Local Langlands for $GL(2)$

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1 Locally Profinite Group

Proposition. *G is a locally profinite group. Let $\psi : G \to \mathbb{C}^\times$ is a group homomorphism. Then following are equivalent:*

(1) ψ is continuous.

(2) $Ker(\psi)$ is open.

If ψ satisfies these conditions and G is the union of its compact open subgroups, then $Im(\psi)$ is contained in the unit cicle $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ in $\mathbb{C}$.

Proof. $(1) \Rightarrow (2)$. let $\mathcal{N}$ be an open neighbourhood of 1 in $\mathbb{C}$. Since ψ is continuous, $\psi^{-1}(\mathcal{N})$ is an open neighbourhood of identity, so it contains a compact open subgroup K. $\psi(K)$ is a subgroup of $\mathcal{N}$. Take $\mathcal{N}$ sufficiently small such that it contains no trivial subgroup of $\mathbb{C}^\times$. Then $\psi(K) = 1$, $K \in Ker(\psi)$. We have following decomposition

$$Ker(\psi) = \bigcup_{g \in Ker(\psi)} gK$$

which means $Ker(\psi)$ is open.

$(2) \Rightarrow (1)$. Given an open subset $U \in \mathbb{C}^\times$, take $g \in \psi^{-1}(U)$. Then $g \in gKer(\psi) \subset \psi^{-1}(U)$ is open. So ψ is continuous.

Let K be a compact subgroup of G, $\psi(K)$ is compact so $\psi(K)$ is contained in S^1 since S^1 is the maximal compact subgroup of $\mathbb{C}^\times$. Therefore $Im(\psi) \subset S^1$. □

Lemma. *S^1 is the unique maximal compact subgroup of $\mathbb{C}^\times$.*

Proof. S^1 is bounded and closed in $\mathbb{C}^\times$ so it is compact.

Claim: If $K \subset \mathbb{C}^\times$ is compact, then $K \subset S^1$. Let $z \in K$.

1. if $|z| > 1$, then $|z|^n \to \infty$ when $n \to \infty$ which contradicts to the boundness of K.

2. if $|z| < 1$, the sequence $\{z^n\}_{n \in \mathbb{N}} \to 0$, then $0 \in K$ since K is closed. It is a contradiction.

Therefore S^1 is the unique maximal compact subgroup of $\mathbb{C}^\times$. □

2 Smooth Representation of Locally Profinite Group

2.1 Let G be a locally profinite group and (π, V) be a complex representation of G. The representation (π, V) is called *smooth* if for every $v \in V$, there is a compact open subgroup K of G(depending on v) such that $\pi(x)v = v$ for all $x \in K$. Equivalent, if V^K denotes the space of $\pi(K)$-fixed vectors in V, then

$$V = \bigcup_K V^K$$

where K ranges over the compact open subgroup of G.

Generally, we deal with the smooth representation of infinite dimension.

Proposition. *Let G be a locally profinite group and let (π, V) be a smooth representation of G. Then following conditions are equivalent:*

(1) V is the sum of its irreducible G-subspaces.

(2) V is the direct sum of a family of irreducible G-subspaces.

(3) any G-subspace of V has a G-complement in V.

Proof. $(1) \Rightarrow (2)$. We take a family $\{U_i : i \in I\}$ of irreducible G-subspace U_i of V such that $V = \sum_{i \in I} U_i$.

Step 1 : **Construct the set** Z

$$Z = \{J \subset I \mid \sum_{j \in J} U_j \ \ is \ \ direct \ \ sum\}$$

It is nonempty since the single set $\{i\} \in Z$.

Step 2 : **Check that Z is inductive ordered by inclusion**: Suppose $\{J_a : a \in A\}$ is a totally order subset of Z, then $\overline{J} := \bigcup_{a \in A} J_a \in Z$.

Suppose $\bigcup_{j \in \overline{J}} U_j$ is not direct sum, then there is a finite set $S \in \overline{J}$ such that $\bigcup_{s \in S} U_s$ is not direct sum. This implies $S \in J_a$ for some a since $\overline{J}$ is the union of totally ordered subset. So we get a contradiction which means $\overline{J} \in Z$.

Step 3 : **Zorn Lemma** : Z is nonempty and inductively oredered, so it has a maximal element $J_0 \in Z$.

Step 4 : $V = \bigoplus_{j \in J_0} U_j$.

If $v \notin \bigoplus_{j \in J_0} U_j$, then there are finite $i_1, i_2 \cdots i_n$ such that $v \in U_{i_1} + U_{i_2} \cdots U_{i_n}$.

If some $U_{i_k} \notin \bigoplus_{j \in J_0} U_j$, then add i_k into J_0 we can get a larger element than J_0 (this is a contradiction). So all $U_{i_k} \in \bigoplus_{j \in J_0} U_j$. Namely $V = \bigoplus_{j \in J_0} U_j$.

$(2) \Rightarrow (3)$. Let W be a G-subspace of V. By (2), we can assume $V = \bigoplus_{i \in I} U_i$ for a family $(U_i)_{i \in I}$ of irreducible G-subspaces of V. As the proof of (1). Define a set $\mathcal{J}$

$$\mathcal{J} = \{J \subset I \mid W \cap \sum_{j \in J} U_j = 0\}$$

We can prove that $\mathcal{J}$ is nonempty and inductively oredered so it has a maximal element J. So $X = W + \bigoplus_{j \in J} U_j$ is a direct sum. If $X \neq V$, then there $U_i \not\subset X$, so $U_i \cap X = \emptyset$ since U_i is irreducible. Then $X + U_i$ is a direct sum which is a contradiction. Therefore $V = \bigoplus_{j \in J} U_j \bigoplus W$.

$(3) \Rightarrow (1)$. Let V_0 be the sum of all irreducible G-subspaces of V and $V = V_0 \bigoplus W$ for some G-subspace W of V. Assume $W \neq 0$, take any $w \in W$, then $W_1 := \{\pi(g)w \mid g \in G\} \subset W$ is a G-subspace. By Zorn Lemma, W_1 has a maximal G-subspace W_0, then W_1/W_0 is irreducible. By (3), we have some G-subspace U such that $V = V_0 \bigoplus W_0 \bigoplus U$. Let $\psi : V \to U$ be the projection map. **Claim**:

$$W_1/W_0 \cong \psi(W_1)$$

This is because that $\psi\mid_{W_1}$ is injective. Specifically, $Ker(\psi\mid_{W_1}) = W_1 \cap (W_0 \bigoplus U) = W_0$.

Now $\psi(W_1) \subset U$ is a irreducible G-subspace, so $\psi(W_1) \subset V_0$ which is a contradiction since $U \cap V_0 = \emptyset$. Therefore $W = 0$.

3 Measure and Duality

3.1 Haar measure

Let G be a locally profinite group. Let $C_c^\infty(G)$ be the space of functions $f : G \to \mathbb{C}$ which are locally constant and of compact support. A equivalent definition is that $C_c^\infty(G)$ is the space of compactly supported complex-valued function on G which is right and left smooth for some compact open subgroup K of G.

If $f \in C_c^\infty(G)$ is locally constant and compactly supported, then $\forall g \in supp(f)$, there is a compact open subgroup K_g such that $f \mid gK_g$ is constant. By compactness, $supp(f)$ is covered by finite cover $\{gK_g\}_{g \in S}$, take $K_1 = \cap_{g \in S} K_g$, then f is right-invariant for K_1. Similarly, f is left-invariant for some compact open subgroup K_2. Therefore f is left and right-invariant for $K_1 \cap K_2$. Conversely, f is smooth implies that it is locally constant.

There are two obvious ways to define a representation of G on $C_c^\infty(G)$, corresponding to a left and right translation. These are

$$\begin{aligned}\lambda, \rho : G &\to GL(C_c^\infty(G))\\ \lambda(g)f(x) &= f(g^{-1}x),\\ \rho(g)f(x) &= f(xg).\end{aligned}$$

for $x, g \in G$. They are smooth representations.

Definition. *A right Haar integral on G is a non-zero linear functional*

$$I : C_c^\infty(G) \longrightarrow \mathbb{C}$$

such that

(1) $I(\rho(g)f) = I(f),\ g \in G.$

(2) $I(f) \geq$ *for any* $f \geq 0$ *in* $C_c^\infty(G)$.

Proposition. *There exists a right Haar integral* I *on* G*. Moreover, if there is another Haar integral* $I^{'}$ *on G, then* $I^{'} = cI$ *for some constant* $c > 0$.

Proof. For every $K \subset G$ compact open, let ${}^K C_c^\infty(G)$ be the subspace of $C_c^\infty(G)$ fixed by $\lambda(K)$. Then the characteristic functions $\{\chi_{Kg}\}_{g \in G/K}$ is a basis of ${}^K C_c^\infty(G)$. Define $I_K :^K C_c^\infty(G) \to \mathbb{C}$ by $I_K(\chi_{Kg}) = 1$, then I_K satisfies $I_K \geq 0$ whenever $f \geq 0$ in ${}^K C_c^\infty(G)$. Since $\rho(h)\chi_{Kg} = \chi_{Kgh^{-1}}$ for $h \in G$, I_K is $\rho(G)$-invariant.

Now we can construct a Haar measure I based on I_K. Fix a compact open subgroup $K \subset G$ and let $K_j \subset K$ be a family of compact open subgroups of K such that $\cap_j K_j = 1$. By definition of smoothness,

$$C_c^\infty(G) = \bigcup_{j \geq 1} C_c^\infty(G)^{K_j}$$

Now for $f \in C_c^\infty(G)$, $f \in C_c^\infty(G)^{K_j}$ for some j. Define

$$I(f) = \frac{1}{[K : K_j]} I_{K_j}(f)$$

and we can prove $I(f)$ does not depend on the index j. Clearly, I is just the right Haar integral that we expect. We need to prove the uniqueness. This is the following lemma. $\square$

Lemma. *Viewing* $\mathbb{C}$ *as the trivial G-space, we have*

$$dim_{\mathbb{C}} \mathrm{Hom}_G({}^K C_c^\infty(G), \mathbb{C}) = 1$$

for every compact open $K \subset G$.

Proof. Notice that

$$^{K}C_c^\infty(G) = c - Ind_K^G 1_K$$

So

$$dim\mathrm{Hom}_G(^{K}C_c^\infty(G), \mathbb{C}) = \mathrm{Hom}_K(1_K, 1_K)$$

$\square$

By Riesz Representation Theorem of locally compact group, there is a positive Borel measure μ_G on G such that

$$I(f) = \int_G f(g)d\mu_G(g), \quad f \in C_c^\infty(G)$$

And if χ_K is the characteristic function of K for a compact open $K \subset G$, then $\mu_G(E) = I(\chi_E)$ which is referred by chapter 2 of [**?**].

Remark Next, let V be a complex vector space, and consider the space $C_c^\infty(G; V)$ of locally constant, compactly supported functions $f : G \to V$. This is isomorphic to $C_c^\infty(G) \otimes V$ by the following homomorphism

$$\begin{array}{ccc} & C_c^\infty(G) \otimes V \xrightarrow{\quad\cong\quad} C_c^\infty(G; V) & \\ \psi & \sum_i f_i \otimes v_i \longrightarrow g \mapsto \sum_i f_i(g)v_i & \\ \psi^{-1} & \sum_j \chi_{U_j} \otimes u_j \longleftarrow f & \end{array}$$

If $f \in C_c^\infty(G; V)$, there are finite many compact open subgroups $U_j \subset G$ covering $supp(f)$ such that f is constant on U_j. Denote $f \mid U_j = u_j$. This is the definition of ψ^{-1}.

If μ_G is a left Haar measure on G, there is a unique linear map $I_V : C_c^\infty(G; V) \to V$ such that

$$I_V(f \otimes v) = \int_G f(g)d\mu_G(g) \cdot v$$

4 Hecke Algebra

There is a more general version of 2.3. We start with a compact open subgroup K of G and $\rho \in \widehat{K}$. We consider the function e_ρ which is defined by

$$e_\rho = \begin{cases} \frac{dim\rho}{\mu(K)} tr\rho(x^{-1}), & x \in K \\ 0, & x \neq K \end{cases}$$

If $K^{'} = ker(\rho)$, then $K^{'}$ contains at least one compact open subgroup of K. So $[K : K^{'}]$ is finite.

Proposition. *Define* $\mathcal{H}(K, K^{'}) = e_{K'} * \mathcal{H}(K) * e_{K'}$. *Then the homomorphism*

$$\mathcal{H}(K, K^{'}) \xrightarrow{\quad\phi\quad} \mathbb{C}[K/K^{'}]$$

$$f \longmapsto \sum_{g \in K/K'} f(g) \cdot gK^{'}$$

is an algebra isomorphism.

Proof. **Step 1** : Check ϕ is a homomorphism, namely $\phi(f_1 * f_2) = \phi(f_1) \cdot \phi(f_2)$

$$\begin{aligned}
\phi(f_1 * f_2) &= \sum_{g \in K/K'} (f_1 * f_2)(g) \cdot gK^{'} \\
&= \sum_{g \in K/K'} (f_1 * f_2)(g) \cdot gK^{'} \\
&= \sum_{g \in K/K'} \int_K f_1(k) f_2(k^{-1}g) dk \cdot gK^{'} \\
&= \sum_{g \in K/K'} \sum_{h \in K/K'} \int_{hK'} f_1(k) f_2(k^{-1}g) dk \cdot gK^{'} \\
(1) &= \sum_{g \in K/K'} \sum_{h \in K/K'} f_1(h) f_2(h^{-1}g) \cdot gK^{'}
\end{aligned}$$

and

$$\begin{aligned}
\phi(f_1) \cdot \phi(f_2) &= (\sum_{h \in K/K'} f_1(h) \cdot hK^{'})(\sum_{g \in K/K'} f_2(g) \cdot gK^{'}) \\
&= \sum_{g \in K/K'} \sum_{h \in K/K'} f_1(h) f_2(g) \cdot hgK^{'} \\
&= \sum_{g \in K/K'} \sum_{h \in K/K'} f_1(h) f_2(h^{-1}g) \cdot gK^{'}
\end{aligned}$$

The last step is to do the following substitution $hg \to g$, and (1) is because that $f(k_1^{'} k k_2^{'}) = f(k)$ for all $k_1^{'}, k_2^{'} \in K^{'}$.

Step 2 Check ϕ is bijective.

Injective : if $f(g) = 0$ for all $g \in K/K^{'}$, then $f = 0$ by the left and right invariant of f.

Surjective: if $\sum_{g \in K/K'} a_g \cdot gK^{'} \in \mathbb{C}[K/K^{'}]$, then define $f(K^{'} gK^{'}) = f(gK^{'}) = a_g$ for all $g \in K/K^{'}$(since $K^{'}$ is a normal subgroup of K). Then $f \in \mathcal{H}(K, K^{'})$ and $\phi(f) = \sum_{g \in K/K'} a_g \cdot gK^{'}$. ∎

By this isomorphism, we have two results. Before stating the results, we need a lemma.

Lemma. *Let G be a finite group. $\rho : G \to GL(V)$ is a irreducible representation. Then for any representation $G \to GL(W)$, $\pi(e_\rho^{'})$ is the projection $V \to V^\rho$. Here*

$$\pi(e_\rho^{'}) := \frac{dim\rho}{|G|} \sum_{g \in G} tr\rho(g^{-1})g$$

and

$$\pi(e_\rho^{'}) \cdot v := \frac{dim\rho}{|G|} \sum_{g \in G} tr\rho(g^{-1})\pi(g)v$$

Proof. We need to prove

$$\pi(e_\rho^{'})v = \begin{cases} v, & v \in V^\rho \\ 0 & v \notin V^\rho \end{cases}$$

Notice V has following decomposition

$$V = \bigoplus_{\sigma \in \widehat{G}} V^\sigma$$

So if $v \in (\sigma_i, V_i)$, then

$$\begin{aligned} \pi(e_\rho^{'}) \cdot v &= \frac{dim\rho}{|G|} \sum_{g \in G} tr\rho(g^{-1})\sigma_i(g)v \\ &= \frac{dim\rho}{|G|} \sum_{g \in G} tr\rho(g^{-1})\lambda v \end{aligned}$$

where $\lambda \in \mathbb{C}^*$ But

$$\sum_{g \in G} tr\rho(g^{-1})tr\sigma_i(g) = \begin{cases} 0, & \rho \ncong \sigma_i \\ |G| & \rho \cong \sigma_i \end{cases}$$

So

$$\sum_{g \in G} tr\rho(g^{-1})\lambda = \begin{cases} 0, & \rho \ncong \sigma_i \\ \frac{|G|}{dim\rho} & \rho \cong \sigma_i \end{cases}$$

This is just the desired result. □

Corollary. *Notation as above. Then:*

(1) Then function $e_\rho \in \mathcal{H}(G)$ is idempotent.

(2) If (π, V) is a smooth representation of G, then $\pi(e_\rho)$ is the K-projection $V \to V^\rho$.

Proof. (1) This is a corollary of Schur orthogonality relation of finite group. We only need to prove $\phi(e_\rho)$ is idempotent.

(2)The idea is that we find an open normal compact subgroup $K_1 \in ker(\rho)$ of K, then analysis the finite group K/K_1 and use the corresponding result of finite group.

Fix $v \in V^\rho$ and let $\rho : K \to GL(W)$ be the irreducible smooth representation such that $v \in W$, then there is a open compact subgroup $K_v \subset K$ such that $\rho(K_v)v = v$. Take

$$K_1 = \bigcap_{g \in K} gK_v g^{-1}$$

then K_1 is an open normal compact subgroup of K(we finally prove it.) and $v \in (V^{K_1})^\rho$. Now $\rho : K/K_1 \to GL(W^{K_1})$ is a irreducible representation . For $\pi \mid K : K/K_1 \to GL(V^{K_1})$, we have $\pi(e'_\rho)v \in (V^{K_1})^\rho \in V^\rho$. But in this case $\pi(e'_\rho)v$ is just $\pi(e_\rho)v$.

$$\begin{aligned}
\pi(e_\rho)v &= \int_K e_\rho(k)\pi(k)vdk \\
&= \int_K e_\rho(k)\pi(k)vdk \\
&= \frac{dim\rho}{\mu(K)} \sum_{k \in K/K_1} \int_{kK_1} tr(\rho(g^{-1}))\pi(g)vdg \\
&= \frac{dim\rho}{\mu(K)} \sum_{k \in K/K_1} \int_{kK_1} tr(\rho(g^{-1}))\pi(g)vdg \\
&= \frac{dim\rho}{\mu(K)} (\sum_{k \in K/K_1} \mu(K_1) tr(\rho(k^{-1}))\pi(k)v) \\
&= \frac{dim\rho}{|K/K_1|} (\sum_{k \in K/K_1} tr(\rho(k^{-1}))\pi(k)v) \\
&= \pi(e'_\rho)v
\end{aligned}$$

Therefore we have proved that $\pi(e_\rho)$ is identity on V^ρ.

Now we prove that if $v \in V^\sigma$ and $\sigma \ncong \rho$, then $\pi(e_\rho)v = 0$. The prove is similar to above lemma , we notice that $\pi(e_\rho)v = \pi(e'_\rho)v = 0$.

