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CLASSIFICATION OF TYPICAL OF BERNSTEIN COMPONENT FOR $GL_2(F)$

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1 Introduction

This document is a personal study note based on the appendix of the paper by Breuil and Mézard ([BM]). To illustrate it, we fix some notations.

Notation: F is a non-Archimedean local field, \mathfrak{o}_F is its valuation ring, \mathfrak{p}_F is the maximal ideal of \mathfrak{o}_F . Set $G = GL_n(F)$, $K = GL_n(\mathfrak{o}_F)$, and let $\mathfrak{R}(G)$ be the category of smooth complex representation of G , $Irr(G)$ be the category of irreducible smooth complex representation of (G) . For other groups, we define similarly.

Recall that J. Bernstein (see [Del]) gave a decomposition of the category of smooth representations of G into a product of indecomposable subcategories. Namely

Theorem 1.0.1. The category $\mathfrak{R}(G)$ decomposes as a direct product

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

To comprehend this theorem, we must clarify the notation $\mathfrak{R}^{\mathfrak{s}}(G)$ and $\mathfrak{B}(G)$, as established by the following theorem.

Theorem 1.0.2. (Jacquet) Let π be an irreducible smooth representation of G , then there exists a Levi subgroup L and an irreducible supercuspidal representation σ of L such that π is a subrepresentation of $Ind_P^G \sigma$. Here, P can be any parabolic subgroup of G with Levi component L .

Remark 1.0.3. The representation π determines the pair (L, σ) up to G -conjugacy. We refers to (L, σ) as the support of π .

Explanation of Theorem 1.1: The idea is that we imposes an equivalence relation on the set of pairs (L, σ) by deeming two such pair (L_i, σ_i) to be *inertially equivalent* if there an element $g \in G$ and an unramified character χ of L_2 (means $\chi|_{L_2 \cap K} = 1$) such that $L_2 = g^{-1}L_1g$ and $\sigma_2 \otimes \chi \cong \sigma_1^g$. One then can define the inertial support $\mathfrak{L}(\pi)$ of an irreducible representation π to be the inertial equivalence class of the support of π . If the inclusion

$$\pi \hookrightarrow Ind_P^G \sigma$$

is obvious, we also denote $\mathfrak{L}(\pi)$ by $[L, \sigma]_G$.

Given an inertial equivalence class \mathfrak{s} , one defines a full subcategory $\mathfrak{R}^{\mathfrak{s}}(G)$ of $\mathfrak{R}(G)$ by deeming that the objects of $\mathfrak{R}^{\mathfrak{s}}(G)$ are the smooth representations of G such that all of whose irreducible quotients have inertial support \mathfrak{s} . Then let \mathfrak{s} run over the set $\mathfrak{B}(G)$ of all inertial equivalence classes, we have Theorem 1.0.1.

To understand the subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$, Bushnell and Kutzko introduced the theory of types in [BK1].

The idea is to identity $\mathfrak{R}^{\mathfrak{s}}(G)$ as the category of modules over a Hecke algebra in the following way. Find a pair (K_1, ρ) where K_1 is a compact open subgroup of G and $\rho \in$

$Irr(K_1)$ such that for any $\pi \in Irr(G)$

$$\pi \in \mathfrak{R}^s(G) \Leftrightarrow \text{Hom}_{K_1}(\rho, \pi) \neq 0$$

In this case, set $\mathcal{H}(G, \rho) = \text{End}_G(\text{Ind}_{K_1}^G \rho)$ and we have an equivalence of categories

$$\mathfrak{R}^s(G) \longleftrightarrow \mathcal{H}(G, \rho) - \text{Mod}$$

This motivates the following definition..

Definition 1.0.4. (1) Given an inertial equivalence class $\mathfrak{s} \in \mathfrak{B}(G)$. A pair (K_1, ρ) where K_1 is a compact open subgroup of G and $\rho \in Irr(K_1)$ is called a **typical** for $\mathfrak{R}^s(G)$, if for any $\pi \in Irr(G)$,

$$\text{Hom}_{K_1}(\rho, \pi) \neq 0 \Rightarrow \pi \in \mathfrak{R}^s(G)$$

and there exists at least one $\pi \in \mathfrak{R}^s(G)$ such that $\text{Hom}_{K_1}(\rho, \pi) \neq 0$. And the pair is call a **type** for $\mathfrak{R}^s(G)$, if for any $\pi \in Irr(G)$,

$$\pi \in \mathfrak{R}^s(G) \Leftrightarrow \text{Hom}_{K_1}(\rho, \pi) \neq 0$$

(2) Denote the category of irreducible smooth representation of $GL_2(F)$ by $\mathcal{A}_F(2)$, then a component of $\pi \in \mathcal{A}_F(2)$ is $\mathfrak{R}^s(G)$ where s is the inertial equivalence class $\mathfrak{L}(\pi)$.

Remark 1.0.5. If s is a component for $\pi \in \mathcal{A}_F(2)$ and χ is a character of F^\times . Then ρ is a typical (resp. is a type) for s if and only if $(\chi \circ \det) \otimes \rho$ is a typical (resp. is a type) for the component of $(\chi \circ \det) \otimes \pi$. Indeed, then multiplicity of $(\chi \circ \det) \otimes \rho$ in $(\chi \circ \det \otimes \pi' |_{K_1})$ is equal to that of ρ in $\pi' |_{K_1}$ for any $\pi' \in \mathcal{A}_F(2)$.

From now on, we call $\mathfrak{R}^s(G)$ the component associated to \mathfrak{s} .

We aim to classify the typicals and types within each component for $G = GL_2(F)$.

First we need to find Levi and parabolic subgroups in $GL_2(F)$. Up to conjugacy, there is two Levi subgroups, the diagonal matrix $T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a, b \in F^\times$ and the whole group $G = GL_2(F)$. Corresponding we have two parabolic subgroups, the Borel subgroup $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in G$ and G .

Definition 1.0.6. Give $\mathfrak{s} \in \mathfrak{B}(G)$. Then we say

- (1) $\mathfrak{R}^{\mathfrak{s}}(G)$ is a supercuspidal component if \mathfrak{s} is the inertial equivalence class $[G, \pi]_G$ for some irreducible supercuspidal representation $\pi \in Irr(G)$. Notice that there are several different supercuspidal components.
- (2) $\mathfrak{R}^{\mathfrak{s}}(G)$ is the trivial component if $\mathfrak{s} = [T, 1_T]$.
- (3) $\mathfrak{R}^{\mathfrak{s}}(G)$ is a principal component if $\mathfrak{s} = [T, \chi_1 \otimes \chi_2]$ which is not inertially equivalent to $[T, 1_T]$. As before, there are several different principal components.

Proof. (1) follows by the equivalent definitions of supercuspidal. (3) is trivial. For (2), we know $St_G \cong St_G^\vee$ and two exact sequences

$$0 \rightarrow \phi_G \rightarrow Ind_B^G(\phi \cdot 1_T) \rightarrow \phi_G \otimes St_G \rightarrow 0$$

and

$$0 \rightarrow \phi_G \otimes St_G^\vee \rightarrow Ind_B^G(\phi \cdot \delta_B^{-1}) \rightarrow \phi_G \rightarrow 0$$

where $\phi_G := \phi \circ \det$. Since (T, ϕ) and $(T, \phi \cdot \delta_B^{-1})$ are inertially equivalent, the result holds. \square

The appendix of [BM] features the following main theorem:

Theorem 1.0.7. Notation as Proposition 1.5. We have

- (1) If s is a supercuspidal component, then there exists a unique (up to isomorphism) smooth irreducible representation ρ of $K = GL_2(\mathfrak{o})$ which is a type for s , and it occurs with multiplicity 1 in every element of s .
- (2) If s is the trivial component (denoted for simplicity by s_0), then up to isomorphism, there are exactly two smooth representations of $K = GL_2(\mathfrak{o})$ which are typical for s_0 . Neither of these is a type.
- (3) Finally, if s is a principal component, then up to isomorphism, there is exactly one smooth irreducible representation of $K = GL_2(\mathfrak{o})$ which is a typical for s , except when the cardinality of \mathbf{k}_F is 2, in this case, up to isomorphism, there are two smooth irreducible representations of $K = GL_2(\mathfrak{o})$ which are typical for s . Moreover, in all cases, every smooth irreducible representation of K which is a typical for s is a type for s and appears with multiplicity 1 in every element of s .

Organization : In Chapter 2, we will establish results for the principal and trivial components, following the approach in W. Casselman [Ca2].

In Chapter 3, we prove results for supercuspidal component, following by P.C.Kutzko[Ku1].

2 Principal components and the trivial component

2.1 Preliminary

In what follows, we denote by \mathfrak{o}_F the ring of integers of F , by \mathfrak{p}_F its maximal ideal, $\mathbf{k}_F = \mathfrak{o}_F/\mathfrak{p}_F$ its residue field, $q = q_F$ the cardinality of \mathbf{k}_F and $K = GL_2(\mathfrak{o}_F)$. We also denote by $U_F = U_F^0$ the group of units of \mathfrak{o}_F , which is filtered by its subgroups $U_F^i = 1 + \mathfrak{p}_F^i$ for an integer $i \geq 1$. We also fix a uniformizer ϖ_F of F and an additive character ψ of F which is trivial on \mathfrak{p}_F but non-trivial on \mathfrak{o}_F .

