# Billiards, Surfaces and Continued Fractions

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## The problems in this text

This style of problems is based on the curriculum at Phillips Exeter Academy, a private high school in Exeter, NH. Diana Davis wrote these problems specifically for this course, adapted from her *Billiards, Surfaces, and Geometry* course at Williams College in Fall 2016. Anyone is welcome to use this text, and these problems, so long as you do not sell the result for profit. If you create your own text using these problems, please give appropriate attribution, as I am doing here.

## About the course

This is a course for mostly high school mathematics teachers. This course met for two hours a day for five days. The first hour was spent discussing the homework problems from this text, and the second hour was spent discussing issues of how to teach a discussion-based math course and working at the board in groups on the next night's problems.

## To the Student

**Contents**: As you work through this book, you will discover that various topics about geometry, surfaces and billiards have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on ellipses. The curriculum is problem-centered, rather than topic-centered. Techniques, definitions and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records — there are no boxes containing important theorems. You should keep a growing list of important terms in the *Reference* at the end of this book.

**Your homework**: Each page of this book contains the homework assignment for one night. The first day of class, we will work on the problems on page 1, and your homework is page 2; on the second day of class, we will discuss the problems on page 2, and your homework will be page 3, and so on. Problems with a star * are optional. You should plan to spend at least 30 minutes each night solving problems for this class.

**Comments on problem-solving**: You should approach each problem as an exploration. Reading each question carefully is essential, especially since definitions, highlighted in italics, are routinely inserted into the problem texts. It is important to make large, clear, accurate diagrams, and paper models, whenever appropriate. Useful strategies to keep in mind are: create an easier problem, work backwards, and recall a similar problem. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day.

Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

## A note on class discussion

Please be patient with me, and with your classmates, and most of all with yourself, as everyone adapts to working and learning in this method. I have carefully constructed this set of problems, thinking hard about each problem and how they all connect and build the ideas, step by step. Often you will see the connections. Sometimes you won't, but a classmate will, and will explain it to you. Occasionally, everyone will miss the connection and I will have to rewrite future problems to reflect that. This is the first time any class has used these problems, so there are bound to be a few errors, of omission or of overexplanation.

You might wonder, what is my job as your teacher? Part of my job is to give you good problems to think about, which are in this book. During class, my job is to help you learn to talk about math with each other, and help you build a set of problem-solving strategies. At the beginning, I will give you lots of pointers, and as you improve your skills I won't need to help as much.

One way of describing this method is "the student bears the laboring oar." This is a metaphor: You are rowing the boat; you are not merely along for the ride. You do the work, and in this way you do the learning. The next page gives some ideas for ways that you can do this work of moving the "boat," which is our class and your learning, forward.

Just remember that we are all in this together. Our goal is for each student to learn the ideas and skills of geometry, surfaces and billiards, really learn them — and along the way I will learn new things, too. That's the beauty of this teaching and learning method, that it recognizes the humanity in each of us, and allows us to communicate authentically, person to person.

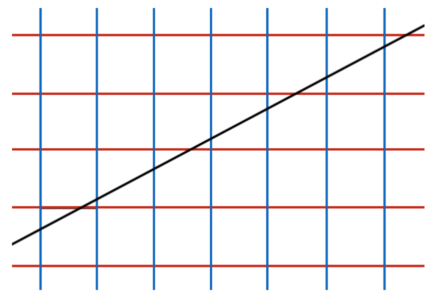
A college course should teach more than the curriculum: students should also learn something about how to be good people and good citizens. Many math courses, in addition to their mathematical content, teach the values of hard work and perseverance. In this course, through talking with your classmates about math, and struggling together to solve hard problems, you will learn that each person has something to contribute, and that the solution may come from the person you thought was least likely. This is a good life lesson.

