Elementary Probabilities

Contents

Exercise 1:

Jacques says to Paul: "if I have a one in six chance of getting a six by rolling a die, then I double my chances by rolling it twice". Is Jacques right? Why?

(Construct a probabilistic model and translate the question asked in terms of calculating the probability of an event.)

Solution

$\Omega = \{(die_1, die_2)\}$ where $die_i = \{1, 2, 3, 4, 5, 6\}, i = \{1, 2\}$; $|\Omega| = 36$

Option 1

Here is the table indicating favorable results:

die 1 \ die 2	1	2	3	4	5	6
1						X
2						X
3						X
4						X
5						X
6	X	X	X	X	X	X

The number of favorable results = 11, the number of possible results = 36.

$\frac{11}{36} < 2 \times \frac{1}{6}$. Therefore, Jacques is not completely right, even though he increases his chances of getting a six by rolling a die twice. Jacques neglected the possibility of getting a double 6.

Option 2 (complementary event)

Let $A = \{\text{having at least 1 six in the 2 rolls}\}$. Then $\bar{A} = \{\text{having no six in the 2 rolls}\}$.

$$\mathbb{P}(\bar{A}) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$$

Then:

$$\mathbb{P}(A) = 1 - \mathbb{P}(\bar{A}) = 1 - \frac{25}{36} = \frac{11}{36}$$

Option 3

$A =$ "*having a six on the* 1*st roll* $(6, \cdot)$"

$B =$ "*having a six on the* 2*nd roll* $(\cdot, 6)$"

$A \cap B = (6, 6)$

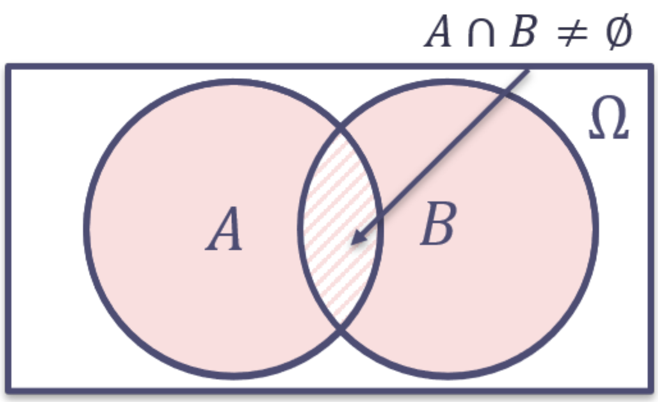

Figure 1: Venn diagram: union

$A \cup B =$ "A *or* B"

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$\mathbb{P}(A) = 1/6$

$\mathbb{P}(B) = 1/6$

$\mathbb{P}(A \cap B) = 1/6 \times 1/6 = 1/36$

Then:

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 1/6 + 1/6 - 1/36 = 6/36 + 6/36 - 1/36 = 11/36$

Exercise 2:

The Prince of Tuscany once asked Galileo: why when throwing 2 dice do we get the sum 7 more often than the sum 6, although these two sums are obtained in three different ways?

(Same indication.)

Solution

Here is the table indicating the sum of results from rolling 2 dice:

die 1 \ die 2	1	2	3	4	5	6
1	2	3	4	5	**6**	**7**
2	3	4	5	**6**	**7**	8
3	4	5	**6**	**7**	8	9
4	5	**6**	**7**	8	9	10

die 1 \ die 2	1	2	3	4	5	6
5	**6**	**7**	8	9	10	11
6	**7**	8	9	10	11	12

Note that there are 5 ways to obtain the sum 6 among 36 possible results, and 6 ways to obtain the sum 7.

Let's express the problem in terms of events:

$A =$ "*obtaining the sum* 6" $= \{(die_1, die_2) \mid die_1 + die_2 = 6\}$

$B =$ "*obtaining the sum* 7" $= \{(die_1, die_2) \mid die_1 + die_2 = 7\}$

$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{5}{36}$

$\mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{6}{36}$

$\mathbb{P}(B) > \mathbb{P}(A)$

Exercise 3:

In a group of n people, what is the probability that the birthdays of at least two of them fall on the same day? From how many people does this probability exceed 1/2? (We do not take leap years into account.)