For each character ϵ_0 of U_F , we denote by $s(\epsilon_0)$ the component of $\mathcal{A}_F(2)$ that contains $\pi(\tilde{\epsilon}_0, 1)$ for every character $\tilde{\epsilon}_0$ of F^\times whose restriction to U_F is ϵ_0 . If $\epsilon_0 = 1$, then $s(\epsilon_0) = s_0$ is the trivial component. Every principal component of $\mathcal{A}_F(2)$ is obtained, by twisting with a character of F^\times , from a component of the form $s(\epsilon_0)$ (Remark 1.0.5). Hence it suffices to determine the typical representations for these components $s(\epsilon_0)$.

Let us fix a character ϵ_0 of U_F and denote its Artin conductor $c(\epsilon_0)$ (namely N_0 is the minimal nonnegative integer N such that $\epsilon_0 \mid U_F^N = 1$) by $\mathfrak{p}_F^{N_0}$, the integer N_0 is called the exponent of ϵ_0 . If χ_1, χ_2 are two characters of F^\times such that $\chi_1 \mid U_F = \epsilon_0$ and $\chi_2 \mid U_F = 1$, the restriction to K of the parabolic induction of $\chi_1 \otimes \chi_2$ depends only on ϵ_0 and is described, according to [Ca2] p.311, as follows. For each integer $N \geq 1$, we denote by $K(N)$ the group $1 + M_2(\mathfrak{p}_F^N)$ and we set $K(0) = K$. For each integer $N \geq 0$, we denote by $K_0(N)$ the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in K such that $c \in \mathfrak{p}_F^N$. We have $K_0(0) = K(0) = K$. For any integer $N \geq N_0$, we set $Ind_N(\epsilon_0) := Ind_{K_0(N)}^K(\epsilon)$ where ϵ is the character of $K_0(N)$ defined by $\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \epsilon_0(a)$. For $N \geq N_0 + 1$, we define by $u_N(\epsilon_0)$ the complement of $Ind_{N-1}(\epsilon_0)$ in $Ind_N(\epsilon_0)$ and set $u_{N_0}(\epsilon_0) = Ind_{N_0}(\epsilon_0)$. Then W.Casselman proved that we have

- Proposition 2.1.1** (Proposition 1 of [Ca2]). (1) $u_N(\epsilon_0)$ is irreducible for every integer $N \geq N_0$.
- (2) For $N \geq N_0 + 1$, $u_N(\epsilon_0)$ is the unique irreducible representation of K , up to isomorphism, which is trivial on $K(N)$ but not on $K(N-1)$ and satisfies $\text{Hom}_{K_0(N)}(\epsilon, u_N(\epsilon_0)) \neq 0$. Moreover $\text{Hom}_{K_0(N_0)}(\epsilon, u_{N_0}(\epsilon_0)) \neq 0$.
- (3) If $\epsilon \neq 1$, then $\dim u_{N_0}(\epsilon_0) = (q+1)q^{N_0-1}$. And for $N \geq N_0 + 1$, $\dim u_N(\epsilon_0) = (q+1)(q-1)q^{N-2}$.

Proof. (1) Notice that from [BK1], 2.5 we have an isomorphism of \mathbb{C} -algebras

$$\mathcal{H}(K, \rho) \cong \text{End}_K(c\text{-}Ind_{K_0(N)}^K \rho) \cong \text{End}_K(Ind_{K_0(N)}^K \rho)$$

for any character ρ of $K_0(N)$. Here

$$\mathcal{H}(K, \rho) = \{\phi : K \rightarrow \mathbb{C} \mid \phi(k_1 k k_2) = \rho(k_1) \phi(k) \rho(k_2), \forall k_1, k_2 \in K_0(N), k \in K\}$$

[Ca2], Lemma 1 says that set $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for any $N \geq 1$ we have following double

cosets decomposition

$$K = K_0(N)wK_0(N) \bigcup \left(\bigcup_{m=1}^N K_0(N) \begin{pmatrix} 1 & 0 \\ \pi^m \backslash \pi^{m+1} & 1 \end{pmatrix} K_0(N) \right)$$

Hence

$$\dim \text{Hom}_K(u_{N_0}(\epsilon_0), u_{N_0}(\epsilon_0)) = \dim \text{End}_K(\text{Ind}_{K_0(N)}^K \epsilon) = \dim \mathcal{H}(K, \epsilon)$$

where ϵ is the character of $K_0(N_0)$. We can take specific k_1, k_2 to prove

$$\phi(w) = \phi\left(\begin{pmatrix} 1 & 0 \\ \pi^m \backslash \pi^{m+1} & 1 \end{pmatrix}\right) = 0$$

for $m = 1, 2, \dots, N_0 - 1$, namely only non-zero possibility is $\phi(1)$. Therefore $\dim \mathcal{H}(K, \epsilon) = 1$ and $u_{N_0}(\epsilon_0)$ is irreducible.

For $N \geq N_0 + 1$, coset decomposition implies

$$\dim \text{End}_K(\text{Ind}_{K_0(N)}^K \epsilon) = \dim \text{End}_K(\text{Ind}_{K_0(N-1)}^K \epsilon) + 1$$

which means $u_N(\epsilon_0)$ is irreducible.

(2) Clearly, $u_N(\epsilon_0)$ satisfied these condition. We need to prove the uniqueness.

If $\pi \in \text{Irr}(K)$ satisfies the condition, then

$$\text{Hom}_K(\pi, \text{Ind}_{K_0(N)}^K \epsilon_0) = \text{Hom}_{K_0(N)}(\pi, \epsilon_0) \neq 0$$

We need to prove $\text{Hom}_K(\pi, \text{Ind}_{K_0(N-1)}^K \epsilon_0) = 0$. If otherwise, there exists $v \in V_\pi$ such that

$$\pi(k)v = \epsilon_0(k)v, \quad \forall k \in K_0(N-1)$$

But $N-1 \geq N_0$, so $\epsilon_0|_{K(N-1)} = 1$. Therefore

$$\pi(k)v = v$$

for all $k \in K(N-1)$. Since π is irreducible and $K(N-1)$ is normal in K , $\pi(k)v = v$ for all $v \in V_\pi$ and $k \in K(N-1)$. This is a contradiction.

(3) Notice that $[K : K_0(N)] = (q+1)q^{N-1}$ for any $N \geq 1$.

□

2.2 Trivial component

First we consider the case where $\epsilon_0 = 1$, that is $s(\epsilon_0) = s_0$. Then $u_0(1) = 1_K$ is the trivial representation of K , and $u_1(1)$ is obtained by inflation from $GL_2(\mathbf{k}_F) \cong K/K(1)$ since $u_1(1)|_{K(1)} = 1$.

We have following exact sequence

$$0 \rightarrow u_0(1) = 1_K \rightarrow \text{Ind}_I^K 1_I \rightarrow u_1(1) \rightarrow 0$$

$\dim \text{Ind}_I^K 1_I = [K : I] = q + 1$, so $\dim u_1(1) = q$ which means that $u_1(1)$ is the inflation of the Steinberg representation of $GL_2(\mathbf{k}_F)$. Here $I = K_0(1)$ is the standard Iwahori subgroup.

We have the following result for the trivial component s_0 :

Proposition 2.2.1. Notation as above, $u_0(1)$ and $u_1(1)$ are typical for s_0 . Neither of these two is a type for s_0 .

To prove this, we need the concept of conductor for an irreducible smooth admissible representation π of $G = GL_2(F)$ of infinite-dimension. Let us recall the important Theorem in [Ca1]:

Theorem 2.2.2 (Theorem 1 of [Ca1]). Let π be an irreducible admissible infinite-dimensional representation of G . Then there exist a largest ideal $\mathfrak{p}^{c(\pi)} (c(\pi) \geq 1)$ of \mathfrak{o}_F such that the space of all non-zero vectors v with

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \omega_\pi(a)v, \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{p}^{c(\pi)})$$

is non-empty. In this case, the space has dimension 1, and we call the ideal $\mathfrak{p}^{c(\pi)}$ the conductor of π .

Proof. Overall, there are three cases.

