This book is for anyone who wishes to illustrate their mathematical ideas, which in our experience means everyone. It is organized by *material*, rather than by subject area, and purposefully emphasizes the *process* of creating things, including discussions of failures that occurred along the way. As a result, the reader can learn from the experiences of those who came before, and will be inspired to create their own illustrations.

Topics illustrated within include prime numbers, fractals, the Klein bottle, Borromean rings, tilings, space-filling curves, knot theory, billiards, complex dynamics, algebraic surfaces, groups and prime ideals, the Riemann zeta function, quadratic fields, hyperbolic space, and hyperbolic 3-manifolds. Everyone who opens this book should find a type of mathematics with which they identify.

As a bonus, each contributor explains the mathematics behind their illustration at an accessible level, so that all readers can appreciate the beauty of both the object itself and the mathematics behind it.

ISBN 978-1-4704-6122-5

9 781470 461225

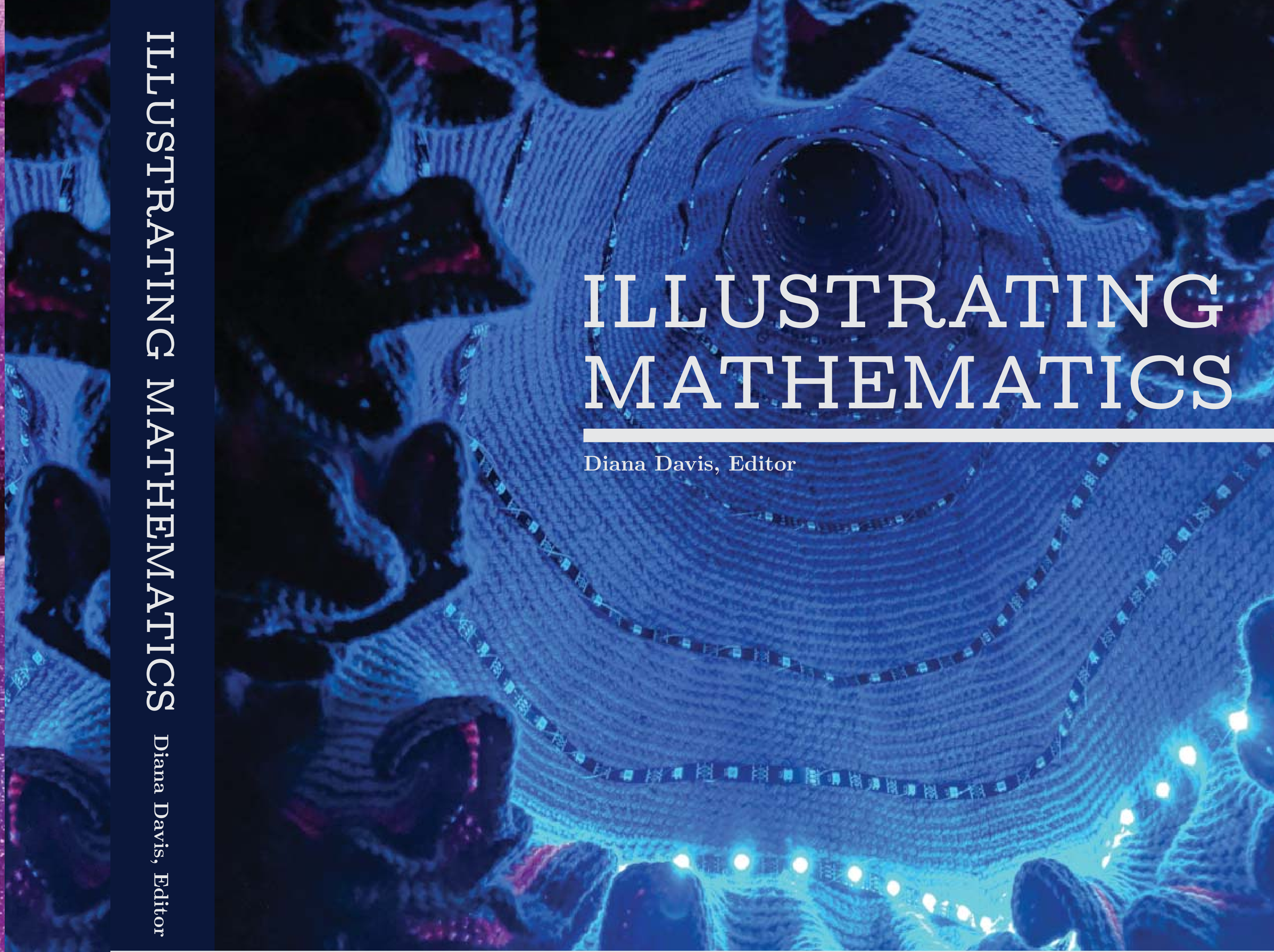
MBK/135

For additional information and updates on this book, visit **www.ams.org/bookpages/mbk-135**

AMS
www.ams.org

ILLUSTRATING MATHEMATICS Diana Davis, Editor

AMS



ILLUSTRATING MATHEMATICS

Diana Davis, Editor

AMS
AMERICAN MATHEMATICAL SOCIETY

ILLUSTRATING MATHEMATICS

Diana Davis, Editor

AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island

Designed by Margo Angelopoulos

2010 Mathematics Subject Classification. Primary 00A09, 00A06, 97U60, 97U80, 00B20.

For additional information and updates on this book, visit
www.ams.org/bookpages/mbk-135

Library of Congress Cataloging-in-Publication Data
Cataloging-in-Publication Data has been applied for by the AMS.
See http://www.loc.gov/publish/cip/.

Visit the AMS home page at https://www.ams.org/
10 9 8 7 6 5 4 3 2 1 25 24 23 22 21 20

CONTENTS

INTRODUCTION

It is said that an interpreter – a person who translates from one language to another – does not exchange words for words, but rather exchanges meaning for meaning. As mathematicians, a large part of our job is to explain things to others. We may use words and symbols to do this, but our job is not primarily to convey words and symbols; like an interpreter's, it is to convey *meaning*.

There are many ways to communicate meaning. As the great mathematician William Thurston said, "Mathematics is an art of human understanding. Mathematical concepts are abstract, so it ends up that there are many different ways that they can sit in our brains. A given mathematical concept might be primarily a symbolic equation, a picture, a rhythmic pattern, a short movie – or best of all, an integrated combination of several different representations."

The purpose of this book, then, is to help you find a good representation of the mathematical concept you wish to illustrate. This book does three things:

- Showcases the great variety of materials for illustrating mathematics,
- Gives voice to people's stories about illustrating their mathematics, so that we can learn from their experience, and
- Shows the variety of ways that different people use the same materials in very different ways.

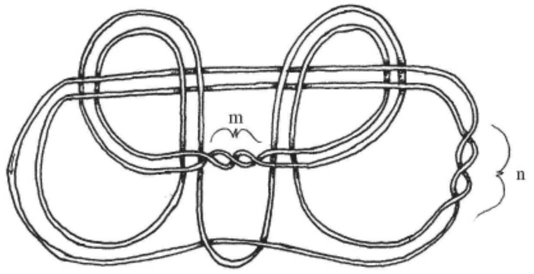
In addition, it will introduce you to many of the amazing people who spent time at the Institute for Computational and Experimental Research Mathematics (ICERM) in fall 2019 for the Illustrating Mathematics program, in an attempt to capture some of the creative and generous spirit that flowed through our days there.

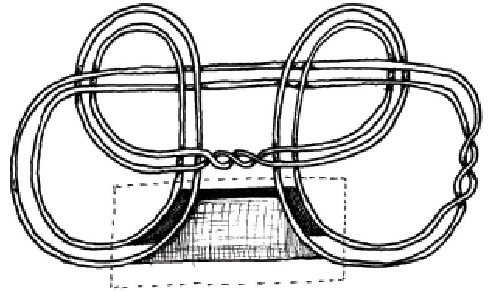
Diana Davis
May 1, 2020
Bures-sur-Yvette, France

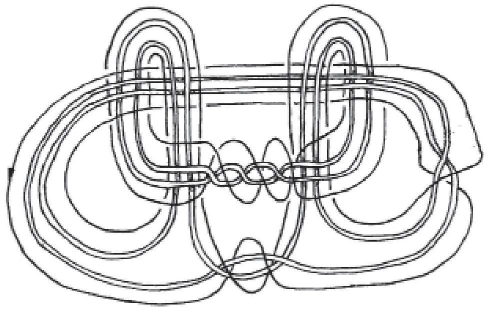
Illustrating Mathematics semester program participants, ICERM, Fall 2019.

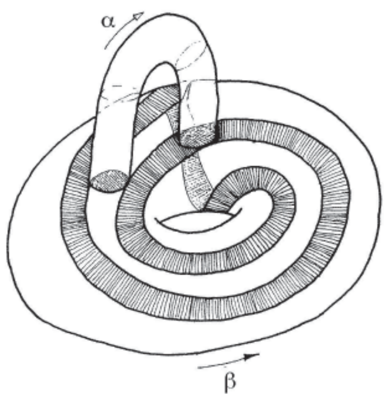
DRAWINGS

The adage that a picture is worth a thousand words is certainly true in mathematics, in which one carefully chosen figure can eliminate the need for many lines of exposition. For most of us, our most frequent way of illustrating mathematics is by hand – on chalkboards, scrap paper, paper napkins or, increasingly, whiteboards and electronic tablets.

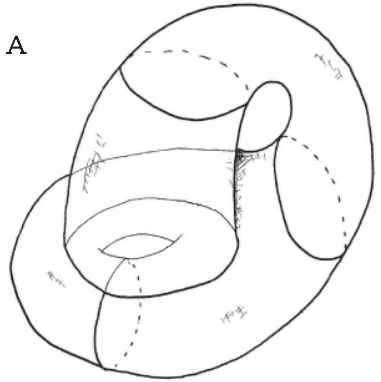




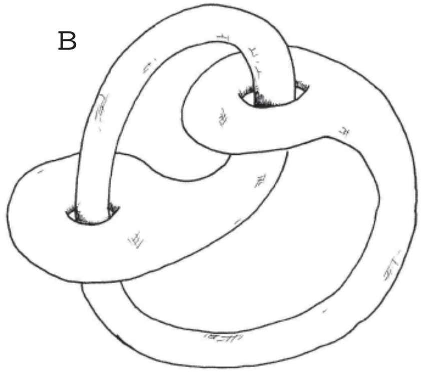




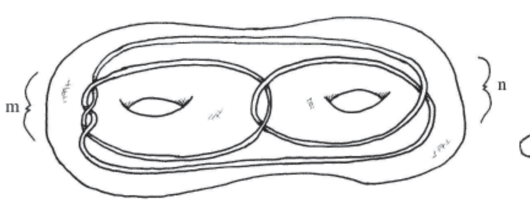
A



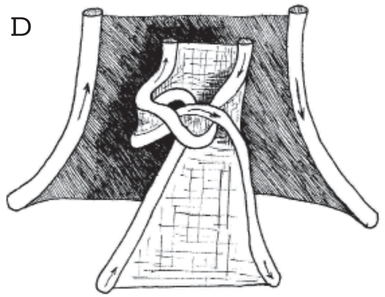
B



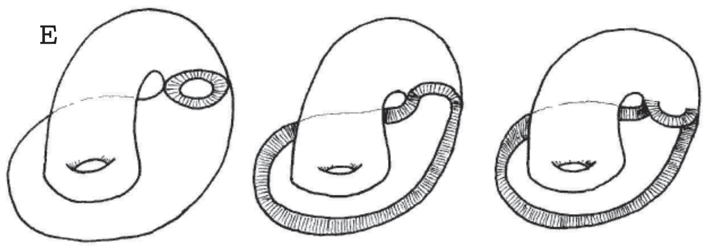
C



D



E



These drawings are from my first solo paper, "Bundles, handcuffs, and local freedom." They illustrate the existence of a knot in the 3-sphere whose complement is hyperbolic, that admits a fibration over the circle, and whose group contains a subgroup that is locally free (finitely generated subgroups are free) and not free. This answered a question from James Anderson: He had observed that if such a thing could not exist, then there would be counterexamples to Thurston's Virtual Fibration Conjecture (which we now know to be true due to the remarkable work of Ian Agol and Dani Wise).

The fundamental group of the 2-complex X (A) contains a subgroup that is locally free and not free. The way to construct the knots is to find a fibered knot in the complement of X so that X's group injects into the knot's. The complement of X is the handlebody (B). The knot is inside the handlebody (C). The knot is built by taking a square knot (which is fibered), cabling (which is again fibered), and then "plumbing." Plumbing is the process of gluing a twisted band along a square in a fiber (D). The rest of the figures are part of the proof that the knot is hyperbolic, and that you can perform surgeries to get closed hyperbolic fibered manifolds with groups having locally free non-free subgroups.

One year while I was in graduate school, Bob Gompf was teaching a course in 4-manifolds that I had lost the thread of. I didn't want to be rude, so I kept attending. So there I was in class, doodling and thinking about Anderson's question. I knew that connect sums, cabling, and plumbing preserved fibering. I sketched the complement of X and saw the square knot there. I cabled it for fun. I wanted the result to be hyperbolic, so I needed to do something else. So I plumbed a little band on to get rid of the essential torus that was ruining hyperbolicity, and I left class with a theorem. It took a while to prove that the examples were hyperbolic, but I had found them. Getting lost in a lecture isn't always bad!

I drew the pictures by hand since I didn't know how to do it any other way!

AUTUMN KENT

University of Wisconsin

pen drawings

In December of 1971, the grad students at Berkeley invited me (I was also heavily bearded with long hair) to paint math frescoes on the corridor wall separating their offices from the elevator foyer. While I was milling around before painting, one of the grad students, Bill Thurston, came up to ask, "Do you think this is interesting to paint?" It was a complicated maze-like looking smooth one-dimensional object encircling three points in the plane. I asked, "What is it?" and was astonished to hear, "It is a simple closed curve." I said, "You bet it's interesting!"

So we proceeded to spend several hours painting this curve on the wall. It was a great learning and bonding experience. For such a curve drawing to look good, it has to be drawn in sections of short, parallel, slightly curved strands that are subsequently smoothly spliced together. It was natural and automatic to do it in terms of bunches of strands at a time – as an approximate foliation – and then connect them up at the end. Thus some years later in '76, when Bill gave an impromptu three-hour lecture about his theory of surface transformations, I absorbed it painlessly at a heuristic level after the experience of several hours of painting in '71.

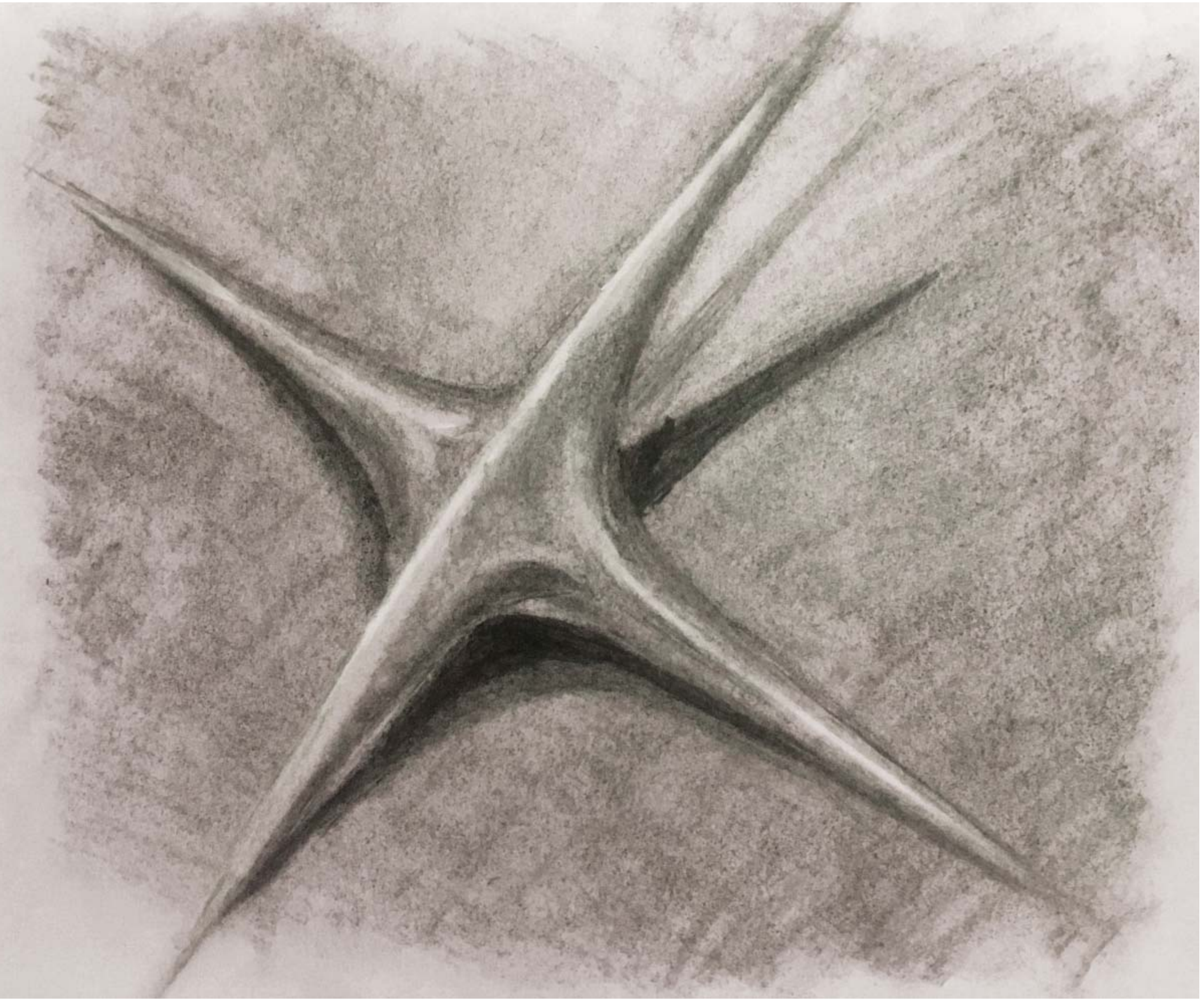
When I asked how Thurston got such curves, he said by successively applying to a given simple curve a pair of Dehn twists along intersecting curves. The "wall curve painting," two meters high and four meters wide, dated and signed, lasted on that Berkeley wall with periodic restoration for almost four decades before finally being painted over.

DENNIS SULLIVAN &
WILLIAM THURSTON

University of California, Berkeley

painted mural

Further information: Lee Mosher, *What Is... A Train Track?* Notices of the AMS 50(3), March 2019, pp. 354–356.



In his work on 3-manifolds, William Thurston proved that there exist exactly eight "geometries" that serve as models for any compact 3D object. The Euclidean geometry that children learn at school is one of them. Others, like *spherical geometry* or *hyperbolic geometry*, are well known to mathematicians. Still others are more exotic, e.g., the Nil and the Sol geometries.

The underlying topological space of the Nil and the Sol geometries is standard 3D Euclidean space. However, the way of measuring distance between two points is not the usual one from the Euclidean geometry. A natural question is:

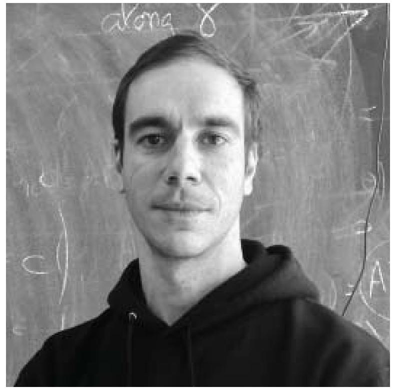
What is the shape of a sphere in each of these geometries?

This is a picture of a sphere in Sol geometry. In other words, every point on the surface of this shape is the same distance from the central point, using the Sol distance metric.

This project originated from a course given by Rich Schwartz during the Illustrating Mathematics semester program at ICERM in fall 2019. Rich showed us beautiful animations of geodesics and spheres in the Nil and Sol geometries. I became so intrigued by these objects that I decided to make physical models of them. As the shapes of the spheres can be generated by a computer, 3D printing was the easiest and fastest way to achieve this goal. The spheres have such an elegant shape that I used them later to practice charcoal drawing!

Later, these 3D-printed models turned out to be very useful for another project that we worked on at ICERM. The goal was to produce a virtual reality program of the eight Thurston geometries. Having the 3D-printed spheres in our hands helped us to check that the pictures generated by the computer made sense, and track the bugs in the code.

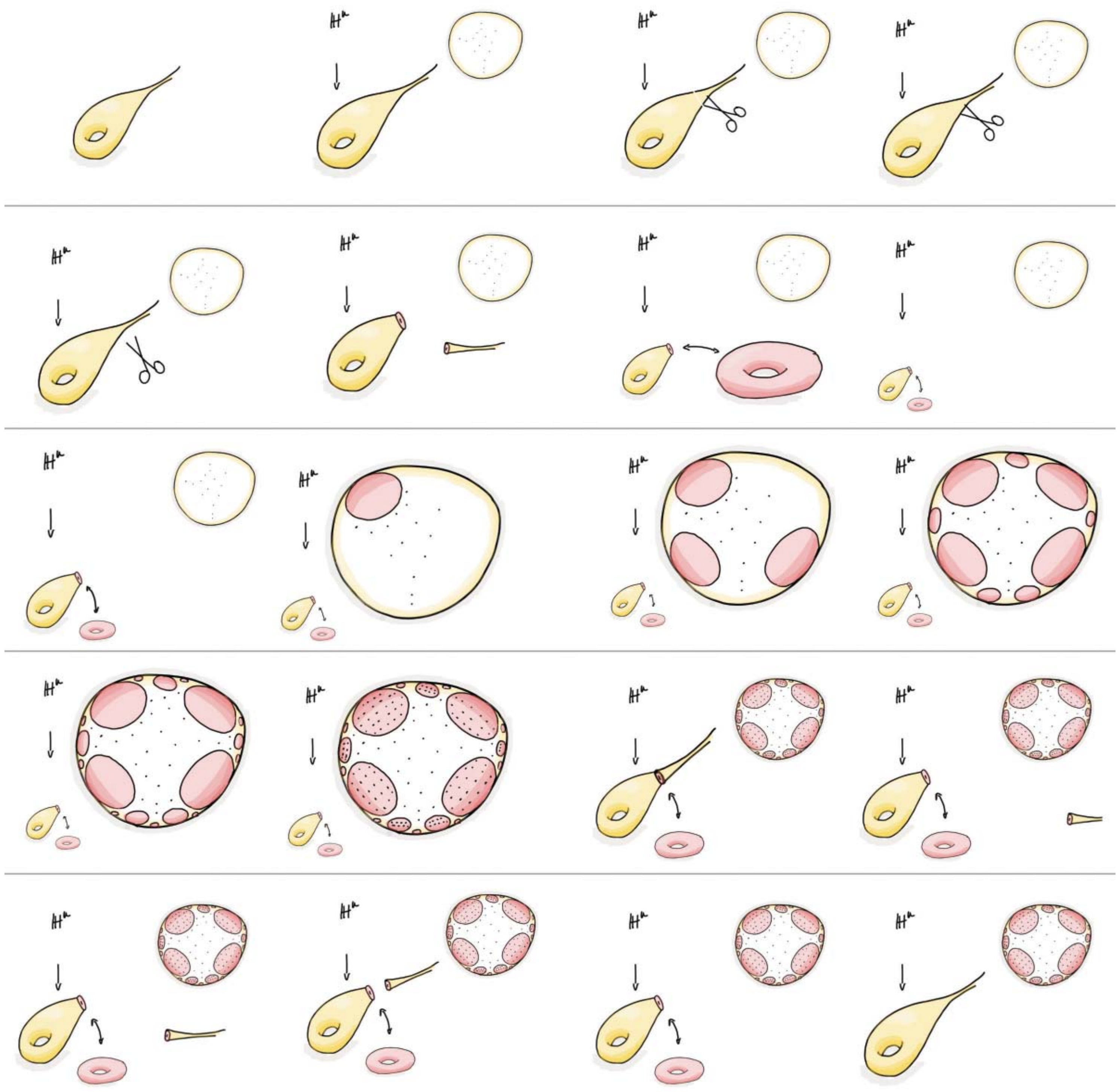
An article by Matei P. Coiculescu and Richard Evan Schwartz about spheres in Sol:
http://www.math.brown.edu/~res/Papers/sol.pdf



RÉMI COULON

CNRS / Université de Rennes 1

charcoal on paper



This animated gif shows a construction of *truncated hyperbolic space*, or *neutered space*. I made it using Notability on an iPad, hand drawing all 50 pictures individually. I like drawing, and I can do it easily on my iPad while traveling and on the go.

For a non-compact hyperbolic 3-manifold with finite volume, the universal cover is hyperbolic 3-space, and the fundamental group of that manifold acts on it but not geometrically, as the quotient is that non-compact manifold. When we cut off the cusp, since the cusp retracts on a slice of itself, it does not change the fundamental group. Cutting the cusp, we see that the slice is a flat torus whose universal cover is the flat plane.

This flat plane, in hyperbolic space, sits as a sphere centered at infinity, on the horizon: a *horosphere*. Cutting the cusp amounts to, at the universal cover level, removing an infinite collection of disjoint *horoballs*, which are the interiors of those horospheres. Now the fundamental group acts on the truncated hyperbolic space geometrically.

The terminology “neutered space” was coined by Benson Farb in 1994, and it is close to perfect: short, illustrative and easy to remember, and it encodes in two words the whole construction and most of the assumptions. The violence associated with that word, however, made it difficult to use and is at odds with the actual precision of the procedure: the horoballs’ removal is precisely encoded by the group action. In 1999, Martin Bridson and André Haefliger used the terminology “truncated hyperbolic space,” which is in fact quite accurate in the upper half-plane model.

View animation: https://math.unice.fr/~indira/GIFS/Horo.gif
More of my animated gifs: https://math.unice.fr/~indira/Mygifs.html

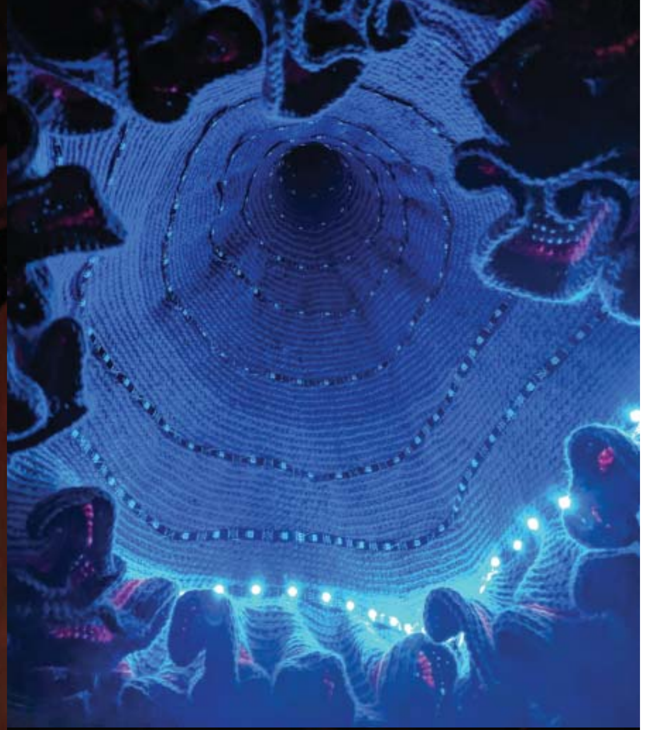
INDIRA CHATTERJI

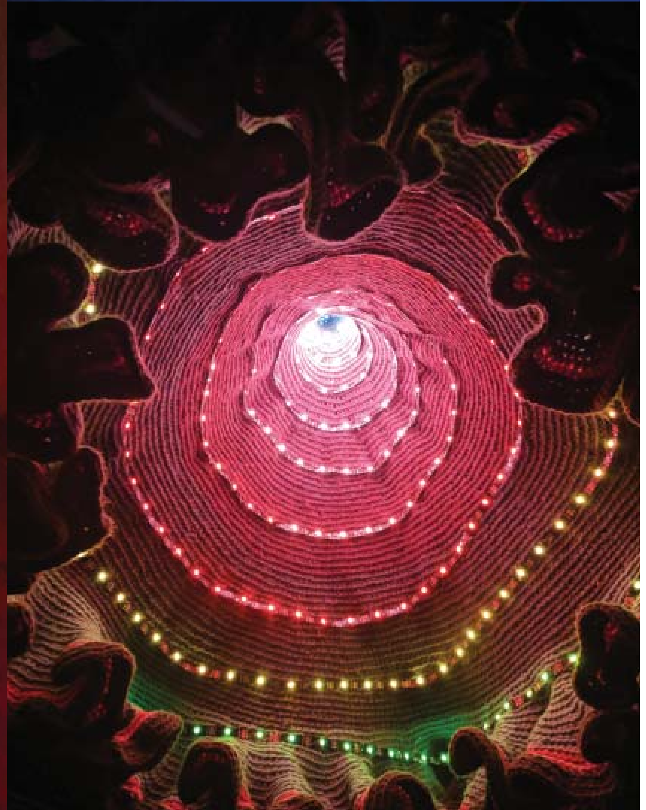
University of Nice

hand-drawn animated gif image

PAPER & FIBER ARTS

Paper is by far the most abundant resource in most mathematicians' offices, and it is our most readily available tool for creating mathematical illustrations. Most of us created the Platonic solids out of paper at some point in our early education, and have created numerous other models throughout our mathematical careers. For creating objects that people will handle and manipulate, and especially for those that require some stretch, sewn or knitted fabrics may work better than paper.





The Fabric of Spacetime is an installation serving as an interactive model of a young universe, combining crochet and electronics to create a dynamic and luminescent experience. The main physical component is a large, hand-crocheted hyperbolic manifold, where the number of stitches in each row increased at an exponential rate. In this way, the circumference grows exponentially faster than the length, introducing negative curvature and resulting in the many folds. This technique was pioneered in 1997 by Daina Taimina, a mathematician at Cornell. The curvature is analogous to the geometry of a very young universe (much less than one second old), where the spatial dimensions grew exponentially with respect to the time dimension, introducing curvature in the geometry of spacetime itself.

Sewed into the fabric model are 264 individually programmable neopixel LEDs, forming a spiral pattern around the inside, and mounted throughout the room are six servo motors, each connected by fishing wire to a fold in the crocheted model. There is also a PIR motion sensor directed toward the underbelly of the piece. All of these are wired to an Arduino MEGA microcontroller.

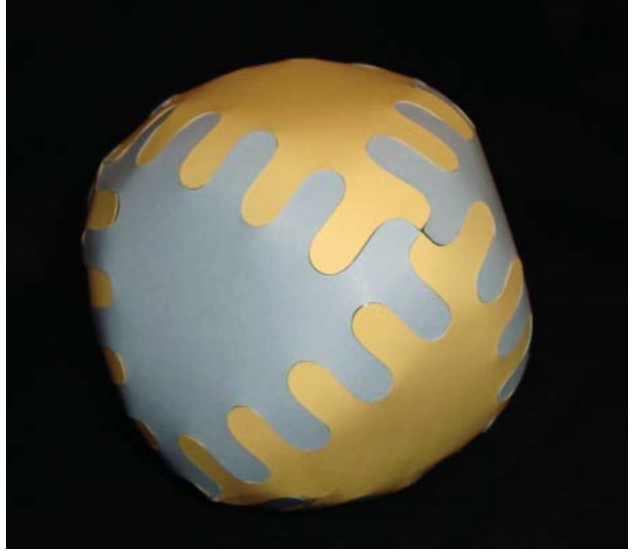
While undisturbed, the servos pull the model open and closed in a regular breathing motion, shifting to red while opening and to blue while closing. This is in homage to the cosmic red shift and blue shift of the universe: since red has the longest wavelength of the visible spectrum, things moving away from us at relativistic velocities gain a slight red tint in color, and, since blue has the shortest wavelenth, things moving towards us gain a slight blue tint. In this way, the color shift of the piece represents the actual shift that would be visible in an expanding and contracting universe. When the viewer walks under the piece, they trigger chaotic motion and lights.

I (Gabriel) find crochet to be a very flexible medium, both literally and figuratively, and I am fascinated at the way it can create various geometries and symmetry patterns, and also how easy it is to combine with electronics. Using crochet, I really tangibly felt the intensity of exponential growth. To create hyperbolic crochet, you add stitches in such a way that the circumference of the crocheted piece grows exponentially. In practice, this means the workflow really starts to slow down. This piece took hundreds of hours of crochet and weighs 10 pounds. The circumference of the piece at the widest part is over 80 feet (all hidden in the folds).

GABRIEL DORFSMAN-HOPKINS & MEGHAN MAYNARD

ICERM; Independent artist

crocheted yarn and LEDs



Essentially, my project illustrates a version of the Gauss-Bonnet Theorem. The question I explored (with Bill Thurston) was the following:

> *How can we make well-fitting "clothing," out of rigid material, for a sphere and other surfaces?*

Our first try at making clothing for a sphere was the version using "fat triangles" to make an octahedral pattern (top left). This is the pattern made from eight "triangles" that have edges made from circle arcs, that make right-angled triangles. This sort of pattern works great when made from cloth, which has some stretch, but when made from paper, created a "sphere" with large flat faces. By making longer, meandering seams, the curvature was distributed over a larger portion of the sphere, which made a very satisfying sphere (bottom). We made an intermediate sphere to further illustrate how seams can distribute the curvature (top right).

Paper was a fun, inexpensive medium to work with. Also, craft paper cutters are relatively inexpensive and can easily cut complicated curves. Through this project, I learned about planar curvature of curves and Gaussian curvature of surfaces.

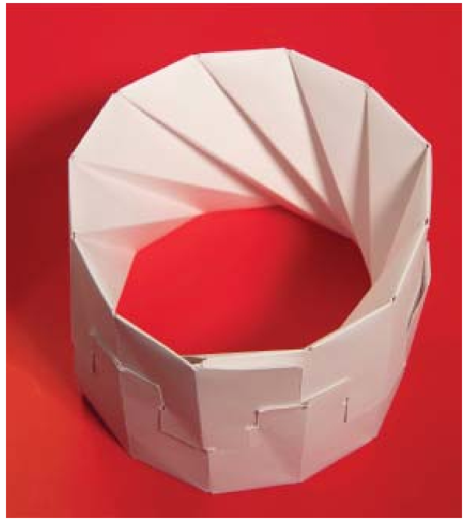
KELLY DELP

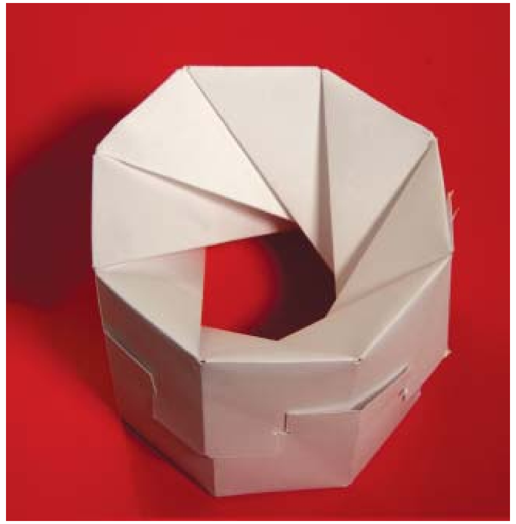
Cornell University

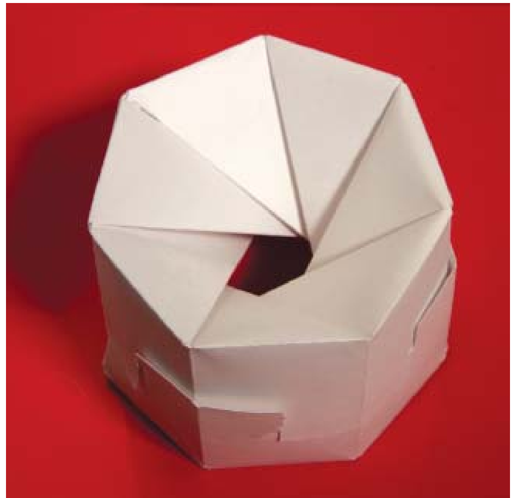
electronically cut paper

Further information:

Kelly Delp and Bill Thurston, *Playing with surfaces: spheres, monkey pants, and zippergons.* Proceedings of Bridges 2011: Mathematics, Music, Art, Architecture, Culture (2011), pp. 1–8.







To make a torus from a rectangle of paper, glue a pair of opposite sides together to form a cylindrical tube, then glue the two ends of that tube together. The second gluing requires squashing the cylinder, resulting in a doubled-up cylindrical tube. It is somewhat unsatisfying, as it encloses no volume, making it far from the "typical torus" formed by a tire or a lifebuoy, which could be described as a "circular tube around a circle."

The mathematician's dream is a non-squashed polyhedral torus, enclosing volume like the "typical torus," yet *everywhere flat* in the sense that each vertex has a total angle of 360°. A common belief is that these wishes cannot be satisfied simultaneously. Yet Dmitry Burago and Victor Zalgaller showed that it *can* be done, as we learned from Henry Segerman and his 3D-printed "hinged flat torus"! Having learned that, we started exploring paper models for such tori.

One description of our model starts from a prism on a regular n-gon, with its bottom and top faces removed. For some integer k less than half of n, consider triangles joining top vertices to bottom vertices: each top vertex is joined to the next top vertex and to the bottom vertex k steps ahead; each bottom vertex is joined to the previous bottom vertex and to the top vertex k steps back. These triangles are all congruent. Together with the outside of the prism, they form a polyhedral torus where, at each vertex, the three angles of congruent triangles meet on the inside and two right angles meet on the outside, making a total of 360°.

For a paper layout, cut along the outer prism's half-height line, along one inner triangle edge, and along the two outer vertical half edges from the ends of that inner edge to the first cut. The pictures show examples of such tori for various choices of n and k, and paper layouts at various stages of folding up. Extra flaps allow us to assemble the tori without glue or tape.

Instead of cutting by hand, we wrote a script using SageMath, a piece of free software for mathematics. Given a choice of n and k, it outputs an SVG file, from which a mechanical or laser cutter can cut along some lines (to cut out the shape) and score along others (to make it easier to fold). Given the cut and scored piece of paper, it still takes some practice to assemble the torus.

Further information: https://im.icerm.brown.edu/portfolio/paper-flat-tori/

ALBA MARINA MÁLAGA SABOGAL & SAMUEL LELIÈVRE

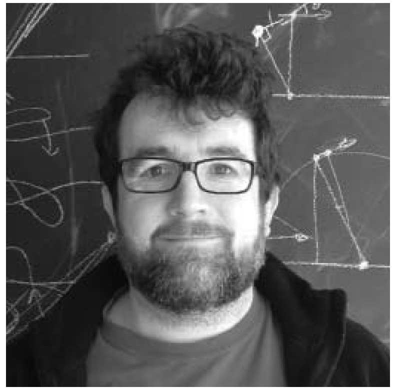
ICERM; Université Paris-Saclay

folded paper

Due to some fortunate events, I found myself with a laser cutter. I was happily using it to make tiles and other flat things, but what I really wanted was to take the leap to making 3D things. Bent and folded paper was an obvious option, and I worked on many different things, some more successful than others – different objects, and different ways of connecting them. Then I woke up one morning and somehow several things had come together in my head: I had an idea for a connector, and something to try to make. I have to confess that this happens quite a lot, and most of the time the ideas turn out to be wrong or not quite work, but this time they were perfect. *Curvahedra* was born!

This process of initial play and discovery meant that I had something; the next question was to ask what I could illustrate with it. The way that the system connects up to make loops controls the holonomy ("turning"), and, thanks to the Gauss-Bonnet theorem, the holonomy controls the curvature of the surface.

This illustration is, of course, a direct result of my initial motivation, to use 2D cutting to produce true 3D objects. I hope this illustrates that sometimes it is worth playing and developing, rather than pushing directly for a desired outcome.



EDMUND HARRISS

University of Arkansas

laser-cut mylar

For more information and to purchase curvahedra: http://curvahedra.com

I make crocheted models of basic hyperbolic surfaces such as a disc or a cylinder. The models carry triangulations of surfaces and demonstrate several properties: first, that the sum of angles in a hyperbolic triangle is less than 180 degrees, and second, that the circumference growth is faster than that of a Euclidean circle.

I prepared the first crochet model to demonstrate some ideas in a Riemannian geometry class I was teaching. It was a solid hyperbolic disc, and I wanted to demonstrate the growth of a ratio of the circumference to the radius of a disc. I found that the process of crocheting a solid hyperbolic disk took too long, because it is exponential in time and in yarn consumption. So, since I had lots of experience with crochet, I started experimenting with different stitches and techniques. I suddenly realized that using a triangular pattern gave a triangulation of the surface that showcases many other visual properties.

I made these objects to illustrate the *Nash-Kuiper C^1 embedding theorem*, which roughly implies that if you have a 2D surface that is oriented and closed, then there is a way to embed it into an arbitrarily small ball of Euclidean 3-space, in such a way that the surface is not stretched or compressed at all.

The crocheted objects serve as models of a hyperbolic disk that is embedded into 3-space in a way that preserves all of its intrinsic properties. It is clear that a crochet model cannot be placed inside of an arbitrarily small Euclidean ball, because the yarn has some thickness and volume. Still, each model demonstrates smooth "curving" inside of some ball and "tries" to fill the whole space inside of that ball. As a crochet model is extended by adding more rows, it will look more and more like a ball, and will be increasingly thick and dense.

For these models, the stretching properties of yarn are important. I tried several types of yarn, and acrylic and wool work best. Making the pattern was the most time-consuming part. I experimented a lot with which pattern returns a look that is most similar to that of equilateral triangles.

Further information and crocheting instructions:

Maria Trnkova, *Hyperbolic flowers*, preprint (2021), arXiv:1910.05900. Accepted for publication in an upcoming issue of "Journal of Mathematics and Arts".

MARIA TRNKOVA

University of California, Davis

crocheted yarn

The *triacontahedron* is a Catalan solid that has thirty congruent rhombic faces. Its dual polyhedron is the icosidodecahedron, which is the Archimedean solid obtained by deeply truncating the dodecahedron or, alternatively, deeply truncating the icosahedron.

To make a paper model of the triacontahedron, one can easily find a *net* on the internet – a flat connected version of all the faces that, when assembled, gives the polyhedron – to download, print, cut out, and tape together. In this case, I wanted a version where the faces are decorated, and where I could print out or laser-cut each piece. To be able to do that cleanly, I affixed tabs to the edges. Yet tabs introduce an issue, because they require space. Acute angles between adjacent faces of any given net might not allow for the insertion of tabs. I therefore dissected the net into pieces so that on each piece no adjacent faces have exterior edges making acute angles. To make matters more mathematically interesting and less demanding to produce, I made all of the pieces congruent.