Solution

We consider that a year is composed of 365 days. $\Omega = \{n \text{ birth dates}\}$ where each birth date can have a value $\{1, ..., 365\}$. It is an ordered sequence with repetition (because birthdays can fall on the same day), so $|\Omega| = 365^n$

$A =$ "*at least* 2 *people among* n *have their birthday on the same day*"

Having "at least", we will reason by the complementary event, because $\mathbb{P}(A) = 1 - \mathbb{P}(\bar{A})$:

$\bar{A} =$ "$the\ birthday\ of\ each\ person\ among\ n\ is\ different$".

In this case:

- the 1st person can have their birthday any day, that is 365 possibilities out of 365: 365/365
- for the 2nd person, the number of possible choices is limited to 364 to not be the same day as the 1st person, that is: 364/365
- for the 3rd person: 363/365
- for the nth person: $\frac{365-n+1}{365}$

Then:

$\mathbb{P}(\bar{A}) = \frac{365}{365} \times \frac{364}{365} \times ... \times \frac{365-n+1}{365} = \frac{365!}{(365-n)! \times 365^n}$

In other words, it is an ordered sequence without repetition $\Rightarrow$ arrangements without repetition of n among 365: $A_{365}^n = \frac{365!}{(365-n)!}$. This is the number of favorable outcomes for $\bar{A}$. The number of possible outcomes $|\Omega| = 365^n$. Then

$$\mathbb{P}(\bar{A}) = \frac{\frac{365!}{(365-n)!}}{365^n} = \frac{365!}{365^n(365-n)!}$$

Therefore:

$\mathbb{P}(A) = 1 - \mathbb{P}(\bar{A}) = 1 - \frac{365!}{(365-n)! \times 365^n}$

To find n such that $\mathbb{P}(A) > 1/2$:

$1 - \frac{365!}{(365-n)! \times 365^n} > 1/2$

$1/2 > \frac{365!}{(365-n)! \times 365^n}$

```r
for(i in 2:50){
  # calculate P(A')
  prod365 <- 365/365
  if(i-1 >= 1){
    for(j in 1:i-1){
      prod365 <- prod365*(365-j)/365
    }
  }
  #message("P(A') = ", prod365)
  pA <- 1 - prod365
  message("n = ", i, ": ", "P(A') = ", prod365, ", P(A) = ", pA)
}
```

```
## n = 2: P(A') = 0.997260273972603, P(A) = 0.00273972602739725

## n = 3: P(A') = 0.991795834115219, P(A) = 0.00820416588478146

## n = 4: P(A') = 0.98364408753345, P(A) = 0.0163559124665503

## n = 5: P(A') = 0.972864426300206, P(A) = 0.0271355736997937

## n = 6: P(A') = 0.959537516350888, P(A) = 0.0404624836491116

## n = 7: P(A') = 0.943764296904024, P(A) = 0.0562357030959756

## n = 8: P(A') = 0.925664707648331, P(A) = 0.0743352923516692

## n = 9: P(A') = 0.905376166110833, P(A) = 0.094623833889167

## n = 10: P(A') = 0.883051822288922, P(A) = 0.116948177711078

## n = 11: P(A') = 0.858858621678267, P(A) = 0.141141378321733

## n = 12: P(A') = 0.832975211161935, P(A) = 0.167024788838065

## n = 13: P(A') = 0.80558972476757, P(A) = 0.19441027523243

## n = 14: P(A') = 0.776897487995027, P(A) = 0.223102512004973

## n = 15: P(A') = 0.747098680236313, P(A) = 0.252901319763687

## n = 16: P(A') = 0.71639599474715, P(A) = 0.28360400525285

## n = 17: P(A') = 0.684992334703439, P(A) = 0.315007665296561

## n = 18: P(A') = 0.653088582128211, P(A) = 0.346911417871789

## n = 19: P(A') = 0.620881473968463, P(A) = 0.379118526031537

## n = 20: P(A') = 0.58856161641942, P(A) = 0.41143838358058

## n = 21: P(A') = 0.556311664834794, P(A) = 0.443688335165206

## n = 22: P(A') = 0.52430469233745, P(A) = 0.47569530766255

## n = 23: P(A') = 0.492702765676014, P(A) = 0.507297234323986

## n = 24: P(A') = 0.461655742085471, P(A) = 0.538344257914529

## n = 25: P(A') = 0.431300296030536, P(A) = 0.568699703969464

## n = 26: P(A') = 0.401759179864061, P(A) = 0.598240820135939
```

```
## n = 27: P(A') = 0.373140717736758, P(A) = 0.626859282263242
## n = 28: P(A') = 0.345538527657601, P(A) = 0.654461472342399
## n = 29: P(A') = 0.319031462522223, P(A) = 0.680968537477777
## n = 30: P(A') = 0.293683757280731, P(A) = 0.706316242719269
## n = 31: P(A') = 0.269545366271356, P(A) = 0.730454633728644
## n = 32: P(A') = 0.246652472149679, P(A) = 0.753347527850321
## n = 33: P(A') = 0.225028145824228, P(A) = 0.774971854175772
## n = 34: P(A') = 0.204683135379846, P(A) = 0.795316864620154
## n = 35: P(A') = 0.185616761125285, P(A) = 0.814383238874715
## n = 36: P(A') = 0.16781789362012, P(A) = 0.83218210637988
## n = 37: P(A') = 0.151265991783615, P(A) = 0.848734008216385
## n = 38: P(A') = 0.135932178917879, P(A) = 0.864067821082121
## n = 39: P(A') = 0.121780335633278, P(A) = 0.878219664366722
## n = 40: P(A') = 0.108768190182051, P(A) = 0.891231809817949
## n = 41: P(A') = 0.0968483885182646, P(A) = 0.903151611481735
## n = 42: P(A') = 0.0859695284381308, P(A) = 0.914030471561869
## n = 43: P(A') = 0.0760771443438801, P(A) = 0.92392285565612
## n = 44: P(A') = 0.0671146314485737, P(A) = 0.932885368551426
## n = 45: P(A') = 0.0590241005342251, P(A) = 0.940975899465775
## n = 46: P(A') = 0.0517471566327453, P(A) = 0.948252843367255
## n = 47: P(A') = 0.0452255971667007, P(A) = 0.954774402833299
## n = 48: P(A') = 0.0394020271205776, P(A) = 0.960597972879422
## n = 49: P(A') = 0.0342203906773235, P(A) = 0.965779609322676
## n = 50: P(A') = 0.0296264204220116, P(A) = 0.970373579577988
```

