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CLASSIFICATION OF TYPICAL OF BERNSTEIN COMPONENT FOR $GL_2(F)$

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1 Introduction

This document is a personal study note based on the appendix of the paper by Breuil and Mézard ([BM]). To illustrate it, we fix some notations.

Notation: F is a non-Archimedean local field, $\mathfrak{o}_F$ is its valuation ring, $\mathfrak{p}_F$ is the maximal ideal of $\mathfrak{o}_F$. Set $G = GL_n(F)$, $K = GL_n(\mathfrak{o}_F)$, and let $\mathfrak{R}(G)$ be the category of smooth complex representation of G, $Irr(G)$ be the category of irreducible smooth complex representation of (G). For other groups, we define similarly.

Recall that J. Bernstein (see [Del]) gave a decomposition of the category of smooth representations of G into a product of indecomposable subcategories. Namely

Theorem 1.0.1. The category $\mathfrak{R}(G)$ decomposes as a direct product

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

To comprehend this theorem, we must clarify the notation $\mathfrak{R}^{\mathfrak{s}}(G)$ and $\mathfrak{B}(G)$, as established by the following theorem.

Theorem 1.0.2. (Jacquet) Let π be an irreducible smooth representation of G, then there exists a Levi subgroup L and an irreducible supercuspidal representation σ of L such that π is a subrepresentation of $Ind_P^G \sigma$. Here, P can be any parabolic subgroup of G with Levi component L.

Remark 1.0.3. The representation π determines the pair (L, σ) up to G-conjugacy. We refers to (L, σ) as the support of π.

Explanation of Theorem 1.1: The idea is that we imposes an equivalence relation on the set of pairs (L, σ) by deeming two such pair (L_i, σ_i) to be *inertially equivalent* if there an element $g \in G$ and an unramified character χ of L_2(means $\chi \mid_{L_2 \cap K} = 1$) such that $L_2 = g^{-1} L_1 g$ and $\sigma_2 \otimes \chi \cong \sigma_1^g$. One then can define the inertial support $\mathfrak{L}(\pi)$ of an irreducible representation π to be the inertial equivalence class of the support of π. If the inclusion

$$\pi \hookrightarrow Ind_P^G \sigma$$

is obvious, we also denote $\mathfrak{L}(\pi)$ by $[L, \sigma]_G$.

Given an inertial equivalence class $\mathfrak{s}$, one defines a full subcategory $\mathfrak{R}^{\mathfrak{s}}(G)$ of $\mathfrak{R}(G)$ by deeming that the objects of $\mathfrak{R}^{\mathfrak{s}}(G)$ are the smooth representations of G such that all of whose irreducible quotients have inertial support $\mathfrak{s}$. Then let $\mathfrak{s}$ run over the set $\mathfrak{B}(G)$ of all inertial equivalence classes, we have Theorem 1.0.1.

To understand the subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$, Bushnell and Kutzko introduced the theory of types in [BK1].

The idea is to identity $\mathfrak{R}^{\mathfrak{s}}(G)$ as the category of modules over a Hecke algebra in the following way. Find a pair (K_1, ρ) where K_1 is a compact open subgroup of G and $\rho \in$

$Irr(K_1)$ such that for any $\pi \in Irr(G)$

$$\pi \in \mathfrak{R}^{\mathfrak{s}}(G) \Leftrightarrow \mathrm{Hom}_{K_1}(\rho, \pi) \neq 0$$

In this case, set $\mathcal{H}(G, \rho) = End_G(Ind_{K_1}^G \rho)$ and we have an equivalence of categories

$$\mathfrak{R}^{\mathfrak{s}}(G) \longleftrightarrow \mathcal{H}(G, \rho) - Mod$$

This motivates the following definition..

Definition 1.0.4. (1) Given an inertial equivalence class $\mathfrak{s} \in \mathfrak{B}(G)$. A pair (K_1, ρ) where K_1 is a compact open subgroup of G and $\rho \in Irr(K_1)$ is called a **typical** for $\mathfrak{R}^{\mathfrak{s}}(G)$, if for any $\pi \in Irr(G)$,

$$\mathrm{Hom}_{K_1}(\rho, \pi) \neq 0 \Rightarrow \pi \in \mathfrak{R}^{\mathfrak{s}}(G)$$

and there exists at least one $\pi \in \mathfrak{R}^{\mathfrak{s}}(G)$ such that $\mathrm{Hom}_{K_1}(\rho, \pi) \neq 0$. And the pair is call a **type** for $\mathfrak{R}^{\mathfrak{s}}(G)$, if for any $\pi \in Irr(G)$,

$$\pi \in \mathfrak{R}^{\mathfrak{s}}(G) \Leftrightarrow \mathrm{Hom}_{K_1}(\rho, \pi) \neq 0$$

(2) Denote the category of irreducible smooth representation of $GL_2(F)$ by $\mathcal{A}_F(2)$, then a component of $\pi \in \mathcal{A}_F(2)$ is $\mathfrak{R}^s(G)$ where s is the inertial equivalence class $\mathfrak{L}(\pi)$.

Remark 1.0.5. If s is a component for $\pi \in \mathcal{A}_F(2)$ and χ is a character of $F^\times$. Then ρ is a typical (resp. is a type) for s if and only if $(\chi \circ \det) \otimes \rho$ is a typical (resp. is a type) for the component of $(\chi \circ \det) \otimes \pi$. Indeed, then multiplicity of $(\chi \circ \det) \otimes \rho$ in $(\chi \circ \det \otimes \pi' \mid_{K_1})$ is equal to that of ρ in $\pi' \mid_{K_1}$ for any $\pi' \in \mathcal{A}_F(2)$.

From now on, we call $\mathfrak{R}^{\mathfrak{s}}(G)$ the component associated to $\mathfrak{s}$.

We aim to classify the typicals and types within each component for $G = GL_2(F)$.

First we need to find Levi and parabolic subgroups in $GL_2(F)$. Up to conjugacy, there is two Levi subgroups, the diagonal matrix $T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in F^\times$ and the whole group $G = GL_2(F)$. Corresponding we have two parabolic subgroups , the Borel subgroup $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in G$ and G.

Definition 1.0.6. Give $\mathfrak{s} \in \mathfrak{B}(G)$. Then we say

(1) $\mathfrak{R}^{\mathfrak{s}}(G)$ is a supercuspidal component if $\mathfrak{s}$ is the inertial equivalence class $[G, \pi]_G$ for some irreducible supercuspidal representation $\pi \in Irr(G)$. Notice that there are several different supercuspidal components.

(2) $\mathfrak{R}^{\mathfrak{s}}(G)$ is the trivial component if $\mathfrak{s} = [T, 1_T]$.

(3) $\mathfrak{R}^{\mathfrak{s}}(G)$ is a principal component if $\mathfrak{s} = [T, \chi_1 \otimes \chi_2]$ which is not inertially equivalent to $[T, 1_T]$. As before, there are several different principal components.

Proof. (1) follows by the equivalent definitions of supercuspidal. (3) is trivial. For (2), we know $St_G \cong St_G^{\vee}$ and two exact sequences

$$0 \to \phi_G \to Ind_B^G(\phi \cdot 1_T) \to \phi_G \otimes St_G \to 0$$

and

$$0 \to \phi_G \otimes St_G^{\vee} \to Ind_B^G(\phi \cdot \delta_B^{-1}) \to \phi_G \to 0$$

where $\phi_G := \phi \circ \det$. Since (T, ϕ) and $(T, \phi \cdot \delta_B^{-1})$ are inertially equivalent, the result holds. $\square$

The appendix of [BM] features the following main theorem:

Theorem 1.0.7. Notation as Proposition 1.5. We have

(1) If s is a supercuspidal component , then there exists a unique (up to isomorphism) smooth irreducible representation ρ of $K = GL_2(\mathfrak{o})$ which is a type for s, and it occurs with multiplicity 1 in every element of s.

(2) If s is the trivial component (denoted for simplicity by s_0), then up to isomorphism, there are exactly two smooth representations of $K = GL_2(\mathfrak{o})$ which are typical for s_0. Neither of these is a type.

