

**Capital Normal University**

Department of Mathematics

# **Perverse Sheaves and the Decomposition Theorem**

Master Thesis

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## 1 Introduction

Perverse sheaves was introduced in the work [3] of Joseph Bernstein, Alexander Beilinson, and Pierre Deligne (1982) as a consequence of the Riemann-Hilbert correspondence, which establishes a connection between the derived categories of regular holonomic  $D$ -modules and constructible sheaves.

Specifically, if  $X$  is a smooth complex algebraic variety, the Riemann-Hilbert correspondence

says

$$D_{reg,hol}^b(\mathcal{D}_X) \cong D_{cons}^b(X^{an}, \mathbb{C})$$

The left is the subcategory of the bounded category of  $\mathcal{D}_X$ -module, such that the cohomology is regular and holonomic. The right is the derived category of bounded constructible complex.

Now notice that the left hand side has an abelian subcategory  $Mod_{reg,holo}(\mathcal{D}_X)$ . It should correspond to an abelian subcategory on the right side. This is just perverse sheaves which is the core of a  $t$ -structure on a certain triangulated category.

This article is a summary of the main results in [3].

**Organization.** In §2 we review some preliminary on  $t$ -structures and their gluing, and introduce the important category  $D_c^b(X, \overline{\mathbb{Q}_\ell})$ .

In §3 we introduce the perverse  $t$ -structure to define perverse sheaves, then talk some functors relative to it. Finally give some results about the intermediate extension.

In §4 we give the definition of  $\tau$ -mixed complex in  $D_c^b(X, \overline{\mathbb{Q}_\ell})$  which is the extension of ordinary definition for  $\ell$ -adic sheaves, and prove the perverse  $t$ -structure preserves mixed sheaves, namely it induces a  $t$ -structure on the subcategory  $D_m^b(X, \overline{\mathbb{Q}_\ell})$  of  $D_c^b(X, \overline{\mathbb{Q}_\ell})$ . Finally state the important weight filtration theorem of perverse sheaves.

In §5 we prove some corollaries by weight filtration. Finally prove the decomposition theorem in [3].

## 2 Triangulated category and $t$ -structure

### 2.1 $t$ -structure

**Definition 2.1.1.** A  $t$ -structure in a triangulated category  $\mathcal{D}$  consists two strictly full subcategories  $\mathcal{D}^{\leq 0}$  and  $\mathcal{D}^{\geq 0}$  of  $\mathcal{D}$ , such that with the definitions  $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$  and  $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$  we have

1.  $Hom(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$ .
2.  $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$  and  $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$ .
3. For every object  $E$  in  $\mathcal{D}$ , there exists a distinguished triangle  $(A, E, B)$  with  $A \in \mathcal{D}^{\leq 0}$  and  $B \in \mathcal{D}^{\geq 1}$ .

Set  $\mathcal{D}^\heartsuit = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ . This is called the core of the  $t$ -structure.

**Example.**  $\mathcal{A}$  is an abelian category,  $\mathcal{D} = D(\mathcal{A})$ . Then

$$\mathcal{D}^{\leq 0} = \{K \in \mathcal{D} \mid H^i(K) = 0, \forall i > 0\}$$

$$\mathcal{D}^{\geq 0} = \{K \in \mathcal{D} \mid H^i(K) = 0, \forall i < 0\}$$

$(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a  $t$ -structure on  $D(\mathcal{A})$ .

*Proof.*  $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$  and  $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$  are trivial.

As for  $\text{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$ , assume  $K \in \mathcal{D}^{\leq 0}$  and  $L \in \mathcal{D}^{\geq 1}$ . After replacing  $K$  and  $L$  by isomorphic objects in  $D(\mathcal{A})$ , we can assume that  $K^m = 0$  for  $m > 0$  and  $L^m = 0$  for  $m \leq 0$ . Let  $u : K \rightarrow L$  be a morphism in  $D(\mathcal{A})$ , then we have morphisms  $f : K \rightarrow Z$  and  $s : L \rightarrow Z$  in  $K(\mathcal{A})$  such that  $s$  is a quasi-isomorphism and  $u = s^{-1} \circ f$  in  $D(\mathcal{A})$ . Since  $L^m = 0$  for  $m \leq 0$ , the morphism  $s' : \tau^{\geq 1} L = L \rightarrow Z' = \tau^{\geq 1} Z$  is also a quasi-isomorphism. we have following commutative diagram:

$$\begin{array}{ccccc} & & Z & & \\ & f \nearrow & \downarrow & \nwarrow s & \\ K & \xrightarrow{\tau^{\geq 1} f} & Z' & \xleftarrow{s'} & L \\ & \searrow \tau^{\geq 1} f & \uparrow id & \swarrow s' & \\ & & Z' & & \end{array}$$

So  $s^{-1} \circ f = s'^{-1} \circ \tau^{\geq n+1} f$  as morphism in  $D(\mathcal{A})$ . But  $K^m = 0$  for  $m \geq 1$  implies that  $\tau^{\geq n+1} f = 0$ , so  $u = 0$ .

Finally, if  $K \in D(\mathcal{A})$ , we have that

$$\tau^{\leq 0} K \rightarrow K \rightarrow \tau^{\geq 1} K \rightarrow \tau^{\leq 0} K[1]$$

is an exact triangle. This is from the short exact sequence of chain complexes

$$0 \rightarrow \tau^{\leq 0} K \rightarrow K \rightarrow \tau^{\geq 1} K \rightarrow 0$$

□

**Proposition 2.1.2.** Let  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a  $t$ -structure on a triangle category  $\mathcal{D}$ . Then

1. The inclusion  $\mathcal{D}^{\leq n} \rightarrow \mathcal{D}$  has a right adjoint  $\tau^{\leq n} : \mathcal{D} \rightarrow \mathcal{D}^{\leq n}$ . In particular, there is a canonical map  $\tau^{\leq n} K \rightarrow K$ , which is universal in the sense that if there is something in  $\mathcal{D}^{\leq n}$  mapping to  $K$ , then this map factors uniquely through  $\tau^{\leq n} K$ .
2. The inclusion  $\mathcal{D}^{\geq n} \rightarrow \mathcal{D}$  has a left adjoint  $\tau^{\geq n} : \mathcal{D} \rightarrow \mathcal{D}^{\geq n}$ , and there is a canonical map  $K \rightarrow \tau^{\geq n} K$ .

3. For each object  $X \in \mathcal{D}$ , there is a unique morphism  $\delta : \tau^{\geq n+1} K \rightarrow \tau^{\leq n} K[1]$  which makes the following sequence into an exact triangle:

$$\tau^{\leq n} K \rightarrow K \rightarrow \tau^{\geq n+1} K \xrightarrow{\delta} \tau^{\leq n} K[1]$$

In fact,  $\delta$  is functorial in  $K$ .

*Proof.* 1. By shifting, we can assume  $n = 0$ . Fix  $K \in \mathcal{D}$ . Axiom 3 of  $t$ -structure gives the exact triangle

$$K_0 \rightarrow K \rightarrow K^1 \xrightarrow{\delta} K_0[1]$$

such that  $K_0 \in \mathcal{D}^{\leq 0}$  and  $K^1 \in \mathcal{D}^{\geq 1}$ . Now fix some  $Y \in \mathcal{D}^{\leq 0}$ , then apply  $\text{Hom}(Y, -)$  to get :

$$\begin{array}{ccccccc} \text{Hom}(Y, K^1[-1]) & \longrightarrow & \text{Hom}(Y, K_0) & \xrightarrow{\cong} & \text{Hom}(Y, K) & \longrightarrow & \text{Hom}(Y, K^1) \\ \parallel & & & & & & \parallel \\ 0 & & & & & & 0 \end{array}$$

the left 0 is because  $\text{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 2}) = 0$  and right is just  $\text{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$ , so set  $\tau^{\leq 0} K = K_0$ .

2. Similarly:  $\tau^{\geq 1} K = K^1$ .

3. Reduce to  $n = 0$ ,

$$K_0 \rightarrow K \rightarrow K^1 \xrightarrow{\delta} K_0[1]$$

we need to prove  $\delta$  is unique. Because  $\text{Hom}(K_0[1], K^1) = 0$ , this follows by following lemma.

□

**Lemma 2.1.3.** Suppose  $\mathcal{D}$  is a triangulated category, and for  $i \in \{1, 2\}$ , the sequences

$$K \xrightarrow{k} L \xrightarrow{m} M \xrightarrow{d_i} K[1]$$

are two exact triangles. Assume  $\text{Hom}(K[1], M) = 0$ , then  $d_1 = d_2$ .

*Proof.* By the Cone axiom (TR4), we get a commutative diagram

$$\begin{array}{ccccccc} K & \xrightarrow{k} & L & \xrightarrow{l} & M & \xrightarrow{d_1} & K[1] \\ \downarrow id & & \downarrow id & & \downarrow m & & \downarrow id \\ K & \xrightarrow{k} & L & \xrightarrow{l} & M & \xrightarrow{d_2} & K[1] \end{array}$$

so  $ml = l$ , namely  $(id_M - m)l = 0$ .  $id_M - m \in Hom(M, M)$ , applying the long exact sequence for  $Hom(-, M)$  to the first row, we have the followig exact sequence

$$Hom(K[1], M) \xrightarrow{d_1^*} Hom(M, M) \xrightarrow{l^*} Hom(L, M) \xrightarrow{k^*} Hom(K, M)$$

there exists  $e : K[1] \rightarrow M$  such that  $e \circ d_1 = id_M - m$ , but  $e = 0$  as  $Hom(K[1], M) = 0$ , so  $id_M = m$  which implies  $d_1 = d_2$ .

