```
In[∘]:= Clear["Global`*"]
```

For what follows, Pi stands for stochastic transition matrices, A captures the O(1) terms, B captures the O(1/n) terms.

- **Define the 10x10 stochastic matrices PiBF, for Big Families, PiWF for Wright Fisher and PiMIX for the mixed model**

$$\mathrm{PiBF} = \Big\{\Big\{(-1+x)^2\,(1+2x) + \frac{(-1+x)^2\left(-24-350x+239x^2\right)}{4n},$$

$$-2\,(-1+x)\,x^2 + \frac{16-60x-18x^2+121x^3-59x^4}{4n},\ \frac{x^4}{4}+\frac{x^2-x^4}{8n},$$

$$-\frac{1}{4}\,(-1+x)\,x^2 + \frac{4-15x-x^2+23x^3-11x^4}{8n},\ -\frac{1}{8}\,(-2+x)\,x^3 + \frac{7\,(-1+x)\,x^3}{8n},\ -\frac{1}{4}\,(-1+x)\,x^2 +$$

$$\frac{4-15x-x^2+23x^3-11x^4}{8n},\ -\frac{1}{8}\,(-2+x)\,x^3 + \frac{7\,(-1+x)\,x^3}{8n},\ \frac{x^4}{32}+\frac{x^2\left(2-3x+x^2\right)}{16n},$$

$$-\frac{(-1+x)\,x^3}{16n}+\frac{x^4}{32},\ -\frac{1}{16}\,x^2\left(-8+x^2\right) + \frac{8+754x-1794x^2+1383x^3-351x^4}{8n}\Big\},$$

$$\Big\{\frac{1}{2}-\frac{3}{n}-\frac{13x}{2n}-\frac{3x^2}{2}+\frac{22x^2}{n}+x^3-\frac{25x^3}{2n},\ \frac{1}{2}+\frac{1}{2n}-\frac{25x}{2n}+\frac{77x^2}{4n}-\frac{x^3}{2}-\frac{29x^3}{4n},$$

$$\frac{x^2}{2}-\frac{3x^2}{4n}-\frac{x^3}{4}+\frac{3x^3}{4n},\ \frac{1}{2n}-\frac{x}{n}+\frac{x^2}{8}+\frac{3x^2}{16n}-\frac{x^3}{8}+\frac{5x^3}{16n},\ \frac{1}{2n}-\frac{x}{n}+\frac{x^2}{8}+\frac{x^2}{2n},$$

$$\frac{1}{2n}-\frac{x}{n}+\frac{x^2}{8}+\frac{3x^2}{16n}-\frac{x^3}{8}+\frac{5x^3}{16n},\ \frac{1}{2n}-\frac{x}{n}+\frac{x^2}{8}+\frac{5x^2}{8n}-\frac{x^3}{8n},\ \frac{x^2}{16n}+\frac{x^3}{32}-\frac{x^3}{16n},$$

$$\frac{3x^2}{32n}+\frac{x^3}{32}-\frac{3x^3}{32n},\ \frac{1}{2n}+\frac{23x}{n}+\frac{x^2}{2}-\frac{1349x^2}{32n}-\frac{x^3}{16}+\frac{597x^3}{32n}\Big\},$$

$$\Big\{\frac{1}{4}-\frac{3}{2n}+\frac{x}{2n}-\frac{x^2}{4}+\frac{x^2}{n},\ \frac{1}{2}-\frac{1}{2n}-\frac{x}{2n}-\frac{x^2}{2}+\frac{x^2}{n},\ \frac{1}{4}+\frac{1}{4n}-\frac{x}{4n},\ \frac{1}{8n}-\frac{x}{8n},\ \frac{1}{4n}-\frac{x}{4n}+\frac{x^2}{8},$$

$$\frac{1}{8n}-\frac{x}{8n},\ \frac{1}{4n}-\frac{x}{4n}+\frac{x^2}{8},\ \frac{x^2}{32},\ \frac{1}{16n}-\frac{x}{16n}+\frac{x^2}{32},\ \frac{15}{16n}+\frac{17x}{16n}+\frac{7x^2}{16}-\frac{2x^2}{n}\Big\},$$