## Discussion Skills

1. Contribute to the class every day
2. Speak to classmates, not to the instructor
3. Put up a difficult problem, even if not correct
4. Use other students' names
5. Ask questions
6. Answer other students' questions
7. Suggest an alternate solution method
8. Draw a picture or build a model
9. Connect to a similar problem
10. Summarize the discussion of a problem

# Billiards, Surfaces and Continued Fractions

**1.** Draw a line on an infinite square grid, and record each time the line crosses a horizontal or vertical edge. We will assume that the direction of travel along a line is always left to right. We could record the line to the right with the sequence $\ldots {\color{blue}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{blue}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{blue}\bullet}{\color{red}\bullet} \ldots$, or we could assign $A$ to horizontal and $B$ to vertical edges, and record it as $\ldots BABBABBABBA \ldots$.



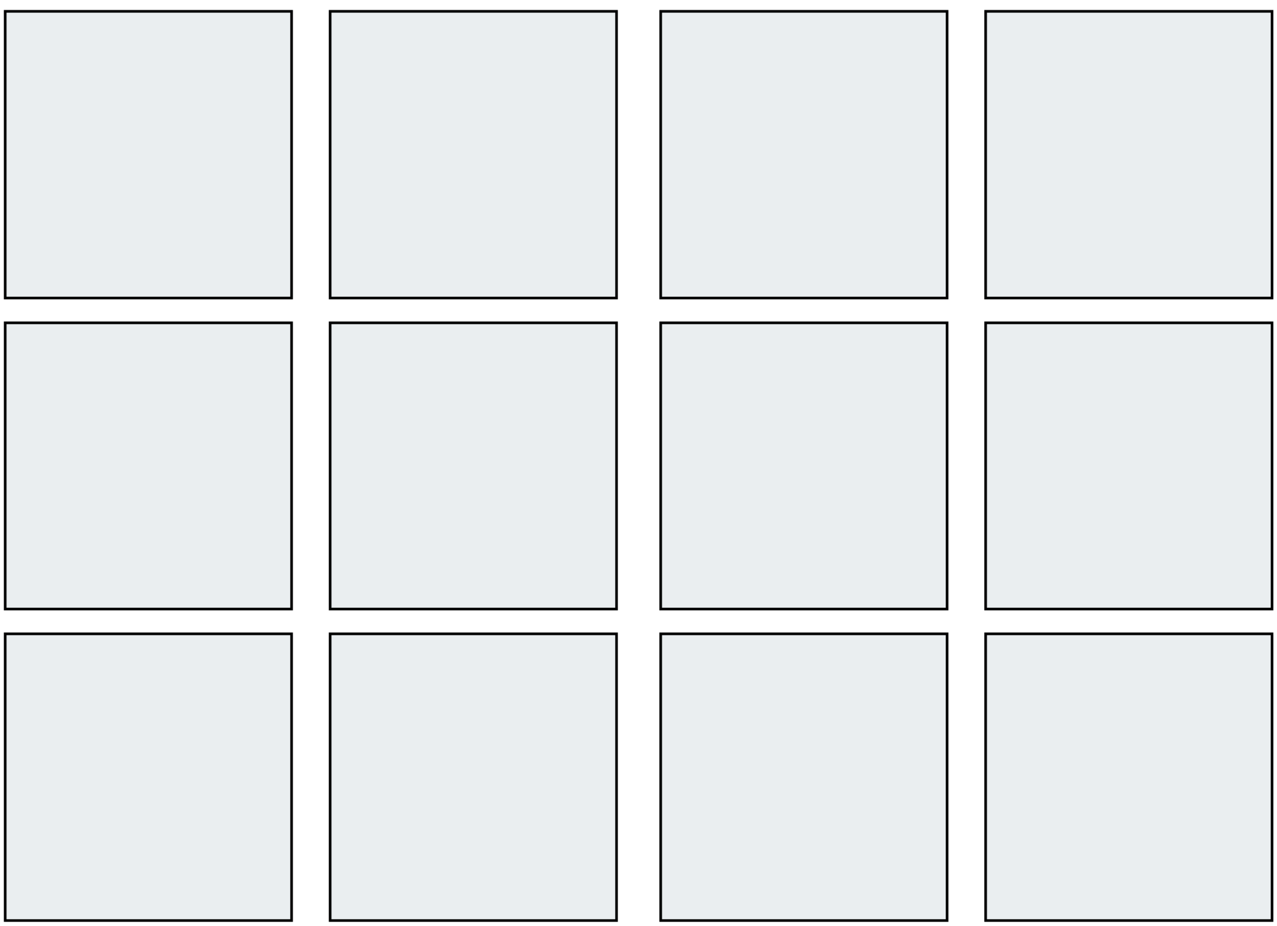
**(a)** What is the slope of the line in the picture?

