目录

1 Introduction

The goal of the note is to prove the Riemann hypothesis part of the Weil conjecture.

1.1 Weil Conjecture

We fix the following notation. Denote by $k = \mathbb{F}_q$ the finite field of q elements. $\overline{k}$ an algebraic closure of k and k_n a degree n extension of k. A variety is a scheme X over k such that X is integral and the structure morphism $X \to Spec(k)$ is separated and of finite type.

X_0 will denote a variety over k, and $X := X_0 \times_k \overline{k}$. The idea is to compute how many F_{q^r}-points there are on X_0. Denote $N_r = \#X_0(F_{q^r})$ For this, we introduce a formal power series.

Definition 1.1.1 *The zeta function for X_0 is defined as the formal power series*

$$Z(X_0,t) = exp(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}) \in \mathbb{Q}[[t]].$$

Example 1.1.2. Let $X_0 = Spec(F_q)$. Then

$$Z(X_0,t) = exp(\sum_{r=1}^{\infty} 1 \cdot \frac{t^r}{r}) = \frac{1}{1-t}.$$

Example 1.1.3. Let $X_0 = P^1$. Then

$$Z(X_0,t) = exp(\sum_{r=1}^{\infty} (1+q^r)\frac{t^r}{r}) = \frac{1}{(1-t)(1-qt)}.$$

These two are rational functions, Weil also compute several other complicated examples. This made him formulate the following conjecture:

Conjecture 1.1 *Let X_0 be a smooth projective variety over k of dimension n, and $Z(t) = Z(X_0,t)$.*

1. Rationality: $Z(t)$ is a rational function.

2. Function equation:

$$Z(\frac{1}{q^n t}) = \pm q^{nE/2} t^E \cdot Z(t).$$

Here $E = \Delta \cdot \Delta$, where $\Delta \subseteq X_0 \times X_0$ is the diagonal.

3. Riemann hypothesis: The rational function has a special form:

$$Z(t) = \frac{P_1(t)P_3(t)\dots P_{2n-1}(t)}{P_0(t)P_2(t)\dots P_{2n}(t)}$$

where each $P_i(t)$ satisfies the following properties:

(a) $P_0(t) = 1-t \in \mathbb{Z}[t]$.

(b) $P_{2n}(t) = 1-q^n t \in \mathbb{Z}[t]$.

(c) For $1 \leq i \leq 2n-1$, we have

$$P_i(t) = \prod_j (1-\alpha_{ij}t) \in \mathbb{Z}[t].$$

where each α_{ij} is an algebraic integer, and $|\alpha_{ij}| = q^{i/2}$. $|\cdot|$ denotes the complex norm for ant embedding of $\mathbb{Z}[\alpha_{ij}]$ in $\mathbb{C}$.

We mainly focus on (3), (1) and (2) are covered in general courses on etale cohomology.

1.2 etale cohomology

2 Weil sheaves

2.1 Several Frobenius

Now, fix a scheme X_0 of finite type over k, $X := X_0 \times_k \overline{k}$ with $\pi : X \to X_0$ the projection morphism. Let $F : \alpha \mapsto \alpha^{1/q}$ be the *geometric* frobenius automorphism of $\overline{k}$.

Definition 2.1.1 *The base change $F_X := id\ X_0 \times_k F$ acts as an automorphism of X. I will call this the* ***Galois-theoretic geometric Frobenius*** *automorphism of X to emphasize that this is coming from $Gal(\overline{k}/k)$.*

There are several other Frobenius morphisms:

Definition 2.1.2 *The* ***absolute Frobenius*** *endmorphism of an F_{q^r}-scheme Y is the morphism $\sigma_Y : Y \to Y$ which is the identity on the underlying topological space $|Y|$ and which is the map $\alpha \mapsto \alpha^q$ on the structure sheaf.*

Definition 2.1.3 *The* ***relative Frobenius*** *endomorphism of X is the morphism $Fr_X : \sigma_{X_0} \times_k id_{\overline{k}}$.*

Proposition 2.1.4 $\sigma_X \circ F_X = Fr_X$

Now we analysis the action of these frobenius on cohomology group.

For any X-scheme Y, let $Y(q/X) = Y \times_{X,\sigma_X} X$. We have following Cartesian diagram

$$\begin{array}{ccc} Y^{(q/X)} & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_2}\downarrow & & \downarrow \\ X & \xrightarrow[\sigma_X]{} & X \end{array}$$

where π_1 and π_2 are the projections. We have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\sigma_Y} & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow[\sigma_X]{} & X \end{array}$$

We define the *Frobenius morphism* $Fr_{Y/X}$ of Y relative to X to be the X-morphism $Fr_{Y/X} : Y \to Y(q/X)$ such that $\pi_1 Fr_{Y/X} = \sigma_Y$.

$$\begin{tikzcd}
Y \arrow[dr, "Fr_{Y/X}"] \arrow[drr, "\sigma_Y"] \arrow[ddr] & & \\
& Y(q/X) \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & Y \arrow[d] \\
& X \arrow[r, "\sigma_X"] & X
\end{tikzcd}$$

Proposition 2.1.5 *The relative Frobenius morphism $Fr_{Y/X} : Y \to Y(q/X)$ is integral, radiciel, and surjective. If Y is etale over X, then $Fr_{Y/X}$ is an isomorphism.*

Proof. Clearly σ_X, σ_Y and π_1 are integral, radiciel and surjective. So $Fr_{Y/X}$ has the same property. If Y is etale over X, then $Fr_{Y/X}$ is also etale. Therefore $Fr_{Y/X}$ is an isomorphism.

For any character p scheme S. Let $\mathscr{G}$ be sheaf on S. For any etale S-scheme U, since $Fr_{U/S}$ is an isomorphism, the restriction map $\mathscr{G}(U^{(q/S)}) \to \mathscr{G}(U)$ is an isomorphism. Thus we have an isomorphism $\sigma_{S*}\mathscr{G} \to \mathscr{G}$. Its inverse $\mathscr{G} \to \sigma_{S*}\mathscr{G}$ defines a morphism

$$\sigma_{\mathscr{G}}^* : \sigma_S^*\mathscr{G} \to \mathscr{G}.$$

We also define a morphism $\sigma_{\mathscr{G}}^* : \sigma_S^* K \to K$ for any object K in the derived category $D(S) = D(S, \mathbb{Z})$ of sheaves of abelian group on S.

Proposition 2.1.6 *Let $K \in obD^+(X)$. Then composite*

$$H^i(X,K) \to H^i(X, \sigma_X^* K) \overset{\sigma_K^*}{\to} H^i(X,K)$$

is identity for each i, where the first homomorphism is the composite

$$H^i(X,K) \overset{adj}{\to} H^i(X, R\sigma_{X*}\sigma_X^* K) \cong H^i(X, \sigma_X^* K).$$

Let $\mathscr{G}_0$ be a sheaf on X_0, let $\pi : X \to X_0$ be the projection, and let $\mathscr{G} = \pi^*\mathscr{G}_0$ be the inverse image of $\mathscr{G}_0$. Define

$$Fr_{\mathscr{G}_0}^* : Fr_X^*\mathscr{G} \to \mathscr{G}$$

to be the morphism induced from $\sigma_{\mathscr{G}_0}^* : \sigma_X^*\mathscr{G}_0 \to \mathscr{G}_0$ by base change, that is, the composite

$$Fr_X^*\mathscr{G} = Fr_X^*\pi^*\mathscr{G}_0 \cong \pi^*\sigma_{X_0}^*\mathscr{G}_0 \overset{\pi^*(\sigma_{\mathscr{G}_0}^*)}{\to} \pi^*\mathscr{G}_0 = \mathscr{G}.$$

For any $K_0 \in obD(X_0)$, we can define $Fr^*_{K_0} : Fr^*_K \to K$ where $K = \pi^* K_0$. On the other hand, we have an isomorphism $F_X^* \mathscr{G} \cong \mathscr{G}$ define as the composite

$$F_X^* \mathscr{G} = F_X^* \pi^* \mathscr{G}_0 \cong (\pi \circ F_X)^* \mathscr{G}_0 = \pi^* \mathscr{G}_0 = \mathscr{G}.$$

Proposition 2.1.7 *Notation as above. For any $K_0 \in obD^+(X_0)$, the composite denote by FR^*_K*

$$H^i(X,K) \to H^i(X, Fr_X^* K) \overset{Fr^*_{K_0}}{\to} H^i(X,K)$$

and the composite

$$H^i(X,K) \to H^i(X, F_X^* K) \cong H^i(X,K)$$

are the same.