One wants to prove that the core has nice properties, but first, we need some extra facts about the truncation functors:

**Lemma 2.1.4.** 1.  $\tau^{\leq n}(K[m]) \cong (\tau^{\leq n+m}(K))[m]$

$$2. \tau^{\geq n}(K[m]) \cong (\tau^{\geq n+m}(K))[m]$$

$$3. K \in \mathcal{D}^{\leq n} \iff \tau^{\leq n}(K) \cong K \iff \tau^{\geq n+1}(K) = 0$$

$$4. K \in \mathcal{D}^{\geq n} \iff \tau^{\geq n}(K) \cong K \iff \tau^{\leq n-1}(K) = 0$$

$$5. \text{ For } a < b \in \mathbb{Z}, \tau^{\leq b} \circ \tau^{\leq a} = \tau^{\leq a} = \tau^{\leq a} \circ \tau^{\leq b}, \tau^{\geq b} \circ \tau^{\geq a} = \tau^{\geq b} = \tau^{\geq a} \circ \tau^{\geq b} \text{ and } \tau^{\leq a} \circ \tau^{\geq b} = 0 = \tau^{\geq b} \circ \tau^{\leq a}$$

6. For  $a, b \in \mathbb{Z}$ , there is a canonical isomorphism

$$\tau^{\leq a} \circ \tau^{\geq b} \cong \tau^{\geq b} \circ \tau^{\leq a}$$

**Definition 2.1.5.** Define  $H^0 : \mathcal{D} \rightarrow \mathcal{D}^\heartsuit$ , where  $K \mapsto (\tau^{\leq 0} \circ \tau^{\geq 0})(K) \cong (\tau^{\geq 0} \circ \tau^{\leq 0})(K)$ , and also  $H^n(K) = H^0(K[n])$

**Lemma 2.1.6.** For all  $K \in \mathcal{D}$ , there is an exact triangle

$$H^n(K)[-n] \rightarrow \tau^{\geq n}(K) \rightarrow \tau^{\geq n+1}(K) \rightarrow H^n(K)[-n+1]$$

*Proof.*

$$H^n(K)[-n] = H^0(K[n])[-n] = (\tau^{\leq 0} \circ \tau^{\geq 0})(K[n]) = (\tau^{\leq 0}(\tau^{\geq n}K)[n]) = \tau^{\leq n}(\tau^{\geq n}K)$$

Using standard triangle for  $\tau^{\geq n}(K)$ :

$$\begin{array}{ccccccc} \tau^{\leq n}(\tau^{\geq n}(K)) & \longrightarrow & \tau^{\geq n}(K) & \longrightarrow & \tau^{\geq n+1}(\tau^{\geq n}(K)) & \longrightarrow & \tau^{\leq n}(\tau^{\geq n}(K))[1] \\ \parallel & & & & \parallel & & \\ H^n(K)[-n] & & & & \tau^{\geq n+1}(K) & & \end{array}$$

**Lemma 2.1.7.** If  $K \rightarrow L \rightarrow M \rightarrow K[1]$  is exact, and  $K, M \in \mathcal{D}^\vee$ , then  $L \in \mathcal{D}^\vee$ .

*Proof.* By 3 of lemma 2.1.4, if  $K, M \in \mathcal{D}^{\leq 0}$ , then  $H^i(K) = H^i(M) = 0$  for  $i \geq 1$ , so by the long exact sequence of cohomology,  $H^i(L) = 0$  for  $i \geq 1$ . Namely  $L \in \mathcal{D}^{\leq 0}$ . Duality, if  $K, M \in \mathcal{D}^{\geq 0}$ , then  $L \in \mathcal{D}^{\geq 0}$ . Thus  $L \in \mathcal{D}^\vee$ .

Now I can state the following nice result, this is kind of a miracle.

**Theorem 2.1.8.**  $\mathcal{D}^\vee$  is an abelian category.

*Proof.*  $K, L \in \mathcal{D}^\vee$  implies  $K \oplus L \in \mathcal{D}^\vee$  by the last lemma. Zero object is 0, and  $\text{Hom}(K, L)$  is a abelian group. So  $\mathcal{D}^\vee$  is additive.

For a morphism  $f : K \rightarrow L$  in  $\mathcal{D}^\vee$ . We want to find the kernel and cokernel of this morphism. Extend to an exact triangle

$$K \rightarrow L \rightarrow M \rightarrow K[1]$$

Clearly  $M \in \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -1}$ , the prove is similar to last lemma.

**Step1.** (Existence of kernel and cokernel ).

*Claim:*  $H^0(M) = \tau^{\geq 0}(M) := \text{Coker}(f)$ ,  $H^{-1}(M) = H^0(M[-1]) = \tau^{\leq 0}(M[-1]) := \text{Ker}(f)$

Fix  $W \in \mathcal{D}^\vee$ , we want to check the universal property of  $\text{Coker}(f)$ . Apply  $\text{Hom}(-, W)$  to the exact triangle above:

$$\text{Hom}(K[1], W) \rightarrow \text{Hom}(M, W) \rightarrow \text{Hom}(L, W) \rightarrow \text{Hom}(K, W)$$

Now  $K[1] \in \mathcal{D}^{\leq -1}$ , so  $\text{Hom}(K[1], W) = 0$  and by adjoint of truncation we have  $\text{Hom}(M, W) = \text{Hom}(\tau^{\geq 0}(M), W)$ . So we get a short exact sequence:

$$0 \rightarrow \text{Hom}(\tau^{\geq 0}(M), W) \rightarrow \text{Hom}(L, W) \rightarrow \text{Hom}(K, W)$$

This implies  $L \rightarrow \tau^{\geq 0}(M) = H^0(M)$  is a cokernel of  $f$ . Dually,

$$\tau^{\leq 0}(M[-1]) \rightarrow M[-1] \rightarrow K$$

is a kernel of  $f$ .

**Step 2.**(Factorization Property.) Namely we need to prove the morphism  $\bar{f}$  is isomorphism for any  $f$ .

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ \downarrow & & \downarrow \\ \text{coim}(f) & \xrightarrow{\bar{f}} & \text{im}(f) \end{array}$$

Here  $\text{coim}(f) := \text{coker}(\ker(f))$  and  $\text{im}(f) := \ker(\text{coker}(f))$ .

**We will find an object  $Z \in \mathcal{D}^\heartsuit$  such that  $Z \cong \text{coim}(f)$  and  $Z \cong \text{im}(f)$ , then the result follows.** Specifically, use octaeder axiom, get following commutative diagram:

$$\begin{array}{ccccccc}
 & & \ker(f)[-1] & \xrightarrow{\text{equal}} & \ker(f)[-1] & & \\
 & & \uparrow & & \uparrow & & \\
 L[-1] & \dashrightarrow & \text{coker}(f)[-1] & \dashrightarrow & Z & \dashrightarrow & L \\
 \parallel & & \uparrow & & \uparrow & & \parallel \\
 L[-1] & \longrightarrow & M[-1] & \longrightarrow & K & \longrightarrow & L \\
 & & \uparrow & & \uparrow & & \\
 & & \ker(f) & \xrightarrow{\text{equal}} & \ker(f) & & 
 \end{array}$$

The first column exact triangle is just

$$M \rightarrow \tau^{\geq 0} M \cong \text{coker}(f) \rightarrow \tau^{\leq -1} M \cong \tau^{\leq 0}(M[-1]) \cong \ker(f)$$

Now, note the first row,  $L \in \mathcal{D}^{\geq 0}$  and  $\text{Coker}(f)[-1] \in \mathcal{D}^\heartsuit[-1] \in \mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$ , by extension lemma, we get  $Z \in \mathcal{D}^{\geq 0}$ . Similarly,  $Z \in \mathcal{D}^{\leq 0}$ . Thus  $Z$  is an object of  $\mathcal{D}^\heartsuit$

$$Z \in \mathcal{D}^\heartsuit$$

The first isomorphism  $Z \cong \text{coker}(\ker(f))$  comes from the distinguished triangle  $(\ker(f), K, Z)$ , this implies  $\text{coker}(\ker(f) \rightarrow K) = \tau^{\geq 0}(Z) = Z$ .

The second isomorphism  $Z \cong \ker(\text{coker}(f))$  comes from the distinguished triangle  $(L, \text{coker}(f), Z[1])$ , this implies  $\ker(\text{coker}(f)) = \tau^{\leq 0} Z[1] = \tau^{\leq 0} Z = Z$ .  $\square$

**Theorem 2.1.9.** The functor  $H^0 : \mathcal{D} \rightarrow \mathcal{D}^\heartsuit$  is a cohomological functor.

*Proof.* Fix an exact triangle  $K \rightarrow L \rightarrow M \rightarrow K[1]$  in  $\mathcal{D}$ . We want to show that  $H^0(K) \rightarrow H^0(L) \rightarrow H^0(M)$  is exact in  $\mathcal{D}^\heartsuit$ . This require some steps

**Step 1.** Assume  $K, L, M \in \mathcal{D}^{\geq 0}$ .

Claim :  $0 \rightarrow H^0(K) \rightarrow H^0(L) \rightarrow H^0(M)$  is exact.

*Proof.* Let  $A \in \mathcal{D}^\heartsuit$ . Notice that  $\text{Hom}(A, K) = \text{Hom}(A, H^0(K))$ , since  $A \in \mathcal{D}^{\leq 0}$  and  $K \in \mathcal{D}^{\geq 0}$ . The same applies to  $L$  and  $M$ . Now applying  $\text{Hom}(A, -)$ , we get the long exact sequence

$$\begin{array}{ccccccc}
 \text{Hom}(A, M[-1]) & \longrightarrow & \text{Hom}(A, K) & \longrightarrow & \text{Hom}(A, L) & \longrightarrow & \text{Hom}(A, M) \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 0 & \longrightarrow & \text{Hom}(A, H^0(K)) & \longrightarrow & \text{Hom}(A, H^0(L)) & \longrightarrow & \text{Hom}(A, H^0(M))
 \end{array}$$



Here  $\text{Hom}(A, M[-1]) = 0$  since  $A \in \mathcal{D}^{\leq 0}$ ,  $M[-1] \in \mathcal{D}^{\geq 1}$ . By Yoneda lemma, we are done.

**Step 2.** Assume  $M \in \mathcal{D}^{\geq 0}$ .

Claim :  $0 \rightarrow H^0(K) \rightarrow H^0(L) \rightarrow H^0(M)$  is exact.

*Proof.* We have an exact triangle  $K \rightarrow L \rightarrow M \rightarrow K[1]$ . Apply the functor  $\tau^{\leq -1}$  to the triangle, we get  $\tau^{\leq -1}(K) = \tau^{\leq -1}(L)$ . By octahedral axiom, we get following diagram

$$\begin{array}{ccccc}
 \tau^{\geq 0}(K) & \dashrightarrow & \tau^{\geq 0}(L) & \dashrightarrow & M \\
 \uparrow & & \uparrow & & \uparrow \\
 K & \longrightarrow & L & \longrightarrow & M \\
 \uparrow & & \uparrow & & \\
 \tau^{\leq -1}(K) & \longrightarrow & \tau^{\leq -1}L & & 
 \end{array}$$

Now apply step 1 to the top row.