- (1) if π is supercuspidal, then $c(\pi) = -n_1$ where n_1 is the unique integer such that $C_{n_1}(1) \neq 0 (n_1 \leq -2)$ in Proposition 2.23 of [JL].
- (2) if $\pi = \text{Ind}_B^G(\delta_B^{-1/2} \chi_1 \otimes \chi_2)$ where χ_1 and χ_2 have conductor $\mathfrak{p}_F^{n_1}$ and $\mathfrak{p}_F^{n_2}$ respectively, then

$$c(\pi) = \begin{cases} 1 & \text{if } n_1 = n_2 = 0. \\ n_1 + n_2 & \text{if otherwise.} \end{cases}$$

- (3) If $\pi = \phi \circ \det \otimes \text{St}_G = \sigma(\phi\alpha^{1/2}, \phi\alpha^{-1/2})$ is the special representation where $\sigma(\phi\alpha^{1/2}, \phi\alpha^{-1/2})$ is the subrepresentation associated the invariant subspace $\mathcal{B}_s(\phi\alpha^{1/2}, \phi\alpha^{-1/2})$ ([JL] Theorem 3.3). There are two cases.

if $\phi|_{U_F} = 1$, then $c(\pi) = 1$.

if $\phi|_{U_F} \neq 1$ and has conductor $\mathfrak{p}^n (n \geq 1)$, then $c(\pi) = 2n$.

□

With this definition, W.Casselman proves following Theorem in [Ca2]:

Theorem 2.2.3 (Theorem 1 of [Ca2]). Let $\pi \in \text{Irr}(G)$ with conductor $\mathfrak{p}^{c(\pi)}$ where $c(\pi) \geq 1$, ω_π the central character of π and $\eta_0 = \omega_\pi|_{U_F}$. Then the complement in $\pi|_K$ of the space fixed by $K(c(\pi) - 1)$ is the representation $\sum_{N \geq c(\pi)} u_N(\eta_0)$.

The proofs of Theorems 2.2.2 and 2.2.3 will be deferred to the subsections 2.6 and 2.7. They are originally due to W. Casselman; For more details, we refer the reader to [Ca1] and [Ca2].

Proof of Proposition 2.2.1:

1 : 1_K is a typical for s_0 . Find an unramified character ϕ of F^\times , then $\pi = \phi \circ \det \otimes St_G \in s_0$ does not contain 1_K since $St_G^K = 0$, thus 1_K is not a type. We know $\text{Hom}_K(1_K, 1_G) \neq 0$ with $1_G \in s_0$, to prove that 1_K is a typical for s_0 , we need to check for any $\pi \in \text{Irr}(G)$,

$$\text{Hom}_K(1_K, \pi) \neq 0 \Rightarrow \pi \in s_0$$

First [BH] 14.3 proposition says that π is non-cuspidal, thus we only need to prove that if π is principal, then $\pi \in s_0$. Assume $\pi = \text{Ind}_B^G \chi_1 \otimes \chi_2$ is a principal representation. Then $\pi \hookrightarrow \text{Ind}_K^G 1_K$. Hence for all $f \in \text{Ind}_B^G \chi_1 \otimes \chi_2$, we have

$$f(bg) = (\chi_1 \otimes \chi_2)(b)f(g) = 1_K(b)f(g) = f(g) \quad \forall b \in B \cap K$$

Namely $\chi_1 \otimes \chi_2|_{B \cap K} = 1$, this means $[T, \chi_1 \otimes \chi_2]$ is inertially equivalent to $[T, 1_T]$. This implies $\pi \in s_0$.

In addition, [BH] 17.10 exercise (2) implies that 1_K appears in all principal representations and characters in s_0 .

2 : $u_1(1)$ is a typical for s_0 . We need to prove for any $\pi \in \text{Irr}(G)$,

$$\text{Hom}_K(u_1(1), \pi) \neq 0 \Rightarrow \pi \in s_0$$

If $\text{Hom}_K(u_1(1), \pi) \neq 0$, $u_1(1)$ is irreducible and $\text{Hom}_I(1_I, u_1(1)) \neq 0$ implies there exists a non-zero vector $v \in V_\pi$ such that

$$\pi(g)v = v \quad \forall g \in I \tag{1}$$

which means that $c(\pi) = 1$ if π is infinite-dimensional. Therefore [BH] 11.5 Theorem and 14.3 Proposition imply that π is not cuspidal, thus it suffices to prove that π is of the form $\text{Ind}_B^G \chi_1 \otimes \chi_2$ (χ_1, χ_2 are unramified) or $\phi \circ \det \otimes St_G$ (ϕ is unramified). if $\pi = \text{Ind}_B^G \sigma_1 \otimes \sigma_2$ is a principal representation, then $\sigma_1 \sigma_2|_{U_F} = 1$ and Theorem 2.2.2 implies σ_i ($i = 1, 2$) are unramified. Similarly, if $\pi = \phi \circ \det \otimes St_G$, π must be unramified.

Notice that $\text{Hom}_K(u_1(1), \phi \circ \det) = 0$ for any unramified character ϕ of F^\times because of the dimension. And by Theorem 2.2.2 and 2.2.3, any special representation or principal representation in s_0 has conductor \mathfrak{p} , hence contains $u_1(1)$.

As for $u_N(1)$ ($N \geq 2$), we have

Proposition 2.2.4. $u_N(1)$ is not typical for s_0 providing $N \geq 2$.

This is the direct corollary of the following Proposition.

Proposition 2.2.5. Let π be an irreducible smooth supercuspidal representation with $\ell(\pi) = 0$ (namely it contains the trivial character of $K(1)$) and such that $\omega_\pi|_{U_F} = \epsilon_0 = 1$. Then $c(\pi) = 2$ and $u_N(1)$ appears in π for $N \geq 2$.

Proof. 11.5 Theorem of [BH] guarantees the existence of π since $\pi|_Z$ can be any character. Then by 14.3 Proposition of [BH], $c(\pi) \neq 1$.

We need to prove $c(\pi) = 2$. Namely there is a non-zero vector v such that

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = v, \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{p}^2)$$

Taking $\begin{pmatrix} \varpi_F^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ -conjugation, this is equivalent to

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = v, \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{K} := \begin{pmatrix} U_F & \mathfrak{p}_F \\ \mathfrak{p}_F & U_F \end{pmatrix}$$

We have the following commutative diagram.

$$\begin{array}{ccc} K & \xrightarrow{\pi} & GL(V^{K_1}) \\ \downarrow & \nearrow \tilde{\pi} & \\ K/K_1 \cong GL_2(\mathbf{k}_F) & & \end{array}$$

and the image of \bar{K} in $GL_2(\mathbf{k}_F)$ is $T_{\mathbf{k}_F} := \begin{pmatrix} \mathbf{k}_F^\times & 0 \\ 0 & \mathbf{k}_F^\times \end{pmatrix}$. Taking an irreducible component ρ (it is cuspidal) of $\tilde{\pi}$, it is sufficient to prove

$$\text{Hom}_{T_{\mathbf{k}_F}}(\rho, 1_{T_{\mathbf{k}_F}}) \neq 0$$

But

$$\langle \chi_\rho, \chi_{1_{T_{\mathbf{k}_F}}} \rangle = \frac{1}{q-1} \sum_{a \in \mathbf{k}_F^\times} \text{tr} \rho(a) = q-1 \neq 0$$

The first equality holds because the trace vanishes on non-central elements by 6.4 Theorem of [BH], the second equality follows by $\rho(a) = \tilde{\pi}(a) = 1$ for all $a \in \mathbf{k}_F$. Hence the result holds. \square

So far, we have shown that the trivial component s_0 has two typicals : $u_0(1)$ and $u_1(1)$. And neither $u_0(1)$ or $u_1(1)$ is a type.

2.3 Principle Component: $q = 3$ non-typical case

For $\epsilon_0 \neq 1$, we need to examine whether $u_N(\epsilon_0)$ for $\epsilon_0 \neq 1$ and $N \geq N_0$ are typical or not.

The result is that

Theorem 2.3.1. For $\epsilon_0 \neq 1$, we have

- (1) If $q \geq 3$, then $u_{N_0}(\epsilon_0)$ is the unique type for $s(\epsilon_0)$. And for $N \geq N_0 + 1$, $u_N(\epsilon_0)$ is not a typical for $s(\epsilon_0)$.
- (2) If $q = 2$, then $u_{N_0}(\epsilon_0)$ and u_{N_0+1} are type for $s(\epsilon_0)$. And for $N \geq N_0 + 2$, $u_N(\epsilon_0)$ is not a typical for $s(\epsilon_0)$.

In this subsection, We will prove (1) of Theorem 2.3.1 for $N \geq N_0 + 1$. Thus, in this subsection, we assume $q \geq 3$. The first proposition is :

Proposition 2.3.2. For $\epsilon_0 \neq 1$ and $N_0 = 1$ (this case does not exist for $q = 2$), $u_N(\epsilon_0) (N \geq 2)$ is not typical for $s(\epsilon_0)$.

