Work supported in part by the National Science Foundation grant PHY-89-15286.

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### 1. Introduction

The Liouville field theory approach to 2-D quantum gravity and non-critical string theory arises naturally from the geometry of the 2-D worldsheet [1]. Yet, despite the classical integrability of the theory, its quantization is still incompletely understood. Progress was made by assuming free field operator product expansions to compute the scaling dimensions of exponential operators [2,3]. The results agree with the worldsheet lightcone gauge approach [4], with matrix models [5,6] and topological field theory [7]. Still, Liouville dynamics is not completely equivalent to that of free field theory in general [2,8,9]. More recently, it was proposed to push the free field approach yet one step further, and to compute correlation functions as in free field theory but with the Liouville exponential interaction interpreted as a screening charge [10,11,12,13]. The three point function obtained in this way agrees with matrix model predictions.

In the present paper, we generalize calculations of [10,11] to the case of arbitrary external states in the Kac table of bosonic minimal models [14]. The integration over the constant Liouville mode, which yields the screening charge interpretation to the Liouville interaction, gives rise to a single screening insertion in the Liouville sector. This is to be contrasted with the approach of [12] where the presence of the two Liouville screening charges is postulated. Still, our final results agree with those of [12] and also with matrix model calculations.

The number of screening charges (s) brought down by integrating out the constant Liouville mode is not in general an integer in this approach. In [10] it was proposed to rearrange the explicit answer for integer s in a form that formally makes sense for all complex s. The justification for this procedure is incomplete. Two meromorphic functions in s that agree on all positive integers need not be the same throughout the complex plane, unless further information on their behavior at infinity provided. We shall justify the continuation procedure here by showing that the asymptotics for  $|s| \to \infty$  of the Liouville functional integral and of the explicit answer in terms of  $\Gamma$  functions in fact agree as well. To do so, we use a semi-classical approximation to the Liouville functional integral precisely valid for  $|s| \to \infty$ . This proof completes the justification of the formal continuation procedure in s.

The great advantage of the Liouville approach to 2-D quantum gravity is that whatever techniques were used successfully in the bosonic theory can be readily extended to the super-Liouville case. This is particularly important since no matrix or topological field theory formulations for 2-D quantum supergravity are presently available. The calculation of the three point function for super-minimal models is carried out in a companion paper [15].

#### 2. General Three Point Functions in Minimal Models

We apply the Liouville field theory approach to minimal conformal field theory models coupled to two dimensional quantum gravity [3]. A general N-point correlation function, evaluated on a genus p worldsheet is given by

$$\left\langle \prod_{j=1}^{N} \psi_{j} \right\rangle = \int_{\mathcal{M}_{p}} dm \int D_{\hat{g}} \phi \, e^{-S_{L}} \prod_{j=1}^{N} e^{\beta(h_{j})\phi(z_{j})} \left\langle \prod_{j=1}^{N} \psi_{r'_{j}r_{j}}^{\text{matter}}(z_{j}) \right\rangle_{\hat{g},p} \tag{2.1}$$

Here, the Liouville action is given by \*

$$S_L = \frac{1}{4\pi} \int \sqrt{\hat{g}} \left[ \frac{1}{2} \phi \Delta_{\hat{g}} \phi - \kappa R_{\hat{g}} \phi + \frac{\mu}{\alpha^2} e^{\alpha \phi} \right]$$
 (2.2)

The measure  $D_{\hat{g}}\phi$  is the translation invariant Lebesgue measure on  $\phi$ , which yields a quantization of 2-D gravity provided  $\kappa$  and  $\alpha$  are related to the matter central charge. [2,3,16,17]

$$3\kappa^2 = 25 - c \qquad \text{and} \qquad \alpha^2 + \kappa\alpha + 2 = 0 \tag{2.3}$$

The gravity-matter vertex operators  $\psi_j$  are constructed by

$$\psi_j = \psi_j^{\text{matter}} e^{\beta(h_j)\phi}$$
  $\beta(h_j) = \frac{-\sqrt{25 - c} + \sqrt{1 - c + 24h_j}}{2\sqrt{3}}$  (2.4)

The requirement that these operators be physical (i.e. dimension (1,1)) determines  $\beta(h_j)$  in the above formula [4,3]. The matter operators  $\psi_{r'_j r_j}^{\text{matter}}$  are spin zero primary fields of conformal dimension  $h_j$ . The matter correlation function is evaluated for metric  $\hat{g}$  and genus p within the matter conformal field theory [18].

We shall henceforth specialize to minimal models, not necessarily unitary. In section 3, we shall provide a proof of the validity of the formal continuation method, which is complete only for minimal models.

$$c = 1 - \frac{6}{q(q+1)} \qquad h_{r'r} = \frac{(qr' - (q+1)r)^2 - 1}{4q(q+1)}$$
 (2.5)

Here q is a rational number for general minimal models [14] and  $q = 2, 3, 4, \ldots$  for unitary

<sup>\*</sup> We have included a factor of  $1/\alpha^2$  in  $\mu$  which will turn out to be convenient in section 3.

models [19]. We also have

$$\alpha = \beta(h_{11}) = -\sqrt{\frac{2q}{(q+1)}}$$
 $\kappa = (2q+1)\sqrt{\frac{2}{q(q+1)}}$ 
(2.6)

and

$$\beta(h_{r'r}) = \frac{-(2q+1) + |qr' - (q+1)r|}{\sqrt{2q(q+1)}}$$
(2.7)