(3) Finally, if s is a principal component, then up to isomorphism, there is exactly one smooth irreducible representation of $K = GL_2(\mathfrak{o})$ which is a typical for s, except when the cardinality of $\mathbf{k}_F$ is 2, in this case, up to isomorphism, there are two smooth irreducible representations of $K = GL_2(\mathfrak{o})$ which are typical for s. Moreover, in all cases, every smooth irreducible representation of K which is a typical for s is a type for s and appears with multiplicity 1 in every element of s.

Organization : In Chapter 2, we will establish results for the principal and trivial components, following the approach in W. Casselman [Ca2].

In Chapter 3, we prove results for supercuspidal component, following by P.C.Kutzko[Ku1].

2 Principal components and the trivial component

2.1 Peliminary

In what follows, we denote by $\mathfrak{o}_F$ the ring of integers of F, by $\mathfrak{p}_F$ its maximal ideal, $\mathbf{k}_F = \mathfrak{o}_F/\mathfrak{p}_F$ its residue field, $q = q_F$ the cardinality of $\mathbf{k}_F$ and $K = GL_2(\mathfrak{o}_F)$. We also denote by $U_F = U_F^0$ the group of units of $\mathfrak{o}_F$, which is filtered by its subgroups $U_F^i = 1 + \mathfrak{p}_F^i$ for an integer $i \geq 1$. We also fix a uniformizer ϖ_F of F and an additive character ψ of F which is trivial on $\mathfrak{p}_F$ but non-trivial on $\mathfrak{o}_F$.

For each character ϵ_0 of U_F, we denote by $s(\epsilon_0)$ the component of $\mathcal{A}_F(2)$ that contains $\pi(\tilde{\epsilon_0}, 1)$ for every character $\tilde{\epsilon_0}$ of $F^\times$ whose restriction to U_F is ϵ_0. If $\epsilon_0 = 1$, then $s(\epsilon_0) = s_0$ is the trivial component. Every principal component of $\mathcal{A}_F(2)$ is obtained, by twisting with a character of $F^\times$, from a component of the form $s(\epsilon_0)$ (Remark 1.0.5). Hence it suffices to determine the typical representations for these components $s(\epsilon_0)$.

Let us fix a character ϵ_0 of U_F and denote its Artin conductor(namely N_0 is the minimal nonegative integer N such that $\epsilon_0 \mid U_F^N = 1$) by $\mathfrak{p}_F^{N_0}$, the integer N_0 is called the exponent of ϵ_0. If χ_1,χ_2 are two characters of $F^\times$ such that $\chi_1 \mid U_F = \epsilon_0$ and $\chi_2 \mid U_F = 1$, the restriction to K of the parabolic induction of $\chi_1 \otimes \chi_2$ depends only on ϵ_0 and is described, according to [Ca2] p.311, as follows. For each integer $N \geq 1$, we denote by $K(N)$ the group $1 + M_2(\mathfrak{p}_F^N)$ and we set $K(0) = K$. For each integer $N \geq 0$, we denote by $K_0(N)$ the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in K such that $c \in \mathfrak{p}_F^N$. We have $K_0(0) = K(0) = K$. For any integer $N \geq N_0$, we set $Ind_N(\epsilon_0) := Ind_{K_0(N)}^K(\epsilon)$ where ϵ is the character of $K_0(N)$ defined by $\epsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \epsilon_0(a)$. For $N \geq N_0 + 1$, we define by $u_N(\epsilon_0)$ the complement of $Ind_{N-1}(\epsilon_0)$ in $Ind_N(\epsilon_0)$ and set $u_{N_0}(\epsilon_0) = Ind_{N_0}(\epsilon_0)$. Then W.Casselman proved that we have

Proposition 2.1.1 (Proposition 1 of [Ca2])**.** (1) $u_N(\epsilon_0)$ is irreducible for every integer $N \geq N_0$.

(2) For $N \geq N_0 + 1$, $u_N(\epsilon_0)$ is the unique irreducible representation of K, up to isomorphism , which is trivial on $K(N)$ but not on $K(N-1)$ and satisfies $\mathrm{Hom}_{K_0(N)}(\epsilon, u_N(\epsilon_0)) \neq 0$. Moreover $\mathrm{Hom}_{K_0(N_0)}(\epsilon, u_{N_0}(\epsilon_0)) \neq 0$

(3) If $\epsilon \neq 1$, then $\dim u_{N_0}(\epsilon_0) = (q+1)q^{N_0-1}$. And for $N \geq N_0 + 1$, $\dim u_N(\epsilon_0) = (q+1)(q-1)q^{N-2}$.

Proof. (1) Notice that from [BK1],2.5 we have an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{H}(K,\rho) \cong End_K(c\text{-}Ind_{K_0(N)}^K \rho) \cong End_K(Ind_{K_0(N)}^K \rho)$$

for any character ρ of $K_0(N)$. Here

$$\mathcal{H}(K,\rho) = \{\phi : K \to \mathbb{C} \mid \phi(k_1 k k_2) = \rho(k_1)\phi(k)\rho(k_2), \forall k_1, k_2 \in K_0(N), k \in K\}$$

[Ca2],Lemma 1 says that set $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for any $N \geq 1$ we have following double cosets decomposition

$$K = K_0(N)wK_0(N)\bigcup(\bigcup_{m=1}^{N} K_0(N)\begin{pmatrix} 1 & 0 \\ \pi^m\backslash\pi^{m+1} & 1 \end{pmatrix} K_0(N))$$

Hence

$$\dim \mathrm{Hom}_K(u_{N_0}(\epsilon_0), u_{N_0}(\epsilon_0)) = \dim End_K(Ind_{K_0(N)}^K \epsilon) = \dim \mathcal{H}(K, \epsilon)$$

where ϵ is tha character of $K_0(N_0)$. We can take specific k_1, k_2 to prove

$$\phi(w) = \phi(\begin{pmatrix} 1 & 0 \\ \pi^m\backslash\pi^{m+1} & 1 \end{pmatrix}) = 0$$

for $m = 1, 2\cdots, N_0-1$, namely only non-zero possibility is $\phi(1)$. Therefore $\dim \mathcal{H}(K, \epsilon) = 1$ and $u_{N_0}(\epsilon_0)$ is irreducible.

For $N \geq N_0 + 1$, coset decomposition implies

$$\dim End_K(Ind_{K_0(N)}^K \epsilon) = \dim End_K(Ind_{K_0(N-1)}^K \epsilon) + 1$$

which means $u_N(\epsilon_0)$ is irreducible.

(2) Clearly, $u_N(\epsilon_0)$ satisfied these condition. We need to prove the uniqueness.

If $\pi \in Irr(K)$ satisfies the condition, then

$$\mathrm{Hom}_K(\pi, Ind_{K_0(N)}^K \epsilon_0) = \mathrm{Hom}_{K_0(N)}(\pi, \epsilon_0) \neq 0$$

We need to prove $\mathrm{Hom}_K(\pi, Ind_{K_0(N-1)}^K \epsilon_0) = 0$. If otherwise, there exists $v \in V_\pi$ such that

$$\pi(k)v = \epsilon_0(k)v, \quad \forall k \in K_0(N-1)$$

But $N - 1 \geq N_0$, so $\epsilon_0 \mid_{K(N-1)}= 1$. Therefore

$$\pi(k)v = v$$

for all $k \in K(N-1)$. Since π is irreducible and $K(N-1)$ is normal in K, $\pi(k)v = v$ for all $v \in V_\pi$ and $k \in K(N-1)$. This is a contradiction.

(3) Notice that $[K : K_0(N)] = (q + 1)q^{N-1}$ for any $N \geq 1$.

□

2.2 Trivial component

First we consider the case where $\epsilon_0 = 1$, that is $s(\epsilon_0) = s_0$. Then $u_0(1) = 1_K$ is the trivial representation of K, and $u_1(1)$ is obtained by inflation from $GL_2(\mathbf{k_F}) \cong K/K(1)$ since $u_1(1) \mid_{K(1)}= 1$.

We have following exact sequence

$$0 \to u_0(1) = 1_K \to Ind_I^K 1_I \to u_1(1) \to 0$$

$\dim Ind_I^K 1_I = [K:I] = q+1$, so $\dim u_1(1) = q$ which means that $u_1(1)$ is the inflation of the steinberg representation of $GL_2(\mathbf{k_F})$. Here $I = K_0(1)$ is the standard Iwahori subgroup.