Finally, we prove that K_1 is open normal compact in K and $K_1 \in ker(\rho)$. Clearly $K_1$is normal in K. $Stab(K_v) := \{g \in K \mid gK_v g^{-1} = K_v\}$ is the stablizer of K_v under the conjugate action. Notice $K_v \subset Stab(K_v)$, so $|Orb(K_v)| = [K : Stab(K_v)] \mid [K : K_v]$. But $[K : K_1]$ is finite, this means that K_1 is the intersection of finite many open compact groups, so it is open compact.

Take $h \in K_1$, then for any $\rho(g)v$, $\rho(h)\rho(g)v = \rho(gkg^{-1})\rho(g)v = \rho(g)v$ since $k \in K_v$. So $\rho(h)$ act identity on $W = span\{\rho(g)v \mid g \in K\}$ which means that $K_1 \in ker(\rho)$. ∎

3 Induced Representation of Linear Group

7 Linear Group over Local Fields

Proposition. *The module δ_B of B is given by:*

$$\delta_B : tn \mapsto \|t_2/t_1\| \quad n \in N, t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in T$$

Proof. Setting

$$c = sm, \quad m \in N, s = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \in T \tag{7.6.1}$$

Then

$$\int_B \Phi(bc)d\mu_B(b) = \int_T \int_N \Phi(ts \cdot s^{-1}ns \cdot m)d\mu_N(n)d\mu_T(t)$$

Notice that

$$\int_N \Phi(ts \cdot s^{-1}ns \cdot m)d\mu_N(n) = \int_N \Phi(ts \cdot s^{-1}ns)d\mu_N(n)$$

since $s^{-1}ns \in N$ and μ_N is right invariant. We know

$$N \xrightarrow{\cong} F$$
$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \longrightarrow x$$

which identifies μ_N with a haar measure μ_F on F. So

$$\int_N \Phi(ts \cdot s^{-1}ns)d\mu_N(n) = \int_F \Phi(ts \cdot \begin{pmatrix} 1 & s_1^{-1}xs_2 \\ 0 & 1 \end{pmatrix})d\mu_F(x)$$

Now do the substitution $x' = s_1^{-1}xs_2$, then $x = s_1x's_2^{-1}$ and $d\mu_F(x) = \|s_1s_2^{-1}\|d\mu_F(x')$. We get

$$\begin{aligned}
\int_F \Phi(ts \cdot \begin{pmatrix} 1 & s_1^{-1}xs_2 \\ 0 & 1 \end{pmatrix})d\mu_F(x) &= \int_F \Phi(ts \cdot \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix})\|s_1s_2^{-1}\|d\mu_F(x') \\
&= \|s_1s_2^{-1}\| \int_N \Phi(tsn')d\mu_N(n') \\
&= \|s_1s_2^{-1}\| \int_N \Phi(tsn)d\mu_N(n)
\end{aligned}$$

where $n^{'} = \begin{pmatrix} 1 & x^{'} \\ 0 & 1 \end{pmatrix}$. Therefore

$$\begin{aligned} \int_B \Phi(bc) d\mu_B(b) &= \int_T (\int_N \Phi(ts \cdot s^{-1} ns \cdot m) d\mu_N(n)) d\mu_T(t) \\ &= \|s_1 s_2^{-1}\| \int_T \int_N \Phi(tsn) d\mu_N(n) d\mu_T(t) \\ (1) &= \|s_1 s_2^{-1}\| \int_T \int_N \Phi(tn) d\mu_N(n) d\mu_T(t) \\ &= \|s_1 s_2^{-1}\| \int_B \Phi(b) d\mu_B(b) \end{aligned}$$

$(7.6.1)$ holds by doing the substitution $t \to ts^{-1}$ in μ_T.

By definition of δ_B. We have

$$\delta_B(c) = \|s_1 s_2^{-1}\|^{-1} = \|s_2 s_1^{-1}\|$$

The result holds. $\square$

8 Representation of the mirabolic group

Lemma. *let μ_N be a haar measure on N and ϑ a character of N.*

(1) Let (π, V) be a smooth representation of N and $v \in V$. Then $v \in V(\vartheta)$ if and only if there is a compact subgroup N_0 of N such that

$$\int_{N_0} \vartheta^{-1}(n) \pi(n) v d\mu_N(n) = 0$$

(2) Then functor $(\pi, V) \to V_\vartheta$ is exact functor from $Rep(N)$ to the category of complex vector space.

Proof. (1)We first assume that χ is the trivial character of N.

The group $N \cong F$ is the union of increasing sequence of compact open subgroups, so if

$$v = \sum_{i=1}^{r} c_i (v_i - \pi(n_i) v_i)$$

Then there is a compact open subgroup N_0 of N containing all the n_i. This N_0 satisfies the result.

(2) Let $(\pi_i, V_i)(i = 1,2,3)$ be three smooth representations of N such that the following sequence is exact

$$0 \to V_1 \xrightarrow{f} V_2 \xrightarrow{h} V_3 \to 0$$

This induces a sequence of complex vector space

$$0 \to (V_1)_\vartheta \xrightarrow{\bar{f}} (V_2)_\vartheta \xrightarrow{\bar{h}} (V_3)_\vartheta \to 0$$

We want to prove it is exact. Clearly $\bar{h}$ is surjective and $im(\bar{f}) \subset ker(\bar{h})$, we need to prove $ker(\bar{f}) = 0$ and $ker(\bar{h}) \subset im(\bar{f})$.

if $v_1 \in (V_1)_\vartheta$ such that $\bar{f}(v_1) \in V_2(\vartheta)$. Then by (1), there is a compact subgroup N_0 of N such that

$$0 = \int_{N_0} \vartheta^{-1}(n)\pi_2(n)f(v_1)d\mu_N(n) = f(\int_{N_0} \vartheta^{-1}(n)\pi_1(n)v_1 d\mu_N(n))$$

which means

$$\int_{N_0} \vartheta^{-1}(n)\pi_1(n)v_1 d\mu_N(n) = 0$$

There $v_1 \in V_1(\vartheta)$.

Take $v_2 \in (V_2)_\vartheta$ such that $\bar{h}(v_2) \in V_3(\vartheta)$, then

$$0 = \int_{N_0} \vartheta^{-1}(n)\pi_3(n)h(v_2)d\mu_N(n) = h(\int_{N_0} \vartheta^{-1}(n)\pi_2(n)v_2 d\mu_N(n))$$

since $ker(h) = im(f)$, there is $v_1 \in V_1$ such that

$$\int_{N_0} \vartheta^{-1}(n)\pi_2(n)v_2 d\mu_N(n) = f(v_1)$$

We claim that $v_2 - f(v_1) \in V_2(\vartheta)$ which mean $\bar{f}(v_1) = v_2$. So we need to prove

$$\int_{N_0} \vartheta^{-1}(n)\pi_2(n)(v_2 - f(v_1))d\mu_N(n) = 0$$

which is equivalent to

$$\int_{N_0} \vartheta^{-1}(n)\pi_2(n)f(v_1)d\mu_N(n) = f(v_1)$$

Take a compact open subgroup N_1 of N fixing v_2 and $f(v_1)$. Then

$$\int_{N_0} \vartheta^{-1}(n)\pi_2(n)v_2 d\mu_N(n) = \sum_{i=1}^{m} |g_i N_0 \cap N_1 \cap ker(\vartheta)| \vartheta^{-1}(g_i)\pi_2(g_i)v_2 = f(v_1) \qquad (8.1.1)$$

Take a $\vartheta^{-1}(g_j)\pi_2(g_j)$ acting on both sides of $(8.1.1)$, we have

$$\begin{aligned}
\vartheta^{-1}(g_j)\pi_2(g_j)f(v_1) &= \sum_{i=1}^{m} |g_iN_0 \cap N_1 \cap ker(\vartheta)|\vartheta^{-1}(g_jg_i)\pi_2(g_jg_i)v_2 \\
&= \sum_{i=1}^{m} |g_iN_0 \cap N_1 \cap ker(\vartheta)|\vartheta^{-1}(g_i)\pi_2(g_i)v_2 \\
&= f(v_1)
\end{aligned}$$

for any $1 \leq j \leq m$. Therefore

$$\begin{aligned}
\int_{N_0} \vartheta^{-1}(n)\pi_2(n)f(v_1)d\mu_N(n) &= \sum_{i=1}^{m} |g_iN_0 \cap N_1 \cap ker(\vartheta)|\vartheta^{-1}(g_i)\pi_2(g_i)f(v_1) \\
&= \sum_{i=1}^{m} |g_iN_0 \cap N_1 \cap ker(\vartheta)|f(v_1) \\
&= f(v_1)
\end{aligned}$$

□

9 Jacquet Module and Classification of non-cuspidal representation

9.1.

Proposition. *Let (π, V) be a irreducible smooth representation of G. The following are equivalent:*

(1) The Jacquet module V_N is non-zero.

(2) The representation π is isomorphic to a G-subspace of a representation $Ind_B^G\chi$, for some character χ of T.

Proof. Suppose (2) holds, then

$$\mathrm{Hom}_T(\pi_N, \chi) \cong \mathrm{Hom}_G(\pi, Ind_B^G\chi) \neq 0$$

which means $V_N \neq 0$.

To prove $(1) \Rightarrow (2)$. We will prove that V_N has an irreducible T-quotient which is a character.

Choose $0 \neq v \in V$. Since π is irreducible, $V = \langle \pi(g)v \mid g \in G \rangle$. Denote $K = GL_2(\mathfrak{o})$, then v is fixed by a open subgroup $K^{'}$ of K of finite index. Set

$$K = \bigcup_{i=1}^{m} g_iK^{'}$$

Since $G = BK$, the elements $\pi(g_1)v, \pi(g_2)v \cdots, \pi(g_m)v$ generate V over B, and their images generate V_N over T.

Therefore V_N is finitely generated as a representation of T. We choose a minimal generating set $\{u_1, \cdots, u_t\}$, $t \geq 1$. Construct the set Z:

$$Z = \{U \text{ is a T subspace of } V_N \mid u_1, u_2 \cdots, u_{t-1} \in U, u_t \notin U\}$$

Then Z is inductive ordered by inclusion. By Zorn Lemma, it has a maximal T-space W such that V_N/W is an irreducible T-representation. It must be a character since T is commutative. $\square$

9.3.

Lemma. ***(Restriction-Induction Lemma)***. *Let (σ, U) be a smooth representation of T and $(\Sigma, X) = Ind_B^G \sigma$. There is an exact sequence of representations over T*

$$0 \longrightarrow \sigma^w \otimes \delta_B^{-1} \longrightarrow \Sigma_N \xrightarrow{\alpha_\sigma} \sigma \longrightarrow 0$$

Proof. α_σ is the map $X \to U$ by $f \mapsto f(1)$. It is surjective。

Set $V = Ker(\alpha_\sigma)$. We have the following exact sequence of representations over B

$$0 \longrightarrow V \longrightarrow \Sigma \xrightarrow{\alpha_\sigma} \sigma \longrightarrow 0$$

Applying the Jacquet functor we have the exact sequence of representations over T.

$$0 \longrightarrow V_N \longrightarrow \Sigma_N \xrightarrow{\alpha_\sigma} \sigma \longrightarrow 0$$

We recall that $G = B \cup BwN$. Thus a function $f \in X$ lies in V if and only if $supp(f) \subset BwN$. More precisely:

Lemma. *Let $f \in X$, then $f \in V$ if and only if there is a compact open subgroup N_0 of N(depending on f) such that $supp(f) \subset BwN_0$*

Proof. If $f(1) = 0$. Since f is G-smooth, f vanishes on $BN_0^{'}$ where $N_0^{'}$ is some compact open subgroup of $N^{'}$. Notice that

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix} w \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \in Bw \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}$$

which means that $supp(f) \subset BwN_0$ for compact open subgroup $N_0 \subset N$. $\square$

Let $f \in V$. By above, we can define a function $f_N : T \to U$ by

$$f_N(x) = \int_N f(xwn)dn = \sigma(x)f_N(1), \ \ \forall x \in T.$$

Then $f \to f_N(1)$ induces a bijective map $V_N \to U$.

Proof. □

Take $t = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \in T$ and $f \in V$, we have

$$\begin{aligned}
(tf)_N(x) &= \int_N f(xwnt)d\mu_N(n) \\
&= \int_N f(xwt \cdot t^{-1}nt)d\mu_N(n) \\
&= \int_F f(xwt \cdot \begin{pmatrix} 1 & s^{-1}ns_2 \\ 0 & 1 \end{pmatrix})d\mu_F(n) \\
&= \delta_B^{-1}(t) \int_N f(xwtn)d\mu_N(n) \\
&= \delta_B^{-1}(t) \int_N f(xt^w wn)d\mu_N(n) \\
&= \delta_B^{-1}(t) \int_N f(t^w xwn)d\mu_N(n) \\
&= \delta^{-1}(t)\sigma(t^w)f_N(x) \\
&= \delta^{-1}(t)\sigma^w(t)f_N(x)
\end{aligned}$$

Thus $f \to f_N(1)$ is a T-homomorphism $V \to \sigma^w \otimes \delta_B^{-1}$. it is also a N-representation which is trivial. So it is a B-homomorphism. This induces a T-isomorphism $V_N \cong \sigma^w \otimes \delta_B^{-1}$. □

9.4. The irreducible representations of G exhibit a helpful finiteness property:

Proposition. *Let (π, V) be an irreducible smooth representation of G which is non-cuspidal. Then π is admissible.*

Proof. By definition, $V_N \neq 0$. 9.1 Proposition says that π is equivalent to a subrepresentation of $X = Ind_B^G \chi$ for some character χ of T. Therefore it is enough to prove that X is admissible.

Fix a compact open subgroup K_0 of G, we can assume $K_0 \subset K = GL_2(\mathfrak{o})$. The subspace X^{K_0} of X consists of the functions $f : G \to \mathbb{C}$ satisfying

$$f(bgk) = \chi(b)f(g), \quad \forall b \in B, g \in G, k \in K_0 \tag{9.4.1}$$

$G = BK$ implies the set $B\backslash G/K_0$ is finite, and every double sets BgK_0 supports at most a 1-dimensional space of functions satisfying (9.4.1)(if $supp(f_1)$ and $supp(f_2)$ are contained in BgK_0, then $f_1 = cf_2$ for some $c \in \mathbb{C}^\times$.) Thus X^{K_0} is finite-dimensional. $\square$

9.5. We introduce another notation. If (π, V) is a smooth representation of G and ϕ is a character of $F^\times$, we define a smooth representation $(\phi\pi, V)$ of G by setting

$$\phi\pi(g) = \phi(det(g))\pi(g), \quad \forall g \in G$$

we call $\phi\pi$ the twist of π by ϕ.

Similarly, if $\chi = \chi_1 \otimes \chi_2$ is a character of T and ϕ is a character of $F^\times$, then we define $\phi \cdot \chi := \phi\chi_1 \otimes \phi\chi_2$. We can inflate $\phi \cdot \chi$ to a B-representation which is trivial on N, then we have following isomorphism of G-representation

$$Ind_B^G(\phi \cdot \chi) \cong \phi Ind_B^G \chi.$$

Proof. Define the following map

$$\begin{aligned} \Phi : Ind_B^G(\phi \cdot \chi) &\to \phi Ind_B^G \chi \\ f &\mapsto (g \mapsto \phi(det(g^{-1}))f(g) \end{aligned}$$

Clearlt, Φ is bijective. We need to verifty it is a G-map, namely

$$\Phi(\Sigma(g_1)f) = \phi(det(g_1))\Sigma(g_1)\Phi(f)$$

which is trivial. We omit it. $\square$

9.6 we now give a precise account of the structure of representation of the form $Ind_B^G \chi$.

Proposition. ***(Irreducibility Criterion)***. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T, and set $(\Sigma, X) = Ind_B^G \chi$. Then we have*

1. *The representation $Ind_B^G \chi$ is reducible if and only if $\chi_1\chi_2^{-1} = 1$ or $\chi_1\chi_2^{-1}(x) = \|x\|^2$ for $x \in F^\times$.*
2. *Suppose that $Ind_B^G \chi$ is reducible. Then :*
   - *(1) the $G-$composition length of X is 2;*
   - *(2) one composition factor of X has dimension 1, the other is of infinite dimension;*
   - *(3) X has a 1-dimension G-subspace if and only if $\chi_1\chi_2^{-1} = 1$*
   - *(4) X has a 1-dimension G-quotient if and only if $\chi_1\chi_2^{-1}(x) = \|x\|^2$ for $x \in F^\times$*

We use this proposition to classify irreducible non-cuspidal representation in 9.11. The proof of the proposition will occpy $9.7 - 9.9$.

9.7 We use the notation of 9.7. Let

$$V = \{f \in X \mid f(1) = 0\}$$

This is a B-space of X and we have an exact sequence

$$0 \to V \to X \to \mathbb{C} \to 0$$

where the 1-dimensional subspace $\mathbb{C} \cong X/V$ carries the character χ of T. By the Restriction-Induction Lemma (9.3), $V_N \cong \delta_B^{-1}\chi^w$.

Proposition. *Let W be the kernel $V(N)$ of the canonical map $V \to V_N$. Then W is irreducible over B.*

Proof. We can check $V(N)$ is a B-subspace of V. Indeed, we will prove W is an irreducible representation of the mirabolic group M. Observe that by $(8.1.2)$, $W = W(N)$ and $W_N = 0$. We need a lemma before proving the proposition.

Lemma. *For $f \in V$, define a function $f_N \in C_c^\infty(N)$ by $f_N(n) = f(wn)$, $n \in N$. The map*

$$\begin{aligned} V &\longrightarrow C_c^\infty(N), \\ f &\longmapsto f_N. \end{aligned}$$

is an N-isomorphism.

Proof. Clear, it is well-define and injective. □

Return to the proposition. For $\phi \in C_c^\infty(N)$ and $a \in F^\times$, we define $a\phi$ by

$$a\phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) = \chi_2(a)\phi(\begin{pmatrix} 1 & a^{-1}x \\ 0 & 1 \end{pmatrix})$$

This give an action of $F^\times$ on $C_c^\infty(N)$ which we regard as a representation of S. Combine the natural action of N and this action of S, $C_c^\infty(N)$ is a **smooth representation** of M.

Let ϑ be a non-trivial character of N. The map $f \to \vartheta f$ is a linear automorphism of $C_c^\infty(N)$ carrying $V(N)$ to $V(\vartheta)$(use equivalent condition of 8.1 Lemma to check).

Proof. If $f \in V(N)$, then

$$\int_{N_0} \Sigma(n) f dn = 0$$

for some open compact subgroup $N_0 \subset N$. If N_1 fixes f, then by 3.1 Remark. we have

$$\sum_{i=1}^{m} |g_i N_0 \cap N_1 \cap ker(\vartheta)| \Sigma(g_i) f = 0$$

where $\{g_i\}$ are the coset representatives. So for any $n_1 \in N$, we have

$$\sum_{i=1}^{m} f(n_1 g_i) = 0$$

Now

$$\int_{N_0} \vartheta^{-1}(n)\Sigma(n)\vartheta f dn = \sum_{i=1}^{m} |g_i N_0 \cap N_1 \cap ker(\vartheta)| \Sigma(g_i)\vartheta^{-1}(g_i)\vartheta f$$

since $N_0 \cap N_1 \cap ker(\vartheta)$ fixes ϑf. So it is enough to prove

$$\sum_{i=1}^{m} \Sigma(g_i)\vartheta^{-1}(g_i)\vartheta f = 0$$

Take any $n_1 \in N$,

$$\begin{aligned}
\sum_{i=1}^{m} \Sigma(g_i)\vartheta^{-1}(g_i)\vartheta f(n_1) &= \sum_{i=1}^{m} \vartheta^{-1}(g_i)\vartheta(n_1 g_i) f(n_1 g_i) \\
&= \sum_{i=1}^{m} \vartheta(n_1) f(n_1 g_i) \\
&= \vartheta(n_1) \sum_{i=1}^{m} f(n_1 g_i) \\
&= 0
\end{aligned}$$

□

Since $V_N = V/V(N)$ has dimension 1, dim $V_\vartheta = 1$ which means $V_\vartheta \cong \vartheta$ since V_ϑ is a direct sum of copies of ϑ . But we know $W_\vartheta \cong V_\vartheta$. Therefore 8.3 theorem implies $W = W(N) \cong$ c-$Ind_N^M \vartheta$ which by 8.2 corollary is irreducible over M.