Following [10], we split the integration over  $\phi$  into a constant mode and a piece orthogonal to constants on the worldsheet

$$\int D_{\hat{g}} \phi \, e^{-S_L} \prod_{j=1}^N e^{\beta(h_j)\phi(z_j)} = \left(\frac{\mu}{4\pi\alpha^2}\right)^s \frac{\Gamma(-s)}{\alpha} \int D_{\hat{g}}' \phi \, e^{-S_L'} \left(\int \sqrt{\hat{g}} \, e^{\alpha\phi}\right)^s \prod_{j=1}^N e^{\beta(h_j)\phi(z_j)}$$
(2.8)

Here  $D'_{\hat{q}}\phi$  denotes the integration over the modes orthogonal to the constant mode and

$$S_L' = \frac{1}{4\pi} \int \sqrt{\hat{g}} \left[ \frac{1}{2} \phi \Delta_{\hat{g}} \phi - \kappa R_{\hat{g}} \phi \right]$$
 (2.9)

The variable s is the total scaling dimension

$$s = -\frac{\kappa}{\alpha}(1-p) - \sum_{j=1}^{N} \frac{\beta(h_j)}{\alpha}$$
 (2.10)

where p is again the genus of the worldsheet. In general, s is not an integer; for minimal models it is a rational number.

We shall now restrict our attention to the three point function on the sphere (p=0). To facilitate the computation, we shall concentrate the curvature of the sphere at  $z=\infty$ . In this case, the computation reduces to evaluating correlation functions of free fields on the plane with the flat metric. We may put the three fields at  $(z_1, z_2, z_3) = (0, 1, \infty)$ . Then the correlation function of the scalar field  $\phi$  reduces to

$$\int D_{\hat{g}} \phi \, e^{-S_L} \prod_{j=1}^{3} e^{\beta(h_j)\phi(z_j)}$$

$$= \left(\frac{\mu}{4\pi\alpha^2}\right)^s \frac{\Gamma(-s)}{\alpha} \int \prod_{i=1}^{s} d^2 w_i |w_i|^{-2\alpha\beta(h_1)} |1 - w_i|^{-2\alpha\beta(h_2)} \prod_{\substack{i < j \\ i,j=1}}^{s} |w_i - w_j|^{-2\alpha^2}.$$
(2.11)

As such, the above formula can only make sense when s is integer. We will need to continue in s. The general integral of this type has been calculated in [18]. Henceforth we shall be

using the notation

$$\Delta(x) \equiv \Gamma(x)/\Gamma(1-x)$$
  $S(x) \equiv \frac{\sin \pi x}{\pi}$  (2.12)

In terms of  $\Delta$ , we obtain

$$\mathcal{J}_{n'n}(a'_{1}, a'_{2}; a_{1}, a_{2}; \rho', \rho) \equiv \frac{1}{n'!n!} \int \prod_{i=1}^{n'} d^{2}w'_{i}|w'_{i}|^{2a'_{1}}|1 - w'_{i}|^{2a'_{2}} \int \prod_{i=1}^{n} d^{2}w_{i}|w_{i}|^{2a_{1}}|1 - w_{i}|^{2a_{2}} \\
\times \prod_{\substack{i < j \\ i,j=1}}^{n'} |w'_{i} - w'_{j}|^{4\rho'} \prod_{\substack{i < j \\ i,j=1}}^{n} |w_{i} - w_{j}|^{4\rho} \prod_{i=1}^{n'} \prod_{j=1}^{n} |w'_{i} - w_{j}|^{-4} \\
= \pi^{n+n'} \rho^{-4nn'} \Delta^{-n'}(\rho') \Delta^{-n}(\rho) \prod_{i=1}^{n'} \Delta(-n+i\rho') \prod_{i=1}^{n} \Delta(i\rho) \\
\times \prod_{i=0}^{n'-1} \prod_{j=1}^{3} \Delta(1-n+a'_{j}+i\rho') \prod_{i=0}^{n-1} \prod_{j=1}^{3} \Delta(1+a_{j}+i\rho) \tag{2.13}$$

Here

$$\sum a_i' = 2n - 2 - (2n' - 2)\rho' \qquad \sum a_i = 2n' - 2 - (2n - 2)\rho \qquad (2.14)$$

Using this integral formula for (n', n) = (0, s), we find

$$\int D_{\hat{g}} \phi \, e^{-S_L} \prod_{j=1}^{3} e^{\beta(h_j)\phi(z_j)} = \left(\frac{\mu}{4\alpha^2}\right)^s \frac{\Gamma(-s)\Gamma(s+1)}{\alpha} \Delta^{-s}(-\rho') \prod_{\substack{(x',x)=(0,0)\\ (r'_j,r_j)}} \prod_{i=1}^{s} \Delta(|x'\rho'-x|-i\rho')$$
(2.15)

Here, we used the notation  $\rho' \equiv \alpha^2/2$ .

