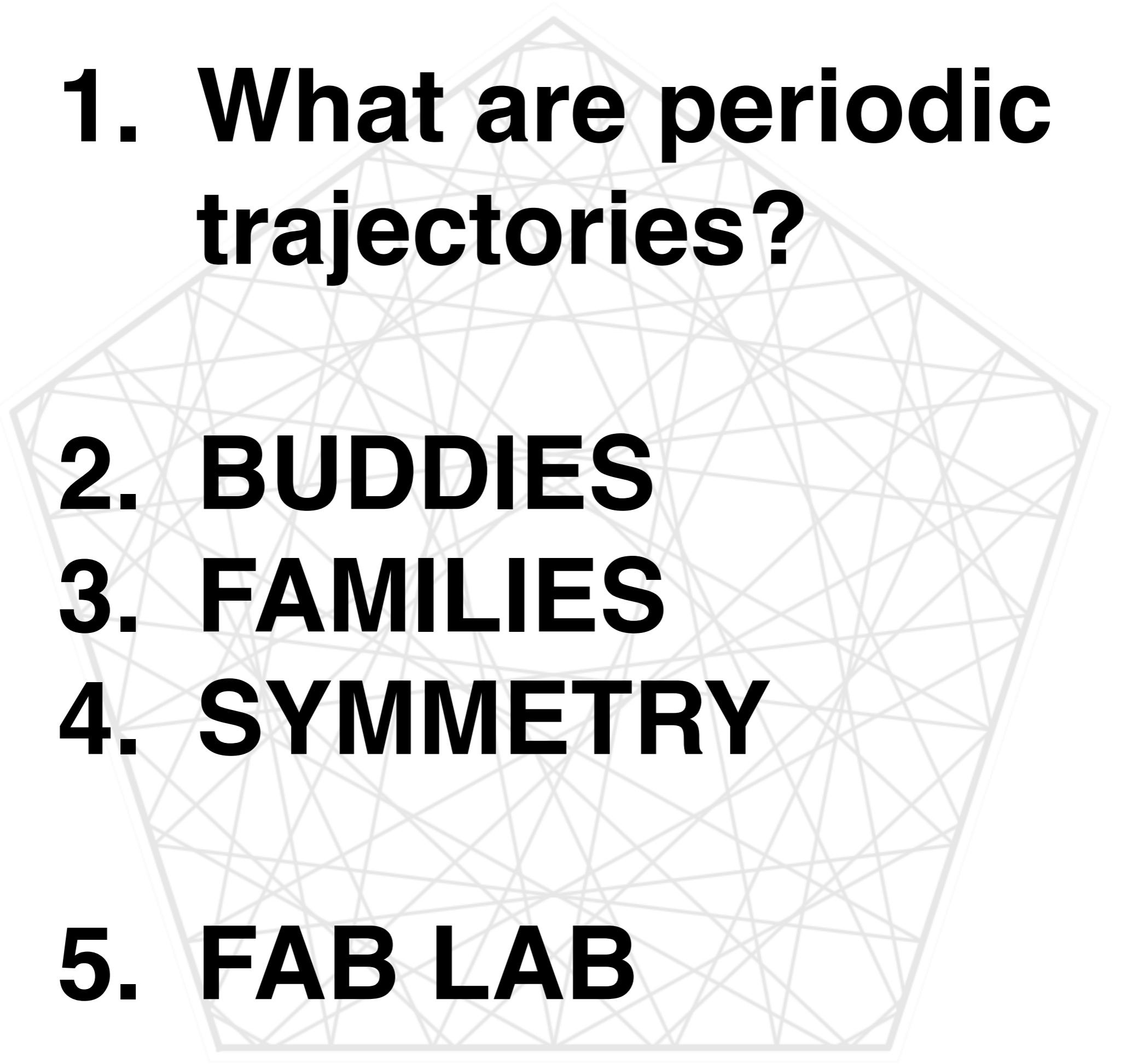


# **Periodic trajectories on the regular pentagon**

**Diana Davis  
Mathematics**

**January 31, 2018**



**1. What are periodic  
trajectories?**

**2. BUDDIES**

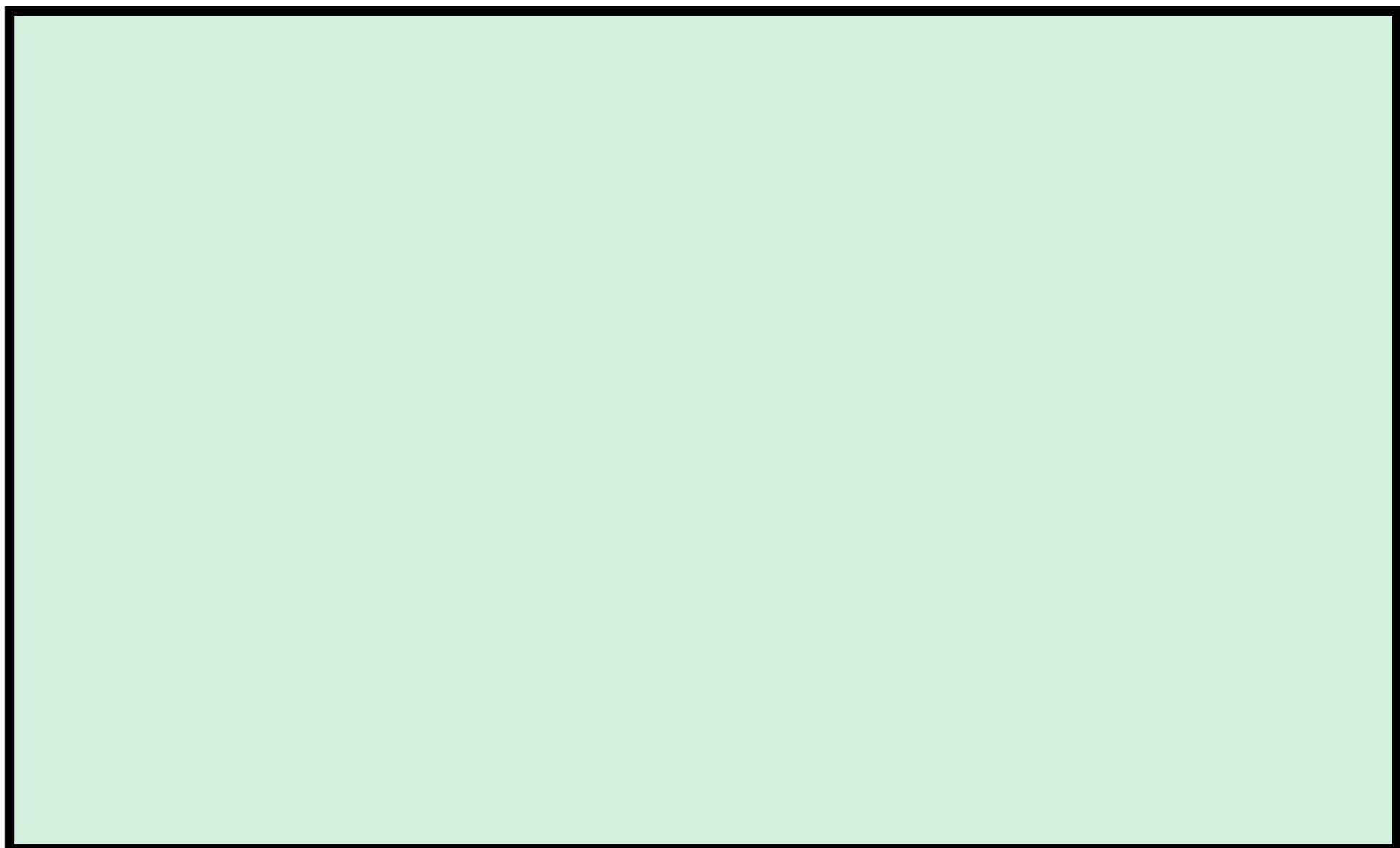