**Step 3.** Assume  $K \in \mathcal{D}^{\leq 0}$ .

Claim :  $H^0(K) \rightarrow H^0(L) \rightarrow H^0(M) \rightarrow 0$  is exact.

*Proof.* this is the opposite version of step 2. The proof is similar, we omit it.

**Step 4.** In general case, use octaeder axiom, we get following commutative diagram

$$\begin{array}{ccccc}
 \tau^{\geq 1}K & \dashrightarrow & W & \dashrightarrow & M \\
 \uparrow & & \uparrow & & \parallel \\
 K & \longrightarrow & L & \longrightarrow & M \\
 \uparrow & & \uparrow & & \\
 \tau^{\leq 0}K & \xrightarrow{\text{equal}} & \tau^{\leq 0}K & & 
 \end{array}$$

So we have the exact triangle

$$W \rightarrow M \rightarrow (\tau^{\geq 1}K)[1]$$

by step 2, this gives the exact sequence

$$0 \rightarrow H^0(W) \rightarrow H^0(M) \rightarrow H^0(\tau^{\geq 1}(K)[1])$$

By step 3, the exact triangle  $\tau^{\leq 0}K \rightarrow L \rightarrow W$  gives the exact sequece

$$H^0(K) = H^0(\tau^{\leq 0}(K)) \rightarrow H^0(L) \rightarrow H^0(W) \rightarrow 0$$

Together the two exact sequences, we have that  $H^0(K) \rightarrow H^0(L) \rightarrow H^0(M)$  is exact.  $\square$

## 2.2 The category $D_c^b(X_0, \overline{\mathbb{Q}_\ell})$

For the basic knowledge of  $l$ -adic sheaves, please refer to chapter 10 of [4].

Fix some notation. Let  $\ell$  be a prime number,  $\mathbb{Z}_\ell \simeq \varprojlim_n \mathbb{Z}/\ell^n \mathbb{Z}$ ,  $\mathbb{Q}_\ell$  its fraction field,  $E$  a finite extension of  $\mathbb{Q}_\ell$ , and  $R$  is the integral closure of  $E$  in  $\mathbb{Z}_\ell$ . Fix a uniformizer  $\lambda$  of  $R$ , denote  $R/(\lambda^n)$  by  $R_n$ .

Let  $D(X, R_n)$  be the derived category of sheaf of  $R_n$ -module on  $X$ . Denote by  $D_{ctf}^b(X, R_n)$  the full subcategory of  $D^b(X, R_n)$  consisting of objects  $\mathcal{F}^\bullet$  with finite Tor-dimension, and such that  $\mathcal{H}^i(\mathcal{F}^\bullet)$  are constructible for all  $i$ .

Recall that we say  $\mathcal{F}^\bullet \in D^b(X, R_n)$  has finite Tor-dimension if there is an integer  $n$  such that  $\text{Tor}_i(\mathcal{F}^\bullet, M) = 0$  for any  $i > n$  and any constant sheaf of  $R_n$ -module  $M$ . We now construct the category  $D_c^b(X, R)$

$$D_c^b(X, R) = \varprojlim_n D_{ctf}^b(X, R_n)$$

**Object.** First, an object of  $D_c^b(X, R)$  is a collection

$$K = K^\bullet = (K_n^\bullet)_{n \geq 1}$$

of complexes  $K_n^\bullet$  in  $D_{ctf}^b(X, R_n)$  together with quasi-isomorphisms

$$\phi_{n+1} : K_{n+1}^\bullet \otimes_{R_{n+1}}^L R_n \cong K_n^\bullet$$

in the categories  $D_c^b(X, R_n)$ . The  $i$ -th cohomology sheaf of  $K^\bullet$  is defined by  $\mathcal{H}^i(K^\bullet) = (\mathcal{H}^i(K_n^\bullet))_{n \geq 1}$ .

**Morphism.** For two objects of  $D_c^b(X, R)$  represented by projective systems  $K^\bullet = (K_n^\bullet)_{n \geq 1}$  and  $L^\bullet = (L_n^\bullet)_{n \geq 1}$  we put

$$\text{Hom}_{D_c^b(X, R)}(K^\bullet, L^\bullet) = \varprojlim_n \text{Hom}_{D_c^b(X, R_n)}(K_n^\bullet, L_n^\bullet)$$

In other words, a homomorphism  $\psi : K^\bullet \rightarrow L^\bullet$  in  $\text{Hom}_{D_c^b(X, R)}(K^\bullet, L^\bullet)$  is a family  $\psi = (\psi_n)_{n \geq 1}$  of morphisms  $\psi_n : K_n^\bullet \rightarrow L_n^\bullet$  in the derived categories  $D_c^b(X, R_n)$  such that the following diagrams for  $n \geq 1$  commute

$$\begin{array}{ccc} \psi_{n+1} \otimes_{R_{n+1}}^L R_n : & K_{n+1}^\bullet \otimes_{R_{n+1}}^L R_n & \longrightarrow L_{n+1}^\bullet \otimes_{R_{n+1}}^L R_n \\ & \downarrow \cong \psi_{r+1}^K & \downarrow \cong \psi_{r+1}^L \\ \psi_n : & K_n^\bullet & \longrightarrow L_n^\bullet \end{array}$$

In order to work with this definition of  $D_c^b(X, R)$ , we give a remark:

**Remark :** By Chapter 6 Proposition 6.4.6 of [4], any object in  $D_{ctf}^b(X, R_n)$  is isomorphic to a

bounded complex of constructible flat sheaves of  $R_n$ -module. So we can suppose that all complexes  $K_n^\bullet$  are bounded constructible flat complexes. Flatness implies

$$K_{n+1}^\bullet \otimes_{R_{n+1}}^L R_n = K_{n+1}^\bullet \otimes_{R_{n+1}} R_n$$

We state an important fact without proof.

**Theorem 2.2.1.** ([1] Chapter 2 Lemma 5.5) Let  $K^\bullet = (K_n^\bullet)_{n \geq 1}$  be a object of  $D_c^b(X, R)$ . Then its cohomology

$$\mathcal{H}^i(K^\bullet) = (\mathcal{H}^i(K_n^\bullet))_{n \geq 1}$$

are A-R  $\lambda$ -adic sheaves and  $\mathcal{H}^i(K^\bullet) = 0$  for  $i$  sufficient large or small.

Recall a  $\lambda$ -adic sheaf on a noetherian scheme has an open dense subset such that the restriction of this sheaf to the open set is lisse.

**Corollary 2.2.2.**  $\mathcal{H}^i(K^\bullet)$  is a lisse  $\lambda$ -adic sheaf on an open dense subscheme  $U \subset X$ .

**Definition 2.2.3.** (The **standard t-structure**) We define two subcategories on  $D_c^b(X, R)$ .

$$D_c^b(X, R)^{\leq 0} = \{K^\bullet \in D_c^b(X, R) \mid \mathcal{H}^i(K^\bullet) = 0 \forall i \geq 1\}$$

$$D_c^b(X, R)^{\geq 0} = \{K^\bullet \in D_c^b(X, R) \mid \mathcal{H}^i(K^\bullet) = 0 \forall i \leq -1\}$$

**Theorem 2.2.4.**  $D_c^b(X, R)^{\leq 0}$  and  $D_c^b(X, R)^{\geq 0}$  define a  $t$ -structure on  $D(X, R)$ . The core of this **standard**  $t$ -structure is the full subcategory of  $D_c^b(X, R)$

$$Core(standard) = \{K^\bullet \in D_c^b(X, R) \mid \mathcal{H}^i(K^\bullet) = 0, \text{ for } i \neq 0\}$$

The functor

$$\begin{aligned} Core(standard) &\longrightarrow \{\lambda\text{-adic sheaves}\} \\ K^\bullet &\longrightarrow \mathcal{H}^0(K^\bullet) \end{aligned}$$

defines a equivalence of categories between the core of the standard  $t$ -structure and the abelian category of  $\lambda$ -adic sheaves on  $X$ .

The category

$$D_c^b(X, E)$$

is deduced from  $D_c^b(X, R)$  by "localization" in the sense of derived category, the category  $D_c^b(X, \overline{\mathbb{Q}_\ell})$  is defined by

$$D_c^b(X, \overline{\mathbb{Q}_\ell}) = \varinjlim_{E \subset \overline{\mathbb{Q}_\ell}} D_c^b(X, E)$$

(See [4],page 560.)

**Remark:** Notice that above theorem and standard  $t$ -structure can extend to  $D_c^b(X, \overline{\mathbb{Q}_\ell})$ .

## 2.3 glueing of $t$ -structures

From now, for a scheme  $X$ , we denote  $D_c^b(X, \overline{\mathbb{Q}_\ell})$  by  $D_c^b(X)$ . We now talk the glueing of  $t$ -structure. Specifically, given a scheme  $X$  and an open subscheme  $U$

$$j : U \rightarrow X$$

Let

$$i : Z \rightarrow X$$

be the closed complement of  $U$  in  $X$ .

Let  $T(U)$  be a full triangulated subcategory of  $D_c^b(U)$  and  $T(Z)$  be a full triangulated subcategory of  $D_c^b(Z)$ . Suppose we have  $t$ -structures on the categories  $T(U)$  and  $T(Z)$ , which are denoted by  $(T^{\leq 0}(U), T^{\geq 0}(U))$  and  $(T^{\leq 0}(Z), T^{\geq 0}(Z))$ , then we can glue these  $t$ -structures together to obtain a new  $t$ -structure on the full subcategory  $T(X, U)$  of  $D_c^b(X)$ , defined by

$$T(X, U) = \{K \in D_c^b(X) \mid j^*K \in T(U), i^*K \in T(Z), i^!K \in T(Z)\}$$

**Definition 2.3.1.**

$$T^{\leq 0}(X, U) := \{K \in T(X, U) \mid j^*K \in T^{\leq 0}(U), i^*K \in T^{\leq 0}(Z)\}$$

$$T^{\geq 0}(X, U) := \{K \in T(X, U) \mid j^*K \in T^{\geq 0}(U), i^!K \in T^{\geq 0}(Z)\}$$

**Proposition 2.3.2.** Notation is as above,  $(T^{\leq 0}(X, U), T^{\geq 0}(X, U))$  is a  $t$ -structure on  $T(X, U)$ .