We have the following result for the trivial component s_0:

Proposition 2.2.1. Notation as above, $u_0(1)$ and $u_1(1)$ are typicals for s_0. Neither of these two is a type for s_0.

To prove this, we need the concept of conductor for an irreducible smooth admissible representation π of $G = GL_2(F)$ of infinite-dimension. Let us recall the important Theorem in [Ca1]:

Theorem 2.2.2 (Theorem 1 of [Ca1])**.** Let π be an irreducible admissible infinite-dimensional representation of G. Then there exist a largest ideal $\mathfrak{p}^{c(\pi)}(c(\pi) \geq 1)$ of $\mathfrak{o}_F$ such that the space of all non-zero vectors v with

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \omega_\pi(a)v, \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{p}^{c(\pi)})$$

is non-empty. In this case, the space has dimension 1, and we call the ideal $\mathfrak{p}^{c(\pi)}$ the conductor of π.

Proof. Overall, there are three cases.

(1) if π is supercuspidal, then $c(\pi) = -n_1$ where n_1 is the unique integer such that $C_{n_1}(1) \neq 0 (n_1 \leq -2)$ in Proposition 2.23 of [JL].

(2) if $\pi = Ind_B^G(\delta_B^{-1/2}\chi_1 \otimes \chi_2)$ where χ_1 and χ_2 have conductor $\mathfrak{p}_F^{n_1}$ and $\mathfrak{p}_F^{n_2}$ respectively, then

$$c(\pi) = \begin{cases} 1 & if \quad n_1 = n_2 = 0. \\ n_1 + n_2 & if \quad otherwise. \end{cases}$$

(3) If $\pi = \phi \circ \det \otimes St_G = \sigma(\phi\alpha^{1/2}, \phi\alpha^{-1/2})$ is the special representation where $\sigma(\phi\alpha^{1/2}, \phi\alpha^{-1/2})$ is the subrepresentation associated the invariant subspace $\mathscr{B}_s(\phi\alpha^{1/2}, \phi\alpha^{-1/2})$([JL] Theorem 3.3). There are two cases.

if $\phi \mid_{U_F} = 1$, then $c(\pi) = 1$.

if $\phi \mid_{U_F} \neq 1$ and has conductor $\mathfrak{p}^n (n \geq 1)$, then $c(\pi) = 2n$.

$\square$

With this definition, W.Casselman proves following Theorem in [Ca2]:

Theorem 2.2.3 (Theorem 1 of [Ca2])**.** Let $\pi \in Irr(G)$ with conductor $\mathfrak{p}^{c(\pi)}$ where $c(\pi) \geq 1$, ω_π the central character of π and $\eta_0 = \omega_\pi \mid_{U_F}$. Then the complement in $\pi \mid_K$ of the space fixed by $K(c(\pi) - 1)$ is the representation $\sum_{N \geq c(\pi)} u_N(\eta_0)$.

The proofs of Theorems 2.2.2 and 2.2.3 will be deferred to the subsection 2.6. They are originally due to W. Casselman; For more details, we refer the reader to [Ca1] and [Ca2].

Proof of Proposition 2.2.1:

1 : 1_K is a typical for s_0. Find an unramified character ϕ of $F^\times$, then $\pi = \phi \circ \det \otimes St_G \in s_0$ does not contain 1_K since $St_G^K = 0$, thus 1_K is not a type. We know $\mathrm{Hom}_K(1_K, 1_G) \neq 0$ with $1_G \in s_0$, to prove that 1_K is a typical for s_0, we need to check for any $\pi \in Irr(G)$,

$$\mathrm{Hom}_K(1_K, \pi) \neq 0 \Rightarrow \pi \in s_0$$

First [BH] 14.3 proposition says that π is non-cuspidal, thus we only need to prove that if π is principal, then $\pi \in s_0$. Assume $\pi = Ind_B^G \chi_1 \otimes \chi_2$ is a principal representation. Then $\pi \hookrightarrow Ind_K^G 1_K$. Hence for all $f \in Ind_B^G \chi_1 \otimes \chi_2$, we have

$$f(bg) = (\chi_1 \otimes \chi_2)(b) f(g) = 1_K(b) f(g) = f(g) \quad \forall b \in B \cap K$$

Namely $\chi_1 \otimes \chi_2 \mid_{B\cap K} = 1$, this means $[T, \chi_1 \otimes \chi_2]$ is inertially equivalent to $[T, 1_T]$. This implies $\pi \in s_0$.

In addition, [BH] 17.10 exercise (2) implies that 1_K appears in all principal representations and characters in s_0.

2 : $u_1(1)$ is a typical for s_0. We need to prove for any $\pi \in Irr(G)$,

$$\mathrm{Hom}_K(u_1(1), \pi) \neq 0 \Rightarrow \pi \in s_0$$

If $\mathrm{Hom}_K(u_1(1), \pi) \neq 0$, $u_1(1)$ is irreducible and $Hom_I(1_I, u_1(1)) \neq 0$ implies there exists a non-zero vector $v \in V_\pi$ such that

$$\pi(g)v = v \quad \forall g \in I \tag{1}$$

which means that $c(\pi) = 1$ if π is infinite-dimensional. Therefore [BH] 11.5 Theorem and 14.3 Proposition imply that π is not cuspidad, thus it suffices to prove that π is of the form $Ind_B^G \chi_1 \otimes \chi_2 (\chi_1,\ \chi_2$ are unramified)or $\phi \circ \det \otimes St_G(\phi$ is unramified). if $\pi = Ind_B^G \sigma_1 \otimes \sigma_2$ is a principal representation, then $\sigma_1\sigma_2 \mid_{U_F} = 1$ and Theorem 2.2.2 implies $\sigma_i (i = 1, 2)$ are unramified. Similarly, if $\pi = \phi \circ \det \otimes St_G$, π must be unramified.

Notice that $\mathrm{Hom}_K(u_1(1), \phi \circ \det) = 0$ for any unramified character ϕ of $F^\times$ because of the dimension. And by Theorem 2.2.2 and 2.2.3, any special representation or principal representation in s_0 has conductor $\mathfrak{p}$, hence contains $u_1(1)$.

As for $u_N(1)(N \geq 2)$, we have

Proposition 2.2.4. $u_N(1)$ is not typical for s_0 providing $N \geq 2$.

This is the direct corollary of the following Proposition.

Proposition 2.2.5. Let π be an irreducible smooth supercuspidal representation with $\ell(\pi) = 0$(namely it contains the trivial character of $K(1)$) and such that $\omega_\pi \mid_{U_F} = \epsilon_0 = 1$. Then $c(\pi) = 2$ and $u_N(1)$ appears in π for $N \geq 2$.

Proof. 11.5 Theorem of [BH] guarantees the existence of π since $\pi \mid_Z$ can be any character. Then by 14.3 Proposition of [BH], $c(\pi) \neq 1$.

We need to prove $c(\pi) = 2$. Namely there is a non-zero vector v such that

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = v, \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{p}^2)$$

Taking $\begin{pmatrix} \varpi_F^{-1} & 0 \\ 0 & 1 \end{pmatrix}$-conjugation, this is equivalent to

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = v, \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{K} := \begin{pmatrix} U_F & \mathfrak{p}_F \\ \mathfrak{p}_F & U_F \end{pmatrix}$$

We have the following commutative diagram.

$$\begin{tikzcd} K \arrow[r, "\pi"] \arrow[d] & GL(V^{K_1}) \\ K/K_1 \cong GL_2(\mathbf{k}_F) \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

and the image of $\bar{K}$ in $GL_2(\mathbf{k}_F)$ is $T_{\mathbf{k}_F} := \begin{pmatrix} \mathbf{k}_F^\times & 0 \\ 0 & \mathbf{k}_F^\times \end{pmatrix}$. Taking an irreducible component ρ(it is cuspidal) of $\tilde{\pi}$, it is sufficient to prove

$$\mathrm{Hom}_{T_{\mathbf{k}_F}}(\rho, 1_{T_{\mathbf{k}_F}}) \neq 0$$

But

$$\langle \chi_\rho, \chi_{1_{T_{\mathbf{k}_F}}} \rangle = \frac{1}{q-1} \sum_{a \in \mathbf{k}_F^\times} tr\rho(a) = q - 1 \neq 0$$

The first equality holds because the trace vanishes on non-central elements by 6.4 Theorem of [BH], the second equality follows by $\rho(a) = \tilde{\pi}(a) = 1$ for all $a \in \mathbf{k}_F$. Hence the result holds. $\square$

So far, we have shown that the trivial component s_0 has two typicals : $u_0(1)$ and $u_1(1)$. And neither $u_0(1)$ or $u_1(1)$ is a type.