Proof. We employ the same approach as that used in Proposition 2.2.5. Take an irreducible smooth supercuspidal representation π with $\ell(\pi) = 0$ such that $\omega_\pi|_{U_F} = \epsilon_0$. Specifically, deem $\epsilon_0 = \epsilon_0 \otimes 1$ as a character of T_F and $T_{\mathbf{k}_F}$. We have

$$\langle \chi_\rho, \chi_{\epsilon_0 \otimes 1} \rangle = \frac{1}{q-1} \sum_{a \in \mathbf{k}_F^\times} |tr \epsilon_0(a)|^2 = (q-1) \sum_{a \in \mathbf{k}_F^\times} |\epsilon_0(a)|^2 \neq 0$$

which means $c(\pi) = 2$. Hence $u_N(\epsilon_0) (N \geq 2)$ appear in π , thus they are not typical for $s(\epsilon_0)$. \square

For $\epsilon_0 \neq 1$ and $N_0 \geq 2$. Let η be a non-trivial character of U_F with conductor \mathfrak{p} , and χ_1, χ_2 be two characters of F^\times such that $\chi_1|_{U_F} = \eta \epsilon_0$ and $\chi_2|_{U_F} = \eta^{-1}$. Then we have

Proposition 2.3.3. The principal representation $\pi(\chi_1, \chi_2)$ does not belong to the component $s(\epsilon_0)$ and its conductor is $\mathfrak{p}_F^{N_0+1}$.

Proof. By Theorem 2.2, the conductor is $\mathfrak{p}_F^{N_0+1}$. Since $\chi_1 \neq \chi_2$ and $\tilde{\epsilon}_0 \neq 1$, $\chi_1 \otimes \chi_2$ and $\tilde{\epsilon}_0 \otimes 1$ are supercuspidal representations of T . Hence the support of $\pi(\chi_1, \chi_2)$ and $\pi(\tilde{\epsilon}_0, 1)$ are $(T, \chi_1 \otimes \chi_2)$ and $(T, \tilde{\epsilon}_0 \otimes 1)$, a straightforward calculation shows that they are not inertially equivalent. Therefore $\pi(\chi_1, \chi_2)$ does not belong to $s(\epsilon_0)$. \square

Corollary 2.3.4. For $N \geq N_0 + 1$, $u_N(\epsilon_0)$ is not typical for $s(\epsilon_0)$.

Proof. Just use Theorem 2.2.3. \square

So far, we have proven that for $\epsilon_0 \neq 1$, $q = 3$, and $N \geq N_0 + 1$, $u_N(\epsilon_0)$ is not typical for $s(\epsilon_0)$.

2.4 Principle Component: $q = 2$ non-typical case

Now assume $\epsilon_0 \neq 1$, $q = 2$ (so $N_0 \geq 2$). We need to determine whether $u_N(\epsilon_0)$ is typical or not for $N \geq N_0 + 1$.

First if $N_0 \geq 3$, we can choose a character η of U_F with conductor \mathfrak{p}_F^2 and construct $\pi = \pi(\chi_1, \chi_2)$ with conductor $\mathfrak{p}_F^{N_0+2}$ as in the Proposition 2.3.3. Then we have :

Proposition 2.4.1. The representation $\pi(\chi_1, \chi_2)$ does not belong to the component $s(\epsilon_0)$, and $u_N(\epsilon_0)$ is not typical for $N \geq N_0 + 2$.

Proof. The method is the same as Proposition 2.3.3. □

If $N_0 = 2$, then ϵ_0 itself is the unique character of U_F with conductor \mathfrak{p}_F^2 , since U_F/U_F^2 has unique non-trivial character. In this case $\pi(\epsilon_0^2 = 1, \epsilon_0^{-1})$ belongs to $s(\epsilon_0)$, thus we cannot use the same trick. The solution is following.

Proposition 2.4.2. Take an unramified quadratic extension E/F , then there exists a character θ of E^\times with conductor \mathfrak{p}_E^2 which is not stable under the action of $\text{Gal}(E/F)$ and such that $\theta|_{U_F} = \epsilon_0$. In this case, the representation $\pi(\theta)$ associated to $\text{Ind}_{W_E}^{W_F}(\theta)$ by local langlands correspondence is supercuspidal with conductor \mathfrak{p}^4 . Therefore $u_N(\epsilon_0)$ is not typical for $N \geq 4$.

Hence we have prove that for $\epsilon_0 \neq 1$, $q = 2$ and $N \geq N_0 + 2$, $u_N(\epsilon_0)$ is not typical for $s(\epsilon_0)$.

2.5 Principle Component: type case

To complete the proof of Theorem 2.3.1, we need to prove

Proposition 2.5.1. For $\epsilon_0 \neq 1$ (so $N_0 \geq 1$).

- (1) for any q , $u_{N_0}(\epsilon_0)$ is a type for $s(\epsilon_0)$.
- (2) If $q = 2$, then $u_{N_0+1}(\epsilon_0)$ is also a type for $s(\epsilon_0)$.

We first prove a Lemma:

Lemma 2.5.2. Let (π, V) be an irreducible smooth representation of G in which $(u_N(\epsilon_0), W)$ appears. Then $c(\pi) \leq N$.

Proof. By Proposition 2.1.1, $\text{Hom}_{K_0(N)}(\epsilon_0, u_N(\epsilon_0)) \neq 0$. Thus there exists $0 \neq w \in W$ such that

$$(u_N(\epsilon_0))(g)w = w \quad \forall g \in \begin{pmatrix} U_F^N & \mathfrak{o}_F \\ \mathfrak{p}_F^N & U_F \end{pmatrix}$$

Take $0 \neq f \in \text{Hom}_K(u_N(\epsilon_0), \pi|_K)$, then $f(w) \neq 0$ since $u_N(\epsilon_0)$ is irreducible. Hence we have

$$\pi(g)f(w) = f(w) \quad \forall g \in \begin{pmatrix} U_F^N & \mathfrak{o}_F \\ \mathfrak{p}_F^N & U_F \end{pmatrix}$$

which means that $c(\pi) \leq N$. □

Proof for (1) of Proposition 2.5.1: we want to show that $u_{N_0}(\epsilon_0)$ does not appear in the principal series of a component which is different from $s(\epsilon_0)$, nor in supercuspidal representations or special representations. Assume $\pi \in \text{Irr}(G)$ contains $u_{N_0}(\epsilon_0)$.

if $\pi = \pi(\chi_1, \chi_2)$ is a principal series, then the sum of the exponents of χ_1 and χ_2 is at most N_0 . But we also have $\chi_1 \chi_2|_{U_F} = \epsilon_0$, which implies that χ_1 or χ_2 has exponent of at least N_0 , so χ_1 or χ_2 has exponent 0, which means that $\pi(\chi_1, \chi_2)$ belongs to the component $s(\epsilon_0)$.

If $\pi = \phi \circ \det \otimes st_G$ is a special representation. [BH] 14.4 Example says that $\dim V^I = 1$ if $(\rho, V) = St_G$. This means $St_G|_{U_F} = 1$, thus $\phi \circ \det|_{U_F} = \epsilon_0$. Assume ϕ has conductor \mathfrak{p}^n , then $n \geq 1$ since $N_0 \geq 1$. Theorem 2.2.2 implies $2n \leq N_0$, namely $n \leq 2n - 1 \leq N_0 - 1$ which means $\phi \circ \det|_{U_F^{N_0-1}} = 1$. This contradicts the fact that the conductor of ϵ_0 is \mathfrak{p}^{N_0} .

If π is supercuspidal, we have $\omega_\pi|_{U_F} = \epsilon_0$, so $c(\pi) \geq 2N_0 > N_0$. Hence by theorem 2.2.3, $u_{N_0}(\epsilon_0)$ appears in the subspace of π fixed by $K(c(\pi) - 1)$. But according to [[Ca2], Theorem 2], this space does not contain any non-zero vector fixed by $\begin{pmatrix} 1 & \mathfrak{o}_F \\ 0 & 1 \end{pmatrix}$, so it cannot contain $u_N(\epsilon_0)$ by the same argument in Lemma 2.5.2.

Since $\epsilon_0 \neq 1$, $\text{Ind}_B^G \tilde{\epsilon}_0 \otimes 1$ is irreducible. Thus any $\pi \in s(\epsilon_0)$ has conductor \mathfrak{p}^{N_0} , so by Theorem 2.2.3, $u_{N_0}(\epsilon_0)$ appears with multiplicity 1 in all elements of $s(\epsilon_0)$, hence it is a type for $s(\epsilon_0)$.

Proof for (2) of Proposition 2.5.1: If $q = 2$, then $N_0 \geq 2$. Assume $\pi \in \text{Irr}(G)$ contains $u_{N_0}(\epsilon_0)$. The proof is similar to (1).

if $\pi = \pi(\chi_1, \chi_2)$ is a principal series, then the sum of the exponents of χ_1 and χ_2 is at most $N_0 + 1$. But we also have $\chi_1 \chi_2|_{U_F} = \epsilon_0$, which implies that χ_1 or χ_2 has exponent of at least N_0 , so χ_1 or χ_2 has exponent 0 or 1. Since $q = 2$, $U_F \cong U_F^1$, thus exponent 1 cannot occur, which means that $\pi(\chi_1, \chi_2)$ belongs to the component $s(\epsilon_0)$.

If $\pi = \phi \circ \det \otimes st_G$ is a special representation. [BH] 14.4 Example says that $\dim V^I = 1$ if $(\rho, V) = St_G$. This means $St_G|_{U_F} = 1$, thus $\phi \circ \det|_{U_F} = \epsilon_0$. Assume ϕ has conductor \mathfrak{p}^n , then $n \geq 2$ since $q = 2$. Theorem 2.2 implies $2n \leq N_0 + 1$, namely $n \leq 2n - 2 \leq N_0 - 1$ which means $\phi \circ \det|_{U_F^{N_0-1}} = 1$. This contradicts the fact that the conductor of ϵ_0 is \mathfrak{p}^{N_0} .