Note that this dissection condition came from the medium: paper is an inexpensive way to construct a solid, and it is easy to decorate. I added the congruence condition so that the number of unique files created for the printer or laser cutter is kept to a minimum. Each face could be cut individually with tabs on all four sides, but it is preferable to have more faces connected, because then the paper is automatically hinged between these faces, which constrains their relationship in space and creates structural integrity.

With ten three-face pieces cut and ready, it seems simple to assemble the triacontahedron. Surprisingly, there are many ways to begin assembling with the pieces and get stuck so that the desired polyhedron cannot be completed. The reader might wonder why we don't use five- or six-face pieces, for then only six or five copies would have been needed, respectively. As with many projects in mathematical art, this enterprise has proven to provide mathematically interesting questions as well as illustrations.

For more information on the rhombic triacontahedron:
http://mathworld.wolfram.com/RhombicTriacontahedron.html

CAROLYN YACKEL

Mercer University

laser-cut paper

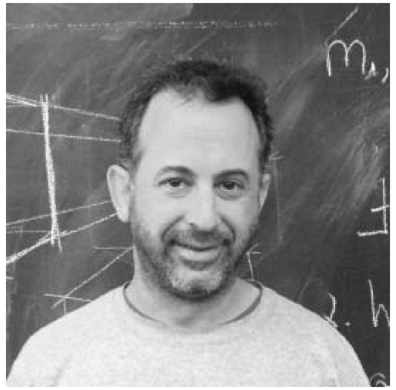
We are making paper with constant negative curvature, in order to demonstrate various features of the hyperbolic plane. A flat piece of paper has zero curvature, a sphere has constant positive curvature, and our paper has constant negative curvature. Our project builds on earlier work of Roger Alperin, Barry Hayes, and Robert Lang.

We hope that this paper can serve as an instructional tool for college-level geometry courses. Our goal is to explore hyperbolic versions of many of the things that we can make from flat paper: straight-edge and compass constructions, tilings, origami, and so on. Paper is a natural medium for this, as, unlike fabric, it has very little elasticity (so the intrinsic geometry is preserved when it is smoothly deformed) and it holds a crease (so we can generalize ideas and methods of flat paper folding to get new hyperbolic designs).

We learned a lot of math in the process of creating this paper! For example, we needed to learn how to isometrically parameterize the pseudosphere (top right) and related constant-curvature surfaces such as Dini's surface (left) from the upper half-plane model of hyperbolic space. This project also enabled us to experience hyperbolic geometry in a physical way, which offered interesting surprises despite our existing theoretical understanding. For instance, we knew from local isometry theorems that the paper should fold over onto itself and fit perfectly, yet when holding the paper in our hands it was hard to believe it would actually work. (It does!)

We are still trying to figure out the best way to make this paper. Our first attempt used papier mâché, but it was too stiff. Following advice the of John Edmark, Rotem Tamir, and others, we improved our process by making flat strips from pulp, then introducing curvature by pressing the strips onto a pseudospherical mold (top right) while they were still wet. We hope to make larger, sturdier, more robust paper by starting with higher-quality fibers. On the other hand, we have learned that Alba Málaga Sabogal has had success making hyperbolic paper from toilet paper fibers.

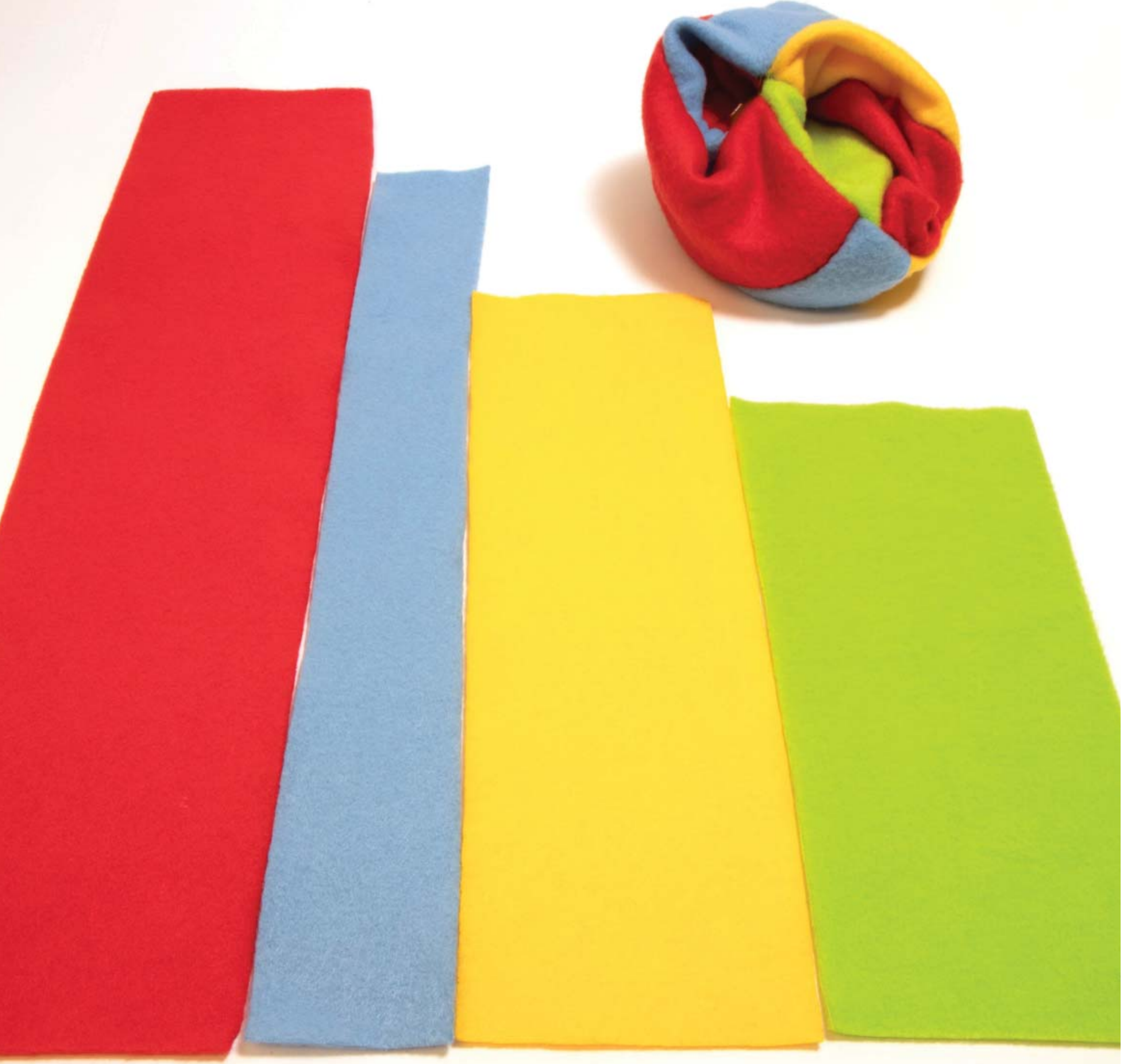
Video showing local isometries, rotating and translating paper on the surfaces:
https://www.youtube.com/watch?v=JRd928WpY9w



AARON ABRAMS &
STEPAN PAUL

Washington & Lee University;
Harvard University

homemade paper, 3D-printed plastic, molded silicone



For many years, I have used sewing and quilting as ways to understand geometric structures. The first structures I made were crocheted hyperbolic planes. I was really fascinated with the idea of making them *homogeneous* and *isotropic*, meaning that there is no special direction or location. I was crocheting at a conference and a friend of mine, Stu Ramsden, suggested that I try making a Klein quartic, which is a bit like a Platonic solid. A *Platonic solid* is a solid body made of copies of a regular polygon (a triangle, square, or pentagon) with the same number of polygons meeting at each corner. For example, the dodecahedron is made of 12 pentagons where three pentagons meet at each corner. The Klein quartic is made of heptagons – a polygon with seven sides – that meet together in groups of three at each corner. This quilt (not pictured) gave me the ability to touch the Klein quartic, and also visualize symmetries in a way that digital models don't.

As part of the Illustrating Mathematics program at ICERM in fall 2019, Rich Schwartz, a math professor from Brown, gave a course called Geometry and Illustration. He covered many topics in this course, including *translation surfaces*, which are surfaces that are made from a flat (Euclidean) polygon with a set of rules that determine which sides are glued together. During a discussion over lunch, Pierre Arnoux mentioned that his colleague Maki Furukado had made a beanbag model of a translation surface that has genus 2, meaning that it has two holes. It was very delicate, so I offered to make him another one out of fleece, so it could be easily manipulated. Working with Pierre, I created the surface from four rectangular strips of fleece. This type of zippered rectangle construction is called a suspension surface, because it is the suspension of an interval exchange transformation.

There are other ways to make a genus-2 translation surface. Several of my colleagues at ICERM were studying billiards problems on pentagons (see page 157), using the *double pentagon* translation surface. This surface is made by taking two pentagons with one pair of sides glued together, forming a polygon with eight sides, and gluing the other pairs of edges that are parallel. I made this surface, too, but it was much harder to manipulate than the zippered rectangular surface. It only has one vertex, which means that all of the corners of both pentagons all meet at a single point. These two physical versions of genus-2 translation surfaces helped me to understand not only what translation surfaces are, but how and why mathematicians find them useful when they are studying very different types of problems.

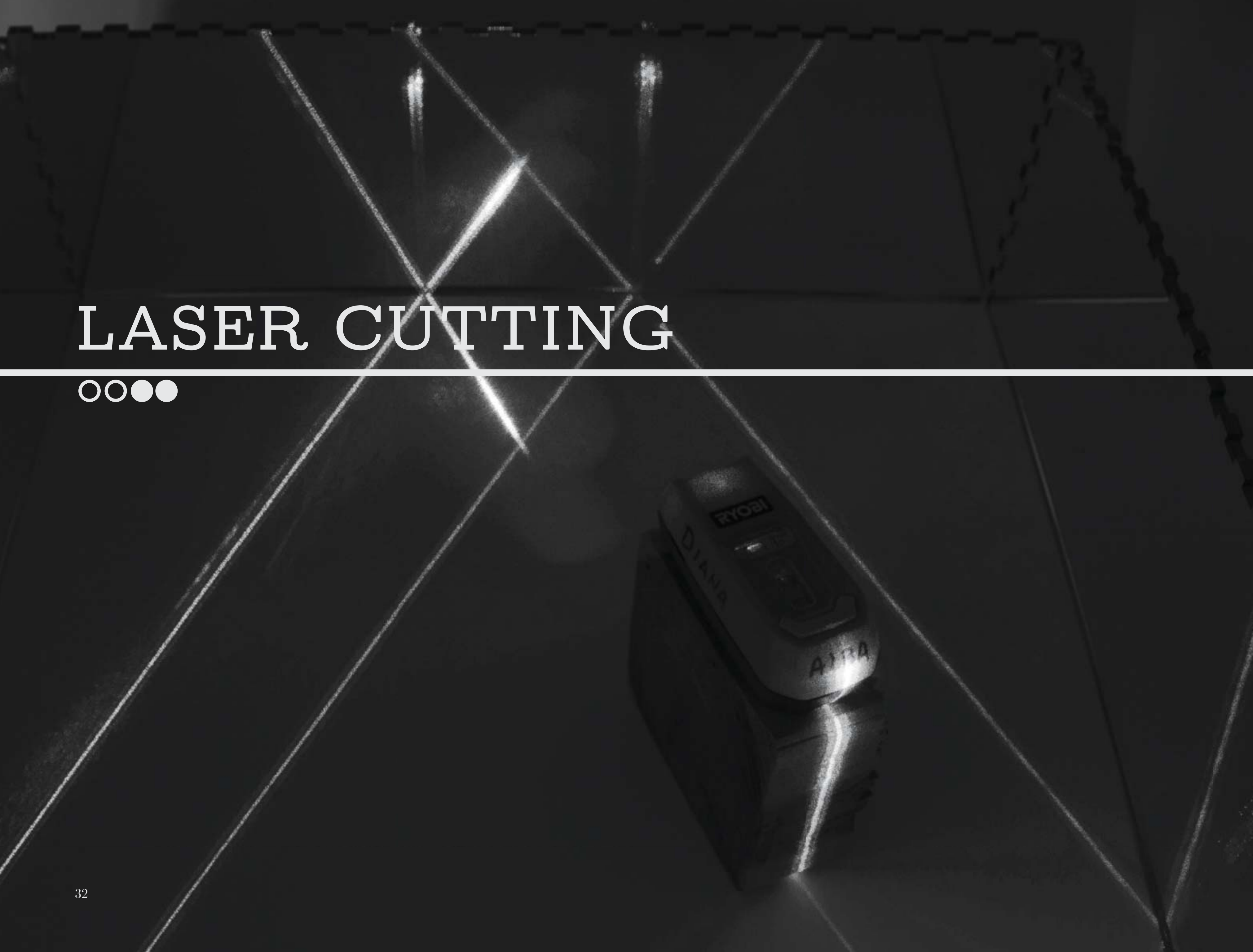
SABETTA MATSUMOTO

Georgia Institute of Technology

hand-sewn fleece and rice

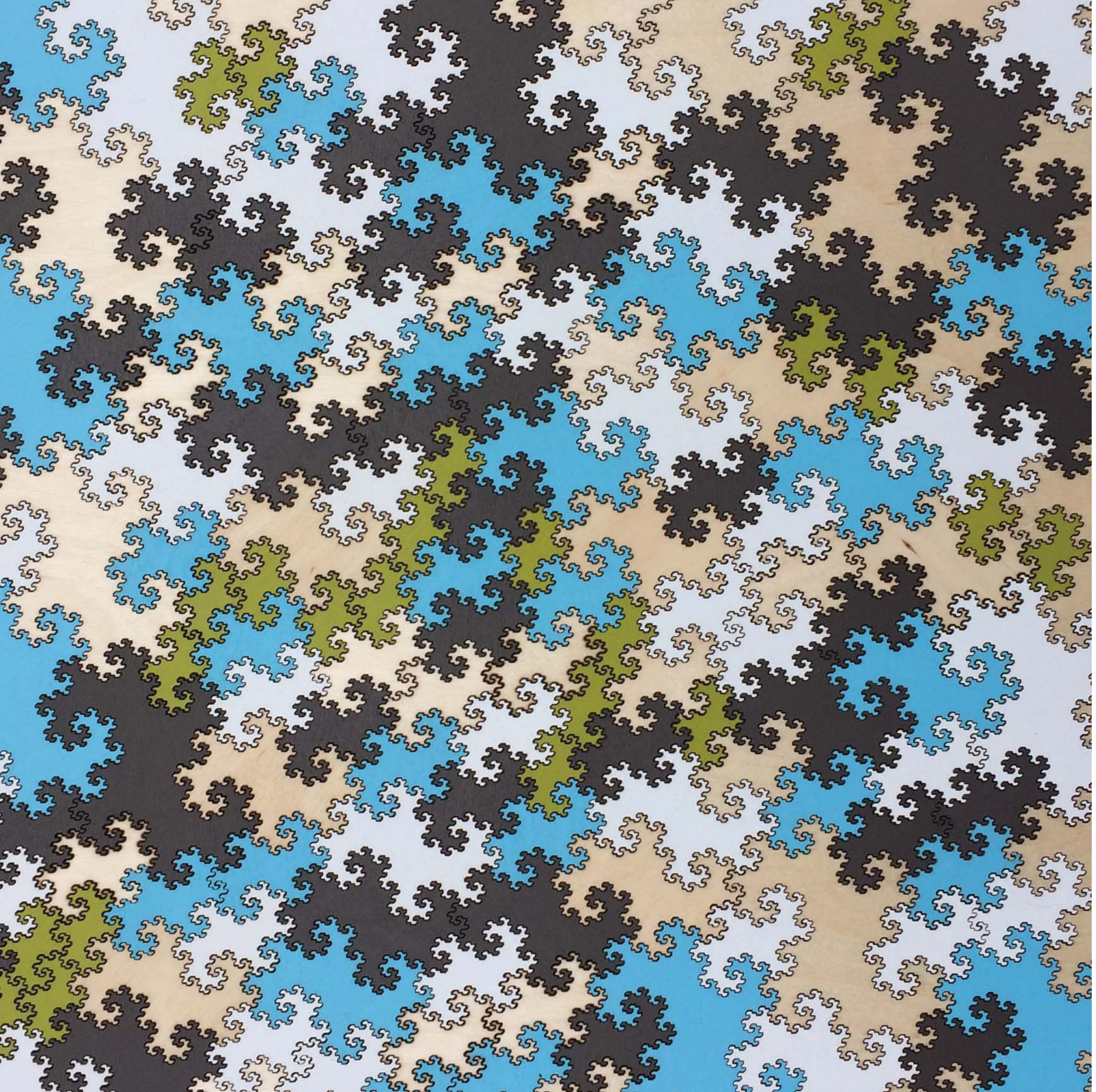
LASER CUTTING

○○●●



Since mathematical objects are infinitely precise, it is always a little disappointing to cut out some pieces with scissors or a bandsaw, fit them together, and find that they are not quite right. The laser cutter eliminates much of this issue because it can reproduce a resolution of up to 600 DPI, creating cuts, folds, curves, tabs, holes, and other mathematical elements in, for all practical purposes, *exactly* the right spot.

Of all the techniques presented here, the laser cutter is the most dangerous, as it can melt and warp materials, create toxic fumes, and start fires. It must be watched at all times when in use, and it is not suitable for children. And yet, if you wish to create a physical manifestation of any 2D object, the laser cutter is almost certainly the best tool.



This is a tessellation with twin dragons. The twin dragon is a 2-reptile fractal, meaning that it can be replicated from two smaller copies of itself. It illustrates that all pieces of the same size are translations of each other. We look for things that are mathematically well-known, beautiful, and accessible and engaging to the general public, and the twin dragon fits our criteria well.

We chose, as we often do, painted wood. People like the feel of wood, and we love the flexibility of an unlimited color palette, which we get by painting the wood before we cut it.

We first made *terdragons*, which are closely related 3-reptiles. Tessellated terdragon pieces of a given size appear in up to six orientations. We were surprised to discover that twin dragons of a given size all have the same orientation.

We could only find descriptions of how to construct the interior of the twin dragon, and we used Adobe Illustrator to trace the boundary, but it wasn't good enough to create interlocking tiles. We were fortunate to be introduced to Dylan Thurston via Sarah Koch while Dylan was visiting the University of Michigan. We discussed the problem, and Dylan devised a clever substitution scheme for constructing the twin dragon starting with a single skewed hexagon. From there, it was obvious how to use an L-system to directly construct a boundary with whatever level of detail we wanted, and the pieces interlocked perfectly.

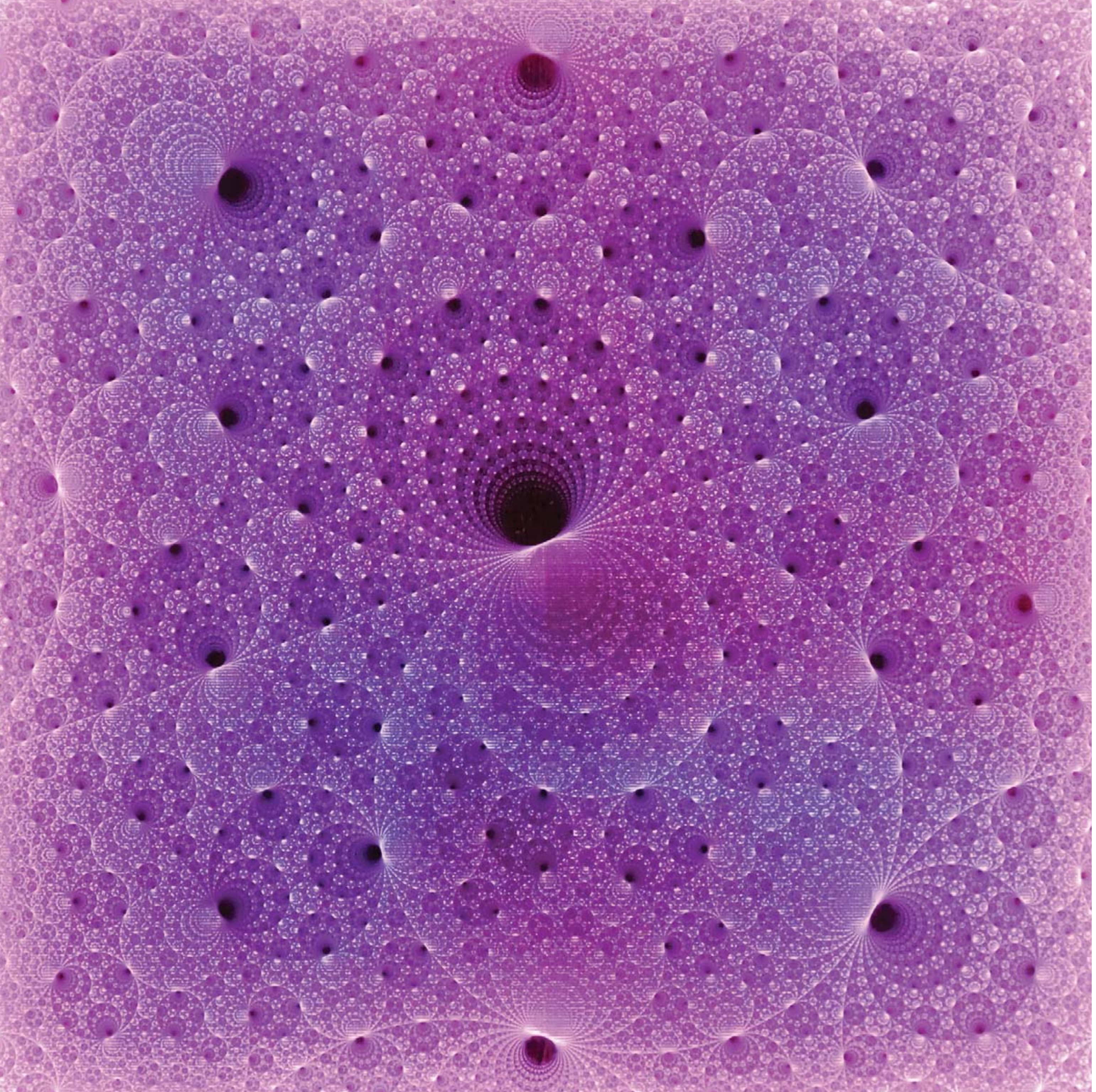
Our first efforts at cutting actually had too much detail. The pieces had fantastical lacy edges that interlocked, but as you tiled with them, the lacy bits would break off. We then dialed in just enough detail to make the pieces sturdy.

Further information:
https://www.cherryarbordesign.com/tessellating-with-twin-dragon-fractals/

HEIDI ROBB & PETER BENSON

Cherry Arbor Design, LLC

laser-cut painted wood



There are many possibilities for the shape of the universe in which we live. To understand these better, topologists often imagine a surface that divides the universe into simpler regions. Imagine a surface that is somewhat translucent and is sitting in the middle of one of these possible universes. As we look out, we see an amazing picture. Because of the way both the universe and the surface fold back on themselves, we see different shades of light and dark wherever we look, creating an amazingly complex and self-similar pattern.

In this piece, we – I, Saul Schleimer, and Henry Segerman – show one such view of one such possible universe, laser-cut in acrylic and side-lit with LED strips. In particular, this picture represents a small part of the boundary of *3D hyperbolic space*. We view this space as coming from the universal cover of some 3-manifold, containing a topologically significant surface. We created a depth map based on how many lifts of the surface you pass through as you move out a fixed distance from some point of hyperbolic space toward the boundary.

This is a laser-etched piece of acrylic, with carefully designed lighting to illuminate the etching. I had never tried edge-lighting an acrylic relief, and thought this would be a perfect way to create a visualization of our results. I was surprised at how beautiful it turned out!

There were a lot of experiments to get the lighting effect correct. First, I built a custom frame that hides an LED strip light. I then learned that the acrylic needed to have a clear boundary where it met the frame so that light propagated through it. Then I needed to cut the acrylic again, make modifications to the frame, and so on.

DAVID BACHMAN

Pitzer College

laser-cut acrylic

These lizards illustrate a tiling of the plane by congruent pieces, following a similar picture created by the mathematical artist M.C. Escher in 1943. I have loved tilings for years, so when I learned to use a laser cutter, I wanted to create this beautiful tiling, as a sort of puzzle for myself and my students.

This tiling is based on the tiling by regular hexagons: As in the hexagon tiling, you can check that here, every tile touches six others. Also, around each lizard there are three points of rotational symmetry of order three: at the left cheek, the left foot, and the right knee.

This tiling is *three-colorable*, meaning that if you have three colors of tiles, you can assemble the tiling in such a way that no two tiles of the same color touch each other. I chose to make lizards in three contrasting colors, so that it would be possible to assemble them in a three-colored way, as shown.

I created a file to cut a sheet of acrylic plastic into 66 lizards, and then ran the same file on plastic sheets of three different colors. I engraved some details on one side of the lizards, which both gives them some personality and helps with orientation: in order to fit together, all of the lizards must either have the engraved side up or the engraved side down. The width of the laser (the *kerf*) is small but nonzero, so there is a bit of wiggle room between the pieces, making them easy to put together.

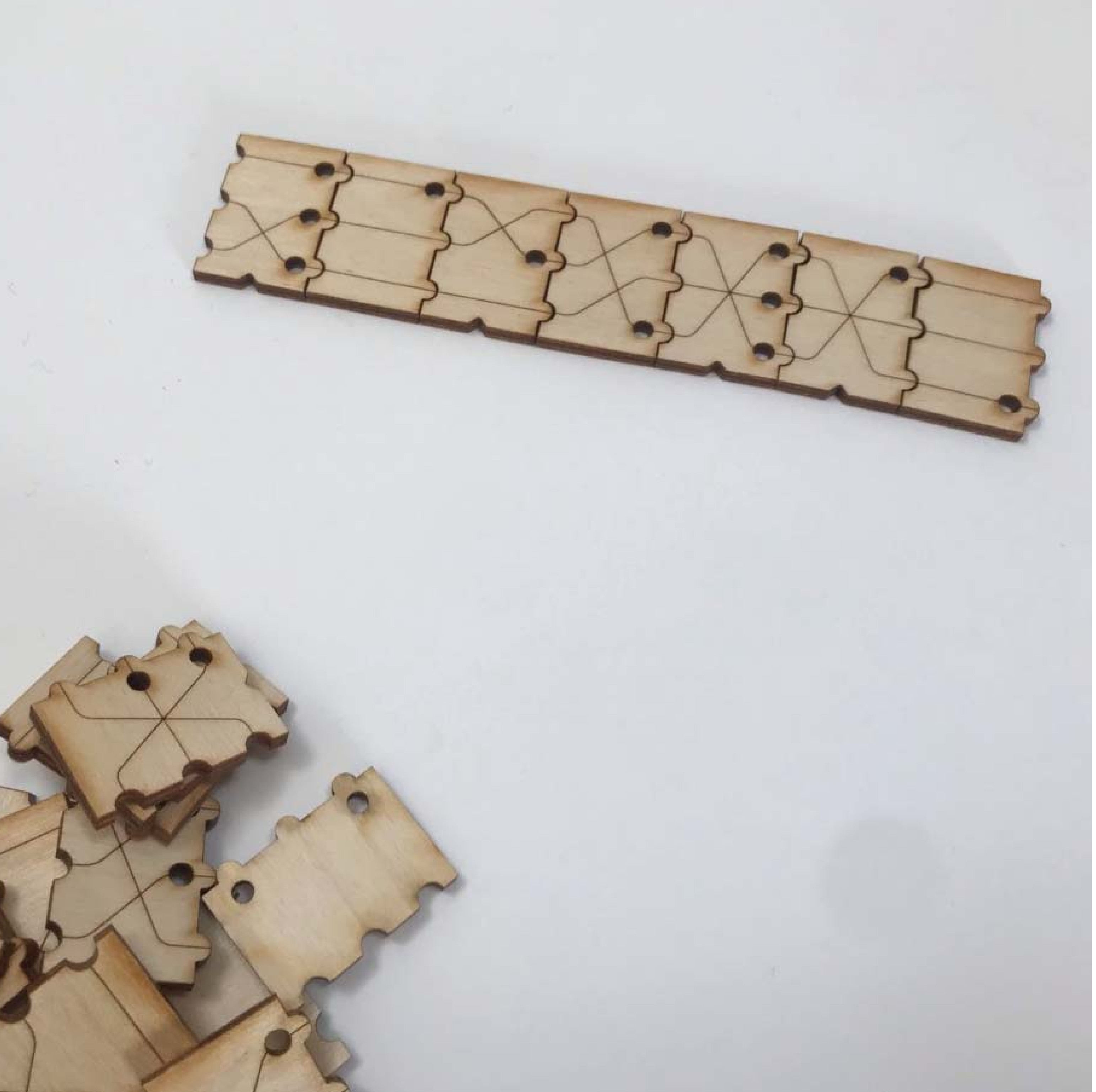
I knew that the tiling was three-colorable, but I didn't realize until I put it together that all of the lizards of a given color would be in the same orientation: for example, the pink lizards in the picture are all facing to the left. I also didn't expect how delightful it would be to assemble the tiling. In the same way that coloring a coloring book is said to approximate a meditative experience, fitting the lizards together is also a calming, satisfyingly tactile process. This is especially true when assembling them in a three-coloring, because after you choose how to put together the first two tiles, the rest of the tiling is determined.

DIANA DAVIS

Swarthmore College

laser-cut acrylic

Escher's original artwork: M.C. Escher, *Reptiles*. Lithograph print, 1943.



These tiles originated in a project with Jonah Ostroff and Lucas Van Meter in which we created variations of the classic card game Set for non-abelian groups. Such a game requires illustrations of group actions that can be easily understood, and so we made these tiles, along with versions for several other non-abelian groups, in order to experiment with the design of these non-abelian Set games.

This medium allows for tactile interaction with what is usually a purely abstract concept. By physically rearranging the wooden pieces, the players can test and visualize the multiplication in this group; small wooden pieces are ideal for this type of manipulation. We have also tried playing with paper cards, and, while the mathematical content is the same, the laser-cut wooden tiles are much more aesthetically pleasing to touch and move.

This particular collection of wooden tiles is a way to visualize the semi-direct product of the symmetric group S_3 acting on three copies of $\mathbf{Z}/2\mathbf{Z}$. Each tile represents one group element: the three lines give an element of S_3, and the three dots give an element of $(\mathbf{Z}/2\mathbf{Z})^3$. We can illustrate the group action by concatenating a series of tiles and then reading off the resulting element from the lines formed between the tiles and the dots, modulo 2, along each line.

I learned lots of interesting details about permutation groups and their group actions by creating this illustration. For example, I discovered that if you order the three transpositions in S_3, then the first two transpositions always compose to the same 3-cycle as the second two transpositions. More generally, I found that playing with these tiles gives an intuitive understanding of the algebra used to construct their underlying groups, especially in the case of semi-direct products.

More variations on the classic game Set:
https://people.maths.bris.ac.uk/~zx18363/set.html
A video of my talk about this project:
https://icerm.brown.edu/video_archive/?play=2063
Further information:
Cathy Hsu, Jonah Ostroff and Lucas Van Meter, *Projectivizing Set*. Math Horizons, 27:4 (2020), pp. 12–15.

CATHERINE HSU

University of Bristol

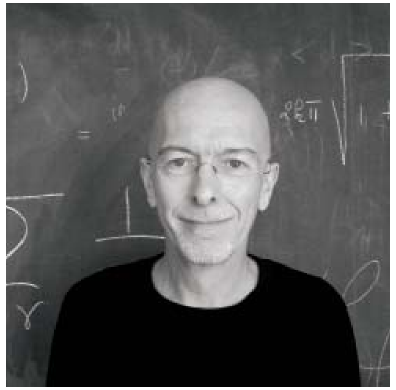
laser-cut wood

This is a regular dodecahedron composed of 12 regular pentagonal faces, three meeting at each vertex. It is one of the five Platonic solids. Plato claimed that it was "used for arranging the constellations on the whole heaven," and so it was associated with the universe, representing mystery and meditation. During the time I was building the piece, I was learning about how to think of its group of symmetries – the alternating group A_5, which I was teaching in class as an abstract entity – in geometric terms.

My dodecahedron is built from 12 flat pieces of fiber board cut in a laser cutter. I chose laser-cut wood because it is an organic material, and as such it has a very different, warmer presence than, say, 3D-printed plastic or metal, inviting one to touch the piece. Wood presented certain challenges, given the objective: a construction in which the pieces are assembled without glue, held by the springiness of the wood that desires to return to its flat state. The solution uses a technique called *kerf cutting* where special incisions have been made to allow the wood to bend, in order to form the shape of the solid in three dimensions.

I coded all the cuts myself using elementary geometry – scalings, translations, rotations. Even now I am amazed at the strange outcomes of applying transformations in the wrong order.

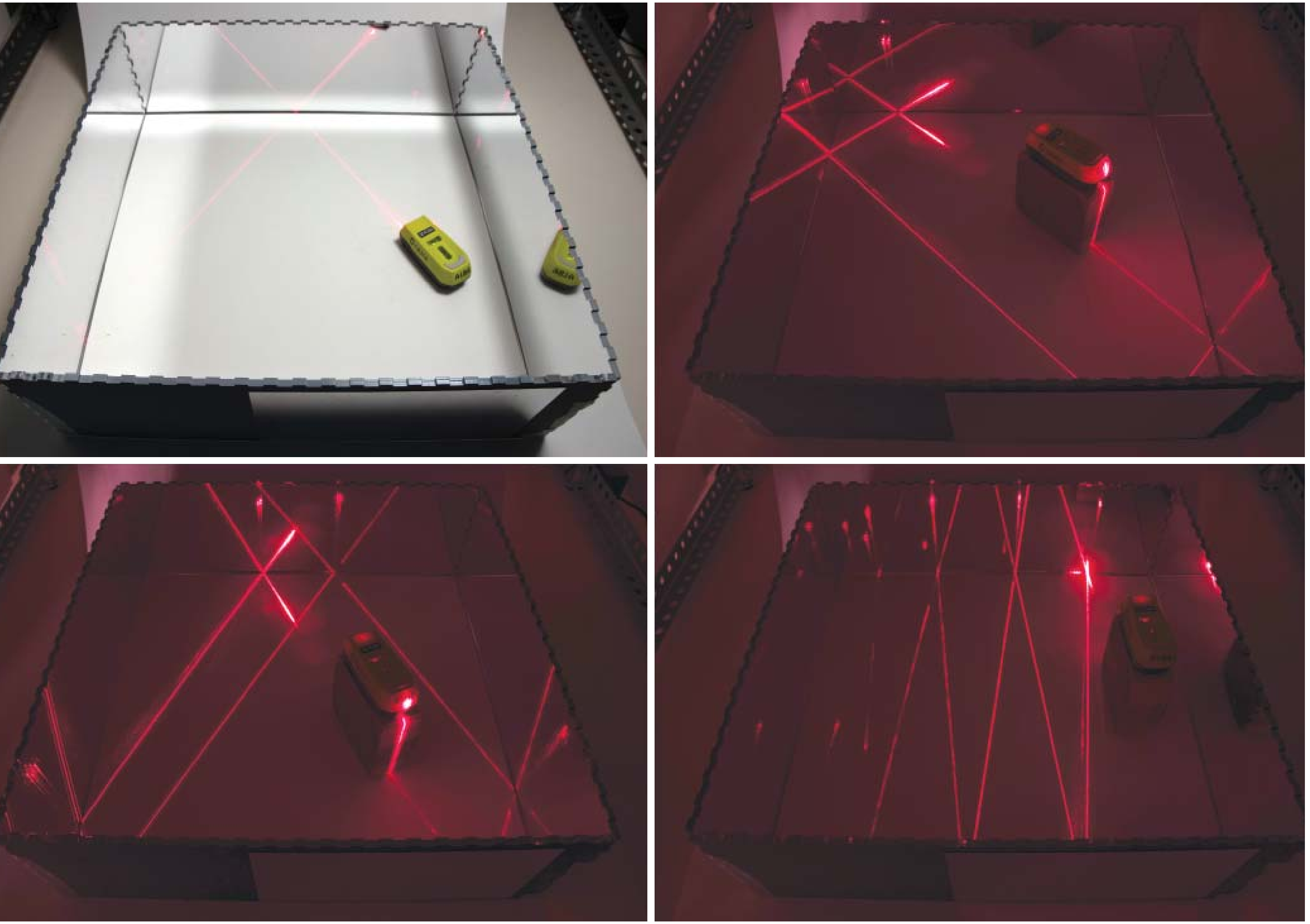
This piece is the result of much frustration through trial and error – an application of the dichotomy method for finding a solution among infinitely many choices. There are several parameters one must tweak in order to achieve the desired result: the width of the strip between pentagons, the number and size of the kerf cuts, the size and position of the jigsaw joints where two pieces are attached.



GREG MCSHANE

Université Grenoble Alpes

laser-cut fiber board



This mirror box illustrates billiards on a square billiard table. Here, instead of a ball bouncing off of the sides, a laser beam reflects off of mirrors, which allows us to see many segments of its path all at once. The picture shows two periodic paths, of period 4 (upper right) and 6 (lower left). We have made several versions of this table over the years, and we have brought them to many math outreach activities. We have found that a 50cm x 50cm box works well; any larger, and the laser beam becomes too weak after just a few bounces; any smaller, and the laser takes up too much space.

Since we wanted to be able to pack up the box and carry it easily, we made it in seven pieces: four mirrored walls and three interlocking pieces that fit together to make a large square base (not shown). The corners of the walls have rectangular protrusions that interlock, and there are the same rectangular protrusions at the bottom of each wall (shown here at the top) that fit into rectangular holes in the base. Fitting the walls into holes in the base ensures that the box is perfectly square, and not just a rhombus. Unfortunately, we made the base out of cheap plywood, which warped significantly after a few weeks, making it unusable with the laser.

There are several considerations when cutting mirrors with a laser cutter. First, if you cut them mirror side *up*, the laser beam is reflected back into the machine, and may damage the machine, so best practices say to cut it mirror side *down*.

Second, there are two kinds of mirrors. *Back mirrors*, which have a mirrored surface coated with glass, are by far the most common; we used them for the box in the picture. They are cheap and widely available, and the mirrored surface is protected by glass, so if you touch it, you can easily clean it. The downsides are that (1) passing through the glass dilutes the strength of the laser, so it becomes invisible after a few bounces, and (2) the laser is not bouncing off of the surface of the wall, but is bouncing off of the mirrored surface a few millimeters behind the wall, which introduces error that accumulates after several bounces. *Front mirrors*, which have the mirrored surface on the face of the material, are much more expensive and you must take care to keep them clean, but they provide an excellent reflective surface that avoids the two issues described above.

Further information and laser cutting files:
https://imaginary.org/hands-on/mirror-rooms

ALBA MARINA
MÁLAGA SABOGAL &
SAMUEL LELIÈVRE

ICERM; Université Paris-Saclay

laser-cut mirrors, carpenter's laser

This project illustrates real projective triangle tilings. Much as squares, equilateral triangles, or regular hexagons tile the 2D plane in a regular way, we can make a regular tiling of other spaces as well. The mathematical artist M.C. Escher famously made pictures showing triangle tilings of *hyperbolic space*, imagined as a disk in which straight lines are given by arcs of circles (upper right). Here we look at similar tilings, in *projective* geometry instead of hyperbolic geometry. I illustrated these as a stack of coasters, with each coaster displaying a way to tile a portion of projective space with triangles. Furthermore, the tilings in the set are all deformations of each other, meaning that, mathematically, we can interpolate from one tiling to the next through a continuous path of tilings. Some sense of this deformation is given by laying all the coasters out sequentially (bottom).

These tilings form a continuous family, so it would be best to experience this via 2D "stills" from different times during the deformation. Coasters turned out to be a fun way to do this. I also created an animation showing the continuous process (see link below).

Compared to working only via computer, there is definitely something to be said about producing physical objects. Looking at some members of the woodcut family as others were printing allowed me to notice new features about the differences between nearby tilings, and get a better sense of what is geometrically happening when one tiling deforms into another.

It took some trial and error to get the mathematical image in my mind and the results produced by the laser cutter to match up. As in hyperbolic space, the triangles here get smaller and smaller as they approach the outer rim. This leads to complications when a triangle is smaller than the laser beam width! After making a few test cuts that were either burned around the border (too many triangles) or had gaps in the interior (too few triangles), I took a new approach and rewrote the code to automatically discard triangles with area or height below a certain threshold. This allowed me to generate a large number of triangles, making sure to get all of the important information for the picture, while ensuring that the machine was only given instructions to engrave triangles big enough to be resolved by the laser.

