Capital Normal University

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Perverse Sheaves and the Decomposition Theorem

Master Thesis

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1 Introduction

Perverse sheaves was introduced in the work [3] of Joseph Bernstein, Alexander Beilinson, and Pierre Deligne (1982) as a consequence of the Riemann-Hilbert correspondence, which establishes a connection between the derived categories of regular holonomic D-modules and constructible sheaves.

Specifically, if X is a smooth complex algebraic variety, the Riemann-Hilbert correspondence says

$$D^b_{reg,hol}(\mathcal{D}_X) \cong D^b_{cons}(X^{an}, \mathbb{C})$$

The left is the subcategory of the bounded category of $\mathcal{D}_X$-module, such that the cohomology is regular and holomonic. The right is the derived category of bounded constructible complex.

Now notice that the left hand side has an abelian subcategory $Mod_{reg,holo}(\mathcal{D}_X)$. It should correspond to an abelian subcategory on the right side. This is just perverse sheaves which is the core of a t-structure on a certain triangulated category.

This article is a summary of the main results in [3].

Organization. In §2 we review some preliminary on t-structures and their gluing, and introduce the important category $D^b_c(X, \overline{\mathbb{Q}_\ell})$.

In §3 we introduce the perverse t-structure to define perverse sheaves, then talk some functors relative to it. Finally give some results about the intermediate extension.

In §4 we give the definition of τ-mixed complex in $D^b_c(X, \overline{\mathbb{Q}_\ell})$ which is the extension of ordinary definition for ℓ-adic sheaves, and prove the perverse t-structure preserves mixed sheaves, namely it induces a t-structure on the subcategory $D^b_m(X, \overline{\mathbb{Q}_\ell})$ of of $D^b_c(X, \overline{\mathbb{Q}_\ell})$. Finally state the important weight filtration theorem of perverse sheaves.

In §5 we prove some corollaries by weight filtration. Finally prove the decomposition theorem in [3].

2 Triangulated category and t-structure

2.1 t-structure

Definition 2.1.1. A ***t*-structure** in a triangulated category $\mathscr{D}$ consists two strictly full subcategories $\mathscr{D}^{\le 0}$ and $\mathscr{D}^{\ge 0}$ of $\mathscr{D}$, such that with the definitions $\mathscr{D}^{\le n} = \mathscr{D}^{\le 0}[-n]$ and $\mathscr{D}^{\ge n} = \mathscr{D}^{\ge 0}[-n]$ we have

1. $Hom(\mathscr{D}^{\le 0}, \mathscr{D}^{\ge 1}) = 0$.
2. $\mathscr{D}^{\le 0} \subset \mathscr{D}^{\le 1}$ and $\mathscr{D}^{\ge 1} \subset \mathscr{D}^{\ge 0}$.
3. For every object E in $\mathscr{D}$, there exists a distinguished triangle (A, E, B) with $A \in \mathscr{D}^{\le 0}$ and $B \in \mathscr{D}^{\ge 1}$.

Set $\mathscr{D}^{\heartsuit} = \mathscr{D}^{\le 0} \cap \mathscr{D}^{\ge 0}$. This is called the core of the t-structure.

Example. $\mathscr{A}$ is an abelian category, $\mathscr{D} = D(\mathscr{A})$. Then

$$\mathscr{D}^{\leq 0} = \{K \in \mathscr{D} \mid H^i(K) = 0, \forall i > 0\}$$

$$\mathscr{D}^{\geq 0} = \{K \in \mathscr{D} \mid H^i(K) = 0, \forall i < 0\}$$

$(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ is a t-structure on $D(\mathscr{A})$.

Proof. $\mathscr{D}^{\leq 0} \subset \mathscr{D}^{\leq 1}$ and $\mathscr{D}^{\geq 1} \subset \mathscr{D}^{\geq 0}$ are trivial.

As for $Hom(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 1}) = 0$. assume $K \in \mathscr{D}^{\leq 0}$ and $L \in \mathscr{D}^{\geq 1}$. After replacing K and L by isomorphic objects in $D(\mathscr{A})$, we can assume that $K^m = 0$ for $m > 0$ and $L^m = 0$ for $m \leq 0$. Let $u : K \to L$ be a morphism in $D(\mathscr{A})$, then we have morphisms $f : K \to Z$ and $s : L \to Z$ in $K(\mathscr{A})$ such that s is a quasi-isomorphism and $u = s^{-1} \circ f$ in $D(\mathscr{A})$. Since $L^m = 0$ for $m \leq 0$, the morphism $s' : \tau^{\geq 1} L = L \to Z' = \tau^{\geq 1} Z$ is also a quasi-isomorphism. we have following commutative diagram:

$$\begin{tikzcd}
 & Z \arrow[d] & \\
K \arrow[ur, "f"] \arrow[r, "\tau^{\geq 1} f"'] \arrow[dr, "\tau^{\geq 1} f"'] & Z' & L \arrow[ul, "s"'] \arrow[l, "s'"'] \arrow[dl, "s'"] \\
 & Z' \arrow[u, "id"'] &
\end{tikzcd}$$

So $s^{-1} \circ f = s'^{-1} \circ \tau^{\geq n+1} f$ as morphism in $D(\mathscr{A})$. But $K^m = 0$ for $m \geq 1$ implies that $\tau^{\geq n+1} f = 0$, so $u = 0$.

Finally, if $K \in D(\mathscr{A})$, we have that

$$\tau^{\leq 0} K \to K \to \tau^{\geq 1} K \to \tau^{\leq 0} K[1]$$

is an exact triangle. This is from the short exact sequence of chain complexes

$$0 \to \tau^{\leq 0} K \to K \to \tau^{\geq 1} K \to 0$$

□

Proposition 2.1.2. Let $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ is a t-structure on a triangle category $\mathscr{D}$. Then

1. The inclusion $\mathscr{D}^{\leq n} \to \mathscr{D}$ has a right adjoint $\tau^{\leq n} : \mathscr{D} \to \mathscr{D}^{\leq n}$. In particular, there is a canonical map $\tau^{\leq n} K \to K$, which is universal in the sense that if there is something in $\mathscr{D}^{\leq n}$ mapping to K, then this map factors uniquely through $\tau^{\leq n} K$.

2. The inclusion $\mathscr{D}^{\geq n} \to \mathscr{D}$ has a left adjoint $\tau^{\geq n} : \mathscr{D} \to \mathscr{D}^{\geq n}$, and there is a canonical map $K \to \tau^{\geq n} K$.

3. For each object $X \in \mathscr{D}$, there is a unique morphism $\delta : \tau^{\geq n+1}K \to \tau^{\leq n}K[1]$ which makes the following sequence into an exact triangle:

$$\tau^{\leq n}K \to K \to \tau^{\geq n+1}K \xrightarrow{\delta} \tau^{\leq n}K[1]$$

In fact, δ is functorial in K.

Proof. 1. By shifting, we can assume $n = 0$. Fix $K \in \mathscr{D}$. Axiom 3 of t-structure gives the exact triangle

$$K_0 \to K \to K^1 \xrightarrow{\delta} K_0[1]$$

such that $K_0 \in \mathscr{D}^{\leq 0}$ and $K^1 \in \mathscr{D}^{\geq 1}$. Now fix some $Y \in \mathscr{D}^{\leq 0}$, then apply $Hom(Y, -)$ to get :

$$\begin{tikzcd} Hom(Y, K^1[-1]) \arrow[r] \arrow[d, phantom, "\shortparallel"] & Hom(Y, K_0) \arrow[r, "\cong"] & Hom(Y, K) \arrow[r] & Hom(Y, K^1) \arrow[d, phantom, "\shortparallel"] \\ 0 & & & 0 \end{tikzcd}$$

the left 0 is because $Hom(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 2}) = 0$ and right is just $Hom(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 1}) = 0$, so set $\tau^{\leq 0}K = K_0$.

2. Similarly: $\tau^{\geq 1}K = K^1$.

3. Reduce to $n = 0$,

$$K_0 \to K \to K^1 \xrightarrow{\delta} K_0[1]$$

we need to prove δ is unique. Because $Hom(K_0[1], K^1) = 0$, this follows by following lemma.

□

Lemma 2.1.3. Suppose $\mathscr{D}$ is a triangulated category, and for $i \in \{1, 2\}$, the sequences

$$K \xrightarrow{k} L \xrightarrow{m} M \xrightarrow{d_i} K[1]$$

are two exact triangles. Assume $Hom(K[1], M) = 0$, then $d_1 = d_2$.

Proof. By the Cone axiom (TR4), we get a commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "id"] & L \arrow[r, "l"] \arrow[d, dashed, "id"] & M \arrow[r, "d_1"] \arrow[d, dashed, "m"] & K[1] \arrow[d, "id"] \\ K \arrow[r, "k"'] & L \arrow[r, "l"'] & M \arrow[r, "d_2"'] & K[1] \end{tikzcd}$$

so $ml = l$, namely $(id_M - m)l = 0$. $id_M - m \in Hom(M, M)$, applying the long exact sequence for $Hom(-, M)$ to the first row, we have the followig exact sequence

$$Hom(K[1], M) \xrightarrow{d_1^*} Hom(M, M) \xrightarrow{l^*} Hom(L, M) \xrightarrow{k^*} Hom(K, M)$$

there exists $e : K[1] \to M$ such that $e \circ d_1 = id_M - m$, but $e = 0$ as $Hom(K[1], M) = 0$, so $id_M = m$ which implies $d_1 = d_2$.

One wants to prove that the core has nice properties, but first, we need some extra facts about the truncation functors:

Lemma 2.1.4.