If π is supercuspidal, we have $\omega_\pi|_{U_F} = \epsilon_0$, so $c(\pi) \geq 2N_0 > N_0 + 1$. Hence by theorem 2.2.3, $u_{N_0+1}(\epsilon_0)$ appears in the subspace of π fixed by $K(c(\pi) - 1)$. The remainder of the argument proceeds in the same manner as in (1).

2.6 The first Theorem

We start with the proof of the Theorem 2.2.2 which is the main result of [Ca1]. recall that

Theorem 2.6.1 (Theorem 1 of [Ca1]). Let π be an irreducible admissible infinite-dimensional representation of G . Then there exist a largest ideal $\mathfrak{p}^{c(\pi)}(c(\pi) \geq 1)$ of \mathfrak{o}_F such that the space of all non-zero vectors v with

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \omega_\pi(a)v, \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{p}^{c(\pi)})$$

is non-empty. In this case, the space has dimension 1, and we call the ideal $\mathfrak{p}^{c(\pi)}$ the conductor of π .

Proof. 1. Assume $\pi = \text{Ind}_B^G(\delta_B^{-1/2} \chi_1 \otimes \chi_2)$ is a principal representation. Then $\omega_\pi = \chi_1 \chi_2$. Recall that $\mathcal{B}(\chi_1, \chi_2)$ is the set of all locally constant functions f on $GL_2(F)$ such that

$$f \left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g \right) = \chi_1(a) \chi_2(b) \cdot |a/b|^{1/2} \cdot f(g)$$

for all $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in B$, $g \in GL_2(F)$. A important fact is that the restriction map $f \mapsto f|_K$ is a K -isomorphism of $\mathcal{B}(\chi_1, \chi_2)$ with the set of all functions f on K such that

$$f \left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g \right) = \chi_1(a) \chi_2(b) \cdot f(g)$$

for all $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in B \cap K$, $g \in K$.

Now notice that the space of functions we seek to determine is that of function on K satisfying

$$f \left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) = \chi_1(a) \chi_2(b) \chi_1 \chi_2(a') f(g)$$

for all $g \in K$, $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in B \cap K$, $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(\mathfrak{c})$ for an arbitrary ideal \mathfrak{c} of \mathfrak{o}_F .

Clearly, if this set is not empty, then $\chi_1|_{1+\mathfrak{c}} = \chi_2|_{1+\mathfrak{c}} = 1$, thus χ_1 and χ_2 define a character of $(\mathfrak{o}_F/\mathfrak{c})^\times \cong \mathfrak{o}_F^\times/(1+\mathfrak{c})$. On the other hand, since the principal congruence subgroup $\Gamma(\mathfrak{c}) := \{g \in K \mid g \equiv 1 \pmod{\mathfrak{c}}\}$ is normal in K and $\Gamma_0(\mathfrak{c})$. We can prove that above space is isomorphic to the set of all functions ϕ on the residue group $GL_2(\mathfrak{o}_F/\mathfrak{c})$ satisfying

$$\phi \left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \right) = \chi_1(a) \chi_2(b) \chi_1 \chi_2(a') \phi(g) \quad (2)$$

for all $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \in \bar{B}$ where \bar{B} is the image of $B \cap K$ in $GL_2(\mathfrak{o}_F/\mathfrak{c})$. Hence ϕ is completely determined on a double coset $\bar{B}g\bar{B}$. Indeed, we have the following result.

Lemma 2.6.2.

$$GL_2(\mathfrak{o}_F/\mathfrak{c}) = \bigcup_{i=0}^j \bar{B} \cdot \begin{pmatrix} 1 & 0 \\ \varpi_F^i & 1 \end{pmatrix} \cdot \bar{B}$$

where $\mathfrak{c} = (\varpi_F^j)$, $\begin{pmatrix} 1 & 0 \\ \varpi_F^0 & 1 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and this union is disjoint.

Proof. Notice that any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ with $c = \gamma \cdot \pi^i$ for some $\gamma \in U_F$ and $i > 0$ lies in $(B \cap K) \cdot \begin{pmatrix} 1 & 0 \\ \pi^i & 1 \end{pmatrix} \cdot (B \cap K)$ since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi^i & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{d}{\gamma} - \frac{b}{a}\pi^i \end{pmatrix}$$

Similarly, any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ with $c \in U_F$ lies in $(B \cap K) \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot (B \cap K) = (B \cap K) \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot (B \cap K)$ since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & -c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & \frac{bc-ad}{c} \end{pmatrix}$$

□

Therefore, we reduce to the following question: given the ideal \mathfrak{c} and two characters χ_1 and χ_2 of $(\mathfrak{o}_F/\mathfrak{c})^\times$, on which double cosets $\bar{B} \cdot \begin{pmatrix} 1 & 0 \\ \varpi_F^i & 1 \end{pmatrix} \cdot \bar{B}$ do functions ϕ satisfying the equality of (2)?

Proposition 2.6.3. There is a function ϕ on $GL_2(\mathfrak{o}_F/\mathfrak{c})$ satisfies (2) on the

coset $\bar{B} \cdot \begin{pmatrix} 1 & 0 \\ \varpi_F^i & 1 \end{pmatrix} \cdot \bar{B}$ if and only if

- (1) ϖ_F^i lies in the conductor of χ_1
- (2) $\mathfrak{c}\varpi_F^{-i}$ is contained in the conductor of χ_2 .

Hence if $c(\chi_1) = \mathfrak{p}_F^{n_1}$ and $c(\chi_2) = \mathfrak{p}_F^{n_2}$, then $\mathfrak{c} = \mathfrak{p}_F^{n_1+n_2}$ satisfies the Theorem.

Proof. (\Rightarrow) : If there exists this ϕ , then take $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \in GL_2(\mathfrak{o}_F/\mathfrak{c})$ such that

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi_F^i & 1 \end{pmatrix} \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varpi_F^i & 1 \end{pmatrix} \quad (3)$$

We must have $\chi_1(a)\chi_2(b) = \chi_1(a')\chi_2(a')$. But (3) implies

$$\begin{aligned} b &\equiv b' \pmod{\varpi_F^i} \\ a &\equiv a' \pmod{\varpi_F^i} \\ a' &\equiv b \pmod{\mathfrak{c}\varpi_F i^{-i}} \\ b - b' &\equiv a' - a \pmod{\mathfrak{c}} \end{aligned}$$

Take $b = a' = 1$, we have $\chi_1(\frac{a}{a'}) = 1$ namely $\chi_1|_{1+\mathfrak{p}_F^i} = 1$. Similarly, $\chi_2|_{1+\mathfrak{c}\pi^{-i}} = 1$.

(\Leftarrow) If these two condition holds, we define a function ϕ by

$$\phi\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi_F^i & 1 \end{pmatrix} \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \chi_1(aa')\chi_2(ba')\phi\left(\begin{pmatrix} 1 & 0 \\ \varpi_F^i & 1 \end{pmatrix}\right).$$

We only need to check it is well-define which is guaranteed by the two conditions. \square

2. If $\pi = (\chi \circ \det) \otimes St_G$ is a special representation.
3. If π is a supercuspidal representation. We first introduce a lemma.

Lemma 2.6.4. Let $\mathfrak{c} = (\varpi_F^m)$ be any proper integral ideal of \mathfrak{o}_F , χ_1 and χ_2 be characters of \mathfrak{o}_F^\times of conductors contains \mathfrak{c} . π is a representation of $GL_2(F)$, set $H = \begin{pmatrix} 0 & 1 \\ -\varpi_F^m & 0 \end{pmatrix}$. Then the following conditions on a vector v in the representation space are equivalent:

(a)

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \chi_1(a)\chi_2(d)v, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{c})$$

(b) (1)

$$\pi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} v = \chi_1(a)\chi_2(d)v, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B \cap K \quad \text{and}$$

(2)

$$\pi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} Hv = \chi_1(d)\chi_2(a)Hv, \quad \forall \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B \cap K$$

Proof. Since H normalizes $\Gamma_0(\mathfrak{c})$, (a) implies (b) is immediate. Conversely, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{c})$, then $c = \gamma\varpi_F^m$ for some $\gamma \in U_F$, and we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (ad - bc)d^{-1} & b \\ 0 & d \end{pmatrix} H^{-1} \begin{pmatrix} 1 & -d^{-1}\gamma \\ 0 & 1 \end{pmatrix} H$$

Thus if (b) holds, we have

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \pi \begin{pmatrix} (ad-bc)d^{-1} & b \\ 0 & d \end{pmatrix} v = \chi_1(a)\chi_2(d)v$$

since $(ad-bc)d^{-1} = a - bcd^{-1} \equiv a \pmod{\mathfrak{c}}$.

To continue, we shall determine the dimension of all vectors v satisfying the Theorem, not just for the particular ideal $c(\pi)$, but for any integral ideal \mathfrak{c} .