The three point function on the sphere,  $\left\langle \prod_{j=1}^{3} \psi_{r'_{j}r_{j}}^{\text{matter}}(z_{j}) \right\rangle_{\hat{g}}$ , may be computed using the method of Dotsenko and Fateev [18]. An arbitrary primary field in the Kac table  $\{\psi_{r'r}^{\text{matter}}, \ 1 \leq r' \leq q+1, 1 \leq r \leq q\}$  may be expressed using a free field  $\varphi$  as  $^{\star}$ 

$$\psi_{r'r}^{\text{matter}} = e^{i\alpha_{r'r}\varphi}, \qquad \alpha_{r'r} \equiv \frac{1}{2} \left[ (1 - r')\alpha_{-} + (1 - r)\alpha_{+} \right], \qquad \alpha_{-} = -\frac{2}{\alpha_{+}} = \alpha \qquad (2.16)$$

with the action

$$S_{\varphi} = \frac{1}{4\pi} \int \sqrt{\hat{g}} \left[ \frac{1}{2} \varphi \Delta \varphi + i\alpha_0 R_{\hat{g}} \varphi + e^{i\alpha_- \varphi} + e^{i\alpha_+ \varphi} \right]$$
 (2.17)

We shall put the background charge  $2\alpha_0 \equiv \alpha_+ + \alpha_-$  at infinity. We note in passing that this representation of the primary field is two-fold degenerate, a fact we shall utilize later.

We normalized the scalar fields  $\phi$ ,  $\varphi$  according to the standard convention,  $\phi(z)\phi(z') = -\ln|z-z'|^2 + \mathcal{O}(1)$ . This is different from the normalization of [18] by a factor of  $\sqrt{2}$ .

To satisfy charge conservation including the charge at infinity, we need to put in "screening charges",  $\int d^2z \exp(i\alpha_{\pm}\varphi)$ , which have conformal dimension (0,0). Then the three point function is

$$\left\langle \prod_{j=1}^{3} \psi_{r'_{j}r_{j}}^{\text{matter}}(z_{j}) \right\rangle_{\hat{g}} = \int D_{\hat{g}} \varphi e^{-S_{\varphi}} \prod_{j=1}^{3} e^{i\alpha_{r'_{j}r_{j}}\varphi(z_{j})} \frac{1}{n'!n!} \left( \int d^{2}w' e^{i\alpha_{-}\varphi(w')} \right)^{n'} \left( \int d^{2}w e^{i\alpha_{+}\varphi(w)} \right)^{n}$$

$$= \mathcal{J}_{n'n}(\alpha_{-}\alpha_{r'_{1}r_{1}}, \alpha_{-}\alpha_{r'_{2}r_{2}}; \alpha_{+}\alpha_{r'_{1}r_{1}}, \alpha_{+}\alpha_{r'_{2}r_{2}}; \rho', \rho)$$

$$(2.18)$$

Here,

$$2n' \equiv \sum_{j=1}^{3} r'_{j} - 1, \quad 2n \equiv \sum_{j=1}^{3} r_{j} - 1, \quad \rho' = \alpha_{-}^{2}/2, \quad \rho = \alpha_{+}^{2}/2$$
 (2.19)

Using the integral formula (2.13) we obtain

$$\left\langle \prod_{j=1}^{3} \psi_{r'_{j}r_{j}}^{\text{matter}}(z_{j}) \right\rangle_{\hat{g}} = \pi^{n+n'} \rho^{-4nn'} \Delta^{-n'}(\rho') \Delta^{-n}(\rho)$$

$$\times \prod_{\substack{(x',x)=(0,0)\\(r'_{j},r_{j})}} \prod_{i=1}^{n'} \Delta(x-n+(-x'+i)\rho') \prod_{i=1}^{n} \Delta(x'+(-x+i)\rho)$$
(2.20)

This form has the advantage that it is manifestly symmetric under the interchange of three fields, unlike the formula obtained in [18]. It differs from their formula by a normalization factor associated with the three fields. However, it suffers from the ambiguity of the type 0/0 when r', r are integers. These ambiguities only exist in factors which may be absorbed in the normalization of the external fields. Furthermore, when we combine the matter part with the gravitational part, these ambiguities cancel out. We may take r', r to be integer when we have combined these factors.

Combining the gravitational part of the correlation function, (2.15), and the conformal field theory part, (2.20), the three point function of the minimal models coupled to two dimensional gravity is

$$\left\langle \prod_{j=1}^{3} \psi_{j} \right\rangle = \left( \frac{\mu}{4\alpha^{2}} \right)^{s} \pi^{n+n'} \rho^{-4nn'-2s} \alpha^{-1} \Delta^{-n'+s} (\rho') \Delta^{-n} (\rho) \Gamma(-s) \Gamma(s+1)$$

$$\times \prod_{\substack{(x',x)=(0,0)\\ (r'_{j},r_{j})}} \prod_{i=1}^{s} \Delta(|x-x'\rho'|-i\rho') \prod_{i=1}^{n'} \Delta(x-x'\rho'-n+i\rho') \prod_{i=1}^{n} \Delta(x'-x\rho+i\rho)$$
(2.21)

We shall now reduce the product over s factors, using the following basic rearrangement

formula

$$\prod_{k=0}^{n-1} \Delta(\frac{x}{n} + k\frac{m}{n}) = (\frac{m}{n})^{2x-m-n+mn} \prod_{l=0}^{m-1} \Delta(\frac{x}{m} + l\frac{n}{m})$$
 (2.22)

In a minimal model, two operators in the Kac table represent the same field. Namely,  $\psi_{r',r}^{\text{matter}}$  and  $\psi_{q+1-r',q-r}^{\text{matter}}$  represent the same field. The three point functions as computed above obey fusion rules which are *not* invariant under the reflection  $(r',r) \leftrightarrow (q+1-r',q-r)$  when applied to one of the three fields. However, the three point function is invariant when the above reflection is applied to two fields simultaneously (up to the normalization of the fields). Since  $(r'\rho'-r) \leftrightarrow -(r'\rho'-r)$  under this operation,  $r_j - r'_j \rho'$  may be assumed to be all of the same sign for the three point function without any loss in generality. We shall treat the two cases when they are all positive and all negative separately.