1. $\tau^{\leq n}(K[m]) \cong (\tau^{\leq n+m}(K))[m]$
2. $\tau^{\geq n}(K[m]) \cong (\tau^{\geq n+m}(K))[m]$
3. $K \in \mathscr{D}^{\leq n} \Longleftrightarrow \tau^{\leq n}(K) \cong K \Longleftrightarrow \tau^{\geq n+1}(K) = 0$
4. $K \in \mathscr{D}^{\geq n} \Longleftrightarrow \tau^{\geq n}(K) \cong K \Longleftrightarrow \tau^{\leq n-1}(K) = 0$
5. For $a < b \in \mathbb{Z}$, $\tau^{\leq b} \circ \tau^{\leq a} = \tau^{\leq a} = \tau^{\leq a} \circ \tau^{\leq b}$, $\tau^{\geq b} \circ \tau^{\geq a} = \tau^{\geq b} = \tau^{\geq a} \circ \tau^{\geq b}$and $\tau^{\leq a} \circ \tau^{\geq b} = 0 = \tau^{\geq b} \circ \tau^{\leq a}$
6. For $a, b \in \mathbb{Z}$, there is a canonical isomorphism

$$\tau^{\leq a} \circ \tau^{\geq b} \cong \tau^{\geq b} \circ \tau^{\leq a}$$

Definition 2.1.5. Define $H^0 : \mathscr{D} \to \mathscr{D}^\heartsuit$, where $K \mapsto (\tau^{\leq 0} \circ \tau^{\geq 0})(K) \cong (\tau^{\geq 0} \circ \tau^{\leq 0})(K)$, and also $H^n(K) = H^0(K[n])$

Lemma 2.1.6. For all $K \in \mathscr{D}$, there is an exact triangle

$$H^n(K)[-n] \to \tau^{\geq n}(K) \to \tau^{\geq n+1}(K) \to H^n(K)[-n+1]$$

Proof.

$$H^n(K)[-n] = H^0(K[n])[-n] = (\tau^{\leq 0} \circ \tau^{\geq 0}(K[n])) = (\tau^{\leq 0}(\tau^{\geq n}K)[n]) = \tau^{\leq n}(\tau^{\geq n}K)$$

Using standard triangle for $\tau^{\geq n}(K)$:

$$\begin{tikzcd}
\tau^{\leq n}(\tau^{\geq n}(K)) \arrow[r] \arrow[d, equal] & \tau^{\geq n}(K) \arrow[r] & \tau^{\geq n+1}(\tau^{\geq n}(K)) \arrow[r] \arrow[d, equal] & \tau^{\leq n}(\tau^{\geq n}(K))[1] \\
H^n(K)[-n] & & \tau^{\geq n+1}(K) &
\end{tikzcd}$$

Lemma 2.1.7. If $K \to L \to M \to K[1]$ is exact, and $K, M \in \mathscr{D}^{\heartsuit}$, then $L \in \mathscr{D}^{\heartsuit}$.

Proof. By 3 of lemma 2.1.4, if $K, M \in \mathscr{D}^{\leq 0}$, then $H^i(K) = H^i(M) = 0$ for $i \geq 1$, so by the long exact sequence of cohomology, $H^i(L) = 0$ for $i \geq 1$. Namely $L \in \mathscr{D}^{\leq 0}$. Duality, if $K, M \in \mathscr{D}^{\geq 0}$, then $L \in \mathscr{D}^{\geq 0}$. Thus $L \in \mathscr{D}^{\heartsuit}$.

Now I can state the following nice result, this is kind of a miracle.

Theorem 2.1.8. $\mathscr{D}^{\heartsuit}$ is an abelian category.

Proof. $K, L \in \mathscr{D}^{\heartsuit}$ implies $K \bigoplus L \in \mathscr{D}^{\heartsuit}$ by the last lemma. Zero object is 0, and $Hom(K, L)$ is a abelian group. So $\mathscr{D}^{\heartsuit}$ is additive.

For a morphism $f : K \to L$ in $\mathscr{D}^{\heartsuit}$. We want to find the kernel and cokernel of this morphism. Extend to an exact triangle

$$K \to L \to M \to K[1]$$

Clearly $M \in \mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq -1}$, the prove is similar to last lemma.

Step1. (Existence of kernel and cokernel).

Claim: $H^0(M) = \tau^{\geq 0}(M) := Coker(f)$, $H^{-1}(M) = H^0(M[-1]) = \tau^{\leq 0}(M[-1]) := Ker(f)$ Fix $W \in \mathscr{D}^{\heartsuit}$, we want to check the universal property of $Coker(f)$. Apply $Hom(-, W)$ to the exact triangle above:

$$Hom(K[1], W) \to Hom(M, W) \to Hom(L, W) \to Hom(K, W)$$

Now $K[1] \in \mathscr{D}^{\leq -1}$, so $Hom(K[1], W) = 0$ and by adjoint of truncation we have $Hom(M, W) = Hom(\tau^{\geq 0}(M), W)$. So we get a short exact sequence:

$$0 \to Hom(\tau^{\geq 0}(M), W) \to Hom(L, W) \to Hom(K, W)$$

This implies $L \to \tau^{\geq 0}(M) = H^0(M)$ is a cokernel of f. Dually,

$$\tau^{\leq 0}(M[-1]) \to M[-1] \to K$$

is a kernel of f.

Step 2.(Factorization Property.) Namely we need to prove the morphism $\overline{f}$ is isomorphism for any f.

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d] & L \arrow[d] \\ coim(f) \arrow[r, "\overline{f}"'] & im(f) \end{tikzcd}$$

Here $coim(f) := coker(ker(f))$ and $im(f) := ker(coker(f))$.

We will find an object $Z \in \mathscr{D}^\heartsuit$ such that $Z \cong coim(f)$ and $Z \cong im(f)$, then the result follows. Specifically, use octaeder axiom, get following commutative diagram:

$$\begin{tikzcd}
 & ker(f)[-1] \arrow[r, equal, "equal"] & ker(f)[-1] & \\
L[-1] \arrow[r, dashed] & coker(f)[-1] \arrow[u] \arrow[r, dashed] & Z \arrow[u] \arrow[r, dashed] & L \\
L[-1] \arrow[u, equal] \arrow[r] & M[-1] \arrow[u] \arrow[r] & K \arrow[u] \arrow[r] & L \arrow[u, equal] \\
 & ker(f) \arrow[u] \arrow[r, equal, "equal"] & ker(f) \arrow[u] &
\end{tikzcd}$$

The first column exact triangle is just

$$M \to \tau^{\geq 0}M \cong coker(f) \to \tau^{\leq -1}M \cong \tau^{\leq 0}(M[-1]) \cong ker(f)$$

Now, note the first row, $L \in \mathscr{D}^{\geq 0}$ and $Coker(f)[-1] \in \mathscr{D}^\heartsuit[-1] \in \mathscr{D}^{\geq 1} \subset \mathscr{D}^{\geq 0}$, by extension lemma, we get $Z \in \mathscr{D}^{\geq 0}$. Similarly, $Z \in \mathscr{D}^{\leq 0}$. Thus Z is an object of $\mathscr{D}^\heartsuit$

$$Z \in \mathscr{D}^\heartsuit$$

The first isomorphism $Z \cong coker(ker(f))$ comes from the distinguished triangle $(ker(f), K, Z)$, this implies $coker(ker(f) \to K) = \tau^{\geq 0}(Z) = Z$.

The second isomorphism $Z \cong ker(coker(f))$ comes from the distinguished triangle $(L, coker(f), Z[1])$, this implies $ker(coker(f)) = \tau^{\leq 0}Z[1][-1] = \tau^{\leq 0}Z = Z$. $\square$

Theorem 2.1.9. The functor $H^0 : \mathscr{D} \to \mathscr{D}^\heartsuit$ is a cohomological functor.

Proof. Fix an exact triangle $K \to L \to M \to K[1]$ in $\mathscr{D}$. We want to show that $H^0(K) \to H^0(L) \to H^0(M)$ is exact in $\mathscr{D}^\heartsuit$. This require some steps

Step 1. Assume $K, L, M \in \mathscr{D}^{\geq 0}$.

Claim : $0 \to H^0(K) \to H^0(L) \to H^0(M)$ is exact.

Proof. Let $A \in \mathscr{D}^\heartsuit$. Notice that $Hom(A, K) = Hom(A, H^0(K))$, since $A \in \mathscr{D}^{\leq 0}$ and $K \in \mathscr{D}^{\geq 0}$. The same applies to L and M. Now applying $Hom(A, -)$, we get the long exact sequence

$$\begin{tikzcd}
Hom(A, M[-1]) \arrow[r] \arrow[d, equal] & Hom(A, K) \arrow[r] \arrow[d, equal] & Hom(A, L) \arrow[r] \arrow[d, equal] & Hom(A, M) \arrow[d, equal] \\
0 \arrow[r] & Hom(A, H^0(K)) \arrow[r] & Hom(A, H^0(L)) \arrow[r] & Hom(A, H^0(M))
\end{tikzcd}$$

Here $Hom(A, M[-1]) = 0$ since $A \in \mathscr{D}^{\leq 0}$, $M[-1] \in \mathscr{D}^{\geq 1}$. By Yoneda lemma, we are done.

Step 2. Assume $M \in \mathscr{D}^{\geq 0}$.

Claim : $0 \to H^0(K) \to H^0(L) \to H^0(M)$ is exact.