Let x be a closed point of X_0 with $[k(x) : k] = n$. Then $\bar{x}$ is a fixed point of Fr^n. Then n-th iteration of $Fr^*_{\mathscr{G}}$ induces a homomorphism $Fr{x}^{n*}_{\bar{x}} : \mathscr{G}_{\bar{x}} \to \mathscr{G}_{\bar{x}}$. Let $f_x : \mathscr{G}_{\bar{x}} \to \mathscr{G}_{\bar{x}}$ be the action on $\mathscr{G}_{\bar{x}}$ of the Frobenius substitution $\alpha \mapsto \alpha^{q^n}$ in $Gal(\bar{k}/k(x))$.

Proposition 2.1.8 *With the above notation, the homomorphism $Fr{x}^{n*}_{\bar{x}} : \mathscr{G}_{\bar{x}} \to \mathscr{G}_{\bar{x}}$ and $f_x : \mathscr{G}_{\bar{x}} \to \mathscr{G}_{\bar{x}}$ are inverse to each other.*

Let X_0 be a compactifiable scheme over $F_q = k$ of character p, A a noetherian $\mathbb{Z}/\ell^n$-algebra with $(\ell, p) = 1$, $K_0 \in obD^b_{ctf}(X_0, A)$. Choose a compactification $\overline{X_0}$ of X_0, let $\overline{X} = \overline{X_0} \otimes_k \bar{k}$, and $j : X \hookrightarrow \overline{X}$ be the open immersion. Still denote by $FR^*_K : R\Gamma_c(X,K) \to R\Gamma_c(X,K)$

$$\begin{aligned} R\Gamma_c(X,K) &\cong R\Gamma(\overline{X}, j_! K) \\ &\to R\Gamma(\overline{X}, Fr^*_{\overline{X}} j_! K) \\ &\cong R\Gamma(\overline{X}, j_! Fr^*_X K) \\ &\overset{Fr^*_{K_0}}{\to} R\Gamma(\overline{X}, j_! K) \\ &\cong R\Gamma_c(X,K) \end{aligned}$$

$$\begin{tikzcd} & X \arrow[r, "j"] \arrow[d, "Fr_X"] & \overline{X} \arrow[d, "Fr_{\overline{X}}"] & \\ X \arrow[r, "\sigma_X"] \arrow[d, "\pi"] & X \arrow[r, "j"] \arrow[d, "\pi"] & \overline{X} \arrow[r] \arrow[d, "\overline{\pi}"] & Spec\bar{k} \arrow[d] \\ X_0 \arrow[r, "\sigma_{X_0}"] & X_0 \arrow[r, "j_0"] & \overline{X_0} \arrow[r] & Speck \end{tikzcd}$$

Proposition 2.1.9 *The action of Fr_X on cohomology group $FR^*_{\mathscr{G}}: H^i_c(X,\mathscr{G}) \to H^i_c(X,\mathscr{G})$ agrees with the Galois action of the automorphism F_X.*

If $\bar{x}$ is an $\bar{k}$-point of X fixd by Fr_X, then $Fr^*_{K_0}: Fr^*_X K \to K$ induces a morphism $Fr_X{}^*_{\bar{x}}: K_{\bar{x}} \to K_{\bar{x}}$. We have following formula.

Theorem 2.1.1 ***(Lefschetz trace formula)****. Let $X^{Fr_X} \cong X_0(F_q)$ be the set of fixed points of Fr_X on $X(\overline{F_q}) \cong X_0(\overline{F_q})$. We have*

$$\sum_{\bar{x}\in X^{Fr_X}} Tr(Fr_X{}^*_{\bar{x}}, K_{\bar{x}}) = Tr(FR^*_K, R\Gamma_c(X,K)).$$

Define the L-function of K_0 to be

$$L(X_0,K_0,s) = \prod_{x\in|X_0|} \frac{1}{det(1-\frac{1}{q^{sd(x)}}f_x^{-1}, K_{\bar{x}})}$$

where

$$\frac{1}{det(1-\frac{1}{q^{sd(x)}}f_x^{-1}, K_{\bar{x}})} = \prod_i det(1-\frac{1}{q^{sd(x)}}f_x^{-1}, \mathscr{H}^i(K_{\bar{x}}))^{(-1)^i}$$

Making the change of variable $t = q^{-s}$, we can also define the L-function as

$$L(X_0,K_0,s) = \prod_{x\in|X_0|} \frac{1}{det(1-t^{d(x)}f_x^{-1}, K_{\bar{x}})}$$

Since $Fr_X{}^{n*}_{\bar{x}} = f_x^{-1}$, we have

$$L(X_0,K_0,t) = \prod_{x\in|X_0|} det(1-t^{d(x)}F_X{}^{deg(x)}_{\bar{x}}, \mathscr{G}_{\bar{x}})^{-1}$$

Theorem 2.1.2 *(Grothendieck). Notation as above, we have*

$$L(X_0,K_0,t) = \prod_i det(1-FR^*_K t, H^i_c(X,K))^{(-1)^{i+1}}$$

Lemma 2.1.3 *Let V be a finite dimensional vector space and F an endomorphism of V. Then*

$$det(1-tF|V)^{-1} = exp(\sum_{r=1}^{\infty} Tr(F^r)\frac{t^r}{r})$$

Proposition 2.1.10 *Let X_0 be projective and smooth over k, of dimension n. Then $P_i(t) = det(1-FR^*_{\mathbb{Q}_\ell}t, H^i(X,\mathbb{Q}_\ell))$*

Proof. Lefschetz trace formula tells us that

$$N_r = \sum_{i=0}^{2n} (-1)^i Tr(FR^{*}_{\mathbb{Q}_\ell}{}^{r}, H^i(X, \mathbb{Q}_\ell))$$

So by Lemma2.1.3the zeta function is just

$$\begin{aligned} Z(X_0,t) &= exp(\sum_{r=1}^{\infty} \sum_{i=0}^{2n} (-1)^i Tr(FR^{*}_{\mathbb{Q}_\ell}{}^{r}, H^i(X, \mathbb{Q}_\ell)) \frac{t^r}{r}) \\ &= \prod_{i=0}^{2n} [exp(\sum_{r=1}^{\infty} Tr(FR^{*}_{\mathbb{Q}_\ell}{}^{r}, H^i(X, \mathbb{Q}_\ell)) \frac{t^r}{r})]^{(-1)^i} \\ &= \prod_{i=0}^{2n} det(1 - FR^{*}_{\mathbb{Q}_\ell} t, H^i(X, \mathbb{Q}_\ell))^{(-1)^{i+1}} \end{aligned}$$

So if we fix an embedding $\tau : \mathbb{Q}_\ell \to \mathbb{C}$, then Riemann hypothesis is equivalent that $\tau(\alpha) = q^{i/2}$ for all eigenvalues α of $FR^{*}_{\mathbb{Q}_\ell}(F_X)$ on $H^i(X, \mathbb{Q}_\ell)$. We will use language of weight for a sheaf to analysis this statement.