From $n = 23$.

Exercise 4:

In a pack of chocolates, there are 50 chocolates, of which 20 are fruit jellies, 20 are dark chocolate and 10 are milk chocolate. I randomly draw 4 chocolates. What is the probability of having 2 milk chocolate chocolates? On average, how many milk chocolates will I draw?

Solution

What is the probability of having 2 milk chocolate chocolates?

$$A = \text{“}2 \textit{ milk chocolate chocolates among } 4 \textit{ selected}\text{”}$$

$P(A) = \frac{\#favorable\ results}{\#possible\ results}$

The total number of possible combinations of choosing 4 chocolates from 50 is: $\binom{50}{4} = \frac{50!}{4!(50-4)!} = \frac{50!}{4!46!} = \frac{50\times49\times48\times47}{4\times3\times2\times1} = 50 \times 49 \times 2 \times 47 = 230,300$

There are 10 milk chocolate chocolates. To choose 2 of them, there are therefore $\binom{10}{2} = \frac{10!}{2!(10-2)!} = \frac{10!}{2!8!} = \frac{10\times 9}{2\times 1} = 50 \times 9 = 45$ combinations.

The other 2 chocolates must be of type(s) different from milk chocolate, that is, we choose 2 from 40: $\binom{40}{2} = \frac{40!}{2!(40-2)!} = \frac{40!}{2!38!} = \frac{40\times 39}{2\times 1} = 20 \times 39 = 780$

Then, the number of favorable results is: $\binom{10}{2} \times \binom{40}{2} = 45 \times 780 = 35,100$

Therefore:

$$P(A) = \frac{\#favorable\ results}{\#possible\ results} = \frac{35,100}{230,300} = 0.152$$

On average, how many milk chocolates will I draw? According to the statement, there are 10 milk chocolate chocolates among 50 in total. We can deduce that the proportion of milk chocolate chocolates is $\frac{10}{50} = \frac{1}{5}$.

Thus, if we draw 4 chocolates keeping the same proportion, on average we will have:

$$\frac{1}{5} \times 4 = \frac{4}{5}$$