**(b)** Record this *cutting sequence* of colors, or of $A$s and $B$s, for lines of slope 1, 2, and another number of your choice. Describe any patterns you notice. What can you predict about the cutting sequence, from the line? *Hint:* don't let a line hit a vertex.

**2.** Consider a ball bouncing around inside a square billiard table. We'll assume that the table has no "pockets" (it's a billiard table, not a pool table!), that the ball is just a point, and that when it hits a wall, it reflects off and the angle of incidence equals the angle of reflection, as in real life.

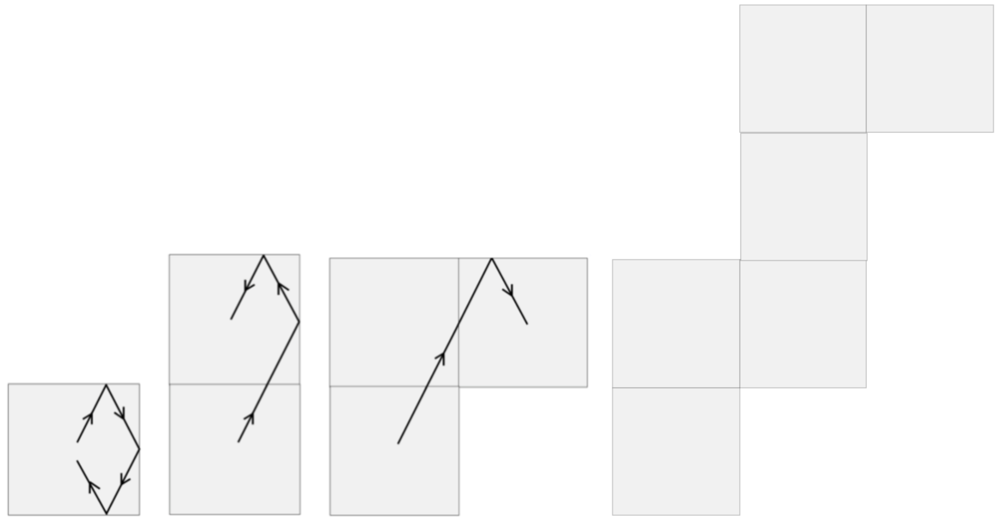
**(a)** A billiard path is called *periodic* if it repeats, and the *period* is the number of bounces before repeating. Construct a periodic billiard path of period 2.

**(b)** For which other periods can you construct periodic paths?



# Billiards, Surfaces and Continued Fractions

**1.** A powerful tool for understanding inner billiards is *unfolding* a trajectory into a straight line, by creating a new copy of the billiard table each time the ball hits an edge. Two steps of the unfolding process are shown for a small piece of trajectory of slope $\pm 2$ in the square.



**(a)** Draw some more steps of the unfolding.

**(b)** Draw the complete billiard path in the square (keep going until it closes up).

**(b)** Use the unfolding to explain why a trajectory with slope 2 yields a periodic billiard trajectory on the square. (We always assume that one edge is horizontal.)