2.2 Deligne's theorem

Definition 2.2.1 *Let β be a real number, $\mathscr{G}_0$ is a $\overline{\mathbb{Q}_\ell}$ sheaf on X_0. Fix $\tau : \overline{\mathbb{Q}_\ell} \cong \mathbb{C}$*

*1. Choose a $\overline{k}$-point $\overline{x} \in X$ lying over $x \in |X_0|$. We say that $\mathscr{G}_0$ is **τ-pure of weight** β if for every $x \in |X_0|$, and all eigenvalues $\alpha \in \overline{\mathbb{Q}_\ell}$ of $F_{X_{\overline{x}}}^{deg(x)}$ on $\mathscr{G}_{0\overline{x}}$, we have*

$$|\tau(\alpha)| = (q^{deg(x)})^{\beta/2}.$$

2.We say $\mathscr{G}_0$ is τ-mixed if there is a finite filtration of subsheaves

$$0 = \mathscr{G}_0^{(0)} \subseteq \mathscr{G}_0^{(1)} \subseteq \cdots \subseteq \mathscr{G}_0^{(r)} = \mathscr{G}_0$$

such that $\mathscr{G}_0^{(j)} / \mathscr{G}_0^{(j-1)}$ is τ-pure of some weight.

In addition

*3. If $V \in Rep^{cont}_{\overline{\mathbb{Q}_\ell}}(Gal(\overline{k}/k))$, then we say V is **τ-pure of weight** β if for all eigenvalues $\alpha \in \overline{\mathbb{Q}_\ell}$ of F_X on V, we have*

$$|\tau(\alpha)| = q^{\beta/2}.$$

Remark. The constant sheaf $\overline{\mathbb{Q}_\ell}$ is pure of weight 0 and the tate twist $\overline{\mathbb{Q}_\ell}(1)$ is pure of weight -2. Since $\chi(F) = \frac{1}{q}$ for the cyclotimic character

$$\chi : Gal(\overline{k}/k) \to \mathbb{Z}_\ell^\times$$

Proposition 2.2.2 *Let X_0 be a smooth, geometrically connected variety of dimension n, $\mathscr{F}$ be a* τ***-pure of weight*** β *lisse $\overline{\mathbb{Q}_\ell}$ sheaf. Then $H^0(X,\mathscr{F}) \supset H^0_c(X,\mathscr{F})$ is* τ***-pure of weight*** β*,*

Proof. For any $x \in |X_0|$, $H^0(X,\mathscr{F}) = \mathscr{F}_{\bar{x}}^{\pi_1(X,\bar{x})} \subseteq \mathscr{F}_{\bar{x}}$. So any eigenvalue of $H^0(X,\mathscr{F})$ is an eigenvalue of $\mathscr{F}_{\bar{x}}$.

Theorem 2.2.1 (*Main theorem of Weil II*) . *Let $f : X_0 \to Y_0$ be a separated morphism of schemes of finite type over F_q. If $\mathscr{G}_0$ is a $\overline{\mathbb{Q}_\ell}$-sheaf on X_0 that is τ-mixed of weight $\leq n$, then for every integer $i \geq 0$ the sheaf $R^i f_! \mathscr{G}_0$ is τ-mixed of weight $\leq n+i$.*

Corollary 1 *If X_0 is a smooth proper variety over F_q of dimension n, and $\mathscr{G}_0$ is τ-pure of weight w, then $R^i f_* \mathscr{G}_0$ is τ- pure of weight $w+i$.*

Proof. We know $(R^i f_* \mathscr{G}_0)_{\bar{x}} = H^i(X,\mathscr{G})$. By theorem 2.2.1 $H^i(X,\mathscr{G})$ is τ-mixed of weight $\leq w+i$. By poincare duality

$$H^i(X,\mathscr{G}) \cong (H^{2n-i}(X,\mathscr{G}^\vee))^\vee(-n)$$

$H^{2n-i}(X,\mathscr{G}^\vee)$ is τ-mixed of weight $\leq -w+2n-i$, so the right side has τ-mixed of weight $\geq w-2n+i+2n = w+i$. Namely $R^i f_* \mathscr{G}_0$ is τ- pure of weight $w+i$.

Remark 2.2.1 *This is immediately implies Weil I(which the case $\mathscr{G}_0 = \overline{\mathbb{Q}_\ell}$, of weight 0). Also notice that we assume that X_0 is only proper , not necessarily projective.*

2.3 Weil sheaf

Definition 2.3.1 *A Weil sheaf $\mathscr{G}_0$ on X_0 consists of a constructible $\overline{\mathbb{Q}_\ell}$-sheaf on X, plus a specified isomorphism $F_{\mathscr{G}_0} : F_X^* \mathscr{G} \to \mathscr{G}$. A lisse Weil sheaf on X_0 is a Weil sheaf $\mathscr{G}_0$ such that the corresponding constructible $\overline{\mathbb{Q}_\ell}$-sheaf $\mathscr{G}$ on X is lisse.*

Notice that every constructible $\overline{\mathbb{Q}_\ell}$-sheaf $\mathscr{G}$ is canonically a Weil sheaf, via the canonical isomorphism $F_X^* \pi^* \mathscr{G}_0 \xrightarrow{\sim} (\pi \circ F_X)^* = \pi^* \mathscr{G}_0$.

Proposition 2.3.2 *Assume that X_0 is geometrically connected and x is a point of X_0. Then the functor $\mathscr{F} \mapsto \mathscr{F}_{\bar{x}}$ defines an equivalence of categories from the category of lisse $\overline{\mathbb{Q}_\ell}$-sheaves to the category of continuous finite-dimensional representation of $\pi_1(X_0,\bar{x})$ over $\overline{\mathbb{Q}_\ell}$.*

Remark 2.3.1 *A $\overline{\mathbb{Q}_\ell}$-representation for $\pi_1(X,\bar{x})$ is a homomorphism $\pi_1(X,\bar{x}) \to GL(V)$ for some finite dimensional $\overline{\mathbb{Q}_\ell}$-vector space V such that we can find a finite extension E of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}_\ell}$ and a finite dimensional E-vector space V_E with a continuous $\pi_1(X,\bar{x})$-action with the property that $V \cong V_E \otimes_E \overline{\mathbb{Q}_\ell}$. And the homomorphism $\pi_1(X,\bar{x}) \to GL(V)$ is the composite*

$$\pi_1(X,\bar{x}) \to GL(V_E) \to GL(V_E \otimes_E \overline{\mathbb{Q}_\ell}) \cong GL(V).$$

Recall that we have the monodromy exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X,\bar{x}) \arrow[r] \arrow[d, "\cong"] & \pi_1(X_0,\bar{x}) \arrow[r] & Gal(\bar{k}/k) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(X,\bar{x}) \arrow[r] & W(X_0,\bar{x}) \arrow[r] \arrow[u, hook] \arrow[dr, "deg"'] & W(\bar{k}/k) \arrow[r] \arrow[u, hook] \arrow[d, "\cong"] & 1 \\
& & & \mathbb{Z} &
\end{tikzcd}$$

where $Gal(\bar{k}/k) \cong \hat{\mathbb{Z}}$ with topological generator F. The Weil group $W(\bar{k}/k)$ is defined to be the infinite cyclic subgroup generated by F.