2.3 Principle Component: $q = 3$ non-typical case

For $\epsilon_0 \neq 1$, we need to examine whether $u_N(\epsilon_0)$ for $\epsilon_0 \neq 1$ and $N \geq N_0$ are typical or not.

The result is that

Theorem 2.3.1. For $\epsilon_0 \neq 1$,we have

(1) If $q \geq 3$, then $u_{N_0}(\epsilon_0)$ is the unique type for $s(\epsilon_0)$. And for $N \geq N_0+1$, $u_N(\epsilon_0)$ is not a typical for $s(\epsilon_0)$.

(2) If $q = 2$, then $u_{N_0}(\epsilon_0)$ and u_{N_0+1} are type for $s(\epsilon_0)$. And for $N \geq N_0 + 2$, $u_N(\epsilon_0)$ is not a typical for $s(\epsilon_0)$.

In this subsection, We will prove (1) of Theorem 2.3.1 for $N \geq N_0 + 1$. Thus, in this subsection, we assume $q \geq 3$. The first proposition is :

Proposition 2.3.2. For $\epsilon_0 \neq 1$ and $N_0 = 1$(this case does not exist for $q = 2$), $u_N(\epsilon_0)(N \geq 2)$ is not typical for $s(\epsilon_0)$.

Proof. We employ the same approach as that used in Proposition 2.2.5. Take an irreducible smooth supercuspidal representation π with $\ell(\pi) = 0$ such that $\omega_\pi \mid_{U_F} = \epsilon_0$. Specifically, deem $\epsilon_0 = \epsilon_0 \otimes 1$ as a character of T_F and $T_{\mathbf{k}_F}$. We have

$$\langle \chi_\rho, \chi_{\epsilon_0 \otimes 1} \rangle = \frac{1}{q-1} \sum_{a \in \mathbf{k}_F^\times} |tr\epsilon_0(a)|^2 = (q-1) \sum_{a \in \mathbf{k}_F^\times} |\epsilon_0(a)|^2 \neq 0$$

which means $c(\pi) = 2$. Hence $u_N(\epsilon_0)(N \geq 2)$ appear in π, thus they are not typical for $s(\epsilon_0)$. □

For $\epsilon_0 \neq 1$ and $N_0 \geq 2$. Let η be a non-trivial character of U_F with conductor $\mathfrak{p}$, and χ_1, χ_2 be two characters of $F^\times$ such that $\chi_1 \mid_{U_F} = \eta\epsilon_0$ and $\chi_2 \mid_{U_F} = \eta^{-1}$. Then we have

Proposition 2.3.3. The principal representation $\pi(\chi_1, \chi_2)$ does not belong to the component $s(\epsilon_0)$ and its conductor is $\mathfrak{p}_F^{N_0+1}$.

Proof. By Theorem 2.2, the conductor is $\mathfrak{p}_F^{N_0+1}$. Since $\chi_1 \neq \chi_2$ and $\tilde{\epsilon_0} \neq 1$, $\chi_1 \otimes \chi_2$ and $\tilde{\epsilon_0} \otimes 1$ are supercuspidal representations of T. Hence the support of $\pi(\chi_1, \chi_2)$ and $\pi(\tilde{\epsilon_0}, 1)$ are $(T, \chi_1 \otimes \chi_2)$ and $(T, \tilde{\epsilon_0} \otimes 1)$, a straightforward calculation shows that they are not inertially equivalent. Therefore $\pi(\chi_1, \chi_2)$ does not belong to $s(\epsilon_0)$. □

Corollary 2.3.4. For $N \geq N_0 + 1$, $u_N(\epsilon_0)$ is not typical for $s(\epsilon_0)$.

Proof. Just use Theorem 2.2.3. □

So far, we have proven that for $\epsilon_0 \neq 1$, $q = 3$, and $N \geq N_0 + 1$, $u_N(\epsilon_0)$ is not typical for $s(\epsilon_0)$.

2.4 Principle Component: $q = 2$ non-typical case

Now assume $\epsilon_0 \neq 1$, $q = 2$ (so $N_0 \geq 2$). We need to determine whether $u_N(\epsilon_0)$ is typical or not for $N \geq N_0 + 1$.

First if $N_0 \geq 3$, we can choose a character η of U_F with conductor $\mathfrak{p}_F^2$ and construct $\pi = \pi(\chi_1, \chi_2)$ with conductor $\mathfrak{p}_F^{N_0+2}$ as in the Proposition 2.3.3. Then we have :

Proposition 2.4.1. The representation $\pi(\chi_1, \chi_2)$ does not belong to the component $s(\epsilon_0)$, and $u_N(\epsilon_0)$ is not typical for $N \geq N_0 + 2$.

Proof. The method is the same as Proposition 2.3.3. $\square$

If $N_0 = 2$, then ϵ_0 itself is the unique character of U_F with conductor $\mathfrak{p}_F^2$ since U_F/U_F^2 has unique non-trivial character. In this case $\pi(\epsilon_0^2 = 1, \epsilon_0^{-1})$ belongs to $s(\epsilon_0)$, thus we cannot use the same trick. The solution is following.

Proposition 2.4.2. Take an unramified quadratic extension E/F, then there exists a character θ of $E^\times$ with conductor $\mathfrak{p}_E^2$ which is not stable under the action of $Gal(E/F)$ and such that $\theta\mid_{U_F} = \epsilon_0$. In this case, the representation $\pi(\theta)$ associated to $Ind_{W_E}^{W_F}(\theta)$ by local langlands correspondence is supercuspidal with conductor $\mathfrak{p}^4$. Therefore $u_N(\epsilon_0)$ is not typical for $N \geq 4$.

Hence we have prove that for $\epsilon_0 \neq 1$, $q = 2$ and $N \geq N_0 + 2$, $u_N(\epsilon_0)$ is not typical for $s(\epsilon_0)$.

2.5 Principle Component: type case

To complete the proof of Theorem 2.3.1, we need to prove

Proposition 2.5.1. For $\epsilon_0 \neq 1$(so $N_0 \geq 1$).

(1) for any q, $u_{N_0}(\epsilon_0)$ is a type for $s(\epsilon_0)$.

(2) If $q = 2$, then $u_{N_0+1}(\epsilon_0)$ is also a type for $s(\epsilon_0)$.

We first prove a Lemma:

Lemma 2.5.2. Let (π, V) be an irreducible smooth representation of G in which $(u_N(\epsilon_0), W)$ appears. Then $c(\pi) \leq N$.

Proof. By Proposition 2.1.1, $\mathrm{Hom}_{K_0(N)}(\epsilon_0, u_N(\epsilon_0)) \neq 0$. Thus there exists $0 \neq w \in W$ such that

$$(u_N(\epsilon_0))(g)w = w \qquad \forall g \in \begin{pmatrix} U_F^N & \mathfrak{o}_F \\ \mathfrak{p}_F^N & U_F \end{pmatrix}$$

Take $0 \neq f \in Hom_K(u_N(\epsilon_0), \pi\mid_K)$, then $f(w) \neq 0$ since $u_N(\epsilon_0)$ is irreducible. Hence we have

$$\pi(g)f(w) = f(w) \qquad \forall g \in \begin{pmatrix} U_F^N & \mathfrak{o}_F \\ \mathfrak{p}_F^N & U_F \end{pmatrix}$$

which means that $c(\pi) \leq N$. $\square$

Proof for (1) *of Proposition* 2.5.1: we want to show that $u_{N_0}(\epsilon_0)$ does not appear in the principal series of a component which is different from $s(\epsilon_0)$, nor in supercuspidal representations or special representations. Assume $\pi \in Irr(G)$ contains $u_{N_0}(\epsilon_0)$.

if $\pi = \pi(\chi_1, \chi_2)$ is a principal series, then the sum of the exponents of χ_1 and χ_2 is at most N_0. But we also have $\chi_1\chi_2 \mid_{U_F} = \epsilon_0$, which implies that χ_1 or χ_2 has exponent of at least N_0, so χ_1 or χ_2 has exponent 0, which means that $\pi(\chi_1, \chi_2)$ belongs to the component $s(\epsilon_0)$.