Proof. We have an exact triangle $K \to L \to M \to K[1]$. Apply the functor $\tau^{\leq -1}$ to the triangle, we get $\tau^{\leq -1}(K) = \tau^{\leq -1}(L)$. By octahedral axiom, we get following diagram

$$\begin{tikzcd}
\tau^{\geq 0}(K) \arrow[r, dashed] & \tau^{\geq 0}(L) \arrow[r, dashed] & M \\
K \arrow[u] \arrow[r] & L \arrow[u] \arrow[r] & M \arrow[u] \\
\tau^{\leq -1}(K) \arrow[u] \arrow[r] & \tau^{\leq -1}L \arrow[u] &
\end{tikzcd}$$

Now apply step 1 to the top row.

Step 3. Assume $K \in \mathscr{D}^{\leq 0}$.

Claim : $H^0(K) \to H^0(L) \to H^0(M) \to 0$ is exact.

Proof. this is the opposite version of step 2. The proof is similar, we omit it.

Step 4. In general case, use octaeder axiom, we get following commutative diagram

$$\begin{tikzcd}
\tau^{\geq 1}K \arrow[r, dashed] & W \arrow[r, dashed] & M \\
K \arrow[u] \arrow[r] & L \arrow[u] \arrow[r] & M \arrow[u, equal] \\
\tau^{\leq 0}K \arrow[u] \arrow[r, equal, "equal"] & \tau^{\leq 0}K \arrow[u] &
\end{tikzcd}$$

So we have the exact triangle

$$W \to M \to (\tau^{\geq 1}K)[1]$$

by step 2, this gives the exact sequence

$$0 \to H^0(W) \to H^0(M) \to H^0(\tau^{\geq 1}(K)[1])$$

By step 3, the exact triangle $\tau^{\leq 0}K \to L \to W$ gives the exact sequece

$$H^0(K) = H^0(\tau^{\leq 0}(K)) \to H^0(L) \to H^0(W) \to 0$$

Together the two exact sequences, we have that $H^0(K) \to H^0(L) \to H^0(M)$ is exact. $\square$

2.2 The category $\mathbf{D}_c^b(X_0, \overline{\mathbb{Q}_\ell})$

For the basic knowledge of l-adic sheaves, please refer to chapter 10 of [4].

Fix some notation. Let ℓ be a prime number, $\mathbb{Z}_\ell \simeq \varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z}$, $\mathbb{Q}_\ell$ its fraction field, E a finite extension of $\mathbb{Q}_\ell$, and R is the integral closure of E in $\mathbb{Z}_\ell$. Fix a uniformizer λ of R, denote $R/(\lambda^n)$ by R_n.

Let $D(X, R_n)$ be the derived category of sheaf of R_n-module on X. Denote by $D^b_{ctf}(X, R_n)$ the full subcategory of $D^b(X, R_n)$ consisting of objects $\mathscr{F}^\bullet$ with finite Tor-dimension, and such that $\mathscr{H}^i(\mathscr{F}^\bullet)$ are constructible for all i.

Recall that we say $\mathscr{F}^\bullet \in D^b(X, R_n)$ has finite Tor-dimension if there is an integer n such that $Tor_i(\mathscr{F}^\bullet, M) = 0$ for any $i > n$ and any constant sheaf of R_n-module M. We now construct the category $D_c^b(X, R)$

$$D_c^b(X, R) = \text{"}\lim_n\text{"} D^b_{ctf}(X, R_n)$$

Object. First, an object of $D_c^b(X, R)$ is a collection

$$K = K^\bullet = (K_n^\bullet)_{n\geq 1}$$

of complexes $K_n^\bullet$ in $D^b_{ctf}(X, R_n)$ together with quasi-isomorphisms

$$\phi_{n+1} : K^\bullet_{n+1} \otimes^L_{R_{n+1}} R_n \cong K_n^\bullet$$

in the categories $D_c^b(X, R_n)$. The **i-th cohomology sheaf** of $K^\bullet$ is defined by $\mathscr{H}^i(K^\bullet) = (\mathscr{H}^i(K_n^\bullet))_{n\geq 1}$.
Morphism. For two objects of $D_c^b(X, R)$ represented by projective systems $K^\bullet = (K_n^\bullet)_{n\geq 1}$ and $L^\bullet = (L_n^\bullet)_{n\geq 1}$ we put

$$Hom_{D_c^b(X,R)}(K^\bullet, L^\bullet) = \varprojlim_n Hom_{D_c^b(X,R_n)}(K_n^\bullet, L_n^\bullet)$$

In other words, a homomorphism $\psi : K^\bullet \to L^\bullet$ in $Hom_{D_c^b(X,R)}(K^\bullet, L^\bullet)$ is a family $\psi = (\psi_n)_{n\geq 1}$ of morphisms $\psi_n : K_n^\bullet \to L_n^\bullet$ in the derived categories $D_c^b(X, R_n)$ such that the following diagrams for $n \geq 1$ commute

$$\begin{array}{lccc}
\psi_{n+1} \otimes^L_{R_{n+1}} R_n : & K^\bullet_{n+1} \otimes^L_{R_{n+1}} R_n & \longrightarrow & L^\bullet_{n+1} \otimes^L_{R_{n+1}} R_n \\
& \downarrow \cong \psi^K_{r+1} & & \downarrow \cong \psi^L_{r+1} \\
\psi_n : & K_n^\bullet & \longrightarrow & L_n^\bullet
\end{array}$$

In order to work with this definition of $D_c^b(X, R)$, we give a remark:
Remark : By Chapter 6 Proposition 6.4.6 of [4], any object in $D^b_{ctf}(X, R_n)$ is isomorphic to a

bounded complex of constructible flat sheaves of R_n-module. So we can suppose that all complexes $K_n^\bullet$ are bounded constructible flat complexes. Flatness implies

$$K_{n+1}^\bullet \otimes^L_{R_{n+1}} R_n = K_{n+1}^\bullet \otimes_{R_{n+1}} R_n$$

We state an important fact without proof.

Theorem 2.2.1. ([1] Chapter 2 Lemma 5.5) Let $K^\bullet = (K_n^\bullet)_{n\geq 1}$ be a object of $D_c^b(X,R)$. Then its cohomology

$$\mathscr{H}^i(K^\bullet) = (\mathscr{H}^i(K_n^\bullet))_{\geq 1}$$

are A-R λ-adic sheaves and $\mathscr{H}^i(K^\bullet) = 0$ for i sufficient large or small.

Recall a λ-adic sheaf on a noetherian scheme has an open dense subset such that the restriction of this sheaf to the open set is lisse.

Corollary 2.2.2. $\mathscr{H}^i(K^\bullet)$ is a lisse λ-adic sheaf on an open dense subscheme $U \subset X$.

Definition 2.2.3. (The **standard t-structure**))We define two subcategories on $D_c^b(X,R)$.

$$D_c^b(X,R)^{\leq 0} = \{K^\bullet \in D_c^b(X,R) \mid \mathscr{H}^i(K^\bullet) = 0 \; \forall i \geq 1\}$$

$$D_c^b(X,R)^{\geq 0} = \{K^\bullet \in D_c^b(X,R) \mid \mathscr{H}^i(K^\bullet) = 0 \; \forall i \leq -1\}$$

Theorem 2.2.4. $D_c^b(X,R)^{\leq 0}$ and $D_c^b(X,R)^{\geq 0}$ define a t-structure on $D(X,R)$. The core of this **standard** t-structure is the full subcategory of $D_c^b(X,R)$

$$Core(stadard) = \{K^\bullet \in D_c^b(X,R) \mid \mathscr{H}^i(K^\bullet) = 0, for\ i \neq 0\}$$

The functor

$$Core(standard) \longrightarrow \{\lambda - adic\ \ sheaves\}$$

$$K^\bullet \longrightarrow \mathscr{H}^0(K^\bullet)$$

defines a equivalence of categories between the core of the standard t-structure and the abelian category of λ-adic sheaves on X.

The category

$$D_c^b(X,E)$$

is deduced from $D_c^b(X,R)$ by "localization" in the sense of derived category , the category $D_c^b(X,\overline{\mathbb{Q}_\ell})$ is defined by

$$D_c^b(X,\overline{\mathbb{Q}_\ell}) = \varinjlim_{E\subset\overline{\mathbb{Q}_\ell}} D_c^b(X,E)$$

(See [4],page 560.)

Remark: Notice that above theorem and standard t-structure can extend to $D_c^b(X,\overline{\mathbb{Q}_\ell})$.

2.3 glueing of t-structures

From now, for a scheme X, we denote $D^b_c(X, \overline{\mathbb{Q}_\ell})$ by $D^b_c(X)$. We now talk the glueig of t-structure. Specificlly, given a scheme X and an open subscheme U

$$j : U \to X$$

Let

$$i : Z \to X$$

be the closed complement of U in X.

Let $T(U)$ be a full triangulated subcategory of $D^b_c(U)$ and $T(Z)$ be a full triangulated subcategory of $D^b_c(Z)$. Suppose we have t-structures on the categories $T(U)$ and $T(Z)$, which are denoted by $(T^{\le 0}(U), T^{\ge 0}(U))$ and $(T^{\le 0}(Z), T^{\ge 0}(Z))$, then we can glue these t-structures together to obtain a new t-structure on the full subcategory $T(X, U)$ of $D^b_c(X)$, defined by

$$T(X,U) = \{K \in D^b_c(X) \mid j^*K \in T(U), i^*K \in T(Y), i^!K \in T(Y)\}$$

Definition 2.3.1.

$$T^{\le 0}(X,U) := \{K \in T(X,U) \mid j^*K \in T^{\le 0}(U), i^*K \in T^{\le 0}(Z)\}$$

$$T^{\ge 0}(X,U) := \{K \in T(X,U) \mid j^*K \in T^{\ge 0}(U), i^!K \in T^{\ge 0}(Z)\}$$

Proposition 2.3.2. Notation is as above, $(T^{\le 0}(X,U), T^{\ge 0}(X,U))$ is a t-structure on $T(X,U)$.