**(c)** Which other slopes yield a periodic billiard trajectory?

**2.** The *continued fraction expansion* gives an expanded expression of a given number. To obtain the continued fraction expansion for a number, say 15/11, we do the following:

$$\frac{15}{11} = 1 + \frac{4}{11} = 1 + \frac{1}{11/4} = 1 + \frac{1}{1 + 7/4} = 1 + \frac{1}{2 + 3/4} = 1 + \cfrac{1}{2 + \cfrac{1}{4/3}} = \mathbf{1} + \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{3}}}}.$$

The idea is to pull off 1s until the number is less than 1, and then take the reciprocal of what is left, and repeat until the reciprocal is a whole number.

Find the continued fraction expansion of 5/7, showing all your steps as above.

**3.** Show that the cutting sequence corresponding to a line of slope 1/2 on the square grid is periodic. Which other slopes yield periodic cutting sequences? What can you say about the period, from the slope?

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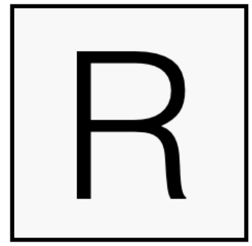
**4*.** Suppose 100 ants are on a log 1 meter long, each moving either to the left or right with unit speed. Assume the ants collide elastically (when they hit each other, each ant immediately turns around and goes the other way), and that when they reach the end of the log, they fall off. What is the longest possible waiting time until all the ants are off the log?

# Billiards, Surfaces and Continued Fractions

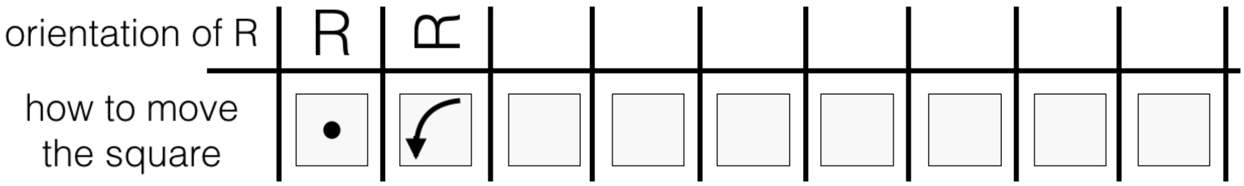
**1.** *Symmetries of the square.* If you turn a square 90° counter-clockwise, it looks the same as before. We call a 90° counter-clockwise rotation a *symmetry* of the square, because after you do it, you have a square just like the original. Let's find all the symmetries of the square.



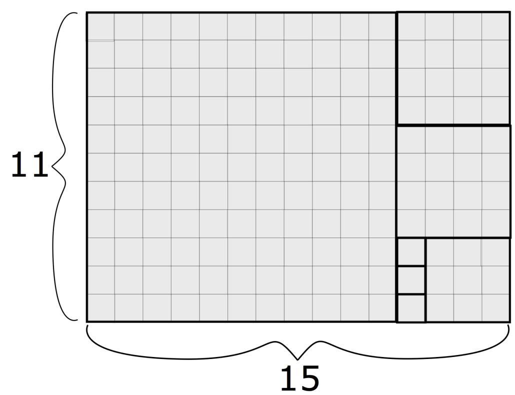

**(a)** Cut out a square and draw an **R** on one side, as shown, and also hold it up to the light and trace through a backwards **R** on the back.

**(b)** How many different symmetries of the square can you find? Record in the first line of the table below the appearance of the **R** for each one.

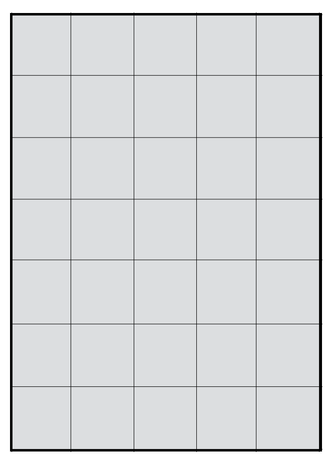
**(c)** In the second line of the table, indicate how to move the square to achieve that position.