If $\pi = \phi \circ \det \otimes st_G$ is a special representation. [BH] 14.4 Example says that $\dim V^I = 1$ if $(\rho, V) = St_G$. This means $St_G \mid_{U_F} = 1$, thus $\phi \circ \det \mid_{U_F} = \epsilon_0$. Assume ϕ has conductor $\mathfrak{p}^n$, then $n \geq 1$ since $N_0 \geq 1$. Theorem 2.2.2 implies $2n \leq N_0$, namely $n \leq 2n-1 \leq N_0 - 1$ which means $\phi \circ \det \mid_{U_F^{N_0-1}} = 1$. This contradicts the fact that the conductor of ϵ_0 is $\mathfrak{p}^{N_0}$.

If π is supercuspidal, we have $\omega_\pi \mid_{U_F} = \epsilon_0$, so $c(\pi) \geq 2N_0 > N_0$. Hence by theorem 2.2.3, $u_{N_0}(\epsilon_0)$ appears in the subspace of π fixed by $K(c(\pi)-1)$. But according to [[Ca2], Theorem 2], this space does not contain any non-zero vector fixed by $\begin{pmatrix} 1 & \mathfrak{o}_F \\ 0 & 1 \end{pmatrix}$, so it cannot contain $u_N(\epsilon_0)$ by the same argument in Lemma 2.5.2.

Since $\epsilon_0 \neq 1$, $Ind_B^G \tilde{\epsilon_0} \otimes 1$ is irreducible. Thus any $\pi \in s(\epsilon_0)$ has conductor $\mathfrak{p}^{N_0}$, so by Theorem 2.2.3, $u_{N_0}(\epsilon_0)$ appears with multiplicity 1 in all elements of $s(\epsilon_0)$, hence it is a type for $s(\epsilon_0)$.

Proof for (2) *of Proposition* 2.5.1: If $q = 2$, then $N_0 \geq 2$. Assume $\pi \in Irr(G)$ contains $u_{N_0}(\epsilon_0)$. The proof is similar to (1).

if $\pi = \pi(\chi_1, \chi_2)$ is a principal series, then the sum of the exponents of χ_1 and χ_2 is at most $N_0 + 1$. But we also have $\chi_1\chi_2 \mid_{U_F} = \epsilon_0$, which implies that χ_1 or χ_2 has exponent of at least N_0, so χ_1 or χ_2 has exponent 0 or 1. Since $q = 2$, $U_F \cong U_F^1$, thus exponent 1 cannot occur, which means that $\pi(\chi_1, \chi_2)$ belongs to the component $s(\epsilon_0)$.

If $\pi = \phi \circ \det \otimes st_G$ is a special representation. [BH] 14.4 Example says that $\dim V^I = 1$ if $(\rho, V) = St_G$. This means $St_G \mid_{U_F} = 1$, thus $\phi \circ \det \mid_{U_F} = \epsilon_0$. Assume ϕ has conductor $\mathfrak{p}^n$, then $n \geq 2$ since $q = 2$. Theorem 2.2 implies $2n \leq N_0 + 1$, namely $n \leq 2n - 2 \leq N_0 - 1$ which means $\phi \circ \det \mid_{U_F^{N_0-1}} = 1$. This contradicts the fact that the conductor of ϵ_0 is $\mathfrak{p}^{N_0}$.

If π is supercuspidal, we have $\omega_\pi \mid_{U_F} = \epsilon_0$, so $c(\pi) \geq 2N_0 > N_0 + 1$. Hence by theorem 2.2.3, $u_{N_0+1}(\epsilon_0)$ appears in the subspace of π fixed by $K(c(\pi)-1)$. The remainder of the argument proceeds in the same manner as in (1).

2.6 Two Theorem of Conductor

A1.We start with the proof of the Theorem 2.2.2.

1. Assume $\pi = Ind_B^G(\delta_B^{-1/2}\chi_1 \otimes \chi_2)$ is a principal representation. Then $\omega_\pi = \chi_1\chi_2$. Recall that $\mathscr{B}(\chi_1, \chi_2)$ is the set of all locally constant functions f on $GL_2(F)$ such that
$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \chi_1(a)\chi_2(b) \cdot |a/b|^{1/2} \cdot f(g)$$
for all $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in B$, $g \in GL_2(F)$. A important fact is that the restriction map $f \mapsto f \mid_K$ is a K-isomorphism of $\mathscr{B}(\chi_1, \chi_2)$ with the set of all functions f on K such

that

$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \chi_1(a)\chi_2(b) \cdot f(g)$$

for all $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in B \cap K$, $g \in K$.

Now notice that the space of functions we seek to dertermine is that of function on K satisfying

$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = \chi_1(a)\chi_2(b)\chi_1\chi_2(a')f(g)$$

for all $g \in K$, $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in B \cap K$, $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(\mathfrak{c})$ for an arbitrary ideal $\mathfrak{c}$ of $\mathfrak{o}_F$.

Clearly, if this set is not empty, then $\chi_1 \mid_{1+\mathfrak{c}} = \chi_2 \mid_{1+\mathfrak{c}} = 1$, thus χ_1 and χ_2 define a character of $(\mathfrak{o}_F/\mathfrak{c})^\times \cong \mathfrak{o}_F^\times/(1+\mathfrak{c})$. On the other hand, since the principal congruence subgroup $\Gamma(\mathfrak{c}) := \{g \in K \mid g \equiv 1(\mod \mathfrak{c})\}$ is normal in K and $\Gamma_0(\mathfrak{c})$. We can prove that above space is isomorphic to the set of all functions ϕ on the residue group $GL_2(\mathfrak{o}_F/\mathfrak{c})$ satisfying

$$\phi\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \chi_1(a)\chi_2(b)\chi_1\chi_2(a')\phi(g) \tag{2}$$

for all $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \in \bar{B}$ where $\bar{B}$ is the image of $B \cap K$ in $GL_2(\mathfrak{o}_F/\mathfrak{c})$. Hence ϕ is completely determined on a double coset $\bar{B}g\bar{B}$. Indeed, we have the following result.

Lemma 2.6.1.

$$GL_2(\mathfrak{o}_F/\mathfrak{c}) = \bigcup_{i=0}^{j} \bar{B} \cdot \begin{pmatrix} 1 & 0 \\ \varpi_F^i & 1 \end{pmatrix} \cdot \bar{B}$$

where $\mathfrak{c} = (\varpi_F^j)$, $\begin{pmatrix} 1 & 0 \\ \varpi_F^0 & 1 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and this union is disjoint.

Proof. Notice that any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ with $c = \gamma \cdot \pi^i$ for some $\gamma \in U_F$ and $i > 0$ lies in $(B \cap K) \cdot \begin{pmatrix} 1 & 0 \\ \pi^i & 1 \end{pmatrix} \cdot (B \cap K)$ since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi^i & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{d}{\gamma} - \frac{b}{a}\pi^i \end{pmatrix}$$

Similarly, any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ with $c \in U_F$ lies in $(B \cap K) \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot (B \cap K) =$

$(B\cap K)\cdot \begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot(B\cap K)$ since

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}1&-a\\0&-c\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&\frac{d}{c}\\0&\frac{bc-ad}{c}\end{pmatrix}$$

□

Therefore, we reduce to the following question: given the ideal $\mathfrak{c}$ and two characters χ_1 and χ_2 of $(\mathfrak{o}_F/\mathfrak{c})^\times$, on which double cosets $\bar{B}\cdot\begin{pmatrix}1&0\\\varpi_F^i&1\end{pmatrix}\cdot\bar{B}$ do functions ϕ satisfying the equality of (2)?

