```
In[]:= Clear["Global`*"]
```

Define the matrices PiBF and PiWF for the transition matrices of the ancestral process for a sample of size 2. BF stands for ``Big Families'' and WF for ``Wright-Fisher''.
Finally let Pimid the transition matrix for the mixed model.

$$\text{In[]:= PiBF} = \left\{\left\{-\frac{1-x^2}{n}+\frac{1}{2}\left(2-x^2\right), \frac{x^2}{4}+\frac{1-x^2}{2\,n}, \frac{x^2}{4}+\frac{1-x^2}{2\,n}\right\}, \left\{1-\frac{1-x}{n}, \frac{1-x}{2\,n}, \frac{1-x}{2\,n}\right\}, \{0, 0, 1\}\right\};$$

```
PiWF = {{1 - 1 / n, 1 / (2 n), 1 / (2 n)}, {1 - 1 / n, 1 / (2 n), 1 / (2 n)}, {0, 0, 1}};
```

$$\text{Pimid} = \frac{l}{n^s} * \left(1-\frac{x^2}{4}\right) * \text{PiBF} + \left(1-\frac{l}{n^s}\right) * \text{PiWF};$$

```
(*the row sum of Pmid is not 1*)

In[]:= Simplify[Total[Pimid, {2}]]
Out[]=
```

$$\left\{1-\frac{1}{4}\,l\,n^{-s}\,x^2, 1-\frac{1}{4}\,l\,n^{-s}\,x^2, 1-\frac{1}{4}\,l\,n^{-s}\,x^2\right\}$$

```
(*in order to apply Mohle's lemma later we need a stochastic matrix*)
```

$$\text{In[]:= Pimidfinal} = \frac{1}{1-\frac{l\,x^2}{4*n^s}} * \text{Pimid};$$

```
In[]:= Simplify[Total[Pimidfinal, {2}]]
Out[]=
{1, 1, 1}

(*the matrix A is defined as:
   A=Limit[Pmidfinal,n→Infinity]*)
(* for simplicity evaluate A for different values of s and infer its true value*)

In[]:= For[s = 0.1, s ≤ 1, s = s + 0.1, Print[Limit[Pimidfinal, n → Infinity]]]
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}

In[]:= Clear[n, s, t, l, x]
```

In[•]:= `A = {{1, 0, 0}, {1, 0, 0}, {0, 0, 1}};`

`(* the matrix B is defined as Bs=Limit[n^s*(Pmidfinal-A),n→Infinity]*)`

`(*Similarly to the matrix A we define B for theta in (0,1) and theta=1*)`

In[•]:= `For[s = 0.1, s ≤ 1, s = s + 0.1, Print[Limit[n^s * (Pimidfinal - A), n → Infinity]]];`

$\{\{l\,x^2\,(-0.5+0.125\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2)\},\ \{0.,\ 0.,\ 0.\},\ \{0,\ 0,\ 0.\}\}$

$\{\{l\,x^2\,(-0.5+0.125\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2)\},\ \{0.,\ 0.,\ 0.\},\ \{0,\ 0,\ 0.\}\}$

$\{\{l\,x^2\,(-0.5+0.125\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2)\},\ \{0.,\ 0.,\ 0.\},\ \{0,\ 0,\ 0.\}\}$

$\{\{l\,x^2\,(-0.5+0.125\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2)\},\ \{0.,\ 0.,\ 0.\},\ \{0,\ 0,\ 0.\}\}$

$\{\{l\,x^2\,(-0.5+0.125\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2)\},\ \{0.,\ 0.,\ 0.\},\ \{0,\ 0,\ 0.\}\}$

$\{\{l\,x^2\,(-0.5+0.125\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2)\},\ \{0.,\ 0.,\ 0.\},\ \{0,\ 0,\ 0.\}\}$

$\{\{l\,x^2\,(-0.5+0.125\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2)\},\ \{0.,\ 0.,\ 0.\},\ \{0,\ 0,\ 0.\}\}$

$\{\{l\,x^2\,(-0.5+0.125\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2)\},\ \{0.,\ 0.,\ 0.\},\ \{0,\ 0,\ 0.\}\}$

$\{\{l\,x^2\,(-0.5+0.125\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2),\ l\,x^2\,(0.25-0.0625\,x^2)\},\ \{0.,\ 0.,\ 0.\},\ \{0,\ 0,\ 0.\}\}$