Animation showing the continuous path of tilings: https://vimeo.com/368068637

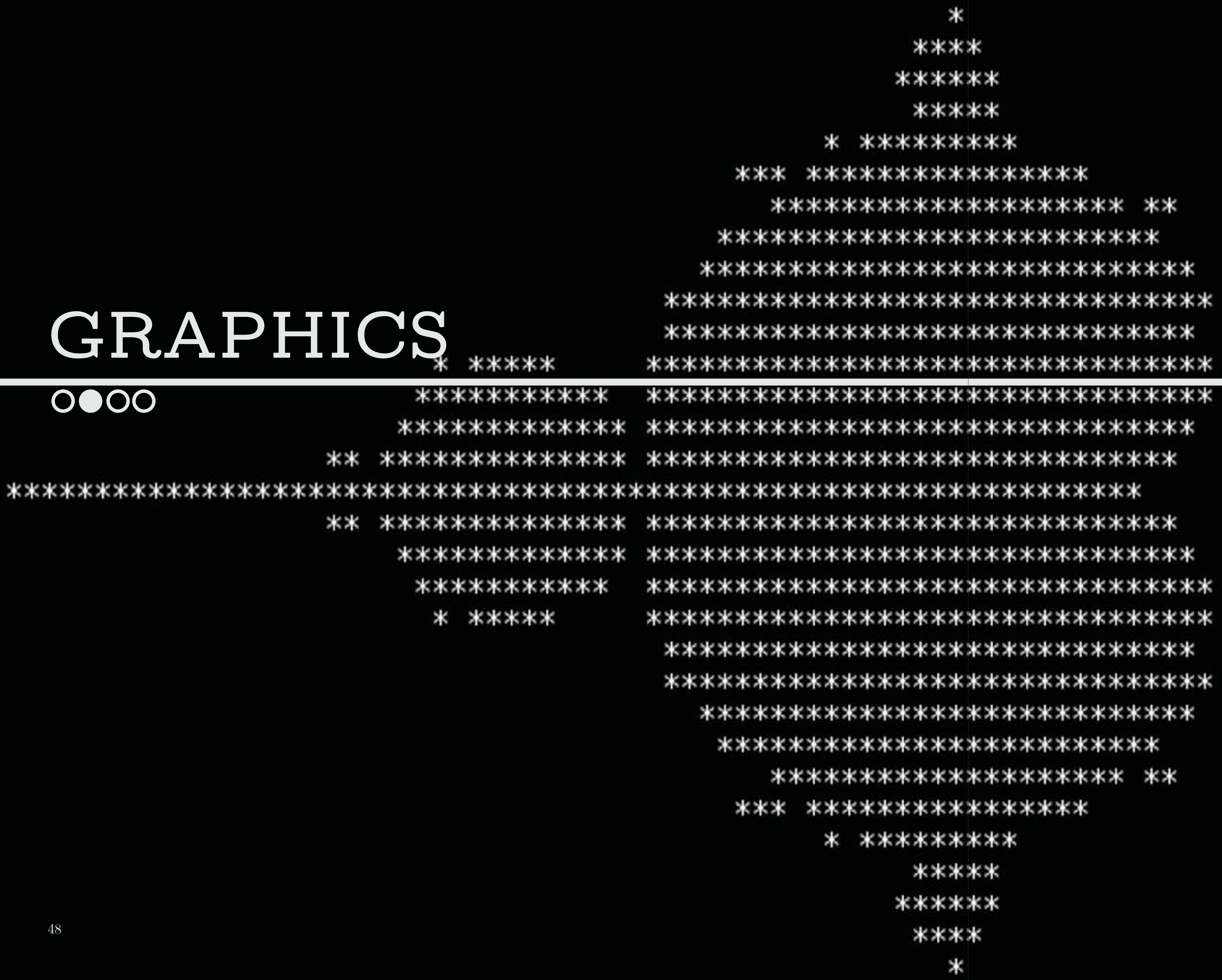
The original procedure for producing the symmetry groups used to create these images comes from: Anton Valerievich Lukyanenko, *Projective deformations of triangle tilings*, master's thesis (2008).

STEVE TRETTEL

ICERM

laser-cut wood

GRAPHICS



One of the most tantalizing mathematical illustrations ever is what I like to call the “asterisk picture” of the Mandelbrot set. Mathematicians had the idea of applying the function $f(z) = z^2 + c$ over and over, starting with the input $z = 0$, and using each output as the next input. They wanted to know what happens to the image, for different points c in the complex plane. Using a very early computer in 1978, they set it to work on this problem, and for each representative point in the box [-2, 2] x [-2, 2], they had the computer print an asterisk if the image stayed bounded, and print nothing if the image went off to infinity. The result was a shape that no one had ever seen before. To this day and beyond, exploring mathematics through computer graphics continues to elicit a sense of surprise and awe.

I am interested in the dynamics of the Riemann zeta function and its relatives. I wanted to prove their fractal nature, and to visualize and characterize them, particularly the neighborhoods of their complex zeros.

This is a unit square of the iteration fractal of the Riemann zeta function centered on the non-trivial zero located at approximately $0.5 + 28979.4096i$. This is one of many examples in which the non-trivial zero lies on an "island" of fractal surface close to, but separated from, the main "arms" that extend up and down the imaginary axis. Each of the protrusions emanating from the "island" (and the protrusions emanating from those protrusions) are scaled and distorted reproductions of the main fractal arms. This is to be expected for a fractal that exhibits self-similarity at all scales.

The iteration fractals of the Riemann zeta function, and its close relatives (the Hurwitz zeta functions and certain Dirichlet L-functions), all exhibit the same essential details: they have discrete clumps of fractal surface, with occasional separated "islands," all connected by points with predictable values, with their complex zeros situated in neighborhoods with predictable characteristics.

Working with computer graphics permits me to make arbitrary changes in scale in order to zoom out or zoom in, concealing or revealing detail, at will. There are two key classes of pixels rendered in the final image: those corresponding to starting values that blow up upon iteration (blue), and those that converge to a fixed point upon iteration (orange). Once I figured out the classification, it was straightforward to choose a color scheme that classified the pixels and reflected their particular iteration counts. Another obvious optimization would be to use interpolation to smooth the transitions between patches of color, but I'm happy with the hard transitions for now.

DAVID RAINFORD

Senior Counsel at Taylor Vinters

computer-generated graphic

Blog posts about this and similar images: https://primepatterns.tumblr.com/

For years, I have struggled to find good techniques for coloring mathematical surfaces with patterns. Coloring a sphere is easy, because it's so round and regular, but the bands in this image have not been easy to color well. My goal is to find *conformal coordinates* on this shape, which means that all the angles in the source pattern are portrayed correctly on the surface. Conformal coordinates make patterns more recognizable. I heard Steve Trettel's talk "What does a torus look like?" and that sent me back to an old project, which involved coloring this set of bands with a wallpaper pattern.

The innovation shown here comes from making the shape in software called Grasshopper (a plug-in for an architectural design software called Rhino). The good news is that I was able to paste a nice wallpaper pattern onto the surface and then make it look nice in Photoshop. The bad news is that the coordinates are not conformal: the pattern is indeed distorted.

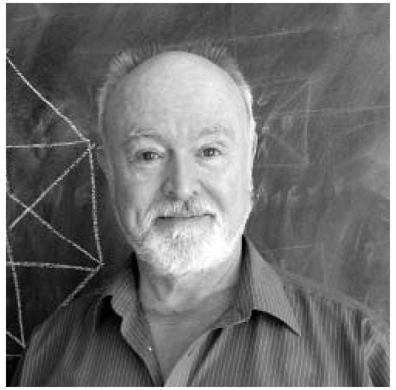
Grasshopper has a growing community of mathematician users who support each other in solving problems, which was very helpful: I had been coloring this shape by coloring a gigantic mesh, with approximately one vertex for each pixel of the coloring pattern. It really bogged down the computer. When I asked the Grasshopper community about it, Daniel Pinker, the creator of Kangaroo (another Rhino plug-in), solved my problem perfectly. I'll be working with this new technique for a long time.

It is still a long road to finding conformal coordinates on this particular structure, but I learned a lot about coloring tori. Surprisingly, the best way to find coordinates on donut shapes is to look for them in four dimensions, where they lie nicely on the 3D sphere.

The alert reader will see that there's an error in the image shown here: the pattern fails to match on the left side of the higher horizontal band. After all my Facebook friends had told me how beautiful it was, two mathematical artists looked at it for two seconds before saying, "Oh, but there's a mistake right there!" It's wonderful to be working in a community of people who take time to look closely at things.

Further information:

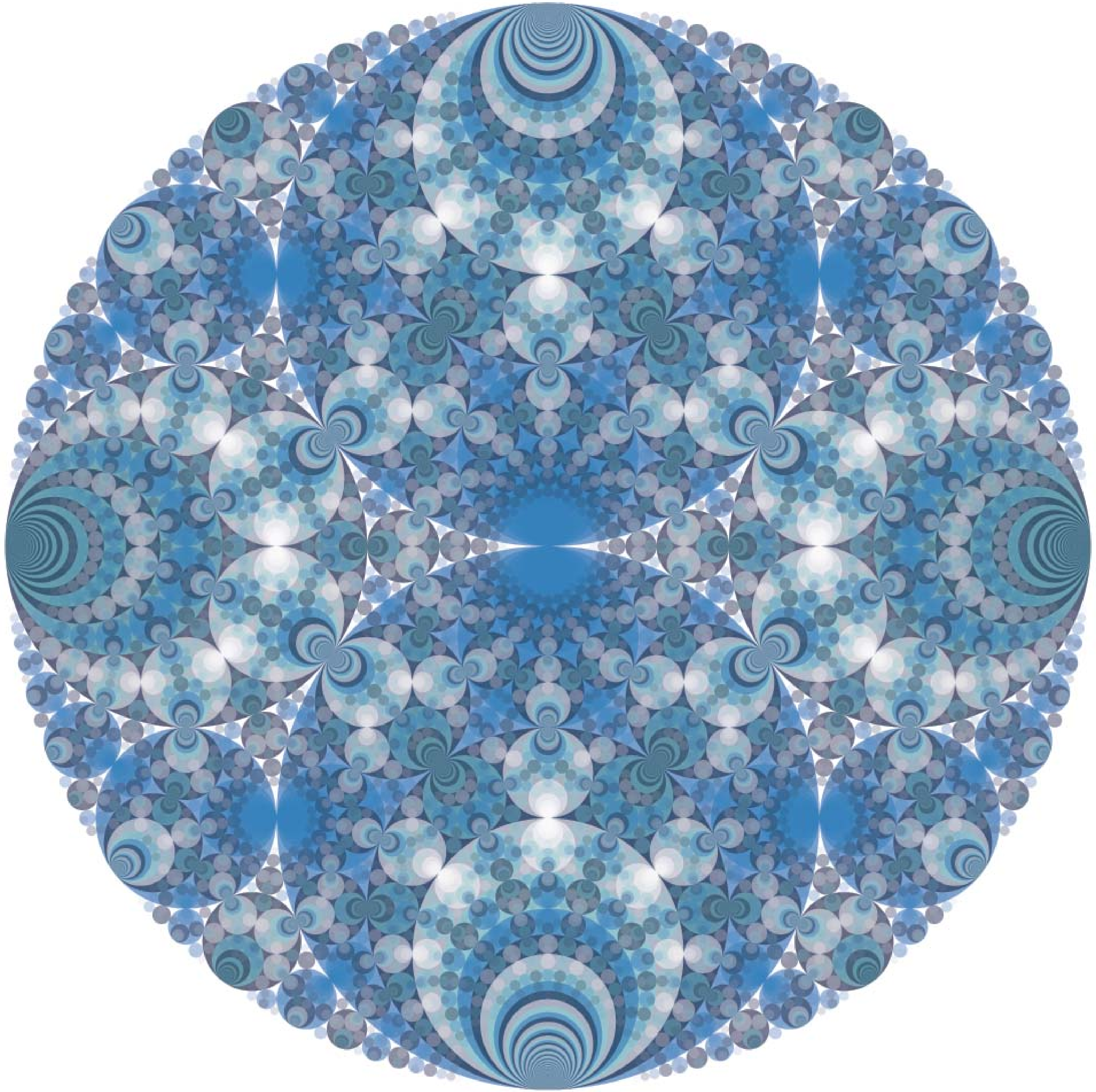
Wilder Boyden and Frank A. Farris, *Polyhedral Symmetry from Bands and Tubes*, Journal of Mathematics and the Arts, to appear.



FRANK A. FARRIS

Santa Clara University

computer-generated graphic

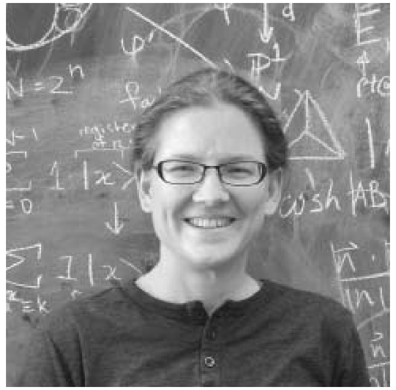


This image shows part of an orbit of a group of matrices over the ring of integers in an imaginary quadratic field. In particular, it is the projective special linear group over the the Gaussian integers. These matrices act as Möbius transformations, taking circles to circles. Asmus Schmidt used such arrangements of circles as a way to break up the complex plane into pieces, for the purposes of a continued fraction algorithm.

Schmidt's original work took place before the advent of powerful computer graphics systems, and his papers included some figures showing only half a dozen or a dozen circles, illustrating only one level of the recursive structure. I wanted to more fully understand the structure, and I wanted to find out what happened in other imaginary quadratic fields. Using computer graphics allowed me to work interactively, changing parameters and redrawing frequently. It was also important to me to be able to print out the pictures and work with colored pencil, ruler, and compass on top of them. For example, in some pictures the human eye picks up "ghost circles" that appear to fit in the geometry but are missing. I measured these with a ruler and compass to conjecture their exact form, and then proved that they exist.

Easy access to exploring these pictures led me to conjecture and prove a variety of things about the relationships between the arithmetic of the imaginary quadratic field and the pictures themselves. For example, the circle arrangement is connected if and only if the field is Euclidean.

The experience of programming the pictures was itself enlightening: for example, a naive algorithm exploring the Cayley tree results in varying resolution in different parts of the picture, and inadvertent congruence conditions add or remove extra layers of circles. Often, errors in coding led to useful insights. Later on, I worked harder to make the pictures more beautiful, and learned about color choices and vector graphics, among other things.

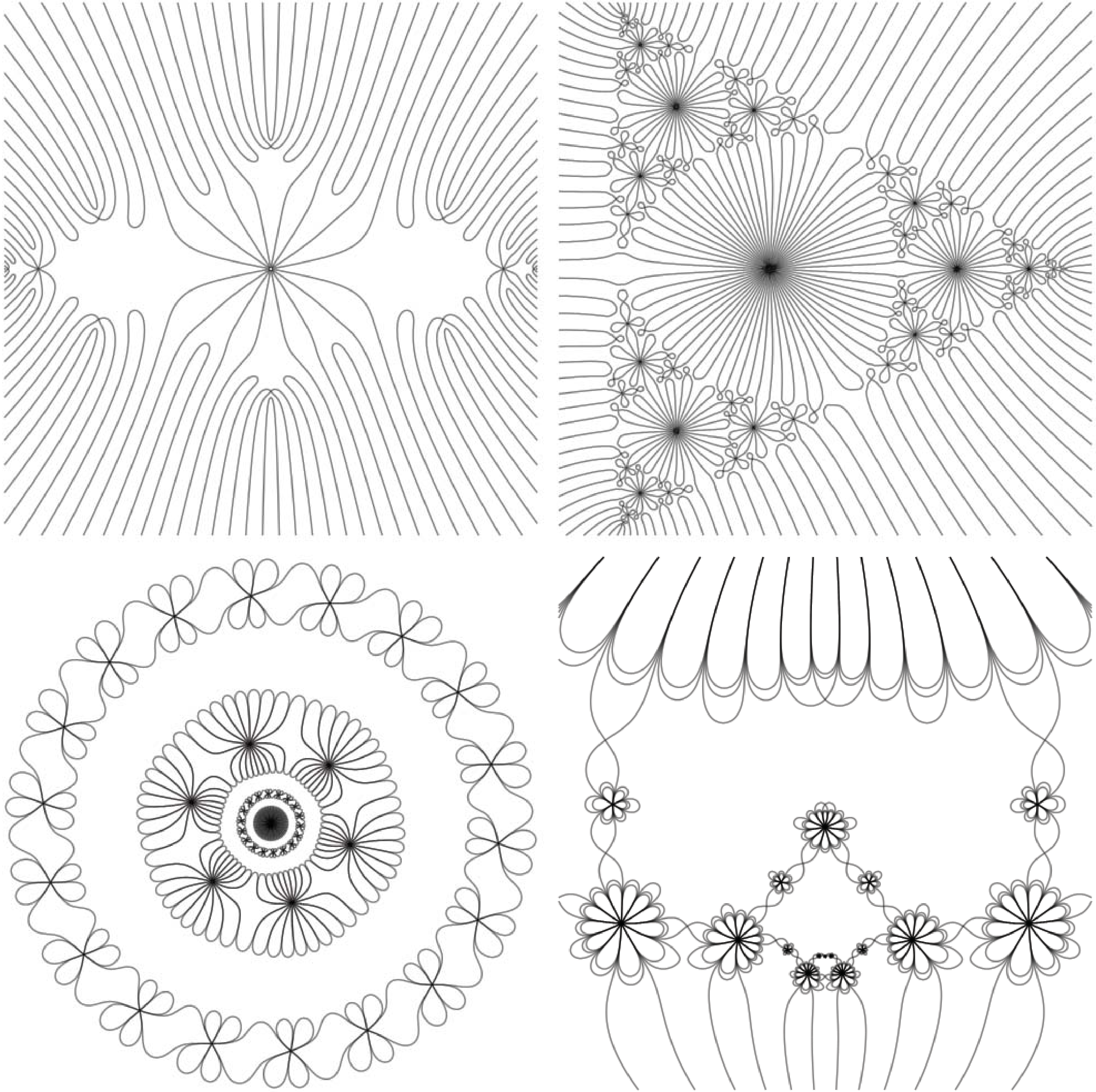


KATHERINE E. STANGE

University of Colorado Boulder

computer-generated graphic

Further information: Katherine E. Stange, *Visualizing the Arithmetic of Imaginary Quadratic Fields*, International Mathematics Research Notices, Volume 2018, Issue 12, June 2018, pp. 3908–3938.



While studying properties of graphs of complex functions, specifically level curves of the real part of iterates of complex functions, we stumbled on the first image shown here (top left). Our initial reaction was that this image looked like something from a coloring book for adults. As we examined this image more closely, we noticed that the Julia set for the function $f(z) = z^2 - 1$ appears to lie in the middle. In addition, the decorations inside the Julia set correspond to critical points for the real part of the function, all of which are saddle points.

From there we decided to explore functions with interesting Julia sets to see if we could find more artistic images, which we made into a coloring book (see below). The image in the top right comes from a function whose Julia set resembles the Sierpinski triangle. The flowers we see are from the poles of the sixth iterate of this rational map. Similarly, the image in the bottom left comes from a function whose Julia set consists of concentric circles, and the poles of its third iterate appear as flowers.

For the image in the bottom right, we started exploring the interplay between zeros and poles of different orders. By experimenting with the window, which cuts off parts of flowers created by poles, we were able to create this image of flowers in a window box.

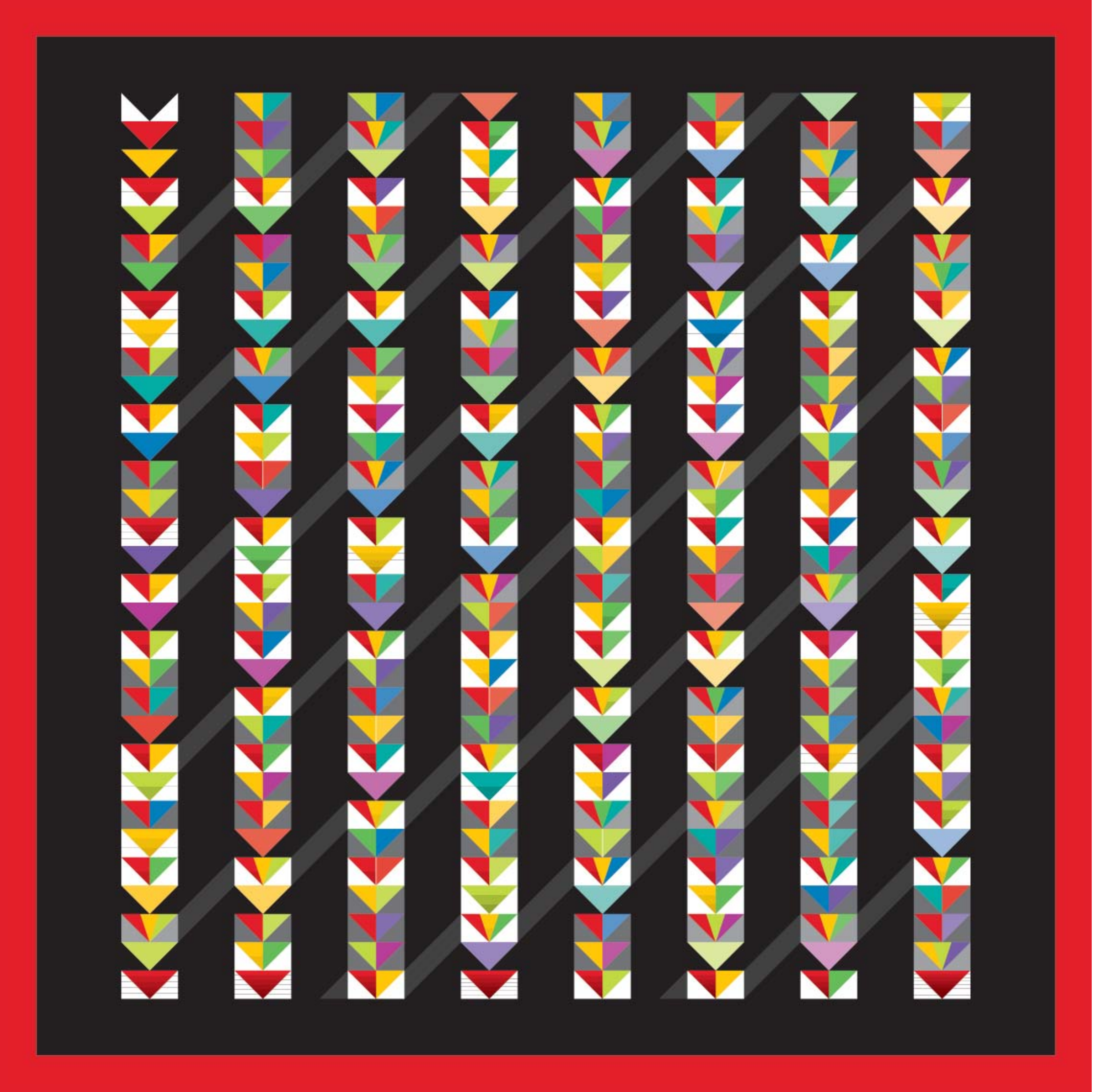
Functions used:
top left: 6th iterate of $z^2 - 1$, level 0 contours
top right: 6th iterate of $z^2 - 0.6/z$, level 0 contours
bottom left: 3rd iterate of $z^3 + 0.0001i/z^3$, level –2 and 2 contours
bottom right: 5th iterate of $2(z - 3)^2(z + 10)/z^3$, level 0 contours

Our coloring book:
Julie Barnes, William Kreahling, and Beth Schaubroeck, *Coloring Book of Complex Function Representations*, Mathematical Association of America, 2017.
Further information:
Julia Barnes, Clinton Curry, Elizabeth Russell and Lisbeth Schaubroeck, *Emerging Julia Sets*, Mathematics Magazine, Vol 88 (2), pp. 91–102, April 2015.
Julia Barnes and Lisbeth Schaubroeck, *Any Way You Slice It, It Comes Up Julia Sets*, Mathematics Magazine, Vol 92 (1), pp. 3–16, February 2019.

JULIE BARNES &
BETH SCHAUBROECK

Western Carolina University;
U.S. Air Force Academy

computer-generated graphics



This print, titled “Prime Goose Chase,” has a structure based on a traditional quilt pattern called Wild Goose Chase, while its content relates to the Fundamental Theorem of Arithmetic. The integers from 1 to 256 are the “geese,” and the prime decomposition of each integer is shown using colored triangles. There are eight columns of numbers, starting with a black triangle representing 1 at the upper left. Solid triangles are used for primes, and each prime is assigned a unique color: 2 = red, 3 = gold, 5 = yellow-green, and so on, until 19 = magenta. As larger primes are needed, more colors are created by adding white to these basic 8 hues. Composite numbers are represented by subdivided triangles. For example, since 6 = 2 x 3, it is half red and half gold. Powers of primes are shown using horizontal strips, in different shades of the base color.

I believe that integers have personalities, largely based on their prime factors. In this design, my goal was to produce a visual table that would show the factoring of individual integers and make it easier to observe number patterns.

This work was produced as a digital print using Adobe Illustrator. This medium provided the precise shapes and range of colors my design required. Developing this piece improved my appreciation of the integers and their inherent rhythms.

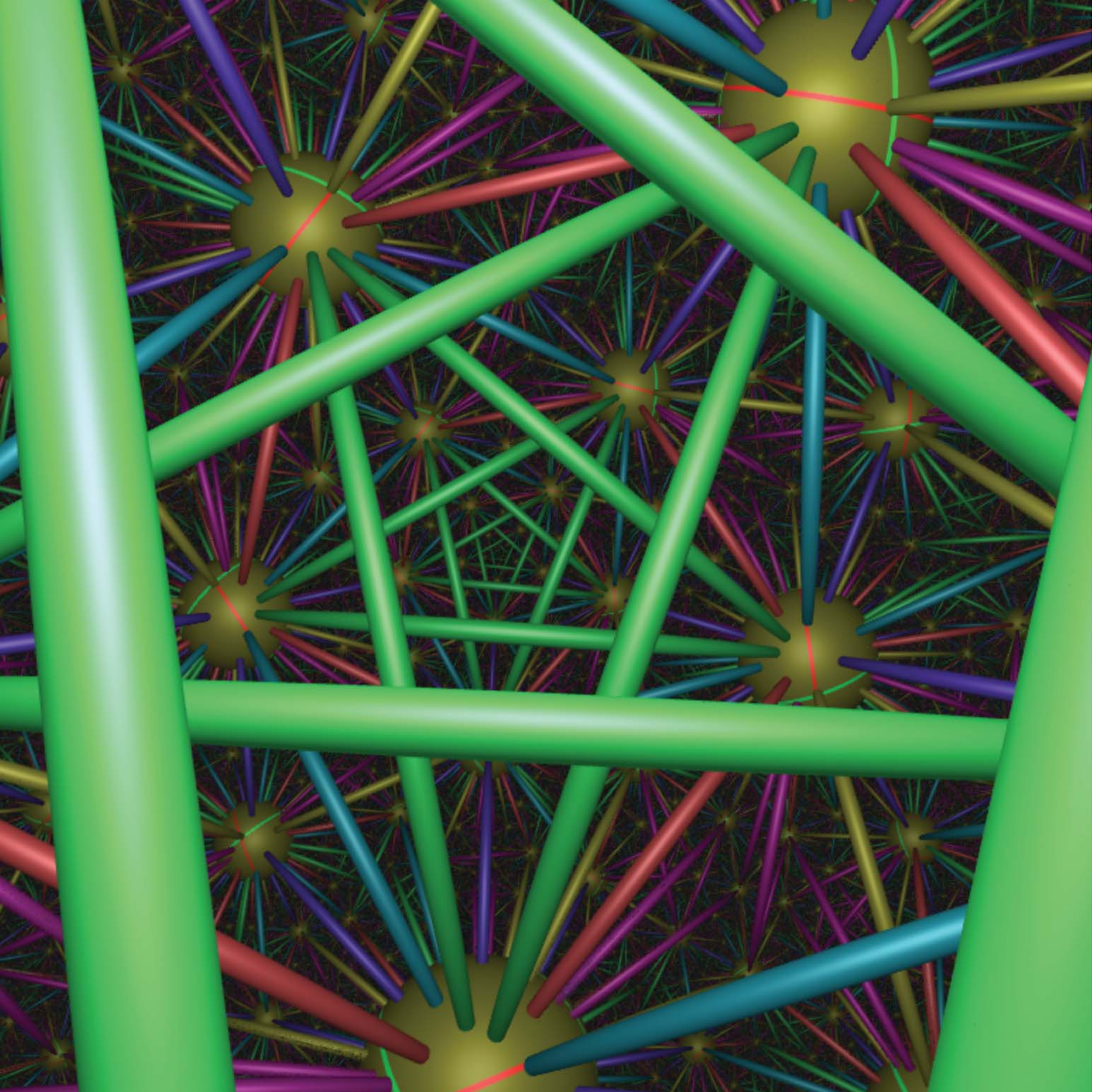
In my first draft, the black spaces between the columns seemed too static. I solved this by adding sloping gray bands connecting multiples of 6. This also illustrated the fact that all primes, after 2 and 3, are plus or minus 1 from a multiple of 6. I enjoy finding visual elements that reinforce mathematical concepts.

MARGARET KEPNER

Independent artist

computer-generated graphic

Gallery of my mathematical artwork: http://mekvisysuals.net



I added a feature to the 3-manifold software SnapPy that shows you the view you have if you were living inside a hyperbolic 3-manifold. I just tried out some random census manifolds to see whether I got an image that would make a nice demo. I was pleasantly surprised to find this one (s431). The image shows the edges of the triangulation and a cusp neighborhood as you look down a closed geodesic in the hyperbolic 3-manifold. Walking down the closed geodesic once rotates the hyperbolic 3-manifold by 2/5 of a turn, and you see a beautiful helix formed by one of the edges.

A 3D print of a tiling by fundamental domains is always distorted in some way because we live in a Euclidean world (at least on human scales). However, the image you see here is the true, undistorted inside view of the hyperbolic 3-manifold.

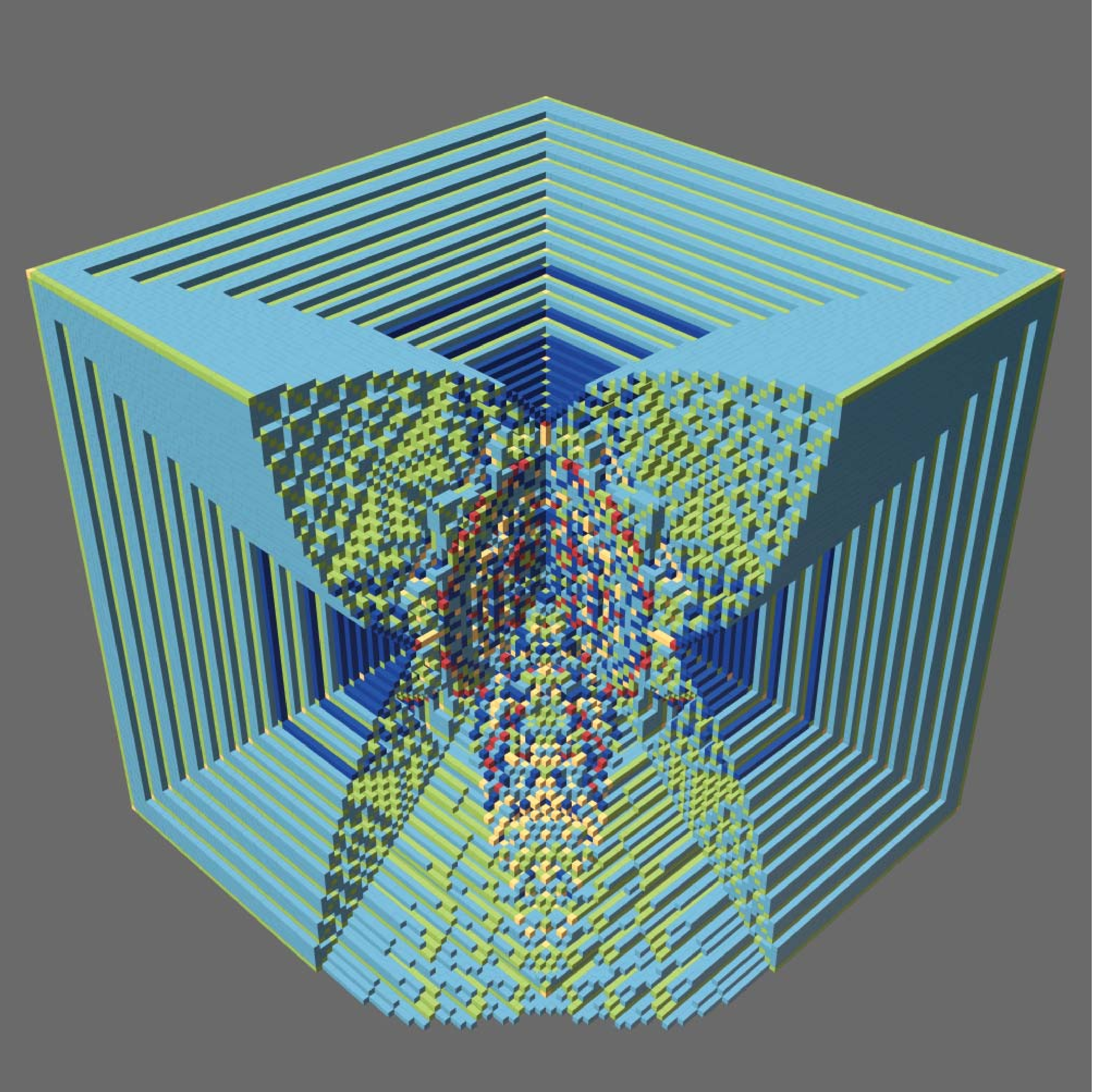
This is still a work in progress. For example, I am still trying to come up with a good way to show which tetrahedra are the same in the manifold. I first tried to do this by coloring a small ball around their incenters, but it turns out that our Euclidean intuition about where we the expect the incenter of a tetrahedron to be is completely off in the hyperbolic world.

MATTHIAS GOERNER

Pixar Animation Studios

computer-generated graphic

To download SnapPy: https://snappy.math.uic.edu/



Chip firing is a process on a *graph*, which is a network of vertices connected by edges. Initially, a stack of "chips" is placed on each vertex. A vertex is allowed to "fire" its chips if it has at least one chip for each vertex to which it is connected (its "neighbors"). When a vertex fires, it sends one chip to each neighbor. We repeat this process until no vertex has enough chips to fire, which we call a "final configuration."

The illustration shows chip firing on a cubical grid in 3D. Every vertex has 6 neighbors, so it needs 6 chips in order to fire. In this example, every vertex except for the origin started with 4 chips, and the origin started with 100,000 chips. The final configuration forms a cube. Vertices with 0, 1, 2, 3, or 4 chips are rendered as small cubes of different colors. Most of the vertices have 5 chips in the final configuration, so we rendered them as transparent, to exhibit the funnel-like shapes going into the cube from each of its faces. The octant of the cube facing the viewer is also transparent, and only half of the lower octant is shown. This is to show the inner structure of the final configuration, which resembles leaves.

A computer-generated rendering allows us to explore both the outer shape and the inner structure of the pattern interactively. Our original plan was to 3D-print the patterns inside the object, but they turned out to be too fragmented, so a 3D print would fall apart. Having a digital model allowed for switching on and off the visibility of certain vertices, which made it easy to identify larger structures within the geometry.

Initially, we represented each vertex by its own cubical geometry object in the software Blender, but this yielded too many objects and crashed the Blender GUI. Therefore, we decided to create only six objects, each representing one of the numbers of remaining chips in the final configuration. Using the clipping plane built into the Blender camera, we could also clip the object and thus show any planar slice of the illustration. However, we wanted spherical, not planar, clipping. Thus, we decided to build more geometries again: six geometries for each "shell" around the origin. That is, all cubes of a fixed radius distance to the origin and of the same color were collected in one object. By hiding the outside shells, it became possible to interactively show the internal layers of the structure.

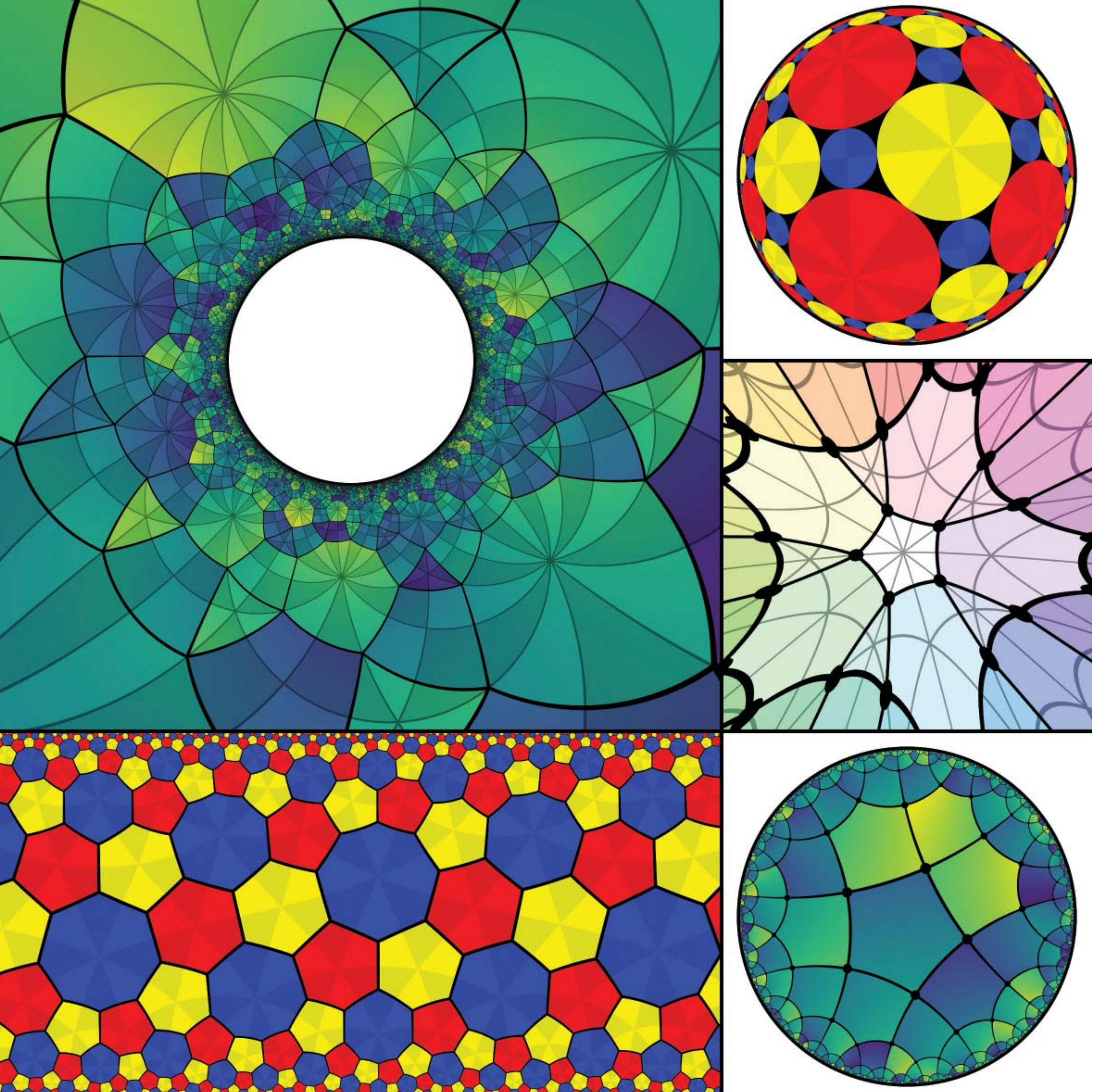
Further information:

Caroline J. Klivans, *The Mathematics of Chip-Firing*, CRC Press (2018).

MARTIN SKRODZKI,
CAROLINE J. KLIVANS &
PEDRO F. FELZENSZWALB

ICERM; Brown University;
Brown University

computer-generated graphic



This illustration shows several tilings of the hyperbolic plane in various different projections, or models. Hyperbolic geometry uses a set of axioms that are slightly modified from the Euclidean geometry that students learn in high school. It enables constructions of tilings that would be impossible in the Euclidean plane, for example the alternating regular hexagons and octagons in the lower-left image. I've been interested in 2D and 3D tilings for many years, and I find the repetition of shapes with varying sizes and curving distortions in hyperbolic tilings to be particularly mesmerizing.

These images are all generated by a *fragment shader* that runs on a computer's graphics processing unit (GPU). I got into this style of programming through the web site shadertoy.com, which hosts shaders for a community of computer graphics hobbyists and professionals. Generating images like these in a pure fragment shader is challenging, because it inverts the typical paradigm of graphics programming: Instead of starting from a description of a polygon's vertices, and coloring in all of the pixels it covers, the program must instead decide independently for each pixel which of the many possible polygons it lies in and color it accordingly. I wrote the shader that generated these images, which you can find in the link below. Figuring out how to write it was a really enjoyable puzzle that unfolded over many months.

One of the biggest lessons I learned along the way was that I had started out using the wrong projection of the hyperbolic plane to do most of my math. Initially, I was using Poincaré disk coordinates to represent points on the plane. In the Poincaré disk model, there are two types of lines: circles orthogonal to the disk and diameters of the disk. The coordinates I was originally using required inefficient and over-complicated code to correctly handle the two different types of lines. Another big problem I ran into was lack of precision close to the edge of the disk. Switching everything over to the upper hyperboloid model led to much simpler code and improved performance, but it took me a while to figure out the best way to program everything. Between a couple of math papers I dug up online and some Twitter correspondence with other graphics programmers, I learned how to perform all of the necessary geometric operations both efficiently and accurately.