**3. FAMILIES**

**4. SYMMETRY**

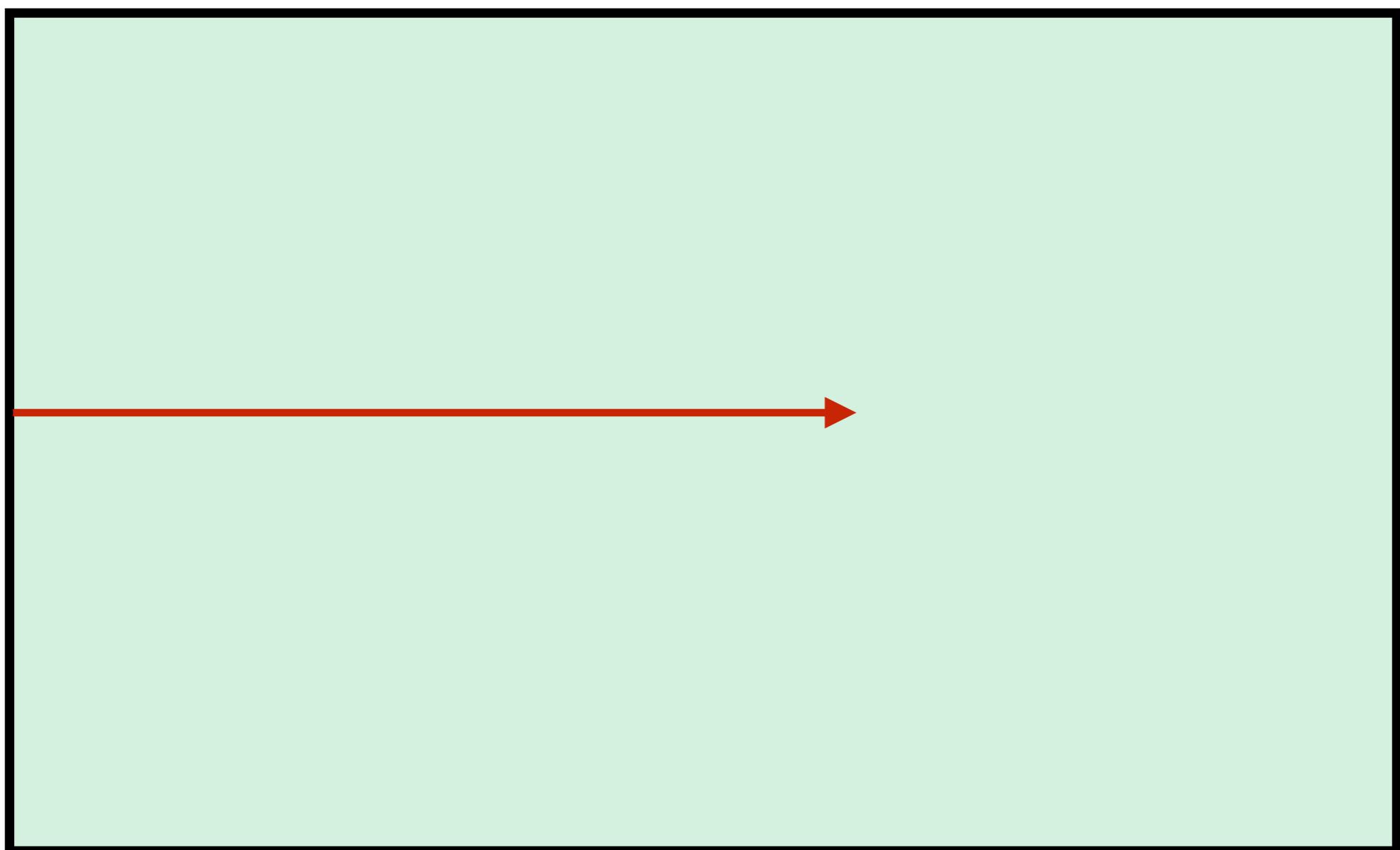
**5. FAB LAB**

**WHAT ARE  
PERIODIC  
TRAJECTORIES?**

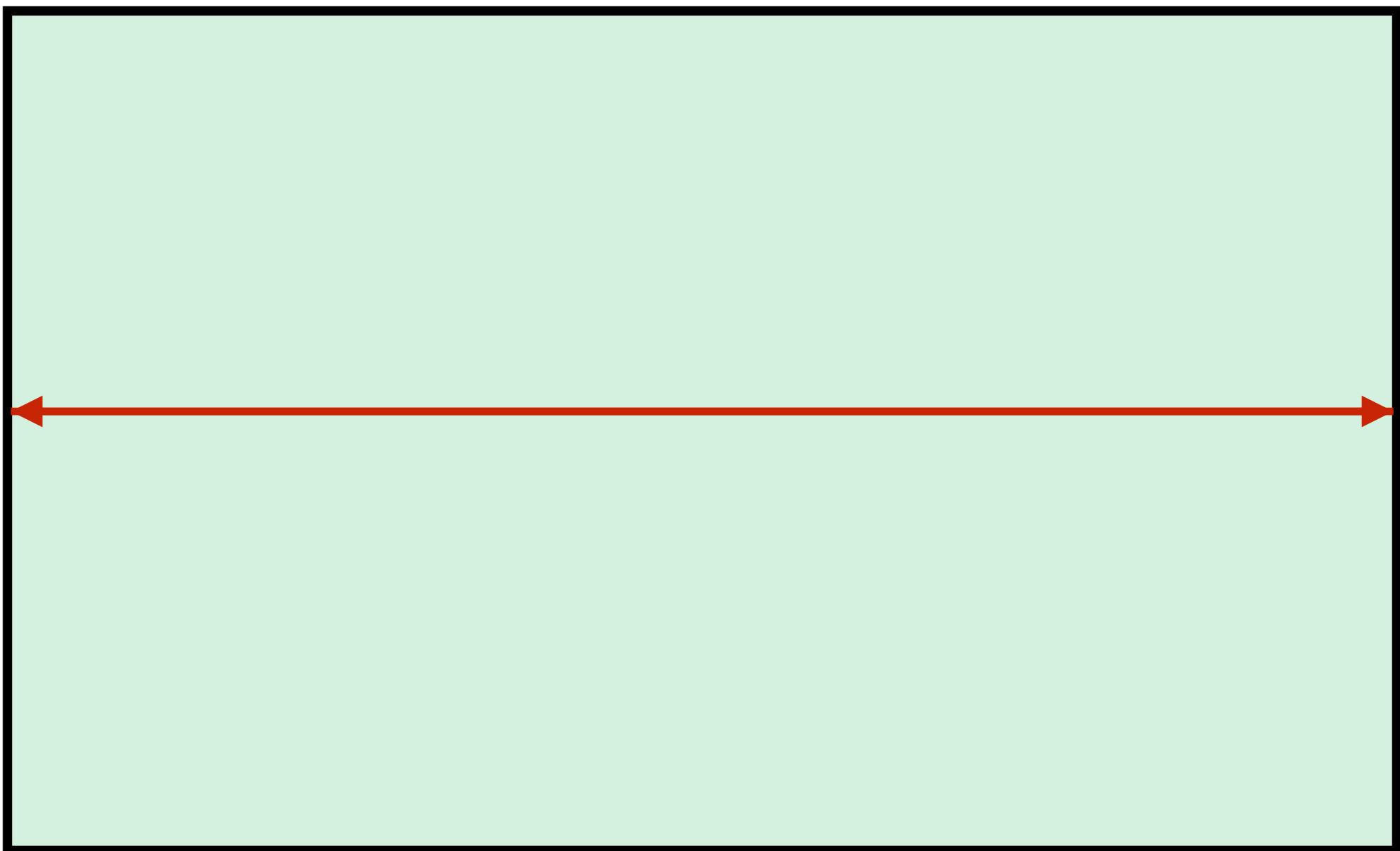
- Standard billiard table (no pockets)
- Shoot the ball in such a way that it comes back to where it started and repeats its path.