When  $r_j - r'_j \rho' \ge 0$  for j = 1, 2, 3, from (2.7) and (2.10), we obtain

$$\rho' = \frac{n}{n' + s + 1} \tag{2.23}$$

Using the basic rearrangement formula (2.22) the following formula holds in this case:

$$\prod_{i=1}^{s} \Delta(y - i\rho') = \rho^{(n+1-2y)(n'+s+1)-n} \Delta^{-1}(y) \prod_{i=1}^{n} \Delta^{-1}((-y+i)\rho) \prod_{i=1}^{n'} \Delta^{-1}(y - n + i\rho') \quad (2.24)$$

Using this formula for y = 0,  $r_j - r'_j \rho'$  and from (2.21), we obtain the formula for the three point function as

$$\left\langle \prod_{j=1}^{3} \psi_{j} \right\rangle = \left( \frac{\mu}{4\alpha^{2}} \right)^{s} \pi^{n+n'} \rho^{2n'-2n+2} \alpha^{-1} \Delta^{-n'+s} (\rho') \Delta^{-n} (\rho) \prod_{j=1}^{3} \Delta^{-1} (r_{j} - r'_{j} \rho')$$
(2.25)

We have set a factor of  $\Gamma(-s)\Gamma(s+1)S(s)$  which is of the form 0/0 when s is integer to be one. This formal procedure is justified by establishing the identity for asymptotic values of s in the semi-classical analysis of the next section. We rescale the external fields as

$$\psi_j \mapsto \pi^{(r_j + r'_j)/2} (4\alpha^2)^{(r'_j - r_j\rho)/2} \rho^{(r'_j - r_j)/2} \Delta^{r_j\rho/2 - r'_j} (\rho') \Delta^{-r_j/2} (\rho) \Delta^{-1} (r_j - r'_j \rho') \psi_j \qquad (2.26)$$

This rescaling is not singular. To see this, note that neither  $\rho$  nor  $\rho'$  are integer valued so that the first two factors of  $\Delta$  are neither zero nor infinity. Next,  $r_j - r'_j \rho'$  is never integer when  $r_j$  and  $r'_j$  belong to the Kac table of a minimal (p, p') model with  $\rho = p/p'$  where p, p' are relatively prime. This follows from the fact that  $1 \le r_j \le p' - 1$  and  $1 \le r'_j \le p - 1$ .

Similarly,  $r'_j - r_j \rho$  is never integer. We furthermore rescale by factors that are independent of the external indices  $(r'_j, r_j)$ , the three point function reduces just to

$$\left\langle \prod_{j=1}^{3} \psi_j \right\rangle = \mu^s \tag{2.27}$$

Similarly, when  $r_j - r'_j \rho' \ge 0$ , we obtain

$$\rho' = \frac{n+1}{n'-s} \tag{2.28}$$

The following formula can be shown to hold in this case using (2.22) again

$$\prod_{i=1}^{s} \Delta(-y - i\rho') = \rho^{(n-2y)(n'-s)-n-1} \Delta^{-1}(-y\rho) \prod_{i=1}^{n} \Delta^{-1}((-y+i)\rho) \prod_{i=1}^{n'} \Delta^{-1}(y - n + i\rho')$$
(2.29)

Using this formula for  $y = 0, r_j - r'_j \rho'$ , the three point function reduces to

$$\left\langle \prod_{j=1}^{3} \psi_{j} \right\rangle = \left( \frac{\mu}{4\alpha^{2}} \right)^{s} \pi^{n+n'} \rho^{2n'-2n-2} \alpha^{-1} \Delta^{-n'+s} (\rho') \Delta^{-n} (\rho) \prod_{j=1}^{3} \Delta^{-1} (r'_{j} - r_{j} \rho)$$
 (2.30)

The same rescaling as before with  $\Delta(r_j - r'_j \rho')$  replaced by  $\Delta(r'_j - r_j \rho)$  again reduces the correlation function to  $\mu^s$  of (2.27).

Differentiating with respect to  $\mu$  is equivalent to bringing down the area operator:

$$\frac{\partial}{\partial \mu} \left\langle \prod_{j=1}^{N} \psi_j \right\rangle = \left\langle \mathbf{1} \prod_{j=1}^{N} \psi_j \right\rangle \tag{2.31}$$

The three point function which is independent of the normalization of the fields may be computed using this relation as

$$\frac{\langle \psi_1 \psi_2 \psi_3 \rangle^2 Z}{\langle \psi_1 \psi_1 \rangle \langle \psi_2 \psi_2 \rangle \langle \psi_3 \psi_3 \rangle} = \frac{\prod_{j=1}^3 \left| qr'_j - (q+1)r_j \right|}{(q+1)(2q+1)}$$
(2.32)

where Z is the partition function of the model on the sphere. This agrees with the three point function computed in the hermitian matrix model [6].

## 3. Formal Continuation in the Number of Liouville Screening Charges

Does the formal continuation procedure in the number of Liouville screening operators s make sense? In the derivation of [10], and again in the preceding section, an expression defined only for integer s like formula (2.15), was manipulated in such a way as to obtain an expression that would make sense for all complex values of s, such as in formula (2.24). This type of manipulation may not in general be permitted, as it is inherently ambiguous up to periodic functions.

In this section, we shall compare the final formula gotten by formal continuation directly with the Liouville functional integral. We know already that these two expressions agree on all positive integer values of s. Furthermore, we clearly have a meromorphic function of s. Thus, in order to show that the formal continued formula indeed yields the answer for the functional integral for all s, it is necessary to verify that the asymptotic behavior as  $|s| \to \infty$  matches.