**2.** Geometrically, way to find the continued fraction expansion of a number $x$ is:




1. Begin with a $1\times x$ rectangle (or $p\times q$ if $x = p/q$).
2. Cut off the largest possible square, as many times as possible. Count how many squares you cut off. If $x > 1$, this number is outside of the fraction; otherwise, it is inside.
3. With the remaining rectangle, cut off the largest possible squares; the number of these is the next number in the continued fraction (refer to Page 2 # 2 to remember how it works).
4. Continue until there is no remaining rectangle. These numbers are the (bold) integers in the denominators of your continued fraction!

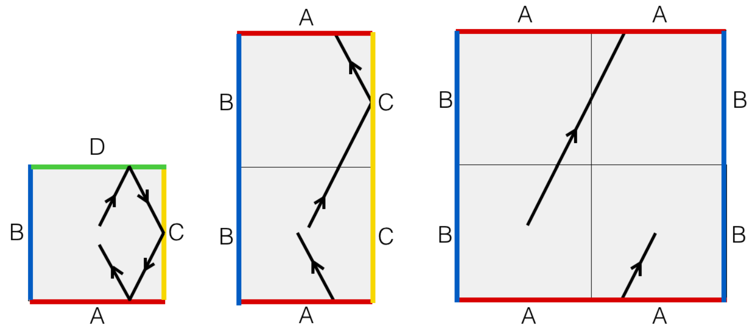


**(a)** Check that the geometric method shown in the $15 \times 11$ figure above agrees with the example in Page 2 # 2.

**(a)** Use the geometric method to compute the continued fraction expansion for 5/7.

**(b)** Explain why this geometric method is equivalent to the fraction method previously explained, for determining the continued fraction expansion.

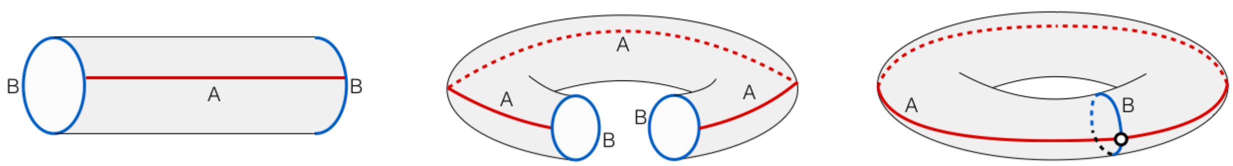
A VERY FUN THIRD PROBLEM IS ON THE NEXT PAGE!




**3.** Here's another way that we can unfold the square billiard table. First, unfold across the top edge of the table, creating another copy in which the ball keeps going straight. The new top edge is just a copy of the bottom edge, so we now label them both $A$ to remember that they are the same. Similarly, we can unfold across the right edge of the table, creating another copy of the unfolded table. The new right edge is a copy of the left edge, so we now label them both $B$. When the trajectory hits the top edge $A$, it reappears in the same place on the bottom edge $A$ and keeps going. Similarly, when the trajectory hits the right edge $B$, it reappears on the left edge $B$.

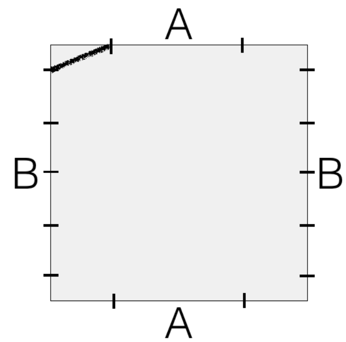
**(a)** Label the top and bottom edges of a sheet of paper $A$, and the left and right edges $B$, and tape the identified edges together to create a surface. What does this surface look like?

**(b)** Explain why, if the paper were very stretchy, the instructions in **(a)** would create the steps in the figure below. The result is called a *torus*, the surface of a donut.




**(c)** The partial billiard trajectory shown on the left part of the top figure repeats after 6 bounces. Sketch in the rest of the trajectory in each of the three square pictures above. What is its corresponding *cutting sequence* for the surface on the right part of the figure?