Definition 2.3.3 *.The Weil group of X_0 , denoted by $W(X_0,\bar{x})$, is the inverse image of $W(\bar{k}/k)$ under the first exact sequence. We call the map $W(X_0,\bar{x}) \to W(\bar{k}/k) \cong \mathbb{Z}$, which the latter isomorphism sending F to 1. We will use σ to denote any degree one element. Clearly, $W(X_0,\bar{x}) = \pi_1(X,\bar{x}) \rtimes \langle\sigma\rangle$ with $\sigma \in \pi_1(X_0,\bar{x})$ which is a degree one element.*

Proposition 2.3.4 *σ act on X by $F_X := id_{X_0} \times_k F$.*

Proof. For each positive integer n, F is an automorphism of k_n. Denote $X_n = X_0 \times_k k_n$, then $\{X_n\}_{n\geq 0}$ is a direct system with affine transition maps, So $X = \varprojlim X_n$. F act on every X_n by $id_{X_0} \times_k F$, these actions are compatible. We are done.

Now we have

Proposition 2.3.5 *Assume that X_0 is geometrically connected and x is a point of X. Then the functor $\mathscr{G}_0 \mapsto \mathscr{G}_{0\bar{x}}$ defines an equivalence of categories from the category of lisse Weil-sheaves to the category of continuous finite-dimensional representation of $W(X_0,\bar{x})$ over $\overline{\mathbb{Q}_\ell}$.*

Proof. We have an action $\pi_1(X,\bar{x}) \to GL(\mathscr{G}_{\bar{x}})$, and need to prove it can factor by $W(X_0,\bar{x})$. This just above proposition.

$$\begin{tikzcd}
\pi_1(X,\bar{x}) \arrow[rr] \arrow[dr] & & GL(\mathscr{G}_{\bar{x}}) \\
& W(X_0,\bar{x}) \arrow[ur] &
\end{tikzcd}$$

Special case. Lisse rank 1 Weil sheaf on $Spec\mathbb{F}_q$ are the same thing as characters

$$\phi : W(\overline{F_q}/F_q) \to \overline{\mathbb{Q}_\ell}^*$$

with $\phi(F) = b$. Conversely, any $b \in \overline{\mathbb{Q}_\ell}^*$ gives a Weil sheaf $\mathscr{L}_b$ on $Spec\mathbb{F}_q$. We wil also us $\mathscr{L}_b$ to denote the pullback of this sheaf to X_0 .

Now we are curious about how different Weil sheaf and general $\overline{\mathbb{Q}_\ell}$-sheaf, following are some criteria. First, we have a criterion for a lisse Weil sheaf to be a $\overline{\mathbb{Q}_\ell}$-sheaf.

Proposition 2.3.6 *A lisse Weil sheaf $\mathscr{G}_0$ on a geometrically connected finite type k-scheme X_0 is an ordinary $\overline{\mathbb{Q}_\ell}$-sheaf if and only if some (or equivalent, any) degree-1 element $\sigma \in W(X_0,\bar{x})$ acts on $\mathscr{G}_{0\bar{x}}$ with eigenvalues which are ℓ-adic units (i.e units of $\mathscr{O}_E$).*

Proof. Denote $V = \mathscr{G}_{\bar{x}}$ with dimension n, then $Aut_{\overline{\mathbb{Q}_\ell}}(V) \cong GL_n(\overline{\mathbb{Q}_\ell})$. By proposition2.3.5, the question is just whether the $\overline{\mathbb{Q}_\ell}$-representation (ρ,V) of $W(X_0,\bar{x})$ given by the action of $W(X_0,\bar{x})$ on V extends to a representation of $\pi_1(X_0,\bar{x})$.
Assume image $\pi_1(X,\bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$ is contained in $GL_n(E_0)$ for some finite extension E_0 of $\mathbb{Q}_\ell$. Then the image of $W(X_0,\bar{x})$ is generated by the images of $\pi_1(X,\bar{x})$ and of σ, so we can increase E_0 to include the matrix coefficients of $\rho(\sigma)$ and its eigenvalues. Then the representation is defined over some finite extension E of $\mathbb{Q}_\ell$. Let $\{e_1, e_2 \cdots e_n\}$ is a basis of E.

$$\begin{tikzcd}
\pi_1(X,\bar{x}) \arrow[r] & W(X_0,\bar{x}) \arrow[r, "\rho"] \arrow[d] & GL_n(E) \arrow[r] & GL_n(\overline{\mathbb{Q}_\ell}) \\
& \pi_1(X_0,\bar{x}) \arrow[urr, dashed, "?"] & &
\end{tikzcd}$$

$(\Rightarrow)$. If the representation can extend to $\pi_1(X_0,\bar{x})$, then there is a $\pi_1(X_0,\bar{x})$-stable lattice $T \cong \mathscr{O}_E^n$, so $\pi_1(X_0,\bar{x})$ factor through $GL_n(\mathscr{O}_E)$, and the eigenvalues of an element of $GL_n(\mathscr{O}_E)$ are ℓ-adic units.

$$\begin{tikzcd}
\pi_1(X_0,\bar{x}) \arrow[rr] \arrow[dr] & & GL_n(E) \\
& GL_n(\mathscr{O}_E) \arrow[ur] &
\end{tikzcd}$$

$(\Leftarrow)$. Note that $\pi_1(X_0,\bar{x})$ is the profinite completion of $W(X_0,\bar{x})$, so any continuous homomorphism from $W(X_0,\bar{x})$ to a profinite group extends to $\pi_1(X_0,\bar{x})$. If $W(X_0,\bar{x})$ stabilizes a lattice T, then the map $W(X_0,\bar{x}) \to GL_n(E)$ factors through the profinite group $GL_n(\mathscr{O}_E)$. So we need to show that if the eigenvalues of $\rho(\sigma)$ are ℓ-adic units, then there is a $W(X_0,\bar{x})$-stable lattice.

We can use multiplicative Jordan decomposition theorem for $\rho(\sigma)$ to write

$$\rho(\sigma) = \sigma_{ss} \cdot \sigma_{uu}$$

with σ_{ss} semisimple and σ_{uu} unipotent. By splitting V into the eigenspaces of σ_{ss} and construct a stable lattice in each subspace, we can assume σ_{ss} acts on V by multiplicating an ℓ-adic unit. Therefore, any lattice is stable for σ_{ss}. Let $\sigma_{uu} = 1+N$, with $N^k = 0$ for some $k \geq 0$. Let $L = \mathscr{O}_E e_1 + \mathscr{O}_E e_2 + \cdots + \mathscr{O}_E e_n$. Then $M = L + \sigma_{uu} L + \cdots + \sigma_{uu}^{k-1} L$ is stable under σ_{uu} and therefore also by $\rho(\sigma)$.

Theorem 2.3.1 *Let X_0 be a finite type scheme over $\mathbb{F}_q$, and let $\mathscr{G}_0 = (F_X^* \mathscr{G} \xrightarrow{\sim} \mathscr{G})$ be a Weil sheaf on X_0. Then*

1. If X_0 is normal and geometrically connected, and if $\mathscr{G}_0$ is irreducible and lisse of rank r, then $\mathscr{G}_0$ is an etale $\overline{\mathbb{Q}_\ell}$-sheaf on X_0 if and only if $\wedge^r \mathscr{G}_0$ is an etale $\overline{\mathbb{Q}_\ell}$-sheaf.