If G is a locally profinite group, H is an open subgroup of G. Λ is an irreducible representation of H. is $c - Ind_{H^{g_1}}^G \Lambda^{g_1}$ isomorphic to $c - Ind_H^G \Lambda$ for $g_1 \in G$. Where $H^{g_1} = g_1^{-1} H g_1$. and Λ^{g_1} is the representation of H^{g_1} defined by $g_1^{-1} h g_1 \mapsto \Lambda(h)$ □

As a direct consequence of the Proposition, we have:

Corollary. *As a representation of B or M, $Ind_B^G \chi$ has composition length 3. Two of the composition factors have dimension 1, and the third is of infinite dimension. In particular, the G-composition length of the representation $Ind_B^G \chi$ is at most 3.*

Proof. By above analysis, $Ind_B^G = \chi \oplus \delta_B^{-1}\chi^w \oplus W$ as B-representation. If Ind_B^G has G-length $l > 3$, since every irreducible G-subrepresentation offers at least one irreducible B-factor, Ind_B^G at least has l irreducible B-factor which is a contradiction. $\square$

9.8 We come to an important result.

Proposition. *Notation as above. The following are equivalent:*

(1) $\chi_1 = \chi_2$*;*

(2) X *has a 1-dimensional* N*-subspace.*

When these conditions hold,

(3) X *has a unique 1-dimensional* N*-subspace* X_0*;*

(4) X_0 *is a* G*-subspace and* $X_0 \cap V = \emptyset$*.*

Proof. If (1) holds, we can assume $\chi_1 = \chi_2 = 1$, then the constant functions span a 1-dimensional G-space of X. Generally, $X = Ind_B^G(\phi \cdot 1) \cong \phi Ind_B^G 1$. Therefore X also has a 1-dimensional N-space.

Conversely, let $X_0 = \{cf \mid c \in \mathbb{C}\}$ be the 1-dimensional N-subspace with generator f. Then N acts on f by a character τ. Take $x \in F^\times$, and consider the identity

$$w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

And we know if $\|x\|$ is sufficiently, then f is fixed under right translation by $\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}$. This means

$$\tau(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})f(w) = f(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) = \chi_1(-1)\chi_1^{-1}\chi_2(x)f(1) \tag{1}$$

Therefore $f(w) = 0$ if and only if $f(1) = 0$. Notice that $f(b) = \chi(b)f(1)$ for $b \in B$, and $f(bwn) = \chi(b)\tau(n)f(w)$ for $b \in B, n \in N$, then $supp(f) = G$ since $f \neq 0$. So $f(1) \neq 0$ and $f \notin V$. $f(tng) = \chi(t)f(g)$ for $t \in T, n \in N, g \in G$. Take $t = g = 1$, we have $f(n) = f(1)$ for all $n \in N$. This implies $\tau(n) = 1$ for all $n \in N$ which means N fixes f under right translation. Now

$$f(w) = \chi_1(-1)\chi_1^{-1}\chi_2(x)f(1)$$

for all $x \in F^\times$ of sufficiently large absolute value. Thus $\chi_1 = \chi_2 = \phi$ and $f(g) = \phi(det(g))f(1)$ which means X_0 is a G-subspace. If $Y_0 = \{ch \mid c \in \mathbb{C}\}$ be another 1-dimensional N-subspace. Similarly, $h(g) = \phi(det(g))h(1)$. Thus

$$h = \frac{h(1)}{g(1)} g$$

which means X_0 is unique.

Remark: Why have we $\chi_1 = \chi_2$? Take two different unit u_1, u_2 such that $\|u_1 \pi^n\|$ and $\|u_1 \pi^n\|$ are sufficiently large, then $\chi_1^{-1}\chi_2(u_1) = \chi_1^{-1}\chi_2(u_2)$. Thus $\chi_1^{-1}\chi_2 \mid U_F = 1$

For $x = \pi^n$, $\chi_1^{-1}\chi_2(\pi^n) = \chi_1^{-1}\chi_2(\pi)^n$ and as n becomes large, $\chi_1^{-1}\chi_2(x)$ is constant , so $\chi_1^{-1}\chi_2(\pi) = 1$. Finally $\chi_1^{-1}\chi_2 = 1$. □

9.9 Now we finish the proof of the Irreducibility Criterion 9.6. Before proving, we need a lemma.

Lemma. *Assume X is reducible. Then its G-length is 2 or 3, and either it has a 1-dimensional N-subspace or its dual a 1-dimensional N-subspace.*

Proof. By 9.7 Corollary, G-length of X is 2 or 3.

1. If G-length of X is 2, then we have following exact sequence of representations of G

$$0 \to V_1 \to X \to V_2 \to 0 \tag{2}$$

   such that V_1 and V_2 are irreducible G-representation. B-length of X is 3 implies only one $V_1 (i = 1, 2)$ is reducible as B-representation and the other is irreducible as B-representation.

   First case, if V_1 is reducible as B-representation, then it at least contains one of χ and $\delta_B^{-1}\chi^w$, so X has a 1-dimensional N-subspace. Second case, if V_2 is reducible as B-representation, we can asume $V_1 \mid_B = W$ since if $V_1 \mid_B$ equals χ or $\delta_B^{-1}\chi^w$, then X has a 1-dimensional N-subspace. So we are in the first case.

   Take dual of (3), we have following exact sequence of representations of G

$$0 \to \check{V_2} \to \check{X} \to \check{V_1} \to 0$$

   such that $\check{V_1}$ and $\check{V_2}$ are irreducible G-representation since V_1 and V_2 are non-cuspidal. By the Duality Theorem of 7.7, $\check{X} \cong Ind_B^G \delta_B^{-1} \check{\chi}$, so the same proof of 9.7 for $\check{X}$ implies that as a B-representation, $\check{X}$ has two i-dimensional factors and one infinite dimensional factor. Since dim $\check{V_2} = 2$, it must be reducible as B-representation. This means $\check{X}$ has a 1-dimensional N-subspace.

2. If X has G-length 3, then the three irreducible G-factor are irreducible B-factor. The result holds.

$\square$

First assume X has a 1-dimensional N-subspace L, then we are in the condition 9.8 and $\chi_1 = \chi_2 = \phi$. We also know G acts on L by $\phi \circ det$ and $V \cap L = \emptyset$. Denote $Y = X/L$, then the canonical morphism

$$\psi : V \to X \to X/L$$

is an B-isomorphism.

Injective : if $\psi(f) \in L$, then $f(g) = \phi(det(g))f(1) = 0$, so $f = 0$.

Surjective : for any $f \in X$, we want find $h \in V$ such that $h - f \in L$. This is trivial, fix $h_1 \in L$, then $h = f - \frac{f(1)}{h_1(1)}h_1 \in V$ such that $h - f \in L$. If X has G-length 3, then Y has G-length 2. However, V has B-length 2 and a unique 1-dimensional B-quotient V_N. This implies Y has a 1-dimensional G-quotient on which G must act by $\phi' \circ det$. This means $\phi' \otimes \phi'$ is a factor of $Y_N \cong \chi^w \delta_B^{-1} \cong \chi\delta_B^{-1}$. Therefore there $0 \neq v \in Y_N$ such that

$$\phi'(det(t))v = \chi\delta_B^{-1}v \ \ \forall t \in T$$

if $t = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, this is just

$$\phi'(ab) = \phi(ab)\frac{\|a\|}{\|b\|} \quad \forall a, b \in F^\times$$

Take $a = 1$ and $b = 1$ respectively, we have $\phi'(a) = \phi(a)\|a\|$ and $\phi'(b) = \phi(b)\frac{1}{\|b\|}$, this means

$$\|x\|^2 = 1 \quad \forall x \in F^\times$$

which is impossible. So X can only have G-length 2. We are in case $2 : (3)$ of 9.6.

Second, if $\check{X}$ has a 1-dimensional N-subspace which is also a G-subspace by 9.8 proposition. So X has a 1-dimensional G-subspace, we are in the case $2 : (4)$ of 9.6. By computing,

$$\delta_B^{-1}\check{\chi}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \chi_1^{-1}(a)\|a\| \cdot \chi_2^{-1}(b)\|b\|^{-1}$$

So the first condition implies

$$\chi_1^{-1}(x)\|x\| = \chi_2^{-1}(x)\|x\|^{-1}, \forall x \in F^\times$$

This is just case $2 : (4)$of 9.6.

So we have proved that X is reducible implies results in Irreducible Criterion. For the converse, just use 9.8 proposition and dual case. Therefore we have proved the Irreducible Criterion.

9.10 To get a classification of the irreducible, non-cuspidal representations of G, we need to investigate the homomorphisms between induced representations:

Proposition. *Let χ and ξ be characters of T. Then*

$$\dim \mathrm{Hom}_G(Ind_B^G\chi, Ind_B^G\xi) = \begin{cases} 1, & \xi = \chi \ \ or \ \ \xi = \chi^w \delta_B^{-1}; \\ 0, & otherwise. \end{cases}$$

Proof. By Frobenius Reciprocity

$$\mathrm{Hom}_G(Ind_B^G\chi, Ind_B^G\xi) \cong \mathrm{Hom}_T((Ind_B^G\chi)_N, \xi).$$

The jacquet module $(Ind_B^G\chi)_N$ fits into the exact sequence

$$0 \to \chi^w\delta_B^{-1} \to (Ind_B^G\chi)_N \to \chi \to 0$$

by 9.3 Lemma.

If $\chi \neq \chi^w\delta_B^{-1}$, then $(Ind_B^G\chi)_N = \chi \oplus \chi^w\delta_B^{-1}$ and the result holds.

If $\chi = \chi^w\delta_B^{-1}$, then $\chi_1(x) = \|x\|\chi_2(x),\ x \in F^\times$. So $Ind_B^G\chi$ is irreducible by the Irreducible Criterion and the result holds. □

9.11 We introduce a new notation. If σ is a smooth representation of T, we define

$$\iota_B^G\sigma = Ind_B^G(\delta_B^{-1/2} \otimes \sigma)$$

In this language, the Irreducible Criterion (9.6) and (9.10) Proposition say:

Lemma. *(1) Let $\chi = \chi_1 \otimes \chi_2$ be a character of T. Then representation $\iota_B^G\chi$ is reducible if and only if $\chi_1\chi_2^{-1}$ is one of the character $x \mapsto \|x\|^{\pm 1}$ of $F^\times$ or, equivalently, $\chi = \phi \cdot \delta_B^{\pm 1/2}$ for some character ϕ of $F^\times$.*

(2) Let χ, ξ be characters of T. The space $\mathrm{Hom}_G(\iota_B^G\chi, \iota_B^G\xi)$ is non-zero if and only if $\xi = \chi$ or $\xi = \chi^w$.

Theorem. ***(Classification Theorem)****. The following is a complete list of the isomorphism classes of irreducible, non-cuspidal representations of G:*

(1) the irreducible induced representations $\iota_B^G\chi$, where $\chi \neq \phi \cdot \delta^{\pm 1/2}$ for any character ϕ of $F^\times$.

(2) the 1-dimensional representations $\phi \circ det$, where ϕ ranges over the characters of $F^\times$.

(3) the special representations $(\phi \circ det) \otimes St_G$, where ϕ ranges over the characters of $F^\times$.

Proof. If $X = Ind_B^G \chi$ is irreducible, by above lemma, (1) holds.

If $Ind_B^G \chi$ is reducible, then there is a character ϕ of $F^\times$ such that $\chi = \phi \cdot 1_T$ or $\chi = \phi \cdot \delta_B^{-1}$.

1. if $\chi = 1_T$. Then irreducible G-quotient of $Ind_B^G 1_T$ is called the *steiberg representation* of G, and is denoted by St_G:

$$0 \to 1_G \to Ind_B^G 1_T \to St_G \to 0 \tag{9.11.1}$$

   its dimension is infinite and $(St_G)_N \cong \delta_B^{-1}$. Map $1_G \to Ind_B^G 1_T$ is given by $c \mapsto (f : g \mapsto c)$, namely the image of a complex number c is a constant function.
   Similarly if $\chi = \phi \cdot 1_T$, apply ϕ to $(9.11.1)$, we have following exact sequence

$$0 \to \phi_G \to Ind_B^G \chi \to \phi_G \otimes St_G \to 0$$

   where $\phi_G = \phi \circ det$.

2. if $\chi = \delta_B^{-1}$. We know $Ind_B^G \delta_B^{-1} \cong \check{X}$. Take dual of $(9.11.1)$, we have following exact sequence

$$0 \to St_G^\vee \to Ind_B^G \delta_B^{-1} \to 1_G \to 0 \tag{9.11.2}$$

   Indeed, we have

$$St_G \cong St_G^\vee$$

   Proof. We have canonical morphism $Ind_B^G 1_T \to Ind_B^G \delta_B^{-1}$. It must contain 1_G in its kernel because otherwise $Ind_B^G \delta_B^{-1}$ has a 1-dimensional subrepresentation which is impossible by the Irreducible Criterion. So we have a morphism $St_G \to Ind_B^G \delta_B^{-1}$, the image is irreducible of infinite dimension, hence contained in $St_G^\vee$. Therefore we have a morphism $St_G \to St_G^\vee$. They are irreducible so $St_G \cong St_G^\vee$. □

   Similarly if $\chi = \phi \cdot \delta_B^{-1}$. Apply ϕ to $(9.11.2)$ We have an exact sequence

$$0 \to \phi_G \otimes St_G^\vee \to Ind_B^G \chi \to \phi_G \to 0$$

Thus theorem holds. □

11 Intertwing: Compact Induction and Cuspidal Representation

Proposition. *Let K be a compact open subgroup of G, let $g \in G$ and $\rho \in \hat{K}$. The following are equivalent :*

(1) There exists $f \in e_\rho * \mathcal{H}(G) * e_\rho$ such that $f \mid KgK \neq 0$;

(2) g intertwines ρ.

Proof. Consider the space $C^\infty(KgK)$ of G-smooth functions on the coset KgK. This carries a smooth representations of $K \times K$ by

$$(k_1, k_2)f : x \longmapsto f(k_1^{-1} x k_2)$$

Let H denote the group of pairs $(k, g^{-1}kg) \in K \times K,\ k \in K \cap gKg^{-1}$. The map $f \mapsto f(g)$ is an H-homomorphism $C^\infty(KgK) \to \mathbb{C}$(with H acting trivially). By Frobenius reciprocity, this induces a $K \times K$-homomorphism

$$\phi : C^\infty(KgK) \to Ind_H^{K\times K} 1_H$$

we show this is an isomorphism.

Denote $V = Ind_H^{K\times K} 1_H$ The condition (1) implies that $e_\rho * C^\infty(KgK) * e_\rho \cong V^{\rho\otimes\hat{\rho}} \neq 0$. Equivalent,

$$\mathrm{Hom}_{K\times K}(\rho \otimes \hat{\rho}, V) \cong \mathrm{Hom}_H(\rho \otimes \hat{\rho}, 1_H) \neq 0$$

the last relation is equivalent the representation $k \mapsto \rho(k) \otimes \hat{\rho}(g^{-1}kg)$ of $K \cap gKg^{-1}$ having a fixed vector, namely $\mathrm{Hom}_{K\cap gKg^{-1}}(\rho^{g^{-1}}, \rho) \neq 0$. This means g^{-1} intertwines ρ which is equivalent to g intertwines ρ.

Remark: Specifically, let W be the representative space of ρ. $\mathrm{Hom}_H(\rho \otimes \hat{\rho}, 1_H) \neq 0$ implies that there exist $0 \neq v_0 = \sum_i u_i \otimes w_i^* \in W \otimes W^*$ such that $\rho(k) \otimes \hat{\rho}(g^{-1}kg)v_0 = v_0$ for all $k \in K \cap gKg^{-1}$ We define following map

$$\begin{aligned} \phi : \quad & W \longrightarrow W \\ & w \longrightarrow \sum_i \langle w_i^*, w\rangle u_i \end{aligned}$$

We will prove

$$\phi(\rho^{g^{-1}}(k)w) = \rho(k)(\phi(w)) \quad \forall k \in K \cap gKg^{-1}. \tag{11.1.1}$$

which mean $\phi \in \mathrm{Hom}_{K\cap gKg^{-1}}(\rho^{g^{-1}}, \rho) \neq 0$.