$$\Big\{1-\frac{6}{n}-\frac{13x}{n}-\frac{5x^2}{2}+\frac{83x^2}{2n}+\frac{3x^3}{2}-\frac{45x^3}{2n},\ \frac{4}{n}-\frac{8x}{n}+2x^2-\frac{x^2}{n}-2x^3+\frac{5x^3}{n},$$

$$\frac{x^2}{n}+\frac{x^3}{2}-\frac{x^3}{n},\ \frac{1}{2n}-\frac{x}{n}+\frac{x^2}{4}-\frac{x^3}{4}+\frac{x^3}{2n},\ 0,\ \frac{1}{2n}-\frac{x}{2n}-\frac{x^2}{4n}+\frac{x^3}{4n},$$

$$\frac{x^2}{4n}+\frac{x^3}{4}-\frac{x^3}{4n},\ \frac{x^2}{4n}-\frac{x^3}{4n},\ 0,\ \frac{1}{n}+\frac{45x}{2n}+\frac{x^2}{4}-\frac{167x^2}{4n}+\frac{73x^3}{4n}\Big\},$$

$$\Big\{0,\ 1-\frac{3}{n}-x^2+\frac{3x^2}{n},\ \frac{1}{n}-\frac{x}{n}+\frac{x^2}{2},\ 0,\ \frac{1}{2n}-\frac{x^2}{2n},\ 0,\ \frac{1}{n}-\frac{x}{n}+\frac{x^2}{4},\ 0,\ 0,\ \frac{1}{2n}+\frac{2x}{n}+\frac{x^2}{4}-\frac{5x^2}{2n}\Big\},$$

$$\Big\{1-\frac{6}{n}-\frac{13x}{n}-\frac{5x^2}{2}+\frac{83x^2}{2n}+\frac{3x^3}{2}-\frac{45x^3}{2n},\ \frac{4}{n}-\frac{8x}{n}+2x^2-\frac{x^2}{n}-2x^3+\frac{5x^3}{n},$$

$$\frac{x^2}{n}+\frac{x^3}{2}-\frac{x^3}{n},\ \frac{1}{2n}-\frac{x}{n}+\frac{x^2}{4}-\frac{x^3}{4}+\frac{x^3}{2n},\ \frac{x^2}{4n}+\frac{x^3}{4}-\frac{x^3}{4n},$$

$$\frac{1}{2n}-\frac{x}{2n}-\frac{x^2}{4n}+\frac{x^3}{4n},\ 0,\ \frac{x^2}{4n}-\frac{x^3}{4n},\ 0,\ \frac{1}{n}+\frac{45\,x}{2n}+\frac{x^2}{4}-\frac{167\,x^2}{4n}+\frac{73\,x^3}{4n}\Big\},$$
$$\Big\{0,\ 1-\frac{3}{n}-x^2+\frac{3x^2}{n},\ \frac{1}{n}-\frac{x}{n}+\frac{x^2}{2},\ 0,\ \frac{1}{n}-\frac{x}{n}+\frac{x^2}{4},\ 0,\ \frac{1}{2n}-\frac{x^2}{2n},\ 0,\ 0,\ \frac{1}{2n}+\frac{2x}{n}+\frac{x^2}{4}-\frac{5x^2}{2n}\Big\},$$
$$\Big\{1-x^2+\frac{-6+2x+4x^2}{n},\ \frac{4-4x}{n},\ x^2,\ \frac{1-x^2}{2n},\ 0,\ \frac{1-x^2}{2n},\ 0,\ 0,\ 0,\ \frac{1+2x-3x^2}{n}\Big\},$$
$$\Big\{0,\ 0,\ 1+\frac{-1+x}{n},\ 0,\ 0,\ 0,\ 0,\ 0,\ \frac{1-x}{4n},\ \frac{3-3x}{4n}\Big\},\ \{0,0,0,0,0,0,0,0,0,1\}\Big\};$$