# Billiards, Surfaces and Continued Fractions



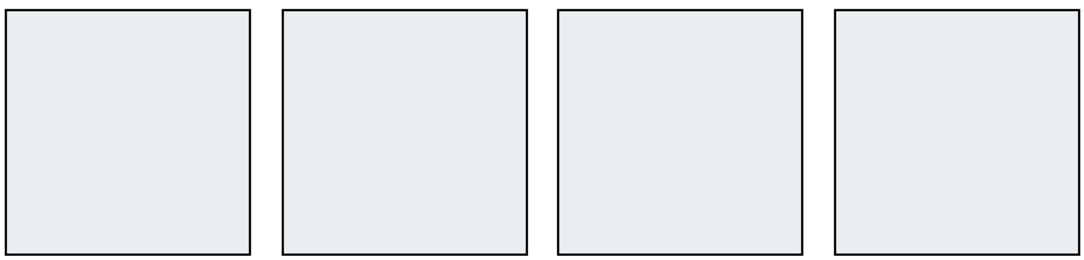

**1.** In Page 3 # 3, we ended up with a trajectory of slope 2 on the *square torus* surface. The picture to the right shows some scratchwork for drawing a trajectory of slope 2/5 on the square torus. Starting at the top-left corner, connect the top mark on the left edge to the left-most mark on the top edge with a line segment, as shown. Then connect the other six pairs with parallel segments, down to the bottom-right corner.

**(a)** Explain why, on the torus surface, these line segments connect up to form a continuous trajectory. Follow the trajectory along, and write down the corresponding cutting sequence of $A$s and $B$s.

**(b)** Exactly where should you place the tick marks so that all of the segments have the same slope? Create an accurate picture for a trajectory of slope 1/2 and then 3/2.

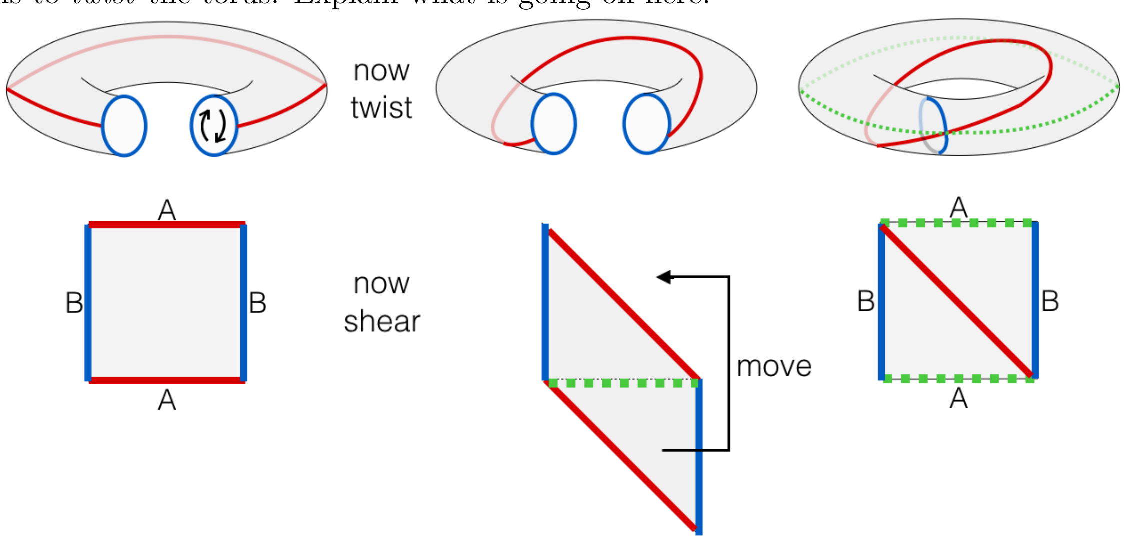
**(c)** Could you draw a trajectory of any other slope, using the same tick marks?