A meromorphic function that vanishes on all positive integers, and also at  $\infty$  must be identically zero. Using an analogous result, the agreement on the integers and the asymptotics, we will have shown the validity of the final answer, and thus of the analytic continuation procedure.

The significance of the large |s| limit is that of a weak coupling expansion. To see this, consider (2.10) expressing  $\beta$  through (2.7)

$$s = \frac{1}{\alpha^2} \left\{ 2 + \alpha^2 + \sum_{j=1}^3 \left( \left| r_j - \frac{\alpha^2}{2} r_j' \right| - \frac{\alpha^2}{2} - 1 \right) \right\}, \qquad s = \frac{2n}{\alpha^2} - n' - 1 \tag{3.1}$$

where the last identity holds when  $r_j$  and  $r'_j$  are fixed and |s| is sufficiently large. Thus, the limit we shall consider corresponds to holding  $r_j$  and  $r'_j$  fixed, but letting  $\alpha \to 0$  or  $c \to -\infty$ . From a field theoretic point of view, this is the semiclassical limit with  $\alpha^2 \sim \hbar$ . The functional integral may now be recast in a form that exhibits the full  $\alpha^2$ -dependence

$$I = \int D_{\hat{g}} \phi \ e^{-\frac{1}{\alpha^2} S_o(\phi) - S_1(\phi)}$$
 (3.2)

where the  $\alpha^2$ -independent actions are given by

$$S_o(\phi) = \frac{1}{4\pi} \int \sqrt{\hat{g}} \left[ \frac{1}{2} \phi \Delta_{\hat{g}} \phi + 2R_{\hat{g}} \phi + \mu e^{\phi} + 4\pi \sum_{j=1}^3 \phi(z) (r_j - 1) \delta_{\hat{g}}(z, z_i) \right]$$
(3.3)

and

$$S_1(\phi) = \frac{1}{4\pi} \int \sqrt{\hat{g}} \ R_{\hat{g}} \phi - \frac{1}{2} \sum_{j=1}^3 \phi(z_j) (r'_j + 1). \tag{3.4}$$

The small  $\alpha^2$ -expansion proceeds by identifying the saddle point  $\phi_o$  of  $S_o$ , evaluating its

classical action, and then expanding in a perturbative series. The final form of the answer is

$$I = e^{-a_o/\alpha^2} \left(\frac{1}{\alpha^2}\right)^{\nu} \left(a_1 + a_2 \alpha^2 + a_3 \alpha^4 + \ldots\right)$$
(3.5)

Here  $a_o$  is essentially  $S_o(\phi_o)$ , a real number dependent only on the  $r_j$ 's,  $2\nu$  is the number of normalizable zero modes of the operator

$$\left. \frac{\delta^2 S_o}{\delta \phi(x) \delta \phi(y)} \right|_{\phi = \phi_o} \tag{3.6}$$

and the coefficients  $a_i$  result from the contribution with *i*-loops.

We begin by constructing the instanton solution  $\phi_o$ . It is prudent to work with the round metric on the sphere, where

$$\hat{g}_{mn} = e^{\sigma} \delta_{mn}$$
  $\sigma = -\ell n \frac{1}{2} (1 + |z|^2)^2$  (3.7)

The saddle point equation

$$\Delta_{\hat{g}}\phi_o + 2R_{\hat{g}} + \mu \ e^{\phi_o} = -4\pi \sum_{j=1}^3 (r_j - 1)\delta_{\hat{g}}(z, z_j)$$
 (3.8)

admits regular real solutions for  $\mu = -|\mu| < 0$ , which is the case we shall study:

$$\phi_o(z) = -\ln \frac{|\mu|}{2} \frac{(1+|A(z)|^2)^2}{(|A'(z)|^2 + \epsilon^2)(1+|z|^2)^2}$$
(3.9)

Here A(z) is a meromorphic function such that A'(z) has a zero of order  $(r_i - 1)$  at  $z_i$  and  $\epsilon$  is an infinitesimal positive constant. For  $\phi_o(z)$  to be regular at  $z = \infty$  (recall the only singularities occur at  $z_i$ 's), we should have

$$A(z) \sim az$$
 as  $|z| \to \infty$   $a \neq 0$  (3.10)

Hence A is the ratio of two polynomials P, Q

$$A(z) = \frac{Q(z)}{P(z)} \qquad \deg Q = \deg P + 1 \tag{3.11}$$

To determine the total number of zeros in A' requires some extra care. There are  $r_1 + r_2 + r_3 - 3 = 2n - 2$  zeros from the  $z_i$ 's, which produce the  $\delta$ -functions on the right hand side of

 $<sup>\</sup>star$  Expansions around classical solutions were studied in [20,17]

(3.8). However, recall that there is also a charge at  $z = \infty$ , due to the curvature term. This is perhaps most easily seen when considering the flat metric on the plane plus a curvature  $\delta$ -function with charge 2 as  $z = \infty$  to make the plane into a sphere. The total number of zeros of A' is then 2n, hence  $\deg Q = n + 1$ . We normalize P such that

$$P(z) = \prod_{k=1}^{n} (z - w_k)$$
 (3.12)

and of course, none of the  $w_k$ 's should equal  $z_i$ .