Corollary 1. *For any smooth, irreducible sheaf $\mathscr{G}_0$, there exists some $\mathscr{L}_b$ and some $\mathscr{F}_0$ an etale $\overline{\mathbb{Q}_\ell}$-sheaf such that $\mathscr{G}_0 \cong \mathscr{F}_0 \otimes \mathscr{L}_b$.*

2. For a general smooth Weil sheaf $\mathscr{G}_0$ on a normal, geometrically connected X_0, there exists a filtration

$$0 = \mathscr{G}_0^{(0)} \subseteq \mathscr{G}_0^{(1)} \subseteq \cdots \subseteq \mathscr{G}_0^{(r)} = \mathscr{G}_0$$

where $\mathscr{G}_0^{(j)} / \mathscr{G}_0^{(j-1)} \cong \mathscr{F}_0^{(j)} \otimes \mathscr{L}_{b_j}$. Here $\mathscr{F}_0^{(j)}$ is lisse etale $\overline{\mathbb{Q}_\ell}$-sheaf, and $\mathscr{L}$ is a Weil sheaf.

Corollary2*.(Grothendieck trace formula for Weil sheaves). Given a smooth weil sheaf $\mathscr{G}_0$ on X_0, define*

$$L(X_0, \mathscr{G}_0, t) = \prod_{x \in |X_0|} det(1 - t^{d(x)} F_{X_{\overline{x}}}^{deg(x)}, \mathscr{G}_{\overline{x}})^{(-1)}$$

Then it can be computed as

$$L(X_0, \mathscr{G}_0, t) = \prod_{i=0}^{2dimX} det(1 - FR_{\mathscr{G}}^* t, H_c^i(X, \mathscr{G}))^{(-1)^{i+1}}$$

Proof. ($1 \Rightarrow$ ***Corollary 1***)we have

$$\wedge^n (\mathscr{G}_0 \otimes \mathscr{L}_{det(\sigma)^{-1/n}}) = \wedge^n (\mathscr{G}_0) \otimes \mathscr{L}_{det(\sigma)^{-1}}$$

where $det(\sigma)$ is the determinant of action of σ on $\mathscr{G}_{\overline{x}}$. The eigenvalues of right side by action of σ is 1, So by proposition2.3.6, we are done.

($2 \Rightarrow$ ***Corollary 2***) In the irreducible case, $\mathscr{G}_0 \cong \mathscr{F}_0 \otimes \mathscr{L}_b$ and $\mathscr{G}_{\overline{x}} \cong \mathscr{F}_{\overline{x}} \otimes \mathscr{L}_{b\overline{x}}$. Then

$$det(1 - t^{d(x)} F{X_{\overline{x}}}^{deg(x)}, \mathscr{G}_{\overline{x}}) = det(1 - t^{d(x)} b^{deg(x)} F{X_{\overline{x}}}^{deg(x)}, \mathscr{F}_{\overline{x}})$$

and

$$det(1 - FR^*_{\mathscr{G}} t, H^i_c(X, \mathscr{G})) = det(1 - FR^*_{\mathscr{G}} tb, H^i_c(X, \mathscr{F}))$$

So by Grothendieck trace formula of etale sheaf, we are done.
For general case, use filtration of (2).

3 Weight

Definition 3.0.1 *Let β be a real number. $\mathscr{G}_0$ is a (Weil) sheaf on X_0. Fix $\tau : \overline{\mathbb{Q}_l} \cong \mathbb{C}$*
*1. Choose a $\overline{k}$-point $\overline{x} \in X$ lying over $x \in |X_0|$. The Weil group $W(\overline{k}/k(x))$ acts on the stalk at $\mathscr{G}_{0\overline{x}}$ via the geometric frobenius $F_x : \mathscr{G}_{0\overline{x}} \to \mathscr{G}_{0\overline{x}}$. We say that $\mathscr{G}_0$ is **τ-pure of weight** β if for every $x \in |X_0|$, and all eigenvalues $\alpha \in \overline{\mathbb{Q}_l}$ of F_x, we have*

$$|\tau(\alpha)| = N(x)^{\beta/2}.$$

2. We say $\mathscr{G}_0$ is τ-mixed if there is a finite filtration of subsheaves

$$0 = \mathscr{G}_0^{(0)} \subseteq \mathscr{G}_0^{(1)} \subseteq \cdots \subseteq \mathscr{G}_0^{(r)} = \mathscr{G}_0$$

such that $\mathscr{G}_0^{(j)} / \mathscr{G}_0^{(j-1)}$ is τ-pure of some weight.
3. $\mathscr{G}_0$ is pure of weight β if it is τ-pure of weight β for all $\tau : \overline{\mathbb{Q}_l} \cong \mathbb{C}$
4. $\mathscr{G}_0$ is mixed if there exists a finite filtration as in (2) such that all quotient are pure.

Definition 3.1.2. For a scheme X_0/k and sheaf $\mathscr{G}_0$ on X_0, we define the **maximal weight** of $\mathscr{G}_0$ (with respect to τ)as

$$\omega(\mathscr{G}_0) := \sup_{x \in |X_0|} \sup_{\alpha\ eigenvalue\ of\ F_x} \frac{log(|\tau(\alpha)|^2)}{log N(x)}$$

For convenience, we define $\omega(0) = -\infty$.

3.1 convergence of L-function

In this section, we prove that the weight of a weil sheaf controls the convergence of its L-function.

Lemma 3.1.1 *Let X_0/k be a scheme. Then we have estimate*

$$|X_0(k_n)| = O(q^{ndimX_0})$$

Lemma 3.1.2 *Let V be a finite dimensional vector space and F an endomorphism of V, and $d \in \mathbf{N}$ is a non-negative integer. Then*

$$\frac{d}{dt}logdet(1-t^dF|V)^{-1} = \sum_{n\geq 1} Tr(F^n)dt^{dn-1}$$

Proof. We have formula by linear algebra

$$det(1-t^dF|V)^{-1} = exp(\sum_{n\geq 1} Tr(F^n)\frac{t^{dn}}{n})$$

taking derivatives, we have

$$\begin{aligned}\frac{d}{dt}det(1-t^dF|V)^{-1} &= (\sum_{n\geq 1} Tr(F^n)dn\frac{t^{dn-1}}{n})\cdot exp(\sum_{n\geq 1} Tr(F^n)\frac{t^{dn}}{n}) \\ &= (\sum_{n\geq 1} Tr(F^n)dt^{dn-1})\cdot det(1-t^dF|V)^{-1}\end{aligned}$$

This is just lemma.

Proposition 3.1.1 *Let $\mathscr{G}_0$ be a sheaf on X_0 and β is a real number such that $w(\mathscr{G}_0) \leq \beta$. Then the L-function*

$$\tau L(X_0,\mathscr{G}_0,t) = \prod_{x\in|X_0|} \tau det(1-t^{d(x)}F_x,\mathscr{G}_{0\bar{x}})$$

converges for all $|t| < q^{-\beta/2-dimX_0}$

Proof. By complex analysis we know the logarithmic derivative of a complex valued function has poles precisely as the original function has poles or zeroes.We will suppress τ, it makes no difference.so taking logarthmic we have

$$\frac{d}{dt}logL(X_0,\mathscr{G}_0,t) = \sum_{x\in|X_0|}\frac{d}{dt}log(det(1-t^{d(x)}F_x,\mathscr{G}_{0\bar{x}})) \tag{1}$$

$$\overset{by(1)}{=} \sum_{x\in|X_0|}\sum_{n\geq 1} d(x)Tr(F_x^n)t^{d(x)n-1} \tag{2}$$

$$= \sum_{n\geq 1}(\sum_{x\in|X_0|:d(x)|n} d(x)(Tr(F_x^{n/d(x)})))t^{n-1}. \tag{3}$$

Let $r := \max_{x\in|X_0|} dim_{\overline{\mathbf{Q_l}}}\mathscr{G}_{0\bar{x}}$, then $w(\mathscr{G}_0) \leq \beta$ implies

$$|Tr(F_x^{n/d(x)})| \leq |r\alpha^{n/d(x)}| \leq \ rq^{n\beta/2}$$

Hence

$$\begin{aligned}
\frac{d}{dt}logL(X_0,\mathscr{G}_0,t) &= \sum_{n\geq 1}(\sum_{x\in|X_0|:d(x)|n} d(x)(Tr(F_x^{n/d(x)})))t^{n-1} \\
&\leq \sum_{n\geq 1}(\sum_{x\in|X_0|:d(x)|n} d(x)\cdot(rq^{n\beta/2}))t^{n-1} \\
&= \sum_{n\geq 1}|X_0(k_n)|\cdot(rq^{n\beta/2}))t^{n-1} \\
&\leq C\cdot rq^{dimX_0+\beta/2}\sum_{n\geq 1}(tq^{dimX_0+\beta/2})^{n-1}
\end{aligned}$$

The last inequality follows Lemma 3.1.1 , since we have

$$|X_0(k_n)| \leq C(q^{ndimX_0})$$

for some constan C. so logarithmic derivative converges for all $|t| < q^{-\beta/2-dimX_0}$. Then $L(X_0,\mathscr{G}_0,t)$ converges for $|t| < q^{-\beta/2-dimX_0}$ Let $N \in \mathbb{Z}_{\geq 1}$ be an integer.