Notice $\rho(k) \otimes \hat{\rho}(g^{-1}kg)v_0 = v_0$ is just $\sum_i \rho(k)u_i \otimes \hat{\rho}(g^{-1}kg)w_i^* = \sum_i u_i \otimes w_i^*$. Define $T_w : W \otimes W^* \to W$ by $T_w(\sum_i u_i \otimes t_i) = \sum_i \langle t_i, w\rangle u_i$ which is a linear map. Therefore

$$T_w(\sum_i \rho(k)u_i \otimes \hat{\rho}(g^{-1}kg)w_i^*) = T_w(\sum_i u_i \otimes w_i^*).$$

This is

$$\sum_i \langle \hat{\rho}(g^{-1}kg)w_i^*, v\rangle \rho(k)u_i = \sum_i \langle w_i^*, v\rangle u_i$$

$$\text{Left hand} = \sum_i \langle w_i^*, \rho(g^{-1}k^{-1}g)v\rangle \rho(k)u_i = \sum_i \langle w_i^*, \rho(g^{-1}kg)v\rangle \rho(k^{-1})u_i$$

We have

$$\sum_i \langle w_i^*, \rho(g^{-1}kg)v\rangle \rho(k^{-1})u_i = \sum_i \langle w_i^*, v\rangle u_i$$

Applying $\rho(k)$ to both hands, we obtain

$$\sum_i \langle w_i^*, \rho(g^{-1}kg)v\rangle u_i = \sum_i \langle w_i^*, v\rangle \rho(k)u_i$$

This is just $(11.1.1)$. $\square$

Lemma. *(Σ, V) is a $K\times K$ representation as above. $\rho \boxtimes \hat{\rho}$ is a irreducible $K\times K$ representation, Then*

$$e_\rho * C^\infty(KgK) * e_\rho \cong \Sigma(e_{\rho\boxtimes\hat{\rho}})V$$

Proof. Take $f \in C^\infty(KgK), m \in K$.Then

$$\begin{aligned} e_\rho * f * e_\rho(m) &= \int_K e_\rho(k_1)(f * e_\rho)(k_1^{-1}m)dk_1 \\ &= \int_K e_\rho(k_1)\int_K f(k_2^{-1})e_\rho(k_2k_1^{-1}m)dk_2dk_1 \\ &= \int_K e_\rho(k_1)\int_K f(k_1^{-1}mk_2^{-1})e_\rho(k_2)dk_2dk_1 \\ &= \int_K\int_K e_\rho(k_1)e_\rho(k_2)f(k_1^{-1}mk_2^{-1})dk_1dk_2 \end{aligned}$$

Therefore

$$\begin{aligned} \phi(e_\rho * f * e_\rho)(b,c) &= ((b,c)e_\rho * f * e_\rho)(g) \\ &= e_\rho * f * e_\rho(b^{-1}gc) \\ &= \int_K\int_K e_\rho(k_1)e_\rho(k_2)f(k_1^{-1}b^{-1}gck_2^{-1})dk_1dk_2 \end{aligned}$$

But

$$\begin{aligned}
\Sigma(e_{\rho\boxtimes\hat{\rho}})\phi(f) &= \int_K\int_K e_{\rho\boxtimes\hat{\rho}}(k_1,k_2)\Sigma(k_1,k_2)f dk_1 dk_2 \\
&= \frac{(dim\rho)^2}{(\mu(K))^2}\int_K\int_K tr((\rho\boxtimes\hat{\rho})(k_1^{-1},k_2^{-1}))\Sigma(k_1,k_2)\phi(f)dk_1dk_2 \\
&= \frac{(dim\rho)^2}{(\mu(K))^2}\int_K\int_K tr(\rho(k_1^{-1}))tr(\hat{\rho}(k_2^{-1}))\Sigma(k_1,k_2)\phi(f)dk_1dk_2 \\
&= \int_K\int_K e_\rho(k_1)e_\rho(k_2^{-1})\Sigma(k_1,k_2)\phi(f)dk_1dk_2 \\
&= \int_K\int_K e_\rho(k_1)e_\rho(k_2)\Sigma(k_1,k_2^{-1})\phi(f)dk_1dk_2
\end{aligned}$$

So

$$\begin{aligned}
\Sigma(e_{\rho\boxtimes\hat{\rho}}\phi(f)(b,c) &= \int_K\int_K e_\rho(k_1)e_\rho(k_2)\Sigma(k_1,k_2^{-1})\phi(f)(b,c)dk_1dk_2 \\
&= \int_K\int_K e_\rho(k_1)e_\rho(k_2)\phi(f)(bk_1,ck_2^{-1})dk_1dk_2 \\
&= \int_K\int_K e_\rho(k_1)e_\rho(k_2)f(k_1^{-1}b^{-1}gck_2^{-1})dk_1dk_2
\end{aligned}$$

Namely

$$\Sigma(e_{\rho\boxtimes\hat{\rho}})\phi(f) = \phi(e_\rho * f * e_\rho) \quad \forall f\in C^\infty(KgK).$$

This is just

$$e_\rho * C^\infty(KgK) * e_\rho \cong \Sigma(e_{\rho\boxtimes\hat{\rho}})V$$

□

11.4. Then central result of this section is :

Theorem. *Let K be an open subgroup of $G = GL_2(F)$, containing and compact modulo Z. Let (ρ, W) be an irreducible smooth representation of K and suppose that an element $g\in G$ intertwines ρ if and only if $g\in K$. Then c–$Ind_K^G\rho$ is irreducible and cuspidal.*

Proof. Denote $(\Sigma, X) = c\text{–}Ind_K^G\rho$. We first show that Σ has a non-zero coefficient which is compactly supported modulo Z.

The groups K, G are unimodular, so the Duality Theorem of 3.5 implies that $\check{X}\cong Ind_K^G\check{\rho}$. The induced representation $Ind_K^G\check{\rho}$ contains $c\text{–}Ind_K^G\check{\rho}$ as a G-subspace. Then canonical K-embedding $\check{W}\to c\text{–}Ind_K^G\rho$ identifies $\check{W}$ witrh the space of functions in $\check{X}$ with support contained in K. We take.

Consequently, we only need to prove that X is irreducible : then it is admissible(7.2 corollary) and apply 7.1 Proposition (2) to show that it is γ-cuspidal, hence cuspidal.

X is the direct sum of its K-isotypic components. Any K-map $\phi : W \to X$ has image contained in X^ρ, since if $im(\phi) \cap X^\sigma \neq \emptyset$ for $\sigma \ncong \rho$ then $\phi = 0$. So

$$\mathrm{Hom}_K(W, X^\rho) \cong \mathrm{Hom}_K(W, X) \cong End_G(X) \cong \mathcal{H}(G, \rho).$$

However, the assumption of intertwining implies $\mathcal{H}(G,\rho) = span\{\chi_{KgK} \mid g \in K\} = span\{\chi_K\}$ which means $dim\mathcal{H}(G,\rho) = 1$. Therefore $dim\mathrm{Hom}_K(W, X^\rho) = 1$, this implies $W = X^\rho$.

Let Y be a non-zero G-subspace of X. Then

$$0 \neq \mathrm{Hom}_G(Y, X) \subset \mathrm{Hom}_G(Y, Ind_K^G \rho) \cong \mathrm{Hom}_K(Y, \rho)$$

Since Y is semisimple over K(2.7 Proposition), we have $Y^\rho \neq 0$. Thus $Y \supset Y^\rho \supset Y \cap W$, but W is irreducible over K. So $W = Y \cap W \subset Y$. Notice that W generates X over G, if $G = \bigcup\limits_i Kg_i$, then $f = \sum\limits_i \pi(g_i^{-1}) f_{w_i}$(notation in 2.5) which $w_i = f(g_i)$ for any $f \in X$. Therefore $Y = X$, namely X is irreducible. □

11.5. We give an application of above result. Let $G = GL_2(F)$, $K = GL_2(\mathfrak{o})$ and $K_1 = 1 + \mathfrak{p}M_2(\mathfrak{o})$. Thus K_1 is an open normal subgroup of K and $K/K_1 \cong GL_2(\mathbf{k})$. We also let I_1 denote the group

$$I_1 = 1 + \begin{pmatrix} \mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$$

which is the inverse image of $N(\mathbf{k})$ consisting upper triangular unipotent matrices in $GL_2(\mathbf{k})$ under the mod-$\mathfrak{p}$ map $GL_2(\mathfrak{o}) \to GL_2(\mathbf{k})$.

Theorem. *Let (π, V) be an irreducible smooth representation of G, and suppose that π contains the trivial character of K_1. Then exactly one of the following holds:*

(1) π contains a representation λ of K, which is inflated from an irreducible cuspidal representation $\bar{\lambda}$ of $GL_2(\mathbf{k})$;

(2) π contains the trivial character of I_1.

In the first case, π is cuspidal and there exists a representation Λ of ZK such that $\Lambda \mid K \cong \lambda$ and

$$\pi \cong c\text{-}Ind_{ZK}^G \Lambda.$$

Proof. Notice K stablizes V^{K_1} since Since $K_1 \lhd K$, we have following diagram

$$\begin{tikzcd} K \arrow[r, "\pi"] \arrow[d] & GL(V^{K_1}) \\ K/K_1 \cong GL_2(\mathbf{k}) \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

So V^{K_1} is a direct sum of irreducible representations λ_i of K which is trivial on K_1 and λ_i is inflated from $GL_2(\mathbf{k})$. Let λ be one of these, inflated from $\tilde{\lambda}$. Then either $\tilde{\lambda}$ is cuspidal or not. In the latter case $\tilde{\lambda}$ contains the trivial character of $N(\mathbf{k})$, so λ contains the trivial character of I_1.

We need to prove the two cases cannot occur together. This follows from the following lemma.

Lemma. *For $i = 1, 2$, let $\tilde{\rho}_i$ be an irreducible representation of $GL_2(\mathbf{k})$ and let ρ_i denote the inflation of $\tilde{\rho}_i$ to a representation of K. Suppose $\tilde{\rho}_1$ is cuspidal. Then we have*

(1) Then representations ρ_i intertwine in G if and only if $\tilde{\rho}_1 \cong \tilde{\rho}_2$

(2) an element $g \in G$ intertwines ρ_1 if and only if $g \in ZK$

Proof. (1)($\Leftarrow$) is clear since $\rho_1 \cong \rho_2$. For ($\Rightarrow$) , let $g \in G$ intertwines ρ_2 with ρ_1, notice interiwining depends on the coset KgK. since the action of Z is a character, this actually depends on the coset $KgZK$. We can assume g of the form

$$g = \begin{pmatrix} \varpi^a & 0 \\ 0 & 1 \end{pmatrix}$$

for some $a \geq 0$. If $a = 0$ then $g = 1$ which means $\rho_1 \cong \rho_2$, so $\tilde{\rho}_1 \cong \tilde{\rho}_2$. Now assume $a \geq 1$, then

$$N = \begin{pmatrix} 1 & \mathfrak{o} \\ 0 & 1 \end{pmatrix} \subset K_1^g \subset K^g \cap K$$

on which ρ_2^g is trivial (since ρ_2 is trivial on K_1). Assume $0 \neq T \in \mathrm{Hom}_{K^g \cap K}(\rho_2^g, \rho_1)$ such that $T\rho_2^g(k) = \rho_1(k)T$ for all $k \in K^g \cap K$, then

$$0 \neq T(v) = \rho_1(k)T(v) \ \ \forall k \in N$$

for some $v \in \rho_2^g$, which means ρ_1 contains the trivial character of N. Therefore $\tilde{\rho}_1$ contains the trivial character of $N(\mathbf{k})$. this is a contradiction since $\tilde{\rho}_1$ is cuspidal.

(2)if $g = zk \in ZK$, let $T = \rho_1(k^{-1})$. Then

$$T\rho_1^g(x) = \rho_1(x)T$$

which means g intertwines ρ_1. Conversely, as above only $KZK = ZK$ can intertwine ρ_1. $\square$

Lemma (1) and 11.1 Proposition 1 imply that the two case cannot occur together. Now assume $\tilde{\lambda}$ is cuspidal. Sure π contains a representation Λ of ZK extending λ(just let $\Lambda\mid_Z = \omega_\pi$).($\Lambda$ is smooth irreducible) So we have a non-trivial ZK-homomorphism $\Lambda \to \pi$, giving a non-trivial G-homomorphism $c\text{-}Ind_{ZK}^G \Lambda \to \pi$. We can prove that Λ satisfies the condition of 11.4 theorem

$$g \in G \ \text{ intertwines } \ \Lambda \ \text{ if and only if } \ g \in ZK$$

The proof is same as above lemma (2)($T = \Lambda(k^{-1})$), so $c\text{-}Ind_{ZK}^G \Lambda$ is irreducible which means $\pi \cong c\text{-}Ind_{ZK}^G \Lambda$. □

4 Cuspidal Representations

12 Chain Orders and Fundamental Strata

12.4 Example: Let E be a F-subalgebra of A which is a quadratic field extension of F. Thus $V = F \oplus F$ is an E-vector space of dimension 1.

Specifically, assume $E = F(\alpha)$ with $\alpha^2 = f \in F$. We can embedd E to A by following map

$$E \to A$$
$$e = a + b\alpha \mapsto \begin{pmatrix} a & bf \\ b & a \end{pmatrix}$$

which preserve multiply of E. Denote the element of V by column vector. For $e = a + b\alpha \in E$ and $v = \begin{pmatrix} x \\ y \end{pmatrix} \in V$, define

$$e \cdot v = \begin{pmatrix} a & bf \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + byf \\ bx + ay \end{pmatrix}$$

Take $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then any $v = \begin{pmatrix} x \\ y \end{pmatrix} \in V$ can be writed by

$$v = (x + y\alpha) \cdot e_1$$

which means V is an E-vector space of dimension 1.

Proposition. *Let E be an F-subalgebra of A such that E/F is a quadratic field extension.*

(1) The set of $\mathfrak{o}_E$-lattice in V forms an $\mathfrak{o}$-lattice chain $\mathcal{L}$, with the property $e_{\mathcal{L}} = e(E/F)$. Further, $\mathcal{L}$ is the unique lattice chain in V which is stable under translation by $E^\times$.

(2) The order $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$ is the unique chain order in A such that $E^\times \subset \mathcal{K}_{\mathfrak{A}}$.

(3) If $\mathfrak{P} = rad\mathfrak{A}$, then $x\mathfrak{A} = \mathfrak{P}^{v_E(x)}$ for all $x \in E^\times$, and $\mathcal{K}_{\mathfrak{A}} = E^\times U_{\mathfrak{A}}$.

Proof. By above analysis, any $\mathfrak{o}_E$-lattice is of form □

12.6 Now we apply these concepts to analyze the representations of G.

Let (π, V) be an irreducible smooth representation of G. Let $\mathcal{S}(\pi)$ denote the set of pair $(\mathfrak{A}, n)$, where $\mathfrak{A}$ is a chain order in A and $n \geq 0$ is an integer, with the condition that π contains the trivial character of $U_{\mathfrak{A}}^{n+1}$. We define the *normalized level* $\ell(\pi)$ of π by

$$\ell(\pi) = \min\{n/e_{\mathfrak{A}} : (\mathfrak{A}, n) \in \mathcal{S}(\pi).\}$$

Proposition. *Let π be an irreducible smooth representation of G; then $\ell(\pi) = 0$ if and only if π contains the trivial character of $U_{\mathfrak{M}}^1$.*

Proof. Just notice that $U_{\mathfrak{M}}^1 \subset U_{\mathfrak{J}}^1$. □

12.7 To deal with the representation π for $\ell(\pi) > 0$, we introduce a new concept. For the remainder of this chapter, we fix a character ψ of F of level 1.

A *stratum* is a triple $(\mathfrak{A}, n, \alpha)$ where $\mathfrak{A}$ is a chain order in A(with radical $\mathfrak{P}$), n is an integer and $a \in \mathfrak{P}^{-n}$.

We say that stratum $(\mathfrak{A}, n, \alpha_1)$,$(\mathfrak{A}, n, \alpha_2)$ are equivalent if $a_1 \equiv a_2 (\mod \mathfrak{P}^{1-n})$.

If $n \geq 1$, we can associate to a stratun $(\mathfrak{A}, n, \alpha)$ the character ψ_a of $U_{\mathfrak{A}}^n$, which is trivial on $U_{\mathfrak{A}}^{n+1}$. By 12.5 Proposition, this character depends only on the equivalence class of the stratum(and the choice of ψ).

Proposition. *Let $(\mathfrak{A}_i, n_i, a_i), i = 1, 2$ be stratum in A. Let $\mathfrak{P}_i = rad\mathfrak{A}_i$ and $g \in G$. Assume $n_i \geq 1$, the following are equivalent:*

(1) The element g intertwines the character ψ_{a_1} of $U_{\mathfrak{A}_1}^{n_1}$ with the character ψ_{a_2} of $U_{\mathfrak{A}_2}^{n_2}$.

(2) $g^{-1}(a_1 + \mathfrak{P}_1^{1-n_1})g \cap (a_2 + \mathfrak{P}_2^{1-n_2}) \neq \emptyset$.

Proof. Take $\mathfrak{A}_3 = g^{-1}\mathfrak{A}_1 g$, its radical $rad\mathfrak{P}_3 = g^{-1}\mathfrak{A}_1 g$. And the character $(\psi_{a_1})^g$ of the group $(U_{\mathfrak{A}_1}^{n_1})^g = U_{\mathfrak{A}_3}^{n_1}$ is associated to the stratun $(\mathfrak{A}_3, n_1, g^{-1}a_1 g)$. So we can reduce to the case $g = 1$.

If (2) holds, take an element a in the section, then $\psi_a = \psi_{a_i}$ on $U_{\mathfrak{A}_i}^{n_i}$, so $\psi_{a_1} = \psi_{a_2} = \psi_a$ on $U_{\mathfrak{A}_1}^{n_1} \cap U_{\mathfrak{A}_2}^{n_2}$.

Conversely, suppose ψ_{a_i} agree on $U_{\mathfrak{A}_1}^{n_1} \cap U_{\mathfrak{A}_2}^{n_2}$, namely

$$\psi_A(a_1 x) = \psi_A(a_2 x), \;\; x \in \mathfrak{P}_1^{n_1} \cap \mathfrak{P}_2^{n_2}$$

In the notation of 12.5, we have

$$(\mathfrak{P}_1^{n_1} \cap \mathfrak{P}_2^{n_2})^* = (\mathfrak{P}_1^{n_1})^* + (\mathfrak{P}_2^{n_2})^* = \mathfrak{P}_1^{1-n_1} + \mathfrak{P}_2^{1-n_2}$$

which means

$$a_1 \equiv a_2 (\mod \mathfrak{P}_1^{1-n_1} + \mathfrak{P}_2^{1-n_2})$$

Therefore there exist $x_i \in \mathfrak{P}_i^{1-n_i}$ such that $a_2 = a_1 + x_1 + x_2$, namely

$$a_2 - x_2 = a_1 + x_1 \in (a_1 + \mathfrak{P}_1^{1-n_1}) \cap (a_2 + \mathfrak{P}_2^{1-n_2})$$

□

When the element g satisfies condition (2) of the Proposition, we say that it intertwines $(\mathfrak{A}_1, n_1, a_1)$ with $(\mathfrak{A}_2, n_2, a_2)$.

12.8 Not all stratum are of equal interest: we have to distinguish a particular class of them.

Definition. *Let $\mathfrak{A}$ be a chain order in A, and set $\mathfrak{P} = rad\mathfrak{A}$. A stratum $(\mathfrak{A}, n, a)$ in A is called fundamental if the coset $a + \mathfrak{P}^{1-n}$ contains no nilpotent element of A.*

This property depends only on the equivalence class of the stratum.

We first need an effective method of recognizing fundamental strata:

Proposition. *Let $(\mathfrak{A}, n, a)$ be a stratum in A with $\mathfrak{P} = rad\mathfrak{A}$. The following are equivalent:*

(1) The coset $a + \mathfrak{P}^{1-n}$ contains a nilpotent element of A.

(2) There is an integer $r \geq 1$ such that $a^r \in \mathfrak{P}^{1-rn}$.

Proof. $(1) \Rightarrow (2)$ is clear.

We assume (2) holds. Nothing changes if we replace $(\mathfrak{A}, n, a)$ by a G-conjugate. So we can reduce to the case $\mathfrak{A} = \mathfrak{M}$ or $\mathfrak{J}$. Simlilarly, nothing changes if we replace $(\mathfrak{A}, n, a)$ by $(\mathfrak{A}, n-e, \varpi a)$ for a prime element ϖ of F, $e = e_{\mathfrak{A}}$. This reduces us to the cases $(\mathfrak{A}, n) = (\mathfrak{M}, 0), (\mathfrak{J}, 0)$ or $(\mathfrak{J}, -1)$.

First case : $(\mathfrak{M}, 0, a)$. $a^r \in \mathfrak{P}$ implies $\bar{a}^r = 0$ in $M_2(\mathbf{k})$. Thus there is $\bar{g} \in GL_2(\mathbf{k})$ such that

$$\bar{g}\bar{a}\bar{g}^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

lifting $\bar{g}$ to $g \in GL_2(\mathfrak{o})$. We have

$$gag^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + A$$

which $A \in \mathfrak{P} = \begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$. Therefore $g^{-1}Ag \in \mathfrak{P}$ and $a - g^{-1}Ag$ is nilpotent.

Second case :$(\mathfrak{J}, 0, a)$. we can assume $a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ by replacing coset representative where $a_1, a_2 \in \mathfrak{o}$. Then $a^r \in \mathfrak{P}_{\mathfrak{J}}$ implies $a_1, a_2 \in \mathfrak{p}$. Therefore $a + \mathfrak{P}_{\mathfrak{J}}$ contains the nilpotent element 0.