From the requirement that only double poles should occur in the expansion of A'(z), we get an equation determining  $w_k^{\dagger}$ 

$$\sum_{i=1}^{3} \frac{r_i - 1}{w_k - z_i} = 2 \sum_{\substack{\ell=1\\\ell \neq k}}^{n} \frac{1}{w_k - w_\ell} \qquad 1 \le k \le n.$$
 (3.13)

This equation governs 3 fixed charges at  $z_i$  with strength  $-\frac{r_i-1}{2}$  and n charges of unit strength that settle into an equilibrium configuration under the forces of 2-D electrostatics. The equation (3.13) is invariant under complex Möbius transformations.

Next, we evaluate the classical action on this saddle point  $\phi_o$ , and we find

$$S_{o}(\phi_{o}) = -4n - 2n \, \ln |\mu| + 2n \, \ln 2 - 2 \sum_{k=1}^{n} \ln \frac{|Q(w_{k})|^{2}}{|a|^{2}} + \frac{1}{2} \sum_{i,j=1}^{3} (r_{i} - 1)(r_{j} - 1) \, \ln (|z_{i} - z_{j}|^{2} + \epsilon^{2}) + 2 \sum_{i=1}^{3} (r_{i} - 1) \, \ln (1 + |z_{i}|^{2})$$

$$(3.14)$$

Notice that this expression is independent of the normalization factor |a|. As it stands, this answer is infinite as  $\epsilon \to 0$ , but of course, we should keep in mind that the self-energies of the charges  $z_i$  must be subtracted. Furthermore,  $e^{-\frac{1}{\alpha^2}S_o(\phi_o)}$  is a three point function in a conformal field theory, so its dependence on the points  $z_i$  is fixed by conformal invariance. We have

$$e^{-\frac{1}{\alpha^2}S_o(\phi_o)} = e^{-\frac{a_o}{\alpha^2}} \prod_{i < j} |z_i - z_j|^{2h_i + 2h_j - 2h_k - 2} \prod_i (g_{z\bar{z}})^{h_i}$$
(3.15)

we obtain the following final expression for  $a_o$ :

$$a_o = -4n - 2n \, \ln |\mu| + 2n \, \ln 2 - \ln |F|^2 \tag{3.16}$$

<sup>†</sup> The point at  $\infty$  disappears in this equation!

with the conformal invariant F given by

$$F = \prod_{k=1}^{n} \frac{Q(w_k)^2}{a^2} \prod_{i < j} (z_i - z_j)^{-(r_i - 1)(r_j - 1) - \frac{1}{2}(r_i^2 - 1) - \frac{1}{2}(r_j^2 - 1) + \frac{1}{2}(r_k^2 - 1)}$$
(3.17)

The quantity  $Q(w_k)$  is also easily calculated from the fact that

$$(Q'P - QP')(z) = a \prod_{i=1}^{3} (z - z_i)^{r_i - 1}$$
(3.18)

so we find

$$Q(w_k)^2 = a^2 (P'(w_k))^{-2} \prod_{i=1}^3 (w_k - z_i)^{2r_i - 2}$$
(3.19)

Using conformal invariance of F, we let  $z_1 = 1, z_2 = -1, z_3 = \infty$ .

$$F = 2^{2n^2 + 2n - 2n(r_1 + r_2)} P(1)^{2r_1 - 2} P(-1)^{2r_2 - 2} \prod_{k=1}^{n} P'(w_k)^{-2}$$
(3.20)

Next, we have to evaluate the solutions  $w_k$  of (3.13) or equivalently the polynomial P. To do so, we slightly generalize a clever trick due to Szego [21]. The Jacobi polynomials  $y = P_n^{(\alpha,\beta)}(x)$  satisfy the following differential equation

$$(1 - x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' + n(n + \alpha + \beta + 1)y = 0.$$
 (3.21)

Hence, at a zero  $w_k$  of  $P_n^{(\alpha,\beta)}$  for  $1 \leq k \leq n$  and with  $w_k \neq \pm 1$  we have

$$\frac{P_n^{''(\alpha,\beta)}(w_k)}{P_n^{'(\alpha,\beta)}(w_k)} = -\frac{1+\alpha}{w_k-1} - \frac{1+\beta}{w_k+1}$$
(3.22)

On the other hand, using the product formula for  $P_n^{(\alpha,\beta)}$  in terms of its zeros

$$\frac{P_n^{''(\alpha,\beta)}(w_k)}{P_n^{'(\alpha,\beta)}(w_k)} = 2\sum_{\ell=1\atop\ell\neq k}^n \frac{1}{w_k - w_\ell}.$$
(3.23)

For  $\alpha = -r_1$  and  $\beta = -r_2$ , (3.22)-(3.23) precisely coincides with (3.13) in which, using conformal invariance, we have moved  $z_1, z_2, z_3$  to  $1, -1, \infty$ . Thus, the solutions to (3.13) are the zeros of Jacobi polynomials  $w_k$ , and P(w) is proportional to  $P_n^{(-r_1, -r_2)}(w)$ .

There is a point that requires clarification here: the Jacobi polynomials  $P_n^{(-r_1,-r_2)}(w)$  can in general have zeros at  $\pm 1$ , when  $r_1$  and  $r_2$  are positive integers. In this case, the above reasoning would not be valid. Thus we may always take n integer, but assume that  $r_i$ 's are slightly away from their integer values. The final answer for  $a_o$  will in fact admit a limit as  $r_i$  tends to integer.

