```
In[ ]:= Clear[s, x, l, PHR, PWF, u, n]
        Clear[s]

        (* First we define the matrices PWF and PHR which give the
          transition probabilities of the ancestral process for a Wright-
         Fisher and a High reproduction generation respectively*)
        (* Parameter x is the same as psi in the paper,
        i.e. the proportion of the population replaced by the highly reproductive
         pair and s is the same as parameter theta, that is, s is the timescale*)

In[1]:= PWF = {{1 - 1/n, 1/(2 n), 1/(2 n)}, {1 - 1/n, 1/(2 n), 1/(2 n)}, {0, 0, 1}};
```

In[2]:= $\text{PHR} = \left\{\left\{-\frac{2-x}{2\,n} + \frac{1}{2}\left(2 - x^2\right), \frac{x^2}{4} + \frac{2-x}{4\,n}, \frac{x^2}{4} + \frac{2-x}{4\,n}\right\}, \left\{1 - \frac{1-x}{n}, \frac{1-x}{2\,n}, \frac{1-x}{2\,n}\right\}, \{0, 0, 1\}\right\};$

```
        (* We now define the mixture matrix *)
```

In[3]:= $\text{Pmix} = \left(1 - \frac{l}{n^s}\right) * \text{PWF} + \frac{l}{n^s} * \text{PHR};$

```
        (*The matrix A, which keeps track of the O(1) terms is defined as *)

In[18]:= A = Limit[Pmix, n → Infinity]
Out[18]=
{{1 if (l | x) ∈ ℝ && s > 1, 0 if (l | x) ∈ ℝ && s > 1, 0 if (l | x) ∈ ℝ && s > 1},
 {1 if (l | x) ∈ ℝ && s > 1, 0 if (l | x) ∈ ℝ && s > 0, 0 if (l | x) ∈ ℝ && s > 0}, {0, 0, 1}}

        (*By running the code you will notice that the
         exact value of A depends on the parameters l, s, x,
        for simplicity calculate A for different values of s and infer its true value*)

In[4]:= For[s = 0.1, s ≤ 1, s = s + 0.1, Print[Limit[Pmix, n → Infinity]]]
        {{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1}}
        {{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1}}
        {{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1}}
        {{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1}}
        {{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1}}
        {{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1}}
        {{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1}}
        {{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1}}
        {{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1}}
        {{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1}}
```

In[5]:=
```
A = {{1, 0, 0}, {1, 0, 0}, {0, 0, 1}};
```

In[6]:=
```
Clear[s]

(* Now similarly to the matrix A we want to define the matrix ``B'' which keeps
   track of the O(1/N) and/or O(1/N^s) terms. The exact value of B depends on
   the timescale we are working with. For 0<s≤1, the matrix B is defined as *)
```

In[7]:=
```
Bs = Limit[n^s * (Pmix - A), n → Infinity]
```

Out[7]=
```
{{-∞ if (l | x) ∈ ℝ && s > 1, ∞ if (l | x) ∈ ℝ && s > 1, ∞ if (l | x) ∈ ℝ && s > 1},
 {-∞ if (l | x) ∈ ℝ && s > 1, ∞ if (l | x) ∈ ℝ && s > 1, ∞ if (l | x) ∈ ℝ && s > 1}, {0, 0, 0}}
```

```
(*Similarly to matrix A,
it's easier to infer the values of B by computing it for different values of s*)
```

In[8]:=
```
For[s = 0.1, s ≤ 1, s = s + 0.1, Print[Limit[n^s * (Pmix - A), n → Infinity]]];
```

$\{\{-0.5\, l\, x^2, 0.25\, l\, x^2, 0.25\, l\, x^2\}, \{0., 0., 0.\}, \{0, 0, 0\}\}$

$\{\{-0.5\, l\, x^2, 0.25\, l\, x^2, 0.25\, l\, x^2\}, \{0., 0., 0.\}, \{0, 0, 0\}\}$

$\{\{-0.5\, l\, x^2, 0.25\, l\, x^2, 0.25\, l\, x^2\}, \{0., 0., 0.\}, \{0, 0, 0\}\}$

$\{\{-0.5\, l\, x^2, 0.25\, l\, x^2, 0.25\, l\, x^2\}, \{0., 0., 0.\}, \{0, 0, 0\}\}$

$\{\{-0.5\, l\, x^2, 0.25\, l\, x^2, 0.25\, l\, x^2\}, \{0., 0., 0.\}, \{0, 0, 0\}\}$

$\{\{-0.5\, l\, x^2, 0.25\, l\, x^2, 0.25\, l\, x^2\}, \{0., 0., 0.\}, \{0, 0, 0\}\}$

$\{\{-0.5\, l\, x^2, 0.25\, l\, x^2, 0.25\, l\, x^2\}, \{0., 0., 0.\}, \{0, 0, 0\}\}$

$\{\{-0.5\, l\, x^2, 0.25\, l\, x^2, 0.25\, l\, x^2\}, \{0., 0., 0.\}, \{0, 0, 0\}\}$

$\{\{-0.5\, l\, x^2, 0.25\, l\, x^2, 0.25\, l\, x^2\}, \{0., 0., 0.\}, \{0, 0, 0\}\}$

$\{\{-1. - 0.5\, l\, x^2, 0.5 + 0.25\, l\, x^2, 0.5 + 0.25\, l\, x^2\}, \{-1., 0.5, 0.5\}, \{0, 0, 0\}\}$

```
(* Define the matrix B1 for theta=1,
i.e. when the time-scale is O(N). Define similalry the matrix
   Bs for theta in (0,1) i.e. when the time-scale is O(N^theta) *)
```

In[9]:=
$$B1 = \left\{\left\{\frac{1}{2}\left(-2 - l\, x^2\right), \frac{1}{4}\left(2 + l\, x^2\right), \frac{1}{4}\left(2 + l\, x^2\right)\right\}, \left\{-1, \frac{1}{2}, \frac{1}{2}\right\}, \{0, 0, 0\}\right\};$$

In[10]:=
$$Bs = \left\{\left\{-\frac{l\, x^2}{2}, \frac{l\, x^2}{4}, \frac{l\, x^2}{4}\right\}, \{0, 0, 0\}, \{0, 0, 0\}\right\};$$

In[11]:=
```
Clear[s]

(*For the final calulcation consider appropriate matrices
 and the coresponding matrix exponentials (this is Mohle's Lemma)*)
```

In[12]:=
```
G1 = A.B1.A;
```

In[13]:=
```
Gs = A.Bs.A;
```

In[14]:=
```
final1 = A.MatrixExp[G1 * t];
```

In[15]:=
```
finals = A.MatrixExp[Gs * t];
```

```
(* The final result (i.e. the first moment) for theta=1 is*)
```

In[16]:=
```
{1, 0, 0}.final1.{{1}, {1}, {0}}
```

Out[16]=

$$\left\{e^{-\frac{1}{4} t \left(2+l x^2\right)}\right\}$$

```
(*Similarly,  for theta is (0,1) is *)
```

In[ ]:=
```
{1, 0, 0}.finals.{{1}, {1}, {0}}
```

Out[ ]=

$$\left\{e^{-\frac{1}{4} l t x^2}\right\}$$