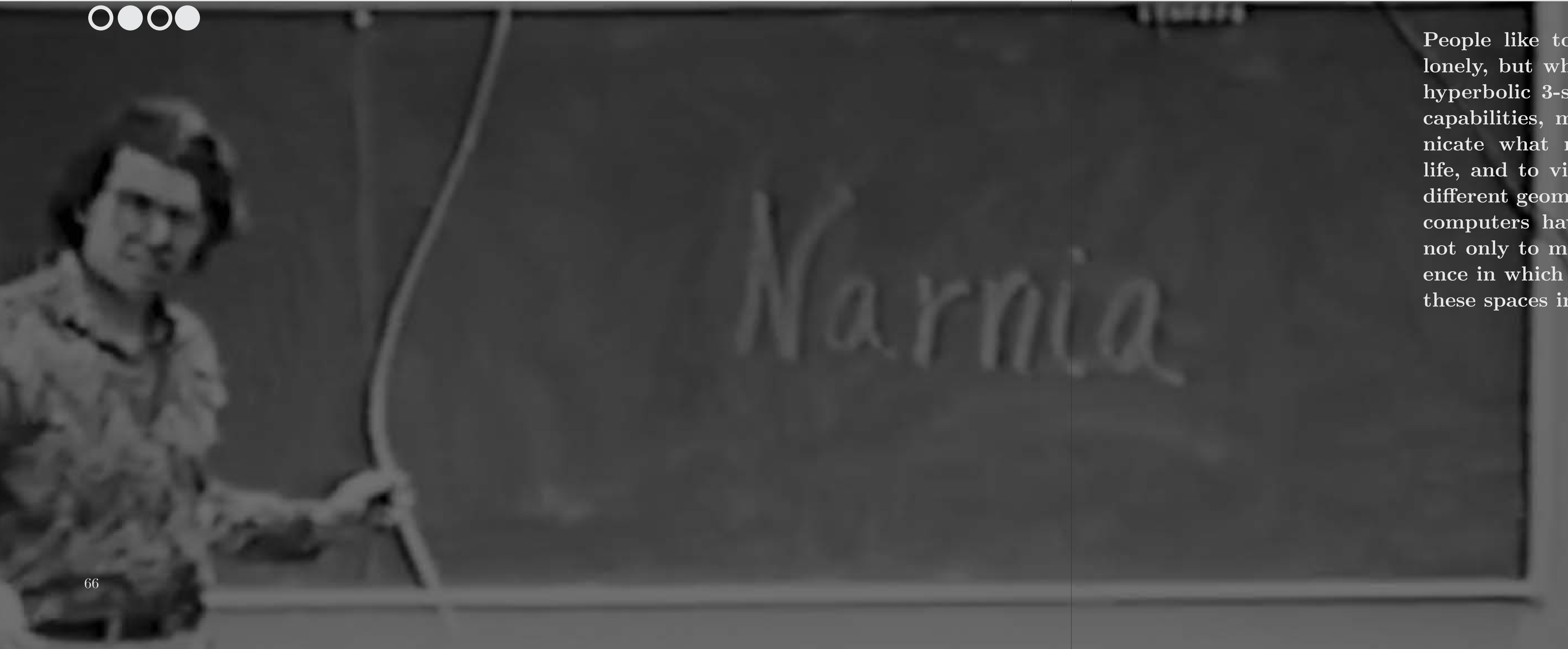
The shader that generated these images: https://www.shadertoy.com/view/3tsSzM

MATT ZUCKER

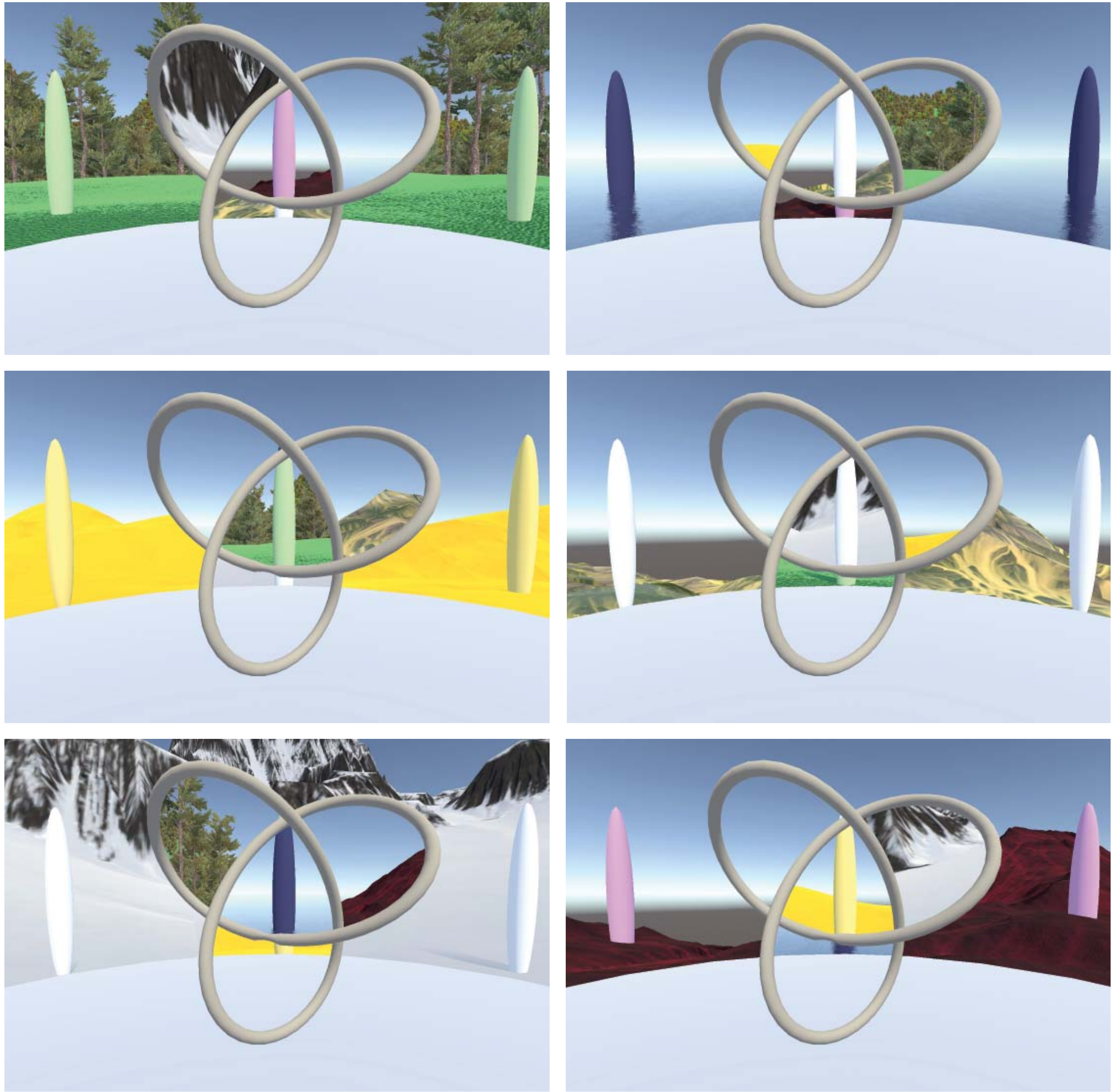
Swarthmore College

computer-generated graphics

VIDEO AND VIRTUAL REALITY



People like to say that hyperbolic space is cold, dark and lonely, but what would it *really* be like to wander around in hyperbolic 3-space? As soon as computers had the requisite capabilities, mathematicians were making videos to communicate what mathematical objects would look like in real life, and to visualize what it would be like to move through different geometries. Now that processors are much faster and computers have excellent graphic capabilities, it is possible not only to make a video, but to make an interactive experience in which people can direct their own movement through these spaces in real time, in virtual reality.



In his video "Knots to Narnia," William Thurston uses large wire knots to demonstrate his concept of knots as portals, where he actually steps through the knot to move back and forth between, in his conception, Earth and Narnia (see previous page). Virtual reality gives us the ability to bring to life this experience of actually stepping into other worlds and *seeing*, not just imagining, what it looks like on the other side.

These pictures show the six-fold branched covering of order two of the trefoil knot, generating the dihedral group of symmetries of the triangle. The three outer loops of the knot correspond to reflections, and the inner region corresponds to the rotation by 120°.

To create this virtual reality experience, I had to learn a lot about cyclic branched covers of knots, as I had to construct them in the software implementation. This led me to discover a construction from Poul Heegaard's dissertation from 1898 that could be implemented to simulate these portals in virtual reality. It consists of gluing a cone to the knot, which serves as the branch cut along which the different worlds are glued together.

At first, I thought about implementing portals as surfaces somehow spanned by the knot. The most obvious choice was to try Seifert surfaces, but this turned out to be a dead end. The next attempt was not to construct the branched covering, but only to simulate it through the knot projection on the screen. This approach used a variant of Reidemeister moves to keep track of the worlds the regions led to, but it turned out to be quite complicated and unstable at the crossings. Luckily, I then found the reference to Heegaard's construction in John Stillwell's *Classical Topology and Combinatorial Group Theory*, and I was able to create the virtual reality world.

It was a challenge to reduce the computational load of the software, which has to compute which world to show for each pixel of both screens in the head-mounted display. I was able to achieve this by using shaders to offload much of the calculations onto the graphics processing unit, thanks to some tips from Roice Nelson. The main computational load is now the rendering of the worlds. I wanted to create worlds that are interesting to look at, but not so interesting that they take away the focus from the knot itself.

Software download and information: https://imaginary.org/program/knotportal
Thurston's original video, "Knots to Narnia":
https://www.youtube.com/watch?v=IKSrBt2kFD4

MORITZ L. SÜMMERMANN
University of Cologne
virtual reality

In 2018, Olga Frolkina of Moscow State University published a proof of the impossibility of packing an uncountably infinite number of Möbius strips into an infinite 3D space. Evelyn Lamb wrote about the proof for *Quanta Magazine*, and I created this animation to accompany the article. This is visual storytelling that hints at Frolkina's result rather than an accurate simulation. Since it's impossible to render an infinite number of anything, we had to make do with "a lot" of Möbius strips in a finite space.

Working in 3D simplifies and expedites certain processes, leaving more time for creativity and ambitious ideas. I would not have tackled this project if I had to draw out dozens of Möbius strips and then animate them frame by frame! Using Cinema 4D, I modeled just one strip, cloned it, and then ran a simulation. Also, in 3D software it's possible to both simulate real-world dynamics accurately and ignore them at will, so you can create situations that are believable but physically impossible – like magic!

Despite being a mathematically-inclined artist, I'm not a mathematician and was not previously aware of this type of packing problem. Although I wouldn't expect anyone to learn much solely from my artwork, I hoped that it would spark intrigue and invite people to read about this fascinating research.

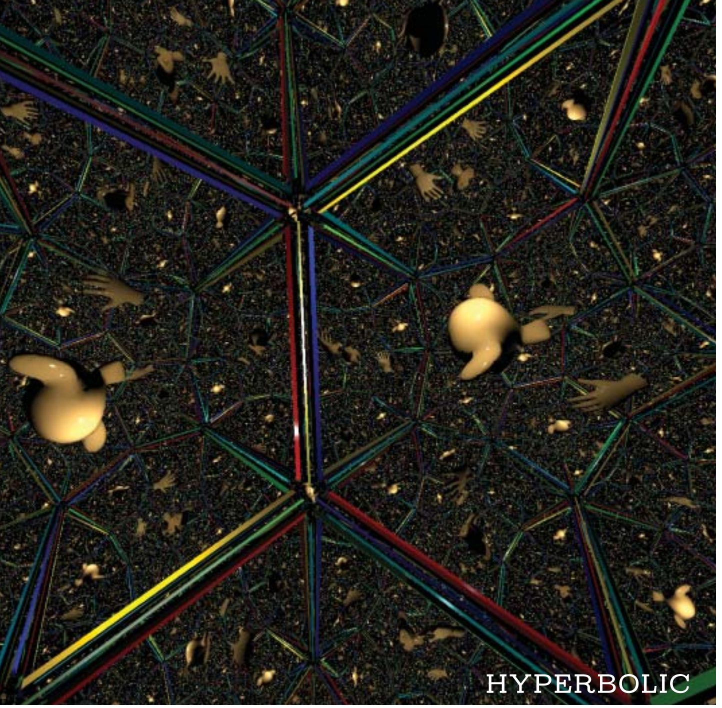
Coincidentally, the difficulties I encountered in creating this illustration echoed the conclusions of the mathematical result – it really is challenging to cram Möbius strips into a 3D space (albeit for different reasons). In theory this should have been straightforward to simulate. In practice, though, all sorts of weird stuff happened. The Möbius strips would glitch and jiggle around wildly, sometimes they'd pop right through the container like ghosts through walls, and getting them to look sufficiently smushed and then pop out dramatically was no picnic either. I had to keep tweaking various parameters to get the whole system to behave. But this is a common enough occurrence – there might as well be some kind of pseudo-scientific "law": "projects that seem simple are usually much more complicated than one might anticipate."

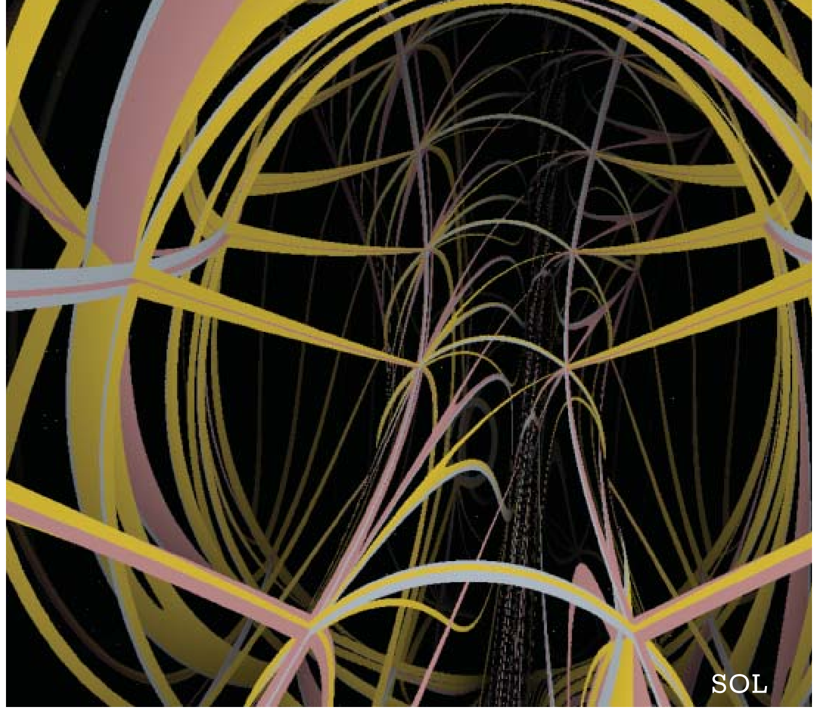
The article with the animation: Evelyn Lamb, *Möbius strips defy a link with infinity.* Quanta Magazine, February 20, 2019.

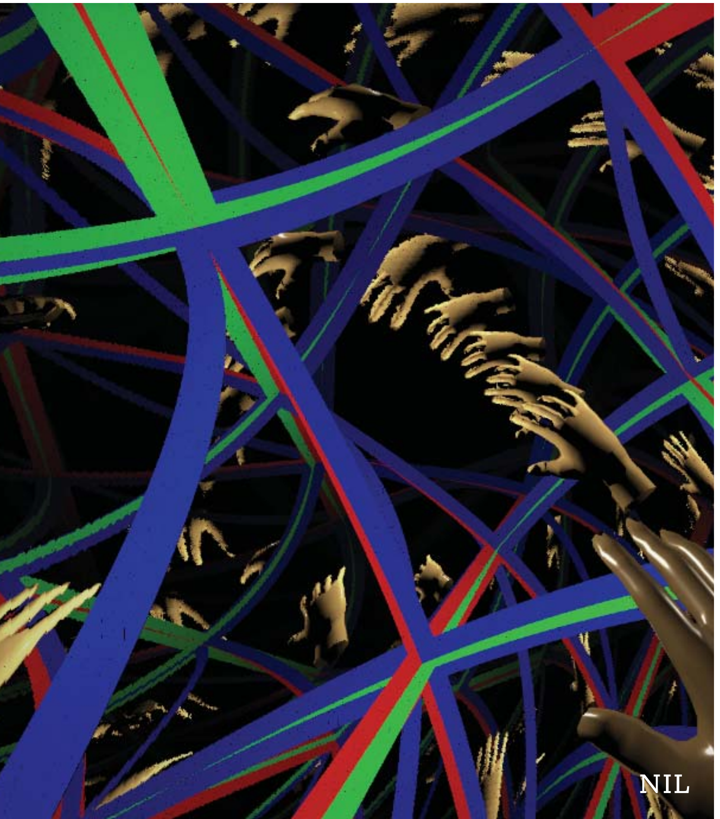
OLENA SHMAHALO

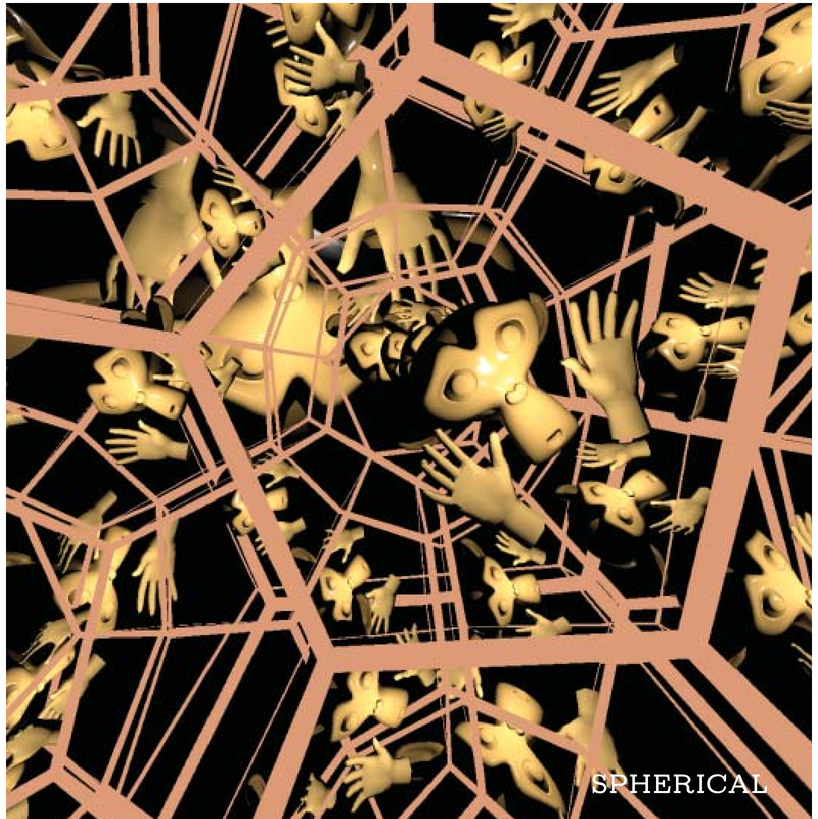
Quanta Magazine

computer-generated animation









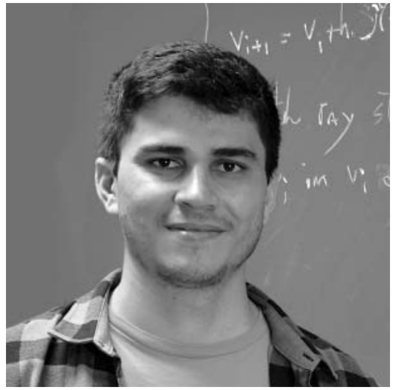
These images are inside views of famous 3-manifolds, with their geometries modeled by Thurston geometries. Such spaces date back to the famous *Thurston geometrization conjecture*, proved in 2003 by Grigori Perelman. The theorem states that every compact 3D manifold decomposes into pieces whose geometry is modeled by one of the eight Thurston geometries. The even more famous Poincaré conjecture is a corollary of this theorem.

It is difficult to visualize such beautiful spaces due to complications imposed by topology and geometry. The main effort to visualize such spaces was at the Geometry Center from 1994 to 1998 under the leadership of William Thurston. This program studied and disseminated modern geometry using interactive visualization based on the traditional rasterization pipeline (from computer graphics). Two problems arise: the scene must be replicated to "unroll" spaces with complicated topology, and to rasterize a scene it is necessary to compute rays between the scene points and camera position (a hard problem). To produce our images we use a ray-tracing framework, RayVR, that overcomes such difficulties by operating intrinsically in the geometry and topology, searching for the scene objects by tracing rays from the viewer toward the pixels. RayVR is a project of the VISGRAF Laboratory that I developed with Vinícius Silva and Luiz Velho.

Our images illustrate four 3-manifolds with their geometries modeled by hyperbolic, spherical, Nil, and Sol geometries – we think these are the most interesting Thurston geometries. Such manifolds are extremely important in the study of 3-manifolds, and we can visualize being immersed in a scene where powerful modern mathematics was developed.

The hyperbolic and spherical manifolds are both obtained by gluing, with appropriate rotations, opposite faces of special regular dodecahedrons in hyperbolic and spherical spaces. A monkey's hand, or face and hands, are added to the scene to give more context. The Nil and Sol manifolds are obtained by identifying faces of the cube. In both cases, two faces are identified by translation, and we identify the remaining faces in such a way to give rise to the Nil and Sol manifolds. The face pairing makes the rays iterate, giving a tessellation of Nil and Sol spaces by cubes.

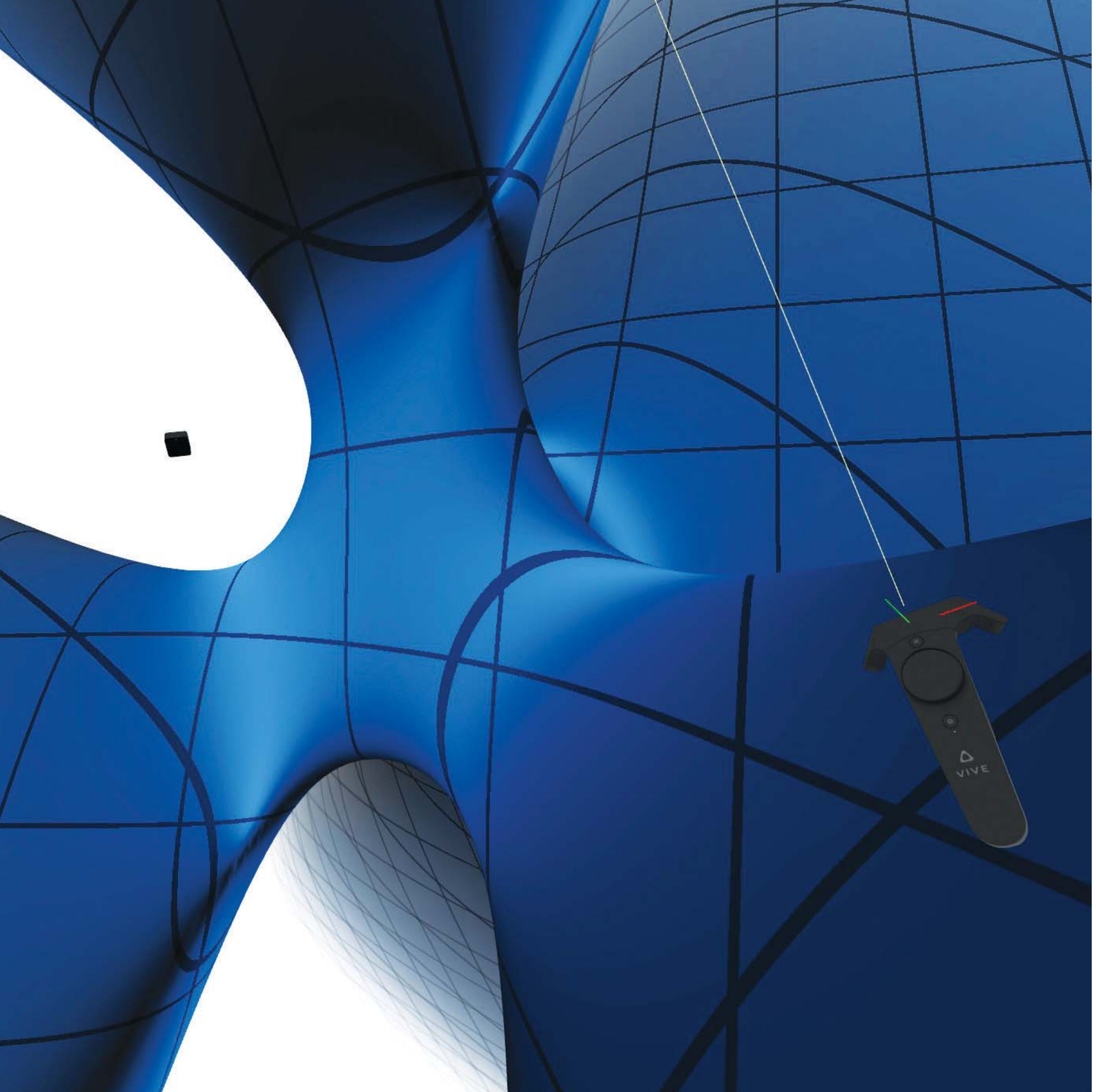
For associated videos, papers, technical reports, and texts:
https://www.visgraf.impa.br/ray-vr/



TIAGO NOVELLO DE BRITO

Instituto de Matemática Pura e Aplicada (IMPA)

virtual reality



Algebraic surfaces are the 2D solution sets of systems of polynomial equations. For example, a cylinder is an algebraic surface, since it can be described as the set of all points (x, y, z) so that $x^2 + y^2 = 1$. Algebraic geometers like to fill in some missing points "at infinity" by considering these surfaces inside of *projective space*. For the cylinder, this adds one extra point, infinitely far away, where all of the straight lines on the cylinder meet up.

When we learn to compute in projective space by hand, we work with several sets of compatible equations in different "affine charts." It can be unclear how the pictures in these charts relate to each other or how they form a geometric whole. The goal of my illustration is to make the projective nature of algebraic surfaces viscerally intuitive, by allowing the user to "click and drag" from one affine chart to another. I do this in virtual reality with an HTC Vive headset.

There seem to be two kinds of geometers: those who imagine that their mathematical objects are something that could fit on your desk, and those who imagine their mathematical objects are room-sized, something you could travel through. I have always been the first kind of geometer, but I have been curious to experience the second point of view. A computer visualization on a monitor reinforces the first point of view – after all, the image on the monitor is only a few inches wide. Virtual reality, on the other hand, allows us to experience the second one.

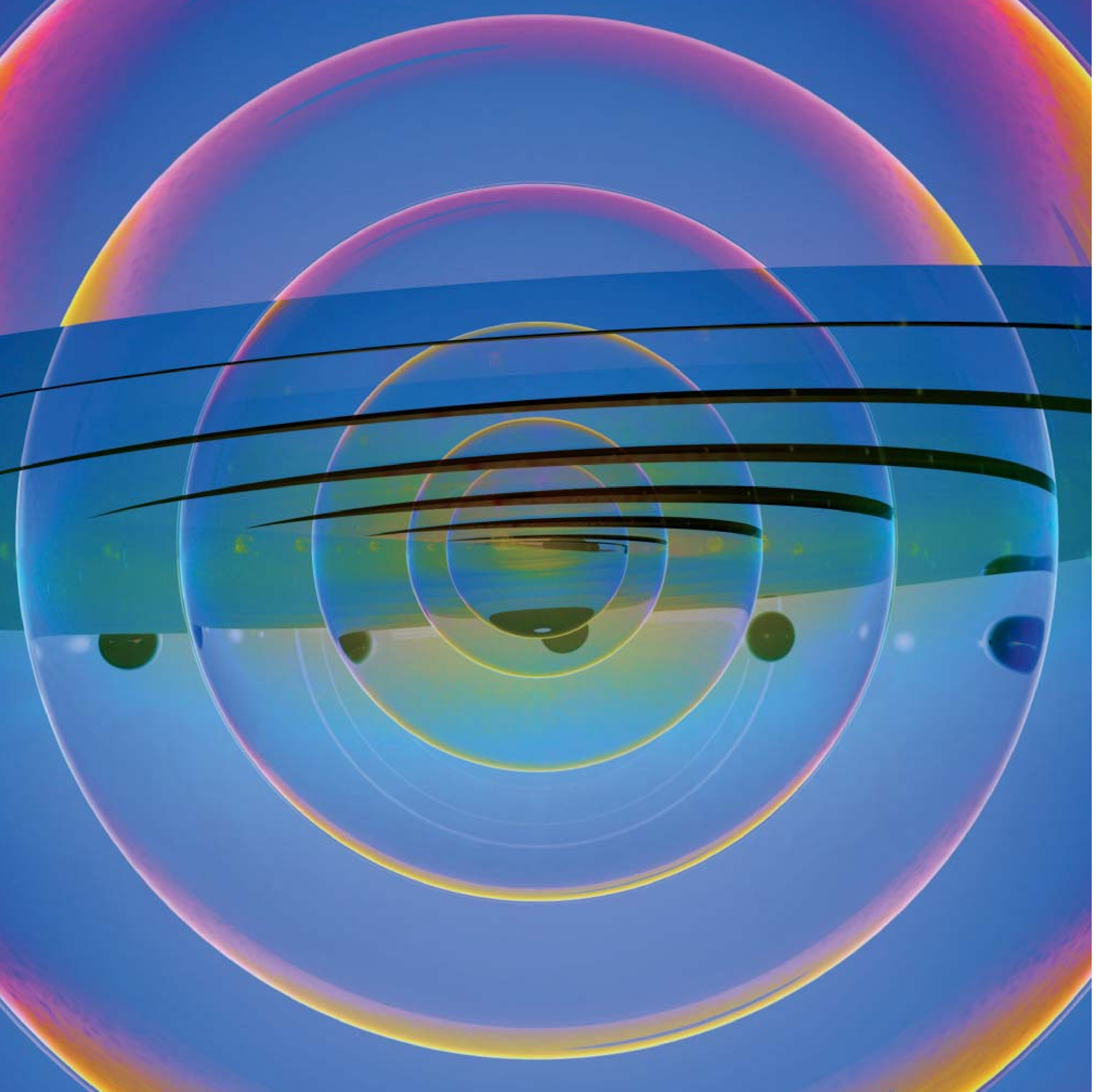
Originally, I displayed the algebraic surface in 2D, and the user manipulated it using a keyboard and mouse. The result was completely unsatisfactory; it was too difficult to understand and control. Switching to virtual reality makes it much easier to get a sense of the shape of the surface, and allows the user to interact with the surface by moving in 3D, which is a much more natural way to interact with something in 3D. Also, when I first got the program running in VR, the surfaces didn't have the coordinate grid on them, which made them into big blue blobs that were hard to read. Adding the grid makes it easier to get a sense of the shape of the surface when it is far away.

Video of VR experience and further information:
https://www.youtube.com/watch?v=MYfUbeiHkfU

SEBASTIAN BOZLEE

University of Colorado Boulder

virtual reality



How do you illustrate high-dimensional spheres? I wanted an image that would convey a sense of interleaving, as equatorial discs are revolved to spin out higher-dimensional balls. I also wanted to capture the beauty and mystery of these objects: the "music of the spheres" that animated the imagination of cosmologists of old.

In my multivariable calculus video course, one of the capstone projects is the computation of volumes of balls of radius r in arbitrary dimensions. The sequence, starting with the familiar πr^2 in 2D and $4/3\pi r^3$ in 3D, becomes an elegant interlaced sequence of recursive formulae in arbitrary dimensions.

Computer graphics has no limits. All the geometric and topological entities I imagine can be cast into colors and forms on the screen. Color is the ultimate hard choice. Given that my calculus YouTube series is called "Calculus BLUE," I changed the initial greyscale images to an oversaturated blue-themed scheme.

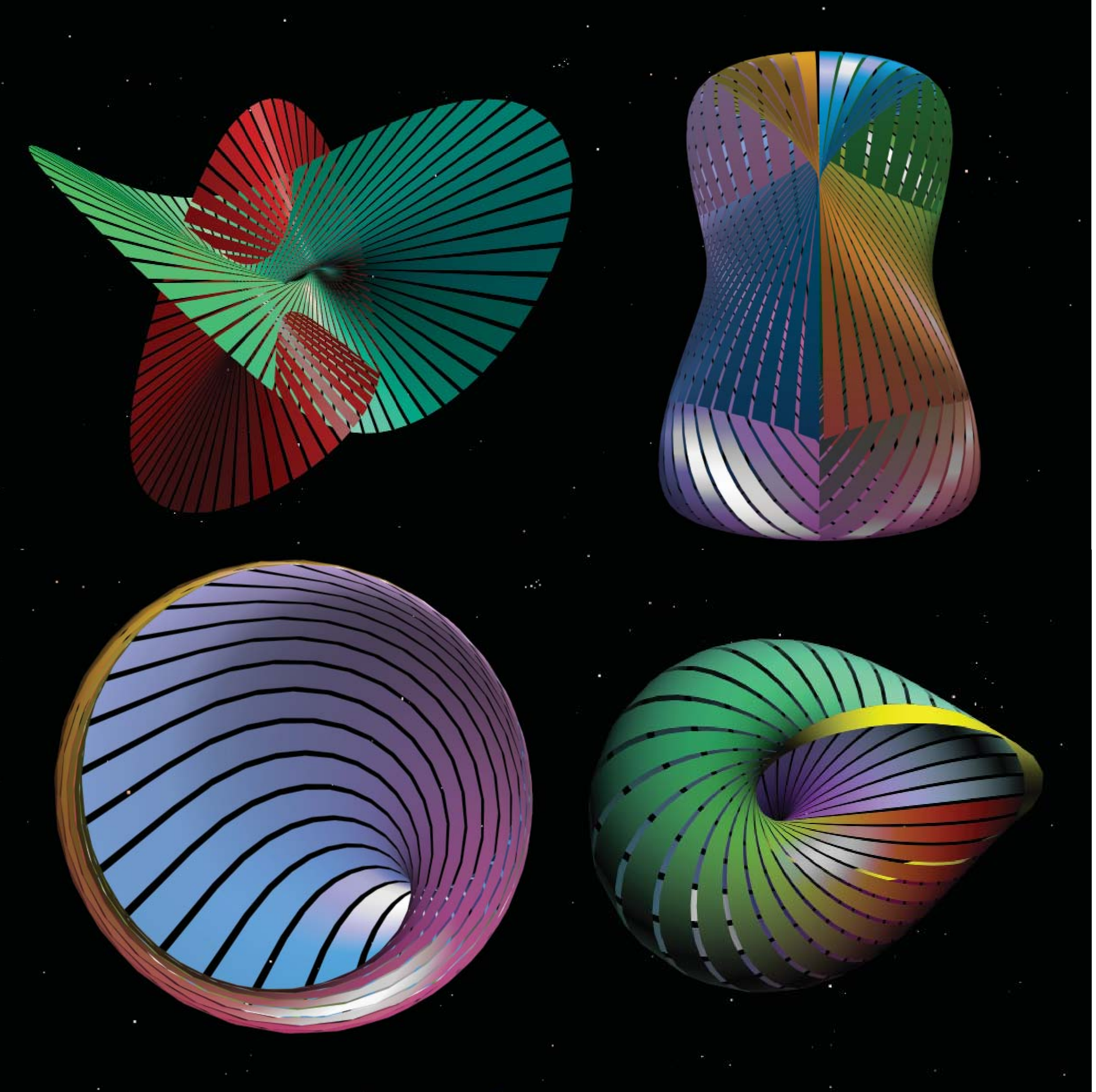
As a result of rendering and, especially, animating this sequence, I fell in love with high-dimensional spheres in a more sensory as opposed to intellectual manner. It's one thing to "do the math" and another altogether to see the thing unfurl.

ROBERT GHRIST

University of Pennsylvania

computer-generated graphic

Calculus BLUE videos: www.youtube.com/c/ProfGhristMath/



Mathematics, unlike other sciences, does not forget its ancestry. My purpose was to reanimate and preserve, if only for one more generation, the mathematical experiments programmed long ago on now-extinct computers in dead languages and with obsolete graphics libraries. The Italians say "traduttore, traditore" – translators are traitors. But David Eck's library makes WebGL read like OpenGL, and some Javascript is reminiscent of C.

I chose to create interactive web animations for two reasons. Most importantly, it has been said that the web browser is the most powerful software running on any device. Furthermore, learning how to download, install, and recompile code has fallen out of fashion. In re-programming the mathematical ideas in Javascript/WebGL from code that no longer compiles, I am realizing anew just how ingenious those original translations from algebraic expressions to computer code by my students and collaborators really were.

The collage shows four real-time interactive computer animations (at the link below) of classical homotopies from the 1990s. Each tells a story of its mathematics and graphical genesis, the better understood if you can manipulate the camera, homotopy, and shapes from your web browser. You are welcome to steal the code, to experiment and improve the animations.

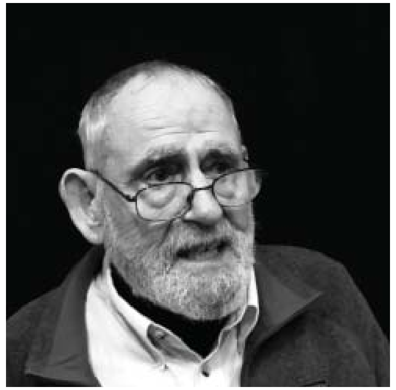
Upper left: This immersed cylinder is almost halfway through its eversion. It is the central topological detail in five homotopies, based on Bernard Morin's pioneering work on sphere eversion. François Apéry and Chris Hartman originally programmed it in C/IrisGL for the CAVE at the NCSA (1992).

Upper right: This image of the Etruscan Venus, a Klein bottle parametrized by a Whitney Excellent mapping, conveys little of the ingenuity of its design and its pinchpoint-cancelling homotopy by Apéry. Watch it morph through deformation into the Roman (Steiner) surface and the Boy's surface.

Lower left: John Dalbec worked out a contraction of an unusual cell complex embedded in space, Zeeman's duncehat, which is contractable but not collapsible. The surface has one vertex, and a single disk-like sheet attached three times to the one edge.

Lower right: A Möbius strip with a circular edge (yellow) morphs through other bubble-like shapes.

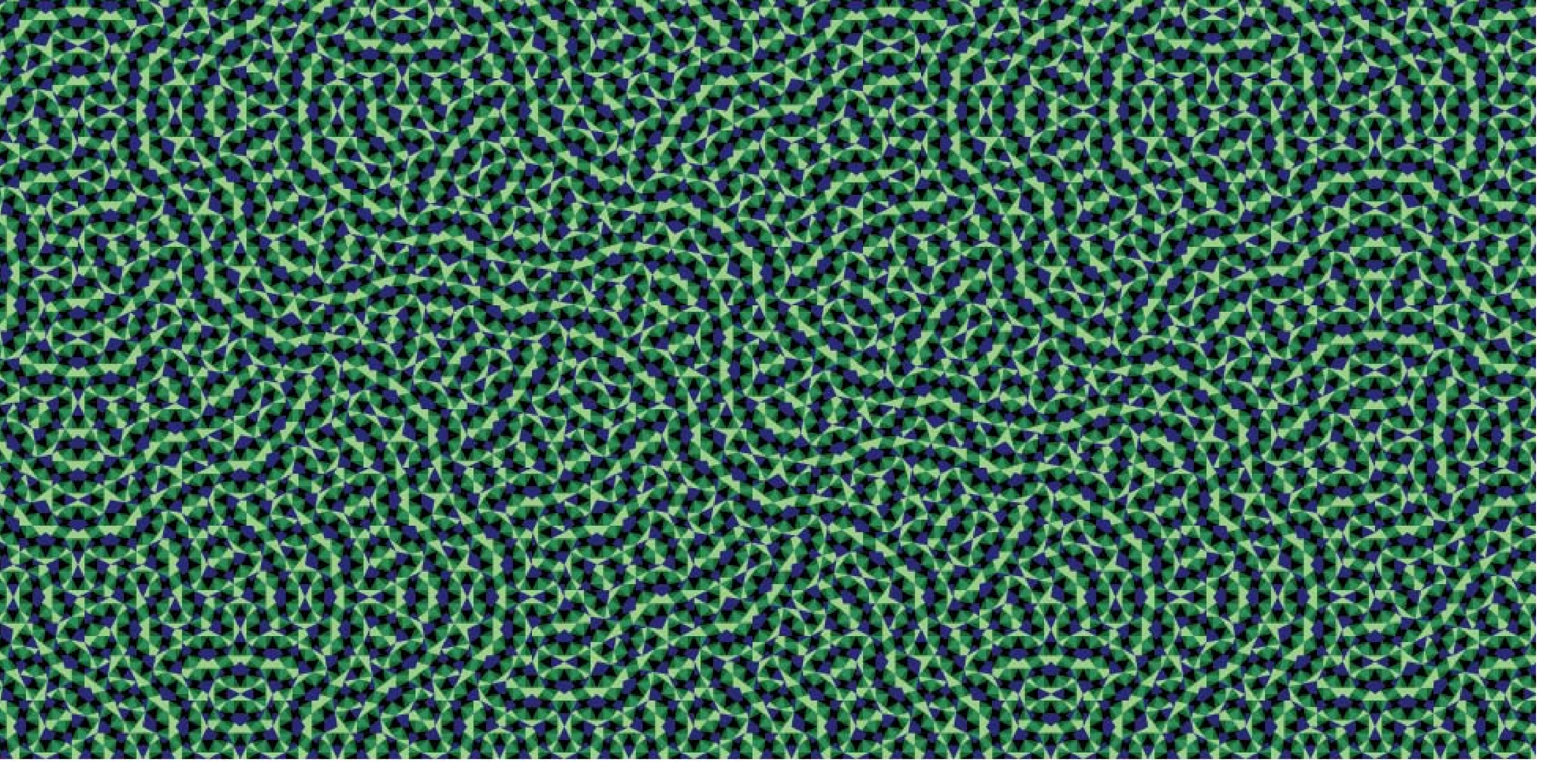
Interactive animations: http://new.math.uiuc.edu/glsimEck/Collage

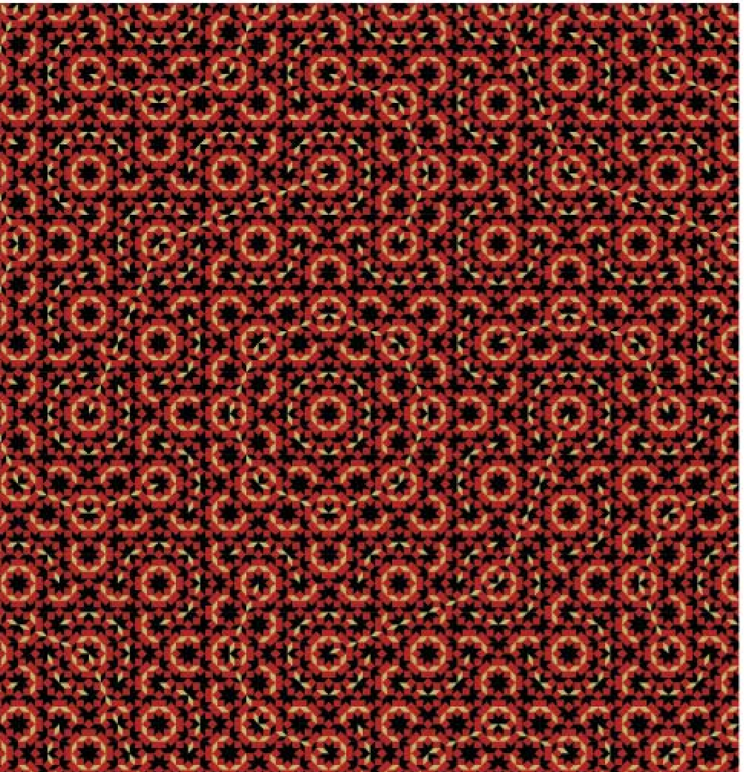


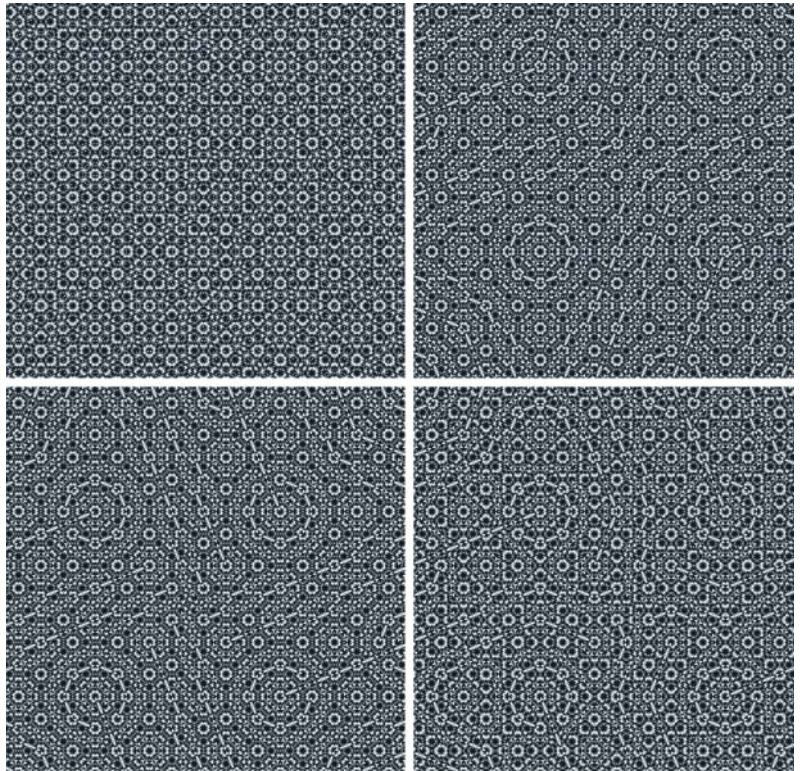
GEORGE K. FRANCIS

University of Illinois at Urbana-Champaign

interactive web animation







Tilings have shaped my journey through mathematics, in my research, outreach, and artistic endeavors. Tilings are also commonly found in our daily lives, making them a convenient starting point for bringing broad and diverse audiences to mathematics.