*Proof.* First, we need to prove orthogonality:  $\text{Hom}(T^{\leq 0}(X, U), T^{\geq 1}(X, U)) = 0$ . Taking  $K \in T^{\leq 0}(X, U)$  and  $L \in T^{\geq 1}(X, U)$ , then using the exact triangle  $j_!j^*K \rightarrow K \rightarrow i_*i^*K$  and applying  $\text{Hom}(-, L)$ , we have

$$\begin{array}{ccccc} \text{Hom}(i_*i^*K, L) & \longrightarrow & \text{Hom}(K, L) & \longrightarrow & \text{Hom}(j_!j^*K, L) \\ \cong \downarrow & & & & \downarrow \\ 0 & = & \text{Hom}(i^*K, i^!L) & & \text{Hom}(j^*K, j^*L) = 0 \end{array}$$

So  $\text{Hom}(K, L) = 0$  (Here the two equality is by definition)

Second,  $T^{\leq 0}(X, U) \subset T^{\leq 1}(X, U)$  and  $T^{\geq 1}(X, U) \subset T^{\geq 0}(X, U)$  is trivial by definition.

Finally, we need to the existence of triangle. Fix  $K \in T(X, U)$ , we have the canonical morphisms  $K \rightarrow j_*j^*K \rightarrow j_*(\tau^{\geq 1}j^*K)$ . Then choose a exact triangle  $Y \rightarrow K \rightarrow j_*(\tau^{\geq 1}j^*K)$ . Similarly,

we have a exact triangle  $A \rightarrow Y \rightarrow i_*(\tau^{\geq 1} i^* Y)$ . By the octaeder axiom TR4a of triangulated categories, we have following two distinguished triangles  $(i_*(\tau^{\geq 1} i^* Y), B, j_*(\tau^{\geq 1} j^* K))$  and  $(A, K, B)$ .

$$\begin{array}{ccccccc}
 & & A[1] & \xrightarrow{\quad \text{equal} \quad} & A[1] & & \\
 & & \uparrow & & \uparrow & & \\
 j_*(\tau^{\geq 1} j^* K)[-1] & \dashrightarrow & i_*(\tau^{\geq 1} i^* Y) & \dashrightarrow & B & \dashrightarrow & j_*(\tau^{\geq 1} j^* K) \\
 \parallel & & \uparrow & & \uparrow & & \parallel \\
 j_*(\tau^{\geq 1} j^* K)[-1] & \longrightarrow & Y & \longrightarrow & K & \longrightarrow & j_*(\tau^{\geq 1} j^* K) \\
 & & \uparrow & & \uparrow & & \\
 & & A & \xrightarrow{\quad \text{equal} \quad} & A & & 
 \end{array}$$

**Claim**  $(A, K, B)$  is the desired triangle. Namely

$$A \in T^{\leq 0}(X, U), B \in T^{\geq 1}(X, U)$$

*Proof.* Note that  $j^* i_* = 0$  and  $i^! j_* = 0$ , so apply  $j^*$  to the triangle  $A \rightarrow Y \rightarrow i_*(\tau^{\geq 1} i^* Y)$ , we have  $j^* A \cong j^* Y$ . Similarly,  $j^* B \cong \tau^{\geq 1} j^* K$ . So we have following isomorphisms of distinguished triangles

$$j^*(A, K, B) \cong (j^* Y, j^* K, \tau^{\geq 1} j^* K) \cong (\tau^{\leq 0} j^* K, j^* K, \tau^{\geq 1} j^* K)$$

Namely  $j^* A \cong \tau^{\leq 0} j^* K \in T^{\leq 0}(U)$  and  $j^* B \cong \tau^{\geq 1} j^* K \in T^{\geq 1}(U)$ .

Do above for  $i^*$  and  $i^!$ , we have

$$i^*(A, Y, i_*(\tau^{\geq 1} i^* Y)) \cong (i^* A, i^* Y, \tau^{\geq 1} i^* Y) \cong (\tau^{\leq 0} i^* Y, i^* Y, \tau^{\geq 1} i^* Y)$$

$$i^!(A, Y, i_*(\tau^{\geq 1} i^* Y)) \cong i^!(A, K, B)$$

Namely  $i^* A \cong \tau^{\leq 0} i^* Y \in T^{\leq 0}(Z)$  and  $i^! B \cong i^! i_*(\tau^{\geq 1} i^* Y) \cong \tau^{\geq 1} i^* Y \in T^{\geq 1}(Z)$ . □

□

### 3 Perverse sheaves

#### 3.1 Verdier Duality

The reference of this section is ([1], 2.7)

**Theorem 3.1.1.** Let

$$f : X \rightarrow S$$

be a compactifiable morphism between finite type scheme over a finite field or algebraically closed field. Then the functor

$$Rf_! : D_c^b(X, \overline{\mathbb{Q}_\ell}) \rightarrow D_c^b(S, \overline{\mathbb{Q}_\ell})$$

have a right adjoint triangulated functor

$$f^! : D_c^b(S, \overline{\mathbb{Q}_\ell}) \rightarrow D_c^b(X, \overline{\mathbb{Q}_\ell})$$

Namely we have functorial isomorphism

$$Hom(K, f^!L) \cong Hom(Rf_!K, L)$$

for all  $K \in D_c^b(X, \overline{\mathbb{Q}_\ell})$  and  $L \in D_c^b(S, \overline{\mathbb{Q}_\ell})$

**Definition 3.1.2.** Let

$$f : X \rightarrow S = Spec(k)$$

be a finite type scheme over the fixed field  $k$  (finite or algebraically closed). The **duality complex** of  $X$  is

$$K_X = f^!(\overline{\mathbb{Q}_\ell}_S) \in D_c^b(X, \overline{\mathbb{Q}_\ell})$$

Then we define the contravariant **dualizing functor** by

$$D_X(L) = R\mathcal{H}om(L, K_X)$$

we often write  $\mathbb{D}L = \mathbb{D}(L) = D_X(L)$  if the scheme  $X$  is fixed.

**Corollary 3.1.3.** (Poincare Duality) Under the assumption of Theorem 3.1.1. If  $K \in D_c^b(X, \overline{\mathbb{Q}_\ell})$ , then the following holds

$$Rf_*(D_X K) = D_S(Rf_! K)$$

**Theorem 3.1.4.** The natural functorial homomorphism

$$K \rightarrow D_X(D_X K)$$

is a canonical isomorphism, namely

$$D_X \circ D_X = id$$

Therefore the dualizing functor defines an anti-equivalence of categories

$$D_X : D_c^b(X, \overline{\mathbb{Q}_\ell}) \rightarrow D_c^b(X, \overline{\mathbb{Q}_\ell})$$

$$Hom(K, L) = Hom(D_X(L), D_X(K))$$

We now collect some formulas, which will be frequently used later.

**Corollary 3.1.5.** Suppose  $f : X \rightarrow S$  is a morphism satisfying the assumptions of **Theorem II.3.2.1**. Then the following formulas hold:

- (a)  $\mathbb{D} \circ \mathbb{D} = \text{id}$
- (b)  $\mathbb{D} \circ Rf_! = Rf_* \circ \mathbb{D}$
- (c)  $\mathbb{D} \circ Rf_* = Rf_! \circ \mathbb{D}$
- (d)  $\mathbb{D} \circ f^* = f^! \circ \mathbb{D}$
- (e)  $\mathbb{D} \circ f^! = f^* \circ \mathbb{D}$
- (f)  $R\mathcal{H}\text{om}(A, B) = \mathbb{D}(A \otimes^L \mathbb{D}(B))$
- (g)  $Rf_!(A \otimes^L f^* B) = Rf_! A \otimes^L B$
- (h)  $f^! R\mathcal{H}\text{om}(A, B) = R\mathcal{H}\text{om}(f^*(A), f^!(B))$

*Proof.* (a) and (b) is true. (c) holds from (a) and (b). (d) and (e) are from (b), (c),  $\text{Hom}(K, L) = \text{Hom}(D_X L, D_X K)$ , and adjoint  $(Rf_!, f^!)$ . For (f), if we put  $C = \mathbb{D}B$ , then  $B = \mathbb{D}C$ . So

$$\begin{aligned} R\mathcal{H}\text{om}(A, B) &= R\mathcal{H}\text{om}(A, \mathbb{D}C) = R\mathcal{H}\text{om}(A, R\mathcal{H}\text{om}(C, K_X)) \\ &= R\mathcal{H}\text{om}(A \otimes^L C, K_X) \\ &= \mathbb{D}(A \otimes^L \mathbb{D}(B)) \end{aligned}$$

(g) and (h) are just similar to corresponding results in etale cohomology. □

### 3.2 Perverse t-structure

Let  $X$  be a scheme over a base field  $k$ , such that  $k$  is either a finite field or a separably closed field. Then  $D_c^b(X, \overline{\mathbb{Q}}_\ell)$  is a triangulated category, for simplicity, we denote it by  $D_c^b(X)$ .

**Definition 3.2.1.** Let  $X$  be a variety. The perverse t-structure on  $X$  is defined by

$$\begin{aligned} {}^p D_c^b(X)^{\leq 0} &= \{K \in D_c^b(X) \mid \dim \text{supp} \mathcal{H}^{-i}(K) \leq i, \forall i \in \mathbb{Z}\} \\ {}^p D_c^b(X)^{\geq 0} &= \{K \in D_c^b(X) \mid \dim \text{supp} \mathcal{H}^{-i}(\mathbb{D}(K)) \leq i, \forall i \in \mathbb{Z}\} \end{aligned}$$

where  $\mathbb{D}(K)$  denotes the Verdier duality of  $K$ . For simplicity, we denote these two subcategories by  ${}^p D^{\leq 0}(X)$  and  ${}^p D^{\geq 0}(X)$ .

The heart of this t-structure is  $Per(X) = {}^pD_c^b(X)^{\leq 0} \cap {}^pD_c^b(X)^{\geq 0}$ . Objects in the heart are called **perverse sheaves**.

**Theorem 3.2.2.** ([3]). This perverse t-structure gives a t-structure on  $D_c^b(X)$ .

To prove this theorem, we need some preparation. First, we need some results about lisse complexes

we consider the effect of pullback of perverse t-structure under open and closed immersion.