It remains to evaluate (3.20). Fortunately, this problem was also already solved by Szego [21], and the final answer is

$$\ln F = \sum_{k=1}^{n} \sum_{x=0, r_1, r_2, r_3} (x-k) \ln |x-k|^2$$
(3.24)

so that

$$a_o = -4n - 2n \ln |\mu| + 2n \ln 2 - 2\sum_{k=1}^n \sum_{x=0,r_i} (x-k) \ln |x-k|^2$$
 (3.25)

Next, we need to determine the number of zero modes of the operator (3.6), which governs the small fluctuation problem

$$\frac{1}{\alpha^2} S_o(\phi_o + \alpha \varphi) = \frac{1}{\alpha^2} S_o(\phi_o) + \frac{1}{4\pi} \int \sqrt{\hat{g}} \left[ \frac{1}{2} \varphi \Delta_{\hat{g}} \varphi - \frac{1}{2} |\mu| e^{\phi_0} \varphi^2 \right] + \mathcal{O}(\alpha)$$
 (3.26)

The small fluctuation problem has zero modes determined by the equation

$$\Delta_{\gamma}\varphi_{o} - 2\varphi_{o} = 0$$
  $\gamma_{mn} = \frac{2|A'|^{2}}{(1+|A|^{2})^{2}}\delta_{mn}$  (3.27)

The metric  $\gamma$  is a round metric on the *n*-covered sphere with curvature 1. Thus (3.27) is an eigenvalue equation for the Laplacian on the sphere (i.e.  $\vec{L}^2$ ) with eigenvalue j(j+1)=2. Hence j=1, and there are 3 normalizable zero modes. Thus  $2\nu=3$  in (3.5). The 1-loop contribution is easy to compute as well; the small fluctuation problem just yields the determinant of  $\vec{L}^2$  on the sphere, which is a number independent of  $r_i$ 's. So the only relevant term comes from evaluating  $S_1(\phi_o)$ . It is easy to see that this corresponds to letting

$$\frac{2n}{\alpha^2} \to s = \frac{2n}{\alpha^2} + 2n', \qquad \frac{1 - r_i}{\alpha} \to \beta_i = \frac{1 - r_i}{\alpha} + \frac{\alpha}{2}(1 + r_i'). \tag{3.28}$$

We shall not need the explicit expression for  $a_1$  here.

It remains to evaluate the asymptotics of the right hand side of (2.15), (2.24) and (2.29). Actually, here it is convenient to recast the formal continuation formula in a form closer to

that obtained originally in [10]

$$\int D\phi e^{-S_L} \prod_{j=1}^{3} e^{\beta(h_j)\phi(z_j)} = (-1)^{n'+1} \left(\frac{-\mu}{4\alpha^2}\right)^s \alpha^{-1} \Delta^s (1+\rho') \rho^{2(2n'n+n'-n+s-1)} 
\times \frac{\prod_{i=1}^{n'} S(i\rho') \Gamma^2 (1+n-i\rho')}{\prod_{i=1}^{n-1} S(i\rho) \prod_{i=1}^{n} \Gamma^2 (i\rho)} 
\times \prod_{j=1}^{3} \Delta^{-\frac{1}{2}} (r_j - r'_j \rho') \frac{\prod_{i \neq r'_j}^{n'} S\left((r'_j - i)\rho'\right) \prod_{i=1}^{n'} \Gamma^2 (1+n-r_j + (r'_j - i)\rho')}{\prod_{i=1}^{n} S\left((r_j - i)\rho\right) \prod_{i=1}^{n} \Gamma^2 (r'_j - (r_j - i)\rho)}$$
(3.29)

There are various potential contributions in this formula that are not of the form (3.5). In each  $\Gamma$  and S-function there are factors of the type  $s^s$  and  $e^{irs}$  with r real. It is straightforward to check that all factors form  $s^s$  cancel, as well as those of the form  $e^{irs}$ , keeping in mind that  $\mu < 0$ . With a little patience, we can evaluate the leading exponential and power asymptotics of the right hand side of (3.29).

$$I = s^{3/2} \mu^s \exp\left\{2s + \frac{2s}{n} \sum_{k=1}^{\infty} \sum_{x=0, r_1, r_2, r_3} (x-k)\ell n |x-k|\right\} \times \mathcal{O}(s^0)$$
 (3.30)

This formula precisely agrees with our semiclassical result. Since the  $\mathcal{O}(s^0)$  term agrees for integer s, it agrees for all s. We conclude that the asymptotics of both sides of equation (3.29) match.

Any zeros and poles on the right hand side of (3.29) can only occur for  $\rho$  rational, and when Re(s) > 0, only for  $\rho$  > 0. This means that we are dealing with a minimal model with  $\rho = p/p'$ , p, p' relatively prime. If  $1 \le r_j \le p' - 1$  and  $1 \le r'_j \le p - 1$ , i.e.  $(r'_j, r_j)$  is in the Kac table of the minimal model, then the right hand side of (3.29) has neither zeros nor poles. To see this, notice in the last line of (3.29) the first factor was already shown to be neither equal to zero nor infinity in a discussion after formula (2.26); whereas the second and third factors are regular by an argument similar to the one after (2.26). The factors on the first and second lines in (3.29) have zeros and poles, but these only occur for Re(s) < 0. Operators corresponding to  $(r'_j, r_j)$  outside the Kac table, may be related by OPE with known singularities to operators within the Kac table plus screening operators. For  $(r'_j, r_j)$  outside the Kac table, this accounts for the zeros and poles on the right hand side of (3.29).