3.2 semicoutinuity thorem of weight

Lemma 3.2.1 *Let X_0/k be a smooth irreducible affine curve, with $U_0 \overset{j_0}{\hookrightarrow} X_0$ a nonempty open subset.Let $\mathscr{G}_0$ be a (weil) sheaf on X_0 such that $\mathscr{G}_0 \to j_{0*}j_0^*\mathscr{G}_0$ is an isomorphism and $j_0^*\mathscr{G}_0$ is lisse. Then*

$$H_c^0(X,\mathscr{G}) = 0$$

Proof. Let $Z \subseteq X$ is proper subvariety, $V = X \setminus Z$. $V \neq \varnothing$ since X is not proper. $H_c^0(X,\mathscr{G}) \subseteq \bigcup_{Z\subseteq X\, proper} H_Z^0(X,\mathscr{G})$. So we need to show

$$H_Z^0(X,\mathscr{G}) := Ker(H^0(X,\mathscr{G}) \to H^0(V,\mathscr{G}\mid_V)) = 0$$

for all proper subvariety Z. by $\mathscr{G}_0 \to j_{0*}j_0^*\mathscr{G}_0$, we rewrite this as

$$H_Z^0(X,\mathscr{G}) = Ker(H^0(U,\mathscr{G}\mid_U) \to H^0(U\cap V,\mathscr{G}\mid_{U\cap V}))$$

The intersection $U \cap V$ is not empty since X is irreducible. Let η is the generic point of U. $\mathscr{G}\mid_U$ is lisse implies for any $u \in U$, the specialization map $\mathscr{G}_{\bar{u}} \to \mathscr{G}_{\bar{\eta}}$ is an isomorphism. Namely any section vanishes on $U \cap V$ also vanishes on U, consequently $H_Z^0(X,\mathscr{G}) = 0$.

Proposition 3.2.1 *(semicontinuity of weight for curve) Let X_0/k be a smooth irreducible curve, with $U_0 \overset{j_0}{\hookrightarrow} X_0$ a nonempty open subset. Denote $S_0 = X_0 \setminus U_0$ to be the complement of U_0. Let $\mathscr{G}_0$ be a (weil) sheaf on X_0 such that the restriction $j_0^*\mathscr{G}_0$ is smooth and $H_S^0(X,\mathscr{G}) = 0$. Then*

$$w(j_0^*\mathscr{G}_0) \leq \beta \quad \Longrightarrow \quad w(\mathscr{G}_0) \leq \beta$$

Proof. We can reduce to 1. $\mathscr{G}_0 \cong j_{0*}j_0^*\mathscr{G}_0$ and 2. X_0 is affine .

1.Denote $i_0 : S_0 \to X_0$. Then the assumption $H_S^0(X,\mathscr{G}) = 0$ implies $\mathscr{G}_0 \hookrightarrow j_{0*}j_0^*\mathscr{G}_0$(since $H^0(S, i^!\mathscr{G}) = H_S^0(X,\mathscr{G}) = 0$, so for all $s \in S$, $(i^!\mathscr{G})_s = 0$). $j_{0*}j_0^*\mathscr{G}$ satisfies the condition of proposition($i^! j_* = 0$), so replace $\mathscr{G}_0$ by $j_{0*}j_0^*\mathscr{G}_0$, we have

$$w(j_0^* j_{0*}j_0^*\mathscr{G}_0) = w(j_0^*\mathscr{G}_0) \leq \beta \Longrightarrow w(j_{0*}j_0^*\mathscr{G}_0) \leq \beta \Longrightarrow \quad w(\mathscr{G}_0) \leq \beta$$

2.If X_0 is projective, take $u \in U$ is closed in X, then $O_{X,u}$ is a dicrete valuation ring, so $X \setminus u$ is affine. we have following commutative diagrams.

$$\begin{array}{ccc} S & \xrightarrow{i'} & X \setminus u \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle p} \\ S & \xrightarrow[i]{} & X \end{array} \qquad\qquad \begin{array}{ccc} U \setminus u & \xrightarrow{j'} & X \setminus u \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[j]{} & X \end{array}$$

Assume proposition is right for X_0 affine , then j_0' and $p_0^*\mathscr{G}_0$ satisfies the proposition. Because by Excision Theorem we have $H_S^0(X \setminus u, p^*\mathscr{G}) = H_S^0(X,\mathscr{G}) = 0$. **So if** $w(j_0^*\mathscr{G}_0) \leq \beta$, then $w(j'^*_0 p_0^*\mathscr{G}_0) = w(p'_0 j_0^*)\mathscr{G}_0 \leq \beta$, by assumption $w(p_0^*\mathscr{G}_0) \leq \beta$. and weight of $\mathscr{G}_0$ at u is also less than β since $u \in U$. Hence $w(\mathscr{G}_0) \leq \beta$.

Now assume $\mathscr{G}_0 \cong j_{0*}j_0^*\mathscr{G}_0$ and X_0 is affine, by lemma3.2.1. $H_c^0(X,\mathscr{G}) = 0$, then the Grothendieck-Lefschetz trace formula implies

$$L(X_0,\mathscr{G}_0,t) = \frac{det(1 - Ft \mid H^1(X,\mathscr{G}))}{det(1 - Ft \mid H^2(X,\mathscr{G}))}$$

Define $\mathscr{F}_0 := j_0^*\mathscr{G}_0$. For $u \in |U_0|$, this corresponds to a representation $V = \mathscr{F}_{\bar{u}}$ of $\pi_1(U,\overline{u})$.

Hence

$$\begin{aligned}
H_c^2(X,\mathscr{G}) &= H_c^2(U,\mathscr{F}) \\
&= H^0(U,\check{\mathscr{F}}(1))^{\vee} \\
&= H^0(U,\check{\mathscr{F}}\otimes\overline{\mathbb{Q}}_\ell(1))^{\vee} \\
&= H^0(U,\check{\mathscr{F}})^{\vee}\otimes\overline{\mathbb{Q}}_\ell(-1) \\
&= ((V^{\vee})^{\pi_1(U,\bar{u})})^{\vee}\otimes\overline{\mathbb{Q}}_\ell(-1) \\
&= (V_{\pi_1(U,\bar{u})})^{\vee}\otimes\overline{\mathbb{Q}}_\ell(-1) \\
&= (V_{\pi_1(U,\bar{u})})\otimes\overline{\mathbb{Q}}_\ell(-1)
\end{aligned}$$

It follows that the poles of $L(X_0,\mathscr{G}_0,t)$ are of form $1/\alpha q$ where α is a eigenvalue of F_u on $V_{\pi_1(U,\bar{u})}$(since geometric frobenius acts by q^{-1} on $\overline{\mathbb{Q}}_\ell(1)$. By definition of coinvariance, $\alpha^{d(u)}$ is an eigenvalue on V. Therefore by assumption $w(\mathscr{F}_0)\leq\beta$, we have $|\tau(\alpha^{d(u)})|\leq q^{d(u)\beta/2}$, i.e

$$|\tau(\frac{1}{\alpha q})| > q^{-\beta/2-1}$$

so $L(X_0,\mathscr{G}_0,t)$ converges for $|t|<q^{-\beta/2-1}$.