The tilings that you see on a bathroom floor, or a terrace paved with bricks, or in a wallpaper pattern are *periodic*: you can find some part of the tiling (often a rectangle) that you could cut out and use as a stamp, repeating the same pattern over and over again to create the entire tiling. Said another way, such tilings are invariant under a translation of the entire plane: you could shift the whole thing by some vector, and the resulting tiling would be the same as the original.

By contrast, the tilings I study and illustrate are *aperiodic*, which means that they don't repeat in any direction. In order to illustrate this property, I use animations, which not only show the difference between periodic and aperiodic tilings – an aperiodic tiling will never overlap with a translate of itself, while a periodic tiling will – but also exhibits their dynamic nature. This dynamic nature of aperiodic tilings is one of my main research interests. It turns out that one can talk about the space containing *all* of the tilings that can be possibly constructed with a given set of tiles. This kind of space, together with the translations of tilings mentioned above, constitute very interesting examples of *hyperbolic dynamical systems*. They are *laminated* spaces similar to solenoids (see page 119), and there are still many aspects about them that remain unknown.

When I started creating graphics and animations for illustrating aspects of my studies, I initially used frame-by-frame animated gifs. Soon after, I started exploring different methods, like Processing software and LiveCode tools. This, in turn, led to a collaboration with a community of digital artists, which I perceive as a great symbiosis.

These graphics and animations were created with Python matplotlib, Python mode for Processing, and ImageMagick.

Animations of aperiodic tilings:
http://penelope.matem.unam.mx/media/photologue/photos/00a.gif
http://penelope.matem.unam.mx/media/photologue/photos/00c.gif
Gallery of more images: http://penelope.matem.unam.mx/galeria/

DARÍO ALATORRE

Institute of Mathematics,
Universidad Nacional Autonoma
de México (UNAM)

aperiodic tiling animations

3D PRINTING

The best way to understand a 3D object is to hold it in your hands. But how can we create an accurate model of a surface or another mathematical object? A hundred years ago, mathematicians made models of surfaces out of plaster, supported by metal rods. Many mathematics departments and mathematical institutes are fortunate to have collections of such surfaces, which are beautiful and instructive. Exactly how people constructed the molds to create these surfaces is somewhat mysterious.

These days, it is possible to 3D-print objects that represent mathematical surfaces, by using math software to plot the surface and then using 3D-printing software to transform the plot into a solid, printable object. Although the process is simple in theory, 3D printing is a relatively new technology, and as such there are usually technical hurdles to resolve along the way, so any project usually requires several failed attempts before the result succeeds in realizing the mathematician's vision.

This set of 3D prints illustrates singular algebraic surfaces. I work on solutions to the problem of physically visualizing nodal singularities, where two or more pieces come together at a single point. My main motivation for 3D-printing them is to illustrate the output of the algorithm for numerical real cellular decomposition implemented in my computer program Bertini_real. It computes a union of "cells," each equipped with a generic point and homotopy that can be used to compute additional points on the real part of a complex variety. This output is naturally 3D-printable.

Using fused filament fabrication (FFF) with thermoplastic polyurethane (TPU) is the right way to produce these objects for a number of reasons. FFF can be done at home or in the office with an inexpensive 3D printer, with no chemically dangerous materials or supplies. The only post-processing step is removing support material, and, since TPU is flexible, printed surfaces can have much thinner connections at nodal singularities without inevitably breaking. Furthermore, the very small earring-sized prints are robust enough to be worn daily, and the skeleton-like object is seemingly delicate but readily able to be carried in a backpack or pocket. I continue to learn about how to engage the public in my research by having cheap, non-breakable objects to show them while wearing earrings at the grocery store. TPU is a challenging material to work with, requiring a high quality hot end and a commitment to printing very slowly, but it's incredibly rewarding.

My early versions of prints of the Barth sextic and other nodal surfaces, such as those coming from the Herwig Hauser gallery of algebraic surfaces like Octdong, broke either during support removal or transport for show. Many of my most experimental polylactic acid (PLA) prints are thus doomed to live forever trapped in their support material, since it's so far inside a cavity in the surface I would destroy it in completing it, or because the material connection between pieces is so small.

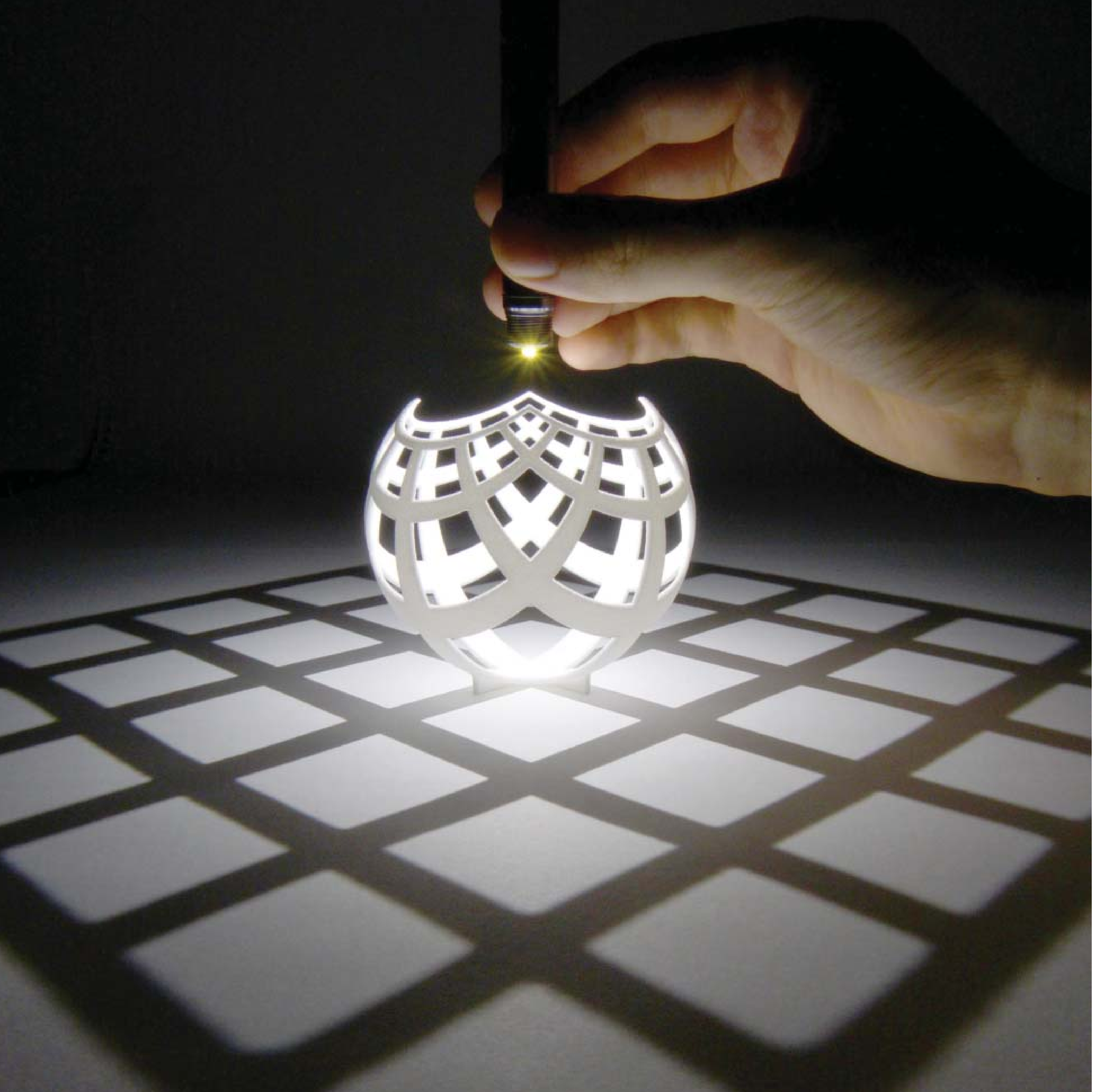
Gallery of more images: https://silviana.org/gallery

Further information: Silviana Amethyst, Daniel J. Bates, Wenrui Hao, Jonathan D. Hauenstein, Andrew J. Sommese, Charles W. Wampler, *Algorithm 976: Bertini_ Real: Numerical Decomposition of Real Algebraic Curves and Surfaces,* ACM Transactions on Mathematical Software, 44(1), [10]. https://doi.org/10.1145/3056528

SILVIANA AMETHYST

University of Wisconsin–Eau Claire

3D-printed plastic



Stereographic projection is a map from the sphere to the plane. Place a sphere on the plane, and draw a straight line from the north pole of the sphere down to the plane. The line goes inside the sphere, hits the sphere, then continues on to hit the plane. Stereographic projection sends the point on the sphere to the point on the plane. This is precisely what the light rays do, so the grid pattern on the plane is the stereographic projection of the pattern on the sphere.

I chose to illustrate this to fill a gap in my book *Visualizing Mathematics with 3D Printing*. I already had some prints showing the "3D shadow" you get as the result of a higher-dimensional version of stereographic projection: from the sphere in 4D space to 3D space. So I needed to explain what stereographic projection is, and the lower-dimensional version seemed like it would be easier to understand.

3D printing has a lot of nice features – it can be very precise, which I needed in order to get a good shadow. A print is also physical – you can touch and play with it, unlike a computer graphics render. Also unlike a computer render, there's no way to "cheat." People are rightly suspicious that some effect could be faked with computer graphics, but they (think they) understand everything about a piece of plastic and a flashlight – so it must be a real phenomenon.

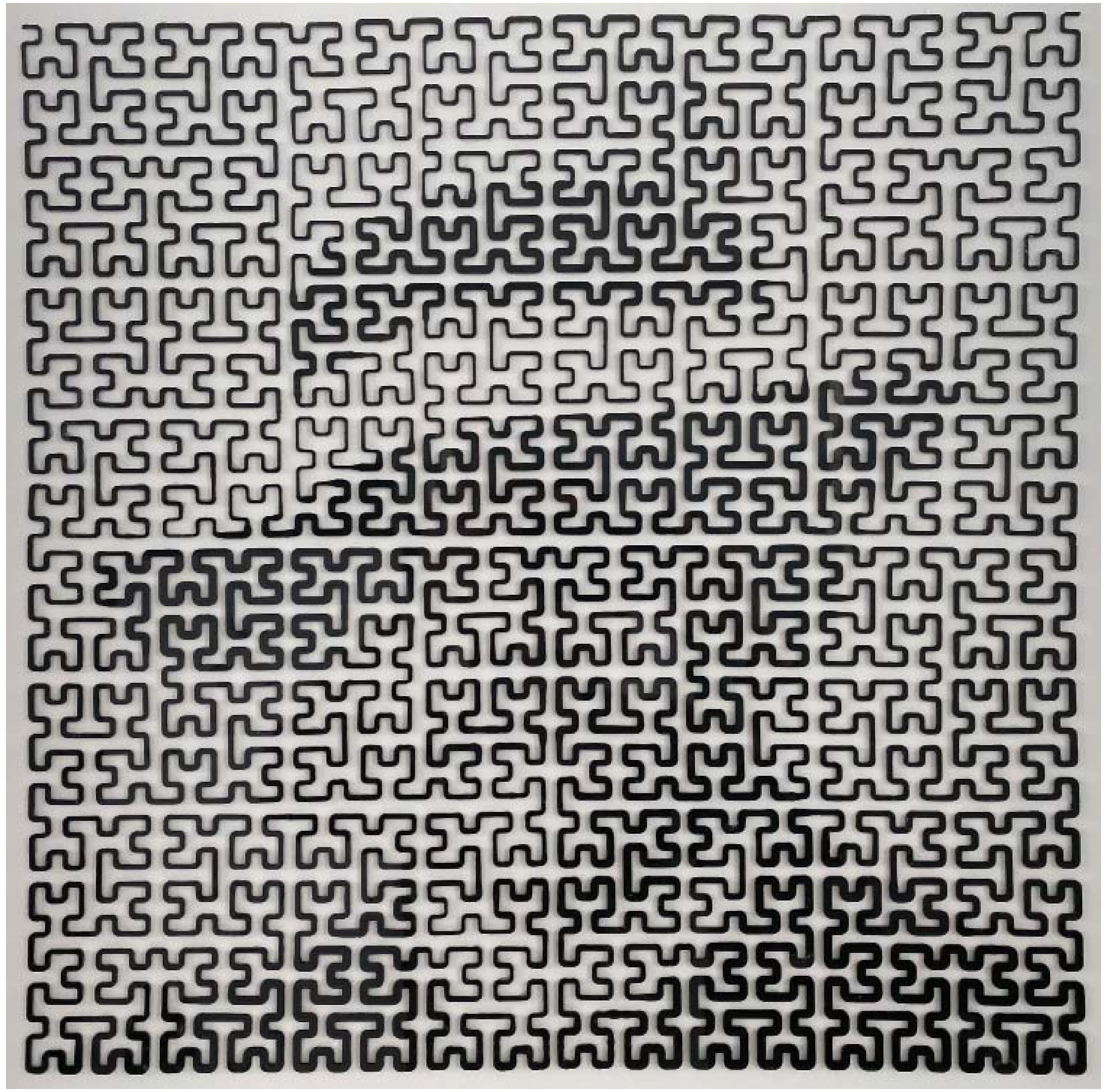
I started with the design on the plane, then made cones from the design up to the north pole. Then I used the cones to cut out the windows. There are some tricky issues to do with the thickness of the sphere. Ideally, the sphere would have zero thickness, but of course that would be impossible to print. Cutting out the cones works, but leaves very sharp angles near the north pole of the sphere. So I needed to cut a second time to remove these sharp angles.

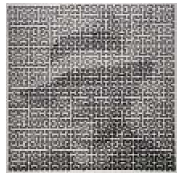
Video of the object in action: https://youtu.be/VX-0Laeczgk
Further information: Henry Segerman, *Visualizing Mathematics with 3D printing*, Johns Hopkins University Press (2016).

HENRY SEGERMAN

Oklahoma State University

3D-printed plastic
...plus a flashlight





More than anything, my work illustrates the human brain's remarkable ability to fill in missing information as it works to make sense of the data it receives from the surrounding world. By varying the thickness of piping generated along the sixth iterate of Hilbert's famous space-filling curve, I have rendered Hilbert's 2D portrait as a 3D object. Using a Python script within the modeling program Rhino, my code reads in an image and uses pixel data to generate rectangular piping along the curve. The width of the rectangular profile of the piping at a point depends on the intensity of the pixel at that point; it is wider when the intensity of the pixel is lower (where the image is darker) and thinner when the intensity of the pixel is higher (where the image is brighter).

I am new to 3D printing, so I chose this medium in large part to gain expertise in the modeling and printing processes of a 3D object. I worked in stages, first learning how to print a Hilbert curve with square piping, then incorporating a rectangular profile with variable width. I used a Python script within Rhino to carry out the process, reading in image data from the famous (cropped) picture of Hilbert in his white hat.

I created this portrait using the sixth iterate of the Hilbert curve, which I constructed in pieces – printing off 16 subsquares of the curve on an Ultimaker 3. Each subsquare is approximately five inches in width, and they were generated using the fourth iterate of the Hilbert curve. Hoping to render a higher resolution of the image, I was initially planning to generate the portrait using the seventh iterate of the curve. However, such an approach would have involved well over 140 hours of printing.

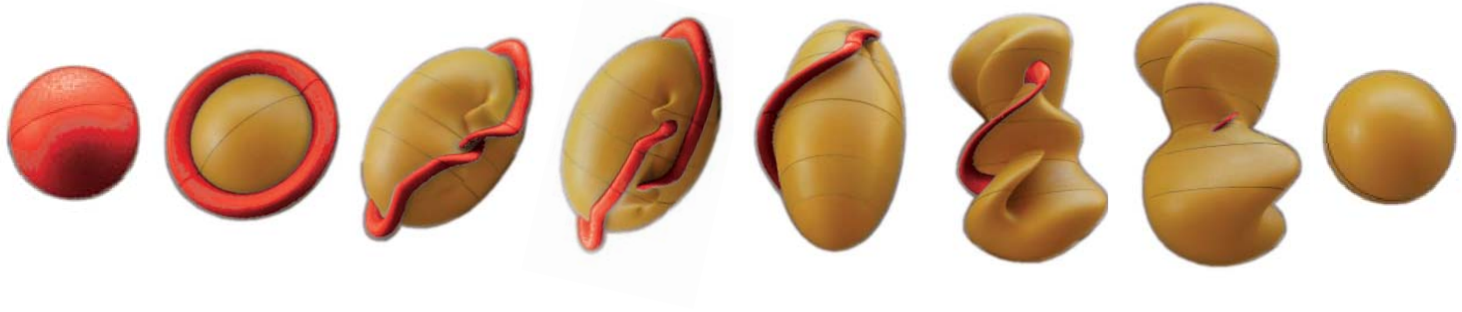
I decided I was happy with a lower resolution, which may require some squinting on the part of the viewer. However, the precision improves dramatically when the image is scaled down (see thumbnail top right). An unexpected discovery was the difference in finish and texture between the top of each printed curve segment and the bottom. Preferring the shiny finish of the bottom, I ultimately decided to print each subsquare of the curve upside down.

Further information: Judy Holdener, *Hilbert's Portrait via his Space-Filling Curve.* Proceedings of Bridges 2020: Mathematics, Music, Art, Architecture, Culture (2020), to appear.

JUDY HOLDENER

Kenyon College

3D-printed plastic



This series of models gives an example of *sphere eversion*, which is the problem of turning a sphere inside-out smoothly, without creating any cusps or creases along the way.

After designing a movie with Jos Leys showing an original way of turning the sphere inside-out, I decided to 3D-print a selection of frames from this animation (bottom). Unlike most physical models of self-intersecting surfaces, I decided not to use cut-out windows or to print a wireframe, but to have a surface that is completely closed. To see what is inside, the viewer opens the object, which is held together with magnets (top). These objects have been designed and parameterized layer by layer, in a tomographic way, so it was natural to slice them horizontally. I ordered a print on Shapeways, glued in the magnets, hand-painted it, and then varnished it to protect the paint.

The 3D-printing technology I chose was selective laser sintering on nylon powder. This material is resistant, thanks to a strong bond between the grains, and has some degree of flexibility. The thickness of the surface is around 1.5 mm, chosen to balance durability and price. The overall size is a balance between price, ease of handling, and the amount of space needed, given a thickness, for the high-curvature parts to render correctly.

Hand-painting is pleasant but tedious, as there are three colors (one for the outside, one for the inside, one for the cuts), and the boundaries between two colors require a lot of precision. This certainly was the most time-consuming part, and was done in my leisure time in the summer break.

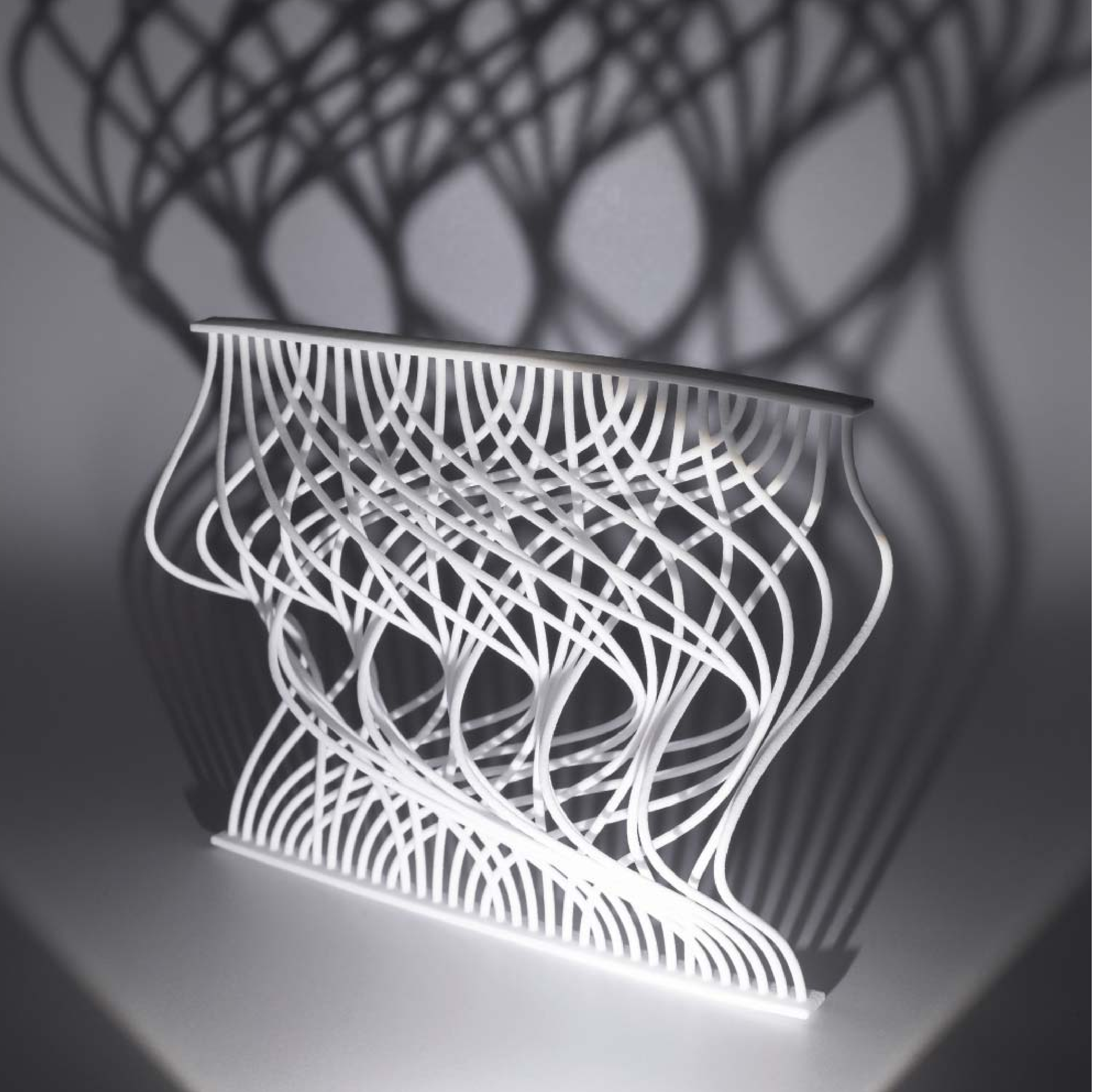
I chose the magnets to be as small as possible while still holding the objects together. After my institution purchased a set of these objects, I created a second series with a better rendering of the shape and a better selection of key moments. This time, I used bigger magnets, because, while handling the first set, they would often come apart spontaneously. On the other hand, the material and varnish did not need adjustment: the models often fell on the floor when we used them for outreach, and they barely got damaged.

Further information: Arnaud Chéritat, *Yet another sphere eversion*, preprint (2014).
John M. Sullivan, *Sphere eversions: From Smale through "The Optiverse,"* Bridges: Mathematical Connections in Art, Music, and Science, pp. 265–274 (1999).

ARNAUD CHÉRITAT

CNRS / Institut de Mathématiques de Toulouse

painted and varnished 3D-printed nylon with magnets



This is a 3D-printed object that illustrates card shuffling. I have previously made 2D visualizations of card shuffling and permutations, so extending this to 3D was a natural continuation.

This particular object illustrates two perfect out-shuffles with 36 cards and six piles. A *perfect shuffle* with *two* piles occurs when a deck of cards is split exactly in two halves and the cards are alternated so that every other card comes from the same half of the deck. An *out-shuffle* is when the top card remains the top card. The principle is the same for six piles.

Each curve, from top to bottom, represents the path of a single card. Together, these curves represent the paths that all of the cards take from start to finish. Notice that each curve starts and ends at the same position along the horizontal bars that hold the curves together. This reflects the fact that the cards are restored to their original order after two perfect out-shuffles. Also observe that the outermost curves remain in the same position, which means that the top and bottom cards each remain in the same position throughout all of the shuffling.

While 2D visualizations of shuffling have unavoidable curve intersections, the third dimension provides the opportunity to avoid these crossings as much as possible. This object is a compromise between avoiding intersections between the curves and making the final object look nice.

I was originally inspired to create these visualizations after attending a talk by Perci Diaconis. I wanted to know if I could better understand the mathematics of card shuffling by aesthetically exploring the various permutations underlying the shuffling methods. My motivation was to make these invisible structures visible, and to create elegant and interesting art in the process.

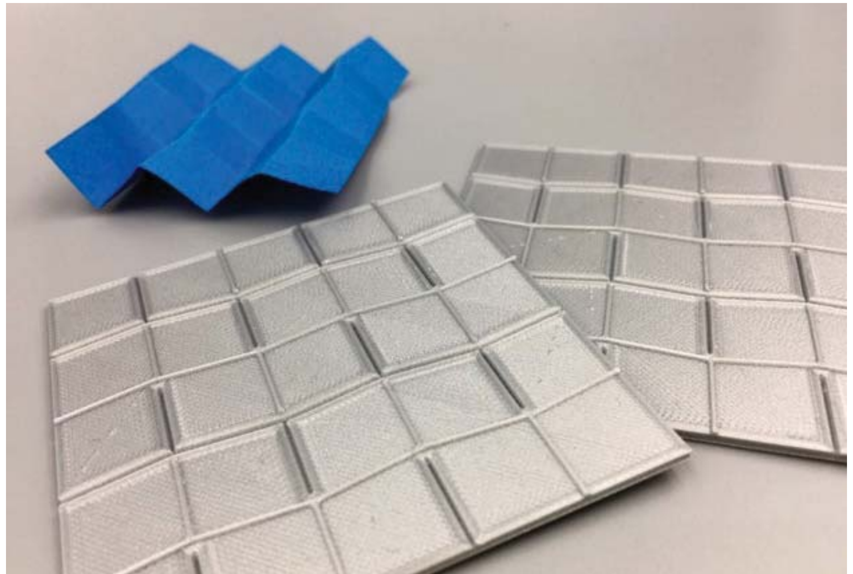
In this case, the curves are calculated with Processing, imported to Rhino, and produced using Shapeways.

Further information:
Roger Antonsen, *Card Shuffling Visualizations*, Proceedings of Bridges 2018: Mathematics, Art, Music, Architecture, Education, Culture, pp. 451–454 (2018).
Eight Ways to Shuffle Sixty-Four Cards: https://rantonse.no/en/art/2018-07-25

ROGER ANTONSEN

University of Oslo

3D-printed plastic



The first picture shows a 3D-printed press for folding a simple Yoshizawa butterfly model, and the second shows a press for constructing an origami model with interesting geometrical properties known as the *Mirua map fold*. We made these presses to enable pre-creasing of the paper for origami models with challenging folds, and to avoid unnecessary creases in the final paper models.

For example, the classic method of creating a Miura map fold requires the folder to make some "universal folds" that bend both ways, to aid in the final construction of the model. With the 3D-printed press, we can avoid these universal folds and produce a stronger model. Another benefit of the Mirua press is that it allows the folder to create folds at exact angles – in this case, 84 and 96 degrees. We also made a press for the Yoshizawa butterfly origami model; the classic origami design involves some partial folds and unnecessary creases, but with the origami press, we can achieve a cleaner and curvier final butterfly.

The press has two pieces that fit together, with the origami paper in between. We printed a star in one corner of each piece to indicate how to orient the two pieces to fit them together correctly. Along each of the lines where we want to make a crease in the paper, one piece has a raised creasing line, and the other has a lowered creasing line, to create a "mountain fold" or "valley fold" in the desired location. 3D printing is a natural fit for these presses, since a laser cutter would not be able to produce both raised and lowered creasing lines.

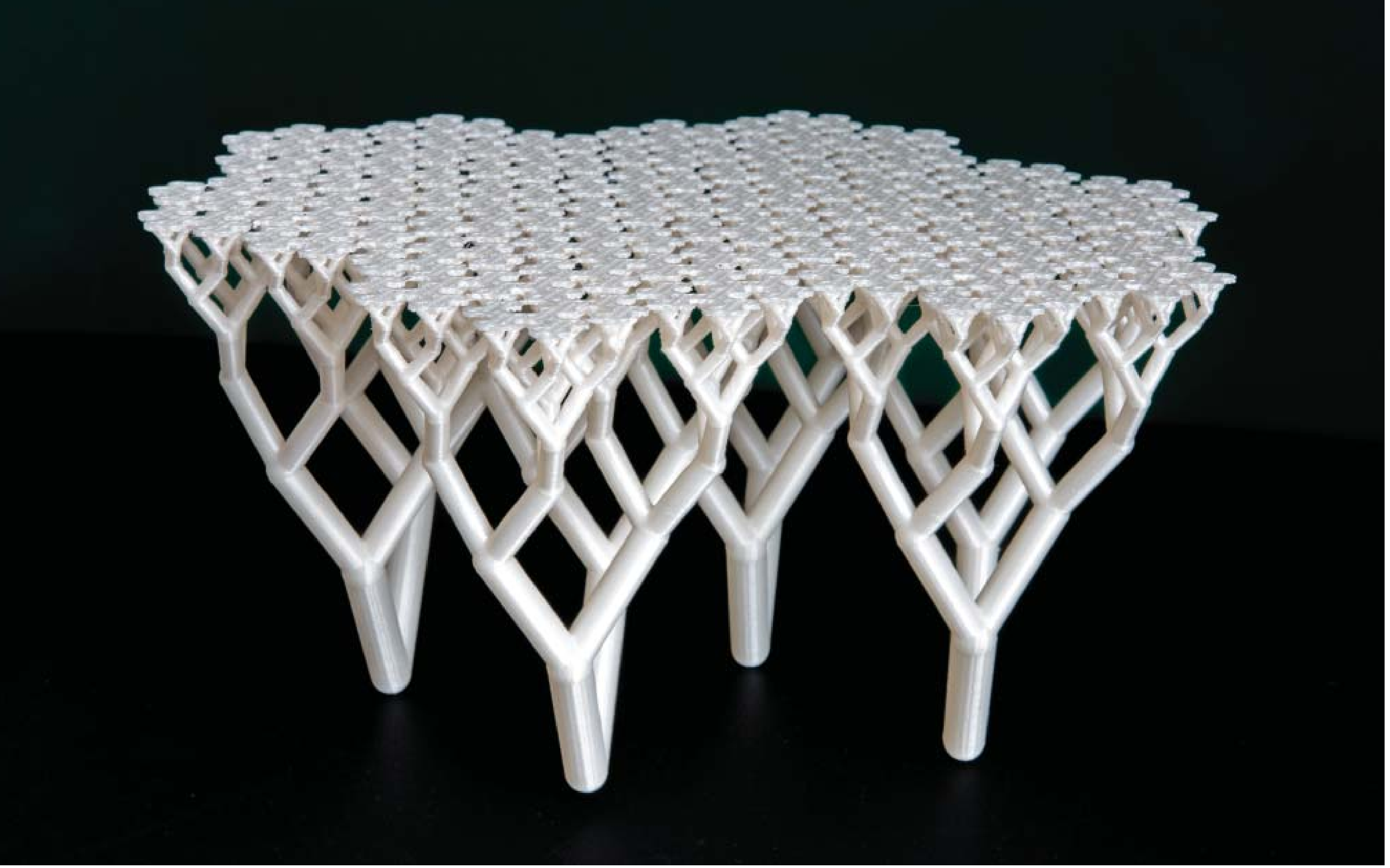
The 3D design for the press went through many iterations so that it could most effectively produce creases in washi origami paper without ripping holes in the paper. The corners where mountain and valley folds intersected were the most challenging in this respect.

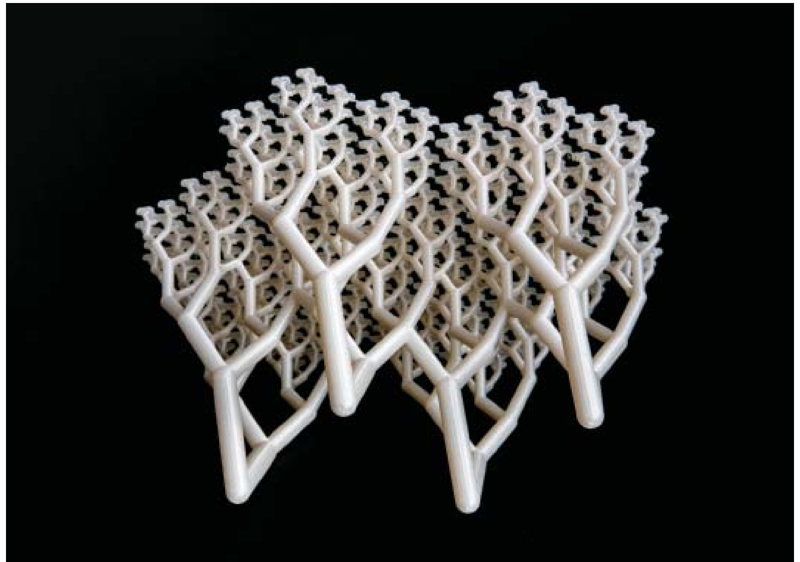
LAURA TAALMAN

James Madison University

3D-printed plastic
and folded paper

For more information, free downloadable files, and folding instructions, see:
https://mathgrrl.com/hacktastic/2019/07/3d-printed-origami-press-miura-fold/
https://mathgrrl.com/hacktastic/2019/08/origami-press-2-yoshizawa-butterfly/





To communicate about mathematics, it is very effective to create physical objects that people can pick up and touch. *Complex trees* are 2D mathematical objects that naturally live in the plane. If at each branching level we add a certain height, this gives the object a third dimension, which illustrates some extra properties that are not visible in the 2D pictures. I 3D-printed fractal trees to make them tangible, to be able to share my work with more people.

I was surprised by the shape of the plane-filling structure of the "canopy" part of the tree that touches the surface of the 3D printing bed (lower left picture). It is a *Rauzy fractal*, an object that comes up in many areas of mathematics (see Pat Hooper's pictures, referenced below). The Rauzy fractal is made of smaller copies of itself, and it tiles the plane, and so this 3D fractal tree is also a modular design that can be used to tile the plane. This example illustrates how the Rauzy fractal can arise as the limit set of an iterated function system of two linear maps, $f_1(z) = 1 + cz$ and $f_2(z) = 1 - cz$, where c is a root of $1 + x + x^2 - x^3$.

3D-printing with CURA software is an art. This object pictured here is a high-quality print, created at ICERM with an Ultimaker 3 printer. It is the result of a sequence of failures and successes. This tree was printed upside-down with a thin *raft* (a solid base layer of plastic) 1mm thick. Applying Magigoo (an adhesive for 3D printing) to the build plate before starting the print was crucial to force the raft and the tree's canopy to stick. Magigoo made the final step of removing this delicate print effortless, since its adhesive properties are designed to release the 3D print once the build plate has cooled.

BERNAT ESPIGULÉ

Universitat de Barcelona & Wolfram Research

3D-printed plastic

Further information: Bernat Espigulé, *Families of connected self-similar sets generated by complex trees*, preprint (2019).
Slide show of related images: www.complextrees.com/imbook
Pat Hooper's artwork showing iterates of the Rauzy fractal: http://wphooper.com/gallery/2019/arnoux_yoccoz.png

Each horizontal slice of this 3D-printed bottle is the Julia set of some quadratic polynomial. Let me explain what this means: In the field of dynamics, one seeks to explain how systems evolve and, in particular, to predict their long-term behavior. For example, when we take a complex number z and repeatedly apply a polynomial $z^2 + c$ to it, where will this take us? For the vast majority of points, one can answer this question with certainty. On the rest of the points, which comprise the Julia set, the question cannot be answered in the same way, because the behavior is "chaotic," just as the weather is inconveniently unpredictable over long periods of time.

In the 1980s, Julia sets were appropriated as a symbol of the inherently chaotic nature of even the simplest systems. Tiny differences in the choice of the initial complex number z yield widely different outcomes, after applying the polynomial often enough. And so, though it is the ethereal frontier of a pair of regions where the future of z is perfectly stable and deterministic, long-term predictions on the Julia set are quite impossible.

Benoît Mandelbrot, who was working for IBM, had access to sophisticated computers, which allowed him to explore the world of Julia sets of quadratic polynomials. In his book, he illustrates the way that the Julia set of the complex polynomial $z^2 + c$ changes with the parameter c, by stacking Julia sets in three dimensions. Inspired by this, we have imitated what he did to obtain our *Mandelbrottle*, stacking 150 Julia sets from a short trip in the complex plane. Starting at zero, we walk around a small circle completely contained in the Mandelbrot set. We return to zero, and pause a short moment there before taking another walk around a slightly smaller circle, and again take another rest at zero. Finally, we hop to another point and walk quickly straight back to zero to make the bottle cap. Our path has to be carefully chosen to obtain just the right profile for a swirling Coke bottle.

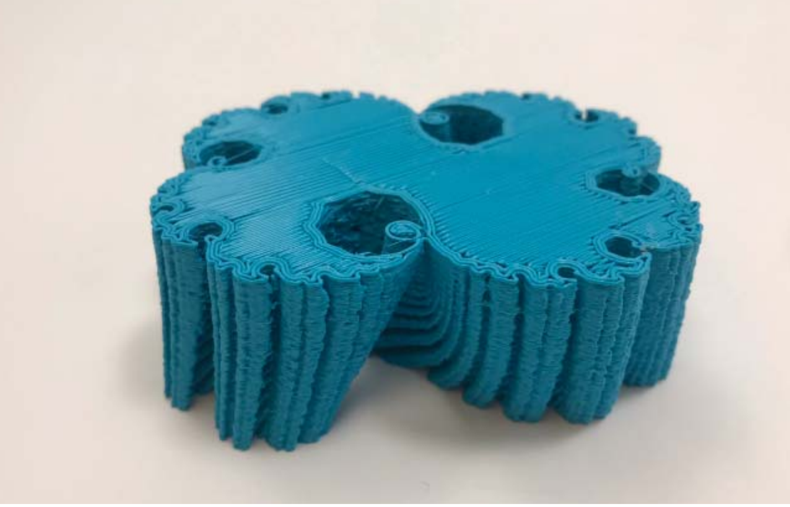
Appropriately, the design of the Coca-Cola bottle dates to about the same time as the work of the French mathematician Gaston Julia a hundred years ago. In Andy Warhol's studio, it was transformed into Pop Art. Through the work of Mandelbrot and others, Julia sets became an icon of what one is tempted to call "math consumption." Representing unavoidable uncertainties and abrupt bifurcations in the apparently banal, Julia sets have appeared on teenage bedroom posters, T-shirts, and album covers.

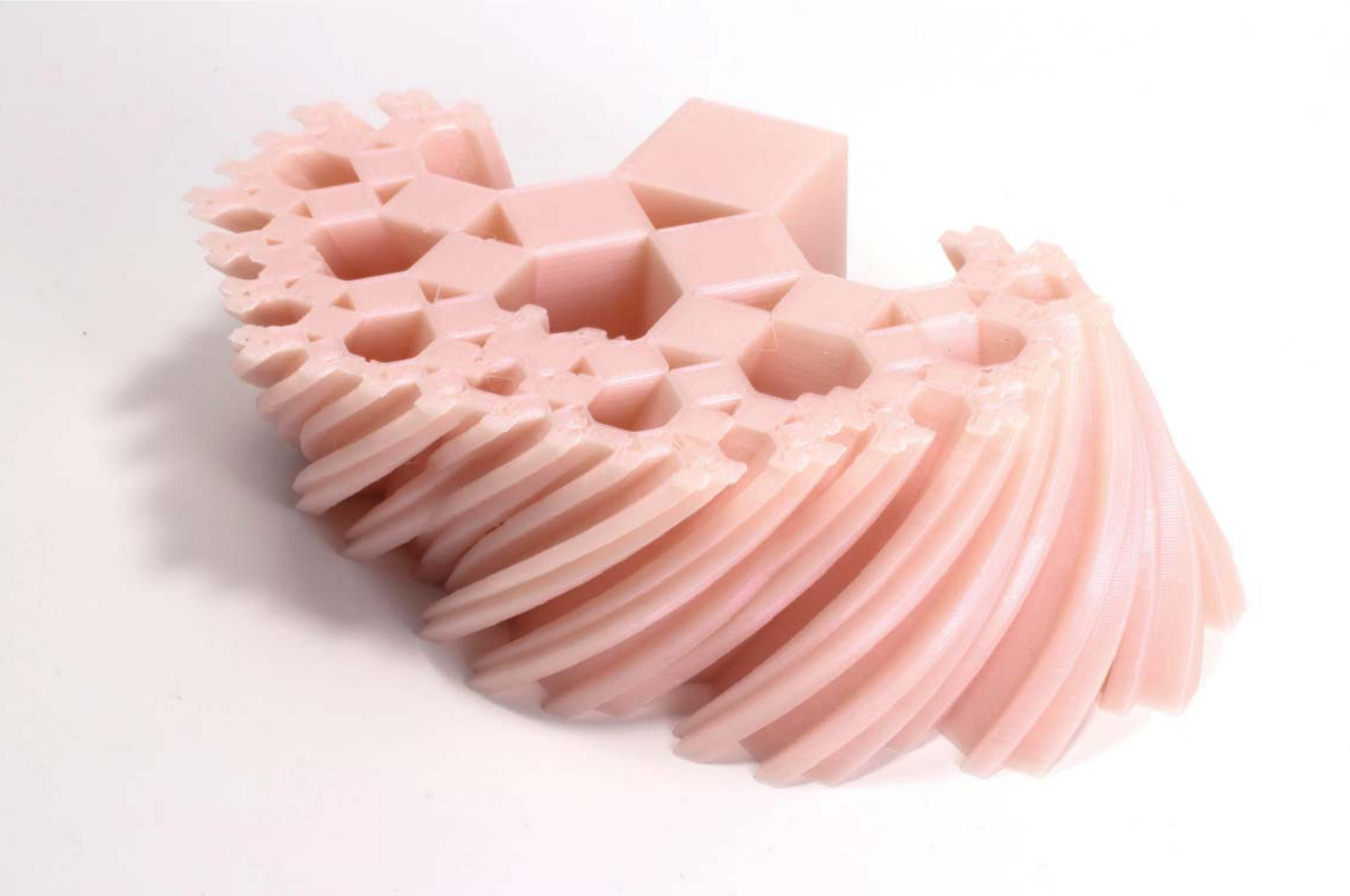
Further information:
Benoît Mandelbrot, *The fractal geometry of nature*, W.H. Freeman (1977).