Proof. First, we need to prove orthogonality: $Hom(T^{\le 0}(X,U), T^{\ge 1}(X,U)) = 0$. Taking $K \in T^{\le 0}(X,U)$ and $L \in T^{\ge 1}(X,U)$, then using the exact triangle $j_!j_*K \to K \to i_*i^*K$ and applying $Hom(-, L)$, we have

$$\begin{array}{ccccc} & Hom(i_*i^*K, L) & \longrightarrow \; Hom(K,L) \; \longrightarrow & Hom(j_!j^*K, L) & \\ & \cong\big\downarrow & & \big\downarrow & \\ 0 \;=\; & Hom(i^*K, i^!L) & & Hom(j^*K, j^*L) & \;=\; 0 \end{array}$$

So $Hom(K, L) = 0$ (Here the two equality is by definition)

Second, $T^{\le 0}(X,U) \subset T^{\le 1}(X,U)$ and $T^{\ge 1}(X,U) \subset T^{\ge 0}(X,U)$ is trivial by definition.

Finally, we need to the existence of triangle. Fix $K \in T(X,U)$, we have the canonical morphisms $K \to j_*j^*K \to j_*(\tau^{\ge 1}j^*K)$.Then choose a exact triangle $Y \to K \to j_*(\tau^{\ge 1}j^*K)$.Similarly, we have a exact triangle $A \to Y \to i_*(\tau^{\geq 1} i^* Y)$. By the octaeder axiom TR4a of triangulated categories, we have following two distinguished triangles$(i_*(\tau^{\geq 1} i^* Y), B, j_*(\tau^{\geq 1} j^* K))$ and (A, K, B).

$$\begin{tikzcd}
 & A[1] \arrow[r, equal, "equal"] & A[1] & \\
j_*(\tau^{\geq 1} j^* K)[-1] \arrow[r, dashed] & i_*(\tau^{\geq 1} i^* Y) \arrow[r, dashed] \arrow[u] & B \arrow[r, dashed] \arrow[u] & j_*(\tau^{\geq 1} j^* K) \\
j_*(\tau^{\geq 1} j^* K)[-1] \arrow[r] \arrow[u, equal] & Y \arrow[r] \arrow[u] & K \arrow[r] \arrow[u] & j_*(\tau^{\geq 1} j^* K) \arrow[u, equal] \\
 & A \arrow[r, equal, "equal"] \arrow[u] & A \arrow[u] &
\end{tikzcd}$$

Claim (A, K, B) is the desired triangle. Namely

$$A \in T^{\leq 0}(X, U), B \in T^{\geq 1}(X, U)$$

Proof. Note that $j^* i_* = 0$ and $i^! j_* = 0$, so apply j^* to the triangle $A \to Y \to i_*(\tau^{\geq 1} i^* Y)$, we have $j^* A \cong j^* Y$. Similarly, $j^* B \cong \tau^{\geq 1} j^* K$. So we have following isomorphisms of distinguished triangles

$$j^*(A, K, B) \cong (j^* Y, j^* K, \tau^{\geq 1} j^* K) \cong (\tau^{\leq 0} j^* K, j^* K, \tau^{\geq 1} j^* K)$$

Namely $j^* A \cong \tau^{\leq 0} j^* K \in T^{\leq 0}(U)$ and $j^* B \cong \tau^{\geq 1} j^* K \in T^{\geq 1}(U)$.

Do above for i^* and $i^!$, we have

$$i^*(A, Y, i_*(\tau^{\geq 1} i^* Y) \cong (i^* A, i^* Y, \tau^{\geq 1} i^* Y) \cong (\tau^{\leq 0} i^* Y, i^* Y, \tau^{\geq 1} i^* Y)$$

$$i^!(A, Y, i_*(\tau^{\geq 1} i^* Y) \cong i^!(A, K, B)$$

Namely $i^* A \cong \tau^{\leq 0} i^* Y \in T^{\leq 0}(Z)$ and $i^! B \cong i^! i_*(\tau^{\geq 1} i^* Y) \cong \tau^{\geq 1} i^* Y \in T^{\geq 1}(Z)$. □

□

3 Perverse sheaves

3.1 Verdier Duality

The reference of this section is ([1], 2.7)

Theorem 3.1.1. Let
$$f : X \to S$$
be a compactifible morphism between finite type scheme over a finite field or algebraically closed field. Then the functor
$$Rf_! : D_c^b(X, \overline{\mathbb{Q}_\ell}) \to D_c^b(S, \overline{\mathbb{Q}_\ell})$$
have a right adjoint triangulated functor
$$f^! : D_c^b(S, \overline{\mathbb{Q}_\ell}) \to D_c^b(X, \overline{\mathbb{Q}_\ell})$$
Namely we have functorial isomorphism
$$Hom(K, f^! L) \cong Hom(Rf_! K, L)$$
for all $K \in D_c^b(X, \overline{\mathbb{Q}_\ell})$ and $L \in D_c^b(S, \overline{\mathbb{Q}_\ell})$

Definition 3.1.2. Let
$$f : X \to S = Spec(k)$$
be a finite type scheme over the fixed field k(finite or algebraically closed).The **duality complex** of X is
$$K_X = f^!(\overline{\mathbb{Q}_\ell}_S) \in D_c^b(X, \overline{\mathbb{Q}_\ell})$$
Then we define the contravariant **dualizing functor** by
$$D_X(L) = R\mathscr{H}om(L, K_X)$$
we often write $\mathbb{D}L = \mathbb{D}(L) = D_X(L)$ if the scheme X is fixed.

Corollary 3.1.3. (Poincare Duality) Under the assumption of Theorem 3.1.1. If $K \in D_c^b(X, \overline{\mathbb{Q}_\ell})$, then the following holds
$$Rf_*(D_X K) = D_S(Rf_! K)$$

Theorem 3.1.4. The natural functorial homomorphism
$$K \to D_X(D_X K)$$
is a canonical isomorphism, namely
$$D_X \circ D_X = id$$
Therefore the dualizing functor defines an anti-equivalence of categories
$$D_X : D_c^b(X, \overline{\mathbb{Q}_\ell}) \to D_c^b(X, \overline{\mathbb{Q}_\ell})$$
$$Hom(K, L) = Hom(D_X(L), D_X(K))$$

We now collect some formulas, which will be frequently used later.

Corollary 3.1.5. Suppose $f : X \to S$ is a morphism satisfying the assumptions of **Theorem II.3.2.1**. Then the following formulas hold:

(a) $\mathbb{D} \circ \mathbb{D} = \mathrm{id}$

(b) $\mathbb{D} \circ Rf_! = Rf_* \circ \mathbb{D}$

(c) $\mathbb{D} \circ Rf_* = Rf_! \circ \mathbb{D}$

(d) $\mathbb{D} \circ f^* = f^! \circ \mathbb{D}$

(e) $\mathbb{D} \circ f^! = f^* \circ \mathbb{D}$

(f) $R\mathscr{H}\mathrm{om}(A, B) = \mathbb{D}\left(A \otimes^L \mathbb{D}(B)\right)$

(g) $Rf_!\left(A \otimes^L f^* B\right) = Rf_! A \otimes^L B$

(h) $f^! R\mathscr{H}\mathrm{om}(A, B) = R\mathscr{H}\mathrm{om}\left(f^*(A), f^!(B)\right)$

Proof. (a) and (b) is true. (c) holds from (a) and (b).(d) and (e) are from (b),(c), $Hom(K, L) = Hom(D_X L, D_X K)$, and adjoint $(Rf_!, f^!)$ .For (f), if we put $C = \mathbb{D}B$, then $B = \mathbb{D}C$. So

$$\begin{aligned} R\mathscr{H}\mathrm{om}(A, B) = R\mathscr{H}\mathrm{om}(A, \mathbb{D}C) &= R\mathscr{H}\mathrm{om}(A, R\mathscr{H}\mathrm{om}(C, K_X)) \\ &= R\mathscr{H}\mathrm{om}(A \otimes^L C, K_X) \\ &= \mathbb{D}(A \otimes^L \mathbb{D}(B)) \end{aligned}$$

(g) and (h) are just similar to corresponding results in etale cohomology. □

3.2 Perverse t-structure

Let X be a scheme over a base field k, such that k is either a finite field or a separably closed field. Then $D^b_c(X, \overline{\mathbb{Q}_\ell})$ is a triangulated category, for simplicity, we denote it by $D^b_c(X)$.

Definition 3.2.1. Let X be a variety. The perverse t-structure on X is defined by

$$\begin{aligned} {}^p D^b_c(X)^{\leq 0} &= \{K \in D^b_c(X) \mid \dim \mathrm{supp} \mathscr{H}^{-i}(K) \leq i, \forall i \in \mathbb{Z}\} \\ {}^p D^b_c(X)^{\geq 0} &= \{K \in D^b_c(X) \mid \dim \mathrm{supp} \mathscr{H}^{-i}(\mathbb{D}(K)) \leq i, , \forall i \in \mathbb{Z}\} \end{aligned}$$

where $\mathbb{D}(K)$ denotes the Verdier duality of K. For simplicity, we denote these two subcategories by ${}^p D^{\leq 0}(X)$ and ${}^p D^{\geq 0}(X)$.

The heart of this t-structure is $Per(X) = {}^pD^b_c(X)^{\le 0} \cap {}^pD^b_c(X)^{\ge 0}$. Objects in the heart are called **perverse sheaves**.

Theorem 3.2.2. ([3]). This perverse t-structure gives a t-structure on $D^b_c(X)$.