$$\text{PiWF}=\Big\{\Big\{1-\frac{6}{n},\ \frac{4}{n},\ 0,\ \frac{1}{2n},\ 0,\ \frac{1}{2n},\ 0,\ 0,\ 0,\ \frac{1}{n}\Big\},$$
$$\Big\{\frac{1}{2}-\frac{3}{n},\ \frac{1}{2}+\frac{1}{2n},\ 0,\ \frac{1}{2n},\ \frac{1}{2n},\ \frac{1}{2n},\ \frac{1}{2n},\ 0,\ 0,\ \frac{1}{2n}\Big\},\ \Big\{\frac{1}{4}-\frac{3}{2n},\ \frac{1}{2}-\frac{1}{2n},\ \frac{1}{4}+\frac{1}{4n},$$
$$\frac{1}{8n},\ \frac{1}{4n},\ \frac{1}{8n},\ \frac{1}{4n},\ 0,\ \frac{1}{16n},\ \frac{15}{16n}\Big\},\ \Big\{1-\frac{6}{n},\ \frac{4}{n},\ 0,\ \frac{1}{2n},\ 0,\ \frac{1}{2n},\ 0,\ 0,\ 0,\ \frac{1}{n}\Big\},$$
$$\Big\{0,\ 1-\frac{3}{n},\ \frac{1}{n},\ 0,\ \frac{1}{n},\ 0,\ \frac{1}{2n},\ 0,\ 0,\ \frac{1}{2n}\Big\},\ \Big\{1-\frac{6}{n},\ \frac{4}{n},\ 0,\ \frac{1}{2n},\ 0,\ \frac{1}{2n},\ 0,\ 0,\ 0,\ \frac{1}{n}\Big\},$$
$$\Big\{0,\ 1-\frac{3}{n},\ \frac{1}{n},\ 0,\ \frac{1}{2n},\ 0,\ \frac{1}{n},\ 0,\ 0,\ \frac{1}{2n}\Big\},\ \Big\{1-\frac{6}{n},\ \frac{4}{n},\ 0,\ \frac{1}{2n},\ 0,\ \frac{1}{2n},\ 0,\ 0,\ 0,\ \frac{1}{n}\Big\},$$
$$\Big\{0,\ 0,\ 1-\frac{1}{n},\ 0,\ 0,\ 0,\ 0,\ 0,\ \frac{1}{4n},\ \frac{3}{4n}\Big\},\ \{0,0,0,0,0,0,0,0,0,1\}\Big\};$$

$$\text{PiMIX}=\left(1-\frac{l}{n^s}\right)*\text{PiWF}+\frac{l}{n^s}*\text{PiBF};$$

- **O(N) timescale**
- Now define the matrices needed for the application of Mohle’s Lemma:

```
AWF = Limit[PiWF, n → Infinity];
ABF = Limit[PiBF, n → Infinity];
Amix = AWF;
Pmix = Limit[MatrixPower[Amix, k], k → Infinity];
BWF = Limit[n * (PiWF - AWF), n → Infinity];
BBF = Limit[n * (PiBF - ABF), n → Infinity];
Bmix = Limit[n * (PiMIX - Amix), n → Infinity];
Bmix = l * (ABF - AWF) + BWF;
Gmix = Pmix.Bmix.Pmix;
```

- Apply Mohle’s Lemma in the O(N) timescale

```
In[15]:= finalmix = Pmix.MatrixExp[t * Gmix];
{{1, 0, 0, 0, 0, 0, 0, 0, 0, 0}}.finalmix.{{1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {0}}
```

Out[16]=

$$\left\{\left\{e^{\frac{1}{16}\,t\,\left(-16-8\,l\,x^2+l\,x^4\right)}\right\}\right\}$$

- **O(N^\theta) timescale**
- Define the matrices needed for the application of Mohle's Lemma:

In[17]:=
```
AWF = Limit[PiWF, n → Infinity];
ABF = Limit[PiBF, n → Infinity];
Amix = AWF;
Pmix = Limit[MatrixPower[Amix, k], k → Infinity];
Bmix = l * (ABF - AWF);
BWF = Limit[n * (PiWF - AWF), n → Infinity];
BBF = Limit[n * (PiBF - ABF), n → Infinity];
```

- Apply Mohle's Lemma in the O(N^\theta) timescale

In[24]:=
```
Gmix = Pmix.Bmix.Pmix;
finalmix = Pmix.MatrixExp[t * Gmix];
{{1, 0, 0, 0, 0, 0, 0, 0, 0, 0}}.finalmix.{{1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {0}}
```

Out[26]=

$$\left\{\left\{e^{\frac{1}{16}\, l\, t\, x^2\, \left(-8+x^2\right)}\right\}\right\}$$