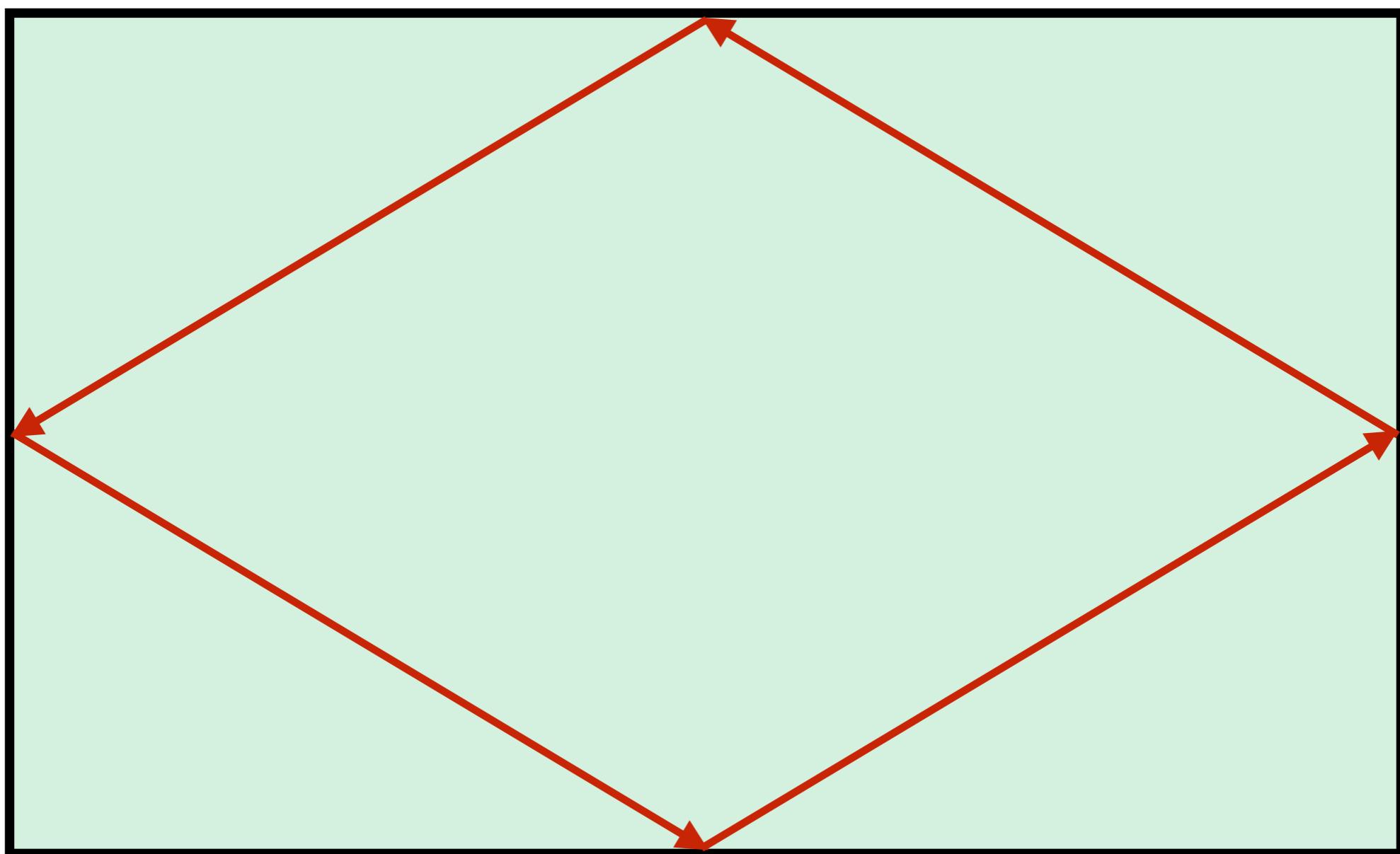
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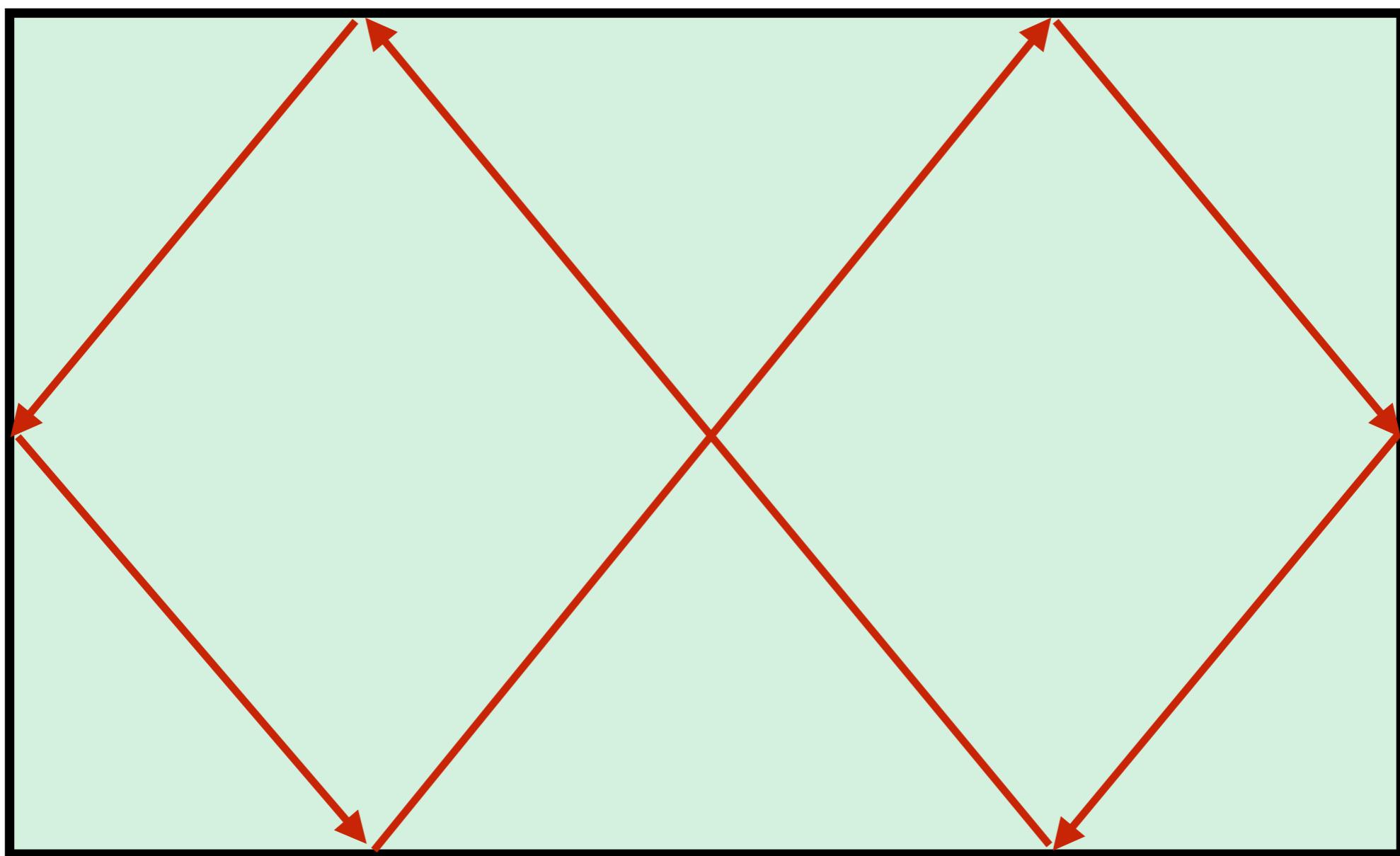
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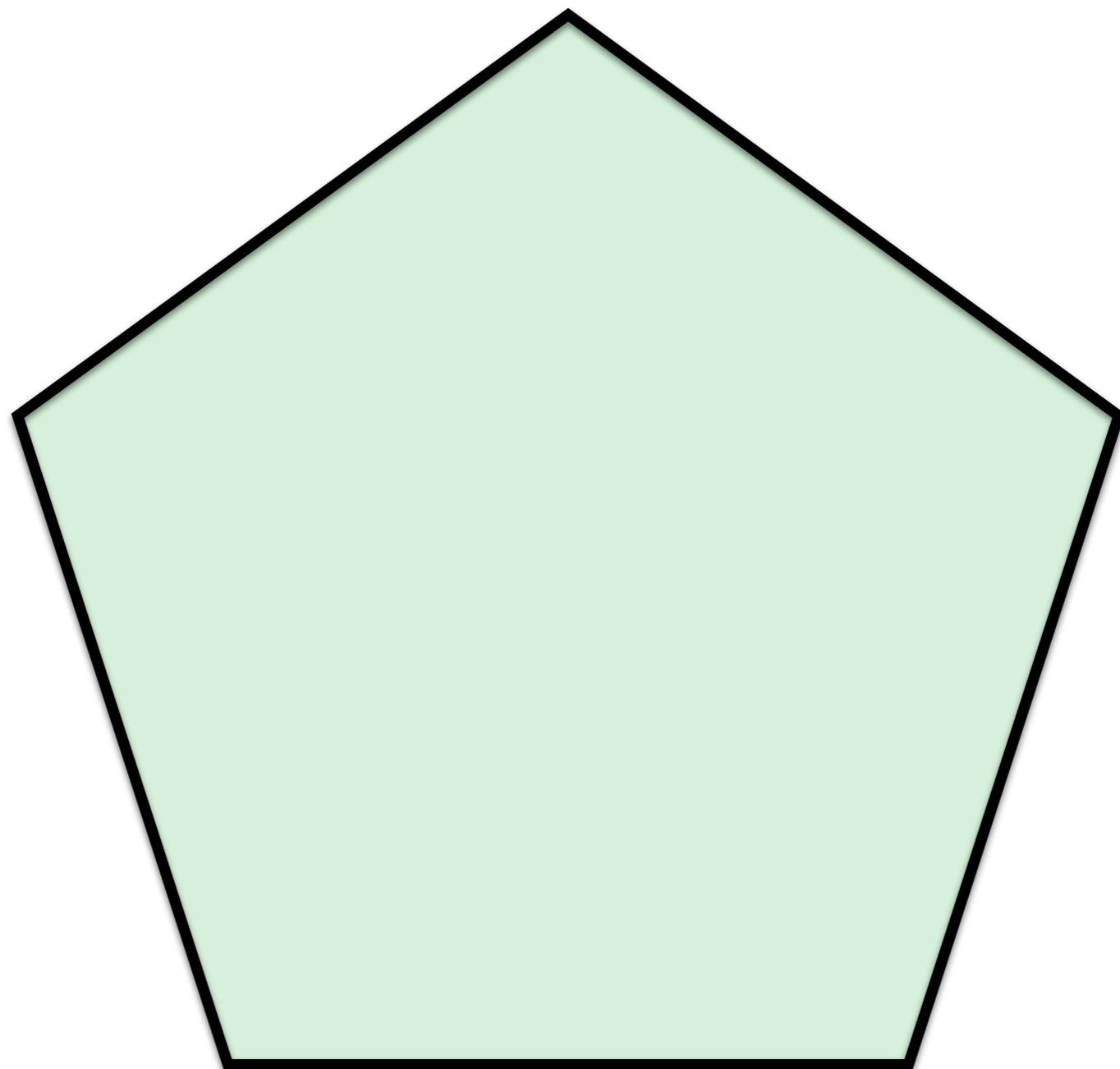