To prove this theorem, we need some preparation. First, we need some results about lisse complexes

we consider the effect of pullback of perverse t-structure under open and closed immersion.

Lemma 3.2.3. Fix $j : U \hookrightarrow X$ open, and $i : Z \hookrightarrow X$ is the closed complement. Fix $K \in D^b_c(X)$. Then we have:

1. $K \in {}^pD^{\le 0}(X) \iff j^*K \in {}^pD^{\le 0}(U),\ i^*K \in {}^pD^{\le 0}(Z)$.
2. $K \in {}^pD^{\ge 0}(X) \iff j^!K = j^*K \in {}^pD^{\ge 0}(U),\ i^!K \in {}^pD^{\ge 0}(Z)$.

Proof. 1. Considering i^* and j^* are exact, which means that they commute with cohomology, we get:

$$\mathrm{Supp}(\mathscr{H}^i(K)) = \mathrm{Supp}(\mathscr{H}^i(j^*K)) \cup \mathrm{Supp}(\mathscr{H}^i(i^*K)).$$

Thus,

$$\dim \mathrm{Supp}(\mathscr{H}^i(K)) = \max\left(\dim \mathrm{Supp}(\mathscr{H}^i(j^*K)), \dim \mathrm{Supp}(\mathscr{H}^i(i^*K))\right).$$

This immediately gives (1).

2. Note that $i^*\mathbb{D}K = \mathbb{D}(i^!K)$, and $j^*\mathbb{D}K = \mathbb{D}(j^*K)$, so (2) is from (1) by duality.

□

To prove Theorem 3.2.2, we need some smooth results.

Suppose X is smooth of dimension d. Then the dualizing complex on X has the form

$$\overline{\mathbb{Q}_\ell}[2d](d)$$

It is a complex K_X, whose cohomology is concentrated in degree $-2d$ and such that its cohomology sheaf $\mathscr{H}^{-2d}(K_X)$ is isomorphic to the smooth sheaf $\overline{\mathbb{Q}_\ell}(d)$. For a smooth sheaf $\mathscr{G}$ on X, the definition of dual sheaf is as the ordinary case

$$\mathscr{G}^\vee = \mathscr{H}om(\mathscr{G}, \overline{\mathbb{Q}_\ell})$$

A sheaf complex $K \in D^b_c(X)$ is called a **smooth complex**, if all its cohomology sheaves $\mathscr{H}^i(K)$ are lisse sheaves on X.

Proposition 3.2.4. 1. Let X be a smooth scheme of dimension d over k and let $K \in D_c^b(X)$ be a smooth complex on X. Then

$$\mathscr{H}^i(\mathbb{D}K) \cong \mathscr{H}^{-i-2d}(K)^\vee(d)$$

2. Suppose X is irreducible and $K \in D_c^b(X)$. Then there exists an open dense smooth subscheme

$$j : U \to X$$

of X, such that

$$j^*(K)$$

is smooth on U.

Corollary 3.2.5. Under the assumption above, i.e X is smooth of dimension d and K is a smooth complex, then

$$K \in {}^pD^{\leq 0}(X) \quad \text{iff} \quad \mathscr{H}^i K = 0\ \forall i > -d$$

$$K \in {}^pD^{\geq 0}(X) \quad \text{iff} \quad \mathscr{H}^i K = 0\ \forall i < -d$$

Namely a smooth complex $K \in Perv(X)$ if and only if

$$K = \mathscr{G}[d]$$

for a smooth sheaf $\mathscr{G}$.

Proof of Theorem 3.2.2 We prove by induction on $d = dim(X)$.

1. If $d = 0$, the X is the union of finitely many point. Then ${}^pD_c^b(X)^{\leq 0} = D_c^b(X)^{\leq 0}$ and ${}^pD_c^b(X)^{\geq 0} = D_c^b(X)^{\geq 0}$ as in Theorem2.2.4. So the perverse t-structure is just the standard t-structure.

2. If $d \geq 1$, let

$$j : U \to X$$

be a nonempty open smooth subscheme of X and let

$$i : Z \to X$$

be its closed complement. Assume $({}^pD^{\leq 0}(Z), {}^pD^{\geq 0}(Z))$ is a perverse t-structure on $T(Z) = D_c^b(Z)$. On the other hand let $T(U)$ be the full subcategory of $D_c^b(U)$, consisting of complexes with smooth cohomology sheaves. From corollary 3.2.5 we know the induced t-structure on $T(U)$ coincides with the standard t-structure up to a shift of degree. By gluing, we get

$$T(X, U) = \{E \in D_c^b(X) \mid j^*E \ \text{ has smooth cohomology sheaves on } U\}$$

By lemma, the glueing t-structure on $T(X,U)$ coincides with the perverse t-structure, which is obtained by restriction from $D^b_c(X)$ to $T(X,U)$.

Now let E be an arbitrary complex in $D^b_c(X)$. Then there always exists an open dense smooth subscheme $U \to X$, such that the restriction of E to U has smooth cohomology sheaves. In other words

$$D^b_c(X) = \bigcup_{U \subset X,\ dense\ open\ ess.smooth} T(X,U)$$

Now we check the three axioms

(1). For complexes $E \in {}^pD^{\leq 0}(X)$ and $E' \in {}^pD^{\geq 1}(X)$, there exists U and U' as above, such that $E \in T^{\leq 0}(X,U)$ and $E' \in T^{\geq 1}(X,U')$. Set $U = U \cap U'$, by definition $E \in T^{\leq 0}(X,U)$ and $E' \in T^{\geq 1}(X,U)$, So $Hom(E,E') = 0$.

(2). Fix $E \in D^b_c(X)$. Suppose $E \in T(X,U)$ for an open subset $U \subset X$. then there $E_1 \in T^{\leq 0}(X,U)$ and $E_2 \in T^{\geq 1}(X,U)$ such that $E_1 \to E \to E_2$ is an exact triangle. But $T^{\leq 0}(X,U) = {}^pD^{\leq 0}(X) \cap T(X,U)$ and $T^{\geq 1}(X,U) = {}^pD^{\geq 1}(X) \cap T(X,U)$, so $E_1 \in {}^pD^{\leq 0}(X)$ and $E_2 \in {}^pD^{\geq 1}(X)$.

3.3 t-exact functor

Definition 3.3.1. Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be two t-structure triangulated categories with heart $\mathscr{D}_1^\heartsuit$ and $\mathscr{D}_2^\heartsuit$ and let $F : \mathscr{D}_1 \to \mathscr{D}_2$ be a morphism of categories. Then we define ${}^pF = H^0 \circ F \circ \epsilon : \mathscr{D}_1^\heartsuit \to \mathscr{D}_2^\heartsuit$.

Definition 3.3.2. Let $F : \mathscr{D}_1 \to \mathscr{D}_2$ be a exact functor of triangulated categories, and assume that there are t-structure$(\mathscr{D}_1^{\leq 0}, \mathscr{D}_1^{\geq 0})$ on $\mathscr{D}_1$ and $(\mathscr{D}_2^{\leq 0}, \mathscr{D}_2^{\geq 0})$ on $\mathscr{D}_2$. Such a functor is called

$$\text{t-right exact iff } F(D^{\leq 0}(A)) \subset D^{\leq 0}(B)$$

and

$$\text{t-left exact iff } F(D^{\geq 0}(A)) \subset D^{\geq 0}(B)$$

Finally, F is exact if it is both t-left and t-right exact.

Now we give some results about the six functors.

Proposition 3.3.3. Let X be a finite type separated scheme over a finite field or algebraically field. Let $j : U \to X$ be an open immersion with closed complement $i : Z \to X$. Consider the perverse t-structure on all schemes. Then

1. $j^*(= j^!), i_* = (i_!)$ are t-exact
2. $j_!, i^*$ are t-right exact.

3. $j_*, i^!$ are t-left exact

4. There are adjoint sequence $({}^p i^*, {}^p i_* = {}^p i_!, {}^p i^!)$, $({}^p j_!, {}^p j^! = {}^p j^*, {}^p j_*)$

5. the compositions ${}^p j^* \circ {}^p i_*, {}^p i^* \circ {}^p j_!, {}^p i^! \circ {}^p j_*$ are zero.

Proof. 1. We prove i_* is t-exact, j^* is similar .

(t- right exact). Specifically, $i^* i_* = id$ implies $i^*(i_* D_Z^{\leq 0}) \subseteq D_Z^{\leq 0}$. $j^* i_* = 0$ implies $j^*(i_* D_Z^{\leq 0}) \subseteq D_U^{\leq 0}$.

(t-left exact). the same argument using $i^! i_* = id$, $j^* i_* = 0$.

2. using $i^* j_! = 0$ and $j^* j_! = id$.

3. same argument with 2.

4. we prove for $({}^p j^*, {}^p j_*)$. Other cases are similar.

Proof. Fix $K \in D_U^\heartsuit$ and $L \in D_X^\heartsuit$. We are interested in

$$Hom_{D_X^\heartsuit}({}^p j_! K, L) = Hom_{D_X^\heartsuit}({}^p H^0(j_! K), L) \overset{full}{=} Hom_{D_X}({}^p H^0(j_! K), L) = Hom_{D_X}(\tau^{\leq 0}(j_! K), L)$$

since $L \in D_X^{\geq 0}$. Now $j_!$ is t-right exact and j^* is t-exact imply $\tau^{\leq 0}(j_! K) = j_! K$ and $j^* L = {}^p H^0(j^* L)$. So we have following equality

$$\begin{aligned} Hom_{D_X}(\tau^{\leq 0}(j_! K), L) &= Hom_{D_X}(j_! K, L) \\ (adjoint) &= Hom_{D_U}(K, j^* L) \\ &= Hom_{D_U}(K, {}^p H^0(j^* L)) \\ (full) &= Hom_{D_U}(K, {}^p j^* L) \\ &= Hom_{D_U^\heartsuit}(K, {}^p j^* L) \end{aligned}$$

Namely $Hom_{D_U^\heartsuit}(K, {}^p j^* L) = Hom_{D_X^\heartsuit}({}^p j_! K, L)$.