Third case :$(\mathfrak{J}, -1, a)$. We can assume $a = \begin{pmatrix} 0 & a_2 \\ \varpi a_1 & 0 \end{pmatrix}$ by replacing coset representative where $a_1, a_2 \in \mathfrak{o}$. Then $a^r \in \mathfrak{P}_{\mathfrak{J}}^{1+r}$ implies $a^{2r} \in \mathfrak{P}^{1+2r}$. But $a^{2r} = (\varpi a_1 a_2)^r I_2$ where I_2 is the unit matrix and $\mathfrak{P}_{\mathfrak{J}}^{1+2r} = \varpi^r \mathfrak{P}_{\mathfrak{J}}$. Hence $a_1 a_2 \in \mathfrak{p}$ which means $a_1 \in \mathfrak{p}$ or $a_2 \in \mathfrak{p}$. Both case imply $a + \mathfrak{P}^2$ contains a nilpotent element. $\square$

Using the calculations in the last proof, we can list the equivalence classes of non-fundamental stratum, up to G-conjugation in G. we say a stratum $(\mathfrak{A}, n, a)$ is trivial if $a \in \mathfrak{P}^{1-n}$ where $\mathfrak{P} = rad\mathfrak{A}$.

Theorem. *Let ϖ be a prime element of F. A non-trivial, non-fundamental stratum in A is equivalent to a G-conjugate of one of the following:*

$$(\mathfrak{M}, n, \varpi^{-n}a), \quad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$(\mathfrak{J}, 2n-1, \varpi^{-n}a), \quad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for some $n \in \mathbb{Z}$.

Proof. By the same calculation as above, we can prove a non-fundamental stratum which is G-conjugate to $(\mathfrak{J}, 2n, a)$ for an integer n must be trivial. Other cases are similar to above, we omit them.(Readers can check by themselves to examine if they know these tricks.) $\square$

12.9 Let (π, V) be an irreducible smooth representation of G. We say that π contains the stratum $(\mathfrak{A}, n, a)$ if $n \geq 1$ and π contains the character ψ_a of $U_{\mathfrak{A}}^n$. Observe that if this is the case, then $n/e_{\mathfrak{A}} \geq \ell(\pi)$ by definition.

Then main result here is :

Theorem. *Let π be an irreducible smooth representation of G and let $(\mathfrak{A}, n, a)$ be a stratum in A, contained in π. The following are equivalent:*

(1) $(\mathfrak{A}, n, a)$ is fundamental;

(2) $\ell(\pi) = n/e_{\mathfrak{A}}$.

In particular, π contains a fundamental stratum if and only if $\ell(\pi) > 0$.

Proof. The first step is :

Lemma. *(1) Let $(\mathfrak{A}, n, a)$ be a non-fundamental stratum in A, and let $\mathfrak{P}$ be the radical of $\mathfrak{A}$. There is a chain order $\mathfrak{A}_1$ in A with radical $\mathfrak{P}_1$, and an integer n_1 such that*

$$a + \mathfrak{P}^{1-n} \subset \mathfrak{P}_1^{-n_1}, \quad \text{and} \quad n_1/e_{\mathfrak{A}_1} < n/e_{\mathfrak{A}}.$$

(2) Let π be an irreducible smooth representation of G, containing a non-fundamental stratum $(\mathfrak{A}, n, a)$. Then we have $\ell(\pi) < n/e_{\mathfrak{A}}$.

Proof. Part (1) is trivial if the stratum $(\mathfrak{A}, n, a)$ is trivial, so we can assume otherwise. The issue is unchanged if taking G-conjugate and replacing $(\mathfrak{A}, n, a)$ by $(\mathfrak{A}, n - e_{\mathfrak{A}}r, \varpi^r a)$. Therefore we can reduce to the cases 12.8 Theorem with $n = 0$. In the first case, we have

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathfrak{P}_M \subset \mathfrak{P}_{\mathfrak{J}}.$$

and in the second case

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathfrak{P}_J^2 \subset \mathfrak{P}_1 = \begin{pmatrix} \mathfrak{p} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$$

where is the radical of $\mathfrak{A}_1$ and

$$\mathfrak{A}_1 = \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix}$$

which is conjugate to $\mathfrak{M}$.

For the second part, we apply (1): $a + \mathfrak{P}^{1-n} \subset \mathfrak{P}_1^{-n_1}$ implies $\mathfrak{P}^{1-n} \subset \mathfrak{P}_1^{-n_1}$, dualizing $\mathfrak{P}_1^{n_1+1} \subset \mathfrak{P}^n$. Therefore $U_{\mathfrak{A}_1}^{n_1+1} \subset U_{\mathfrak{A}}^n$, and the character ψ_a of $U_{\mathfrak{A}}^n$ is trivial on $U_{\mathfrak{A}}^n$. Thus $\ell(\pi) \leq n_1/e_{\mathfrak{A}_1} < n/e_{\mathfrak{A}}$. □

If $\ell(\pi) > 0$, then by definition π contains a stratum $(\mathfrak{A}_1, n_1, a_1)$ with $n_1/e_{\mathfrak{A}_1} = \ell(\pi)$ since $U_{\mathfrak{A}_1}^{n_1+1} \lhd U_{\mathfrak{A}_1}^{n_1}$, hence we can extend the representation. By lemma, this stratum must be fundamental. If π contains another stratum $(\mathfrak{A}_2, n_2, a_2)$, then $(12.7, 11.1)$ it must intertwine with $(\mathfrak{A}_1, n_1, a_1)$.

Proposition. *Let $(\mathfrak{A}_1, n_1, a_1)$ be a fundamental stratum in A. Let $(\mathfrak{A}_2, n_2, a_2)$ be a stratum in A which intertwines with $(\mathfrak{A}_1, n_1, a_1)$. Then $n_2/e_{\mathfrak{A}_2} \geq n_1/e_{\mathfrak{A}_1}$, with equality if and only if $(\mathfrak{A}_2, n_2, a_2)$ is fundamental.*

Proof. For simplicity, denote $e_1 = e_{\mathfrak{A}_1}, e_2 = e_{\mathfrak{A}_2}$.

If $g \in G$ intertwines ψ_{a_2} with ψ_{a_1}, replace $(\mathfrak{A}_2, n_2, a_2)$ by $(g^{-1}\mathfrak{A}_2 g, n_2, g^{-1}a_2 g)$, we can assume $g = 1$. Take $b \in (a_1 + \mathfrak{P}_1^{1-n_1}) \cap (a_2 + \mathfrak{P}_2^{1-n_2}) \neq \emptyset$.

Assume $n_1/e_1 > n_2/e_2$, then $n_1 e_2 > n_2 e_1$. There integer $r \geq 1$ such that

$$b^{e_1 e_2 r} \in \mathfrak{P}_2^{-e_1 e_2 r n_2} = \mathfrak{p}^{-e_1 n_2 r}\mathfrak{A}_2 \subset \mathfrak{p}^{1-e_2 n_1 r}\mathfrak{A}_1 = \mathfrak{P}_1^{e_1 - e_1 e_2 n_1 r} \subset \mathfrak{P}_1^{1-e_1 e_2 n_1 r}$$

contrary to $(\mathfrak{A}_1, n_1, a_1)$ is fundamental.

If $(\mathfrak{A}_2, n_2, a_2)$ is fundamental, symmetry implies $n_2/e_2 \leq n_1/e_1$, and hence $n_2/e_2 = n_1/e_1$. Conversely, suppose $n_2/e_{\mathfrak{A}_2} = n_1/e_{\mathfrak{A}_1}$ but $(\mathfrak{A}_2, n_2, a_2)$ is non-fundamental. Then there exists $r \geq 1$ such that $b^r \in \mathfrak{P}_2^{1-n_2 r}$. Hence

$$b^{2r} \in \mathfrak{P}_2^{2-2n_2 r} = \mathfrak{p}^{\frac{2-2n_2 r}{e_2}}\mathfrak{A}_2$$

There exists an integer $m \geq 1$ such that

$$b^{2rm} \in \mathfrak{p}^{\frac{2m}{e_2} - \frac{2mn_2 r}{e_2}}\mathfrak{A}_2 = \mathfrak{p}^{\frac{2-2rmn_1}{e_1}}\mathfrak{p}^{\frac{2m}{e_2} - \frac{2}{e_1}}\mathfrak{A}_2 \subset \mathfrak{p}^{\frac{2-2rmn_1}{e_1}}\mathfrak{A}_1 = \mathfrak{P}_1^{2-2rmn_1} \subset \mathfrak{P}_1^{1-2rmn_1}$$

contrary to $(\mathfrak{A}_1, n_1, a_1)$ is fundamental. □

In case $\ell(\pi) > 0$, the proof of the theorem is complete. It remains to show that if $\ell(\pi) = 0$, then π cannot contain a fundamental stratum $(\mathfrak{A}, n, a), n \geq 1$. By definition, π contains the trivial character of $U^1_{\mathfrak{M}}$. Since $U^1_{\mathfrak{J}}/U^1_{\mathfrak{M}} \cong \mathbf{k}$, we can consider $\pi \mid U^1_{\mathfrak{J}} : U^1_{\mathfrak{J}} \to GL(V^{U^1_{\mathfrak{M}}})$, then there is a character ϕ of $U^1_{\mathfrak{J}}$ which is trivial on $U^1_{\mathfrak{M}} \supset U^2_{\mathfrak{J}}$ such that $V^\phi \neq 0$. By 12.5 Proposition $\phi = \psi_a$ for some $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathfrak{P}_{\mathfrak{J}}^{-1} = \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{o} & \mathfrak{o} \end{pmatrix}$. We claim that $a_{12} \in \mathfrak{o}$ since $\psi(tr_A(ax)) = 1$ for all $x \in \mathfrak{P}_{\mathfrak{M}}$. so by calculating, $a_{12} \in \mathfrak{o}$. Therefore we can take

$$a \equiv \begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix} \pmod{\mathfrak{J}}$$

for some $a_{21} \in \mathfrak{o}$. Then π contains the stratum $(\mathfrak{J}, 1, a)$ which is non-fundamental. If π contains a stratum $(\mathfrak{A}, n, b)$, then $n/e_{\mathfrak{A}} \geq 1/2$. It cannot be fundamental by lemma 2 since this stratum must intertwine with $(\mathfrak{J}, 1, a)$(11.1Proposition) □

13 Classification of Fundamental Strata

In this section, we give some results of the classification of fundamental stratum. We continue to fix a character ψ of F of level 1.

13.1 The first case is :

Proposition. *(1) Let $(\mathfrak{A}, n, a)$ be a stratum with $e_{\mathfrak{A}} = 2$ and n is odd. Let $\mathfrak{P} = rad\mathfrak{A}$. The stratum $(\mathfrak{A}, n, a)$ is fundamental if and only if $a\mathfrak{A} = \mathfrak{P}^{-n}$ or, equivalently, $a \in \Pi^{-n}U_{\mathfrak{A}}$ for a prime element Π of $\mathfrak{A}$.*

(2) Let π be an irreducible smooth representation of G with $\ell(\pi) > 0$. If $\ell(\pi) = n/2 \notin \mathbb{Z}$, then π contains a fundamental stratum $(\mathfrak{J}, n, a)$.

Proof. The first assertion concerns only the conjugacy class of the stratum, so we can take $\mathfrak{A} = \mathfrak{J}$. As 12.8 Proposition, we can assume $a = \begin{pmatrix} 0 & a_2 \\ \varpi a_1 & 0 \end{pmatrix}$ where $a_1, a_2 \in \mathfrak{o}$, then to prove that $(\mathfrak{A}, n, a)$ is fundamental if and only if $a_1 \notin \mathfrak{p}, a_2 \notin \mathfrak{p}$.

For (2), 12.9 Theorem says that π contains a fundamental stratum; Since $\ell(\pi) \notin \mathbb{Z}$, it must be conjugate to one of the form $(\mathfrak{J}, n, a)$. We know that π contains stratum $(\mathfrak{A}, n, a)$ if and only π contains the stratum $(g^{-1}\mathfrak{A}g, n, g^{-1}ag)$ for any $g \in G$. Hence the result holds. □

Definition. *A ramified simple stratum is a fundamental stratum $(\mathfrak{A}, n, a)$ in which $e_{\mathfrak{A}} = 2$ and n is odd.*

Lemma. *If $0 < \ell(\pi) = n \in \mathbb{Z}$, then π contains a fundamental stratum of the form $(\mathfrak{M}, n, a)$.*

Proof. We only need to prove that if π contains the stratum $(\mathfrak{J}, 2n, a)$, then it contains the stratum $(\mathfrak{M}, n, a)$. Notice that

$$U_{\mathfrak{M}}^{n+1} \subset U_{\mathfrak{J}}^{2n+1} \subset U_{\mathfrak{J}}^{2n} \subset U_{\mathfrak{M}}^{n}.$$

Do the same analysis as the end of 12.9, the character ψ_a of $U_{\mathfrak{J}}^{2n}$ is trivial on $U_{\mathfrak{M}}^{n+1}$ and $U_{\mathfrak{M}}^{n+1} \lhd U_{\mathfrak{M}}^{n}$. Therefore π contains the character ψ_a of $U_{\mathfrak{M}}^{n}$. □

For this reason, there is no need to consider fundamental stratun of the form $(\mathfrak{J}, 2n, a), n \in \mathbb{Z}$.

Corollary. *Let π be an irreducible smooth representation of G with $\ell(\pi) > 0$, then π contains a fundamental stratum $(\mathfrak{A}, n, a)$ such that* $\gcd(n, e_{\mathfrak{A}}) = 1$.

13.2 Consider a stratum $(\mathfrak{A}, n, \alpha)$ in which $e_{\mathfrak{A}} = 1$. We can write $\alpha = \varpi^{-n}\alpha_0$, for some $\alpha_0 \in \mathfrak{A}$. Let $f_\alpha(t) \in \mathfrak{o}[t]$ be the characteristic polynomial of α_0, and let $\tilde{f}_\alpha(t) \in \mathbf{k}[t]$ be its reduction modulo $\mathfrak{p}$.

If we regards the prime element ϖ as fixed, the polynomial $\tilde{f}_\alpha(t)$ only depends on the equivalent class of the stratum $(\mathfrak{A}, n, a)$.

Observe that the stratum $(\mathfrak{A}, n, a)$ is fundamental if and only if $\tilde{f}_\alpha(t) \neq t^2$(12.8 Proposition).

Definition. *Let $(\mathfrak{A}, n, \alpha)$ be a fundamental stratum with $e_{\mathfrak{A}} = 1$. We say that $(\mathfrak{A}, n, a)$ is*

$$\begin{cases} \textit{unramified simple} & \textit{if } \tilde{f}_\alpha(t) \textit{ is irreducible in } \mathbf{k}[t], \\ \textit{split} & \textit{if } \tilde{f}_\alpha(t) \textit{ has distinct roots in } \mathbf{k}, \\ \textit{essentially scalar} & \textit{if } \tilde{f}_\alpha(t) \textit{ has a repeated root in } \mathbf{k}^\times. \end{cases}$$

A stratum $(\mathfrak{A}, n, \alpha)$ is called *simple* if it is either ramified or unramified simple.

Proposition. *(1) A ramified simple stratum cannot intertwine with any fundamental stratum of the form $(\mathfrak{M}, n, \alpha)$.*

(2) Let $(\mathfrak{M}, n, \alpha), (\mathfrak{M}, n, \beta)$ be fundamental strata which intertwine. We then have $\tilde{f}_\alpha(t) = \tilde{f}_\beta(t)$.

Proof. (1) follows from 12.8 Proposition.

For (2), 12.7 Proposition says that there $g \in G$ and $\beta' \in \beta + \mathfrak{P}_{\mathfrak{M}}^{1-n}$ such that $g^{-1}\beta' g \in \alpha + \mathfrak{P}_{\mathfrak{M}}^{1-n}$. The characteristic polynomial of the element $\varpi^n g^{-1}\beta' g$ is the same as that of $\varpi^n \beta'$, so when we reduce it modulo $\mathfrak{p}$, we get $\tilde{f}_\beta(t)$. On the other hand, $g^{-1}\beta' g \in \alpha + \mathfrak{P}_{\mathfrak{M}}^{1-n}$ implies that this reduction is also $\tilde{f}_\alpha(t)$. Thus $\tilde{f}_\alpha(t) = \tilde{f}_\beta(t)$. $\square$

13.3 One of the 13.2 Definition is easy to describe. Recall a notation. If π is an irreducible smooth representation of G and if χ is a character of $F^\times$, then $\chi\pi$ denote the representation $g \mapsto \chi(\det g)\pi(g)$.

Theorem. *Let π be an irreducible smooth representation of G with $\ell(\pi) > 0$. The following are equivalent:*

(1) The representation π contains an essentially scalar stratum $(\mathfrak{M}, n, \alpha)$.

(2) There is a character χ of $F^\times$ such that $\ell(\chi\pi) < \ell(\pi)$.

Proof. Suppose that $(\mathfrak{M}, n, \alpha)$ is an essentially scalar stratum occurring in π. Replacing α by a $U_{\mathfrak{M}}$-conjugate, we can assume

$$\alpha \equiv \varpi^{-n} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \pmod{\mathfrak{p}^{1-n}\mathfrak{M}}.$$

for a prime element ϖ of F and $a \in U_F, b \in \mathfrak{o}$. Let χ be the character of $U_{\mathfrak{M}}$ which is trivial on $U_{\mathfrak{M}}^{n+1}$ corresponding to $-a\varpi^{-n} \in \mathfrak{P}^{-n}$. Then deem it as a character of U_F by resstricting to the center, finally extend to a character of $F^\times$. Hence we have $\chi(1+x) = \psi_A(-a\varpi^{-n}x), x \in \mathfrak{p}^n$. Notice that $\chi \circ \det \mid U_{\mathfrak{M}}^n = \psi_{-a\varpi^{-n}}$, by calculating the representation $\chi\pi$ contains the stratum $(\mathfrak{M}, n, \beta)$, with

$$\beta \equiv \varpi^{-n} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \pmod{\mathfrak{p}^{1-n}\mathfrak{M}}$$

This is not fundamental. So $\ell(\chi\pi) < n$.

Conversely,

□

Corollary. *Let π be an irreducible smooth representation of G such that $0 < \ell(\pi) \leq \ell(\chi\pi)$ for every character χ of $F^\times$. Then one and only one of the following holds:*

(1) π contains a split fundamental stratum.

(2) π contains a ramified simple stratum.

(3) π contains an unramified simple stratum.

Proof. Since $\pi > 0$, π contains a fundamental stratum. By the theorem, this stratum is not essentially scalar. We can assume it is either ramified simple, unramified simple, or split. Proposition 13.2 show that only one of these possibilities can occur. □

13.4

Definition. *An element $\alpha \in G \backslash Z$ is called minimal over F if the sub-algebra $E = F[\alpha]$ of A is a field and setting $n = -v_E(\alpha)$, one of the following holds:*

(1) E/F is totally ramified and n is odd;

(2) E/F is unramified, and for a prime element ϖ of F, the coset $\varpi^n\alpha + \mathfrak{p}_E$ generates the field extension $\mathbf{k}_E/\mathbf{k}$.