GREG MCSHANE

Université Grenoble Alpes

3D-printed plastic





In many different disciplines, we study how families of mathematical objects change, when one of their parameters is changed a little bit. If these objects are all 2D, one can vary the parameter over time and display how the object changes as an animation. Alternatively, with a 3D printer, we can replace the time dimension with the z-axis, stacking each successively deformed layer atop the last. This allows us to hold the entire parametrized family of objects in our hands, giving an entire new dimension to the deformation (literally!). I like to call these "static animations."

My first experiment with static animation was studying the deformation of the Pythagorean tree fractal (bottom). It is constructed by attaching the long edge of a right triangle to the side of a square, then attaching squares to the legs of the right triangle, and repeating this process forever. A right triangle (up to scaling) is determined by one acute angle, and a small shift in that angle corresponds to a small change in the triangle, but the movement reverberates more and more as you progress deeper into the fractal. In the static animation, I vary the angle from 45° to 60° slowly, over the course of more than 300 layers.

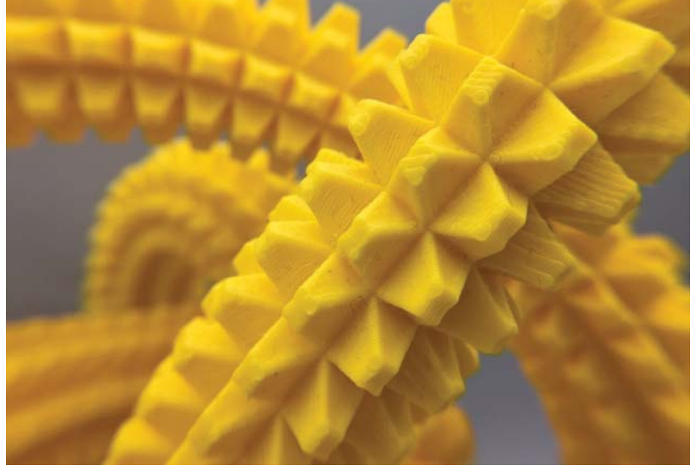
A second experiment is with deformations of Julia sets (top), following a workflow developed jointly with Bernat Espigulé. Defining Julia sets is actually quite easy: Let c be a complex number, and consider the function $f(z) = z^2 + c$. If I start with a point p in the complex plane, and I iterate f – that is, I consider p, $f(p)$, $f(f(p))$, $f(f(f(p)))$, ... and so on – one of two things happens: either it gets infinitely large and escapes to infinity, or else at each iteration it stays relatively close to the origin. The points that stay close to the origin are those in the Julia set J_c.

It turns out that if you change the constant c just a little bit, the set J_c only changes a little bit as well, but it can change in quite interesting ways. Given a path through the complex numbers, we stack the Julia sets corresponding to the points in the path: If we are at a point c in the path, we print the layer J_c. Then if c' is a point just a little bit further down the path, we print $J_{c'}$ as the next layer. In this way, we turn a path through the complex numbers $\boldsymbol{C}$ into a 3D-printed deformation of Julia sets. One can view the combination of Mandelbrot and Julia sets as a sort of fractal subset of $\boldsymbol{C} \times \boldsymbol{C}$ (which is 4D), and these prints are 3D slices of this mysterious and beautiful 4D space.

GABRIEL DORFSMAN-HOPKINS

ICERM

3D-printed plastic



The Perko Pair is a famous pair of knots that for a long time were thought to be different, but later were famously revealed to be the same by Kenneth Perko in 1973. When these knots originally appeared in Rolfsen's Knot Table, they were known as 10_{161} (now "Perko A") and 10_{162} ("Perko B"). The knot we printed is a morph about halfway between Perko A and Perko B, with spikes just because we felt like it, and because we could.

The large size of this model made it too expensive to print with a service, so we had to 3D print it ourselves on a desktop FDM 3D printer. In this size, this model only prints well with either dissolvable supports or so-called "breakaway" supports. Water-soluable dissolvable supports are an option, but they can take days to dissolve, and create a lot of goopy mess. Breakaway supports are really difficult to remove, but they break off with an amazingly good finish on the final model, as you can see in the upper right image.

Being able to examine this knot from all directions in physical space gave us a much better understanding of this knot, as well as its well-known "A" and "B" conformations.

We went through many failed prints of this model before getting the stars to align on this six-day print job for a successful result. Then a year later our cat knocked it off the shelf and we had to do it all over again!

LAURA TAALMAN

James Madison University

3D-printed plastic

Perko's 1973 paper: Kenneth A. Perko Jr., *On the classification of knots.* Proceedings of the American Mathematical Society 45 (1974), 262—266.

For more information and downloadable files:
Perko Knot Reprint: Dissolvable vs. Breakaway Supports: https://mathgrrl.com/hacktastic/2018/12/8132/ and
Giant Spiky Perko Knot: https://mathgrrl.com/hacktastic/2017/05/giant-spiky-perko-k

These objects solve the following minimization problem:

Among all polyhedra with given number of vertices and given volume, find the one with the least surface area.

We call them *Akiyama polyhedra*, after Shigeki Akiyama, who recently took up this old problem and computed solutions for up to 12 vertices. From left to right, bottom to top, these are the Akiyama polyhedra with 4 to 12 vertices, all scaled to the same volume.

In February 2017, I attended a five-week program on dynamical systems at CIRM in Marseille. Between talks, I would show 3D-printed mathematical objects to others and ask for more ideas of things to print. Shigeki told me, “There is a curious 3D body that I would like to see 3D-printed.” He gave me the coordinates for his minimizing polyhedron with eight vertices. I had just peer-reviewed code by Frédéric Chapoton that allowed SageMath to export polyhedra to STL, so I was able to make an STL file (the 3D-printing equivalent of a PDF) from the coordinates. A few months later Shigeki 3D printed his whole series back in Japan. I suggested sharing the STL files on the Imaginary website (see link below) and he agreed, making it easy for me and everyone to use Akiyama polyhedra in many outreach activities. I like to re-tell this story, which highlights how 3D printing connected mathematical research, software, and outreach.

I do a lot of mathematical outreach for children and adults, and I made the Akiyama polyhedra in the picture to bring to math festivals. When you can hold an object in your hand, you get a better understanding of it. For example, in some cases, the symmetry of the shape is obvious (the tetrahedron and icosahedron) but in some cases it is not at all obvious, so it helps to be able to look at it from all sides to find its axis or plane of symmetry.

At ICERM, I printed them with 100% infill, so that they have not only the same volume, but the same weight. Visually, some of the polyhedra look bigger than others, so this allows us to physically feel their same volume. This was easier to do using 3D printing technology than with any other method.

For downloadable printing files: https://imaginary.org/hands-on/minimizing-polyhedra
The original paper: Shigeki Akiyama, *Minimum polyhedron with n vertices*, preprint (2017).

ALBA MARINA
MÁLAGA SABOGAL

ICERM

3D-printed plastic

This object illustrates a small piece of an infinite periodic surface, originally studied by Dami Lee. It's part of a multimedia piece of art called "Example/Exception." The surface can be made by gluing octahedra of two colors (here, grey and white) in the following way: each white octahedron is glued to four greys in such a way that no two greys share an edge, and each grey is glued to two whites on opposite faces. The process of building the surface out of polyhedra can be continued forever, to obtain an infinite surface that has three directions of periodicity. Hence, it is called a *triply periodic polyhedral surface*.

When you quotient by the three directions of periodicity, you get a genus-3 Riemann surface, for which you can explicitly write down the automorphism group, hyperbolic structure, algebraic equation (it is defined over $\mathbf{Q}$), and several distinct flat structures. This work made up much of Lee's PhD thesis.

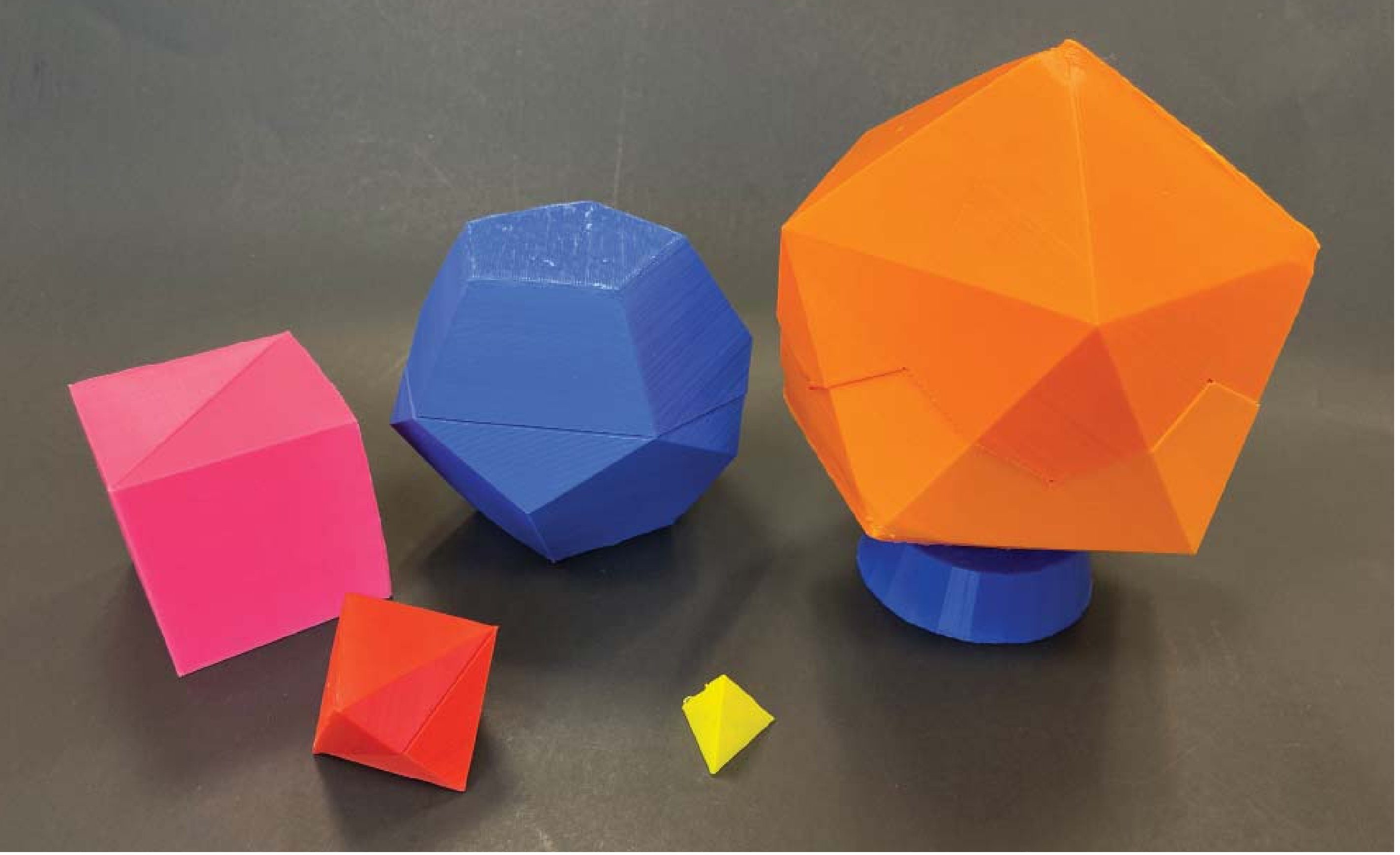
The ease of printing octahedra with a 3D printer and the ability to glue them made it feasible to make this at an appropriate scale. The ability to print in multiple colors also made it possible to show off the two distinct families of octahedra.

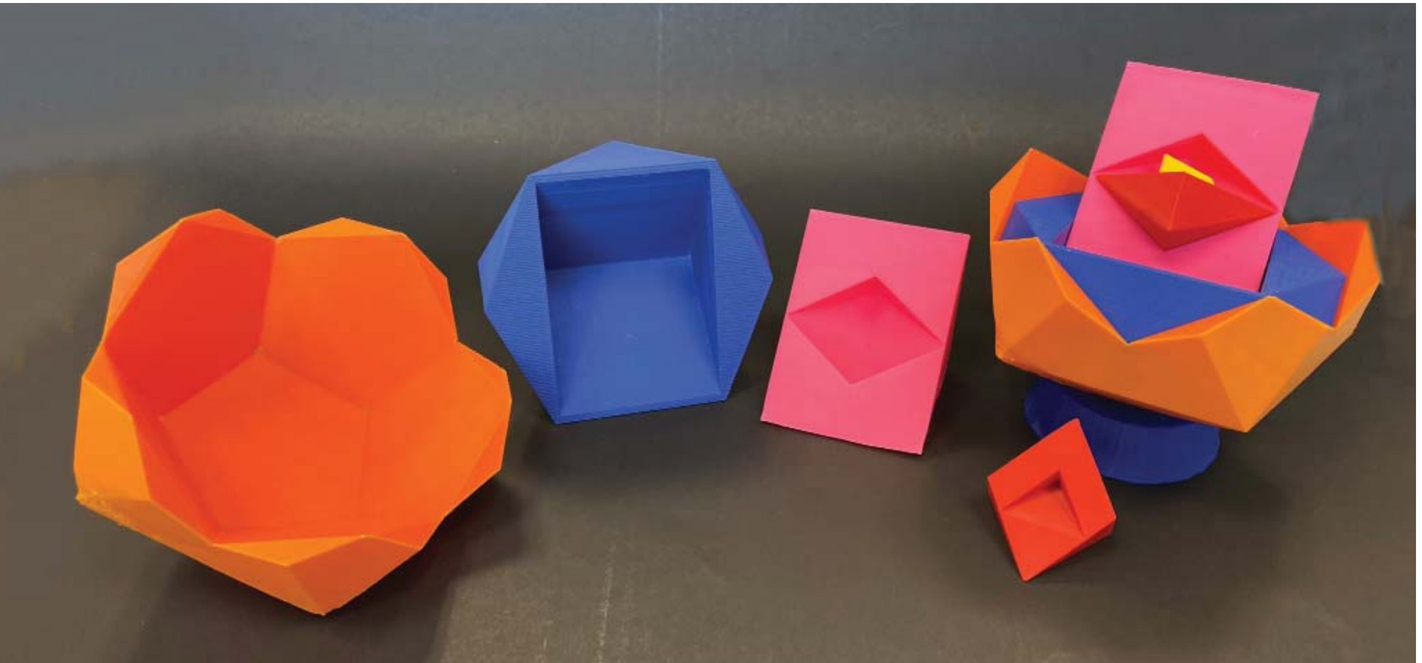
In making the physical object, I learned that the number of octahedra in a large ball grows cubically in the size of the ball, and I started to think about what it would be like to be an ant living on this surface. It leads to thinking about the geometry of cubic differentials. I also discovered that glue is hard to work with, and that, sometimes, something bigger is more fun. I have subsequently made a much bigger version out of Rubik's octahedra.

JAYADEV ATHREYA

University of Washington

3D-printed plastic





This object shows the five platonic solids. We have chosen to nest them like a Russian nesting doll. We specifically directed this object at getting children excited about the Platonic solids and the fact that there are only five of them. In addition, the method of nesting one in another is based on a pairing (or *duality*) between them, which gives rise to a natural way to embed one in another.

The five Platonic solids are classical, and I probably learned of them in high school, but I did not know the fact that they come in pairs with a natural nesting of one into the other until graduate school, and I did not decide to illustrate them until now. This type of thing is ideally suited for 3D printing, in that it involves multiple interlocking 3D objects.

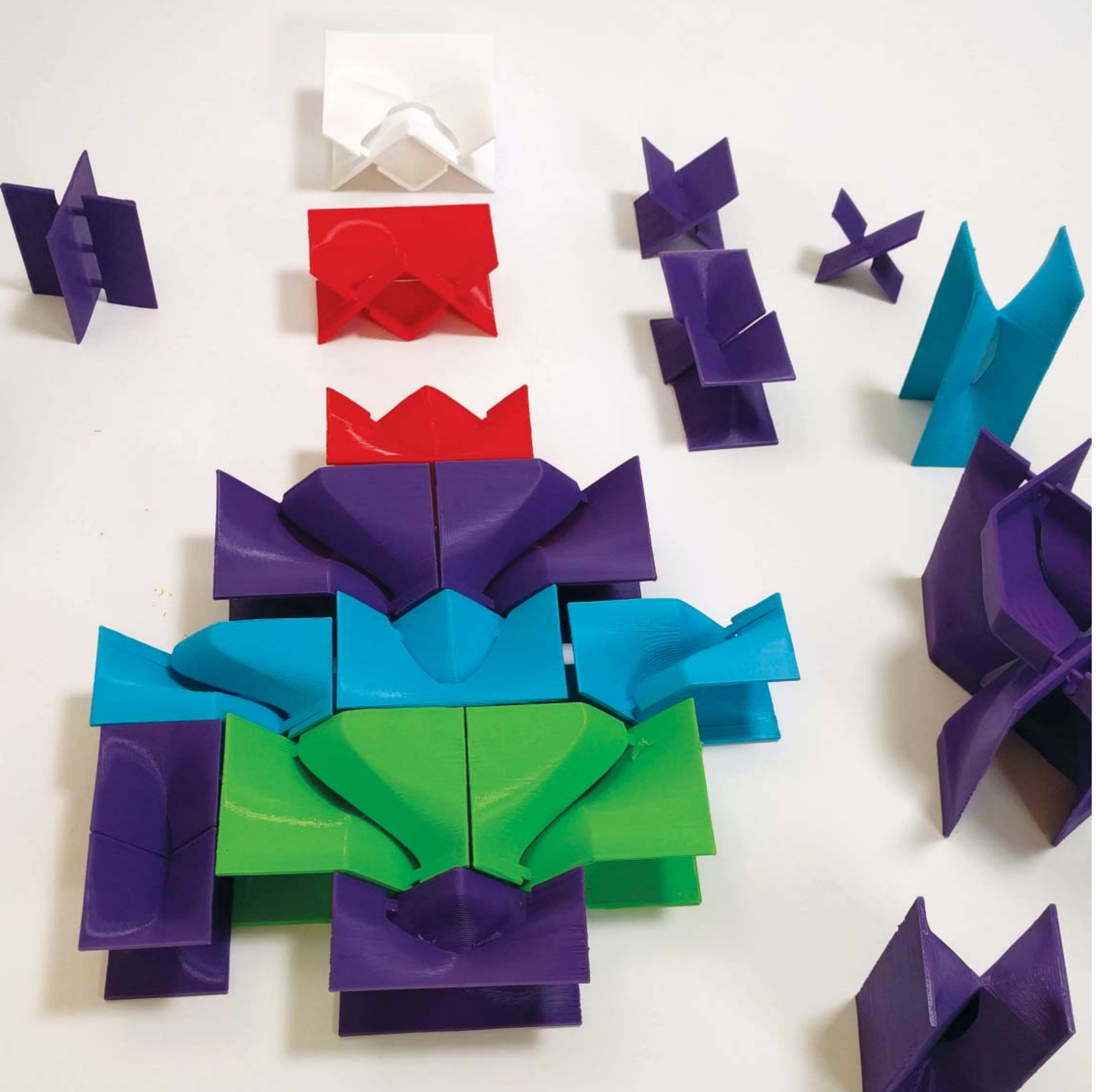
This project is joint work with Jack Love at George Mason University. On our first attempt, we did not put any padding between the outside and inside of the shapes, and they each fell into pieces upon printing. On our second attempt, we did not leave any extra space between the objects, and we could not close each piece to fit around the inside piece. (This 3D-printing issue would not be an issue for a laser-cut object.) On our third attempt, we realized that there was no way to hold it up, and made a customized stand to hold it.

EVELYN SANDER

George Mason University

3D-printed plastic

For a video of the nesting in action:
https://gmumathmaker.blogspot.com/2019/11/nested-platonic-solids.html



These are fundamental building blocks for knotted and braided surfaces in 4D space. My goal was to create, as children's toys, standard-sized models that could be snapped together in a Lego-like fashion, so that it would be possible to make a variety of surface braids, as children's toys. I made these prototypes on a 3D printer.

In the 3D-printing world, I learned to cope with unifying normal vectors. Unsurprisingly, I learned that, in general, curves and surfaces are fictions. My goal is to depict surfaces as they appear when projected into 3D space from 4D space, but to do so, the surfaces that I draw must be thickened to create 3D models that can be sent to the printer. I had to make some effort to accommodate the transition from 2D graphics to 3D graphics. Splines and handles of curves require more care on a 3D palette that is projected to a computer screen. On the other hand, some needed operations became easier. As I move forward, I will have to change my paradigm from *drawing* surfaces to *sculpting* them, as I develop facility with Blender.

J. SCOTT CARTER

University of South Alabama

3D-printed plastic

Further information:

J. Scott Carter and Seiichi Kamada, *How to fold a manifold.* New ideas in low-dimensional topology: Series on knots and everything, World Scientific Publishing (2015), pp. 31–77.

J. Scott Carter and Seiichi Kamada, *3D braids and their descriptions.* Topology and its Applications 196 (2015), pp. 510–521.

These plastic sheets illustrate the graph of a cubic function of two variables, as an instructional tool for multivariable calculus. Concepts in multivariable calculus such as gradient, partial derivatives, chain rule, and optimization can be visualized in terms of the geometry of the graph of such a function, so I wanted to give my students a tool for engaging with these ideas in a tactile and visual way.

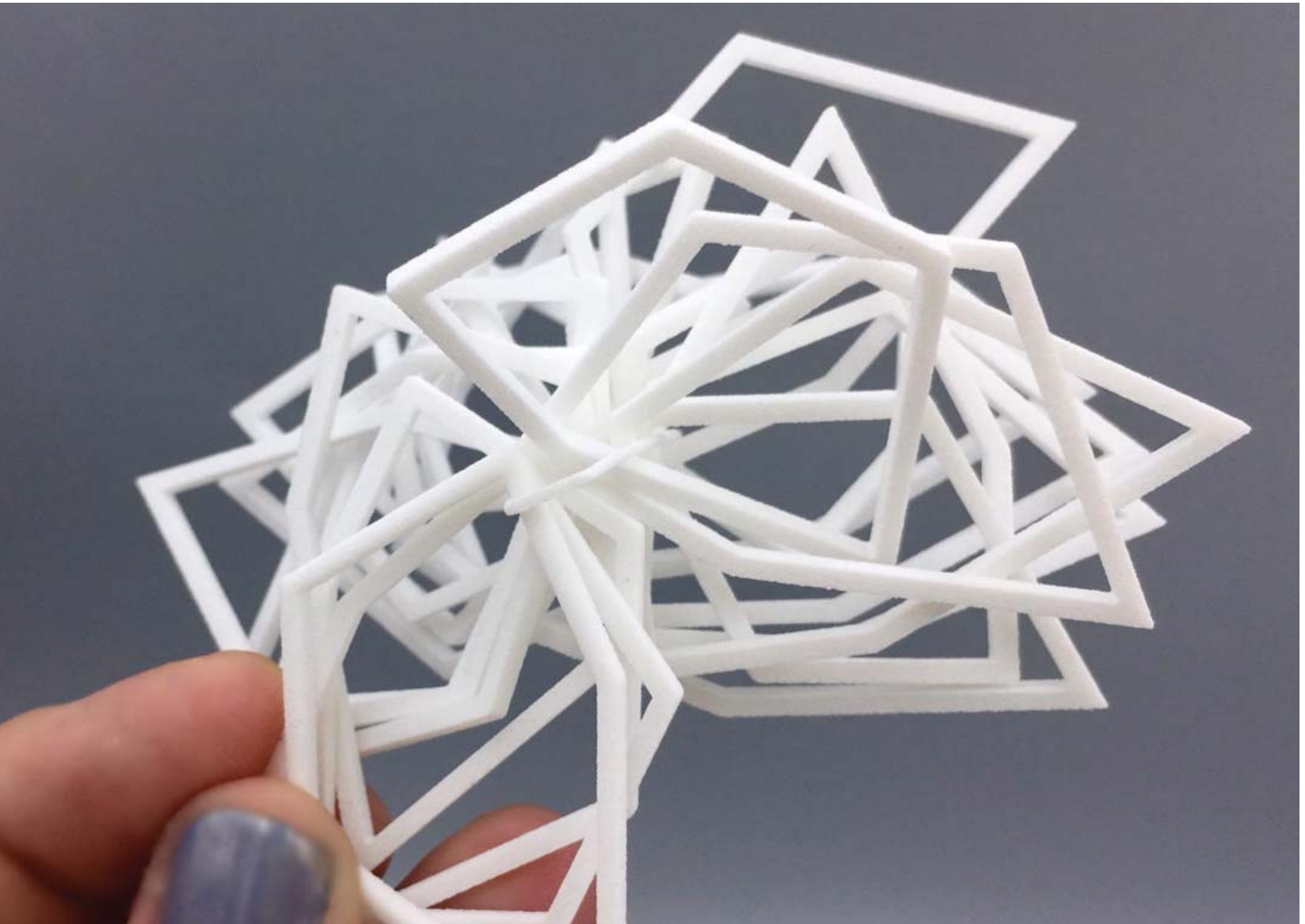
I needed to make 50 copies of the model so that every group of students could interact with the model. Instead of 3D-printing 50 copies of the model, which would be expensive and time-consuming, I 3D-printed one large model, and then thermoformed PETG plastic sheets onto the model to create many copies. Unlike most 3D-printed plastic, PETG plastic is clear, and it works well with dry-erase markers.

I had to experiment with the kind of plastics used for the 3D-printed mold and the thermoplastic sheets used for the surfaces, and also with the design of the mold. The first mold I made tore the thermoplastic along the edges and corners, and got stuck inside the cooling plastic. I had to redesign the mold with rounded edges and with a 2-degree draft along the sides, and I switched from polystyrene thermoplastic sheets (think single-use hot beverage lids) to PETG sheets (think soda bottles). Also I discovered that typical 3D-printing plastics like PLA melt when they come into contact with the heated thermoplastic sheets; Alba Málaga and I destroyed a large model that had taken hours for Alba to 3D-print when it melted on contact with the hot plastic sheets. I then switched to 3D printing with ABS, the plastic that Legos are made of, which has a sufficiently high melting point.

STEPAN PAUL

Harvard University

thermoformed PETG plastic



Regular pentagons do not tessellate (tile) the plane, but it turns out that there are 15 families of non-regular pentagons that do. These pictures show 3D-printed copies of a representative of the "Type 5" family of tessellating pentagons (see tessellation in upper left), discovered by Karl Reinhardt in 1918. The design for these models comes from our open-source "Pentomizer" OpenSCAD code, which can produce all of the representatives of each of the 15 families of tessellating pentagons. The code we wrote for this model allows us to vary the parameters and visually explore the entire space of each tiling pentagon family.

To print the pictured models, we used a 3D-printing service with a professional *selective laser sintering* (SLS) 3D printer. Rather than extruding a thin stream of melted plastic layer by layer like a desktop 3D printer, the SLS method starts with a 3D volume full of plastic powder, and a laser fuses the plastic in the regions required by the 3D object. Afterwards, each resulting 3D-printed object must be sifted out of the remaining powder. To minimize per-part costs, we designed a thin loop into the model so we could connect the pieces together as one unit (upper right and lower pictures).

The models print well on any desktop 3D printer or can be laser cut. 3D printing is not necessarily the best medium for these flat models; it is easier and faster to laser-cut this design if all you want is a handful of pentagons. However, when printed in plastic on a desktop printer, the individual pentagons are slightly flexible and particularly nice. When printed on an SLS printer, the pentagons have a very professional and finished look.

Fun story: One time we printed a full gallon Ziploc bag of one of these pentagons, for an event. Only after printing them all did we realize that our pentagons were degenerate representatives of the family we were printing, and in fact had only four sides.

For additional design tips and downloadable files:
https://mathgrrl.com/hacktastic/2019/07/stacking-3d-models-for-bulk-printing/
https://mathgrrl.com/hacktastic/2019/07/print-all-the-pentagons/

LAURA TAALMAN

James Madison University

3D-printed plastic

These are two so-called *cubic surfaces*: the points satisfy a certain equation in x, y, z of degree three. These are some of the most fascinating objects in the intersection of algebra and geometry, and they are from the 19th century.

I have created many different 3D-printed versions of such objects over the past 20 years. Classical sculptures show them in plaster, and my modern versions, which are exhibited in museums, are usually 3D-printed in white plastics. But these objects in gold-plated brass visualize their fascinating geometry in a much more obvious way, mainly because of the reflections of the material. In particular, in some interesting light situations with lighter and darker colors around, these reflections yield interesting visual effects on the surfaces of the sculpture. The curvature of these surfaces is much more interesting than you might think when just computing them, and printing them in brass allows us to observe these features.

Over the years, I have created many different versions of cubic surface sculptures using 3D printing, until finally, now, I like them. Earlier versions I created consisted of the part of each surface lying inside a certain sphere. This is a natural way to cut a finite part of an infinite object – at least when one does not know much about the object and wants to focus a region around the origin. But I worked with these objects for a long time, and thought a lot about which representation in space I prefer. My final choice is close to the classical one: cutting by a cylinder. But the decision about the position and size of the cylinder is not always an obvious one, and making this decision has been quite time-consuming for some of the cubic surfaces, as has the choice of the exact equations.

For more sculptures, and to purchase them: https://math-sculpture.com/
Further information: Oliver Labs, *Straight Lines on Models of Curved Surfaces*, The Mathematical Intelligencer, Volume 39, (2017), pp. 15–26.

OLIVER LABS

MO-Labs

3D-printed gold-plated brass

The 2-adic solenoid is quite the elusive pop star, featured on the covers of many dynamical systems textbooks but rarely seen as a tangible object in our world. Fortunately, this piece can be held in the palm of your hand! It consists of three solid tori 3D-printed in nickel, representing the geometric construction of the 2-adic solenoid. The second torus is stretched and wrapped inside the first, and the third is stretched twice and wrapped inside the second. The process can be repeated ad infinitum. The intersection of all transformed solid tori gives the solenoid. Taking a look at the cross-section of the piece shows a system of intersecting disks, in which each disk contains two smaller disks. A point in the cross-section of the 2-adic solenoid corresponds to choosing one of two disks at each step, ad infinitum. Hence the cross-section must be a Cantor space.

I chose 3D printing because I wanted the model to be as precise as possible. I chose nickel because I wanted the object to act as a paperweight, sitting heavy atop some papers or in a person's hand.

When making the piece, I knew very little about the actual mathematical object, but the making of the tangible piece spurred a lot of research and discovery. I gave talks explaining the different constructions, which I labeled as *geometric*, *topological,* and *algebraic*. Through this research I learned that the solenoid arises in several mathematical fields and takes on many forms that seem different but are ultimately equivalent dynamical systems.

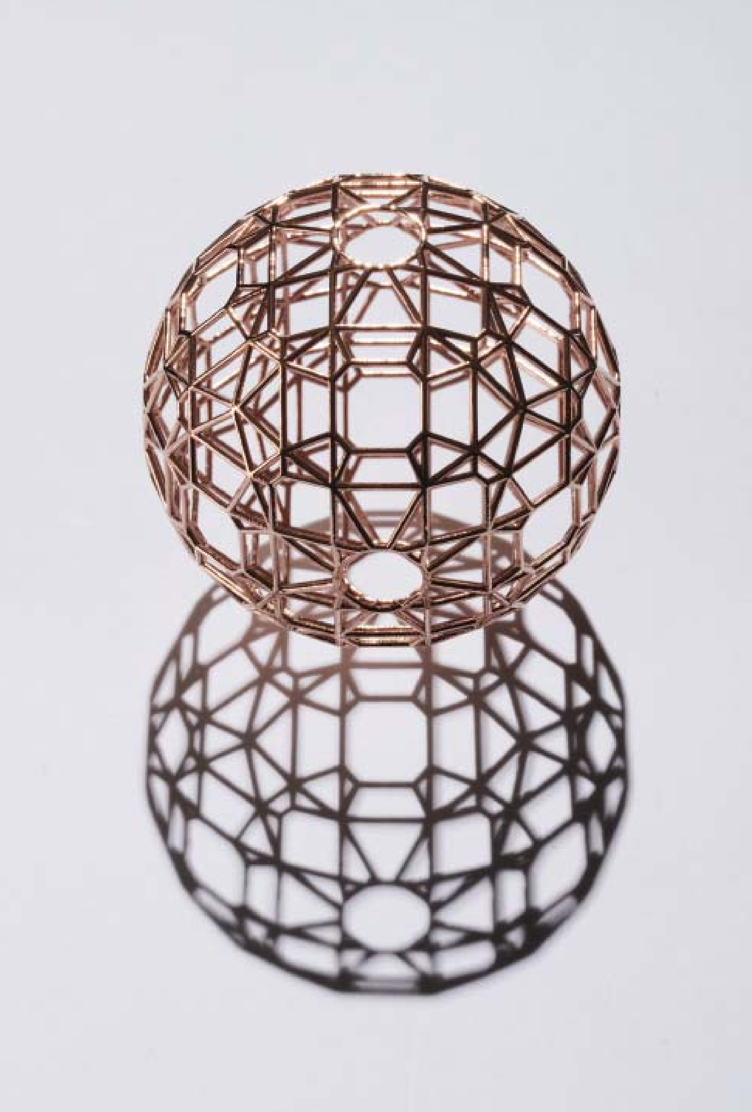
Creating the design in Mathematica did not take too long, once Ian Putnam helped me with the surface formula. Printing a rough sketch model at the University of Victoria's Design Scholarship Commons also happened fairly quickly. But making the model printable in metal through Shapeways was a process of trial and error over the course of many frustrating months. Rendering the model in Mathematica and uploading to Shapeways often took several minutes, with failure being the final result. Ultimately, tinkering and persistence won Shapeways over and my design was printed!

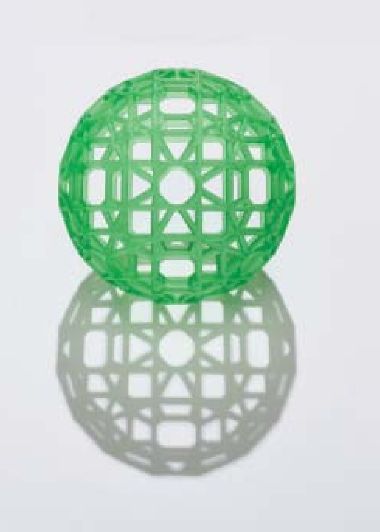
DINA BURIC

University of Victoria

3D-printed nickel

Further information:
http://www.math.uvic.ca/faculty/putnam/ln/Smale_spaces.pdf.
R. Clark Robinson, *An Introduction to Dynamical Systems*, Pearson Prentice Hall (2004)
Files for printing: http://www.math.uvic.ca/~buricd/projects.html.





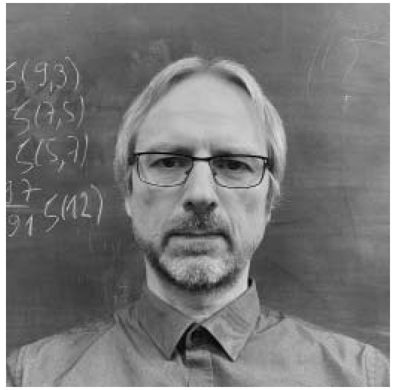



3D hyperbolic space $\mathbf{H}^3$ allows for lots of interesting *group actions*, transformations that return a space to itself. These give rise to periodic tessellations (tilings) of that space, in which one has a finite number of building blocks that are repeated over and over again, with each one occurring infinitely many times. For a 2D analogue in our standard *Euclidean geometry*, think of an infinitely extended bathroom wall with a periodic pattern of tiles, or the work of M.C. Escher (see page 47).

Many of those 3D hyperbolic tessellations arise from number theory. For each natural number d, we can form a *Bianchi group* G_d of 2×2 matrices. Specifically, the entries of G_d are of the form $a + b\sqrt{-d}$ with integers a, b. Each of these matrices maps hyperbolic 3-space onto itself. Since this group action has a special property (it is *discrete*), it provides a periodic tessellation as advertised, with the same finite number of tiles as it has building blocks. Imposing a further technical condition results in a tessellation for which the vertices of all the tiles lie at the boundary of $\mathbf{H}^3$ (it is an *ideal tessellation*), and thereby ensures a close connection to our standard 3D Euclidean geometry. Each of our polyhedra displayed in the pictures is associated to some polyhedron in an ideal tessellation that results from the action of a group G_d as above.

The choice of tessellation arose from my diploma thesis. Don Zagier gave me the topic, and Walter Neumann crucially suggested that I use work of David Epstein and Robert Penner. I computed the first dozens of cases on an Atari ST+ in Omikron Basic (in the late 80s). In recent years, the topic became a nice playground for summer undergraduate research projects in Durham. With the advent of affordable 3D printers, an obvious task was to find ways to visualize—and materialize—the objects. This became easier after my student Josh Inoue discovered a simple way to recover an even more symmetric version of each such polyhedron having all its vertices on a sphere (via a suitable affine transformation, dependent on d). For computing and displaying them, we used software like Pari/GP, Mathematica and OpenSCAD.

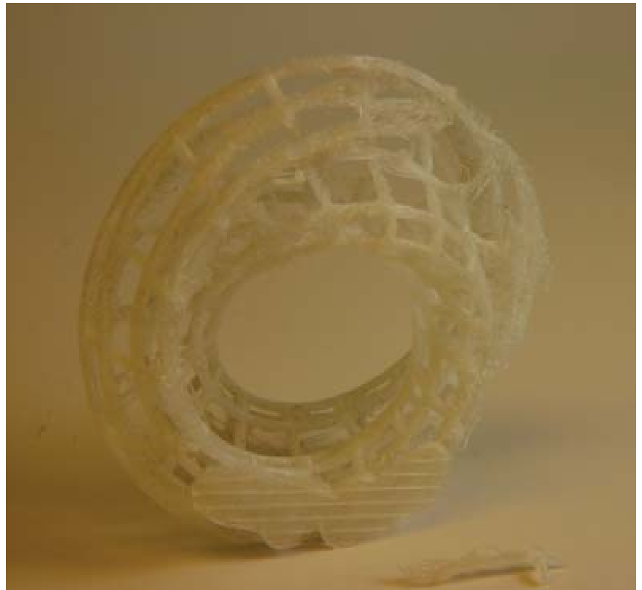
Eventually I was tempted to get the objects materialized in precious metals through a 3D-printing service, a process that turned out to necessitate many iterations. I was so amazed by the beauty of the outcome that I decided to create a whole collection of mathematical jewelry.



HERBERT GANGL

Durham University

3D-printed precious metals and plastic



The *Klein bottle* is one of the simplest examples of a *fiber bundle*: a space built by putting copies of one shape (the *fiber*) at each point of another space (the *base space*) in such a way that adjacent copies glue together nicely to form the total space. The usual embedding of the Klein bottle into 3-space illustrates how to construct the Klein bottle: glue one pair of sides to form a tube, and then glue the other pair with a flip. In contrast, the shape of the Klein bottle that I chose here highlights the fiber bundle structure itself, using a wire frame so that we can see and follow the base space around the figure, and we can see the circular structure of selected fibers as well.

3D printing was a natural choice, since the construction I envisioned was relatively straightforward to implement algorithmically. Where a hand-constructed or sculpted approach would have suffered from variations across the piece, a 3D-printed approach gave it the uniformity I sought.

I printed my first attempt (lower right) on an extrusion printer. A combination of extreme overhangs and issues with temperature control failed that print. Based on that failure, I switched to sintered-type printing methods, and sent my models to Shapeways rather than trying to print them myself. With sintering, the overhangs are supported by the unprinted materials and cease to be a problem. The first instance of that, seen here in grey, was printed in alumide. I also used the features of that printing method to print a disconnected ball in place inside the Klein bottle.

The modeling for the alumide model had aesthetic issues: along a curve winding twice around the Klein bottle, several tubes in the model bundle up together. Furthermore, the fibers intersect themselves, somewhat obscuring the fact that they are all topological circles. Revisiting the modeling script, I created a new version that includes the base space as one of the longitudinal wireframe wires, with the fibers dodging out of each other's way. This way, the circularity of the fibers is more clearly visible, and the self-intersection of the Klein bottle along the base space is visible in the alternating intersections of fibers with the base space circle.

Further information: Mikael Vejdemo-Johansson, *3D printing identification spaces of the square*, preprint (2019).

MIKAEL VEJDEMO-JOHANSSON

CUNY College of Staten Island / CUNY Graduate Center

3D-printed plastic, alumide, and steel

This piece is an illustration of elementary cellular automata, made from 3D-printed ceramic. A *cellular automaton* consists of a grid of cells, in which the state of each cell depends on the states of those around it. Simple examples of cellular automata are *Pascal's triangle*, which has rows of numbers each of which is the sum of the two above it, and John Conway's *Game of Life*, in which each square in a grid is colored white or black at each time step, depending on the colors of the cells around it in the previous time step.

I'm not a mathematician, so I learned everything about cellular automata through this project. Once I was doing the work and the research, I started to be even more interested in exploring complex systems that are based on iterating simple rules. I think about these rule-based mathematical systems as games that can be played for an infinitely long time.

My workflow from the math to the object can be summarized by the following steps: After picking the game rules, we coded them in Sage, using CoCalc (see link below). I worked on coding this with mathematician Sara Billey from the University of Washington, then altered Billey's code-kernel with the help of Daria Mićović. The code generated a 2D plot of a distribution matrix in black and white. We then used a 3D-modeling program, Rhino, to turn this matrix into printable 3D geometries of cubes and cylinders.

This process required careful calculations and many tests, which were reverse engineered from the relationship between the printing layer height, nozzle size, and cell size, and the desired scale of the finished object. A precise reflection of the original matrix was my main goal, reflecting on the algorithmic nature of the first half of the process. On the other hand, I enjoy seeing how physical reality – the qualities of the material and the physics of gravity, time, etc. – makes subtle unexpected changes on the otherwise perfect model.

I work by testing and constantly refining until I have a very solid prototype that can be 3D-printed reliably. Still, I might lose 50% of the pieces because what I'm asking from the fragile porcelain is kind of difficult already.