Proposition 2.6.2. There is a function ϕ on $GL_2(\mathfrak{o}_F/\mathfrak{c})$ satisfies (2) on the coset $\bar{B}\cdot\begin{pmatrix}1&0\\\varpi_F^i&1\end{pmatrix}\cdot\bar{B}$ if and only if

(1) ϖ_F^i lies in the conductor of χ_1

(2) $\mathfrak{c}\varpi_F^{-i}$ is contained in the conductor of χ_2.

Hence if $c(\chi_1)=\mathfrak{p}_F^{n_1}$ and $c(\chi_2)=\mathfrak{p}_F^{n_2}$, then $\mathfrak{c}=\mathfrak{p}_F^{n_1+n_2}$ satisfies the Theorem.

Proof. $(\Rightarrow)$: If there exists this ϕ, then take $\begin{pmatrix}a&x\\0&b\end{pmatrix},\begin{pmatrix}a'&x'\\0&b'\end{pmatrix}\in GL_2(\mathfrak{o}_F/\mathfrak{c})$ such that

$$\begin{pmatrix}a&x\\0&b\end{pmatrix}\begin{pmatrix}1&0\\\varpi_F^i&1\end{pmatrix}\begin{pmatrix}a'&x'\\0&b'\end{pmatrix}=\begin{pmatrix}1&0\\\varpi_F^i&1\end{pmatrix}\tag{3}$$

We must have $\chi_1(a)\chi_2(b)=\chi_1(a')\chi_2(a')$. But (3) implies

$$\begin{aligned}b&\equiv b'(\mod \varpi_F^i)\\a&\equiv a'(\mod \varpi_F^i)\\a'&\equiv b(\mod \mathfrak{c}\varpi_F i^{-i})\\b-b'&\equiv a'-a(\mod \mathfrak{c})\end{aligned}$$

Take $b=a'=1$, we have $\chi_1(\frac{a}{a'})=1$ namely $\chi_1\mid_{1+\mathfrak{p}_F^i}=1$. Similarly, $\chi_2\mid_{1+\mathfrak{c}\pi^{-i}}=1$.

$(\Leftarrow)$ If these two condition holds, we define a function ϕ by

$$\phi\left(\begin{pmatrix}a&x\\0&b\end{pmatrix}\begin{pmatrix}1&0\\\varpi_F^i&1\end{pmatrix}\begin{pmatrix}a'&x'\\0&b'\end{pmatrix}\right)=\chi_1(aa')\chi_2(ba')\phi\left(\begin{pmatrix}1&0\\\varpi_F^i&1\end{pmatrix}\right).$$

We only need to check it is well-define which is guaranteed by the two conditions. □

2. If $\pi=(\chi\circ\det)\otimes St_G$ is a special representation.

3. If π is a supercuspidal representation. We first introduce a lemma.

Lemma 2.6.3. Let $\mathfrak{c} = (\varpi_F^m)$ be any proper integral ideal of $\mathfrak{o}_F$, χ_1 and χ_2 be characters of $\mathfrak{o}_F^\times$ of conductors contains $\mathfrak{c}$. π is a representation of $GL_2(F)$, set $H = \begin{pmatrix} 0 & 1 \\ -\varpi_F^m & 0 \end{pmatrix}$. Then the following conditions on a vector v in the representation space are equivalent:

(a)

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \chi_1(a)\chi_2(d)v, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{c})$$

(b) (1)

$$\pi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} v = \chi_1(a)\chi_2(d)v, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B \cap K \quad \text{and}$$

(2)

$$\pi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} Hv = \chi_1(d)\chi_2(a)Hv, \quad \forall \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B \cap K$$

Proof. Since H normalizes $\Gamma_0(\mathfrak{c})$, (a) implies (b) is immediate. Conversely, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{c})$, then $c = \gamma\varpi_F^m$ for some $\gamma \in U_F$, and we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (ad-bc)d^{-1} & b \\ 0 & d \end{pmatrix} H^{-1} \begin{pmatrix} 1 & -d^{-1}\gamma \\ 0 & 1 \end{pmatrix} H$$

Thus if (b) holds, we have

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \pi \begin{pmatrix} (ad-bc)d^{-1} & b \\ 0 & d \end{pmatrix} v = \chi_1(a)\chi_2(d)v$$

since $(ad-bc)d^{-1} = a - bcd^{-1} \equiv a(\mod \mathfrak{c})$.

To continue, we shall dertermine the dimension of all vectors v satisfying the Theorem, not just for the particular ideal $c(\pi)$, but for any integral ideal $\mathfrak{c}$.

By [JL]p.117 or Lemma 3.9 of re-typeset, there are none vectors which are fixed by all of K, so we may assume $\mathfrak{c}$ is a proper ideal. Then we apply above Lemma in the case $\chi_2 = 1$, so it remains to determine all v satisfying (b) for $\chi_1 = \omega_\pi = \epsilon \text{and} \chi_2 = 1$. To do this, we need to use Kirillov model, please refer to 5.3.2 of [AS]. Embedding (π, V) to $(\xi_\psi, C(F^\times))$ where ψ is a character of F with conductor $\mathfrak{o}_F$. Now (b) is equivelent to there exists $f \in C(F^\times)$ such that

$$\left(\xi_\psi \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} f\right)(\alpha) = \epsilon(a)f(\alpha) \;\; \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B \cap K \tag{4}$$

and

$$\left(\xi_\psi \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} Hf\right)(\alpha) = \epsilon(d)Hf(\alpha) \;\; \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B \cap K \tag{5}$$

We only analyze the first equality (4), the second is similar. Apply [JL] Proposition 2.10, (4) is just

$$\epsilon(b)\psi(b^{-1}x\alpha)f(ab^{-1}\alpha) = \epsilon(a)f(\alpha)$$

Namely

$$\psi(b^{-1}x\alpha)f(u\alpha) = \epsilon(u)f(\alpha)$$

for all $a, b \in U_F$,$x \in \mathfrak{o}_F$, $\alpha \in F^\times$. Since ψ has conductor $\mathfrak{o}_F$, take $u = 1$and $b^{-1}x\alpha$ such that $\psi(b^{-1}x\alpha) \neq 1$, we have $f(\alpha) = 0$ for $\alpha \notin \mathfrak{o}_F$. Thus (4) is equivalent to

$$f(u\alpha) = \epsilon(u)f(\alpha)$$

for all $u \in U_F$, $\alpha \in F^\times$ and $supp(f) \subset \mathfrak{o}_F$. In the language of mellin transform, this is equivalent to

$$\hat{f}_n(v) = 0 \quad \text{unless} \quad n \geq 0 \quad \text{and} \quad v = \epsilon^{-1} |_{\mathfrak{o}_F}$$

Similarly, (5) is equivalent to

$$(\hat{Hf})_n(v) = 0 \quad \text{unless} \quad n \geq 0 \quad \text{and} \quad v = 1$$

But

$$H = \begin{pmatrix} 0 & 1 \\ -\varpi_F^m & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \varpi_F^m & 0 \\ 0 & 1 \end{pmatrix}$$

Hence by [JL] Proposition 2.10,

$$\begin{aligned} \widehat{Hf}(1,t) &= C_{n_1}(1) \cdot t^{n_1} (\pi \widehat{\begin{pmatrix} \varpi_F^m & 0 \\ 0 & 1 \end{pmatrix} f})(\epsilon^{-1}, z_0^{-1}t^{-1}) \\ &= C_{n_1}(1) \cdot t^{n_1} z_0^m t^m f(\epsilon^{-1}, t^{-1}z_0^{-1}) \\ &= C_{n_1}(1) z_0^m f(\epsilon^{-1}, t^{-1}z_0^{-1}) t^{n_1+m} \end{aligned}$$

where n_1 is the unique integer such that $C_{n_1}(1) \neq 0$(see [JL] Proposition 2.21 and 2.23) and $z_0 = \omega_\pi(\varpi_F)$ Hence f satisfies (4) and (5) if and only

$$\hat{f}_n(v) = 0 \quad \text{unless} \quad 0 \leq n \leq n_1 + m \quad \text{and} \quad v = \epsilon^{-1}$$

Hence if there exist such f ,$n_1+m \geq 0$. Notice n_1 is a negative integer, thus $\mathfrak{c} = (\varpi_F^{-n_1})$ satisfies the Theorem. □

3 Supercuspidal Components

In this chapter, we will explain how to identify a type corresponding to a given supercuspidal component.