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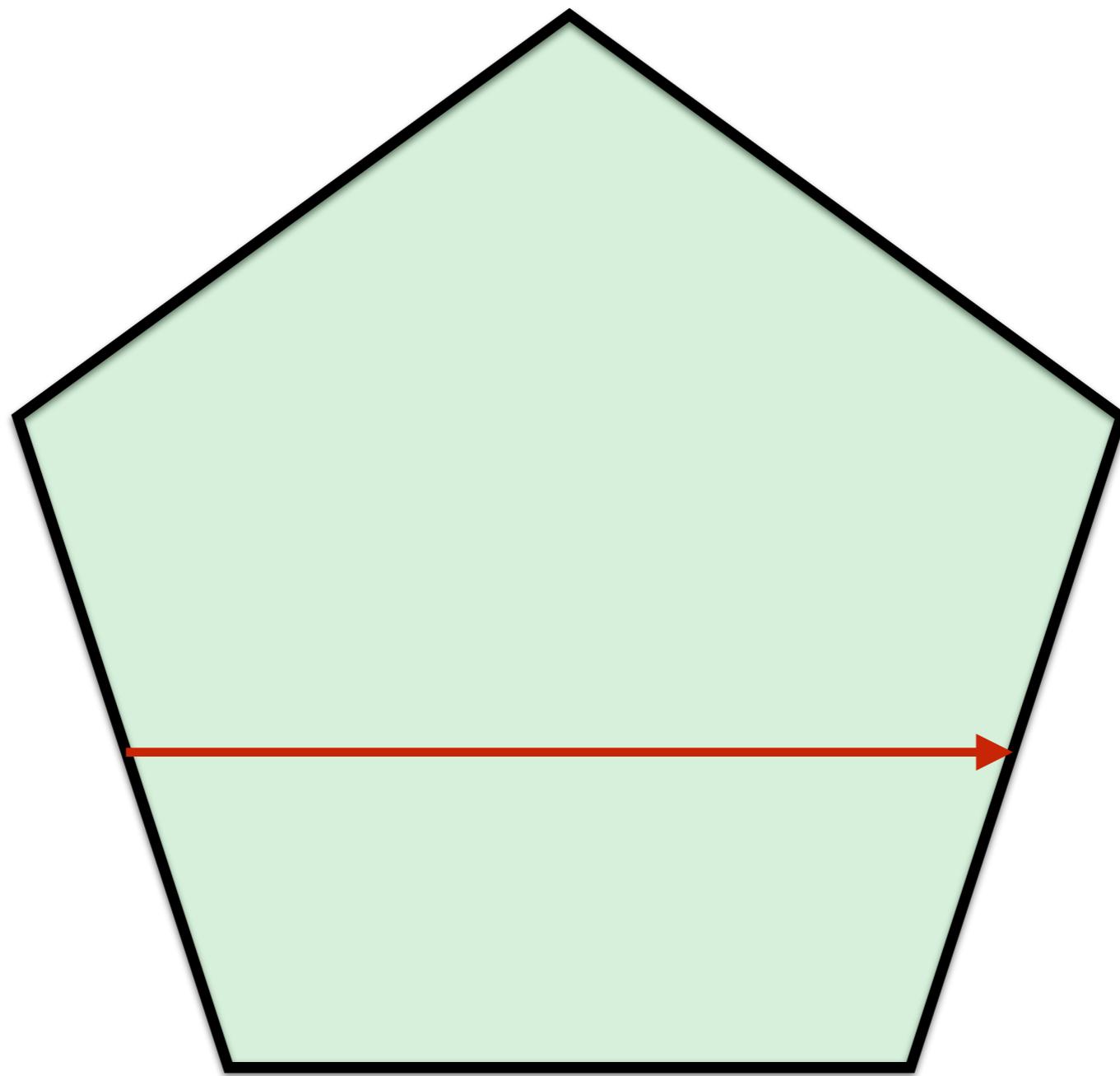
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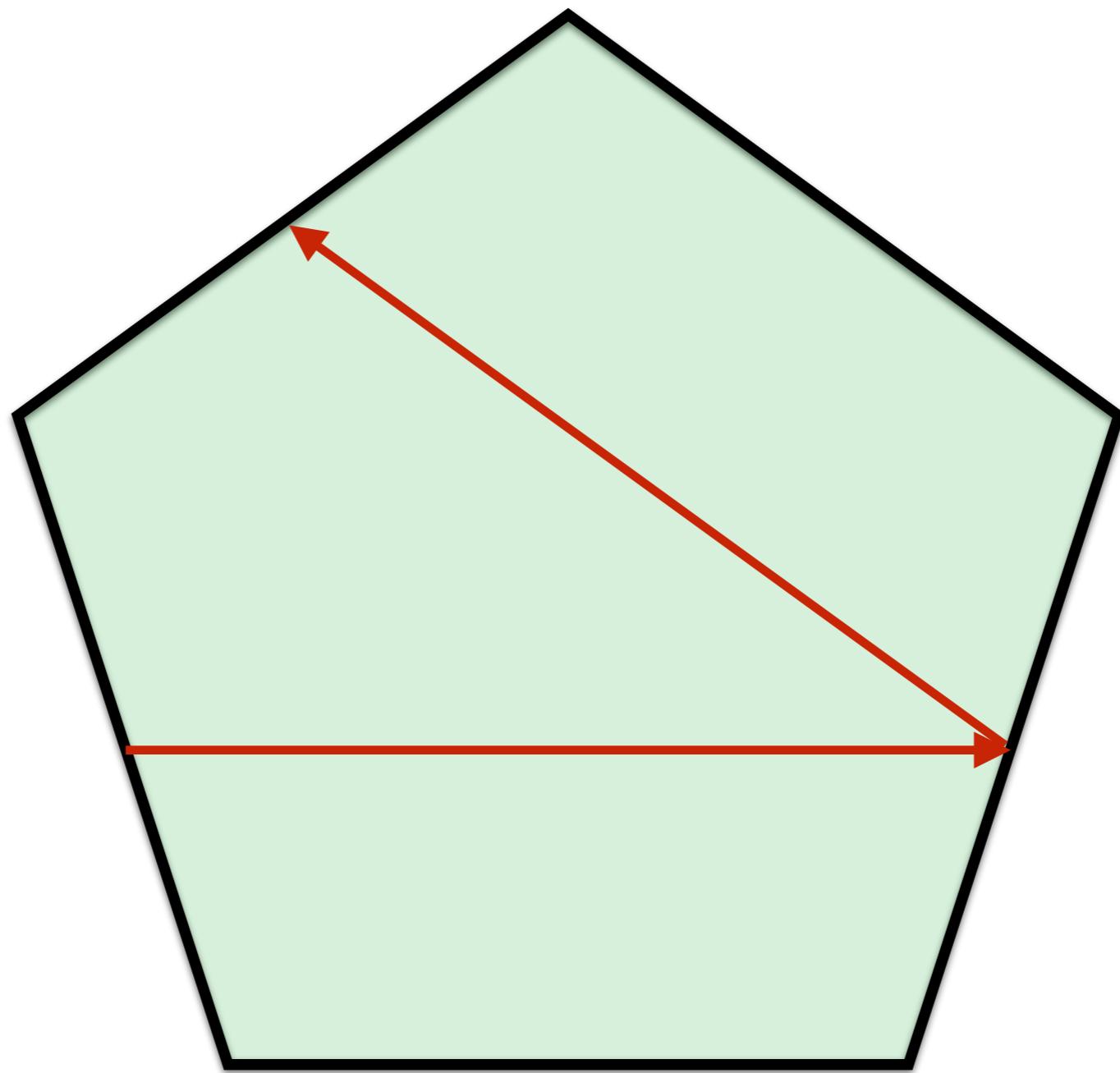
- Pentagonal billiard table
- Shoot the ball in such a way that it comes back to where it started and repeats its path.