$\{\{-1.-0.5\,l\,x^2+0.125\,l\,x^4,\ 0.5+0.25\,l\,x^2-0.0625\,l\,x^4,\ 0.5+0.25\,l\,x^2-0.0625\,l\,x^4\},$
$\{-1.,\ 0.5,\ 0.5\},\ \{0,\ 0,\ 0.\}\}$

In[•]:= `Clear[n, s, t, l, x]`

In[•]:= $B1 = \{\{-1+\frac{1}{8}\,l\,x^2\,(-4+x^2),\ \frac{1}{16}\,(8-l\,x^2\,(-4+x^2)),\ \frac{1}{16}\,(8-l\,x^2\,(-4+x^2))\},$
$\{-1,\ \frac{1}{2},\ \frac{1}{2}\},\ \{0,\ 0,\ 0\}\};$

In[•]:= `TableForm[B1]`

Out[•]//TableForm=

$-1+\frac{1}{8}\,l\,x^2\,(-4+x^2)$	$\frac{1}{16}\,(8-l\,x^2\,(-4+x^2))$	$\frac{1}{16}\,(8-l\,x^2\,(-4+x^2))$
-1	$\frac{1}{2}$	$\frac{1}{2}$
0	0	0

In[•]:= $Bs = \{\{\frac{1}{8}\,l\,x^2\,(-4+x^2),\ \frac{-1}{16}\,l\,x^2\,(-4+x^2),\ \frac{-1}{16}\,l\,x^2\,(-4+x^2)\},\ \{0,\ 0,\ 0\},\ \{0,\ 0,\ 0\}\};$

In[•]:= `TableForm[Bs]`

Out[•]//TableForm=

$\frac{1}{8}\,l\,x^2\,(-4+x^2)$	$-\frac{1}{16}\,l\,x^2\,(-4+x^2)$	$-\frac{1}{16}\,l\,x^2\,(-4+x^2)$
0	0	0
0	0	0

`(*define the matrices G for different values of theta*)`

In[∘]:=
```
G1 = A.B1.A;
Gs = A.Bs.A;
```

In[∘]:=
```
TableForm[G1]
```

Out[∘]//TableForm=

$-1+\frac{1}{8}\,l\,x^2\left(-4+x^2\right)+\frac{1}{16}\left(8-l\,x^2\left(-4+x^2\right)\right)$	0	$\frac{1}{16}\left(8-l\,x^2\left(-4+x^2\right)\right)$
$-1+\frac{1}{8}\,l\,x^2\left(-4+x^2\right)+\frac{1}{16}\left(8-l\,x^2\left(-4+x^2\right)\right)$	0	$\frac{1}{16}\left(8-l\,x^2\left(-4+x^2\right)\right)$
0	0	0

In[∘]:=
```
TableForm[Gs]
```

Out[∘]//TableForm=

$\frac{1}{16}\,l\,x^2\left(-4+x^2\right)$	0	$-\frac{1}{16}\,l\,x^2\left(-4+x^2\right)$
$\frac{1}{16}\,l\,x^2\left(-4+x^2\right)$	0	$-\frac{1}{16}\,l\,x^2\left(-4+x^2\right)$
0	0	0

```
(*and the matrix exponentials*)
```

In[∘]:=
```
expo1 = MatrixExp[G1 * t];
expos = MatrixExp[Gs * t];
```

```
(*so the final matrix we work with is the one
 in \eqref{E_mixed_term_lim} and they are defined as *)
```

In[∘]:=
```
final1 = A.expo1;
finals = A.expos;
```

```
(*thus for theta=1 we have that *)
```

In[∘]:=
```
{1, 0, 0}.final1.{{1}, {1}, {0}}
```

Out[∘]=

$$\left\{e^{\frac{1}{16}\,t\left(-8-4\,l\,x^2+l\,x^4\right)}\right\}$$

```
(*and for theta in (0,1) we have that *)
```

In[∘]:=
```
{1, 0, 0}.finals.{{1}, {1}, {0}}
```

Out[∘]=

$$\left\{e^{\frac{1}{16}\,l\,t\,x^2\left(-4+x^2\right)}\right\}$$

```
(*note, the above results are multiplied by a factor of exp(-l*t*x^2/4) *)
```