The hypothesis $\alpha \notin Z$ implies $[E : F] = 2$.

Lemma. *Let $\alpha \in G$ be minimal over F. Set $E = F[\alpha]$, $n = -v_E(\alpha)$, and choose a prime element ϖ of F. Define*

$$\alpha_0 = \begin{cases} \varpi^{(n+1)/2}\alpha & \textit{if} \quad E/F \quad \textit{is ramified} \\ \varpi^n \alpha & \textit{if} \quad E/F \quad \textit{is unramified} \end{cases} \tag{13.4.1}$$

Then we have $\mathfrak{o}_E = \mathfrak{o}[\alpha_0]$.

Proof. See https://ocw.mit.edu/courses/18-785-number-theory-i-fall-2021/pages/lecture-notes/ Theorem 10.12 and Theorem 11.5. □

There is a close connection between minimal elements and simple strata.

Proposition. *Let $(\mathfrak{A}, n, \alpha)$ be a simple stratum in A. Then:*

(1) α is minimal over F;

(2) $F[\alpha]^\times \subset \mathcal{K}_{\mathfrak{A}}$;

(3) $e(F[\alpha] \mid F) = e_{\mathfrak{A}}$.

(4) every $\alpha^{'} \in \alpha + \mathfrak{P}^{1-n}$ is minimal over F, where $\mathfrak{P} = rad\mathfrak{A}$.

Proof. We fixed a prime element ϖ of F. Suppose first that $(\mathfrak{A}, n, a)$ is ramified. Thus $n = 2m + 1$ is odd, and the element $\alpha_0 = \varpi^{1+m}\alpha$ satisfies $\alpha_0\mathfrak{A} = \mathfrak{P}$(13.1 Proposition). Therefore $v_F(\det \alpha_0) = 1$ and $v_F(tr\alpha_0) \geq 1$, namely the minimal polynomial of α_0 over F is Eisenstein, so $E = F[\alpha] = F[\alpha_0]$ is a ramified quadratic field of F. Moreover,$v_E(\alpha_0) = v_F(\det \alpha_0) = 1$, $v_E(\alpha) = -n$ and n is odd. Thus α is minimal over F.

If $(\mathfrak{A}, n, a)$ is unramified, we put $\alpha_0 = \varpi^n\alpha$. The minimal polynomial $f(t)$ of α_0 over F remains irreducible on reduction modulo $\mathfrak{p}$ hence $E = F[\alpha]$ is an unramified extension of degree 2. And $\mathbf{k}_E = \mathbf{k}(\bar{\alpha_0})$.

For (2), $E^\times = U_E \times (\alpha_0)^{\mathbb{Z}}$. In both two case $\mathfrak{o}_E = \mathfrak{o}[\alpha_0] \subset \mathfrak{A}$, hence $U_E \subset U_{\mathfrak{A}} \subset \mathcal{K}_{\mathfrak{A}}$. In the ramified case, 13.1 Proposition says that $\alpha \in \Pi^{-n}U_{\mathfrak{A}}$, so $\alpha_0 \in \Pi U_{\mathfrak{A}} \in \mathcal{K}_{\mathfrak{A}}$. In the unramified case, we take the decomposition $E^\times = U_E \times (\pi_F)^{\mathbb{Z}}$. Clearly, $E^\times \subset \mathcal{K}_{\mathfrak{A}}$.

(3) is trivial.

For (4), if $\alpha^{'} \in \alpha + \mathfrak{P}^{1-n}$, then $(\mathfrak{A}, n, \alpha^{'})$ is equivalent to $(\mathfrak{A}, n, \alpha)$, hence simple, then do the same argument. □

13.5 Thus simple strata give rise to minimal elements. The converse also holds

Proposition. *Let α be minimal over F. There exists a unique chain order $\mathfrak{A}$ in A such that $\alpha \in \mathcal{K}_{\mathfrak{A}}$. Moreover, $F[\alpha]^\times \in \mathcal{K}_{\mathfrak{A}}$ and if $n = -v_{F[\alpha]}(\alpha)$, the triple $(\mathfrak{A}, n, a)$ is a simple stratum.*

Proof. Put $E = F[\alpha]$ and $n = -v_E(\alpha)$. Let $\mathfrak{A}$ be the unique chain order such that $E^\times \subset \mathcal{K}_{\mathfrak{A}}$. In particular, $\alpha \in \mathcal{K}_{\mathfrak{A}}$.

Define α_0 as in $(13.4.1)$. Let $\mathfrak{B}$ be a chain order with $\alpha \in \mathcal{K}_{\mathfrak{B}}$. We need to prove $\mathfrak{A} = \mathfrak{B}$. We know that $\alpha_0 \in \mathcal{K}_{\mathfrak{B}}$, and by calculating in both cases, $v_F(\det \alpha_0) \geq 0$. Therefore $\alpha_0 \in \mathfrak{B}$ since $\alpha_0 \in F^\times U_{\mathfrak{M}}$ or $\mathcal{K}_{\mathfrak{J}}$. Therefore by 13.4 Lemma $\mathfrak{o}_E \subset \mathfrak{B}$. Do the same argument, we have $E^\times \subset \mathcal{K}_{\mathfrak{B}}$ and therefore $\mathfrak{B} = \mathfrak{A}$.

The final assertion is by definition.

14 Strata and Principal Series

14.1

Proposition. *Let (π, V) be a irreducible smooth representation of G, and suppose that π contains a split fundamental stratum $(\mathfrak{M}, n, \alpha)$. We can take $\alpha \in T$, in this case, the Jacquet module (π_N, V_N) contains the character $\psi_\alpha \mid U_{\mathfrak{M}}^n \cap T$. In particular, $V_N \neq 0$ and π is not cuspidal.*

Proof. If π is an irreducible cuspidal representation of $GL_2(F)$ containg the trivial character of $K(1)$where

$$K(1) = 1 + \begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$$

, then how to prove that there a nonzero v such that $\pi(g)v = v$ for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix}$
By taking a $U_{\mathfrak{M}}$-conjugation, this conjugation does not change $\mathfrak{M}$, so we can assume

$$\alpha = \varpi^{-n} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{14.1.1}$$

where ϖ is a prime element of F and $a, b \in U_F = \mathfrak{o} \cap F^\times$ with $a \not\equiv b \mod \mathfrak{p}$.

Denote $\xi = \psi_\alpha \mid U_{\mathfrak{M}}^n$. Its is enough to prove that the space V^ξ has non-zero image in V_N.

So suppose $V^\xi \in V(N)$, we want to to obtain a contradiction. By 8.1 Lemma, for each $v \in V^\xi$, there is a compact open subgroup $N(v)$ of N such that

$$\int_{N(v)} \pi(u) v du = 0$$

Denote

$$N_j = \begin{pmatrix} 1 & \mathfrak{p}^j \\ 0 & 1 \end{pmatrix}$$

We know π is admissible (10.2 corollary), so V^ξ is finite-dimensional since $V^\xi \subset V^{K_1}$ for any compact open subgroup $K_1 \subset ker(\xi)$. Therefore there exists $j \in \mathbb{Z}$ such that

$$\int_{N_j} \pi(u) v du = 0$$

for all $v \in V^\xi$. We choose j maximal for this property, so there exist $v_1 \in V^\xi$ such that

$$\int_{N_{j+1}} \pi(u) v_1 du \neq 0$$

Set

$$t = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$$

By calculating(noticing $\alpha \in T$), t intertwines ξ, namely ξ and ξ^t agree on

$$Y := U_{\mathfrak{M}}^n \cap t^{-1} U_{\mathfrak{M}}^n t = 1 + \begin{pmatrix} \mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p}^{n+1} & \mathfrak{p}^n \end{pmatrix}.$$

Lemma. *(1) Any irreducible representation (ϕ, V) of $U_{\mathfrak{M}}^n$ containing $\xi \mid_Y$ is of dimension* 1.

(2) Let ϕ be a character of $U_{\mathfrak{M}}^n$ such that $\phi \mid_Y = \xi \mid_Y$. There exists $n_0 \in N_0$ such that $\phi^{n_0} = \xi$.

Proof. (1)Noticing that $V = V^\xi$, so the quotient representation of ϕ on the abelian group $U_{\mathfrak{M}}^n / U_{\mathfrak{M}}^{n+1}$ is irreducible so it is of dimension 1.

(2) We can take $\phi = \psi_\delta$ where

$$\delta \equiv \varpi^{-n} \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \mod \mathfrak{p}^{1-n}\mathfrak{M}.$$

for some $x \in \mathfrak{o}$. Because if

$$\delta \equiv \varpi^{-n} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

for $a_{11}, a_{12}, a_{21}, a_{22} \in \mathfrak{o}$. Then $\phi \mid_Y = \xi \mid_Y$ implies

$$\psi \circ tr(\alpha \begin{pmatrix} \varpi^n b_{11} & \varpi^n b_{12} \\ \varpi^{n+1} b_{21} & \varpi^n b_{22} \end{pmatrix}) = \psi \circ tr(\delta \begin{pmatrix} \varpi^n b_{11} & \varpi^n b_{12} \\ \varpi^{n+1} b_{21} & \varpi^n b_{22} \end{pmatrix}) \text{ for any } \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_2(\mathfrak{o}).$$

which is equivalent to

$$\psi(ab_{11} + bb_{22}) = \psi(a_{11}b_{11} + \varpi a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}) = \psi(a_{11}b_{11} + a_{21}b_{12} + a_{22}b_{22})$$

since ψ has level 1. Take $b_{22} = b_{12} = 0$ we have

$$\psi(ab_{11}) = \psi(a_{11}b_{11})$$

which means $a \equiv a_{11} \mod \mathfrak{p}$. Similarly $b \equiv a_{22} \mod \mathfrak{p}$ and $a_{21} \in \mathfrak{p}$. This means

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mod \mathfrak{p}$$

For $y \in \mathfrak{o}$, we have

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & z \\ 0 & b \end{pmatrix}$$

where $z = x + (b-a)y$. Take $y = \frac{x}{a-b}$ and $n_0 = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, the result holds. $\square$

We consider the vector $v_2 = \pi(t^{-1})v_1$. By the definition of j,

$$\begin{aligned} \int_{N_j} \pi(u)v_2 du &= \int_{N_j} \pi(ut^{-1})v_1 du \\ &= \pi(t^{-1}) \int_{N_j} \pi(tut^{-1})v_1 du \\ &= q\pi(t^{-1}) \int_{N_{j+1}} \pi(u)v_1 du \neq 0 \end{aligned}$$

where $q = \#\mathbf{k}$.

We will prove that if $v_2 \in V^\rho$ for some irreducible representation ρ of $U^n_{\mathfrak{M}}$, then ρ must contain $\xi \mid_Y$. By above lemma, ρ is of dimension 1. Let $v_2 = \pi(t^{-1})v_1 \in V^\rho$. Then

$$\rho(g)v_2 = \pi(g)v_2 \quad \forall g \in U^n_{\mathfrak{M}}$$

and we know

$$\xi(g)v_1 = \pi(g)v_1 \quad \forall g \in U^n_{\mathfrak{M}}$$

So for $g \in Y$,

$$\begin{aligned} \pi(t)\rho(g)v_2 &= \pi(tgt^{-1})v_1 \\ &= \xi(tgt^{-1})v_1 \\ &= \xi(g)v_1 \end{aligned}$$

which means

$$\rho(g)v_2 = \xi(g)v_2$$

for all $g \in Y$. Therefore ρ contains $\xi \mid_Y$.

Let Φ be the set of characters ϕ of $U_{\mathfrak{M}}^n$ which agree with ξ on Y. By above analysis, we have $v_2 = \sum_{\phi\in\Phi} v_\phi$ for certain vectors $v_\phi \in V^\phi$. So there exists $\phi \in \Phi$ such that

$$\int_{N_j} \pi(u)v_\phi du = 0$$

By part (2) of the lemma, $\phi^{n_0} = \xi$ for some $n_0 \in N_0$. Thus $v_3 = \pi(x^{-1})v_\phi \in V^\xi$(just by a same calculating as above) and

$$\int_{N_j} \pi(u)v_3 du \neq 0$$

which contradicts the definition of j. $\square$

14.2 In the opposite direction, we can spot a fundamental stratum in an induced representation $Ind_B^G\chi$:

Proposition. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and set $\Sigma = Ind_B^G\chi$. Let n_i be the level of χ_i.*

(1) If $n = \max(n_1, n_2) > 0$ and $\chi_1\chi_2^{-1} \mid U_F^n \neq 1$, then Σ contains a split fundamental stratum.

(2) If $n_1 = n_2 = n > 0$ and $\chi_1\chi_2^{-1} \mid U_F^n$ is trivial, then Σ contains an essentially scalar fundamental stratum.

(3) If $n_1 = n_2 = 0$, then Σ contains the trivial character of $U_{\mathfrak{J}}^1$.

Proof. (1) We can choose $a_i \in \mathfrak{p}^{-n}$ such that $\chi_i(1+x) = \psi(a_i x)$ for all $x \in \mathfrak{p}^n$. Then $a_1 \not\equiv a_2 \mod \mathfrak{p}^{1-n}$. Now set

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \qquad N_n^{'} = \begin{pmatrix} 1 & 0 \\ \mathfrak{p}^n & 1 \end{pmatrix}$$

The triple $(\mathfrak{M}, n, a)$ is a split fundamental stratum. Define $f \in Ind_B^G\chi$ by

$$f(g) = \begin{cases} \chi(b) & if \quad g = bn^{'} \in BN_n^{'} \\ 0 & if \quad g \notin BN_n^{'} \end{cases}$$

Then f has support $BU_{\mathfrak{M}}^n = BN_n^{'}$ and is fixed by $U_{\mathfrak{M}}^{n+1}$(we prove it in the end). We claim that $\Sigma(u)f = \psi_a(u)f$ for all $u \in U_{\mathfrak{M}}^n$ which means Σ contains ψ_a. Take $u = \begin{pmatrix} 1+a_{11} & a_{12} \\ a_{21} & 1+a_{22} \end{pmatrix} \in U_{\mathfrak{M}}^n$, then

$$(\Sigma(u)f)(bn^{'}) = f(bn^{'}u)$$

Let $bn^{'}u = b^{'}n_1^{'}$ where $b^{'} \in B, n_1^{'} \in N_n^{'}$ by Iwahori Decomposition. Therefore

$$(\Sigma(u)f)(bn^{'}) = f(bn^{'}u) = f(b^{'}n_1^{'}) = \chi(b^{'})$$

But

$$\begin{aligned}\psi_a(u)f(bn^{'}) &= \psi \circ tr(a(u-1))\chi(b)\\ &= \psi(a_1a_{11} + a_2a_{22})\chi(b)\\ &= \chi_1(1+a_{11})\chi_2(1+a_{22})\chi(b)\end{aligned}$$

So it is enough to prove

$$\chi(b^{'}) = \chi_1(1+a_{11})\chi_2(1+a_{22})\chi(b)$$

namely

$$\chi(n^{'}un_1^{'\,-1}) = \chi_1(1+a_{11})\chi_2(1+a_{22})$$

If $n^{'} = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ and $n_1^{'\,-1} = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}$, then

$$\begin{aligned}\chi(n^{'}un_1^{'\,-1}) &= \chi_1(1+a_{11}+s_1a_{12})\chi_2(1+a_{22}+sa_{12})\\ &= \chi_1(1+a_{11})\chi_2(1+a_{22})\end{aligned}$$

since $s_1a_{12}, sa_{12} \in \mathfrak{p}^{2n}$.

If $u \in U_{\mathfrak{M}}^{n+1}$, then $a_{11}\ a_{22} \in \mathfrak{p}^{n+1}$. Therefore $\chi(n^{'}un_1^{'\,-1}) = 1$ which means f is fixed by $U_{\mathfrak{M}}^{n+1}$.

(2) The proof is same as (1). But $a_1 \equiv a_2 \mod \mathfrak{p}^{1-\mathfrak{n}}$.

(3) Define $f \in Ind_B^G\chi$ by

$$f(g) = \begin{cases} \chi(b) & if \quad g = bn^{'} \in BN_1^{'} \\ 0 & if \quad g \notin BN_1^{'} \end{cases}$$

Then f has support $BU_{\mathfrak{M}}^1 = BN_1^{'} = BU_{\mathfrak{J}}^1$ and is fixed by $U_{\mathfrak{M}}^1$. We will prove

$$\Sigma(u)f = f$$

for all $u \in U_{\mathfrak{J}}^1$. As above, take $bn^{'} \in BN_1^{'}$, then $bn^{'}u = b^{'}n_1^{'}$ where $n_1^{'} \in N_1^{'}$ by Iwahori Decomposition. Therefore

$$\begin{aligned}\chi(n^{'}un_1^{'\,-1}) &= \chi_1(1+a_{11}+s_1a_{12})\chi_2(1+a_{22}+sa_{12})\\ &= 1\end{aligned}$$

since $a_{11} \in \mathfrak{p}, a_{12} \in \mathfrak{o}, a_{22} \in \mathfrak{p}, s \in \mathfrak{p}, s_1 \in \mathfrak{p}$. It means

$$(\Sigma(u)f)(bn^{'}) = f(bn^{'})$$

So the result holds. $\square$

14.3

Proposition. *Let (π, V) be an irreducible smooth representation containing a character ϕ of I which is trivial on $U_{\mathfrak{J}}^1$. Then the canonical map $V \to V_N$ is injective on the isotypic space V^ϕ. In particular, (π, V) is not cuspidal.*

Proof. We use the notation

$$T^0 = \begin{pmatrix} U_F & 0 \\ 0 & U_F \end{pmatrix}, \ N_j = \begin{pmatrix} 1 & \mathfrak{p}^j \\ 0 & 1 \end{pmatrix}, \ N_j^{'} = \begin{pmatrix} 1 & 0 \\ \mathfrak{p}^j & 1 \end{pmatrix}, \ t = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$$

for $j \in \mathbb{Z}$ and some prime element ϖ of F. $\square$

We defer the proof of this lemma to 14.4. Accepting it for the moment, assume that there exists $v \in V^\phi, v \neq 0$ with zero image in V_N. Thus there exists $j \in \mathbb{Z}$ such that

$$\int_{N_J} \pi(x) v dx = 0. \tag{14.3.1}$$

$\dim V^\phi$ is finite. Therefore we can choose j maximal for the property that there exists $0 \neq v \in V^\phi$ satisfying $(14.3.1)$.

We consider the element $w = \pi(t)v$. Then

$$\int_{N_{j+1}} \pi(x) w dx = 0$$

and

$$\pi(y)w = \phi(y)w, \quad \forall y \in N_1^{'} T^0 N_1 \tag{14.3.2}$$

Take $y = n_1^{'} t_0 n_1 \in N_1^{'} T^0 N_1$, then $t^{-1}yt = t^{-1}n_1^{'}t \cdot t^{-1}t_0 t \cdot t^{-1} n_1 t \in I$, (14.3.2) just say

$$\pi(t^{-1}yt)v = \phi(y)v$$

But we know

$$\pi(g)v = \phi(g)v \quad \forall g \in I$$

So we need to prove

$$\phi(t^{-1}yt) = \phi(y)$$

which is equivalent to

$$\phi(t^{-1}n_1^{'}t \cdot t^{-1}n_1t) = \phi(t^{-1}n_1^{'}n_1t) = \phi(n_1^{'}n_1)$$

This holds since $n_1^{-1}{n_1^{'}}^{-1}t^{-1}n_1^{'}n_1t \in U_{\mathfrak{J}}^1$.