5. j^*, i_* are t-exact, so ${}^p j^* = j^*$ on $D_X^\heartsuit$, ${}^p i_* = i_*$ on $D_Z^\heartsuit$. Then ${}^p j^* \circ {}^p i_* = j^* i_* = 0$. By adjoint, ${}^p i^* \circ {}^p j_! = 0$ and ${}^p i^! \circ {}^p j_* = 0$.

3.4 intermediate extension

As above, let X be a finite type scheme over a finite field or algebraically closed field. Let $j : U \to X$ be an open immersion and $i : Z \to X$ be the closed complement of U. Fix a perverse sheaf K on U, a perverse sheaf $\overline{K}$ on X is called an extension of K, if

$$j^*\overline{K} = K$$

we have following results.

Lemma 3.4.1. With the preceding notations we have following chain of equivalent conditions (1) – (4) for a perverse extension $\overline{K}$ of the perverse sheaf K:

(1) $\overline{K}$ has neither subobjects nor quotients from $i_*Perv(Y)$

(2) ${}^pH^0(i^*\overline{K}) = {}^pH^0(i^!\overline{K}) = 0$

(3) $i^*\overline{K} \in {}^pD^{\leq -1}(Z)$ and $i^!\overline{K} \in {}^pD^{\geq 1}(Z)$

(4) $\overline{K} \cong image({}^pH^0(j_!K) \to {}^pH^0(j_*K))$

If one of these equivalent conditions holds, then there exists an exact triangle

$$(\overline{K}, j_*K, i_*{}^p\tau^{\geq 0}i^*j_*K)$$

Definition 3.4.2. We know that for a perverse sheaves $K \in Perv(U)$, there is a unique (up to quasi isomorphism) perverse sheaves $\overline{K} \in Perv(X)$ which satisfies the conditions in Lemma 3.4.1. This unique extensionn is called the intermediate extension of K, and denote by

$$j_{!*}K$$

which define a functor

$$j_{!*} : Perv(U) \to Perv(X)$$

Corollary 3.4.3. Let $j : U \to X$ be an open immersion, and $K \in Perv(U)$, then

$$\mathbb{D}(j_{!*}K) = j_{!*}(\mathbb{D}K)$$

Proof. We use the characterization (4) above. Let $\overline{L} = \mathbb{D}(j_{!*}K)$, then $j^*\overline{L} = j^*\mathbb{D}(j_{!*}K) = \mathbb{D}(j^*j_{!*}K) = \mathbb{D}K$. So $\overline{L}$ is an extension of $\mathbb{D}K$.

Now $i^*\overline{L} = i^*\mathbb{D}(j_{!*}K) = \mathbb{D}(i^!j_{!*}K)$. $i^!j_{!*}K \in {}^pD^{\geq 1}(Z)$,so $i^*\overline{L} = \mathbb{D}(i^!j_{!*}K) \in {}^pD^{\leq -1}(Z)$.

Similarly, $i^!\overline{L} \in {}^pD^{\geq 1}(Z)$. So $\overline{L} = j_{!*}(\mathbb{D}K)$. □

Corollary 3.4.4. Let $j : U \to X$ be an open immersion with closed complement $i : Z \to X$. Then any simple object $\overline{K} \in Perv(X)$ is either of the form i_*A for a simple object $A \in Perv(Z)$ or of the form $j_{!*}B$ for a simple object $B \in Perv(U)$.

Corollary 3.4.5. A perverse sheaf K on X is simple if and only if it is of the form $K = i_*j_{!*}\mathscr{F}[d]$, for an irreducible closed subscheme $i : Z \to X$, a open dense smooth of dimension d subscheme $j : U \to Z$ of Z and a smooth irreducible $\overline{\mathbb{Q}_\ell}$-sheaf $\mathscr{F}$ on U.

Corollary 3.4.6. The abelian category $Perv(X)$ is artinian and noetherian. (Namely has finite length).

Proof. We have to show, that a perverse sheaf $\overline{K} \in Perv(X)$ can have only finitely many perverse constituents. If $dim(X) = 0$, the category $Perv(X)$ is equivalent to the category of finite dimension $\overline{\mathbb{Q}_\ell}$- vector space, which is artinian and noetherian. By noetherian induction the assertion can assumed to be true for all closed subspaces of smaller dimension. Choose open smooth sense subset $j : U \to X$ such that $K = j^*\overline{K} = \mathscr{F}[d]$ for a smooth sheaf $\mathscr{F}$. We have following exact sequence:

$$\begin{tikzcd}
 & & & 0 & & \\
 & & & i_*{}^pH^0(i^*j_*K) \arrow[u] & & \\
0 \arrow[r] & i_*{}^pH^0(i^!\overline{K}) \arrow[r] & \overline{K} \arrow[r] & {}^pH^0(j_*K) \arrow[r] \arrow[u] & i_*{}^pH^1(i^!\overline{K}) \arrow[r] & 0 \\
 & & & j_{!*}K \arrow[u] & & \\
 & & & 0 \arrow[u] & &
\end{tikzcd}$$

The horizontal sequence is obtained by take $H^0(-)$ for the exact triangle:

$$i_*i^!\overline{K} \to \overline{K} \to j_*j^*\overline{K}$$

notice that i_* commutes with $H^n (n \in \mathbb{Z})$ since it is exact, and $j_*j^*\overline{K} \in D_X^{\geq 0}$, so by Theorem 2.1.9 step2, we have the left 0. The right 0 is just because $\tau^{\geq 1} \circ \tau^{\leq 0} = 0$.

The vertical sequence following the equivalent conditions in Lemma 3.4.1, the apply $H^0(-)$ and notice $i_* {}^p\tau^{\geq 0} i^* j_* K \in D_X^{\geq 0}$, as the same as above, we have desired sequence.

Now analyze above diagram. First see the horizontal exact sequence, by induction, ${}^pH^0(i^!\overline{K})$ has finite length, so is $i_* {}^pH^0(i^!\overline{K})$. So we need to show ${}^pH^0(j_*K)$ is also.

Notice the vertical sequence, $i_* {}^pH^0(i^* j_* K)$ has finite length, the following lemma implies $j_{!*}$ preserves finite length, so we are done. □

Lemma 3.4.7. If $j : U \in X$ is an open immersion, then $j_{!*}$ preserves finite length objects.

4 Mixed complex and weight filtration

4.1 Mixed complex

Recall some notations in [2].

Definition 4.1.1. Let β be a real number. $\mathscr{G}_0$ is a ℓ-adic sheaf on X_0. Fix $\tau : \overline{\mathbb{Q}_l} \cong \mathbb{C}$
1. Choose a $\overline{k}$-point $\overline{x} \in X$ lying over $x \in |X_0|$. The Weil group $W(\overline{k}/k(x))$ acts on the stalk at $\mathscr{G}_{0\overline{x}}$ via the geometric frobenius $F_x : \mathscr{G}_{0\overline{x}} \to \mathscr{G}_{0\overline{x}}$. We say that $\mathscr{G}_0$ is **τ-pure of weight** β if for every $x \in |X_0|$, and all eigenvalues $\alpha \in \overline{\mathbb{Q}_l}$ of F_x, we have

$$|\tau(\alpha)| = N(x)^{\beta/2}.$$

2. We say $\mathscr{G}_0$ is τ-mixed if there is a finite filtration of subsheaves

$$0 = \mathscr{G}_0^{(0)} \subseteq \mathscr{G}_0^{(1)} \subseteq \cdots \subseteq \mathscr{G}_0^{(r)} = \mathscr{G}_0$$

such that $\mathscr{G}_0^{(j)}/\mathscr{G}_0^{(j-1)}$ is τ-pure of some weight.
3. $\mathscr{G}_0$ is pure of weight β if it is τ-pure of weight β for all $\tau : \overline{\mathbb{Q}_l} \cong \mathbb{C}$
4. $\mathscr{G}_0$ is mixed if there exists a finite filtration as in (2) such that all quotient are pure.

Definition 4.1.2. Let X_0 be a scheme of finite type over a $\mathbb{F}_q$. Then an object K_0 of the category $D_c^b(X_0, \overline{\mathbb{Q}_\ell})$ is said to be τ-mixed (mixed), if all its cohomology sheaves $\mathscr{H}^i(K_0)$ are τ-mixed (mixed)sheaves on X_0.The full subcategory of mixed complexes is denoted by $D_m^b(X_0, \overline{\mathbb{Q}_\ell}) \subset D_c^b(X_0, \overline{\mathbb{Q}_\ell})$.

Lemma 4.1.3. The category of mixed sheaves on X_0 is closed under subquotients and extensions.

Proposition 4.1.4. Let X_0 be a scheme of finite type over $\mathbb{F}_q$. Then the category $D_m^b(X_0, \overline{\mathbb{Q}_\ell})$ is a triangulated subcategory stable under the six functor.