On the other hand, we can write

$$L(X_0,\mathscr{G}_0,t)=L(U_0,j_0^*\mathscr{G}_0,t)L(S_0,i_0^*\mathscr{G}_0,t)$$

Moreover, by proposition 3.2.1 $L(U_0,j_0^*\mathscr{G}_0,t)$ also converges for $|t|<q^{-\beta/2-1}$. Therefore, none of the factors $L(S_0,i_0^*\mathscr{G}_0,t)$ can have poles in this region(which is a finite product since $|S_0|$ is finite). If α is an eigenvalue of $F_{\bar{s}}$ on $\mathscr{G}_{0\bar{s}}$, then any $d(s)$-root of $\tau(\frac{1}{\alpha})$ will be a pole. So

$$|\tau(\frac{1}{\alpha})|\geq (q^{-\beta/2-1})^{d(s)}$$

i.e

$$|\tau(\frac{1}{\alpha})|\leq (q^{\beta/2+1})^{d(s)}=N(s)^{\frac{\beta+2}{2}}$$

namely

$$w(\mathscr{G}_0)\leq\beta+2$$

Finally, we use a tensor trick. For any $k\geq 1$, consider $j_{0*}(\mathscr{F}_0^{\otimes k})$, If α is an eigenvalue of F on $j_{0*}(\mathscr{F}_0)$, then α^k is an eigenvalue of

$$(j_{0*}(\mathscr{F}_0))^{\otimes k}$$

Applying previous argument to $j_{0*}(\mathscr{F}_0^{\otimes k})$, we have $w(j_{0*}(\mathscr{F}_0^{\otimes k})) \leq k\beta + 2$ since $w(\mathscr{F}_0^{\otimes k}) \leq k\beta$ Now by the injectivity of the homomorphism

$$(j_{0*}(\mathscr{F}_0))_{\overline{s}}^{\otimes k} \hookrightarrow (j_{0*}(\mathscr{F}_0^{\otimes k}))_{\overline{s}}$$

we have

$$w(j_{0*}(\mathscr{F}_0)) \leq \beta + \frac{k}{2}$$

Since k is arbitrary, we are done.

Lemma 3.2.2 *If X_0 is a normal, irreducible algebraic scheme over k, and $\mathscr{G}_0$ is irreducible and smooth, and $j_0 : U_0 \hookrightarrow X_0$ where is an open subscheme of X_0, then $j_0^*\mathscr{G}_0$ is also irreducible.*

Proof. X_0 is normal implies $\pi_1(U_0, \overline{a}) \twoheadrightarrow \pi_1(X_0, \overline{a})$.. By representation, We know if $G \to G/H \to GL(V)$, such that the second arrow is an irreducible representation, then the compositum is also irreducible.

all of arguments is to prove the following

Theorem 3.2.3 *(Semicontinuity).Let $\mathscr{G}_0$ be a lisse sheaf on a finit type scheme X_0/k and $j_0 : U_0 \hookrightarrow X_0$ be an open subscheme. Then,*

1. $w(\mathscr{G}_0) = w(j_0^*\mathscr{G}_0)$.

2. if $(j_0^*\mathscr{G}_0)$ is τ-pure of weight β, then $\mathscr{G}_0$ is τ-pure of weight β.

3. Let X_0 be normal and irreducible, $\mathscr{G}_0$ be irreducible. If $(j_0^*\mathscr{G}_0)$ is τ-mixed, then $\mathscr{G}_0$ is τ-pure.

4. Suppose that X_0 is connected, $j_0^*\mathscr{G}_0$ is τ-mixed and $\mathscr{G}_0$ is τ-pure of weight β at a single point $x \in |X_0|$, then $\mathscr{G}_0$ is τ-pure of weight β.

Proof.

1. Since $\nu : X_0^\nu \to X_0$ is surjective, $w(\nu^*\mathscr{G}_0) = w(\mathscr{G}_0)$. By take normalization X^ν of X_{0red} we can reduce to that X_0 is a normal geometrically integral scheme.The assumption that $\mathscr{G}_0$ is lisse on X_0 is used to prove $H^0_{X\setminus U}(X, \mathscr{G}) = 0$. If $dimX_0 = 1$, we are done by the semicontinuity theorem for curve above. If $dimX_0 > 1$, $\overline{\{\eta\}} = X_0$ we can connect any point of $X_0 \setminus U_0$ to the generic point η of U_0 by a curve C_0.(GTM52 exercise4.11.)

$$\begin{array}{ccc} U_0 \times_{X_0} C_0 & \xrightarrow{j_0'} & C_0 \\ {\scriptstyle f_0'}\downarrow & & \downarrow{\scriptstyle f_0} \\ U_0 & \xrightarrow[j_0]{} & X_0 \end{array}$$

By curve case we know $w(f_0^*\mathscr{G}_0) = W({j'_0}^* f_0^*\mathscr{G}_0)$. Since $\mathscr{G}_0$ is lisse and the four terms of diagram contain η, so $w(f_0^*\mathscr{G}_0) = w(\mathscr{G}_0)$ and $W({j'_0}^* f_0^*\mathscr{G}_0) = w(j_0^*\mathscr{G}_0)$. Hence $w(\mathscr{G}_0) = w(j_0^*\mathscr{G}_0)$.

2. Since $j_0^! = j_0^*$ is exact, dual commute with j_0^*. Apply (1) to $\mathscr{G}_0$ and $\mathscr{G}_0^\vee$, we have $w(\mathscr{G}_0) = \beta$ and $w(\mathscr{G}_0^\vee) = -\beta$. We are done.

3. Apply lemma 3.2.1： $j_0^*\mathscr{G}_0$ is irreducible, and so it is τ-pure. Then apply (2).

4.we can assume X_0 be normal. Furthermore it is enough to proof the cliam for all the irreducible constituents of$\mathscr{G}_0$, which allows us to assume $\mathscr{G}_0$ is irreducible. Now just apply (2)(3).

Definition 3.2.2 *Let $\mathscr{G}_0$ be a weil sheaf on a finite type scheme X_0/k. Then there is an open dense subscheme $j_0 : U_0 \hookrightarrow X_0$ such that $j_0^*\mathscr{G}_0$ is lisse on U_0. we define $w_{gen}(\mathscr{G}_0) := w(j_0^*\mathscr{G}_0)$.*

Remark, by theorem 3.2.3 the definition is independent of the choice of U_0. Therefore we define a notion of maximal weight for a Weil sheaf not necessarily lisse.

3.3 Real sheaves

Definition 3.3.1 *Let X_0 be a scheme of finite type over k and $\mathscr{G}_0$ a Weil sheaf on X_0. Fix $\tau : \overline{\mathbb{Q}_\ell} \to^{\cong} \mathbb{C}$. We say $\mathscr{G}_0$ is τ-real, if for any $x \in |X_0|$,*

$$\tau det(1 - F_x t \mid \mathscr{G}_{0\bar{x}}) \in \mathbb{R}[t] \subseteq \mathbb{C}[t]$$

Definition 3.3.2 *Furthermore Assume that $\mathscr{G}_0$ is τ-pure of weight β, then we define its τ-complex conjugate to be*

$$\overline{\mathscr{G}_0} := \mathscr{G}_0^\vee \otimes \chi_b$$

where $b \in \mathbb{Q}_\ell^\times$ such that $\tau(b) = q^\beta$.