**(d)** Draw a picture of a *billiard* trajectory with slope ±2/5.



**2.** You showed in Page 3 # 1 that the square has two types of symmetries: four *rotations* by multiples of 90°, and four *reflections*. It turns out that the square *torus* has all of these, plus one more type that is not a symmetry of the square: a *shear*.

The shear is shown below on the square torus on and on the torus surface, where its effect is to *twist* the torus. Explain what is going on here.



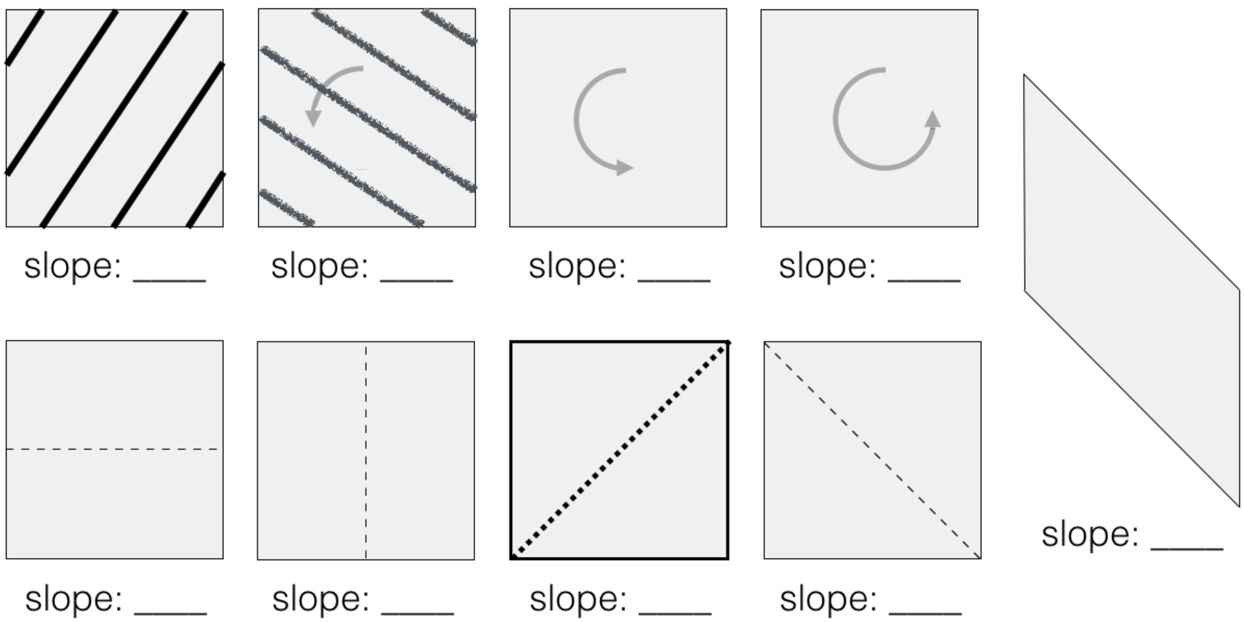

**3.** Find the continued fraction expansions of 3/2 and 5/3 (and, optional: 8/5, 13/8). Describe any patterns you notice. Can you explain the reasons for the patterns you found?

**4*.** We have identified the top and bottom edges, and the left and right edges, of a square to obtain a surface: the square torus. If we identify opposite parallel edges of a parallelogram, what surface do we get?

**5*.** In problem 1, we put 2 marks on edge $A$ and 5 marks on edge $B$ and connected up the marks to create a trajectory with slope 2/5. What if we did the same procedure with 4 marks on edge $A$ and 10 marks on edge $B$?

# Billiards, Surfaces and Continued Fractions

**1.** Given a trajectory on the square torus, we want to know what happens to that trajectory if we apply a symmetry of the surface. To do this, we can sketch the trajectory before and after applying the symmetry. Do so below for each of the eight symmetries of the square, as indicated by the curved arrow or the reflection line, and for the shear. I've done one for you.