3.1 General result

We know that a supercuspidal representation π of $G = GL_2(F)$ is of the form

$$\pi = c - Ind_J^G \lambda$$

where J is a open subgroup of G and compact modulo the center of G, and λ is a smooth irreducible representation of J. Denote the component $[G, \pi]_G$ by s. [BK1] (5.4) Proposition and (5.5) comment (b) proved that

(1) $(J \cap K, \lambda^0)$ is a type for s where $\lambda^0 := \lambda \mid_{J \cap K}$.

(2) $g \in G$ interwines λ^0 if and only if $g \in J$.

Hence [BH] 11.4 Theorem implies $\rho := c - Ind_{J\cap K}^K \lambda^0$ is irreducible and supercuspidal. Thus by Frobenius Reciprocity, (K, ρ) is a type for s.

3.2 Exponent 2

First, let's consider the case where s is the component of a smooth irreducible supercuspidal representation π of level zero(namely contains the trivial character of $K(1)$). In this case, $J = F^\times K$ and ρ is the inflation of an irreducible cuspidal representation of $K/K(1) \cong GL_2(\mathbf{k_F})$. According to Theorem 2.2.3, the complement of ρ in $\pi \mid_K$ is the direct sum of $u_N(\epsilon_0)$ for $N \geq 2$ where $\epsilon_0 = \omega_\pi \mid U_F$. Thus (K, ρ) is the only type representation for s by subsection 3.1, up to isomorphism.

3.3 Even exponent > 2

If π is a smooth irreducible supercuspidal representation of level $\ell(\pi) \geq 1$, then the exponent of its conductor is greater than 3. Thus we will first give the construction of all smooth irreducible supercuspidal representations π of G whose exponent is even and greater than 4.

To do this, choose an unramified quadratic extension E of F and an emdedding of E into $M_2(F)$ such that the image of $U_E = \mathfrak{o}_E^\times$ is contained in K, we can choose this since $N(U_E) = U_F$. Notice taht ψ has level 1 implies that $\psi_E := \psi \circ Tr_{E/F}$ has level 1 as a character of $E^\times$.

Then fix an element $b \in \mathfrak{p}_E^{-n}$ where n is a positive integer, [BH] 1.8 Proposition implies there exists a character θ of $U_E^{[n/2]+1}/U_E^{n+1}$ such that

$$\theta(1 + x) = \psi_E(bx), \quad \forall x \in \mathfrak{p}_E^{[n/2]+1}$$

(1) If n is odd, we set $H = J = E^\times K((n+1)/2)$ and define a character λ of $H = J$ by

$$\lambda(y) = \theta(y) \quad \text{for} \quad y \in E^\times,$$
$$\lambda(1 + x) = \psi \circ tr_A(bx) \quad \text{for} \quad 1 + x \in K((n+1)/2)$$

Then $\pi := c - Ind_J^G \lambda$ is a smooth irreducible supercuspidal representation of G of exponent $2(n+1)$. By 3.1, the representation $\rho = Ind_{J\cap K}^K(\lambda_{J\cap K})$ is irreducible and occurs with multiplicity 1 in all elements of the component of π and is a type for this component. All smooth irreducible minimal supercuspidal representations of G of exponent a multiple of 4 are obtained by this construction.

(2) If n is even, set $H = E^\times K(n/2+1)$, $J = E^\times K(n/2)$ and $J^1 = U_E^1 K(n/2)$. We have $H \subset J$ and $J^1 \subset J$. We define a character η of H by

$$\eta(y) = \theta(y) \quad \text{for} \quad y \in E^\times,$$
$$\eta(1+x) = \psi \circ tr_A(bx) \quad \text{for} \quad 1+x \in K(n/2+1)$$

Refers to [BH] 19.4, in this case, the representation $\pi := c - Ind_J^G \lambda$ is smooth irreducible supercuspidal of exponent $2n+2$. As before, $\rho = Ind_{J\cap K}^K(\lambda \mid_{J\cap K})$ is irreducible and occurs with multiplicity 1 in all elements of the component of π, and is a type for this component. All smooth irreducible supercuspidal representations of G of exponent greater than 4 and congruent to 2 modulo 4 are obtained by this construction.

In above two cases, to see that the irreducible representation ρ of K is the only constituent of $\pi \mid_K$ that is a typical(type) for the component s of π. We need to prove that other constituents of π occur in other component which is different from the component of π.

To analyze $\pi \mid_K$, we decompose G as a disjoint union of double cosets $E^\times KgK$ with $g \in \{\begin{pmatrix} \omega_F^a & 0 \\ 0 & 1 \end{pmatrix}, a \in \mathbb{N}\}$(see [BH], 10.2). Then

$$\pi \mid_K = \bigoplus_g Ind_{K\cap g^{-1}Kg}^K(\rho^g)$$

where $\rho^g(x) = \rho(gxg^{-1})$ for $x \in K \cap g^{-1}Kg$. Let μ be a character of $E^\times$ which is trivial on $F^\times$ of exponent 1. It exists since $E^\times = U_E F^\times$ and $|\mathbf{k_E}| > 1$. Now we can do the same construction as above by replacing θ by $\theta' = \theta\mu$, which gives an irreducible representation λ' of J.

By induction from J to G, $\rho' := Ind_{J\cap K}^K(\lambda_{J\cap K})$ is a type for the component s' of $\pi' := c - Ind_J^G \lambda'$. Notice that $s \neq s'$ since $\mu \mid_{U_E^1} \neq 1$. We claim that for

$$g = \begin{pmatrix} \omega_F^a & 0 \\ 0 & 1 \end{pmatrix}, a \geq 1$$

the representation ρ^g and ρ'^g are equivalent. This implies that ρ is the unique typical (type) representation for s.

Proposition 3.3.1. Notation as above, then $\rho^g \cong \rho'^g$.

Proof. We know that if $H_1 \leq H_2$ are subgroups of G, χ is a representation of H_1, ξ is a representation of H_2 and $g \in G$, then we have

$$Ind_{g^{-1}H_1g}^{g^{-1}H_2g}\chi^g \cong (Ind_{H_1}^{H_2}\chi)^g \quad \text{and} \quad Res_{g^{-1}H_1g}^{g^{-1}H_2g}\xi^g \cong (Res_{H_1}^{H_2}\xi)^g$$

Hence

$$\rho^g = Res_{K\cap g^{-1}Kg}^{g^{-1}Kg}(Ind_{J\cap K}^K \lambda)^g \cong (Res_{K\cap gKg^{-1}}^K(Ind_{J\cap K}^K \lambda))^g$$

and

$$\rho'^g = Res^{g^{-1}Kg}_{K\cap g^{-1}Kg}(Ind^K_{J\cap K}\lambda')^g \cong (Res^K_{K\cap gKg^{-1}}(Ind^K_{J\cap K}\lambda'))^g$$

Thus we reduce to prove

$$Res^K_{K\cap gKg^{-1}}(Ind^K_{J\cap K}\lambda') \cong Res^K_{K\cap gKg^{-1}}(Ind^K_{J\cap K}\lambda)$$

We write K as a disjoint union of double cosets $(K \cap gKg^{-1})h(J \cap K)$, and we want to identify for a fixed h, the representation λ^h and $(\lambda')^h$ of $(K\cap gKg^{-1})\cap h^{-1}(J\cap K)h$. But by the following Lemma 3.3.2, $(K\cap gKg^{-1})\cap h^{-1}(J\cap K)h \subset h^{-1}U_F J^1 h$, and the construction of (λ') implies $\lambda \mid U_F J^1 = (\lambda') \mid U_F J^1$. Thus the result holds. ☐

Lemma 3.3.2. Notation as above, we have $(K\cap gKg^{-1})\cap h^{-1}(J\cap K)h \subset h^{-1}U_F J^1 h$.

Proof. We first need to prove $J \cap K = U_E K(n/2)$. Assume $t = ek \in J \cap K$ where $e \in E^\times, k \in K(n/2)$, then $e = tk^{-1} \in K$. But $E^\times \cap K = U_E$ since $U_E\omega_E^k = U_E\omega_F^k$ for any $k \in \mathbb{Z}$ and $U_E \in K$.