We now put

$$\begin{aligned} u = \pi(e_\phi)w &= \frac{1}{\mu(I)}\int_I \phi(g^{-1})\pi(g)wdg \\ &= \frac{1}{q}\sum_{g_i \in I/N_1T_0N_1^{'}} \phi(g_i^{-1})\pi(g_i)w \\ &= \frac{1}{q}\sum_{g_i \in N_0/N_1} \pi(g_i)w \end{aligned}$$

Thus $u \in V^\phi$ and

$$\int_{N_{j+1}} \pi(x)udx = 0.$$

But

$$\begin{aligned} \pi(f)v &= \int_{ItI} f(g)\pi(g)vdg \\ &= \int_I\int_I \phi(i_1i_2)^{-1}\pi(i_1ti_2)vdi_1di_2 \\ &= \int_I \phi(i_1^{-1})\pi(i_1t)(\int_I \phi(i_2^{-1})\pi(i_2)vdi_2)di_1 \\ &= \mu(I)\int_I \phi(i_1^{-1})\pi(i_1t)vdi_1 \\ &= [\mu(I)]^2u \end{aligned}$$

Therefore by the lemma, $u \neq 0$. since if $u = 0$, then $\pi(f^{-1} * f)v = \pi(e_\phi)v = \mu(I)v = 0$ which is a contradiction.

14.5 We now can characterize the irreducible cuspidal representation in terms of strata:

Theorem. *Let (π, V) be an irreducible representation of G, which satisfies $\ell(\pi) \leq \ell(\chi\pi)$ for every character χ of $F^\times$. The following are equivalent :*

(1) The representation π is cuspidal.

(2) Either

(a) $\ell(\pi) = 0$ and π contains a representation of $U_{\mathfrak{M}} \cong GL_2(\mathfrak{o})$ inflated from an irreducible cuspidal representation of $GL_2(\mathbf{k})$ or

(b) $\ell(\pi) > 0$ and π contains a simple stratum.

Proof. First suppose that $\ell(\pi) = 0$, then the result follows from 11.5 Theorem and 14.3 Proposition.

Now assume $\ell(\pi) > 0$. If π does not contain a simple stratum, then it must contain a split fundamental stratum(13.3 Corollary). By 14.1 Proposition, π is not cuspidal which is a contradiction. Therefore we have shown $(1) \Rightarrow (2)$.

Conversely, assume π is not cuspidal. We identity π with a G-subspace of a representation $\Sigma = Ind_B^G \chi$ for some character $\chi = \chi_1 \otimes \chi_2$ of T. Suppose first that Σ is irreducible. In particular, $\pi = \Sigma$. If some χ_i has level ≥ 1, 14.2 Proposition says that Σ contains either a split or an essentially scalar fundamental stratum. The second possibility is excluded by the hypothesis and 13.3 Theorem, and the first possibility is excluded by 13.3 Corollary and π contains a simple stratum. So both χ_i have level zero. Then $\Sigma = \pi$ contains the trivial character of $U_{\mathfrak{I}}^1$. This implies $\ell(\pi) = 0$, contrary to hypothesis.

We therefore assume Σ is reducible. Thus π is either $\phi \circ det$ or $\phi \circ det \otimes St_G$, for some character ϕ of $F^\times$. If $\pi = \phi \circ det$ such that ϕ has level $l > 0$, define $\chi = \phi^{-1}$, then $\chi\phi = 1$ which implies $\ell(\chi\pi) = 0$, so $\ell(\pi) > \ell(\chi\pi)$ which contradicts the hypothesis. So ϕ has level 0, then π contains the trivial character of $U_{\mathfrak{I}}^1$ which means $\ell(\pi) = 0$. This is a contradiction.

If $(\pi, V) = St_G$, then $\dim V^I = 1$ and $\ell(\pi) = 0$ which is a contradiction.

Proof. Set $(\Sigma, X) = Ind_B^G 1_T$, then (7.3.3) implies $\dim X^I = 2$. By the standard exact sequence of steinberg representation , we have $\dim V^I = 1$. □

If $(\pi, V) = \phi \circ det \otimes St_G$, by the same analysis as above, we can obtain contradiction since St_G does not affect the level of π. □

15 Classification of Cuspidal Representations

15.3 Let $(\mathfrak{A}, n, \alpha)$ be a simple stratum in A with $n \geq 1$ and $E = F[\alpha]$ as in 15.1. We set

$$J_\alpha = E^\times U_{\mathfrak{A}}^{[(n+1)/2]}$$

Thus $J_\alpha \subset \mathcal{K}_{\mathfrak{A}}$ is open in G. It contains and is compact modulo $Z \cong F^\times$.

Theorem. *With the preceding notation, let Λ be an irreducible representation of J_α which contains the character ψ_α of $U_{\mathfrak{A}}^{[n/2]+1}$. Then :*

(1) The restriction of Λ to $U^{[n/2]+1}$ is a multiple of ψ_α.

(2) *The representation*

$$\pi_\Lambda := c\text{-}Ind_{J_\alpha}^G \Lambda$$

is irreducible and cuspidal.

Proof. (1). Denote V by the representation space of Λ. We claim that $\emptyset \neq V^{\psi_\alpha}$ is a J_α-subspace of V hence $V = V^{\psi_\alpha}$. Take $v \in V^{\psi_\alpha}, g \in J_\alpha$, then

$$\begin{aligned}\Lambda(h)\Lambda(g)v &= \Lambda(g)\Lambda(g^{-1}hg)v \\ &= \Lambda(g)\psi_\alpha(g^{-1}hg)v \\ &= \Lambda(g)\psi_\alpha(h)v \\ &= \psi_\alpha(h)\Lambda(g)v\end{aligned}$$

for all $h \in U_{\mathfrak{A}}^{[n/2]+1}$. The second equality is because that J_α normalizes $U_{\mathfrak{A}}^{[n/2]+1}$. The third equality is by 15.1 Theorem which says that J_α normalizes the character ψ_α.

(2). If $g \in G$ intertwines Λ, then it must intertwine ψ_α by (1). Thus $g \in J_\alpha$ by 15.1 Theorem. So the result follows by 11.4 Theorem. □

It will be convenient to have a special notation for this class of representations.

Definition. *Let $(\mathfrak{A}, n, \alpha), n \geq 1$ be a simple stratum. Let $C(\psi_\alpha, \mathfrak{A})$ denote the set of equivalence classes of irreducible representations Λ of the group $J_\alpha = F[\alpha]^\times U_{\mathfrak{A}}^{[(n+1)/2]}$ such that $\Lambda \mid U_{\mathfrak{A}}^{[n/2]+1}$ is a multiple of ψ_α.*

15.4 We have a strong uniqueness property:

Theorem. *For $i = 1, 2$, let $(\mathfrak{A}_i, n_i, \alpha_i)$ be a simple stratum in A, $n_i \geq 1$, and let $\Lambda_i \in C(\psi_{\alpha_i}, \mathfrak{A}_i)$. Suppose that the representations*

$$\pi_{\Lambda_i} = c\text{-}Ind_{J_{\alpha_i}}^G \Lambda_i, \;\; i = 1, 2$$

are equivalent. Then $n_1 = n_2$ and there exist $g \in G$ such that

$$\mathfrak{A}_2 = g^{-1}\mathfrak{A}_1 g, \quad J_{\alpha_2} = g^{-1}J_{\alpha_1}g, \quad \Lambda_2 = \Lambda_1^g$$

If $\mathfrak{A}_1 = \mathfrak{A}_2$, we can choose $g \in U_{\mathfrak{A}_1}$.

Proof. We identity $\pi_{\Lambda_1} = \pi_{\Lambda_2} = \pi$. The representation π contains each simple stratum$(\mathfrak{A}_i, n_i, \alpha_i)$, so the two strata are either both ramified or both unramified (13.3 Corollary). Both $\mathfrak{A}_i$ are conjugate to $\mathfrak{J}$ in the first case, to $\mathfrak{M}$ in the second case. In other words, we can assume $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}$(we will explain this reduce in the following remark). By 12.9 Theorem, $n_i/e_{\mathfrak{A}} = \ell(\pi)$, so $n_1 = n_2 =$

n. The character ψ_{α_i} of $U_{\mathfrak{A}}^{[n/2]+1}$ intertwine in G(11.1 Proposition 1), so are $U_{\mathfrak{A}}$-conjugate(15.2 Theorem). Assume $\psi_{\alpha_2} = \psi_{\alpha_1}^g, g \in U_{\mathfrak{A}}$. Therefore the G-normalizers J_{α_i} of ψ_{α_i} are conjugate under g, namely $J_{\alpha_2} = g^{-1} J_{\alpha_1} g$.

Finally, we need to prove $\Lambda_2 \cong \Lambda_1^g$. As a representation of J_{α_2}, the restriction of Λ_1^g to $U_{\mathfrak{A}}^{[n/2]+1}$ is a multiple of ψ_{α_2}, π contains Λ_2 and Λ_1^g since $c\text{-}Ind_{J_{\alpha_1}}^G \Lambda_1 \cong c\text{-}Ind_{g_1^{-1} J_{\alpha_1} g_1}^G \Lambda_1^{g_1}$. Thus there $h \in G$ intertwines Λ_1^g with Λ_2, and h also intertwines ψ_{α_2} so lies in J_{α_2} by 15.1 Theorem. Therefore $\mathrm{Hom}_{J_{\alpha_2}}(\Lambda_1^g, \Lambda_2) \neq 0$ which means $\Lambda_1^g \cong \Lambda_2$. □

Remark: Why can we assume $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}$ in the proof?

Proof. if $\mathfrak{A}_2 \neq \mathfrak{A}_1$. We can find $g_1 \in G$ such that $\mathfrak{A}_2 = g_1^{-1} \mathfrak{A}_1 g_1$ and $(g_1^{-1} \mathfrak{A}_1 g_1, n_1, g_1^{-1} \alpha_1 g_1)$ is also simple. And $c\text{-}Ind_{J_{\alpha_1}}^G \Lambda_1 \cong c\text{-}Ind_{J_{\alpha_2}}^G \Lambda_2$ implies

$$c\text{-}Ind_{J_{\alpha_2}}^G \Lambda_2 \cong c\text{-}Ind_{g_1^{-1} J_{\alpha_1} g_1}^G \Lambda_1^{g_1}$$

since $c\text{-}Ind_{J_{\alpha_1}}^G \Lambda_1 \cong c\text{-}Ind_{g_1^{-1} J_{\alpha_1} g_1}^G \Lambda_1^{g_1}$, the isomorphism map is $f \mapsto (g \mapsto f(g_1 g))$. By above proof, there $g_2 \in G$ such that

$$\mathfrak{A}_2 = g_2^{-1}(g_1^{-1} \mathfrak{A}_1 g_1) g_2, \quad J_{\alpha_2} = g_2^{-1} J_{g_1^{-1} \alpha_1 g_1} g_2 \quad \Lambda_2 = (\Lambda_1^{g_1})^{g_2}$$

But $g_2^{-1} J_{g_1^{-1} \alpha_1 g_1} g_2 = g_2^{-1} g_1^{-1} J_{\alpha_1} g_1 g_2$ and $(\Lambda_1^{g_1})^{g_2} = \Lambda_1^{g_1 g_2}$. Take $g = g_1 g_2$, then general case holds. We are done. □

15.5 It will be convenient to introduce a new term:

Definition. *A cuspidal type in G is a triple $(\mathfrak{A}, J, \Lambda)$, where $\mathfrak{A}$ is a chain order in A, J is a subgroup of $\mathcal{K}_{\mathfrak{A}}$ and Λ is an irreducible smooth representation of J, satisfying one of the following kinds:*

(1) $\mathfrak{A} \cong \mathfrak{M}$, $J = ZU_{\mathfrak{A}}$, and $\Lambda \mid U_{\mathfrak{A}}$ is the inflation of an irreducible cuspidal representation of the group $U_{\mathfrak{A}}/U_{\mathfrak{A}}^1 \cong GL_2(\mathbf{k})$;

(2) there is a simple stratum $(\mathfrak{A}, n, \alpha)$, $n \geq 1$, such that $J = J_\alpha$ and $\Lambda \in C(\psi_\alpha, \mathfrak{A})$;

(3) there is a triple $(\mathfrak{A}, J, \Lambda_0)$ satisfying (1) or (2), and a character χ of $F^\times$, such that $\Lambda \cong \Lambda_0 \otimes \chi \circ det$.

We come to the main result.

Theorem. *Let π be an irreducible cuspidal representation of $G = GL_2(F)$. There exists a cuspidal type $(\mathfrak{A}, J, \Lambda)$ in G such that $\pi \cong c\text{-}Ind_J^G \Lambda$. The representation π determines $(\mathfrak{A}, J, \Lambda)$ uniquely, up to G-conjugacy.*

Proof. If $\ell(\pi) = 0$, the result is given by 14.5 Theorem and 11.5 Theorem. If π contains another cuspidal type $(\mathfrak{A}_1, J_1, \Lambda_1)$, it must be the first kind of the definition since π donot contains any fundamental strata. Hence $c\text{-}Ind_J^G \Lambda = c\text{-}Ind_{J_1}^G \Lambda_1$, the conjugacy is identity by 11.5 Lemma.

If $\ell(\pi) > 0$. In case (3) of the definition above, we have

$$c\text{-}Ind_J^G(\Lambda_0 \otimes \chi \circ det) \cong \chi c\text{-}Ind_J^G \Lambda_0$$

So it is enough to prove the case where π satisfies $\ell(\pi) \leq \ell(\chi\pi)$ for all character χ of $F^\times$. (Because if it is not, we can decrease the level of π by twisting a character. This process will end after finite steps since $\ell(\pi)$ is finite). By 14.5 Theorem, there is a simple stratum $(\mathfrak{A}, n, \alpha), n \geq 1$ such that π contains the character ψ_α of $U_{\mathfrak{A}}^n$.

$$\begin{tikzcd} U_{\mathfrak{A}}^{[n/2]+1} \arrow[r, "\pi"] \arrow[d] & GL(V^{\psi_\alpha}) \\ U_{\mathfrak{A}}^{[n/2]+1}/U_{\mathfrak{A}}^n \arrow[ur, "\bar{\pi}"'] & \end{tikzcd}$$

Hence $\bar{\pi}$ contains a character ξ of $U_{\mathfrak{A}}^{[n/2]+1}$ such that $\xi \mid U_{\mathfrak{A}}^n = \psi_\alpha$. This means that $\xi = \psi_\beta$ for some $\beta \in \alpha + \mathfrak{P}^{1-n}$. By $13.4, 13.5$ Proposition, $(\mathfrak{A}, n, \beta)$ is also simple. 2.7 Proposition says that $\pi \mid J_\beta$ is semisimple, thus π contains some irreducible smooth representation Λ of J_β such that $\Lambda \mid U_{\mathfrak{A}}^{[n/2]+1}$ contains ψ_β. As before, this restriction is a multiple of ψ_β, so the triple $(\mathfrak{A}, J_\beta, \Lambda)$ is a cuspidal type occurring in π. Since the representation $\pi_\Lambda = c\text{-}Ind_{J_\beta}^G \Lambda$ is irreducible, so $\pi \cong \pi_\Lambda$.

If $(\mathfrak{A}', J', \Lambda')$ is another cuspidal type occurring in π, then $\ell(\pi) > 0$ and 13.3 Corollary imply Λ' is of the second kind of in the definition. Then the uniqueness statement follows by 15.4 Theorem. $\square$

Consequently:

Corollary. ***(Classification Theorem)****. The map*

$$(\mathfrak{A}, J, \Lambda) \longmapsto \pi_\Lambda = c\text{-}Ind_J^G \Lambda$$

induces a bijection between the set of conjugacy class of cuspidal types in G and the set of equivalence classes of irreducible cuspidal representations of G.

Remark. Thus an irreducible cuspidal representation π of G contains a cuspidal type $(\mathfrak{A}, J, \Lambda)$. If $\ell(\pi) = 0$, the type is of the first kind in 15.5 Definition. If $0 < \ell(\pi) \leq \ell(\chi\pi)$ for all character χ of $F^\times$, the type is of the second kind. Otherwise, it is of the third kind.

15.6 Corollary 15.5 reduces the study of cuspidal representations of G to that of cuspidal types in G. Therefore we need to investigate the structure of cuspidal types. Only the second definition need to be argued. Therefore we take a stratum $(\mathfrak{A}, n, \alpha), n \geq 1, E = F[\alpha]$, hope to describe the representation $\Lambda \in C(\psi_\alpha, \mathfrak{A})$.

We first state a result of these unit groups.

Lemma. *Let $(\mathfrak{A}, n, \alpha)$ be a simple strarum, take $E = F[\alpha]$, then we can embedd $\mathfrak{o}_E$ into $\mathfrak{A}$. Hence*

$$E^\times \cap U_{\mathfrak{A}}^m = U_E^m$$

for all integer $m \geq 1$.

Proof. In the unramified case, $\mathfrak{o}_E = \mathfrak{o}[\alpha_0]$ where $\alpha_0 = \varpi^n\alpha$(notation of 12.4).Hence any $a+b\alpha_0 \in \mathfrak{o}_E$ is equal to $a + b\varpi^n\alpha$, and

$$\begin{pmatrix} a & b\varpi^n f \\ b\varpi^n & a \end{pmatrix} \in \mathfrak{M}$$

In the ramified case, $\mathfrak{o}_E = \mathfrak{o}[\alpha_0]$ where $\alpha_0 = \varpi^{[(n+1)/2]}\alpha$. Hence any $a + b\alpha_0 \in \mathfrak{o}_E$ is equal to $a + b\varpi^{[(n+1)/2]}\alpha$, and

$$\begin{pmatrix} a & b\varpi^{(n+1)/2} f \\ b\varpi^{(n+1)/2} & a \end{pmatrix} \in \mathfrak{J}$$

For the final assertion

$$E^\times \cap U_{\mathfrak{A}}^m = \{x \in E^\times \mid x - 1 \in \mathfrak{P}_{\mathfrak{A}}^m\} \subset U_E^m$$

is trivial by calculating valuation of E for both sides. Converse inclusion is because $\mathfrak{p}_E = \mathfrak{P}_{\mathfrak{A}} \cap \mathfrak{o}_E$. $\square$

We need some intermediate groups :

$$H_\alpha^1 = U_E^1 U_{\mathfrak{A}}^{[n/2]+1}, \quad J_\alpha^1 = J_\alpha \cap U_{\mathfrak{A}}^1 = U_E^1 U_{\mathfrak{A}}^{[(n+1)/2]}.$$

Observe that $J_\alpha^1 = H_\alpha^1$ if and only if n is odd.

Proposition. *Suppose that n is odd.*

(1) Every $\Lambda \in C(\psi_\alpha, \mathfrak{A})$ has dimension 1, and

(2) Two characters $\Lambda_1, \Lambda_2 \in C(\psi_\alpha, \mathfrak{A})$ intertwine in G if and only if $\Lambda_1 = \Lambda_2$.

Proof. (1) Let V be the representation space of Λ. Then $\Lambda \mid E^\times$ is an irreducible representation of V since $U_{\mathfrak{A}}^{[n/2]+1} = U_{\mathfrak{A}}^{[(n+1)/2]}$. Hence $\dim V = 1$ since $E^\times$ is abelian.