Proof. To show that $D^b_m(X_0,\overline{\mathbb{Q}_\ell})$ is a triangulated subcategory we have to check the following three things:

1. *The subcategory $D^b_m(X_0,\overline{\mathbb{Q}_\ell})$ is stable under shifts.* This is trivial since $\mathscr{H}^i(K_0[1]) = \mathscr{H}^{i+1}(K_0)$
2. *The subcategory $D^b_m(X_0,\overline{\mathbb{Q}_\ell})$ is stable under isomorphism.* This is trivial.
3. *If there is an exact triangle $X \to Y \to Z \to X[1]$ in* $D^b_c(X_0,\overline{\mathbb{Q}_\ell})$ with $X, Z \in D^b_m(X_0,\overline{\mathbb{Q}_\ell})$ then $Y \in D^b_m(X_0,\overline{\mathbb{Q}_\ell})$.

   take the long exact sequence of cohomology

   $$\cdots \longrightarrow \mathscr{H}^i(X) \xrightarrow{f} \mathscr{H}^i(Y) \xrightarrow{g} \mathscr{H}^i(Z) \longrightarrow \cdots$$

   this induces a short sequence

   $$0 \longrightarrow \mathscr{H}^i(X)/ker(f) \longrightarrow \mathscr{H}^i(Y) \longrightarrow im(g) \longrightarrow 0$$

   by 4.1.3 $\mathscr{H}^i(Y)$ is mixed.

Next, we check the six functors preserve the category $D^b_m(X_0,\overline{\mathbb{Q}_\ell})$. Let Y_0 be a finite type $\mathbb{F}_q$-scheme and $f : X_0 \to Y_0$ be a morphism of scheme. We need the following results by Deligne:

Theorem 4.1.5. (Deligne,[2],Thm 3.3.1) Assume f is separated and $\mathscr{F}_0$ is a mixed sheaf of maximal weight $\leq w$. Then $R^i f_! \mathscr{F}_0$ is a mixed sheaf of maximal weight $\leq w + i$ on Y_0.

Theorem 4.1.6. (Deligne,[2],Thm 6.1.1) Let $\mathscr{F}_0$ be a mixed sheaf on X_0, then $R^i f_* \mathscr{F}_0$ is a mixed sheaf on Y_0.

So Rf_* and $Rf_!$ preserve mixedness. f^* preserves mixedness is trivial. the case $f^!$ is reduced to the case f^* by biduality, using the formula

$$\mathbb{D} \circ f^! = f^* \circ \mathbb{D}$$

and the assertion $\mathbb{D}$ preserves mixedness. The tensor product $\otimes^L$ follows using the kunneth formula

$$\mathscr{H}^i(K_0 \otimes L_0) = \bigoplus_{i+j=n} \mathscr{H}^i(K_0) \otimes \mathscr{H}^j(L_0)$$

So it remains to prove the last case : $\mathbb{D}K_0$ is mixed if K_0 is mixed.

This is clear when $dim(X_0) = 0$, so we ca prove by induction on $dim(X_0)$. Let $j : U_0 \hookrightarrow X_0$ be a smooth dense open subscheme such that j^*K_0 is a smooth complex, let $i : Z_0 \hookrightarrow X_0$ be the closed complment. Because U_0 is smooth, the complex $\mathbb{D}(j^*K_0)$ is mixed by proposition 3.2.4, hence also the complex

$$j_*\mathbb{D}(j^*K_0) = \mathbb{D}(j_!j^*K_0).$$

By induction, $\mathbb{D}(i^*K_0)$ is mixed and so also

$$\mathbb{D}(i_*i^*K_0) = i_*\mathbb{D}(i^*)K_0$$

Now using the exact triangle

$$j_!j_*K_0 \to K_0 \to i_*i^*K_0$$

the duals of left term and right term are mixed, so is the dual of the middle term. □

For the rest content, we refer to 2.12 of [1].

Definition 4.1.7. For a scheme X_0/k and ℓ-adic sheaf $\mathscr{G}_0$ on X_0, we define the **maximal weight** of $\mathscr{G}_0$ (with respect to τ)as

$$w(\mathscr{G}_0) := \sup_{x\in|X_0|} \sup_{\alpha \text{ eigenvalue of } F_x} \frac{log(|\tau(\alpha)|^2)}{logN(x)}$$

For convenience, we define $w(0) = -\infty$.

Definition 4.1.8. Notation as above. For a τ-mixed complex $K_0 \in D^b_m(X_0, \overline{\mathbb{Q}_\ell})$, we define

$$w(K_0) = \max_i(w(\mathscr{H}^i(K_0)) - i)$$

Note by Theorem 2.2.1, this is well-defined.

Definition 4.1.9. For any real number w, we define two subcategories

$$D^b_{\le w} = D^b_{\le w}(X_0) = \{K_0 \in D^b_m(X_0) \mid w(K_0) \le w\}$$

and

$$D^b_{\ge w} = D^b_{\ge w}(X_0) = \{K_0 \in D^b_m(X_0) \mid w(\mathbb{D}(K_0)) \le -w\}$$

Remark. Note that $w(K_0) \ge w$ does not imply $K_0 \in D^b_{\ge w}(X_0)$. But it is shown in Lemma 4.1.11 below, that the other direction is true. Namely $K_0 \in D^b_{\ge w}(X_0)$ imply $w(K_0) \ge w$.

Proposition 4.1.10. (5.1.15 of [3]) Let $K_0, L_0 \in D^b_m(X_0)$. Let w be an integer. Assume that $K_0 \in D^b_{\le w}$ and $L_0 \in D^b_{\ge w}$, then

1. $Ext^i(K, L)^F = 0$ for $i > 0$
2. the morphisms $Ext^i(K_0, L_0) \to Ext^i(K, L)$ is the zero map for $i > 0$

If $L \in D^b_{\ge w+1}$, then $Ext^i(K_0, L_0) = 0$

Lemma 4.1.11. Let $K_0 \in D^b_m(X_0, \overline{\mathbb{Q}_\ell})$ with $w(K_0) = w$. If $K_0 \neq 0$ is nontrivial, then the following holds

$$w(\mathbb{D}K) \geq -w(K_0)$$

In particular, if $K_0 \in D^b_{\geq w}(X_0)$ holds for some w, then $w \leq w(K_0)$.

Definition 4.1.12. A complex $K_0 \in D^b_c(X_0, \overline{\mathbb{Q}_\ell})$ is calld τ**-pure of weight** w if $w(\mathbb{D}K) = -w(K_0)$ holds, i.e if

$$K_0 \in D^b_{\leq w}(X_0) \cap D^b_{\geq w}(X_0)$$

Remark. This definition for complex does ot coincide with the notion of ℓ-adic sheaves in definition 4.1.1.

4.2 The perverse t-structure on mixed complexes

We have introduced the triangulated subcategory $D^b_m(X_0, \overline{\mathbb{Q}_\ell}) \subset D^b_c(X_0, \overline{\mathbb{Q}_\ell})$. We want to show the t-structure on $D^b_c(X_0, \overline{\mathbb{Q}_\ell})$ restricts to one on $D^b_m(X_0, \overline{\mathbb{Q}_\ell})$

Proposition 4.2.1. The perverse truncation functors ${}^p\tau_{\geq 0}$ and ${}^p\tau_{\leq 0}$ preserve the subcategory $D^b_m(X_0, \overline{\mathbb{Q}_\ell}) \subset D^b_c(X_0, \overline{\mathbb{Q}_\ell})$. Namely for $K_0 \in D^b_m(X_0)$, we have

$${}^p\tau^{\leq 0}K_0 \in D^b_m(X_0) \quad \text{and} \quad {}^p\tau^{\geq 0}K_0 \in D^b_m(X_0)$$

Proof. Then prove is by induction on $dim(X_0)$. If $dim(X_0) = 0$, the perverse t-structure is just the standard t-structure so the result holds. Now let $K_0 \in D^b_m(X_0)$ and $j : U_0 \to X_0$ be an smooth open subset such that j^*K_0 is a smooth complex.

We know the perverse truncation of j^*K_0 is mixed, since the perverse truncation of j^*K_0 is just a shift of the ordinary truncation of j^*K_0. By the induction, we know the perverse truncation of i^*K_0 is mixed. Now we claim that i^* and j^* commute with perverse truncation, i.e. that

$${}^p\tau^{\leq 0}i^*K_0 = i^{*p}\tau^{\leq 0}K_0 \tag{1}$$

$${}^p\tau^{\leq 0}j^*K_0 = j^{*p}\tau^{\leq 0}K_0 \tag{2}$$

To check ${}^p\tau^{\leq 0}K_0$ is mixed, we only need to check for $i^{*p}\tau^{\leq 0}K_0$ and $j^{*p}\tau^{\leq 0}K_0$ since mixedness is a stalk condition. It remains to prove (1) and (2).

This follows followig commutative diagrams

$$\begin{array}{ccc} {}^pD^b_c(U_0)^{\leq 0} & \longrightarrow & D^b_c(U_0) \\ \downarrow{\scriptstyle j_!} & & \downarrow{\scriptstyle j_!} \\ {}^pD^b_c(X_0)^{\leq 0} & \longrightarrow & D^b_c(X_0) \end{array} \qquad \begin{array}{ccc} {}^pD^b_c(Z_0)^{\leq 0} & \longrightarrow & D^b_c(Z_0) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ {}^pD^b_c(Z_0)^{\leq 0} & \longrightarrow & D^b_c(X_0) \end{array}$$

by the adjoint , we have the same diagram replacing all the functors with the adjoint functor in the other direction.

The proof for ${}^p\tau^{\geq 0}$ is similar, just need to notice that $\mathbb{D}$ preserve mixed. □

4.3 Weight filtration

Let X_0 be a scheme of finite type over a finite field $k = \mathbb{F}_q$. Fix an isomorphism $\tau : \overline{\mathbb{Q}_\ell} \to \mathbb{C}$. Let K_0 be a τ-mixed complex in $D^b_c(X_0)$. For any τ-mixed complex K_0, we define $w(K_0)$ as in Definition 4.1.8.