More of my work: https://www.timeatihanyi.com/
Information about 3D-printed sculpture: https://www.sliprabbit.org/studio
CoCalc is a web-based platform for computational math, founded by mathematician William Stein: http://cocalc.com/

TIMEA TIHANYI
Slip Rabbit Studio
3D-printed ceramic

I am interested in curved 3D spaces. We're used to flat space, in which two straight lines that are parallel in one location remain parallel forever. In curved spaces, two straight lines that start out parallel will intersect. On the curved surface of the earth, imagine two people on the equator, one in Ecuador and one in Kenya. They start walking due north at the same pace. Initially, their paths are parallel, but they will get closer to one another, and eventually their paths meet at the north pole.

In 3D, there is a space called the *3-sphere* that, like the surface of the familiar 2D sphere, has uniform positive curvature. There are many beautiful geometric structures that live naturally in this space. The three sets shown here are a pair of interlocking Möbius strips (bottom left), a twisted Scherk saddle tower minimal surface (bottom right), and a woven pair of Hopf fibrations on the Clifford torus (top left). Both the Möbius strips and the Scherk saddle tower started as objects in 3D Euclidean space with finite extent in the z-direction. I then placed them along an equator of the 3-sphere, which closed them up into a loop. For the third piece, I took the Hopf fibration – which is a series of circles that are all interlinked with each other – and looked at the circles that live on the Clifford torus. I took two of these Hopf fibrations that are orthogonal to one another and wove them together along the Clifford torus. I then projected all of these objects onto 3D Euclidean space.

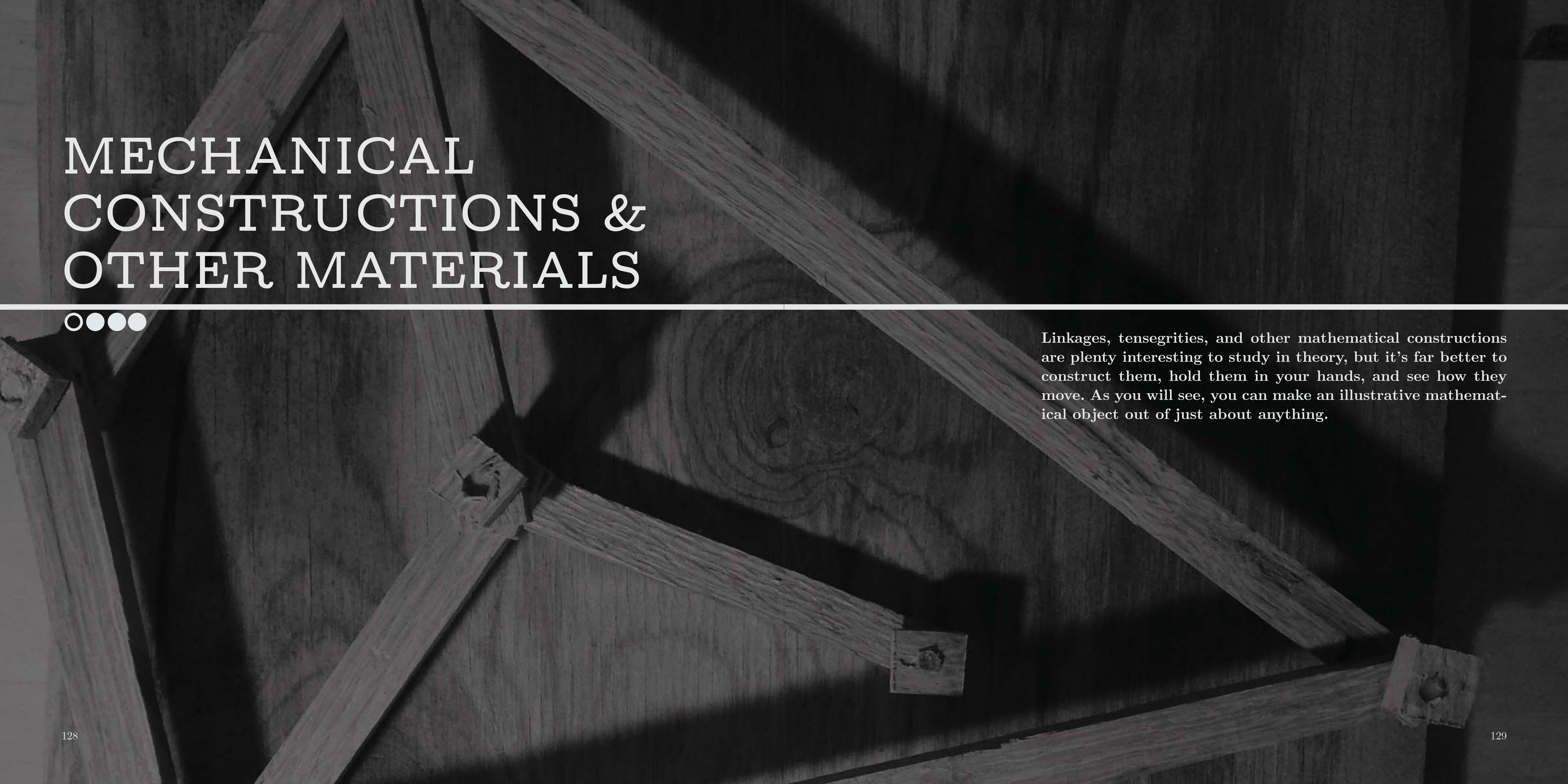
I used Shapeways' cast metal process because it accurately reproduces geometry and creates professionally finished pieces. These pieces are 3D printed in wax and then cast into precious and semi-precious metals. The choice of materials depends on the type of geometry. For instance, if there are multiple interlocking pieces, the piece cannot be cast in or plated with precious metals. My experience with interlocking pieces is that they turn out beautifully, but they require more upkeep to keep them looking new than ones that are made from or plated in gold.

I wanted to bring many of these objects to life in a medium that complements their natural aesthetic simplicity. I chose jewelry for a few reasons: it is simple and elegant while still able to show mathematical details. I love the notion of creating something elegant and feminine out of mathematics. Those of us who are mathematicians see the beauty in everything we do, but it is hard to communicate that outside the community. I believe jewelry is a way to universally share the beauty we see with others.

SABETTA MATSUMOTO

Georgia Institute of Technology

metals cast from 3D printed molds



MECHANICAL CONSTRUCTIONS & OTHER MATERIALS

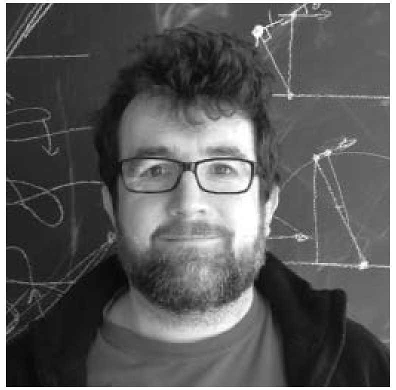
Linkages, tensegrities, and other mathematical constructions are plenty interesting to study in theory, but it's far better to construct them, hold them in your hands, and see how they move. As you will see, you can make an illustrative mathematical object out of just about anything.

This is a hyperboloid, cut from a brick.

This item was the result of a project with Nick Bruscia and Dan Vrana from University at Buffalo and OMAX corporation, who make the water jet. The challenge was simply to explore what was possible with a five-axis water jet. As the cutting "tool" is a straight (well, spiraling) jet of water, *ruled surfaces* are the natural thing to start experimenting with.

A surface is *ruled* if, for every point on the surface, there is a line through that point that is completely contained in the surface. A hyperboloid is a ruled surface, so to cut it out we need the jet of water to follow the ruling lines. This object therefore starts as an illustration of an application of geometry: how to control and think about the forms that can be made by controlling the form of a general tool.

It also presents a compelling version of a classic mathematical object. The surprising material (brick) draws you in, with the question of how it could be made. Even knowing the tool used might not help, as the curved surface seems impossible to make with such a straight tool. This leads directly to an understanding of the behavior of ruled surfaces and the hyperboloid.



EDMUND HARRISS

University of Arkansas

Waterjet-cut brick

This piece shows rational points on a cubic surface. *Rational points* are the points (x, y, z) that lie on the surface, such that x, y and z are all rationals. This is related to Fermat's famous last theorem.

Away from the 27 straight lines contained in the surface, there is only a small finite number of rational points with numerator and denominator below 100 (on the part of the surface shown). To visualize these points in real life, we needed some support material to hold them at their position, because the points themselves are not connected – there is an empty space between them. Laser-in-glass allows us to do this with a support material (glass!) that is almost invisible. This is perfect for getting a good idea of the fascinating structure and geometry of these points.

There is some complicated hidden structure in the positions of the points. Understanding it mathematically is not easy, but visually, one quickly gets a rough feeling for it.

Choosing the correct size for the small balls representing the points was a challenge. Also, choosing a good bound for the numerator and denominator (100) was a lengthy experimental process. Working with Ulrich Derenthal, we essentially had to choose a compromise between having enough points to see certain geometric features and having too many points or too-large balls. Too many tiny balls almost cause the whole cubic surface to be filled.

OLIVER LABS

MO-Labs

laser-in-glass

For more sculptures, and to purchase them: https://math-sculpture.com/

The *Cantor set* is one of the simplest fractal objects. Its ternary representation is obtained by an iterative procedure: start with a segment of length 1 and remove the middle third; for each of the two remaining segments, remove the middle third; and so on. After an infinite number of steps, the remaining object is the Cantor set and looks like dust. For a long time, topologists were intrigued by the embeddings of a Cantor set in 3D Euclidean space. In particular, they asked:

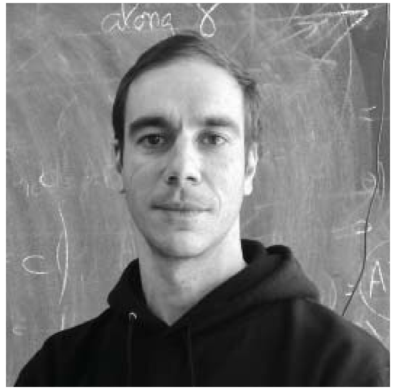
> *Is it possible to find an embedding of the Cantor set in $\boldsymbol{R}^3$ so that there exists a loop that cannot be shrunk to a point without crossing the Cantor set?* or, said differently: *Does there exist an embedding of the Cantor set in $\boldsymbol{R}^3$ whose complement is not simply connected?*

The ternary Cantor set described above does not satisfy this property. Nevertheless, the question was answered positively by Louis Antoine at the beginning of the 20th century. This embedding is now known as *Antoine's necklace*. It is a necklace, whose links are themselves necklaces, whose links are themselves necklaces, whose links...

This piece represents the first four levels of Antoine's necklace. Louis Antoine was a professor at the Université de Rennes 1, where I currently work. When my colleagues discovered that I was interested in illustrating mathematics, they challenged me to realize an Antoine's necklace.

Because the links of the necklace are so intertwined, it was not possible create this object with a 3D printer or a laser cutter. Inspired by the name of the object, I decided to build the object from brass chain. This piece is made of 10,000 links, which I opened and closed individually. The whole process let me experiment with the self-similar structure of fractals. In particular, at the end, a single link is simultaneously used to close one copy of each level of the necklace. The first versions were unicolor. However, it was not easy to distinguish the smallest levels. Using two different platings solves this problem. The next challenge is to build a stand to display the necklace.

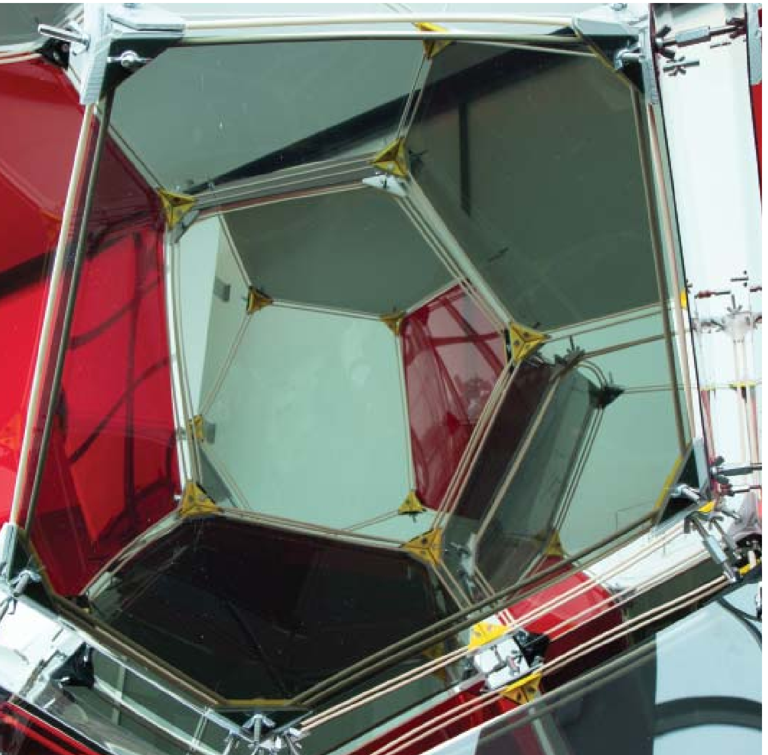
Antoine's original paper:
Louis Antoine, *Sur l'homéomorphisme de deux figures et leurs voisinages.* Journal de Mathématiques Pures et Appliquées 4 (1921), pp. 221–325.

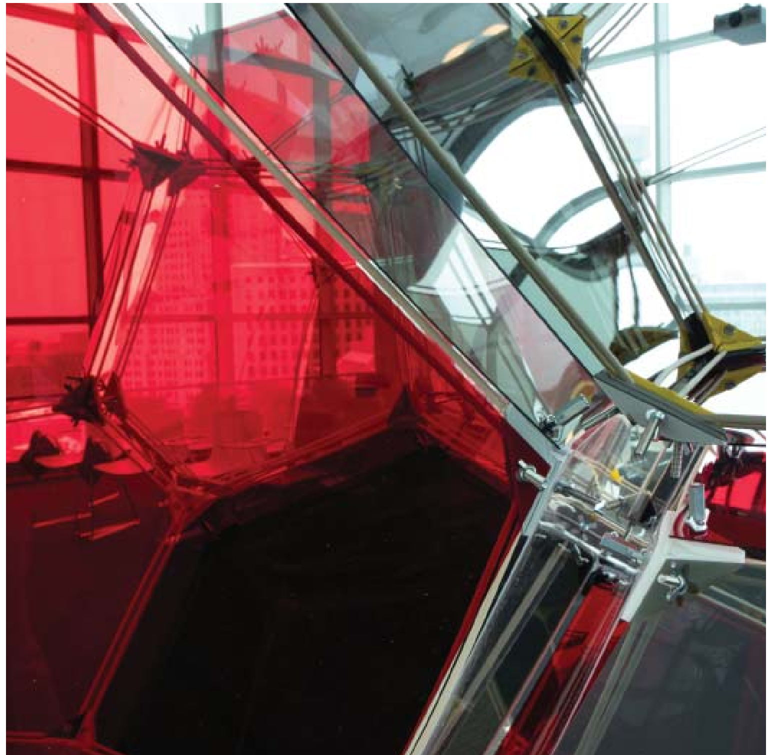


RÉMI COULON

CNRS / Université de Rennes 1

Brass links





This assembly provides a large-scale model of *Weaire-Phelan foam*, which is a candidate answer to the question:

> *How can we enclose and separate infinitely many unit-volume regions ("bubbles") in 3-space, with the minimum possible surface area?*

The structure arises because the surface films of bubbles naturally tend to minimize their surface area. For a single bubble, the minimizing shape is a sphere, but for many tightly-packed bubbles, it's still not clear what the optimal structure should be. I chose to illustrate the Weaire-Phelan foam for a public address on the nature of mathematics:

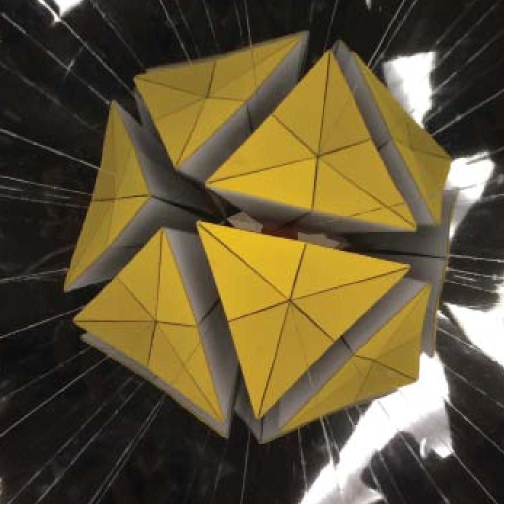
- I wanted to bring out the multi-layered nature of mathematical discovery: each time a pattern is discovered, it may lay the groundwork for deeper, more-encompassing patterns. By analogy, the audience could discover that their "line segments" – wooden dowels distributed prior to the talk – could fit together into pentagons and hexagons, that the pentagons and hexagons could fit together into polyhedra, and that those polyhedra fit together perfectly into a potentially infinite lattice.
- This lattice acts as a gateway to an unsolved problem – *is the Weaire-Phelan foam really the optimal shape for packed bubbles?* – revealing that in mathematics, there are always more patterns to be discovered.
- I wanted the audience to have a satisfying tactile experience assembling the model. There's nothing like touching and manipulating material objects to form a cognitive connection with a subject. Also, the scale of the model was important: I wanted the finished product to be visible throughout the auditorium and to provide the audience with a sense of accomplishment.

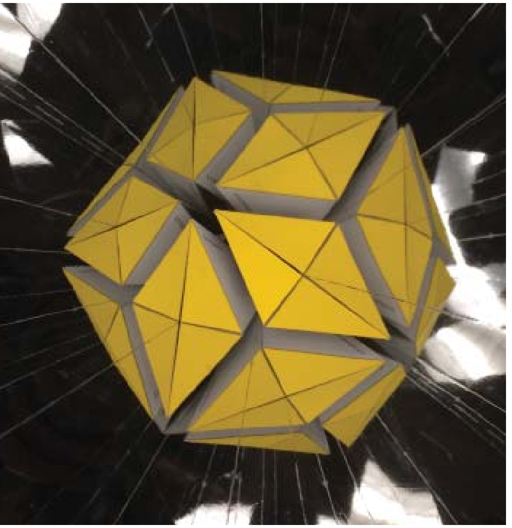
My plan was that faces of adjacent cells that join together in the final lattice would have the same color, whereas adjacent faces of the same cell would have different colors. Unfortunately, I discovered that it's mathematically impossible to accomplish this goal with only four colors, and in fact it may not even be possible with five! The biggest problems I encountered were about stretching and bending: First, the acrylic sheets don't stretch and bend exactly like the surface of a bubble would. Second, the dowels forming the edges of the faces do bend, but my model for the dowels' lengths did not take the bending into account, so I cut the dowels to approximate lengths. I made the holes in the acrylic sheets extra-large to compensate, but they still only fit with difficulty, and some cracking of the acrylic occurred.

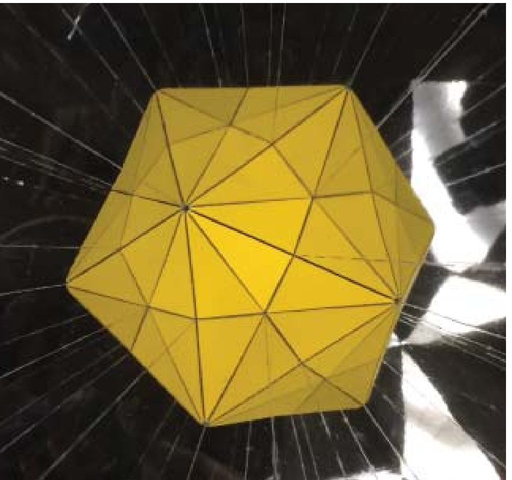
GLEN WHITNEY

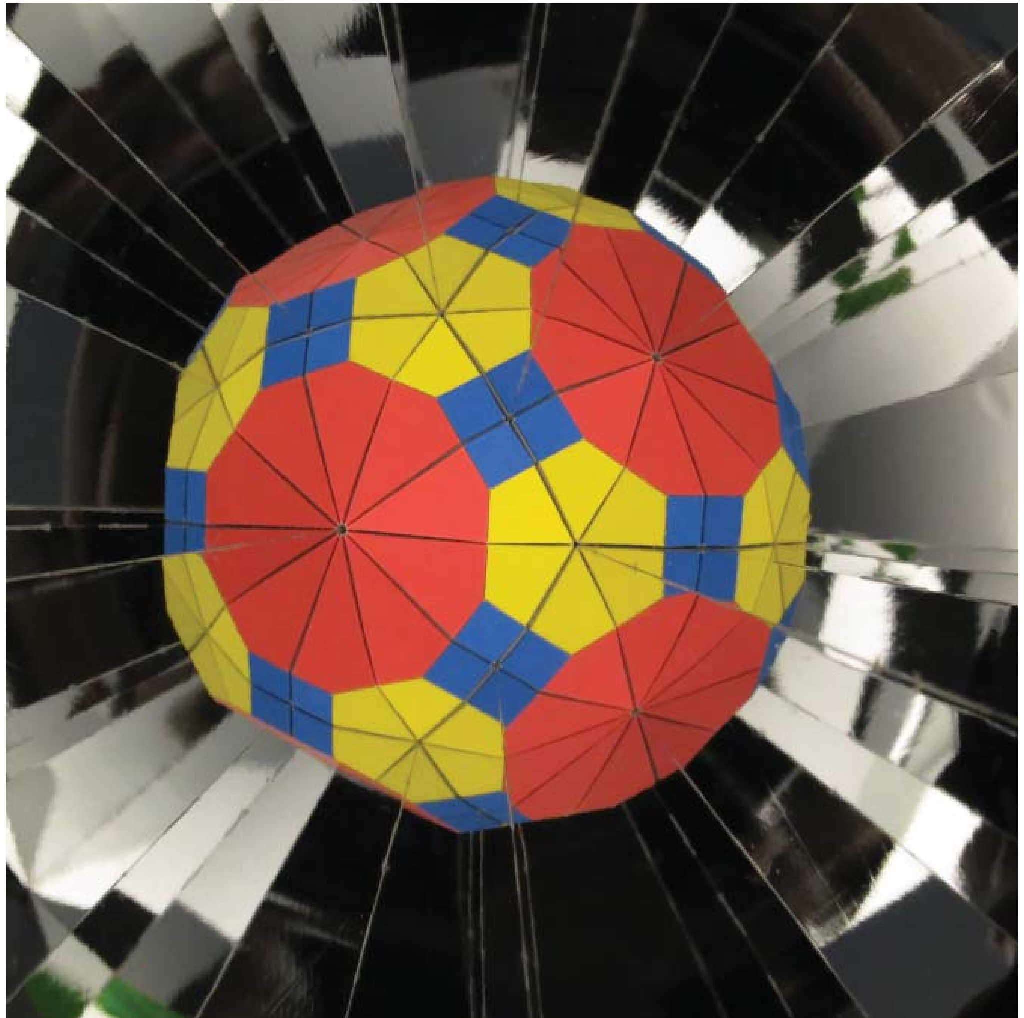
Studio Infinity

acrylic plastic sheets, dowels, 3D-printed brackets, and metal hardware









A traditional kaleidoscope is made from three long, thin, rectangular mirrors, faced inward to form a long, thin, triangular prism inside a cylindrical tube. Small colorful objects in various shapes are placed in the far end of the tube, and the many reflections of the colorful objects in the mirrors creates an image that looks like an infinitely patterned plane. In such a setup, we would say that the *fundamental domain* of the kaleidoscope is a triangular prism and the resulting image has triangular symmetries in the flat plane.

Rather than using rectangular mirrors in a tube to create a flat reflection, I used triangular mirrors coming together at a point (top right). This creates a *polyhedral* reflection, as shown. The three polyhedral kaleidoscopes pictured here allow one to create and view objects with icosahedral, octahedral, and tetrahedral point symmetries.

Here, the fundamental domain of each kaleidoscope is a tetrahedron, and the angles between its faces determine the symmetry group of the reflection image. Using these mirrored fundamental domains allows one to create all of the Platonic solids and most of the Archimedean and Catalan solids. I created the inserts out of card stock (top middle) that fit perfectly into the fundamental domains, to create the associated polyhedra, as shown. One can also create other card stock polyhedra, or use any objects that fit inside.

I chose to use front-surface acrylic mirror and card stock because they are laser-cuttable. I made several iterations of the kaleidoscopes, working to refine the design so that they could easily be fabricated by someone with little or no crafting ability (see instructions at link below). I hope they will be useful tools for others to learn about polyhedral symmetries.

Instructions for creating these:
https://www.instructables.com/id/Easy-to-Make-Polyhedral-Kaleidoscopes/

JOHN EDMARK
Stanford University

laser-cut mirrors

What happens when you attach mirror surfaces to the inside faces of a rhombicosidodecahedron? We set out to answer this question by creating this polyhedron out of Geometiles, placing a spherical camera near its center, and then stereographically projecting the output. In this image, the "pole" of the projection is the face opposite to the one through which the camera was placed. Careful placement of the camera allows us to see reflections of pentagons in other pentagons and squares.

With all the freedom that computer-generated images allow, photography has an almost "retro" appeal. The spherical camera, of course, allows for very unconventional photography by taking in the entire scene around it. Stereographically projecting this view onto a 2D plane gives a unique interpretation of what it's like to live inside the intriguing mirrored polyhedron.

An unintentional byproduct of this project was seeing what happens when there are slight deviations from the stereographic projection. When using a spherical camera inside an *almost* spherical object, these deviations are expressed as the lens of the camera being slightly below or above the center of the object. As a result, the projections look different depending on which pole you project from.

In order to get the best possible image, we had to experiment with the position of the camera in the polyhedron and with the ambient light. The latter is tricky, since light can come into the rhombicosidodecahedron only through the thin crevices between the tiles and the bottom face where the camera is placed.

We say "we" even though Bjoern was taking the pictures at Dartmouth College and Yana was processing the raw images in California. At some point, things got downright comical: Yana was making recommendations on where to shoot the pictures based on pictures of rooms on the Dartmouth campus that she had only seen online.

YANA MOHANTY &
BJOERN MUETZEL

Imathgination LLC;
Dartmouth College

Photography, and mirrors attached to plastic tiles

Further information: https://geometiles.com/archimedean-billiards/
More pictures and videos: https://math.dartmouth.edu/~mutzel/gallery.php

This is a large window, about 30 feet wide (bottom image), installed at the Mathematics Tutoring and Teaching Center at the University of Arkansas. It shows a self-similar hierarchical pattern in the plane, made by a recursive *tiling substitution rule*: all of the arcs are copies of one another, and each arc sits beneath four smaller arcs, always in the same manner. Conversely, each arc is one of four sitting above a larger arc, themselves a group of four upon an even larger one. In the mathematician's mind, this continues forever, smaller and smaller, and larger and larger, ad infinitum, always in a regular manner.

I made this image in Adobe Illustrator. When drawing a mathematical illustration in such a program, the main task is figuring out *how* to do the drawing, coming up with a process that will give the desired result. The mathematical illustrator must be willing to start from scratch several times, refining this process. I drew this window four or five times over, ultimately leading to a simple method that was accurate to 1/100th inch across a nearly 30 foot span.

The complexity of the details of the drawing was severely restricted by its exponential growth each scale of refinement – as drawn, the illustration nearly overwhelmed my computer!

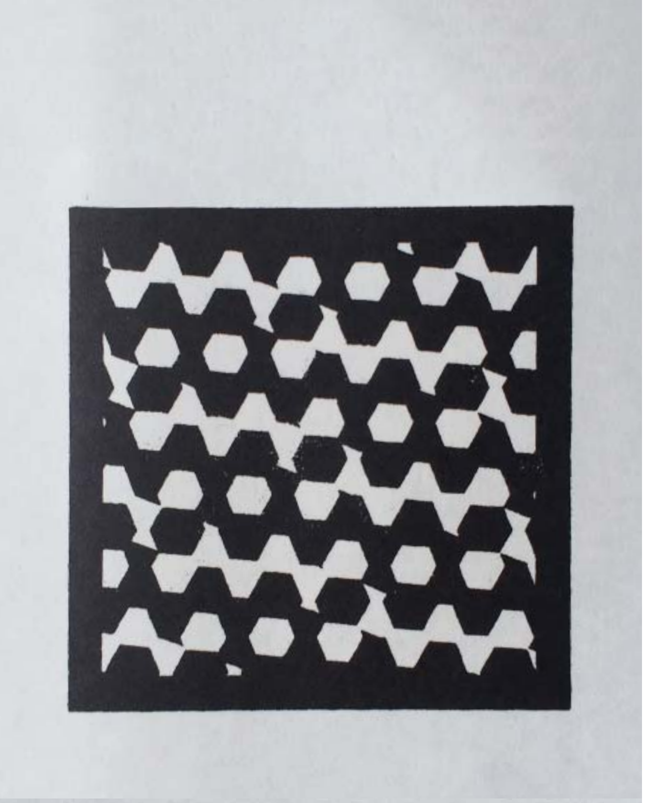
I work with tiling substitution rules in my research, and the opportunity to draw this window led me to explore this one. There's a long literature on how *undecidability* and other aspects of the Theory of Computation arise as matching rules on these hierarchical tilings, and certain aspects of this particular system seem to point in some new directions for "programming" in these tilings.

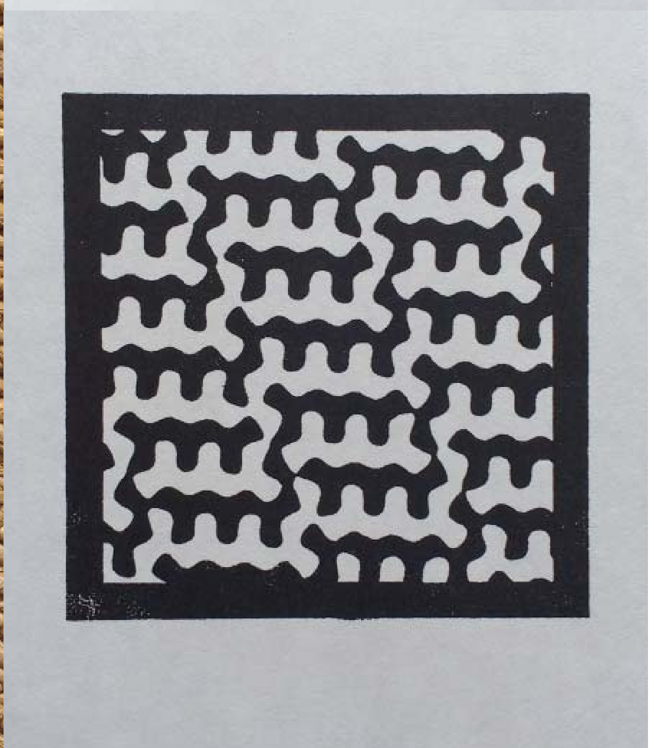
The design was printed in three layers: a sandwich of two layers of colored ink around a layer of white ink. The central white layer was printed in various opacities, reflecting light and color in some areas, and translucent in others, with bright zings of color moving through the window.

CHAIM GOODMAN-STRAUSS

University of Arkansas

laminated glass with printed interlayer





Each of these prints shows a planar slice of a $\mathbf{Z}^3$-periodic function on $\mathbf{R}^3$. The function is negative in the black region and positive in the white one. The slicing plane is irrational, so the pattern never quite repeats. Slices like these show up when physicists study how metals conduct electricity in strong magnetic fields. Sergei Novikov and his coworkers realized in the 1980s that, to know how a metal conducts, you have to know how the connected components of the black and white regions behave. Olga Paris-Romaskevich told me about Novikov's problem and mentioned that she'd love to see computer renderings of the slices. I'd been wanting to experiment with laser-cut block prints, so I was keeping an eye out for striking black and white patterns. I rendered a slice and thought, *This is it*.

I was drawn to block printing because I love M.C. Escher's prints, and digital reinterpretations of his work by folks like Christopher Becker and Vladimir Bulatov have really caught my fancy. It would be wonderful to give those new digital forms all the texture and depth of the originals. When I turned a hyperbolic tiling from my research into a mathematical art poster, I had it screen-printed by hand at a local print shop with that hope in mind. The printers' lovely work encouraged me to try block printing next.

I got lots of tips from volunteers at AS220, the Providence art space where I did most of the laser-cutting and all of the printing. One was to try different inks, papers, and block materials. Switching from Akua intaglio ink to a letterpress ink thinned with soy oil made a big difference, and the printing papers I sampled from the AS220 scrap drawer each made noticeably different impressions.

It took some care to find block materials that seemed safe to laser-cut. Several brands of printmaking linoleum have "no dangerous decomposition products" listed on their Safety Data Sheets. However, many synthetic rubber "easy-cut" blocks contain PVC, which is not laser-safe. I had good luck with both printmaking linoleum and laser-safe MDF.

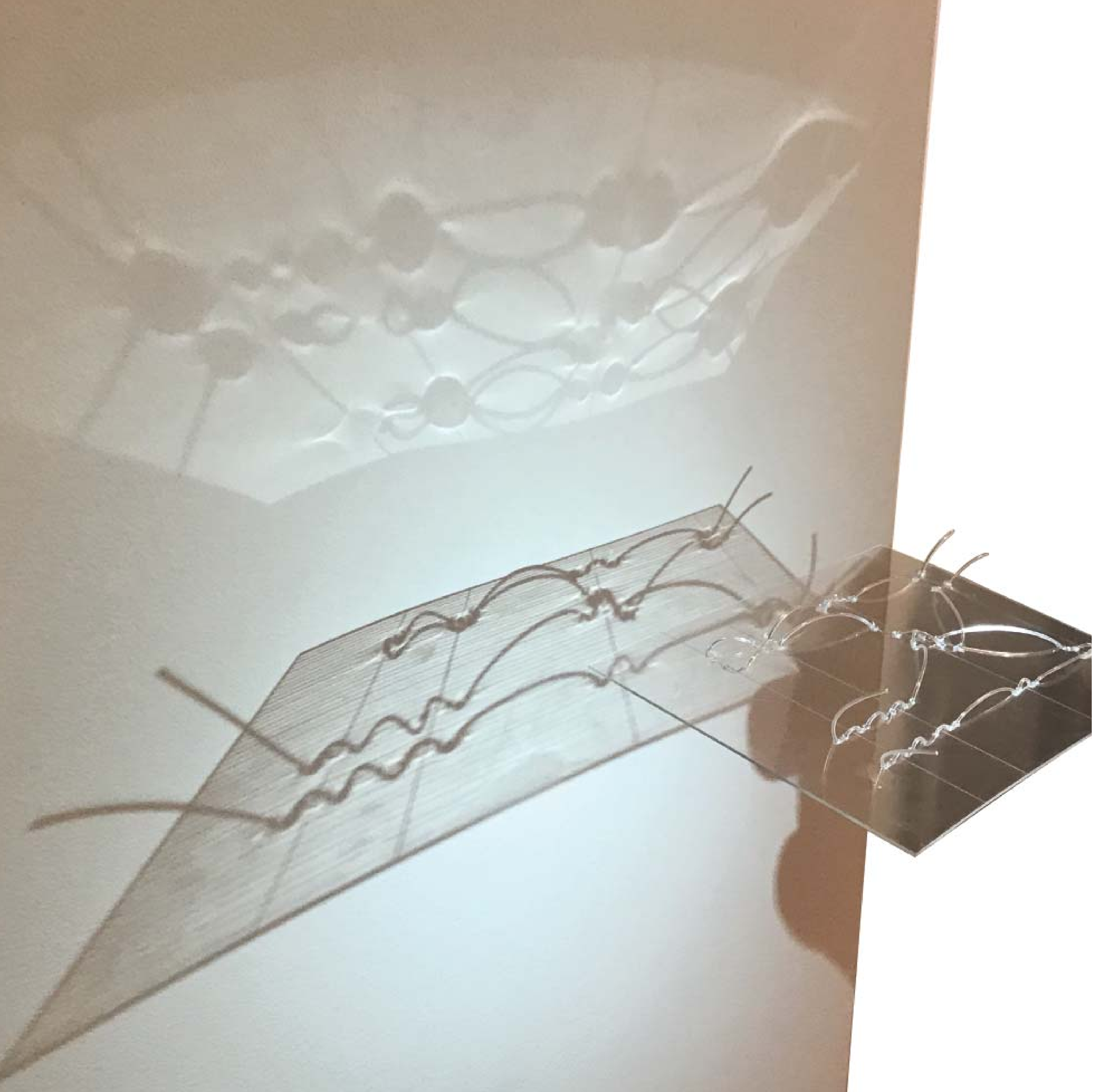
During a laser cutter outage, I bought a carving tool and cut the print on the left by hand. One corner shows a downside of hand-cutting: it takes hours, not minutes, to make a cut, but still only seconds to make a mistake.

Further information: Artur Avila, Pascal Hubert, and Alexandra Skripchenko, *Diffusion for chaotic plane sections of 3-periodic surfaces*. Inventiones Mathematicae, Tome 206 (2016), pp. 109–146.

AARON FENYES

Institut des Hautes Études Scientifiques

laser- and hand-cut block prints



We can think about prime numbers as being the "atoms" of the integers, in the sense that every integer is the product of prime numbers, and prime numbers themselves are indivisible, like the atoms that make up molecules.

A *ring* is a mathematical object that generalizes the notion of integers: it is a group of objects such that, for any two elements of the ring, their sum and product are also elements of the ring. As prime numbers are for the integers, *prime ideals* are in some sense the "atoms" for rings. In this case, we consider the ring $\mathbf{Z}[x]$ of polynomials with integer coefficients and its set of prime ideals, denoted by $\mathrm{Spec}(\mathbf{Z}[x])$.

In *The Red Book of Varieties and Schemes*, David Mumford depicts $\mathrm{Spec}(\mathbf{Z}[x])$, and Gabriel Dorfsman-Hopkins, Sachi Hashimoto and I wanted to understand his picture through actually building it. We considered the medium and technique as a representation of the underlying abstract mathematics.

Translucent materials tend to have a non-formless quality to them, and they have a nice tension with the shadows created when we project light through them. Using two types of translucent materials, we could create tension in the relationship between the string and the polypropylene.

By creating this object, we learned how to interpret the plane used to frame the interplay of the prime ideals in $\mathrm{Spec}(\mathbf{Z}[x])$. This was the first version, and it can be considered an experiment of perception and information.

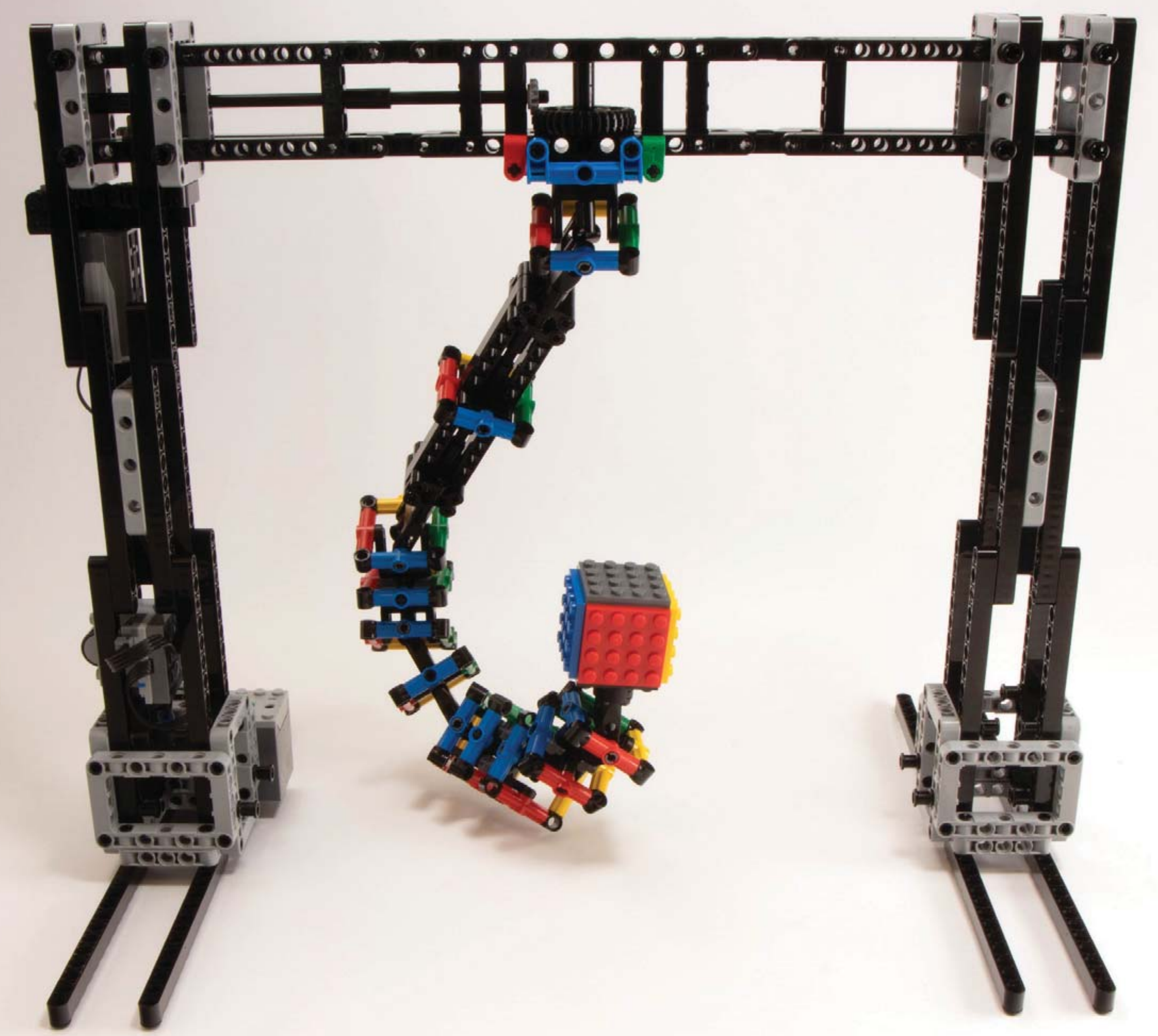
TASHRIKA SHARMA

University of Vienna

Polypropylene and monofilament

Mumford's original book:

David Mumford, *The Red Book of Varieties and Schemes*, mimeographed notes from the Harvard Mathematics Department (1967).



This is a mechanical illustration of the “Dirac belt trick” or “plate trick.” The central cube rotates continuously around a vertical axis, yet it is connected to the fixed outer frame via a sequence of hinge joints, each pivoting by at most 45° each way. The arm returns to its original state after every *two* complete turns of the cube. It can serve as an anti-twist mechanism: one can run a piece of thread or an electrical wire along the arm, fixed to the cube and the frame at its ends, which turns continuously with the arm, yet never gets twisted up. This reflects the two-to-one map from the (simply connected) space of unit quaternions to the (not simply connected) space of 3D rotations. To my knowledge, this is the first mechanical linkage that demonstrates the phenomenon. The rotation can be powered by an electric motor or by turning a handle.