By calculating,

$$K \cap gKg^{-1} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \mid b \in \mathfrak{p}_F\}$$

Take $s \in (K \cap gKg^{-1}) \cap h^{-1}(J \cap K)h$. The embedding $U_E \hookrightarrow K$ induces the embeddings $\mathbf{k}_E^\times \hookrightarrow GL_2(\mathbf{k}_F)$, let $\bar{s}$ be the image of s in $M_2(\mathbf{k}_F)$. We have

(1) The characteristic polynomial of $\bar{s}$ is splitting on $\mathbf{k}_F$.

(2) If set $s = h^{-1}th$ where $t = ek \in J \cap K = U_E K(n/2)$, by (1) the characteristic polynomial $f(x)$ of $\bar{t} = \bar{e}$ is splitting on $\mathbf{k}_F$.

We know $f(x)$ is reducible on $\mathbf{k}_F$ if and only if $\bar{e} \in \mathbf{k}_F^\times$. Thus $\bar{e} \in \mathbf{k}_F^\times$. This means that $e \in U_F U_E^1$. Therefore $t = ek \in U_F U_E^1 K(n/2) = U_F J^1$. This implies $(K \cap gKg^{-1}) \cap h^{-1}(J \cap K)h \subset h^{-1}U_F J^1 h$. ☐

3.4 Odd exponent ≥ 3

We now turn to the study of irreducible smooth supercuspidal representations of G with odd exponent.

We adopt the terminology of chain order in the book [BH]. Let $\mathfrak{J}$ be the chain order

$$\mathfrak{J} = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F \end{pmatrix}$$

and $\mathcal{K}_\mathfrak{J}$ be the normalizer of $\mathfrak{J}$ in G. Then $U_\mathfrak{J}$ is the standard Iwahori group $I = K_0(1)$, and $\mathfrak{P}_\mathfrak{J} = \begin{pmatrix} \mathfrak{p}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}$. For $i \geq 1$, define $U_\mathfrak{J}^n = 1 + \mathfrak{P}_\mathfrak{J}^n$. [BH] 12.3 Exercise says that $\mathcal{K}_\mathfrak{J}$ is also the normalizer of $U_\mathfrak{J}$ and all $U_\mathfrak{J}^n$.

We now choose a ramified quadratic extension E of F, then choose an embedding of E into $M_2(F)$ such that $E^\times \subset \mathcal{K}_\mathfrak{J}$(Taking a G-conjugation for the chain order in [BH] 12.4

Proposition). In this case, $\mathcal{K}_{\mathfrak{J}} = E^\times U_{\mathfrak{J}}$. As before, fix a character θ of $E^\times$ of odd level $n \geq 1$, then there is $b \in \mathfrak{p}_E^{-n}$ such that

$$\theta(1+x) = \psi_E(bx), \quad \forall x \in \mathfrak{p}_E^{[n/2]+1} = \mathfrak{p}_E^{(n+1)/2}$$

Set $J = E^\times U_{\mathfrak{J}}^{(n+1)/2}$, we can define a character λ of J by

$$\lambda(y) = \theta(y) \quad \forall y \in E^\times$$
$$\lambda(1+x) = \psi_E(bx) \quad \forall 1+x \in U_{\mathfrak{J}}^{(n+1)/2}$$

Then the compactly induced representation $\pi = c - Ind_J^G \lambda$ is an irreducible smooth supercuspidal representation of G with exponent $n+2$. As explained in \$ 3.1, the representation $\rho = Ind_{J\cap K}^K(\lambda \mid_{J\cap K})$ is irreducible, appears with mutiplicity in every element of the component of π, and is a type of this component.

As before, we need to prove that the constituents of $\pi \mid_K$ other than ρ are not typical. We will classify the value of $(n+1)/2$.

Theorem 3.4.1. The constituents of $\pi \mid_K$ other than ρ are not typical.

Proof. (1) Suppose first that $(n+1)/2 \geq 2$, namely $n \geq 3$. Let μ be a character of $E^\times$ which is trivial on U_F with exponent 2. We can replace θ by $\theta' = \theta\mu$ in the previous paragraph, yielding a construction of λ', σ', π' and ρ' analogous to the previous one. Then λ and λ' have the same restriction on $U_{\mathfrak{J}}^{(n+1)/2}$. Now If π is equivalent to π', then there $g \in G$ interwines λ with λ' by [BH] 11.1 Proposition, thus g also interwines $\lambda \mid U_{\mathfrak{J}}^{(n+1)/2} = \lambda' \mid U_{\mathfrak{J}}^{(n+1)/2}$ But by [BH] 15.1 Interwining Theorem, the interwining in G of the restriction is J. Thus $g \in J$ and g conjugates λ with λ' which means $\lambda \cong \lambda'$, this is impossible.

We will prove that every constituent of $\pi \mid_K$ other than ρ appears in $\pi' \mid_K$.

Denote $\mathcal{K}_{\mathfrak{J}} = E^\times I$ by K'. Write G as a disjoint union of double cosets KgK', where

$$g = \begin{pmatrix} \varpi_F^a & 0 \\ 0 & 1 \end{pmatrix}, \quad a \geq 1,$$

with the case $a = 1$ corresponding to the class KK'. Set $\tau = \mathrm{Ind}_J^{K'}(\lambda)$ and $\tau = \mathrm{Ind}_J^{K'}(\lambda')$, we have

$$\pi|_K = \bigoplus \mathrm{Ind}_{K\cap g^{-1}K'g}^K(\tau^g) \text{ and } \pi|_K = \bigoplus \mathrm{Ind}_{K\cap g^{-1}K'g}^K(\tau')^g$$

For

$$g = \begin{pmatrix} \varpi_F^a & 0 \\ 0 & 1 \end{pmatrix} \text{ with } a \geq 2,$$

we therefore want to identify $\mathrm{Ind}_{K\cap g^{-1}K'g}^K(\tau^g)$ and $\mathrm{Ind}_{K\cap g^{-1}K'g}^K(\tau')^g$, and for that, it suffices to identify

$$\mathrm{Res}_{K\cap g^{-1}K'g}^{g^{-1}K'g} \tau^g \quad \text{and} \quad \mathrm{Res}_{K\cap g^{-1}K'g}^{g^{-1}K'g} \tau'^g.$$

As in §A.3.7, this amounts to identifying

$$\mathrm{Res}^{K'}_{gKg^{-1}\cap K'}\left(\mathrm{Ind}_J^{K'}(\lambda)\right) \quad \text{and} \quad \mathrm{Res}^{K'}_{gKg^{-1}\cap K'}\left(\mathrm{Ind}_J^{K'}(\lambda')\right).$$

Now write K' as a disjoint union of double cosets $(gKg^{-1}\cap K')hJ$. We wish to identify, for fixed h, the representations λ^h and λ'^h of $gKg^{-1}\cap K'\cap h^{-1}Jh$.

Let ϖ be a uniformizer of $h^{-1}E^\times h$, and let $j = 1+x \in I((n+1)/2)$. If $(1+\varpi)j = y \in gKg^{-1}\cap K'$, then it is a matrix of the form

$$\begin{pmatrix} \alpha & \beta \\ \varpi_F^a\gamma & \delta \end{pmatrix}, \quad \text{with } \alpha,\delta \in U_F,\ \beta,\gamma\in\mathfrak{o}_F,$$

and $\alpha\equiv\delta\equiv 1 \mod \mathfrak{p}_F$, since $y-1$ is topologically nilpotent.

But then

$$\det(y-1) = (\alpha-1)(\delta-1) - \varpi_F^a\beta\gamma$$

has valuation in F at least 2, which is impossible.

It follows that

$$gKg^{-1}\cap K'\cap hJh^{-1} \subset h(1+\mathfrak{p}_E^2)h^{-1}I((n+1)/2)\mathcal{O}_F^\times,$$

and on this group, λ^h and λ'^h coincide, hence the desired result follows.

(2) If $(n+1)/2 = 1$, namely $n=1$. Then π has conductor $\mathfrak{p}_F^3$. According to Theorem 2.2.3, the vectors of π fixed by $K(2)$ forms an irreducible representation of K.

□

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