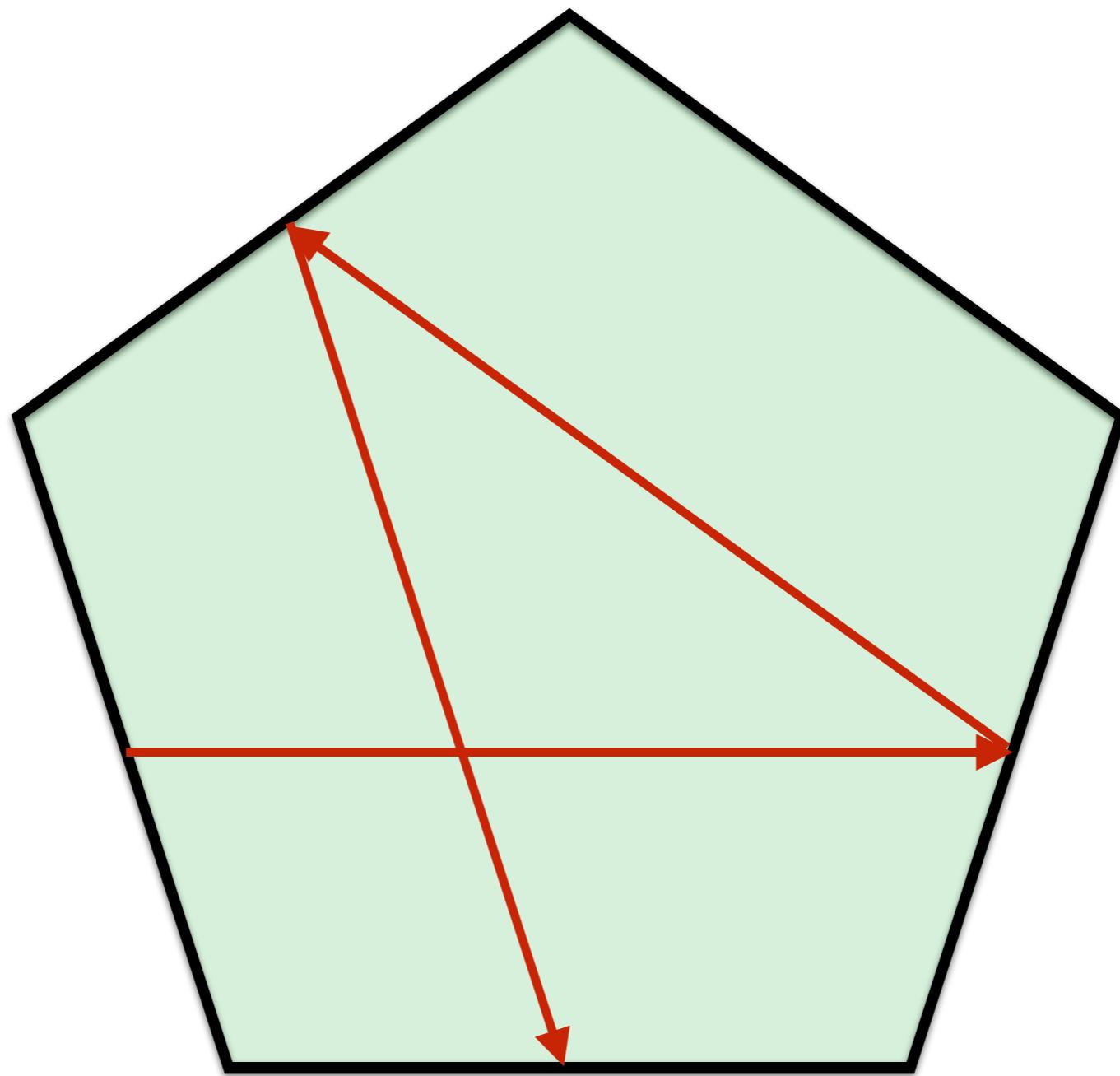
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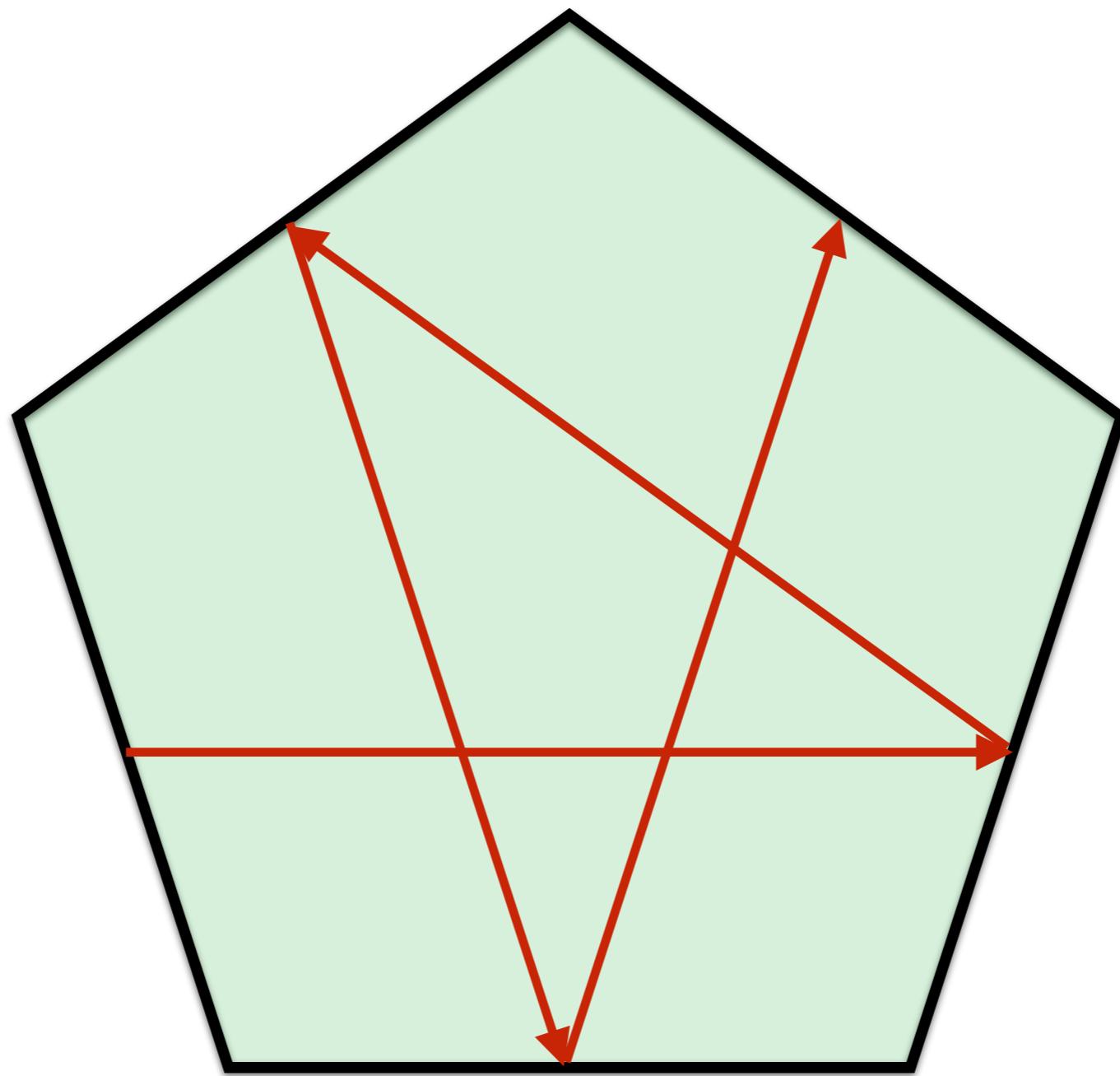
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