**Lemma 3.2.3.** Fix  $j : U \hookrightarrow X$  open, and  $i : Z \hookrightarrow X$  is the closed complement. Fix  $K \in D_c^b(X)$ . Then we have:

1.  $K \in {}^pD^{\leq 0}(X) \iff j^*K \in {}^pD^{\leq 0}(U), i^*K \in {}^pD^{\leq 0}(Z).$
2.  $K \in {}^pD^{\geq 0}(X) \iff j^!K = j^*K \in {}^pD^{\geq 0}(U), i^!K \in {}^pD^{\geq 0}(Z).$

*Proof.* 1. Considering  $i^*$  and  $j^*$  are exact, which means that they commute with cohomology, we get:

$$\text{Supp}(\mathcal{H}^i(K)) = \text{Supp}(\mathcal{H}^i(j^*K)) \cup \text{Supp}(\mathcal{H}^i(i^*K)).$$

Thus,

$$\dim \text{Supp}(\mathcal{H}^i(K)) = \max(\dim \text{Supp}(\mathcal{H}^i(j^*K)), \dim \text{Supp}(\mathcal{H}^i(i^*K))).$$

This immediately gives (1).

2. Note that  $i^*\mathbb{D}K = \mathbb{D}(i^!K)$ , and  $j^*\mathbb{D}K = \mathbb{D}(j^*K)$ , so (2) is from (1) by duality.

□

To prove Theorem 3.2.2, we need some smooth results.

Suppose  $X$  is smooth of dimension  $d$ . Then the dualizing complex on  $X$  has the form

$$\overline{\mathbb{Q}}_\ell[2d](d)$$

It is a complex  $K_X$ , whose cohomology is concentrated in degree  $-2d$  and such that its cohomology sheaf  $\mathcal{H}^{-2d}(K_X)$  is isomorphic to the smooth sheaf  $\overline{\mathbb{Q}}_\ell(d)$ . For a smooth sheaf  $\mathcal{G}$  on  $X$ , the definition of dual sheaf is as the ordinary case

$$\mathcal{G}^\vee = \mathcal{H}om(\mathcal{G}, \overline{\mathbb{Q}}_\ell)$$

A sheaf complex  $K \in D_c^b(X)$  is called a **smooth complex**, if all its cohomology sheaves  $\mathcal{H}^i(K)$  are lisse sheaves on  $X$ .



**Proposition 3.2.4.** 1. Let  $X$  be a smooth scheme of dimension  $d$  over  $k$  and let  $K \in D_c^b(X)$  be a smooth complex on  $X$ . Then

$$\mathcal{H}^i(\mathbb{D}K) \cong \mathcal{H}^{-i-2d}(K)^\vee(d)$$

2. Suppose  $X$  is irreducible and  $K \in D_c^b(X)$ . Then there exists an open dense smooth subscheme

$$j : U \rightarrow X$$

of  $X$ , such that

$$j^*(K)$$

is smooth on  $U$ .

**Corollary 3.2.5.** Under the assumption above, i.e  $X$  is smooth of dimension  $d$  and  $K$  is a smooth complex, then

$$K \in {}^pD^{\leq 0}_c(X) \text{ iff } \mathcal{H}^i K = 0 \ \forall i > -d$$

$$K \in {}^pD^{\geq 0}_c(X) \text{ iff } \mathcal{H}^i K = 0 \ \forall i < -d$$

Namely a smooth complex  $K \in \text{Perv}(X)$  if and only if

$$K = \mathcal{G}[d]$$

for a smooth sheaf  $\mathcal{G}$ .

*Proof of Theorem 3.2.2* We prove by induction on  $d = \dim(X)$ .

1. If  $d = 0$ , the  $X$  is the union of finitely many point. Then  ${}^pD_c^b(X)^{\leq 0} = D_c^b(X)^{\leq 0}$  and  ${}^pD_c^b(X)^{\geq 0} = D_c^b(X)^{\geq 0}$  as in Theorem 2.2.4. So the perverse  $t$ -structure is just the standard  $t$ -structure.

2. If  $d \geq 1$ , let

$$j : U \rightarrow X$$

be a nonempty open smooth subscheme of  $X$  and let

$$i : Z \rightarrow X$$

be its closed complement. Assume  $({}^pD^{\leq 0}(Z), {}^pD^{\geq 0}(Z))$  is a perverse  $t$ -structure on  $T(Z) = D_c^b(Z)$ . On the other hand let  $T(U)$  be the full subcategory of  $D_c^b(U)$ , consisting of complexes with smooth cohomology sheaves. From corollary 3.2.5 we know the induced  $t$ -structure on  $T(U)$  coincides with the standard  $t$ -structure up to a shift of degree. By gluing, we get

$$T(X, U) = \{E \in D_c^b(X) \mid j^*E \text{ has smooth cohomology sheaves on } U\}$$

By lemma, the glueing  $t$ -structure on  $T(X, U)$  coincides with the perverse  $t$ -structure, which is obtained by restriction from  $D_c^b(X)$  to  $T(X, U)$ .

Now let  $E$  be an arbitrary complex in  $D_c^b(X)$ . Then there always exists an open dense smooth subscheme  $U \rightarrow X$ , such that the restriction of  $E$  to  $U$  has smooth cohomology sheaves. In other words

$$D_c^b(X) = \bigcup_{U \subset X, \text{ dense open } \text{ess. smooth}} T(X, U)$$

Now we check the three axioms

(1). For complexes  $E \in {}^p D^{\leq 0}(X)$  and  $E' \in {}^p D^{\geq 1}(X)$ , there exists  $U$  and  $U'$  as above, such that  $E \in T^{\leq 0}(X, U)$  and  $E' \in T^{\geq 1}(X, U')$ . Set  $U = U \cap U'$ , by definition  $E \in T^{\leq 0}(X, U)$  and  $E' \in T^{\geq 1}(X, U)$ , So  $\text{Hom}(E, E') = 0$ .

(2). Fix  $E \in D_c^b(X)$ . Suppose  $E \in T(X, U)$  for an open subset  $U \subset X$ . then there  $E_1 \in T^{\leq 0}(X, U)$  and  $E_2 \in T^{\geq 1}(X, U)$  such that  $E_1 \rightarrow E \rightarrow E_2$  is an exact triangle. But  $T^{\leq 0}(X, U) = {}^p D^{\leq 0}(X) \cap T(X, U)$  and  $T^{\geq 1}(X, U) = {}^p D^{\geq 1}(X) \cap T(X, U)$ , so  $E_1 \in {}^p D^{\leq 0}(X)$  and  $E_2 \in {}^p D^{\geq 1}(X)$ .

### 3.3 t-exact functor

**Definition 3.3.1.** Let  $\mathcal{D}_1$  and  $\mathcal{D}_2$  be two  $t$ -structure triangulated categories with heart  $\mathcal{D}_1^\heartsuit$  and  $\mathcal{D}_2^\heartsuit$  and let  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  be a morphism of categories. Then we define  ${}^p F = H^0 \circ F \circ \epsilon : \mathcal{D}_1^\heartsuit \rightarrow \mathcal{D}_2^\heartsuit$ .

**Definition 3.3.2.** Let  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  be a exact functor of triangulated categories, and assume that there are  $t$ -structure  $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 0})$  on  $\mathcal{D}_1$  and  $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 0})$  on  $\mathcal{D}_2$ . Such a functor is called

$$\text{t-right exact iff } F(D^{\leq 0}(A)) \subset D^{\leq 0}(B)$$

and

$$\text{t-left exact iff } F(D^{\geq 0}(A)) \subset D^{\geq 0}(B)$$

Finally,  $F$  is exact if it is both t-left and t-right exact.

Now we give some results about the six functors.

**Proposition 3.3.3.** Let  $X$  be a finite type separated scheme over a finite field or algebraically field. Let  $j : U \rightarrow X$  be an open immersion with closed complement  $i : Z \rightarrow X$ . Consider the perverse  $t$ -structure on all schemes. Then

1.  $j^*(= j^!), i_* = (i_!)$  are t-exact
2.  $j_!, i^*$  are t-right exact.

3.  $j_*, i^!$  are  $t$ -left exact
4. There are adjoint sequence  $({}^p i^*, {}^p i_* = {}^p i_!, {}^p i^!), ({}^p j_!, {}^p j^! = {}^p j^*, {}^p j_*)$
5. the compositions  ${}^p j^* \circ {}^p i_*, {}^p i^* \circ {}^p j_!, {}^p i^! \circ {}^p j_*$  are zero.

*Proof.* 1. We prove  $i_*$  is  $t$ -exact,  $j^*$  is similar .

( $t$ -right exact). Specifically,  $i^* i_* = id$  implies  $i^*(i_* D_{\bar{Z}}^{\leq 0}) \subseteq D_{\bar{Z}}^{\leq 0}$ .  $j^* i_* = 0$  implies  $j^*(i_* D_{\bar{Z}}^{\leq 0}) \subseteq D_U^{\leq 0}$ .

( $t$ -left exact). the same argument using  $i^! i_* = id, j^* i_* = 0$ .

2. using  $i^* j_! = 0$  and  $j^* j_! = id$ .
3. same argument with 2.
4. we prove for  $({}^p j^*, {}^p j_*)$ . Other cases are similar.

*Proof.* Fix  $K \in D_U^\heartsuit$  and  $L \in D_X^\heartsuit$ . We are interested in

$$Hom_{D_X^\heartsuit}({}^p j_! K, L) = Hom_{D_X^\heartsuit}({}^p H^0(j_! K), L) \stackrel{full}{=} Hom_{D_X}({}^p H^0(j_! K), L) = Hom_{D_X}(\tau^{\leq 0}(j_! K), L)$$

since  $L \in D_X^{\geq 0}$ . Now  $j_!$  is  $t$ -right exact and  $j^*$  is  $t$ -exact imply  $\tau^{\leq 0}(j_! K) = j_! K$  and  $j^* L = {}^p H^0(j^* L)$ . So we have following equality

$$\begin{aligned} Hom_{D_X}(\tau^{\leq 0}(j_! K), L) &= Hom_{D_X}(j_! K, L) \\ (adjoint) &= Hom_{D_U}(K, j^* L) \\ &= Hom_{D_U}(K, {}^p H^0(j^* L)) \\ (full) &= Hom_{D_U}(K, {}^p j^* L) \\ &= Hom_{D_U^\heartsuit}(K, {}^p j^* L) \end{aligned}$$

Namely  $Hom_{D_U^\heartsuit}(K, {}^p j^* L) = Hom_{D_X^\heartsuit}({}^p j_! K, L)$ .