(2) If $g \in G$ intertwines Λ_1 with Λ_2, then it must intertwine ψ_α itself. Thus $g \in J_\alpha$ by 15.1 Theorem. Thus $\Lambda_1 = \Lambda_2$.

□

5 Parametrization of Tame Cuspidal Representations

19 Construction of Cuspidal Representation

In this section, we associate to an admissible pair $(E/F, \chi)$ an irreducible cuspidal representation π_χ of $G = GL_2(F)$.

19.1 We start with the special case of an admissible pair $(E/F, \chi)$ in which χ has level 0. Thus by definition, E/F is unramified.

Lemma. *Let E/F be an unramified quadratic extension, let χ be a character of $E^\times$ of level zero, and let $\sigma \in Gal(E/F), \sigma \neq 1$. The following are equivalent:*

(1) Then pair $(E/F, \chi)$ is admissible;

(2) $\chi \neq \chi^\sigma$;

(3) $\chi \mid U_E \neq \chi^\sigma \mid U_E$.

Proof. Since E/F is unramified, we have $E^\times = F^\times U_E$, so $(2) \Leftrightarrow (3)$. Since E/F is cyclic, Hilbert 90 implies $ker(N_{E/F}) = \{\sigma(x)/x \mid x \in E^\times\}$. Thus χ factors through $N_{E/F}$ if and only if $\chi = \chi^\sigma$. The second condition in the definition of admissible pair is empty in this case, so $(1) \Leftrightarrow (2)$. □

Now return to the admissible pair $(E/F, \chi)$ of level 0, we write $\mathbf{k}_E = \mathfrak{o}_E/\mathfrak{p}_E$: thus $\mathbf{k}_E/\mathbf{k}$ is a quadratic field extension. We choose an F-embedding $E \to A$, and let $\mathfrak{A}$ be the unique chain order with $E^\times \subset \mathcal{K}_{\mathfrak{A}}$(12.4 Proposition). Conjugating by an element of G, we can take $\mathfrak{A} = \mathfrak{M} = M_2(\mathfrak{o})$.

The character $\chi \mid U_E$ is the inflation of a character $\tilde{\chi}$ of $\mathbf{k}_E^\times$ since $U_E/U_E^1 \cong \mathbf{k}_E^\times$. Condition (3) in the lemma is equivalent to $\tilde{\chi}$ being a regular character of $\mathbf{k}_E^\times$(ref 6.4). As in 6.4, $\tilde{\chi}$ give us an irreducible cuspidal representation $\tilde{\lambda} := \pi_{\tilde{\chi}}$ of $GL_2(\mathbf{k})$. Let λ be the inflation of $\tilde{\lambda}$ to a representation of $U_{\mathfrak{M}} = GL_2(\mathfrak{o})$.

$$\begin{tikzcd} GL_2(\mathfrak{o}) \arrow[r, "\lambda"] \arrow[d] & GL(V) \\ GL_2(\mathbf{k}) \arrow[ur, "\tilde{\lambda}"'] & \end{tikzcd}$$

We claim that $V = V^{\chi|U_F}$.

Proof. 6.4 Theorem implies

$$tr\tilde{\lambda}(g) = (q-1)\tilde{\chi}(g), \quad \forall g \in Z_{\mathbf{k}}$$

where $Z_{\mathbf{k}}$ is the center of $GL_2(\mathbf{k})$. By inflating, this is just

$$tr\lambda(g) = (q-1)\chi(g), \quad \forall g \in U_F$$

Assume $\lambda(g)v = \theta(g)v$ for some character θ of U_F, then $(q-1)\theta(g) = (q-1)\chi(g)$ since $\dim V = q-1$, namely $\lambda(g)v = \chi(g)v$ for all $g \in U_F$. $\square$

We therefore extend λ to an irreducible smooth representation Λ of $\mathcal{K}_{\mathfrak{M}} = F^\times U_{\mathfrak{M}} = ZK$(11.4 notation) by deeming that $\Lambda \mid F^\times$ be the direct sum of $q-1$ copies of χ. This is well-defined since on $U_F = F^\times \cap U_{\mathfrak{M}}$, $\lambda = \chi$.

The triple $(\mathfrak{M}, \mathcal{K}_{\mathfrak{M}}, \Lambda)$ is then a cuspidal type. We set

$$\pi_\chi = c\text{-}Ind_{K_{\mathfrak{M}}}^G \Lambda.$$

Thus π_χ is an irreducible cuspidal representation of G such that $\ell(\pi_\chi) = 0$.

Remark: if $(E_1/F, \chi_1) \cong (E_2/F, \chi_2)$,namely there F-isomorphism $j : E_2 \to E_1$such that $\chi_1 = j \circ \chi_2$ then how to prove $\pi_{\chi_1} \cong \pi_{\chi_2}$?.

Write $\mathbb{P}_2(F)_0$ for the set of isomorphism classes of admissible pairs $(E/F, \chi)$ in which χ has level 0. Likewise, let $\mathcal{A}_2^0(F)_0$ denote the set of equivalent classes of irreducible cuspidal representations π of G such that $\ell(\pi) = 0$. The remark gives us following result.

Proposition. *The map $(E/F, \chi) \mapsto \pi_\chi$ induces a bijection*

$$\mathbb{P}_2(F)_0 \xrightarrow{\cong} \mathcal{A}_2^0(F)_0 \tag{19.1.1}$$

Further, if $(E/F, \chi) \in \mathbb{P}_2(F)_0$, then :

(1) If ϕ is a character of $F^\times$ of level 0, then $\pi_{\chi\phi_E} = \phi\pi_\chi$;

(2) If $\pi = \pi_\chi$, then $\omega_\pi = \chi \mid F^\times$;

(3) Then pair $(E/F, \check{\chi})$ is admissible and $\check{\pi_\chi} = \pi_{\check{\chi}}$.

Proof. Given an irreducible cuspidal representation π of G with $\ell(\pi) = 0$, by14.5 Theorem and 11.5 Theorem $\pi \cong c\text{-}Ind_{ZK}^G \Lambda$ where Λ is an irreducible smooth representation of ZK such that $\Lambda \mid K \cong \lambda$. Here λ is a representation of K inflated from an irreducible cuspidal representation $\tilde{\lambda}$ of $GL_2(\mathbf{k})$. $\tilde{\lambda}$ corresponds a regular character $\tilde{\chi}$ of $\mathbf{k}_E^\times$ for an umramified quadratic extension E of F. $\tilde{\chi}$ can be inflated to a level 0 character χ of $E^\times$. By above lemma, the map (19.1.1) is surjective.

To prove injectivity, suppose we have pairs $(E_i/F, \chi_i)$ such that the representations π_{χ_i} are equivalent. The extension E_i are unramified and so F-isomorphic: we can take $E_1 = E_2$. 11.5 Lemma implies the cuspidal representations $\pi_{\tilde{\chi}_i}$ of $GL_2(\mathbf{k})$ are equivalent. So the character $\tilde{\chi_i}$ of $\mathbf{k}_E^\times$ are Galois-conjugate which means $\chi_i \mid U_E$ are Galois-conjugate, and then the pairs $(E/F, \chi_i)$ are F-isomorphic. □

19.2 We now fix a character $\psi \in \widehat{F}$ of level 1. Let $(E/F, \chi)$ be a minimal admissible pair such that χ has level $n \geq 1$. We set $\psi_E = \psi \circ Tr_{E/F}$, $\psi_A = \psi \circ tr_A$.

Next, we choose an element $\alpha \in \mathfrak{p}_E^{-n}$ such that $\chi(1+x) = \psi_E(\alpha x), x \in \mathfrak{p}_E^{[n/2]+1}$(By 1.8 Proposition). Then We choose an F-embedding of E in $A = M_2(F)$ and let $\mathfrak{A}$ be the unique chain order in A such that $E^\times \subset \mathcal{K}_\mathfrak{A}$.(12.4) Then $e_\mathfrak{A} = e(E/F)$ and the triple $(\mathfrak{A}, n, \alpha)$ is a simple stratum.

Attached to the simple stratum $(\mathfrak{A}, n, \alpha)$, we have the subgroups $J_\alpha, J_\alpha^1, H_\alpha^1$ as in \$15. The next step is to define an irreducible representation $\Lambda \in C(\psi_\alpha, \mathfrak{A})$.(notation of 15.5)

19.3 Suppose in this paragraph that $n = 2m+1$ is odd. The desired representation Λ is the character of $J_\alpha = E^\times U_\mathfrak{A}^{m+1}$ given by

$$\Lambda \mid U_\mathfrak{A}^{m+1} = \psi_\alpha, \ \ \Lambda \mid E^\times = \chi$$

Notice that $E^\times \cap U_\mathfrak{A}^{m+1} = U_E^{m+1}$ and $\psi_\alpha \mid U_E^{m+1} = \chi \mid U_E^{m+1}$ since $tr_A \mid E = Tr_{E/F}$, so these two conditions are consistent. Then the triple $(\mathfrak{A}, J_\alpha, \Lambda)$ is a cuspidal type in G, so

$$\pi_\chi = c\text{-}Ind_{J_\alpha}^G \Lambda$$

is an irreducible cuspidal representation of G containing the fundamental stratum $(\mathfrak{A}, n, \alpha)$. Thus

$$\ell(\pi_\chi) = n/e(E/F), \quad \omega_{\pi_\chi} = \chi \mid F^\times$$

19.4 In this paragraph, we assume that $(E/F, \chi)$ is a minimal pair in which χ has even level $n = 2m > 0$. Then it is an unramified simple stratum which means E/F is unramified. We define a character θ of $H_\alpha^1 = U_E^1 U_\mathfrak{A}^{m+1}$by

$$\theta(ux) = \chi(u)\psi_\alpha(x), \quad x \in U_\mathfrak{A}^{m+1}, \ u \in U_E^1.$$

As before, this is well-defined. We let $\eta = \eta_\theta$ be the unique irreducible representation of $J_\alpha^1 = U_E^1 U_{\mathfrak{A}}^m$ which contains θ(15.6 Lemma).

Proposition. *There is a unique irreducible representation $\tilde{\eta}$ of $\mu_E/\mu_F \ltimes J_\alpha^1$ such that $\tilde{\eta} \mid J_\alpha^1 \cong \eta$ and*

$$tr\tilde{\eta}(\zeta u) = -\theta(u)$$

for $u \in H_\alpha^1$ and every $\zeta \in \mu_E/\mu_F,\ \zeta \neq 1$.

We prove this later, in \$22. We need the following consequence:

Corollary. *There is a unique irreducible representation Λ of J_α such that*

(1) $\Lambda \mid J_\alpha^1 \cong \eta$;

(2) $\Lambda \mid F^\times$ is a multiple of $\chi \mid F^\times$;

(3) for every $\zeta \in \mu_E/\mu_F$, we have $tr\Lambda(\zeta) = -\chi(\zeta)$.

Proof. These three condition determine Λ uniquely, we need to prove that it exists.

□

The representation Λ of the corollary lies in $C(\psi_\alpha, \mathfrak{A})$. We define :

$$\pi_\chi = c\text{-}Ind_{J_\alpha}^G \Lambda$$

Thus π is an irreducible cuspidal representation of G satisfying

$$\ell(\pi_\chi) = n,\ \ \omega_{\pi_\chi} = \chi \mid F^\times.$$

19.5 We have to check that the construction of π_χ is independent of choices:

Proposition. *Let $(E/F, \chi)$ be a minimal pair in which χ has positive level. The representation π_χ depends, up to equivalence, only on the isomorphism class of the pair $(E/F, \chi)$. In particular, it is independent of the choices of ψ, α and of the embedding $E \to A$.*

Moreover, if ϕ is a character of $F^\times$ such that $(E/F, \chi\phi_E)$ is also minimal, then $\pi_{\chi\phi_E} = \phi\pi_\chi$.

Proof. □

19.6 Let $(E/F,\chi)$ be an admissible pair. As in 18.2, there is a character ϕ of $F^\times$ and a character $\chi^{'}$ of $E^\times$ such that $(E/F,\chi^{'})$ is minimal and $\chi=\chi^{'}\phi_E$. We define

$$\pi_\chi=\phi\pi_{\chi'}$$

This is independent of the choice of decomposition $\chi=\chi^{'}\phi_E$, by the final assertion of 19.5 proposition. And we have

$$\ell(\pi_\chi)=n/e(E/F),\quad \omega_{\pi_\chi}=\chi\mid F^\times.$$

where n is the level of χ.

Proof. If $\chi^{'}$ has level $m<n$, then $\pi_{\chi'}$ contains the simple stratum $(\mathfrak{A},m,\alpha)$. We can prove that π_χ contains a stratum $(\mathfrak{A},n,\beta)$ which menas $\ell(\pi_\chi)\leq\ell(\pi_{\chi'})$. It must equal by 13.3 Theorem. $\square$

Writing $\mathbb{P}_2(F)$ for the set of isomorphism classes of admissible pair $(E/F,\chi)$ and $\mathcal{A}_2^0(F)$ for the set of equivalence classes of irreducible cuspidal representations of $G=GL_2(F)$, we have a map

$$\begin{aligned}\mathbb{P}_2(F)&\to\mathcal{A}_2^0(F)\\ (E/F,\chi)&\mapsto\pi_\chi\end{aligned}\tag{19.6.1}$$

defined independently of all choices.

20 The Parametrization Theorem

20.1 Let π be an irreducible cuspidal representation of $G=GL_2(F)$. We say that π is unramified if there exists an unramified character $\phi\neq 1$(unramified means $\phi\mid U_F=1$) of $F^\times$ such that $\phi\pi\cong\pi$.

We denote by $\mathcal{A}_2^{nr}(F)$ the set of unramified classes in $\mathcal{A}_2^0(F)$. A representation $\pi\in\mathcal{A}_2^0(F)\backslash\mathcal{A}_2^{nr}(F)$ will be a called totally ramified.

20.2 We come to the main result of this section:

Tame Parametrization Theorem. The map $(E/F,\chi)\mapsto\pi_\chi$ of $(19.6.1)$ induces a bijection

$$\mathbb{P}_2(F)\xrightarrow{\cong}\mathcal{A}_2^0(F)\quad\text{if}\ \ p\neq 2\quad\text{or}$$
$$\mathbb{P}_2(F)\xrightarrow{\cong}\mathcal{A}_2^0(F)\quad\text{if}\ \ p=2$$

If $(E/F,\chi)\in\mathbb{P}_2(F)$, then :

(1) if χ has level $\ell(\chi)$, then $\ell(\pi_\chi)=\ell(\chi)/e(E/F)$;

(2) $\omega_{\pi_\chi} = \chi \mid F^\times$;

(3) The pair $(E/F, \check{\chi})$ is admissible and $\pi_{\check{\chi}} = \check{\pi_\chi}$'

(4) If ϕ is a character of $F^\times$, then $\pi_{\chi\phi_E} = \phi\pi_\chi$.

20.3 Let $\pi \in \mathcal{A}_2^0(F)$.

Lemma. *Let π be an irreducible cuspidal representation of G containing a cuspidal inducing datum $(\mathfrak{A}, \Xi)$. Then representation π is unramified if and only if $\mathfrak{A} \cong \mathfrak{M}$.*

Proof. Let ϕ be an unramified character of $F^\times$ of order 2. If $\mathfrak{A} \cong \mathfrak{M}$, then $det(\mathcal{K}_{\mathfrak{A}}) = det(F^\times U_{\mathfrak{M}}) = (F^\times)^2 U_F \subset ker(\phi)$. Hence $\Xi \otimes \phi \circ det \cong \Xi$, and

$$\phi\pi = \phi \circ det \otimes c\text{-}Ind_{\mathcal{K}_{\mathfrak{A}}}^G(\Xi) = c\text{-}Ind_{\mathcal{K}_{\mathfrak{A}}}^G(\phi \circ det \otimes \Xi) = c\text{-}Ind_{\mathcal{K}_{\mathfrak{A}}}^G(\Xi) = \pi.$$

Conversely, suppose that $\mathfrak{A} \cong \mathfrak{J}$. In this case, $\pi = c\text{-}Ind_J^G \Lambda$ for some cuspidal type $(\mathfrak{J}, J, \Lambda)$ where Λ is a character by 15.6 Proposiotion 1. We know $\phi \circ det$ is not trivial on J. Because if π contains $(\mathfrak{J}, 2m+1, \alpha)$, then $det(\alpha) \in \mathfrak{p}^{-2m+1}$. Thus $\phi\pi \cong \pi$ implies that the charatcers Λ and $\Lambda \otimes \phi \circ det$ intertwine in G , contrary to 15.6 proposition 1. □

Proposition. *Suppose $p \neq 2$, and let $\pi \in \mathcal{A}_2^0(F)$ be totally ramified. Then we have*

(1) There exists a unique character ϕ of $F^\times$, $\phi \neq 1$, such that $\phi\pi \cong \pi$. The character ϕ is ramified, of level 0, and of order 2.

(2) Let $(\mathfrak{A}, n, \alpha)$ be a simple stratum, with $n \geq 1$, and suppose that $\pi = \theta\pi_0$, for a character θ of $F^\times$ and a representation π_0 containing the character $\psi_\alpha \mid U_{\mathfrak{A}}^{[n/2]+1}$. The field $E = F[\alpha]$ satisfies $N_{E/F}(E^\times) = ker\phi$.

Proof. Noticing that nothing changes if we replace π by a twist, so we can assume $\ell(\pi) \leq \ell(\xi\pi)$ for all character ξ of $F^\times$. Then the lemma implies that there is a ramified simple stratum $(\mathfrak{A}, n, \alpha)$ such that π contains the character ψ_α of $U_{\mathfrak{A}}^{[n/2]+1}$. $n = 2m+1$ is odd; Putting $E = F[\alpha]$, we have $J_\alpha = E^\times U_{\mathfrak{A}}^{m+1}$, and the representation π contains a cuspidal type $(\mathfrak{A}, J_\alpha, \Lambda)$.

Since E/F is totally tamely ramified, $det(E^\times) = N_{E/F}(E^\times) \supset U_F^1$. On the other hand, $\det U_{\mathfrak{A}}^{m+1} \subset \det U_{\mathfrak{A}}^1 = U_F^1$. Thus

$$\det J_\alpha = N_{E/F}(E^\times)$$

which is a subgroup of $F^\times$ of index 2 by local class field theory. Let ϕ be the nontrivial character of $F^\times$ such that $\phi \mid N_{E/F}(E^\times) = 1$ and equals -1 on the other coset representative, then $\Lambda \otimes \phi \circ \det = \Lambda$, hence $\pi \cong \phi\pi$. Clearly ϕ is ramified(otherwise, norm map can extend to $F^\times$ which means ϕ is trivial), of level 0 and of order 2.

To prove the uniqueness, let ξ be a character of $F^\times$ such that $\xi\pi \cong \pi$. $\xi^2 = 1$ and p is odd imply that there is no non-trivial homomoprhism $\xi \mid U_F^1 : U_F^1 \to \{1, -1\}$ since U_F^1 is a pro-p group, hence $\xi \mid U_F^1$ is trivial and ξ has level 0. π contains the two character $\Lambda, \Lambda \otimes \xi \circ \det$ of J_α, hence they intertwine in G which means $\Lambda = \Lambda \otimes \xi \circ \det$(15.6 Proposition 1). Thus ξ vanishes on $N_{E/F}(E^\times)$, and therefore ξ is either trivial or equal to ϕ. $\square$

参考文献