By [JL]p.117 or Lemma 3.9 of re-typeset, there are none vectors which are fixed by all of K , so we may assume \mathfrak{c} is a proper ideal. Then we apply above Lemma in the case $\chi_2 = 1$, so it remains to determine all v satisfying (b) for $\chi_1 = \omega_\pi = \epsilon$ and $\chi_2 = 1$. To do this, we need to use Kirillov model, please refer to 5.3.2 of [AS]. Embedding (π, V) to $(\xi_\psi, C(F^\times))$ where ψ is a character of F with conductor \mathfrak{o}_F . Now (b) is equivalent to there exists $f \in C(F^\times)$ such that

$$\left(\xi_\psi \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} f \right) (\alpha) = \epsilon(a)f(\alpha) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B \cap K \quad (4)$$

and

$$\left(\xi_\psi \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} Hf \right) (\alpha) = \epsilon(d)Hf(\alpha) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B \cap K \quad (5)$$

We only analyze the first equality (4), the second is similar. Apply [JL] Proposition 2.10, (4) is just

$$\epsilon(b)\psi(b^{-1}x\alpha)f(ab^{-1}\alpha) = \epsilon(a)f(\alpha)$$

Namely

$$\psi(b^{-1}x\alpha)f(u\alpha) = \epsilon(u)f(\alpha)$$

for all $a, b \in U_F, x \in \mathfrak{o}_F, \alpha \in F^\times$. Since ψ has conductor \mathfrak{o}_F , take $u = 1$ and $b^{-1}x\alpha$ such that $\psi(b^{-1}x\alpha) \neq 1$, we have $f(\alpha) = 0$ for $\alpha \notin \mathfrak{o}_F$. Thus (4) is equivalent to

$$f(u\alpha) = \epsilon(u)f(\alpha)$$

for all $u \in U_F, \alpha \in F^\times$ and $\text{supp}(f) \subset \mathfrak{o}_F$. In the language of mellin transform, this is equivalent to

$$\hat{f}_n(v) = 0 \quad \text{unless} \quad n \geq 0 \quad \text{and} \quad v = \epsilon^{-1}|_{\mathfrak{o}_F}$$

Similarly, (5) is equivalent to

$$(\hat{H}f)_n(v) = 0 \quad \text{unless} \quad n \geq 0 \quad \text{and} \quad v = 1$$

But

$$H = \begin{pmatrix} 0 & 1 \\ -\varpi_F^m & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \varpi_F^m & 0 \\ 0 & 1 \end{pmatrix}$$

Hence by [JL] Proposition 2.10,

$$\begin{aligned}\widehat{Hf}(1, t) &= C_{n_1}(1) \cdot t^{n_1} (\pi \left(\widehat{\begin{pmatrix} \varpi_F^m & 0 \\ 0 & 1 \end{pmatrix}} f \right) (\epsilon^{-1}, z_0^{-1} t^{-1}) \\ &= C_{n_1}(1) \cdot t^{n_1} z_0^m t^m f(\epsilon^{-1}, t^{-1} z_0^{-1}) \\ &= C_{n_1}(1) z_0^m f(\epsilon^{-1}, t^{-1} z_0^{-1}) t^{n_1+m}\end{aligned}$$

where n_1 is the unique integer such that $C_{n_1}(1) \neq 0$ (see [JL] Proposition 2.21 and 2.23) and $z_0 = \omega_\pi(\varpi_F)$. Hence f satisfies (4) and (5) if and only

$$\hat{f}_n(v) = 0 \quad \text{unless} \quad 0 \leq n \leq n_1 + m \quad \text{and} \quad v = \epsilon^{-1}$$

Hence if there exist such f , $n_1 + m \geq 0$. Notice n_1 is a negative integer, thus $\mathbf{c} = (\varpi_F^{-n_1})$ satisfies the Theorem. \square

\square

2.7 The second Theorem

We start the proof of the Theorem 2.2.3 which is the main result of [Ca2]. Recall that it is

Theorem 2.7.1 (Theorem 1 of [Ca2]). Let $(\pi, V) \in Irr(G)$ with conductor $\mathbf{p}^{c(\pi)}$ where $c(\pi) \geq 1$, ω_π the central character of π and $\eta_0 = \omega_\pi|_{U_F}$. Then the complement in $\pi|_K$ of the space fixed by $K(c(\pi) - 1)$ is the representation $\sum_{N \geq c(\pi)} u_N(\eta_0)$.

If π is a principles or a special representation, the proof is standard.

So I assume that π is supercuspidal. The notation about Kirillov and Mellin transform is as above, let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $c(\eta_0) = N_0$.

Lemma 2.7.2. Let π be any representation of G . Then a vector v is fixed by $K(N)$ if and only if both v and $\pi(w)v$ are fixed by $B \cap K(N)$.

Proof. This is just because that $K(N)$ is generated by $B \cap K(N)$ and $w(B \cap K(N))w^{-1}$ which follows from the fact for any $c \in \mathbf{p}^N$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ca^{-1} & a^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & (ad - bc) \end{pmatrix}$$

\square

Corollary 2.7.3. Assume $N \geq N_0$. Then the subspace of π of vectors fixed by $K(N)$ is isomorphic to the subspace of all functions $\hat{f}(v, t)$ such that

1. $\hat{f}(v, t) = 0$ unless $c(v) \leq N$
2. For each v , $\hat{f}(v, t) = \sum_n \hat{f}_n(v) t^n$ with $\hat{f}_n(v) = 0$ unless $-N \leq n \leq n_v + N$.

Proof. 88 □

Proposition 2.7.4. No supercuspidal representation contains $u_{N_0}(\eta_0)$.

Proof. Recall that if σ is any character of U_F and $x \in F$, the Gauss sum $\eta(\sigma, x)$ is defined by

$$\eta(\sigma, x) = \int_{U_F} \sigma(y) \psi(yx) dy$$

We have

- Lemma 2.7.5.** (1) If $\sigma = 1$, then $\eta(\sigma, x) = 0$ unless $x \in \mathfrak{p}^{-1}$
- (2) If $\sigma \neq 1$, then $\eta(\sigma, x) = 0$ unless $\text{ord}_{\mathfrak{p}_F}(x) = -c(\sigma)$

Proof. See [AS] § 1.5.1 proposition 121. □

Now assume a supercuspidal representation $\pi(c(\pi) = -n_1 \geq 2)$ contain $u_{N_0}(\eta_0)$, then $-n_1 = c(\pi) \leq N_0$. In the proof of Proposition 2.23 of [JL], let $v = 1$ and $v_0 = \eta_0$, we have

$$\Sigma_\sigma h(\sigma) := \Sigma_\sigma \eta(\sigma^{-1}, \varpi_F^{n_1+1}) \eta(\sigma^{-1} \eta_0^{-1}, \varpi_F^{n_1+1}) C_{2n_1+2}(\sigma) \neq 0 \quad (6)$$

If $\eta_0 = 1$, then π contains $u_0(1) = 1_K$ which is impossible. If $\eta_0 \neq 1$, then

- (1) If $\sigma = 1$, then $h(1) = 0$ since $n_1 + 1 \neq -N_0 = -c(\eta_0^{-1})$
- (2) If $\sigma^{-1} \eta_0^{-1} = 1$, then $h(\eta_0^{-1}) = 0$ since $n_1 + 1 \neq -N_0 = -c(\eta_0^{-1})$
- (3) By (1) and (2), if (6) holds, we must have $n_1 + 1 = -c(\sigma^{-1})$ and $n_1 + 1 = -c(\sigma \eta_0)$ for $\sigma \neq 1$ and $\sigma \neq \eta_0^{-1}$. Hence $N_0 = c(\eta_0) = c(\sigma^{-1} \sigma \eta_0) = -n_1 - 1$ which is a contradiction.

This completes the proof of Proposition 2.7.4. □

To prove the Theorem, we need the final lemma:

Lemma 2.7.6. ([Ca2] Lemma 4) Let v be a character of U_F .

1. If $c(v) \geq c = c(\pi)$, then $n_v = -2c(v)$
2. If $c(v) < c$, then $n_v \geq 2 - 2c$.

The proof of Theorem 2.7.1: Theorem 2.2.2 implies that $\text{Hom}_{K_0(c(\pi))}(\eta_0, \pi) \neq 0$. Thus for $N \geq c(\pi)$, π contains $u_N(\eta_0)$. If

$$V^{K(N)} = V^{K(N-1)} \bigoplus u_N(\epsilon_0)$$

holds, then the result is true. Hence We remain to prove

$$\begin{aligned} \dim V^{K(N)} &= \dim V^{K(N-1)} + \dim u_N(\eta_0) \\ &= \dim V^{K(N-1)} + (q+1)(q-1)q^{N-2} \end{aligned}$$

By corollary 2.7.3 the difference is just

$$\sum_v \#\{n \mid -N \leq n \leq n_v + N\} - \sum_v \#\{n \mid -N+1 \leq n \leq n_v + N-1\}$$

where the first sum is over v with $c(v) \leq N$, the second sum is over with $c(v) \leq N-1$. Lemma 2.7.6 implies the difference is

$$\begin{aligned} &\sum_{c(v)=N} \#\{n \mid -N \leq n \leq n_v + N\} + \sum_{c(v) \leq N-1} (\#\{n \mid -N \leq n \leq n_v + N\} \\ &\quad - \#\{n \mid -N+1 \leq n \leq n_v + N-1\}) \\ &= \#\{v \mid c(v) = N\} + 2\#\{v \mid c(v) \leq N-1\} \\ &= \#\{v \mid c(v) \leq N\} + \#\{v \mid c(v) \leq N-1\} \\ &= (q-1)q^{N-1} + (q-1)q^{N-2} \\ &= \dim u_N(\eta_0) \end{aligned}$$

3 Supercuspidal Components

In this chapter, we will explain how to identify a type corresponding to a given supercuspidal component.