5.  $j^*, i_*$  are  $t$ -exact, so  ${}^p j^* = j^*$  on  $D_X^\heartsuit$ ,  ${}^p i_* = i_*$  on  $D_Z^\heartsuit$ . Then  ${}^p j^* \circ {}^p i_* = j^* i_* = 0$ . By adjoint,  ${}^p i^* \circ {}^p j_! = 0$  and  ${}^p i^! \circ {}^p j_* = 0$ .

### 3.4 intermediate extension

As above, let  $X$  be a finite type scheme over a finite field or algebraically closed field. Let  $j : U \rightarrow X$  be an open immersion and  $i : Z \rightarrow X$  be the closed complement of  $U$ . Fix a perverse sheaf  $K$  on  $U$ , a perverse sheaf  $\overline{K}$  on  $X$  is called an extension of  $K$ , if

$$j^* \overline{K} = K$$

we have following results.

**Lemma 3.4.1.** With the preceding notations we have following chain of equivalent conditions (1) – (4) for a perverse extension  $\overline{K}$  of the perverse sheaf  $K$ :

- (1)  $\overline{K}$  has neither subobjects nor quotients from  $i_* Perv(Y)$
- (2)  ${}^p H^0(i^* \overline{K}) = {}^p H^0(i^! \overline{K}) = 0$
- (3)  $i^* \overline{K} \in {}^p D^{\leq -1}(Z)$  and  $i^! \overline{K} \in {}^p D^{\geq 1}(Z)$
- (4)  $\overline{K} \cong \text{image}({}^p H^0(j_! K) \rightarrow {}^p H^0(j_* K))$

If one of these equivalent conditions holds, then there exists an exact triangle

$$(\overline{K}, j_* K, i_* {}^p \tau^{\geq 0} i^* j_* K)$$

**Definition 3.4.2.** We know that for a perverse sheaves  $K \in Perv(U)$ , there is a unique (up to quasi isomorphism) perverse sheaves  $\overline{K} \in Perv(X)$  which satisfies the conditions in Lemma 3.4.1. This unique extension is called the intermediate extension of  $K$ , and denote by

$$j_{!*} K$$

which define a functor

$$j_{!*} : Perv(U) \rightarrow Perv(X)$$

**Corollary 3.4.3.** Let  $j : U \rightarrow X$  be an open immersion, and  $K \in Perv(U)$ , then

$$\mathbb{D}(j_{!*} K) = j_{!*}(\mathbb{D}K)$$

*Proof.* We use the characterization (4) above. Let  $\bar{L} = \mathbb{D}(j_{!*}K)$ , then  $j^*\bar{L} = j^*\mathbb{D}(j_{!*}K) = \mathbb{D}(j^*j_{!*}K) = \mathbb{D}K$ . So  $\bar{L}$  is an extension of  $\mathbb{D}K$ .

Now  $i^*\bar{L} = i^*\mathbb{D}(j_{!*}K) = \mathbb{D}(i^!j_{!*}K)$ .  $i^!j_{!*}K \in {}^pD^{\geq 1}(Z)$ , so  $i^*\bar{L} = \mathbb{D}(i^!j_{!*}K) \in {}^pD^{\leq -1}(Z)$ .

Similarly,  $i^!\bar{L} \in {}^pD^{\geq 1}(Z)$ . So  $\bar{L} = j_{!*}(\mathbb{D}K)$ .  $\square$

**Corollary 3.4.4.** Let  $j : U \rightarrow X$  be an open immersion with closed complement  $i : Z \rightarrow X$ . Then any simple object  $\bar{K} \in \text{Perv}(X)$  is either of the form  $i_*A$  for a simple object  $A \in \text{Perv}(Z)$  or of the form  $j_{!*}B$  for a simple object  $B \in \text{Perv}(U)$ .

**Corollary 3.4.5.** A perverse sheaf  $K$  on  $X$  is simple if and only if it is of the form  $K = i_*j_{!*}\mathcal{F}[d]$ , for an irreducible closed subscheme  $i : Z \rightarrow X$ , a open dense smooth of dimension  $d$  subscheme  $j : U \rightarrow Z$  of  $Z$  and a smooth irreducible  $\overline{\mathbb{Q}}_\ell$ -sheaf  $\mathcal{F}$  on  $U$ .

**Corollary 3.4.6.** The abelian category  $\text{Perv}(X)$  is artinian and noetherian. (Namely has finite length).

*Proof.* We have to show, that a perverse sheaf  $\bar{K} \in \text{Perv}(X)$  can have only finitely many perverse constituents. If  $\dim(X) = 0$ , the category  $\text{Perv}(X)$  is equivalent to the category of finite dimension  $\overline{\mathbb{Q}}_\ell$ - vector space, which is artinian and noetherian. By noetherian induction the assertion can assumed to be true for all closed subspaces of smaller dimension. Choose open smooth subset  $j : U \rightarrow X$  such that  $K = j^*\bar{K} = \mathcal{F}[d]$  for a smooth sheaf  $\mathcal{F}$ . We have following exact sequence:

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \uparrow & & & \\
& & & i_*{}^pH^0(i^*j_*K) & & & \\
& & & \uparrow & & & \\
0 & \longrightarrow & i_*{}^pH^0(i^!\bar{K}) & \longrightarrow & \bar{K} & \longrightarrow & {}^pH^0(j_*K) \longrightarrow i_*{}^pH^1(i^!\bar{K}) \longrightarrow 0 \\
& & & \uparrow & & & \\
& & & j_{!*}K & & & \\
& & & \uparrow & & & \\
& & & 0 & & & 
\end{array}$$

The horizontal sequence is obtained by take  $H^0(-)$  for the exact triangle:

$$i_*i^!\bar{K} \rightarrow \bar{K} \rightarrow j_*j^*\bar{K}$$

notice that  $i_*$  commutes with  $H^n(n \in \mathbb{Z})$  since it is exact, and  $j_*j^*\bar{K} \in D_X^{\geq 0}$ , so by Theorem 2.1.9 step2, we have the left 0. The right 0 is just because  $\tau^{\geq 1} \circ \tau^{\leq 0} = 0$ .

The vertical sequence following the equivalent conditions in Lemma 3.4.1, the apply  $H^0(-)$  and notice  $i_*^p \tau^{\geq 0} i^* j_* K \in D_X^{\geq 0}$ , as the same as above, we have desired sequence.

Now analyze above diagram. First see the horizontal exact sequence, by induction,  ${}^p H^0(i^! \bar{K})$  has finite length, so is  $i_*^p H^0(i^! \bar{K})$ . So we need to show  ${}^p H^0(j_* K)$  is also.

Notice the vertical sequence,  $i_*^p H^0(i^* j_* K)$  has finite length, the following lemma implies  $j_{!*}$  preserves finite length, so we are done.  $\square$

**Lemma 3.4.7.** If  $j : U \rightarrow X$  is an open immersion, then  $j_{!*}$  preserves finite length objects.

## 4 Mixed complex and weight filtration

### 4.1 Mixed complex

Recall some notations in [2].

**Definition 4.1.1.** Let  $\beta$  be a real number.  $\mathcal{G}_0$  is a  $\ell$ -adic sheaf on  $X_0$ . Fix  $\tau : \bar{\mathbb{Q}}_\ell \cong \mathbb{C}$

1. Choose a  $\bar{k}$ -point  $\bar{x} \in X$  lying over  $x \in |X_0|$ . The Weil group  $W(\bar{k}/k(x))$  acts on the stalk at  $\mathcal{G}_{0\bar{x}}$  via the geometric frobenius  $F_x : \mathcal{G}_{0\bar{x}} \rightarrow \mathcal{G}_{0\bar{x}}$ . We say that  $\mathcal{G}_0$  is  **$\tau$ -pure of weight  $\beta$**  if for every  $x \in |X_0|$ , and all eigenvalues  $\alpha \in \bar{\mathbb{Q}}_\ell$  of  $F_x$ , we have

$$|\tau(\alpha)| = N(x)^{\beta/2}.$$

2. We say  $\mathcal{G}_0$  is  $\tau$ -mixed if there is a finite filtration of subsheaves

$$0 = \mathcal{G}_0^{(0)} \subseteq \mathcal{G}_0^{(1)} \subseteq \cdots \subseteq \mathcal{G}_0^{(r)} = \mathcal{G}_0$$

such that  $\mathcal{G}_0^{(j)} / \mathcal{G}_0^{(j-1)}$  is  $\tau$ -pure of some weight.

3.  $\mathcal{G}_0$  is pure of weight  $\beta$  if it is  $\tau$ -pure of weight  $\beta$  for all  $\tau : \bar{\mathbb{Q}}_\ell \cong \mathbb{C}$

4.  $\mathcal{G}_0$  is mixed if there exists a finite filtration as in (2) such that all quotient are pure.

**Definition 4.1.2.** Let  $X_0$  be a scheme of finite type over a  $\mathbb{F}_q$ . Then an object  $K_0$  of the category  $D_c^b(X_0, \bar{\mathbb{Q}}_\ell)$  is said to be  $\tau$ -mixed (mixed), if all its cohomology sheaves  $\mathcal{H}^i(K_0)$  are  $\tau$ -mixed (mixed)sheaves on  $X_0$ . The full subcategory of mixed complexes is denoted by  $D_m^b(X_0, \bar{\mathbb{Q}}_\ell) \subset D_c^b(X_0, \bar{\mathbb{Q}}_\ell)$ .

**Lemma 4.1.3.** The category of mixed sheaves on  $X_0$  is closed under subquotients and extensions.

**Proposition 4.1.4.** Let  $X_0$  be a scheme of finite type over  $\mathbb{F}_q$ . Then the category  $D_m^b(X_0, \bar{\mathbb{Q}}_\ell)$  is a triangulated subcategory stable under the six functor.

*Proof.* To show that  $D_m^b(X_0, \overline{\mathbb{Q}_\ell})$  is a triangulated subcategory we have to check the following three things:

1. The subcategory  $D_m^b(X_0, \overline{\mathbb{Q}_\ell})$  is stable under shifts. This is trivial since  $\mathcal{H}^i(K_0[1]) = \mathcal{H}^{i+1}(K_0)$
2. The subcategory  $D_m^b(X_0, \overline{\mathbb{Q}_\ell})$  is stable under isomorphism. This is trivial.
3. If there is an exact triangle  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  in  $D_c^b(X_0, \overline{\mathbb{Q}_\ell})$  with  $X, Z \in D_m^b(X_0, \overline{\mathbb{Q}_\ell})$  then  $Y \in D_m^b(X_0, \overline{\mathbb{Q}_\ell})$ .

take the long exact sequence of cohomology

$$\dots \longrightarrow \mathcal{H}^i(X) \xrightarrow{f} \mathcal{H}^i(Y) \xrightarrow{g} \mathcal{H}^i(Z) \longrightarrow \dots$$

this induces a short sequence

$$0 \longrightarrow \mathcal{H}^i(X)/\ker(f) \longrightarrow \mathcal{H}^i(Y) \longrightarrow \operatorname{im}(g) \longrightarrow 0$$

by 4.1.3  $\mathcal{H}^i(Y)$  is mixed.