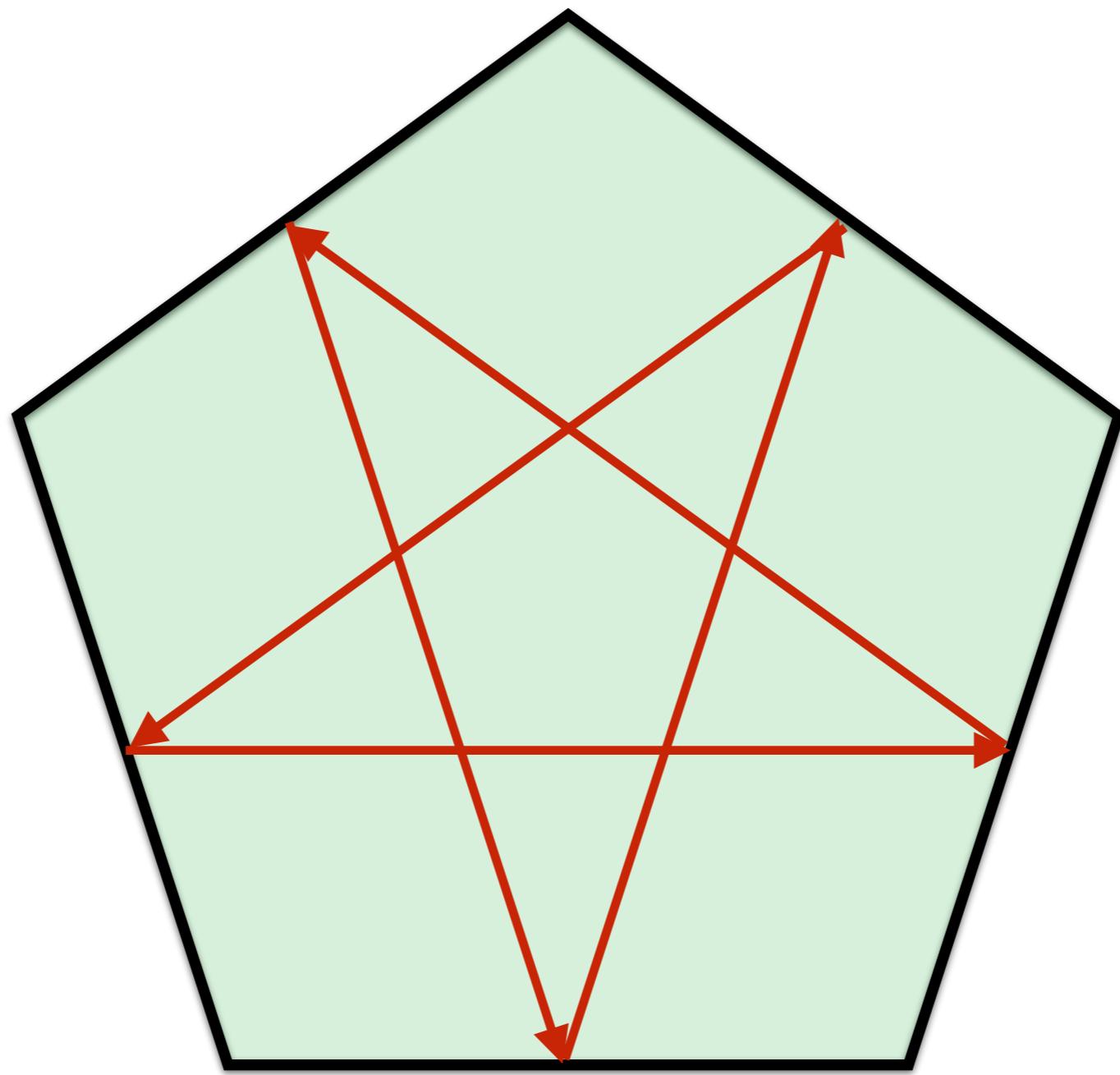
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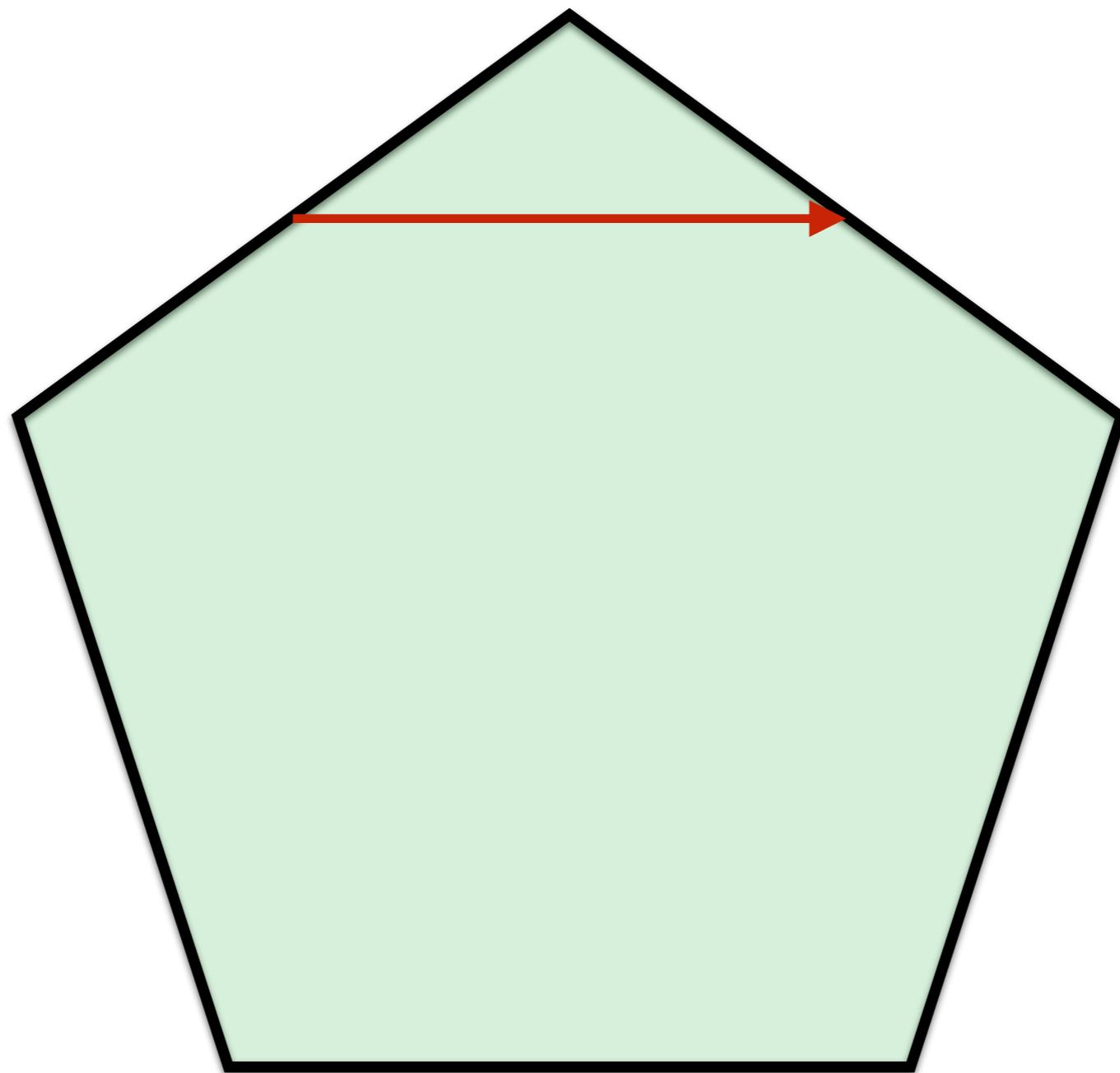
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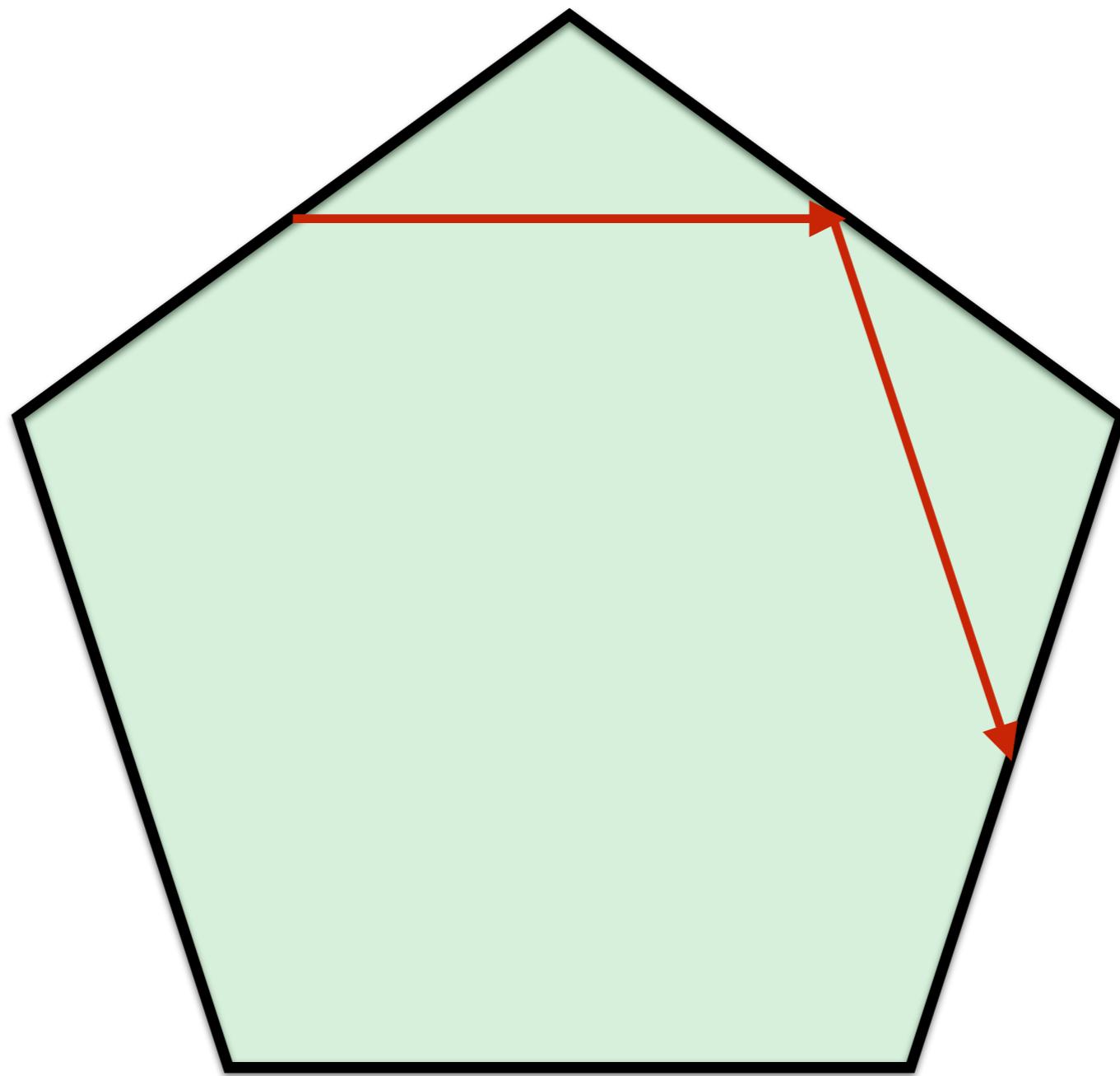