Recall that a τ-mixed complex K_0 is called τ-pure of weight w if $K_0 \in D_{\leq w}(X_0) \cap D_{\geq w}(X_0)$. By definition this is equivalent to $w(K_0) \leq w$ and $w(\mathbb{D}K_0) \leq -w$. we have following result.

Proposition 4.3.1. Let K_0 and L_0 be $\tau-$mixed perverse sheaves on X_0 Then

$$Hom_{Perv(X_0)}(K_0, L_0) = Hom_{Perv(X)}(K, L)^F$$

Lemma 4.3.2. (**Semicontinuity of Weight**)Let $j : U_0 \to X_0$ be an open immersion with closed complement $i : Z_0 \to X_0$. Let $\overline{K_0} \in Perv(X_0)$ be a τ-mixed perverse sheaf on X_0, such that

$$j^*(\overline{K_0}) = K_0, {}^pH^0(i^*(\overline{K_0})) = 0$$

Then

$$w(\overline{K_0}) \leq w(K_0).$$

In particular

$$w(j_{!*}(K_0)) \leq w(K_0).$$

Corollary 4.3.3. Any τ-mixed simple perverse sheaves K_0 on X_0 is τ-pure of weight $w = w(K_0)$.

Lemma 4.3.4. (**Subquotient**). Let $K_0 \in Perv(X_0)$ be a τ-mixed perverse sheaves. then $w(L_0) \leq w(K_0)$ holds for any perverse subquotient L_0 of K_0 in $Perv(X_0)$.

Theorem 4.3.5. (**Weight Filtration**,5.3.5 of [3]) In the abelian category $Perv(X_0)$ any τ-mixed perverse sheaf K_0 on X_0 has a canonical finite increasing τ-weight filtration $W = (K_0^{(w_i)})$

$$0 = K_0^{(-\infty)} \subset K_0^{(w_1)} \subset \cdots \subset K_0^{(w_r)} = K_0$$

such that the quotient $Gr^{(w_i)}(K_0) := K_0^{(w_i)}/K_0^{(w_{i-1})}$ are either zero or τ-pure perverse sheaves of weight w_i such that

$$w_i < w_j \quad \text{for} \quad i < j.$$

Remark. (1)If we demand all quotient $Gr^{(w_i)}(K_0)$ to be nontrivial, then the filtration is uniquely determined.

(2) Let $\phi_0 : K_0 \to L_0$ be a homomorphism of a τ-mixed perverse sheaf K_0 into a τ-mixed perverse sheaf L_0, then there exist weight filtration $K_0^{(w_i)}, L_0^{(w_i)}$ on K, L such that ϕ_0 maps $K_0^{(w_i)}$ to $L_0^{(w_i)}$.

5 Decomposition Theorem

5.1 Deligne's Theorem

Theorem 5.1.1. (5.3.7 of [3]) Let K_0 be a τ-mixed perverse sheaf in $Perv(X_0)$. Then

$$w(K_0) \leq w$$

holds iff for every irreducible dimension d subscheme Z_0 of X_0, there is an open dense subscheme $j : U_0 \to Z_0$ such that

$$w(\mathscr{H}^{-d} K_0 \mid_{U_0}) \leq w - d.$$

Proof. ($\Rightarrow$) is trivial.

Consider the nontrivial direction ($\Leftarrow$). By the weight filtration Theorem 4.3.5, we have a short exact sequence

$$0 \to A_0 \to B_0 \to Q_0 \to 0$$

in $Perv(X_0)$ with Q_0 simple of maximal weight $w(Q_0) = w(B_0)$. Then $Q_0 = i_* j_{!*} C_0$ for a lisse sheaf C_0 on U_0 with $U_0 \xrightarrow{j} Y_0 \xrightarrow{i} X_0$ and Y_0 is irreducible. We need to estimate $w(Q_0)$. Since $A_0 \in {}^pD^{\leq 0}(X_0)$, we have $\dim supp\mathscr{H}^{-d+1}(A_0) \leq d-1$. On the open dense subset $V_0 = U_0 \backslash supp\mathscr{H}^{-d+1}(A_0)$ of U_0, the map $\mathscr{H}^{-d}(B_0) \to \mathscr{H}^{-d}(Q_0)$ is surjective. By shrinking V_0, the surjectivity and the assumption $w(\mathscr{H}^{-d} Q_0 \mid V_0) \leq w - d$. Hence by Lemma 4.3.2, $w(B_0) = w(Q_0) \leq w$. □

Lemma 5.1.2. $K \in D_c^b(X)$ implies $K \in {}^pD_c^b(X)^{\leq n} \cap {}^pD_c^b(X)^{\geq -m}$ for sufficiently large $m, n \gg 0$. Namely for $m, n \gg 0$, $\tau^{\geq n+1} K = 0$ and $\tau^{\leq -m-1} K = 0$.

Proof. Just by definition. □

Corollary 5.1.3. (5.4.1 of [3]) Let K_0 be τ-mixed in $D_c^b(X_0, \overline{\mathbb{Q}_\ell})$, then

$$w(K_0) \leq w \iff w({}^pH^i(K_0)) \leq w + i, \ \ \forall i \in \mathbb{Z}$$

Proof. 1. ($\Longleftarrow$). Note that if $K \to L \to M$ is an exact triangle with $w(K) \leq w$ and $w(M) \leq w$, then $w(L) \in w$. Use the exact sequence ${}^p\tau^{\leq i-1}K_0 \to {}^p\tau^{\leq i}K_0 \to {}^pH^i(K_0)[-i] \to \cdots$, by Lemma 5.1.2, there $m, n \gg 0$ such that ${}^p\tau^{\leq -m-1}K_0 = 0$ and ${}^p\tau^{\leq n}K_0 = K_0$, so by induction, $w({}^p\tau^{\leq i}K_0) \leq w$ for $i \geq -m$ since $w({}^pH^i(K_0)[-i]) \leq w$. Clearly $n > -m$, so $w(K_0) \leq w$.

2. ($\Longrightarrow$). Assume $w(K_0) \leq w$ and $w({}^pH^i(K_0)) \leq w + i$ for all $i > l$. Then $w({}^p\tau^{\geq l+1}K_0) \leq w$, using the $\Longleftarrow$ direction. We will prove $w({}^pH^l(K_0)) \leq w + l$, which inductively will imply

$$w({}^pH^i(K_0)) \leq w + i$$

for all i.

For simplicity we may assume $l = 0$. For some integer $0 \leq d \leq dim(X_0)$. Using the long exact sequence for the exact triangle $({}^p\tau^{\leq 0}K_0, K_0, {}^p\tau^{\geq 1}K_0)$, the weight estimates $w(K_0), w({}^p\tau^{\geq 1}K_0) \leq w$ imply $w({}^p\tau^{\leq 0}K_0) \leq w$. So $w(\mathscr{H}^{-d}({}^p\tau^{\leq 0}K_0)) \leq w-d$. Since $dim\, supp \mathscr{H}^{-d+1}({}^p\tau^{\geq 1}K_0) < d$, this weight estimate and the exact sequence

$$\mathscr{H}^{-d}({}^p\tau^{\leq 0}K_0) \to \mathscr{H}^{-d}({}^pH^0K_0) \to \mathscr{H}^{-d+1}({}^p\tau^{\geq 1}K_0)$$

So take any irreducible subscheme $Y_0 \subset X_0$ of dimension d,

$$U_0 = Y_0 \cap (X_0 \setminus supp\ \mathscr{H}^{-d+1}({}^p\tau^{\geq 1}K_0))$$

is an open subset of X_0 such that $\mathscr{H}^{-d+1}({}^p\tau^{\geq 1}K_0)\ |_{U_0} = 0$. so

$$\mathscr{H}^{-d}({}^p\tau^{\leq 0}K_0) \cong \mathscr{H}^{-d}({}^pH^0K_0)$$

on U_0. Now ${}^pH^0K_0$ satisfies the condition in Theorem 5.1.1, so ${}^pH^0K_0 \leq w$. We are done.

□

Corollary 5.1.4. A τ-mixed complex $K_0 \in D_c^b(X_0, \overline{\mathbb{Q}_\ell})$ is τ-pure of weight w iff all ${}^pH^i(K_0)$ are τ-pure of weight $w + i$.

5.2 Decomposition theorem

Theorem 5.2.1. (Decomposition theorem, 5.4.5 of [3]) Let $K_0 \in D_m^b(X_0, \overline{\mathbb{Q}_\ell})$ be τ-pure of weight w. Consider the base change K of K_0 to the algebraic closure $K \in D_c^b(X, \overline{\mathbb{Q}_\ell})$ where $X = X_0 \times_k \overline{k}$. Then

$$K \cong \bigoplus_i H^i(K)[-i]$$

. Here the number of i is finite, namely it is a finite direct sum.

Proof. As proof in corollary 5.1.3 take an integer i such that ${}^{p}\tau^{\leq i-1}K_0$ is pure of weight w.

$$ {}^{p}\tau^{\leq i-1}K_0 \longrightarrow {}^{p}\tau^{\leq i}K_0 \longrightarrow {}^{p}H^i(K_0)[-i] \longrightarrow \cdots $$

By corollary 5.1.4 $w({}^{p}H^i(K_0)[-i]) = w$, so we can apply proposition 4.1.10 get

$$ Ext^1({}^{p}H^i(K_0)[-i], {}^{p}\tau^{\leq i-1}K_0) = 0 $$

So ${}^{p}\tau^{\leq i}K_0 = {}^{p}\tau^{\leq i-1}K_0 \bigoplus {}^{p}H^i(K_0)[-i]$, which completes the proof by induction since ${}^{p}\tau^{\leq m}K_0 = 0$ for $m \ll 0$. □

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