Next, we check the six functors preserve the category  $D_m^b(X_0, \overline{\mathbb{Q}_\ell})$ . Let  $Y_0$  be a finite type  $\mathbb{F}_q$ -scheme and  $f : X_0 \rightarrow Y_0$  be a morphism of scheme. We need the following results by Deligne:

**Theorem 4.1.5.** (Deligne,[2],Thm 3.3.1) Assume  $f$  is separated and  $\mathcal{F}_0$  is a mixed sheaf of maximal weight  $\leq w$ . Then  $R^i f_! \mathcal{F}_0$  is a mixed sheaf of maximal weight  $\leq w + i$  on  $Y_0$ .

**Theorem 4.1.6.** (Deligne,[2],Thm 6.1.1) Let  $\mathcal{F}_0$  be a mixed sheaf on  $X_0$ , then  $R^i f_* \mathcal{F}_0$  is a mixed sheaf on  $Y_0$ .

So  $Rf_*$  and  $Rf_!$  preserve mixedness.  $f^*$  preserves mixedness is trivial. the case  $f^!$  is reduced to the case  $f^*$  by biduality, using the formula

$$\mathbb{D} \circ f^! = f^* \circ \mathbb{D}$$

and the assertion  $\mathbb{D}$  preserves mixedness. The tensor product  $\otimes^L$  follows using the kunneth formula

$$\mathcal{H}^i(K_0 \otimes L_0) = \bigoplus_{i+j=n} \mathcal{H}^i(K_0) \otimes \mathcal{H}^j(L_0)$$

So it remains to prove the last case :  $\mathbb{D}K_0$  is mixed if  $K_0$  is mixed.

This is clear when  $\dim(X_0) = 0$ , so we can prove by induction on  $\dim(X_0)$ . Let  $j : U_0 \hookrightarrow X_0$  be a smooth dense open subscheme such that  $j^* K_0$  is a smooth complex, let  $i : Z_0 \hookrightarrow X_0$  be the closed complement. Because  $U_0$  is smooth, the complex  $\mathbb{D}(j^* K_0)$  is mixed by proposition 3.2.4, hence also the complex

$$j_* \mathbb{D}(j^* K_0) = \mathbb{D}(j_! j^* K_0).$$

By induction,  $\mathbb{D}(i^*K_0)$  is mixed and so also

$$\mathbb{D}(i_*i^*K_0) = i_*\mathbb{D}(i^*)K_0$$

Now using the exact triangle

$$j_!j_*K_0 \rightarrow K_0 \rightarrow i_*i^*K_0$$

the duals of left term and right term are mixed, so is the dual of the middle term.  $\square$

For the rest content, we refer to 2.12 of [1].

**Definition 4.1.7.** For a scheme  $X_0/k$  and  $\ell$ -adic sheaf  $\mathcal{G}_0$  on  $X_0$ , we define the **maximal weight** of  $\mathcal{G}_0$  (with respect to  $\tau$ ) as

$$w(\mathcal{G}_0) := \sup_{x \in |X_0|} \sup_{\alpha \text{ eigenvalue of } F_x} \frac{\log(|\tau(\alpha)|^2)}{\log N(x)}$$

For convenience, we define  $w(0) = -\infty$ .

**Definition 4.1.8.** Notation as above. For a  $\tau$ -mixed complex  $K_0 \in D_m^b(X_0, \overline{\mathbb{Q}_\ell})$ , we define

$$w(K_0) = \max_i (w(\mathcal{H}^i(K_0)) - i)$$

Note by Theorem 2.2.1, this is well-defined.

**Definition 4.1.9.** For any real number  $w$ , we define two subcategories

$$D_{\leq w}^b = D_{\leq w}^b(X_0) = \{K_0 \in D_m^b(X_0) \mid w(K_0) \leq w\}$$

and

$$D_{\geq w}^b = D_{\geq w}^b(X_0) = \{K_0 \in D_m^b(X_0) \mid w(\mathbb{D}(K_0)) \leq -w\}$$

**Remark.** Note that  $w(K_0) \geq w$  does not imply  $K_0 \in D_{\geq w}^b(X_0)$ . But it is shown in Lemma 4.1.11 below, that the other direction is true. Namely  $K_0 \in D_{\geq w}^b(X_0)$  imply  $w(K_0) \geq w$ .

**Proposition 4.1.10.** (5.1.15 of [3]) Let  $K_0, L_0 \in D_m^b(X_0)$ . Let  $w$  be an integer. Assume that  $K_0 \in D_{\leq w}^b$  and  $L_0 \in D_{\geq w}^b$ , then

1.  $\text{Ext}^i(K, L)^F = 0$  for  $i > 0$
2. the morphisms  $\text{Ext}^i(K_0, L_0) \rightarrow \text{Ext}^i(K, L)$  is the zero map for  $i > 0$

If  $L \in D_{\geq w+1}^b$ , then  $\text{Ext}^i(K_0, L_0) = 0$



**Lemma 4.1.11.** Let  $K_0 \in D_m^b(X_0, \overline{\mathbb{Q}_\ell})$  with  $w(K_0) = w$ . If  $K_0 \neq 0$  is nontrivial, then the following holds

$$w(\mathbb{D}K) \geq -w(K_0)$$

In particular, if  $K_0 \in D_{\geq w}^b(X_0)$  holds for some  $w$ , then  $w \leq w(K_0)$ .

**Definition 4.1.12.** A complex  $K_0 \in D_c^b(X_0, \overline{\mathbb{Q}_\ell})$  is called  $\tau$ -pure of weight  $w$  if  $w(\mathbb{D}K) = -w(K_0)$  holds, i.e if

$$K_0 \in D_{\leq w}^b(X_0) \cap D_{\geq w}^b(X_0)$$

**Remark.** This definition for complex does not coincide with the notion of  $\ell$ -adic sheaves in definition 4.1.1.

## 4.2 The perverse t-structure on mixed complexes

We have introduced the triangulated subcategory  $D_m^b(X_0, \overline{\mathbb{Q}_\ell}) \subset D_c^b(X_0, \overline{\mathbb{Q}_\ell})$ . We want to show the t-structure on  $D_c^b(X_0, \overline{\mathbb{Q}_\ell})$  restricts to one on  $D_m^b(X_0, \overline{\mathbb{Q}_\ell})$

**Proposition 4.2.1.** The perverse truncation functors  ${}^p\tau_{\geq 0}$  and  ${}^p\tau_{\leq 0}$  preserve the subcategory  $D_m^b(X_0, \overline{\mathbb{Q}_\ell}) \subset D_c^b(X_0, \overline{\mathbb{Q}_\ell})$ . Namely for  $K_0 \in D_m^b(X_0)$ , we have

$${}^p\tau^{\leq 0}K_0 \in D_m^b(X_0) \quad \text{and} \quad {}^p\tau^{\geq 0}K_0 \in D_m^b(X_0)$$

*Proof.* Then prove is by induction on  $\dim(X_0)$ . If  $\dim(X_0) = 0$ , the perverse  $t$ -structure is just the standard  $t$ -structure so the result holds. Now let  $K_0 \in D_m^b(X_0)$  and  $j : U_0 \rightarrow X_0$  be an smooth open subset such that  $j^*K_0$  is a smooth complex.

We know the perverse truncation of  $j^*K_0$  is mixed, since the perverse truncation of  $j^*K_0$  is just a shift of the ordinary truncation of  $j^*K_0$ . By the induction, we know the perverse truncation of  $i^*K_0$  is mixed. Now we claim that  $i^*$  and  $j^*$  commute with perverse truncation, i.e. that

$${}^p\tau^{\leq 0}i^*K_0 = i^*{}^p\tau^{\leq 0}K_0 \tag{1}$$

$${}^p\tau^{\leq 0}j^*K_0 = j^*{}^p\tau^{\leq 0}K_0 \tag{2}$$

To check  ${}^p\tau^{\leq 0}K_0$  is mixed, we only need to check for  $i^*{}^p\tau^{\leq 0}K_0$  and  $j^*{}^p\tau^{\leq 0}K_0$  since mixedness is a stalk condition. It remains to prove (1) and (2).

This follows following commutative diagrams

$$\begin{array}{ccc} {}^pD_c^b(U_0)^{\leq 0} & \longrightarrow & D_c^b(U_0) \\ \downarrow j^! & & \downarrow j^! \\ {}^pD_c^b(X_0)^{\leq 0} & \longrightarrow & D_c^b(X_0) \end{array} \quad \begin{array}{ccc} {}^pD_c^b(Z_0)^{\leq 0} & \longrightarrow & D_c^b(Z_0) \\ \downarrow i_* & & \downarrow i_* \\ {}^pD_c^b(Z_0)^{\leq 0} & \longrightarrow & D_c^b(X_0) \end{array}$$

by the adjoint , we have the same diagram replacing all the functors with the adjoint functor in the other direction.

The proof for  ${}^p\tau^{\geq 0}$  is similar, just need to notice that  $\mathbb{D}$  preserve mixed.  $\square$

### 4.3 Weight filtration

Let  $X_0$  be a scheme of finite type over a finite field  $k = \mathbb{F}_q$ . Fix an isomorphism  $\tau : \overline{\mathbb{Q}_\ell} \rightarrow \mathbb{C}$ . Let  $K_0$  be a  $\tau$ -mixed complex in  $D_c^b(X_0)$ . For any  $\tau$ -mixed complex  $K_0$ , we define  $w(K_0)$  as in Definition 4.1.8.

Recall that a  $\tau$ -mixed complex  $K_0$  is called  $\tau$ -pure of weight  $w$  if  $K_0 \in D_{\leq w}(X_0) \cap D_{\geq w}(X_0)$ . By definition this is equivalent to  $w(K_0) \leq w$  and  $w(\mathbb{D}K_0) \leq -w$ . we have following result.