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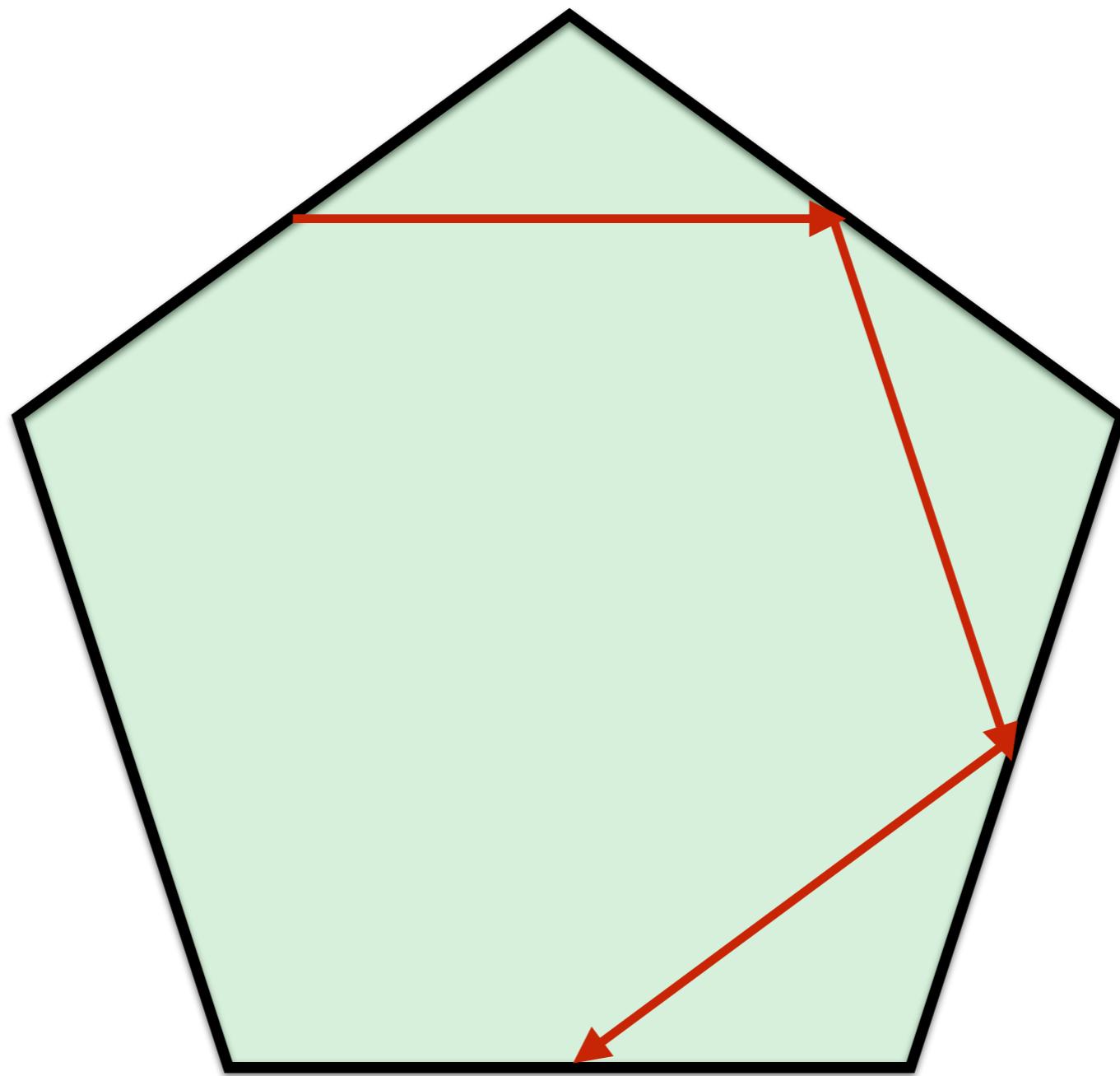
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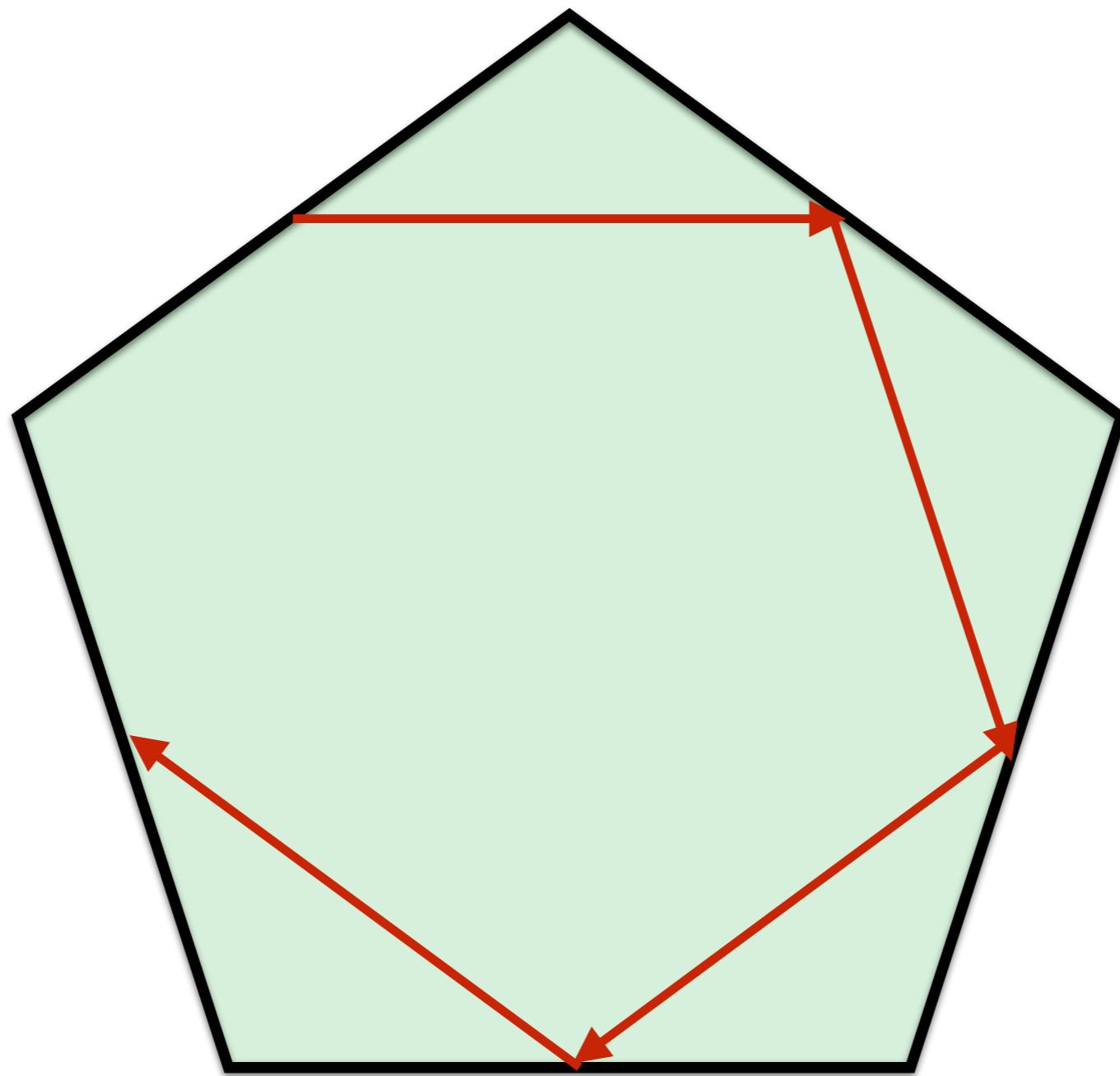