The flip across the positive diagonal is in bold because we will use it later.

**2.** (Continuation) For each symmetry above, make a guess about what it does to a starting slope of the form $p/q$. Can you prove your answer correct?

**3.** Express the number $1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}}$ as a fraction $p/q$.

______________________________.

**4*.** Prove that a trajectory of slope $p/q$ in lowest terms:

**(a)** on the square torus is periodic with period $p + q$,

**(b)** on the square billiard table is periodic with period $2(p + q)$.

**5*.** Find the continued fraction expansion of $\sqrt{2} - 1$. (Use a calculator to deal with the decimals.) Then solve the equation $x = \dfrac{1}{2 + x}$ and explain how these are related.

**6*.** Give an example of a sequence of $A$s and $B$s that is *not* a cutting sequence for any trajectory on the square torus.

## Billiards, Surfaces and Continued Fractions

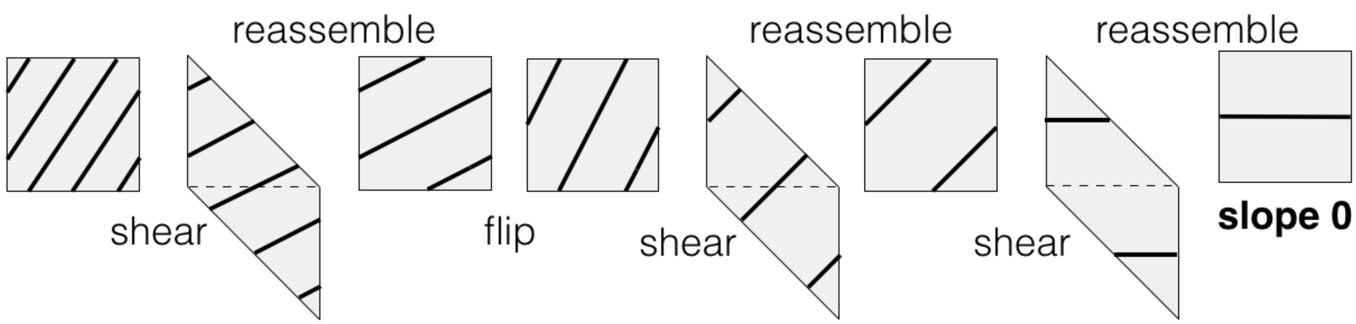
**1.** Starting with a trajectory on the square torus with positive slope, apply the following *algorithm*:

(1) If the slope is greater than or equal to 1, apply the shear.

(2) If the slope is between 0 and 1, apply the flip across the positive diagonal.

(3) If the slope is 0, stop.

An example is shown below.




We can note down the steps we took: shear, flip, shear, shear. We ended with a slope of 0. Work backwards, using this information and your work in Page 5 # 2, to determine the slope of the initial trajectory. Keep track of each step.

*Hint*: The shear decreases the slope by 1; the flip turns a slope into its reciprocal.

**2.** Explain the connection between the preceding problem and continued fraction expansions.

# Reference

**algorithm**: A deterministic recipe for carrying out a task. [6]

**continued fraction expansion**: A way of expressing a real number with a nested fraction. [2]

**cutting sequence**: The sequence of edges that a trajectory crosses, or that a ball hits. [1]

**period**: The number of edge crossings or bounces before a periodic trajectory repeats. [1]

**periodic**: A trajectory is periodic if it repeats. [1]

**reflection**: An orientation-reversing symmetry. [4]

**rotation**: An orientation-preserving symmetry. [4]

**shear**: A symmetry of the square torus that is a twist on the torus surface. [4]

**square torus**: A flat square representation of the torus surface. [3]

**symmetry**: A rigid motion that preserves a shape. [3]

**torus**: A two-dimensional surface that lives in three-dimensional space, and is the surface of a donut. [3]

**unfolding**: A strategy for transforming a billiard trajectory on a polygonal billiard table into a straight-line trajectory on a surface. [2]