Lego is the ideal medium for experimentation with mechanisms. One can go from an idea to a working model in minutes, and then to a more polished art piece with a bit more investment and effort.

I had wondered for some time whether such a linkage was possible. When I had the idea for the particular mechanism, it was not all clear to me whether it would work, even theoretically. The Lego model quickly provided the answer. Moreover, the model has provided new insight into the mathematical and physical phenomenon: the arm can be separated naturally into upper and lower parts, each of which on its own moves in an intuitive way; all that remains is to combine them.

ALEXANDER HOLROYD

University of Bristol

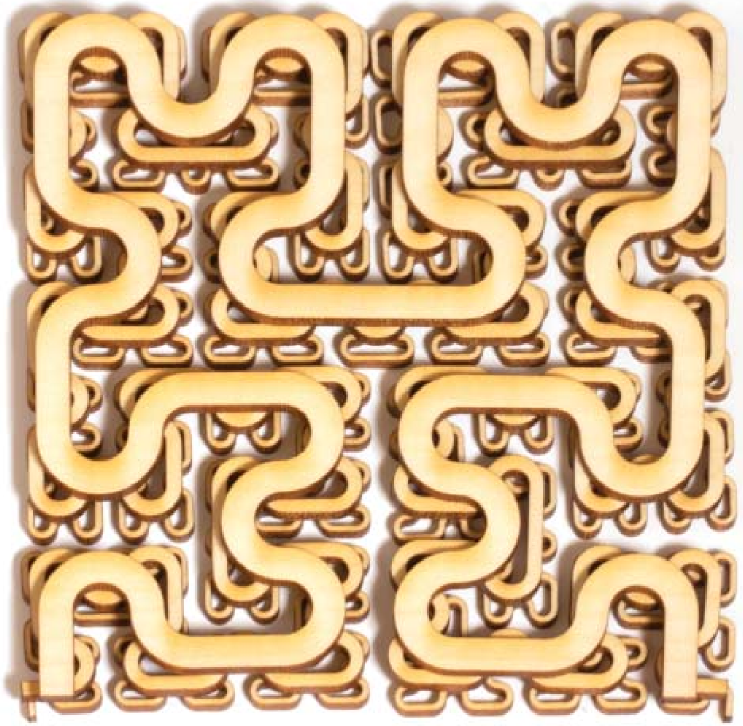
Lego construction pieces

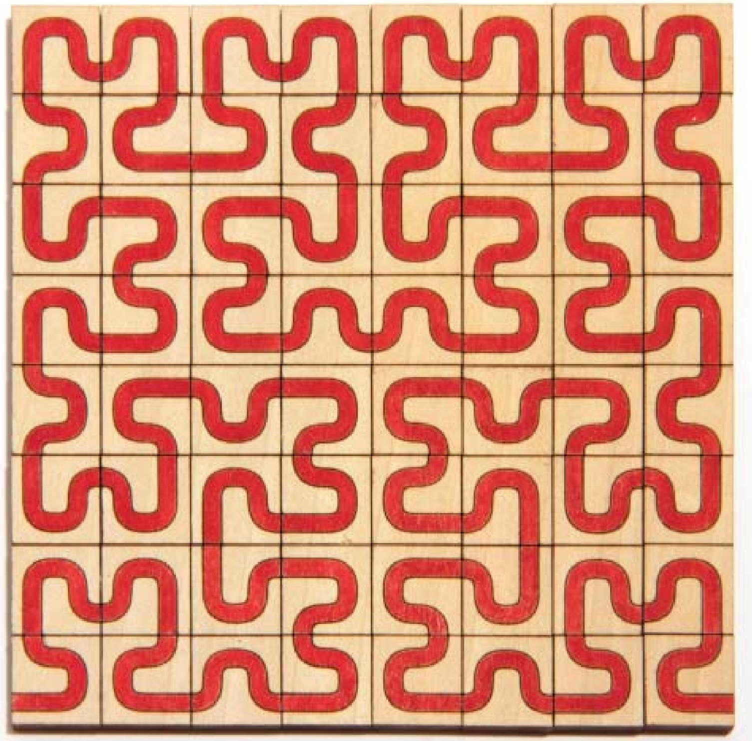
Video of the object in motion: http://youtu.be/byi5Gzjc04Q
Double version with lights: https://youtu.be/1x_oQv_qj_U
Further information: https://en.wikipedia.org/wiki/Plate_trick

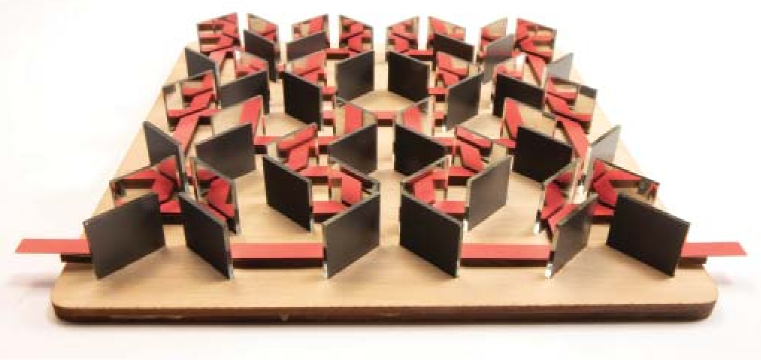
gear into the North Atlantic. One of the biggest disputes on a sword boat is whether to set out or not. Crews have hauled in a fresh gale because their captain misjudged the weather.

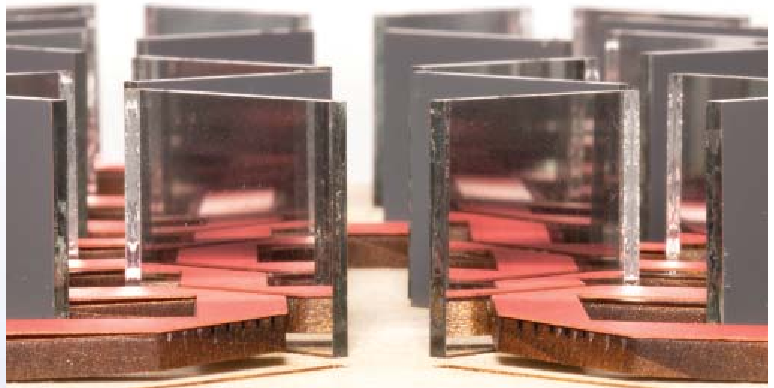
MULTIPLE WAYS TO ILLUSTRATE THE SAME THING

One of the great joys of a career in mathematics is the opportunity to sit with a problem for a long time. Over the months, years and decades, our perspective on a mathematical idea may change several times, so that the way we come to understand it later is very different from the way we initially learned it. When we personalize a problem, an object, or an idea in this way, we own it and, in so doing, we give it new life. After all, no mathematical idea can be said to “exist” in its own right; an idea only exists in the minds of those who understand it.









In order to understand something, we should look at it from different perspectives. The Hilbert curve is a continuous fractal space-filling curve first described by the German mathematician David Hilbert in 1891. There are several different but equivalent ways of defining such curves: One way is "external" or "plotter-based," given by absolute directions: up, down, left, and right. Another is "internal" or "turtle-based," given by commands like "turn left" or "move forward" without any knowledge of absolute direction. Here are some illustrations of both.

Top left: This shows three iterations of the curve, illustrating the recursive step from one level to the next, created with a laser cutter. Laser cutting opens up an enormous space of possibilities by its extreme precision.

Top right: This is a tiling system, with only three types of tiles that trace out the curve. The tiles correspond precisely to the instructions "turn left," "turn right," and "move forward." These wooden tiles were created with a laser cutter and colored by hand.

Bottom left: This is a rendition in paper of a Celtic knot: a 192-crossing, 3-component Brunnian link. Each strand has its own color and is equivalent to the unknot. If any one of the strands is removed, the other two are left unconnected. Because strands are overlapping, I cut this from three different pieces of paper and braided them together. Paper is a fun medium, but also very fragile. I made early versions with a mechanical paper cutter, but had better luck with a laser cutter.

Bottom right: This is a 52-mirror labyrinth, with a "laser beam" that traces out the curve. The top layer of the base has slits precisely cut to hold the 52 mirrors so that their reflective back faces are on the appropriate diagonals. The symbolic "laser beam" consists of a red piece of paper resting on a wooden platform of the same shape. This illustrates the internal point of view, as the "laser beam" has no knowledge of absolute direction, and the mirrors (or their absence) serve as the local commands. Originally I wanted to use an actual laser beam to trace out the curve, but the mirrors absorbed too much light, and the beam was invisible after a dozen bounces.

Further information:

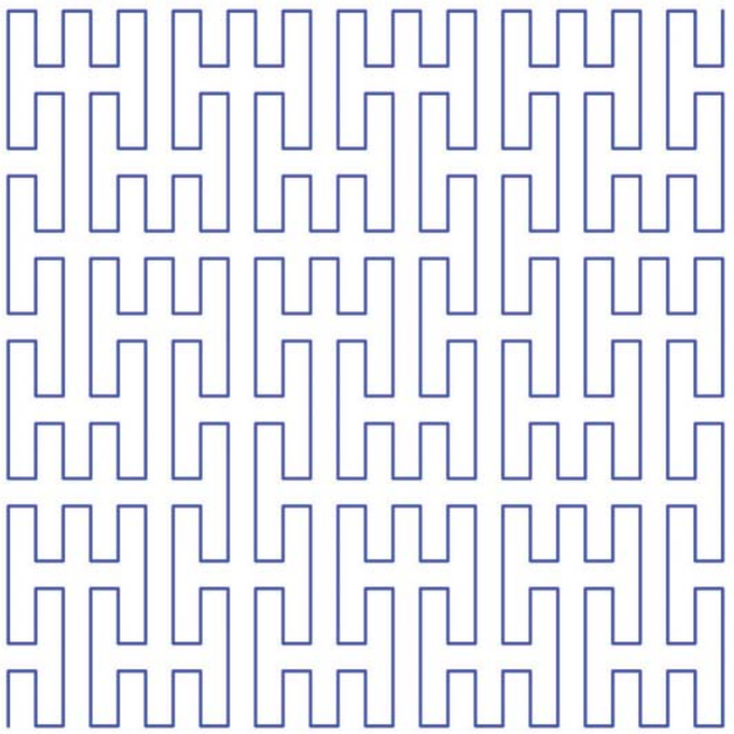
Michael Bader, *An Introduction with Applications in Scientific Computing*, Springer-Verlag Berlin Heidelberg (2013).

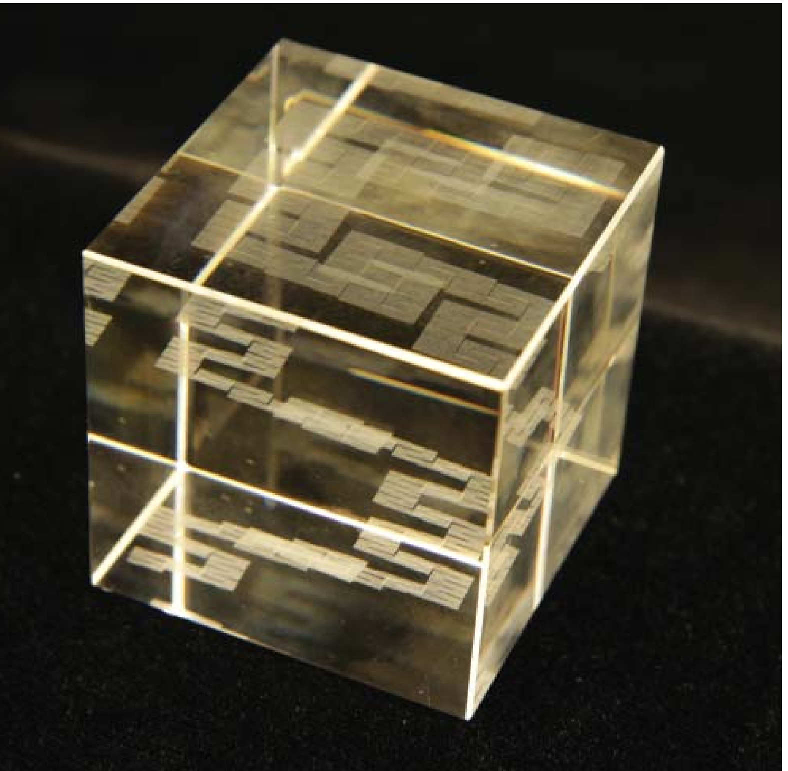
Doug McKenna, *Hilbert Curves: Outside-In and Inside Gone*, Mathemæsthetics, Inc. (2019).

ROGER ANTONSEN

University of Oslo

laser-cut wood, paper, and mirrors



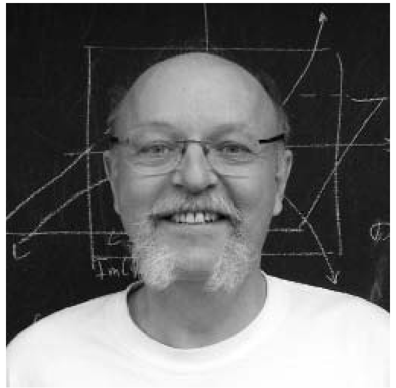



It has always been evident to geometers that curves and surfaces are different objects. But in 1890, Giuseppe Peano, in the famous paper "Sur une courbe qui remplit toute une aire plane," showed a very counterintuitive example of a continuous "space-filling" curve that completely fills the unit square.

Peano's paper has no figures; when I was a student, the conventional wisdom was that such a curve was impossible to represent, as the figure would have been a black square! To get intuition about this curve, the best way was to draw a finite approximation (top left). However, this does not give a good intuition of the curve itself, as it is very homogeneous, and makes it difficult to understand the dynamics of the curve. I had the idea of making a representation in which the color of each point depends on the parameter of the curve; in this way, one can see the movement along the curve as the direction of continuous change of color (top right).

A few years later, Xavier Bressaud suggested that it might be clearer to show the 3D graph of the function, and we made some interesting figures showing perspective views of this graph for a paper in *Experimental Mathematics*. But this representation was not complete: why not built the real object in 3D? There is an obvious problem: a fractal curve in $\mathbf{R}^3$ is too fragile to 3D print. But laser engraving in glass is a technology perfectly adapted to an object of this type, and in 2017, with Eltarr Loukman, an energetic undergraduate student at Aix-Marseille University, we succeeded in creating this object, which shows how a topological line can have Hausdorff dimension 2 and project onto a square in one direction, but not in all directions (bottom left).

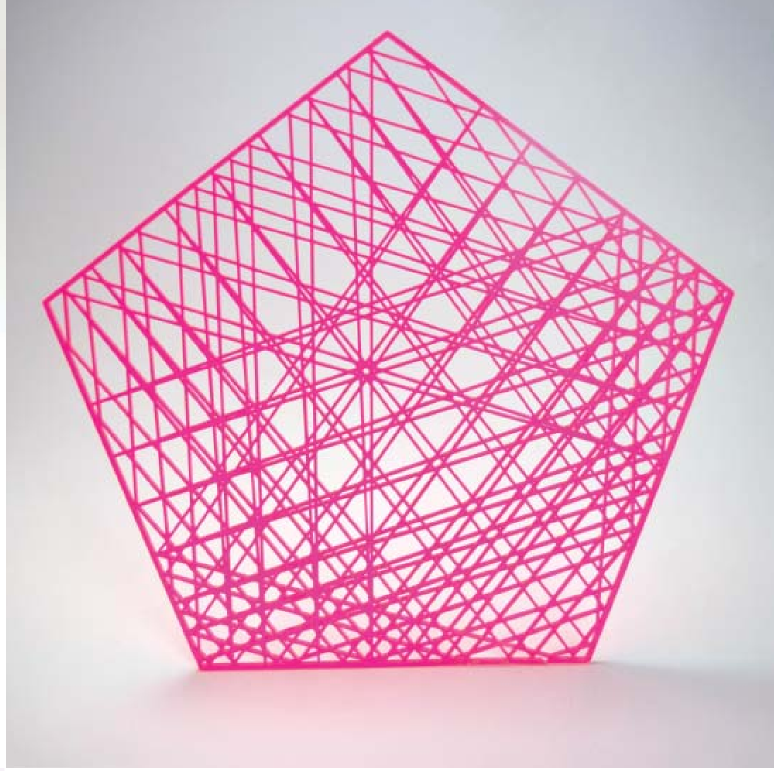
Upon seeing this curve in glass, I realized that the set of points below the graph formed a kind of strange hill, with an ascending path passing through all points of the hill, and that this object could easily be 3D printed, which I did at ICERM this fall (bottom right). This gives another viewpoint on this remarkable object. Similar models can be made for any plane-filling curve, and it will be interesting to make them to get a better intuition about the properties of these curves and the differences between them.

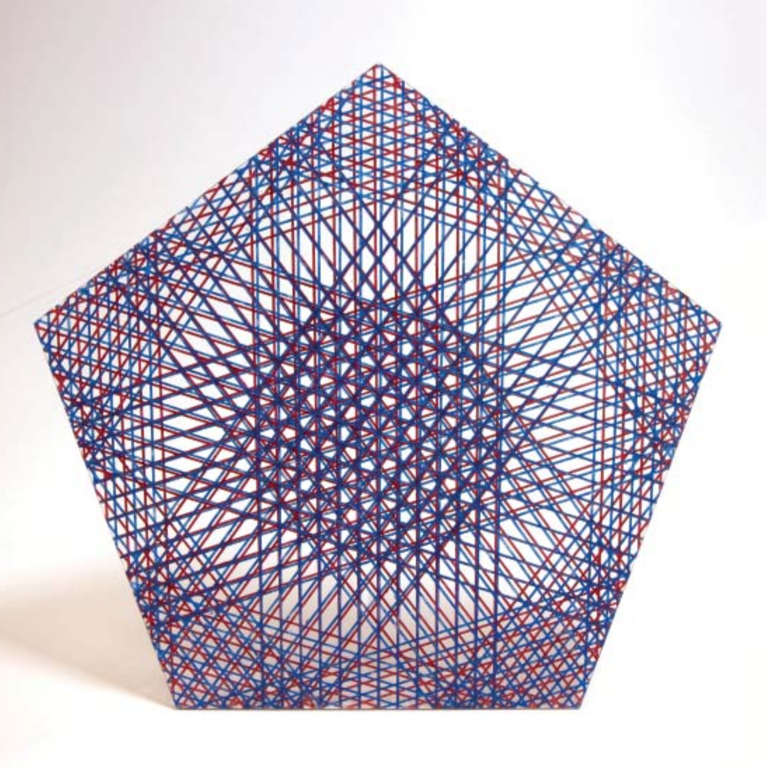


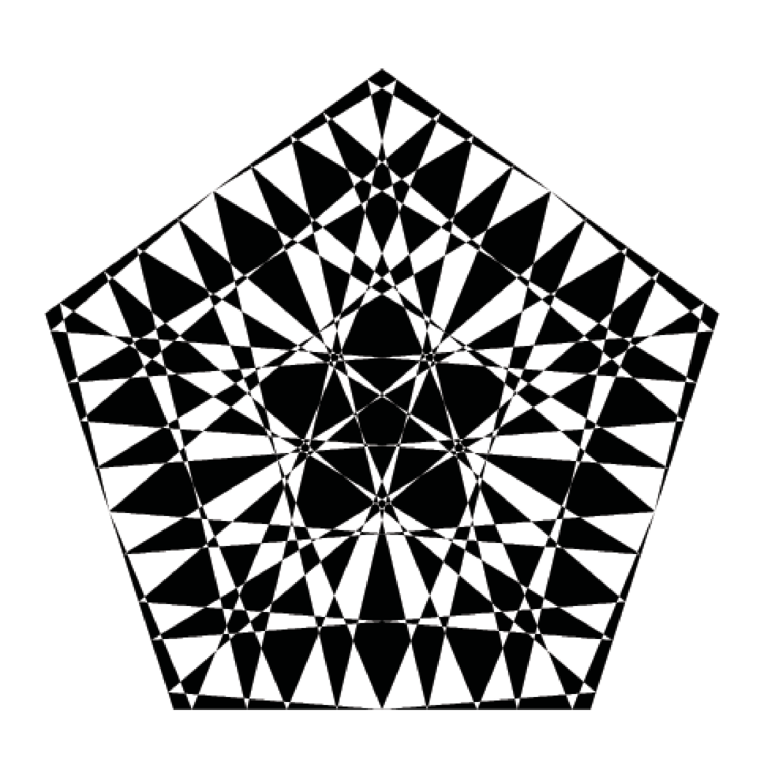
PIERRE ARNOUX

Université d'Aix-Marseille / CNRS

graphics, 3D-printed plastic







These show periodic billiard paths on the regular pentagon: You imagine that you have a pentagon-shaped billiard table, and you shoot the ball so that it bounces around, eventually returning to where it started and repeating the same path forever. We studied and classified all of the periodic billiard paths on the pentagon, of which there are (countably) infinitely many, and we also wrote a program in Sage to draw them.

Over the past couple of years, we have tried multiple ways of illustrating these paths. The simplest way is just to draw the path itself, as in the upper left; this is laser-engraved in wood, which allows you to draw a very intricate path. Incidentally, the laser follows the path of the billiard as it engraves the path, and it's interesting to see the pattern emerge in real time.

Next, we thickened up the path and removed the negative space, as in the upper right; this is 1/8-inch laser-cut acrylic plastic. While this requires a much simpler trajectory, this method emphasizes the path more. Diana has made small versions of this into dangly earrings, which are excellent conversation starters about our research.

For the picture in the lower left, you imagine that when the path hits the edge of the table, it changes color, from red, to blue, to red, etc. To do this, Diana used clear acrylic with protective paper adhered to both sides, and laser-engraved the even-numbered pieces of the path on one side and the odd-numbered ones on the other side. Since the laser beam is very narrow, she engraved ten nearby parallel paths, removing the protective paper in strips. At this point, the protective paper only remained in the negative space. Then she painted one side red and the other side blue, and removed the painted paper, to leave only the painted path. Removing the paper was *much* more painful and tedious than expected, but the result is beautiful.

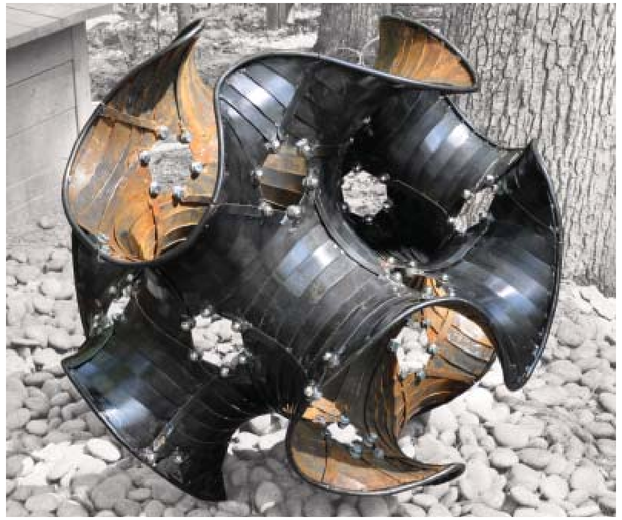
Lastly, we noticed that the path partitions the pentagon in a way that is two-colorable, so we colored the regions black and white (bottom right). We coded this on a deadline: when we started, Diana had a train to catch in 30 minutes, and we managed to produce a working version before she had to leave. This picture de-emphasizes the path but raises interesting questions, like whether the ratio of black to white might approach the golden ratio.

Further information: Diana Davis and Samuel Lelièvre, *Periodic paths on the pentagon, double pentagon and golden L*, preprint (2019).

DIANA DAVIS &
SAMUEL LELIÈVRE

Swarthmore College;
Université Paris-Saclay

laser-cut acrylic and wood, computer-generated graphics



Surfaces with negative curvature are familiar in everyday life: a surface has *negative curvature* at a point if it is saddle-like there, and the more negative it is, the more extreme this saddle is. A surface with negative curvature is ruffly, like lettuce or curly kale, and has a tremendous amount of surface area for the volume it occupies. Though a lot of the mathematics of these surfaces has been well understood for more than 150 years, there remain many open, unexplored questions about just how these surfaces actually sit in space and the dramatic changes they undergo when they are manipulated. Partly, this may be because there aren't many ways to actually build such a surface – crochet (see pages 17 and 25) is one technology – and mathematicians haven't generally played with many physical examples.

Top: This surface of constant negative curvature is made out of annular strips connected edge to edge. Eugenio Beltrami made a model like this one out of paper, more than 150 years ago! But that is an inefficient use of material. It's impossible to cut very many annular strips from a flat sheet without a great deal of waste. Straight strips are far more efficient, but strips of flat materials such as steel or paper can only be bent in the direction perpendicular to the surface, and the strip must remain a geodesic on any surface it sits upon. But foam strips are squishy and can be bent in other ways, so they are just right for this piece. The pieces for this were cut using a router on a CNC machine, which creates a lot of dust, so it would be much better to invest in a CNC knife that cuts cleanly.

Bottom left: This is a spherical piece of the *gyroid*, an infinite periodic minimal surface. The gyroid has one of the more fascinating of the 3D Euclidean discrete symmetries – it is difficult to understand even when you are looking at it! Steel is solid, durable, and upgrades a piece. Mathematical illustrators should know: Welding is easy, and basic metal work is a fast and forgiving medium! (Welding *well* is another matter.) Because straight strips of steel do not bend into annular ones, these straight strips of steel lie along geodesics on the gyroid.

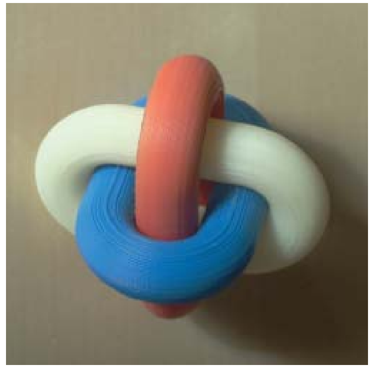
Bottom middle and right: These are surfaces with constant negative curvature made from flat strips of paper and steel, respectively. The steel object is about ten times the height of the paper one!

Further information: Daina Taimina, *Crocheting Adventures with Hyperbolic Planes: Tactile Mathematics, Art and Craft for All to Explore*, CRC Press (2018).

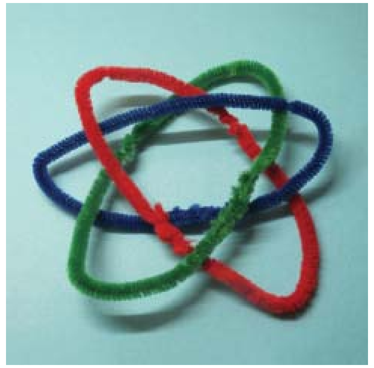
CHAIM GOODMAN-STRAUSS

University of Arkansas

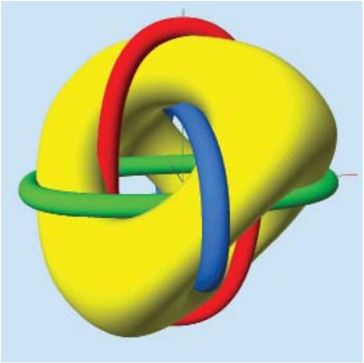
Ethylene-vinyl acetate (EVA) foam, steel, paper



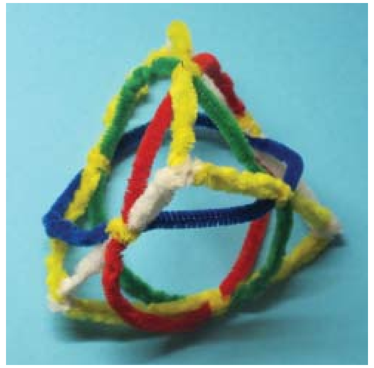
A



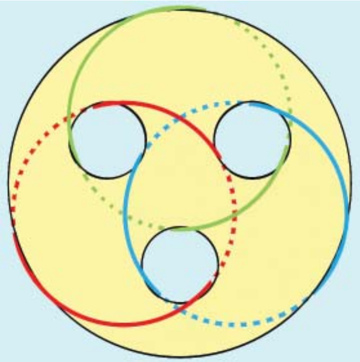
B



C

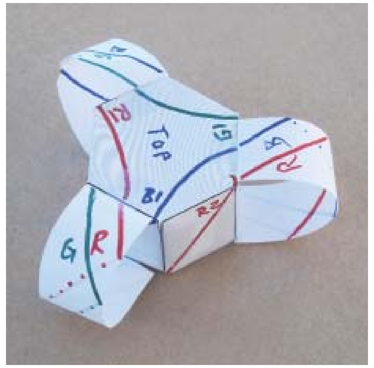


D



E

F

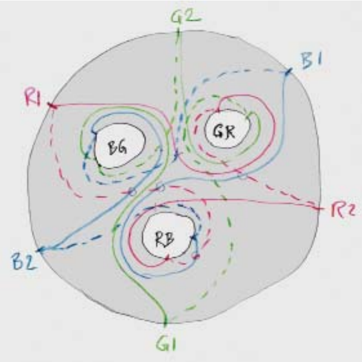


G

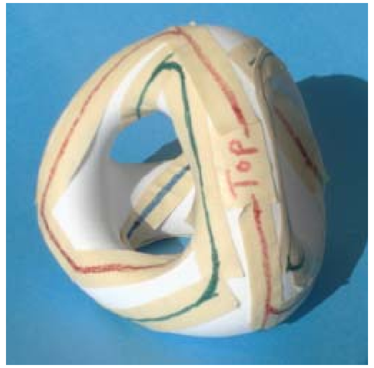
H

I

J



K



L

These objects all depict the three *Borromean rings*, which have the property that, though no two rings are interlinked, all three cannot be unlinked. I created these models to study how the embedding of the three rings will change, when the surface of the handlebody is *everted* (turned inside-out) by pulling the whole surface through a small puncture in its skin.

A. The Borromean rings in a solid, unmovable arrangement. Even though no pair of ovals is interlinked, the whole assembly cannot be taken apart.
B. The Borromean rings made out of pipe cleaners, which allows one to study how the rings can be rearranged without opening any one of them.
C. The most symmetrical way of embedding the Borromean rings in a genus-3 handlebody. This graphic shows the three entangled ovals placed onto the surface of a *Tetrus*, a fattened and nicely rounded tetrahedral wire frame.
D. A flexible version of (C), in which you can push a tetrahedron vertex through the opposite tunnel, changing the order of the circling rings.
E. A drawing showing the same embedding on a 3-hole torus, obtained by flattening the Tetrus and emphasizing the 3-fold rotational symmetry.
F. The goal is to make a model that can by physically everted. For this structure, it is known how to do this. Marker pen traces on a 3D-printed model show the embedding equivalent to that of models (C) through (E).
G. It was quite difficult to translate the embedding in Figure (E) onto the prism model (F). We made this paper model to experiment with twisting the handles, to give insight as to how many times each loop winds around.
H. This "4-pillar pagoda" is an alternative way to re-shape (C), resulting in a shape that can be everted through a puncture placed at the "North pole."
I. A model of the prism with three handles made from cylindrical newspaper bags. They are pasted together with their top and bottom ends, respectively, to form two four-way tubular junctions. In the top junction, at the "North pole," a star-shaped cut serves as an expandable puncture.
J. The same model after the whole skin has been pulled through the puncture. The prism and each handle turn into half-length "reversed socks," ending in a "mouth" as can be found in the standard depiction of a Klein bottle.
K. A sketch of the everted embedding from (J). Eversion changes the embedding: originally, each ring passes once through each of the three holes; after eversion, only two of the three rings visit each tunnel.
L. The alternative embedding resulting from everting model (H), which breaks the 3-fold symmetry. Four of the six handles carry all three rings, and the remaining pair of opposite edges each carry just two colors.

CARLO SÉQUIN

Univeristy of California, Berkeley

3D-printed plastic, pipe cleaners, folded paper, computer-generated graphics, plastic bags

ACKNOWLEDGMENTS

Thanks to Eriko Hironaka for working with the AMS to make this project happen. Thanks to Marcia Almeida for being our AMS liaison. Hannah Kushnick helpfully found over 300 ways to improve this book. Katie Vokes contributed many ideas for the chapter 8 puzzle collage. The idea for the image of the Alexander horned sphere came from Curtis McMullen's talk at ICERM. Thank you to Edmund Harriss for providing a professional-level photography setup to photograph people's objects and for taking many of the beautiful pictures in this book.

Many thanks to ICERM for hosting the wonderful Illustrating Mathematics semester program in fall 2019 and to the program's funders, including the National Science Foundation (DMS-1439786), the Simons Foundation Targeted Grants for Institutes, and the Alfred P. Sloan Foundation. I would also like to thank David Bachman, Kelly Delp, David Dumas, Saul Schleimer, Richard Schwartz, Henry Segerman, Katherine Stange, and Laura Taalman for organizing the program. Thank you to the Institut des Hautes Études Scientifiques (IHES) for hosting me as an invited research visitor for three months during the coronavirus pandemic of 2020, during which I finished this book.

Thank you to everyone who passed through the ICERM semester program, for your kindness and generosity of spirit, and especially to all those who contributed work for this book. A special thanks to all of the contributors for responding to my questions and returning their consent forms, despite the global pandemic.

Finally, a multitude of thanks to Margo Angelopoulos for editing every image, typesetting every word, and laying out every page of this book, and for making the composing of this book a thoroughly enjoyable collaborative experience.

IMAGE CREDITS

Object photographs/images courtesy of the contributor(s):

Coulon sketch, Chatterji, Dorfsman-Hopkins/Maynard, Delp, Robb/Benson, Bachman, Hsu, McShane polyhedron, Rainford, Farris, Stange, Goerner, Felzenszwalb/Klivans/Skrodzki, Sümmermann, Novello de Brito, Bozlee, Ghrist, Francis, Alatorre, Amethyst, Segerman, Holdener, Chéritat*, Antonsen shuffling, Taalman origami, Espigulé, McShane bottle*, Taalman knot, Sander, Carter, Paul, Taalman pentagons, Buric, Gangl, Vejdemo-Johansson, Tihanyi, Matsumoto jewelry, Labs glass, Edmark, Mohanty/Muetzel, Sharma, Arnoux line and color graphics, Davis/Lelièvre graphic, Goodman-Strauss bottom three images*, Séquin.

Object photographs by Edmund Harriss:

Málaga Sabogal/Lelièvre flat tori, Harriss curvahedra, Yackel, Abrams/Paul, Matsumoto fleece, Davis lizards*, Málaga Sabogal/Lelièvre laser box, Trettel, Dorfsman-Hopkins Julia set, Málaga Sabogal, Harriss brick, Coulon necklace, Whitney, Holroyd, Antonsen curves, Arnoux glass and 3D-printed objects, Davis/Lelièvre pentagons.

Contributor photographs courtesy of the contributor(s):

Kent, Chatterji, Maynard, Trnkova, Paul, Matsumoto, Bachman, Davis, Hsu, Rainford, Barnes, Schaubroeck, Kepner, Felzenszwalb/Klivans/Skrodzki*, Zucker, Sümmermann, Shmahalo, Novello de Brito, Ghrist, Francis, Alatorre, Amethyst, Segerman, Chéritat, Taalman, Espigulé, Gangl, Vejdemo-Johansson, Tihanyi, Mohanty, Muetzel, Holroyd, Goodman-Strauss, Fenyes, Séquin.

Contributor photographs by Diana Davis:

Coulon, Dorfsman-Hopkins, Delp, Málaga Sabogal/Lelièvre, Harriss, Yackel, Abrams, Robb/Benson, McShane, Trettel, Farris, Stange, Goerner, Bozlee, Holdener, Antonsen, Málaga Sabogal, Sander, Carter, Buric, Whitney, Edmark, Sharma, Arnoux, Lelièvre.

Additional photo and image credits:

Kent: Reprinted by permission from Springer Nature: Geometriae Dedicata, 106, *Bundles, handcuffs, and local freedom*, Autumn Kent, 2004, https://www.springer.com/journal/10711/.

Sullivan/Thurston: Curve picture by Ken Ribet, used with permission. Image of Sullivan courtesy of SUNY Stony Brook Office of Communications. Reprinted with permission. Image of Thurston courtesy of michael pirrocco.

Coulon sketch: © 2019 Rémi COULON Some Rights Reserved.

Málaga Sabogal/Lelièvre flat tori: © Alba Marina Málaga Sabogal and Samuel Lelièvre. Some rights reserved.

Trnkova: © 2020 by the author.

Davis lizards: © Diana Davis. Some rights reserved.

Málaga Sabogal/Lelièvre laser box: © Alba Marina Málaga Sabogal and Samuel Lelièvre. Some rights reserved.

Barnes/Schaubroeck: Federal employee; no rights to transfer. Top left and top right images first published in *Coloring Book of Complex Function Representations* by Julie Barnes, William Kreahling, and Beth Schaubroeck, in 2017, published by the Mathematical Association of America. © 2017 held by the AMS. Bottom right image first appeared in *AMS Page a Day Calendar* by Evelyn Lamb in 2019, published by the AMS. Used with permission of the authors.

Kepner: © 2011 Margaret Kepner. All rights reserved.

Zucker: © 2020 Matthew Zucker. Some rights reserved.

Shmahalo: Illustration originally published February 20, 2019 in Quanta Magazine. Reprinted with permission.

Amethyst: © Silviana Amythest. Some rights reserved.

Segerman: © 2020 by Henry Segerman.

Antonsen shuffling: © Roger Antonsen. Some rights reserved.

Taalman (3 works): © Laura Taalman. Some rights reserved.

Málaga Sabogal polyhedra: © Alba Marina Málaga Sabogal. Some rights reserved.

Athreya: images of object by Dan Paz, used with permission.

Carter: © J. Scott Carter. Some rights reserved.

Labs gold plated brass photos and portrait: © Jonas Otte - artefont.de. Used with permission.

Coulon necklace: © 2019 Rémi COULON Some Rights Reserved.

Goodman-Strauss mural picture: © 2020 University of Arkansas. Photo: Russell Cothren. Used with permission.

Goodman-Strauss mural page: © 2020 Chaim Goodman-Strauss. All rights reserved.

Fenyes: © 2020 Aaron Fenyes.

Antonsen curves: © Roger Antonsen. Some rights reserved.

Davis/Lelièvre pentagons: © Diana Davis and Samuel Lelièvre. Some rights reserved.

Goodman-Strauss surface page: © 2020 Chaim Goodman-Strauss. All rights reserved.

Goodman-Strauss top surface picture: © National Museum of Mathematics. Used with permission.

Photo of ICERM participants by ICERM Staff. Used with permission.

Section background images:

Drawings: from Dale Rolfsen, *Knots and Links*, AMS Chelsea Publishing (2003), used with permission.

Paper & fiber arts: by Diana Davis

Laser cutting: by Edmund Harris

Graphics: from Wikipedia, public domain image

Video and virtual reality: Still frame from "Knots to Narnia" video courtesy of Anthony Phillips

3D printing: by Alba Marina Málaga Sabogal

Mechanical constructions: by Diana Davis

Multiple ways: Collage © 2020 by Margo Angelopoulos and Diana Davis. Background satellite image by NASA, reuse permitted. Images of octopus, spider and eightfold path symbol are public domain. Quotation from Sebastian Junger, *The Perfect Storm*. Photographs of flower and stop sign by Diana Davis. All other graphics by Margo Angelopoulos.

Cover images:

Dorfsman-Hopkins/Maynard (front cover, and back cover right): used with permission.

Tihanyi (back cover left): used with permission.

Bachman (back cover middle): Artwork by David Bachman, based on work with Saul Schleimer and Henry Segerman

* Contributed photographs substantially modified by Margo Angelopoulos

CONSENT TO PUBLISH

The following authors have assigned copyright to the AMS.

Coulon sketch, Chatterji, Dorfsman-Hopkins/Maynard, Delp, Robb/Benson, Bachman, Hsu, McShane, Rainford, Farris, Stange, Goerner, Felzenszwalb/Klivans/Skrodzki, Sümmermann, Novello de Brito, Bozlee, Ghrist, Francis, Alatorre, Amethyst, Segerman, Holdener, Chéritat, Antonsen, Taalman, Espigulé, Sander, Carter, Paul, Buric, Gangl, Vejdemo-Johansson, Tihanyi, Matsumoto, Labs, Edmark, Mohanty/Muetzel, Sharma, Arnoux, Davis/Lelièvre, Goodman-Strauss, Séquin.

Section text:

Introductory remarks for each section by Diana Davis. All rights reserved.

The following authors have assigned copyright as indicated.

Coulon sketch: © 2019 Rémi COULON Some Rights Reserved.
Málaga Sabogal/Lelièvre flat tori: © Alba Marina Málaga Sabogal and Samuel Lelièvre. Some rights reserved.
Trnkova: © 2020 by the author.
Davis lizards: © Diana Davis. Some rights reserved.
Málaga Sabogal/Lelièvre laser box: © Alba Marina Málaga Sabogal and Samuel Lelièvre. Some rights reserved.
Barnes/Schaubroeck: Federal employee; no rights to transfer. Top left and top right images first published in *Coloring Book of Complex Function Representations* by Julie Barnes, William Kreahling, and Beth Schaubroeck, in 2017, published by the Mathematical Association of America. © 2017 held by the AMS. Bottom right image first appeared in *AMS Page a Day Calendar* by Evelyn Lamb in 2019, published by the AMS. Used with permission of the authors.
Kepner: © 2011 Margaret Kepner. All rights reserved.
Zucker: © 2020 Matthew Zucker. Some rights reserved.
Amethyst: © Silviana Amythest. Some rights reserved.
Segerman: © 2020 by Henry Segerman.
Antonsen shuffling: © Roger Antonsen. Some rights reserved.
Taalman (3 works): © Laura Taalman. Some rights reserved.
Málaga Sabogal polyhedra: © Alba Marina Málaga Sabogal. Some rights reserved.
Carter: © J. Scott Carter. Some rights reserved.
Labs gold plated brass photos and portrait: © Jonas Otte - artefont.de. Used with permission.
Coulon necklace: © 2019 Rémi COULON Some Rights Reserved.
Goodman-Strauss mural page: © 2020 Chaim Goodman-Strauss. All rights reserved.
Fenyes: © 2020 Aaron Fenyes.
Antonsen curves: © Roger Antonsen. Some rights reserved.
Davis/Lelièvre pentagons: © Diana Davis and Samuel Lelièvre. Some rights reserved.
Goodman-Strauss surface page: © 2020 Chaim Goodman-Strauss. All rights reserved.

INDEX