**Proposition 4.3.1.** Let  $K_0$  and  $L_0$  be  $\tau$ -mixed perverse sheaves on  $X_0$  Then

$$Hom_{Perv(X_0)}(K_0, L_0) = Hom_{Perv(X)}(K, L)^F$$

**Lemma 4.3.2. (Semicontinuity of Weight)** Let  $j : U_0 \rightarrow X_0$  be an open immersion with closed complement  $i : Z_0 \rightarrow X_0$ . Let  $\overline{K_0} \in Perv(X_0)$  be a  $\tau$ -mixed perverse sheaf on  $X_0$ , such that

$$j^*(\overline{K_0}) = K_0, {}^pH^0(i^*(\overline{K_0})) = 0$$

Then

$$w(\overline{K_0}) \leq w(K_0).$$

In particular

$$w(j_{!*}(K_0)) \leq w(K_0).$$

**Corollary 4.3.3.** Any  $\tau$ -mixed simple perverse sheaves  $K_0$  on  $X_0$  is  $\tau$ -pure of weight  $w = w(K_0)$ .

**Lemma 4.3.4. (Subquotient).** Let  $K_0 \in Perv(X_0)$  be a  $\tau$ -mixed perverse sheaves. then  $w(L_0) \leq w(K_0)$  holds for any perverse subquotient  $L_0$  of  $K_0$  in  $Perv(X_0)$ .

**Theorem 4.3.5. (Weight Filtration, 5.3.5 of [3])** In the abelian category  $Perv(X_0)$  any  $\tau$ -mixed perverse sheaf  $K_0$  on  $X_0$  has a canonical finite increasing  $\tau$ -weight filtration  $W = (K_0^{(w_i)})$

$$0 = K_0^{(-\infty)} \subset K_0^{(w_1)} \subset \dots \subset K_0^{(w_r)} = K_0$$

such that the quotient  $Gr^{(w_i)}(K_0) := K_0^{(w_i)} / K_0^{(w_{i-1})}$  are either zero or  $\tau$ -pure perverse sheaves of weight  $w_i$  such that

$$w_i < w_j \quad \text{for } i < j.$$

**Remark.** (1) If we demand all quotient  $Gr^{(w_i)}(K_0)$  to be nontrivial, then the filtration is uniquely determined.

(2) Let  $\phi_0 : K_0 \rightarrow L_0$  be a homomorphism of a  $\tau$ -mixed perverse sheaf  $K_0$  into a  $\tau$ -mixed perverse sheaf  $L_0$ , then there exist weight filtration  $K_0^{(w_i)}, L_0^{(w_i)}$  on  $K, L$  such that  $\phi_0$  maps  $K_0^{(w_i)}$  to  $L_0^{(w_i)}$ .

## 5 Decomposition Theorem

### 5.1 Deligne's Theorem

**Theorem 5.1.1.** (5.3.7 of [3]) Let  $K_0$  be a  $\tau$ -mixed perverse sheaf in  $Perv(X_0)$ . Then

$$w(K_0) \leq w$$

holds iff for every irreducible dimension  $d$  subscheme  $Z_0$  of  $X_0$ , there is an open dense subscheme  $j : U_0 \rightarrow Z_0$  such that

$$w(\mathcal{H}^{-d}K_0|_{U_0}) \leq w - d.$$

*Proof.*  $(\Rightarrow)$  is trivial.

Consider the nontrivial direction  $(\Leftarrow)$ . By the weight filtration Theorem 4.3.5, we have a short exact sequence

$$0 \rightarrow A_0 \rightarrow B_0 \rightarrow Q_0 \rightarrow 0$$

in  $Perv(X_0)$  with  $Q_0$  simple of maximal weight  $w(Q_0) = w(B_0)$ . Then  $Q_0 = i_*j_{!*}C_0$  for a lisse sheaf  $C_0$  on  $U_0$  with  $U_0 \xrightarrow{j} Y_0 \xrightarrow{i} X_0$  and  $Y_0$  is irreducible. We need to estimate  $w(Q_0)$ . Since  $A_0 \in {}^pD^{\leq 0}(X_0)$ , we have  $\dim \text{supp } \mathcal{H}^{-d+1}(A_0) \leq d - 1$ . On the open dense subset  $V_0 = U_0 \setminus \text{supp } \mathcal{H}^{-d+1}(A_0)$  of  $U_0$ , the map  $\mathcal{H}^{-d}(B_0) \rightarrow \mathcal{H}^{-d}(Q_0)$  is surjective. By shrinking  $V_0$ , the surjectivity and the assumption  $w(\mathcal{H}^{-d}Q_0|_{V_0}) \leq w - d$ . Hence by Lemma 4.3.2,  $w(B_0) = w(Q_0) \leq w$ .  $\square$

**Lemma 5.1.2.**  $K \in D_c^b(X)$  implies  $K \in {}^pD_c^b(X)^{\leq n} \cap {}^pD_c^b(X)^{\geq -m}$  for sufficiently large  $m, n \gg 0$ . Namely for  $m, n \gg 0$ ,  $\tau^{\geq n+1}K = 0$  and  $\tau^{\leq -m-1}K = 0$ .

*Proof.* Just by definition.  $\square$

**Corollary 5.1.3.** (5.4.1 of [3]) Let  $K_0$  be  $\tau$ -mixed in  $D_c^b(X_0, \overline{\mathbb{Q}}_\ell)$ , then

$$w(K_0) \leq w \iff w({}^pH^i(K_0)) \leq w + i, \quad \forall i \in \mathbb{Z}$$

*Proof.* 1. ( $\Leftarrow$ ). Note that if  $K \rightarrow L \rightarrow M$  is an exact triangle with  $w(K) \leq w$  and  $w(M) \leq w$ , then  $w(L) \in w$ . Use the exact sequence  ${}^p\tau^{\leq i-1}K_0 \rightarrow {}^p\tau^{\leq i}K_0 \rightarrow {}^pH^i(K_0)[-i] \rightarrow \cdots$ , by Lemma 5.1.2, there  $m, n \gg 0$  such that  ${}^p\tau^{\leq -m-1}K_0 = 0$  and  ${}^p\tau^{\leq n}K_0 = K_0$ , so by induction,  $w({}^p\tau^{\leq i}K_0) \leq w$  for  $i \geq -m$  since  $w({}^pH^i(K_0)[-i]) \leq w$ . Clearly  $n > -m$ , so  $w(K_0) \leq w$ .

2. ( $\Rightarrow$ ). Assume  $w(K_0) \leq w$  and  $w({}^pH^i(K_0)) \leq w + i$  for all  $i > l$ . Then  $w({}^p\tau^{\geq l+1}K_0) \leq w$ , using the  $\Leftarrow$  direction. We will prove  $w({}^pH^l(K_0)) \leq w + l$ , which inductively will imply

$$w({}^pH^i(K_0)) \leq w + i$$

for all  $i$ .

For simplicity we may assume  $l = 0$ . For some integer  $0 \leq d \leq \dim(X_0)$ . Using the long exact sequence for the exact triangle  $({}^p\tau^{\leq 0}K_0, K_0, {}^p\tau^{\geq 1}K_0)$ , the weight estimates  $w(K_0), w({}^p\tau^{\geq 1}K_0) \leq w$  imply  $w({}^p\tau^{\leq 0}K_0) \leq w$ . So  $w(\mathcal{H}^{-d}({}^p\tau^{\leq 0}K_0)) \leq w - d$ . Since  $\dim \text{supp } \mathcal{H}^{-d+1}({}^p\tau^{\geq 1}K_0) < d$ , this weight estimate and the exact sequence

$$\mathcal{H}^{-d}({}^p\tau^{\leq 0}K_0) \rightarrow \mathcal{H}^{-d}({}^pH^0K_0) \rightarrow \mathcal{H}^{-d+1}({}^p\tau^{\geq 1}K_0)$$

So take any irreducible subscheme  $Y_0 \subset X_0$  of dimension  $d$ ,

$$U_0 = Y_0 \cap (X_0 \setminus \text{supp } \mathcal{H}^{-d+1}({}^p\tau^{\geq 1}K_0))$$

is an open subset of  $X_0$  such that  $\mathcal{H}^{-d+1}({}^p\tau^{\geq 1}K_0)|_{U_0} = 0$ . so

$$\mathcal{H}^{-d}({}^p\tau^{\leq 0}K_0) \cong \mathcal{H}^{-d}({}^pH^0K_0)$$

on  $U_0$ . Now  ${}^pH^0K_0$  satisfies the condition in Theorem 5.1.1, so  ${}^pH^0K_0 \leq w$ . We are done.  $\square$

**Corollary 5.1.4.** A  $\tau$ -mixed complex  $K_0 \in D_c^b(X_0, \overline{\mathbb{Q}}_\ell)$  is  $\tau$ -pure of weight  $w$  iff all  ${}^pH^i(K_0)$  are  $\tau$ -pure of weight  $w + i$ .

## 5.2 Decomposition theorem

**Theorem 5.2.1.** (Decomposition theorem, 5.4.5 of [3]) Let  $K_0 \in D_m^b(X_0, \overline{\mathbb{Q}}_\ell)$  be  $\tau$ -pure of weight  $w$ . Consider the base change  $K$  of  $K_0$  to the algebraic closure  $K \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$  where  $X = X_0 \times_k \bar{k}$ . Then

$$K \cong \bigoplus_i H^i(K)[-i]$$

. Here the number of  $i$  is finite, namely it is a finite direct sum.

*Proof.* As proof in corollary 5.1.3 take an integer  $i$  such that  ${}^p\tau^{\leq i-1}K_0$  is pure of weight  $w$ .

$${}^p\tau^{\leq i-1}K_0 \rightarrow {}^p\tau^{\leq i}K_0 \rightarrow {}^pH^i(K_0)[-i] \rightarrow \cdots$$

By corollary 5.1.4  $w({}^pH^i(K_0)[-i]) = w$ , so we can apply proposition 4.1.10 get

$$\text{Ext}^1({}^pH^i(K_0)[-i], {}^p\tau^{\leq i-1}K_0) = 0$$

So  ${}^p\tau^{\leq i}K_0 = {}^p\tau^{\leq i-1}K_0 \oplus {}^pH^i(K_0)[-i]$ , which completes the proof by induction since  ${}^p\tau^{\leq m}K_0 = 0$  for  $m \ll 0$ .  $\square$

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