Lemma 3.3.1 *If $\mathscr{G}_0$ is τ-pure of weight β, then $\mathscr{G}_0 \oplus \overline{\mathscr{G}_0}$ is τ-pure of weight β and τ-real.*

*Proof.*If $\alpha_1,\alpha_2\ldots\alpha_n$ are eigenvalues of $\mathscr{G}_{0\bar{x}}$ by the action of F_x, then $\alpha_1^{-1},\alpha_2^{-1}\ldots\alpha_n^{-1}$ are eigenvalues of $\mathscr{G}_{0\bar{x}}^\vee$. The eigenvalue of χ_b under action of F_x is just $b^{d(x)}$.Therefore

$$\begin{aligned}
\tau det(1 - F_x t \mid (\mathscr{G}_0 \oplus \overline{\mathscr{G}_0})_{\bar{x}}) &= \tau(1-\alpha_1 t)(1-\alpha_2 t)\ldots(1-\alpha_n t)(1-\frac{b^{d(x)}}{\alpha_1}t)(1-\frac{b^{d(x)}}{\alpha_2}t)\ldots(1-\frac{b^{d(x)}}{\alpha_n}t) \\
&= (1-\tau(\alpha_1)t)(1-\tau(\alpha_2)t)\ldots(1-\tau(\alpha_n)t)(1-\frac{q^{\beta d(x)}}{\tau(\alpha_1)}t)(1-\frac{q^{\beta d(x)}}{\tau(\alpha_2)}t)\ldots(1-\frac{q^{\beta d(x)}}{\tau(\alpha_n)}t) \\
&= (1-\tau(\alpha_1)t)(1-\tau(\alpha_2)t)\ldots(1-\tau(\alpha_n)t)(1-\overline{\tau(\alpha_1)}t)(1-\overline{\tau(\alpha_2)}t)\ldots(1-\overline{\tau(\alpha_n)}t)
\end{aligned}$$

Clearly, $\tau det(1 - F_x t \mid (\mathscr{G}_0 \oplus \overline{\mathscr{G}_0})_{\bar{x}}) \in \mathbb{R}[t]$. And weight of χ_b is 2β. We are done.

3.4 radius of convergence

The goal of this section is to give an alternate description for what the max weight $w(\mathscr{G}_0)$ is, at least in the case that $\mathscr{G}_0$ is a τ-mixed sheaf on a smooth curve. It turns out that $w(\mathscr{G}_0)$ determined the radius of convergence of a certain power series we now introduce.

Definition 3.4.1 *The main definition is the following function:*

$$f^{\mathscr{G}_0} = f_n^{\mathscr{G}_0} : \begin{cases} X_0(k_n) \to \mathbb{C} \\ \qquad \bar{x} \mapsto \tau Tr(F_x^{n/d(x)}, \mathscr{G}_{0\bar{x}}) \end{cases}$$

where $x \in |X_0|$, and $\bar{x}$ is a geometric point lying over it:

$$\begin{tikzcd} Spec\bar{k} \arrow[d] \arrow[drr, dotted, "\bar{x}"] & & \\ Speck_n \arrow[rr, "x"] \arrow[dr] & & X_0 \arrow[dl] \\ & Speck & \end{tikzcd}$$

Definition 3.4.2 *For any two functions f, $g : X_0(k_n) \to \mathbb{C}$, we define their inner product by*

$$(f,g)_n = \sum_{y \in X_0(k_n)} f(y)\overline{g(y)}$$

and norm

$$\|f\|_n^2 = (f,f)_n$$

Now since

$$(f^{\mathscr{G}_0},1)_n = \sum_{y \in X_0(k_n)} f^{\mathscr{G}_0}(y) = \sum_{y \in X_0(k_n)} \tau Tr(F_x^{n/d(x)}, \mathscr{G}_{0\bar{x}}) = \sum_{\substack{x \in |X_0| \\ d(x)|n}} d(x)\tau Tr(F_x^{n/d(x)}, \mathscr{G}_{0\bar{x}})$$

so 2 becomes

$$\frac{d}{dt} logL(X_0, \mathscr{G}_0, t) = \sum_{n \geq 1} \Big(\sum_{\substack{x \in |X_0| \\ d(x)|n}} d(x)(Tr(F_x^{n/d(x)}, \mathscr{G}_{0\bar{x}})) \Big) t^{n-1} = \sum_{n \geq 1} (f^{\mathscr{G}_0}, 1)_n t^{n-1}$$

Definition 3.4.3 *We define*

$$\phi^{\mathscr{G}_0}(t) = \sum_{n=1}^{\infty} \|f^{\mathscr{G}_0}\|_n^2 \cdot t^{n-1}$$

A reason for introducing $\phi^{\mathscr{G}_0}(t)$ is because it might work better with the Fourier transform, which will come later. We want to determine its convergence radius.

Lemma 3.4.1 *There is a constant C independent from n such that*

$$\|f^{\mathscr{G}_0}\|_n^2 \leq C \cdot q^{n(w(\mathscr{G}_0)+dimX_0)}$$

for all $n \in \mathbb{Z}_{\geq 1}$, so $\phi^{\mathscr{G}_0}(t)$ converges for $|t| < q^{-w(\mathscr{G}_0)-dimX_0}$.

Proof. First

$$|f^{\mathscr{G}_0}(x)|^2 = |\tau Tr(F_x^{n/d(x)})|^2 \leq r^2 \cdot q^{n \cdot w(\mathscr{G}_0)} \quad \text{where } \ r := \max_{x \in |X_0|} dim_{\overline{\mathbf{Q}_l}} \mathscr{G}_{0\bar{x}}$$

so

$$\|f^{\mathscr{G}_0}\|_n^2 = \sum_{x \in X_0(k_n)} |f^{\mathscr{G}_0}(x)|^2 \leq \#X_0(k_n) \cdot r^2 \cdot q^{n \cdot w(\mathscr{G}_0)} \leq C \cdot q^{n(w(\mathscr{G}_0)+dimX_0)}$$

We want to know whether $q^{-w(\mathscr{G}_0)-dimX_0}$ is exactly the radius of convergence. The main is that this is in fact is the radius of convergence, in some nice cases.

Before stating, we introduce a new notation:

Definition 3.4.4 *We define the L^2-norm of a sheaf $\mathscr{G}_0$ as*

$$\|\mathscr{G}_0\| = sup\{\rho \in \mathbb{R} \mid \lim_n sup \frac{\|f^{\mathscr{G}_0}\|_n^2}{q^{n(\rho+dimX_0)}} > 0\}$$

Note that by above discussion, we always have

$$\|\mathscr{G}_0\| \leq w(\mathscr{G}_0).$$

The following theorem tells us that we sometimes can get the opposite inequality.

Theorem 3.4.2 *(Radius of Convergence). Let $\mathscr{G}_0$ be a τ-mixed sheaf on a finite type scheme X_0/k of $dimX_0 \leq 1$. Then we have:*

(1) $\|\mathscr{G}_0\| = max\{w(\mathscr{G}_0) - 1, w_{gen}(\mathscr{G}_0)\}$

(2) Assume that X_0 is a smooth curve. If $H_S^0(X, \mathscr{G}_0) = 0$ for all closed subsets S of X, then

$$\|\mathscr{G}_0\| = w(\mathscr{G}_0).$$

Proof. Part (2) follows (1) and Proposition 3.2.1. Therefore we need to prove (1) .