3.1 General result

We know that a supercuspidal representation π of $G = GL_2(F)$ is of the form

$$\pi = c - \text{Ind}_J^G \lambda$$

where J is a open subgroup of G and compact modulo the center of G , and λ is a smooth irreducible representation of J . Denote the component $[G, \pi]_G$ by s . [BH] 15.7 Proposition proved that

- (1) $(J \cap K, \lambda^0)$ is an irreducible representation where $\lambda^0 := \lambda|_{J \cap K}$.
- (2) $g \in G$ intertwines λ^0 if and only if $g \in J$.
- (3) λ^0 appears in π with multiplicity 1 for any $\pi \in \text{Irr}(G)$ containing λ^0 .

Hence [BH] 11.4 Theorem implies $\rho := c - \text{Ind}_{J \cap K}^K \lambda^0$ is irreducible and supercuspidal, moreover, it occurs in π' with multiplicity 1 for any $\pi' \in s$.

[BK1] Proposition 5.4 proved $(J \cap K, \lambda^0)$ is a type for s , thus by Frobenius Reciprocity, (K, ρ) is a type for s and appears in π' with multiplicity 1 for any $\pi' \in s$.

3.2 Exponent 2

First, let's consider the case where s is the component of a smooth irreducible supercuspidal representation π of level zero (namely contains the trivial character of $K(1)$). In this case, $J = F^\times K$ and $\rho = \lambda^0$ is the inflation of an irreducible cuspidal representation of $K/K(1) \cong GL_2(\mathbf{k}_F)$. According to Theorem 2.2.3, the complement of ρ in $\pi|_K$ is the direct sum of $u_N(\epsilon_0)$ for $N \geq 2$ where $\epsilon_0 = \omega_\pi|_{U_F}$. Thus (K, ρ) is the only type representation for s by subsection 3.1, up to isomorphism.

3.3 Even exponent > 2

If π is a smooth irreducible supercuspidal representation of level $\ell(\pi) \geq 1$, then the exponent of its conductor is greater than 3. Thus we will first give the construction of all smooth irreducible supercuspidal representations π of G whose exponent is even and greater than 4.

To do this, choose an unramified quadratic extension E of F and an embedding of E into $M_2(F)$ such that the image of $U_E = \mathfrak{o}_E^\times$ is contained in K , we can choose this since $N(U_E) = U_F$. Notice that ψ has level 1 implies that $\psi_E := \psi \circ \text{Tr}_{E/F}$ has level 1 as a character of E^\times .

Then fix an element $b \in \mathfrak{p}_E^{-n}$ where n is a positive integer, [BH] 1.8 Proposition implies there exists a character θ of $U_E^{[n/2]+1}/U_E^{n+1}$ such that

$$\theta(1+x) = \psi_E(bx), \quad \forall x \in \mathfrak{p}_E^{[n/2]+1}$$

(1) If n is odd, we set $H = J = E^\times K((n+1)/2)$ and define a character λ of $H = J$ by

$$\begin{aligned} \lambda(y) &= \theta(y) \quad \text{for } y \in E^\times, \\ \lambda(1+x) &= \psi \circ \text{tr}_A(bx) \quad \text{for } 1+x \in K((n+1)/2) \end{aligned}$$

Here $\text{tr}_A : A := M_2(F) \rightarrow F$ is defined by $\text{tr}_A \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+d$. Then $\pi := c - \text{Ind}_J^G \lambda$ is a smooth irreducible supercuspidal representation of G of exponent $2(n+1)$. By 3.1, the representation $\rho = \text{Ind}_{J \cap K}^K(\lambda|_{J \cap K})$ is irreducible and occurs with multiplicity 1 in all elements of the component of π and is a type for this component. All smooth irreducible minimal supercuspidal representations of G of exponent a multiple of 4 are obtained by this construction.

(2) If n is even, set $H = E^\times K(n/2+1)$, $J = E^\times K(n/2)$ and $J^1 = U_E^1 K(n/2)$. We have $H \subset J$ and $J^1 \subset J$. We define a character η of H by

$$\begin{aligned} \eta(y) &= \theta(y) \quad \text{for } y \in E^\times, \\ \eta(1+x) &= \psi \circ \text{tr}_A(bx) \quad \text{for } 1+x \in K(n/2+1) \end{aligned}$$

Refers to [BH] 19.4, in this case, the representation $\pi := c - \text{Ind}_J^G \lambda$ is smooth irreducible supercuspidal of exponent $2n+2$. As before, $\rho = \text{Ind}_{J \cap K}^K(\lambda|_{J \cap K})$ is irreducible and occurs with multiplicity 1 in all elements of the component of π , and is a type for this component. All smooth minimal irreducible supercuspidal representations of G of exponent greater than 4 and congruent to 2 modulo 4 are obtained by this construction.

In above two cases, to see that the irreducible representation ρ of K is the only constituent of $\pi|_K$ that is a typical(type) for the component s of π . We need to prove that other constituents of π occur in other component which is different from the component of π .

To analyze $\pi|_K$, we decompose G as a disjoint union of double cosets $E^\times K g K$ with $g \in \left\{ \begin{pmatrix} \omega_F^a & 0 \\ 0 & 1 \end{pmatrix}, a \in \mathbb{N} \right\}$ (see [BH], 10.2). Then

$$\pi|_K = \bigoplus_g \text{Ind}_{K \cap g^{-1}Kg}^K(\rho^g)$$

where $\rho^g(x) = \rho(gxg^{-1})$ for $x \in K \cap g^{-1}Kg$. Let μ be a character of E^\times which is trivial on F^\times of exponent 1. It exists since $E^\times = U_E F^\times$ and $|\mathbf{k}_E| > 1$. Now we can do the same construction as above by replacing θ by $\theta' = \theta\mu$, which gives an irreducible representation λ' of J .

By induction from J to G , $\rho' := \text{Ind}_{J \cap K}^K(\lambda|_{J \cap K})$ is a type for the component s' of $\pi' := c - \text{Ind}_J^G \lambda'$. Notice that $s \neq s'$ since $\mu|_{U_E^1} \neq 1$. We claim that for

$$g = \begin{pmatrix} \omega_F^a & 0 \\ 0 & 1 \end{pmatrix}, a \geq 1$$

the representation ρ^g and ρ'^g are equivalent. This implies that ρ is the unique typical (type) representation for s .

Proposition 3.3.1. Notation as above, then $\rho^g \cong \rho'^g$.

Proof. We know that if $H_1 \leq H_2$ are subgroups of G , χ is a representation of H_1 , ξ is a representation of H_2 and $g \in G$, then we have

$$\text{Ind}_{g^{-1}H_1g}^{g^{-1}H_2g} \chi^g \cong (\text{Ind}_{H_1}^{H_2} \chi)^g \quad \text{and} \quad \text{Res}_{g^{-1}H_1g}^{g^{-1}H_2g} \xi^g \cong (\text{Res}_{H_1}^{H_2} \xi)^g$$

Hence

$$\rho^g = \text{Res}_{K \cap g^{-1}Kg}^{g^{-1}Kg} (\text{Ind}_{J \cap K}^K \lambda)^g \cong (\text{Res}_{K \cap gKg^{-1}}^K (\text{Ind}_{J \cap K}^K \lambda))^g$$

and

$$\rho'^g = \text{Res}_{K \cap g^{-1}Kg}^{g^{-1}Kg} (\text{Ind}_{J \cap K}^K \lambda')^g \cong (\text{Res}_{K \cap gKg^{-1}}^K (\text{Ind}_{J \cap K}^K \lambda'))^g$$

Thus we reduce to prove

$$\text{Res}_{K \cap gKg^{-1}}^K (\text{Ind}_{J \cap K}^K \lambda') \cong \text{Res}_{K \cap gKg^{-1}}^K (\text{Ind}_{J \cap K}^K \lambda)$$

We write K as a disjoint union of double cosets $(K \cap gKg^{-1})h(J \cap K)$, and we want to

identify for a fixed h , the representation λ^h and $(\lambda')^h$ of $(K \cap gKg^{-1}) \cap h^{-1}(J \cap K)h$. But by the following Lemma 3.3.2, $(K \cap gKg^{-1}) \cap h^{-1}(J \cap K)h \subset h^{-1}U_F J^1 h$, and the construction of (λ') implies $\lambda \mid U_F J^1 = (\lambda') \mid U_F J^1$. Thus the result holds. \square

Lemma 3.3.2. Notation as above, we have $(K \cap gKg^{-1}) \cap h^{-1}(J \cap K)h \subset h^{-1}U_F J^1 h$.