On the left hand side of (3.29), we have a well–defined Euclidean functional integral for all values Re(s) > 0 provided  $(r'_j, r_j)$  belongs to the Kac table and no singularities occur as a function of coupling constant. Outside the Kac table, new ultraviolet divergences occur governed by the OPE discussed above.

To conclude the argument, we divide both sides by the asymptotic  $|s| \to \infty$  behavior of (3.30), so that both tend to a constant as  $|s| \to \infty$ . Since both sides agree on s integer, these constants are the same.

The difference of both sides is now a meromorphic function for Re(s) > 0, tends to zero as  $|s| \to \infty$  and vanishes for integer s. In addition, we have shown above that for  $(r'_j, r_j)$  inside the Kac table, both sides have no poles in Re(s) > 0 and hence are holomorphic in Re(s) > 0. For  $(r'_j, r_j)$  outside the Kac table, we argued that the singularities on both sides are identical, so that the difference is again holomorphic when Re(s) > 0. By a standard theorem of complex analysis [22], such a function vanishes identically and hence we have shown the validity of (3.29).

### 4. Summary

We have shown that the formal continuation procedure proposed in [10] in the case of the 3-point function holds in the case of minimal models, directly from semiclassical methods on the Liouville functional integral. This increases our confidence in the procedure considerably, and puts the free field, screening operator approach to Liouville theory on a footing close to equal with the Dotsenko-Fateev approach to calculating correlation functions in minimal models.

We generalized the calculation of [10,11] to the case of arbitrary external physical states in minimal models using just Liouville theory. We obtain the same answer as in [12] who uses two screening operator instead of the Liouville theory. It would be very interesting to generalize this approach to compute N-point correlation functions on the sphere and on the torus. The generalization to supergravity is in a companion paper [15].

#### Acknowledgments

One of us (E.D) acknowledges useful discussions with Jean-Loup Gervais, Mark Goulian and David Gross and the hospitality of the Aspen Center for Physics where part of this work was carried out.

### REFERENCES

- 1. A.M. Polyakov, Phys. Lett. B103 (1981) 207
- 2. T. Curtright and C. Thorn, Phys. Rev. Lett. 48 (1982) 1309
  - E. D'Hoker and R. Jackiw, *Phys. Rev.* D26 (1982) 3517
  - J.-L. Gervais and A. Neveu, Nucl. Phys. B209 (1982) 125; B224 (1983) 329
- 3. F. David and E. Guitter, Euro. Phys. Lett. 3 (1987) 1169
  - F. David, Mod. Phys. Lett. A3 (1988) 1651
  - J. Distler and H. Kawai, Nucl. Phys. B321 (1988) 509
- 4. A. M. Polyakov, Mod. Phys. Lett. A2 (1987) 893 V. Knizhnik, A. M. Polyakov and A. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819
- 5. E. Brézin and V. Kazakov, Phys. Lett. B236 (1990) 144
  - M. Douglas and S. Shenker, *Nucl. Phys.* B335 (1990) 635
  - D. J. Gross, A. A. Migdal, Phys. Rev. Lett. 64 (1990) 127
- 6. P. Di Francesco, D. Kutasov, Nucl. Phys. B342 (1990) 475
- 7. E. Witten, Nucl. Phys. B340 (1990) 280
  - J. Distler, Nucl. Phys. B342 (1990) 523
  - R. Dijkgraaf and E. Witten, Nucl. Phys. B342 (1990) 486
  - E. Verlinde and H. Verlinde, IASSNS-HEP-90/40 (1990) preprint
  - M. Fukuma, H. Kawai, R. Nakayama, Int. J. of Mod. Phys. A6 (1991) 1385
  - R. Dijkgraaf, E. Verlinde and H. Verlinde, PUPT-1184 (1990) preprint
  - J. Distler and P. Nelson, UPR-0438T, PUPT-1201 (1990) preprint
- 8. E. Braaten, T. Curtright, G. Ghandour and C. Thorn, *Ann. Phys.* (NY) 147 (1983) 365; 153 (1984)
  - J.-L. Gervais and A. Neveu, Nucl. Phys. B257 (1985) 59; B264 (1986) 557
- 9. J. Polchinski, Texas preprint UTTG-1990 (1990)
  - N. Seiberg, Rutgers preprint RU-90-29
- 10. M. Goulian, M. Li, Phys. Rev. Lett. 66 (1991) 2051
- 11. Y. Kitazawa, HUTP-91/A013 (1991) preprint
- 12. Vl. S. Dotsenko, PAR-LPTHE 91-18 (1991) preprint
- P. Di Francesco and D. Kutasov, Phys. Lett. B261 (1991) 385
   J.G. Russo, SU-ITP-895 (1991) preprint
- 14. A. Belavin, A. M. Polyakov and A. Zamolodchikov, Nucl. Phys. B241 (1984) 333
- 15. K. Aoki and E. D'Hoker, UCLA/91/TEP/33 (1991) preprint
- 16. N. Mavromatos and J. Miramontes, Mod. Phys. Lett. A4 (1989) 1849
  - E. D'Hoker and P. S. Kurzepa, Mod. Phys. Lett. A5 (1990) 1411
- 17. E. D'Hoker, Mod. Phys. Lett. A6 (1991) 745

- 18. Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. B240 (1984) 312; B251 (1985) 691
- 19. D. Friedan, Z, Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575
- 20. E. D'Hoker, D. Z. Freedman and R. Jackiw, Phys. Rev. D28 (1983) 2583
- 21. G. Szegö, "Orthogonal Polynomials" Am. Math. Soc. XXIII, 1959
- 22. R. Boas, "Entire Functions", Academic Press, 1954