Proof. We first need to prove $J \cap K = U_E K(n/2)$. Assume $t = ek \in J \cap K$ where $e \in E^\times, k \in K(n/2)$, then $e = tk^{-1} \in K$. But $E^\times \cap K = U_E$ since $U_E \omega_E^k = U_E \omega_F^k$ for any $k \in \mathbb{Z}$ and $U_E \in K$.

By calculating,

$$K \cap gKg^{-1} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \mid b \in \mathfrak{p}_F \right\}$$

Take $s \in (K \cap gKg^{-1}) \cap h^{-1}(J \cap K)h$. The embedding $U_E \hookrightarrow K$ induces the embeddings $\mathbf{k}_E^\times \hookrightarrow GL_2(\mathbf{k}_F)$, let \bar{s} be the image of s in $M_2(\mathbf{k}_F)$. We have

- (1) The characteristic polynomial of \bar{s} is splitting on \mathbf{k}_F .
- (2) If set $s = h^{-1}th$ where $t = ek \in J \cap K = U_E K(n/2)$, by (1) the characteristic polynomial $f(x)$ of $\bar{t} = \bar{e}$ is splitting on \mathbf{k}_F .

We know $f(x)$ is reducible on \mathbf{k}_F if and only if $\bar{e} \in \mathbf{k}_F^\times$. Thus $\bar{e} \in \mathbf{k}_F^\times$. This means that $e \in U_F U_E^1$. Therefore $t = ek \in U_F U_E^1 K(n/2) = U_F J^1$. This implies $(K \cap gKg^{-1}) \cap h^{-1}(J \cap K)h \subset h^{-1}U_F J^1 h$. \square

3.4 Odd exponent ≥ 3

We now turn to the study of irreducible smooth supercuspidal representations of G with odd exponent.

We adopt the terminology of chain order in the book [BH]. Let \mathfrak{J} be the chain order

$$\mathfrak{J} = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F \end{pmatrix}$$

and $\mathcal{K}_{\mathfrak{J}}$ be the normalizer of \mathfrak{J} in G . Then $U_{\mathfrak{J}}$ is the standard Iwahori group $I = K_0(1)$, and $\mathfrak{P}_{\mathfrak{J}} = \begin{pmatrix} \mathfrak{p}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}$. For $i \geq 1$, define $U_{\mathfrak{J}}^n = 1 + \mathfrak{P}_{\mathfrak{J}}^n$. [BH] 12.3 Exercise says that $\mathcal{K}_{\mathfrak{J}}$ is also the normalizer of $U_{\mathfrak{J}}$ and all $U_{\mathfrak{J}}^n$.

We now choose a ramified quadratic extension E of F , then choose an embedding of E into $M_2(F)$ such that $E^\times \subset \mathcal{K}_{\mathfrak{J}}$ (Taking a G -conjugation for the chain order in [BH] 12.4 Proposition). In this case, $\mathcal{K}_{\mathfrak{J}} = E^\times U_{\mathfrak{J}}$. As before, fix a character θ of E^\times of odd level $n \geq 1$, then there is $b \in \mathfrak{p}_E^{-n}$ such that

$$\theta(1+x) = \psi_E(bx), \quad \forall x \in \mathfrak{p}_E^{[n/2]+1} = \mathfrak{p}_E^{(n+1)/2}$$

Set $J = E^\times U_3^{(n+1)/2}$, we can define a character λ of J by

$$\begin{aligned}\lambda(y) &= \theta(y) \quad \forall y \in E^\times \\ \lambda(1+x) &= \psi_E(bx) \quad \forall 1+x \in U_3^{(n+1)/2}\end{aligned}$$

Then the compactly induced representation $\pi = c - \text{Ind}_J^G \lambda$ is an irreducible smooth supercuspidal representation of G with exponent $n+2$. As explained in § 3.1, the representation $\rho = \text{Ind}_{J \cap K}^K (\lambda|_{J \cap K})$ is irreducible, appears with multiplicity in every element of the component of π , and is a type of this component.

As before, we need to prove that the constituents of $\pi|_K$ other than ρ are not typical. We will classify the value of $(n+1)/2$.

Theorem 3.4.1. The constituents of $\pi|_K$ other than ρ are not typical.

Proof. (1) Suppose first that $(n+1)/2 \geq 2$, namely $n \geq 3$. Let μ be a character of E^\times which is trivial on U_F with exponent 2. We can replace θ by $\theta' = \theta\mu$ in the previous paragraph, yielding a construction of λ', σ', π' and ρ' analogous to the previous one. Then λ and λ' have the same restriction on $U_3^{(n+1)/2}$. Now If π is equivalent to π' , then there $g \in G$ intertwines λ with λ' by [BH] 11.1 Proposition, thus g also intertwines $\lambda|_{U_3^{(n+1)/2}}$ with $\lambda'|_{U_3^{(n+1)/2}}$. But by [BH] 15.1 Interwining Theorem, the interwining in G of the restriction is J . Thus $g \in J$ and g conjugates λ with λ' which means $\lambda \cong \lambda'$, this is impossible.

We will prove that every constituent of $\pi|_K$ other than ρ appears in $\pi'|_K$.

Denote $\mathcal{K}_3 = E^\times I$ by K' . Write G as a disjoint union of double cosets KgK' , where

$$g = \begin{pmatrix} \varpi_F^a & 0 \\ 0 & 1 \end{pmatrix}, \quad a \geq 1,$$

with the case $a = 1$ corresponding to the class KK' . Set $\tau = \text{Ind}_J^{K'}(\lambda)$ and $\tau' = \text{Ind}_J^{K'}(\lambda')$, we have

$$\pi|_K = \bigoplus \text{Ind}_{K \cap g^{-1}K'g}^K(\tau^g) \quad \text{and} \quad \pi|_K = \bigoplus \text{Ind}_{K \cap g^{-1}K'g}^K(\tau')^g$$

For

$$g = \begin{pmatrix} \varpi_F^a & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with } a \geq 2,$$

we therefore want to identify $\text{Ind}_{K \cap g^{-1}K'g}^K(\tau^g)$ and $\text{Ind}_{K \cap g^{-1}K'g}^K(\tau')^g$, and for that, it suffices to identify

$$\text{Res}_{K \cap g^{-1}K'g}^{g^{-1}K'g} \tau^g \quad \text{and} \quad \text{Res}_{K \cap g^{-1}K'g}^{g^{-1}K'g} \tau'^g.$$

As in §A.3.7, this amounts to identifying

$$\text{Res}_{gKg^{-1} \cap K'}^{K'} \left(\text{Ind}_J^{K'}(\lambda) \right) \quad \text{and} \quad \text{Res}_{gKg^{-1} \cap K'}^{K'} \left(\text{Ind}_J^{K'}(\lambda') \right).$$

Now write K' as a disjoint union of double cosets $(gKg^{-1} \cap K')hJ$. We wish to identify, for fixed h , the representations λ^h and λ'^h of $gKg^{-1} \cap K' \cap h^{-1}Jh$.

Let ϖ be a uniformizer of $h^{-1}E^\times h$, and let $j = 1 + x \in I((n+1)/2)$. If $(1 + \varpi)j = y \in gKg^{-1} \cap K'$, then it is a matrix of the form

$$\begin{pmatrix} \alpha & \beta \\ \varpi_F^a \gamma & \delta \end{pmatrix}, \quad \text{with } \alpha, \delta \in U_F, \beta, \gamma \in \mathfrak{o}_F,$$

and $\alpha \equiv \delta \equiv 1 \pmod{\mathfrak{p}_F}$, since $y - 1$ is topologically nilpotent.

But then

$$\det(y - 1) = (\alpha - 1)(\delta - 1) - \varpi_F^a \beta \gamma$$

has valuation in F at least 2, which is impossible.

It follows that

$$gKg^{-1} \cap K' \cap hJh^{-1} \subset h(1 + \mathfrak{p}_E^2)h^{-1}I((n+1)/2)\mathcal{O}_F^\times,$$

and on this group, λ^h and λ'^h coincide, hence the desired result follows.

- (2) If $(n+1)/2 = 1$, namely $n = 1$. Then π has conductor \mathfrak{p}_F^3 . According to [Ca2] Theorem 3, the vectors of π fixed by $K(2)$ form an irreducible representation of K . Since $J \cap K$ contains $K(2)$, this representation is just ρ . Hence by Theorem 2.2.3

$$\pi|_K \cong \rho \bigoplus_{N \geq 3} u_N(\epsilon_0)$$

where $\epsilon_0 := \omega_\pi|_{U_F}$. This means ρ is the unique typical for s .

□

All in all, if s is a supercuspidal component, then it has a unique type.

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