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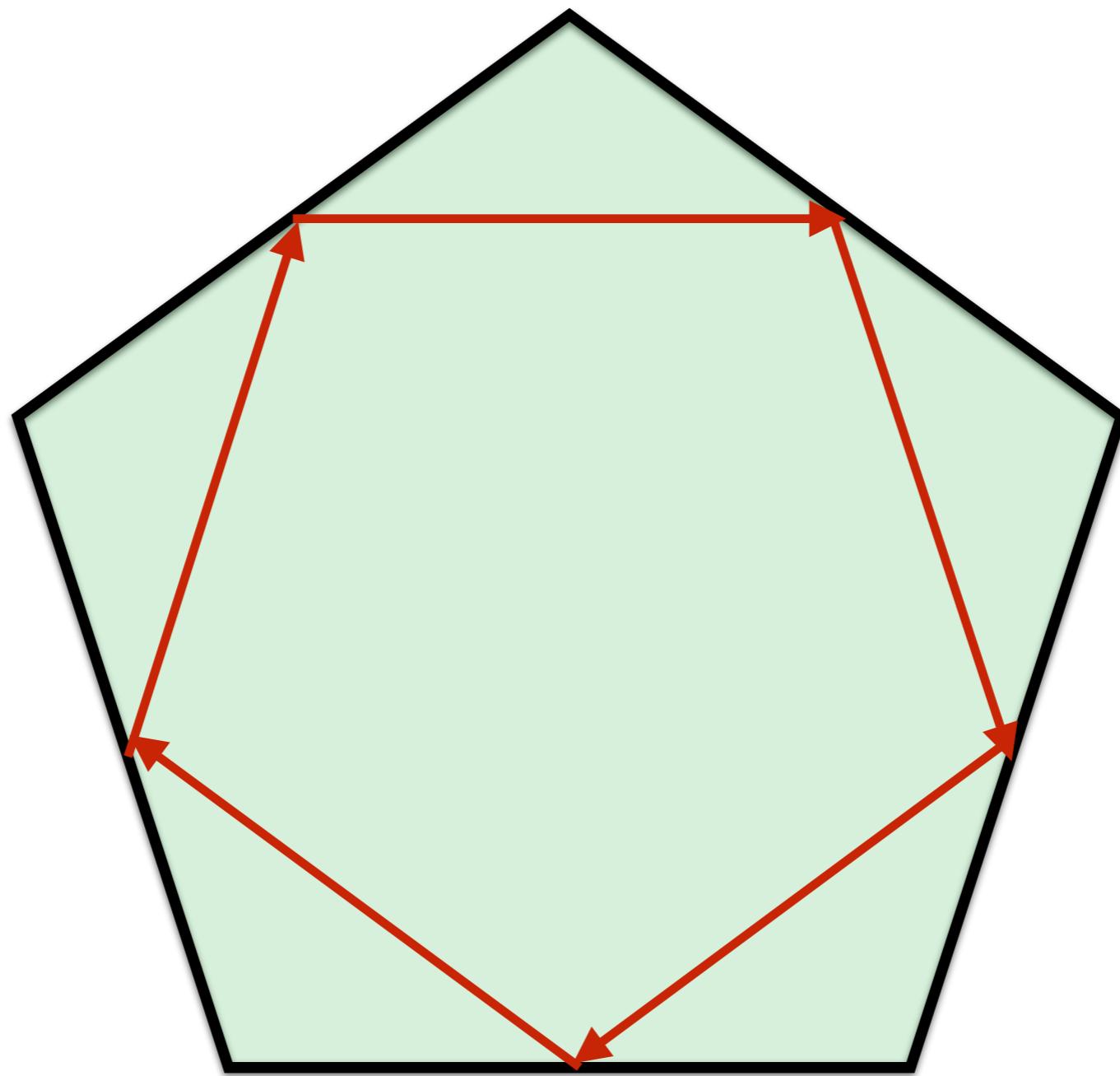
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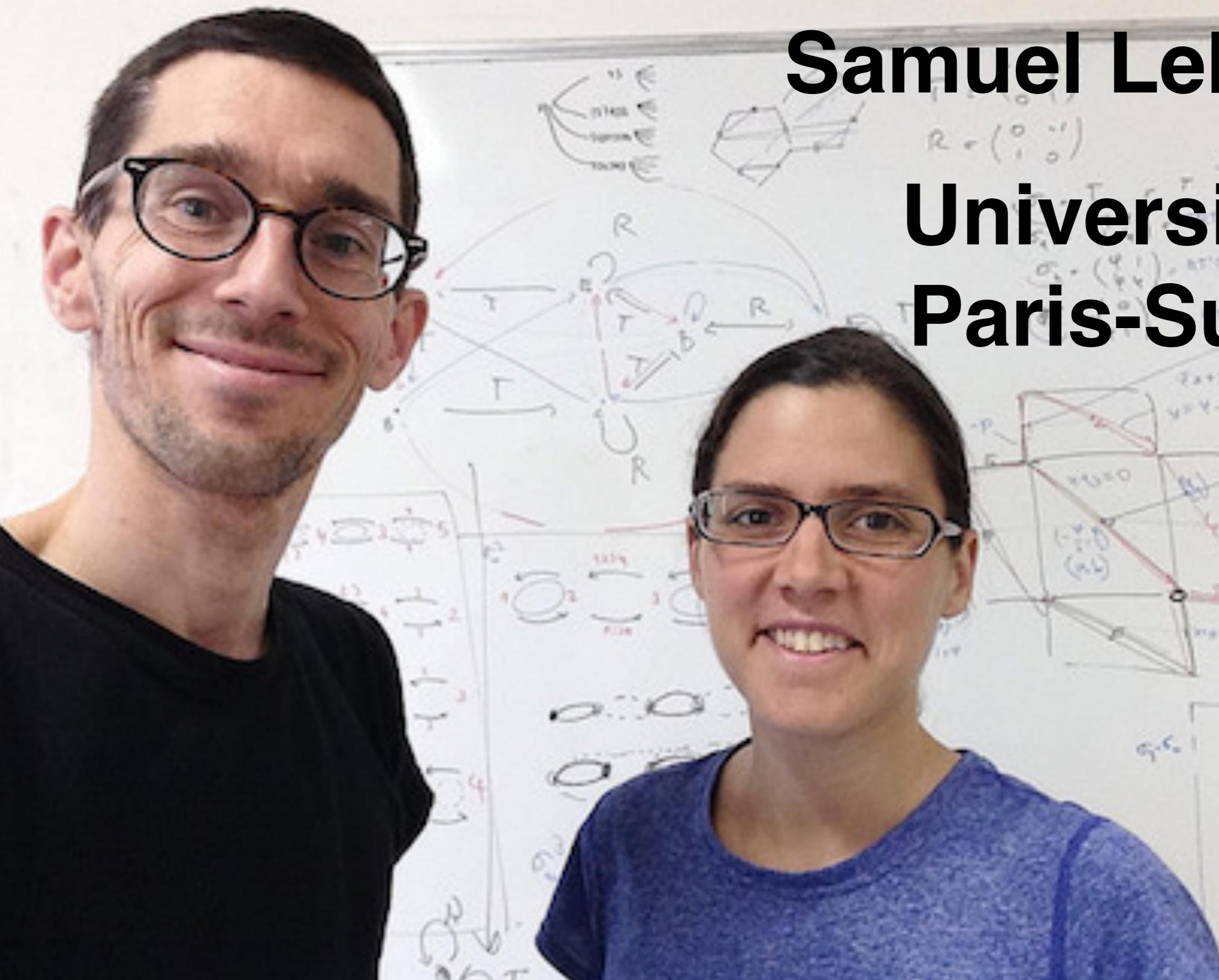


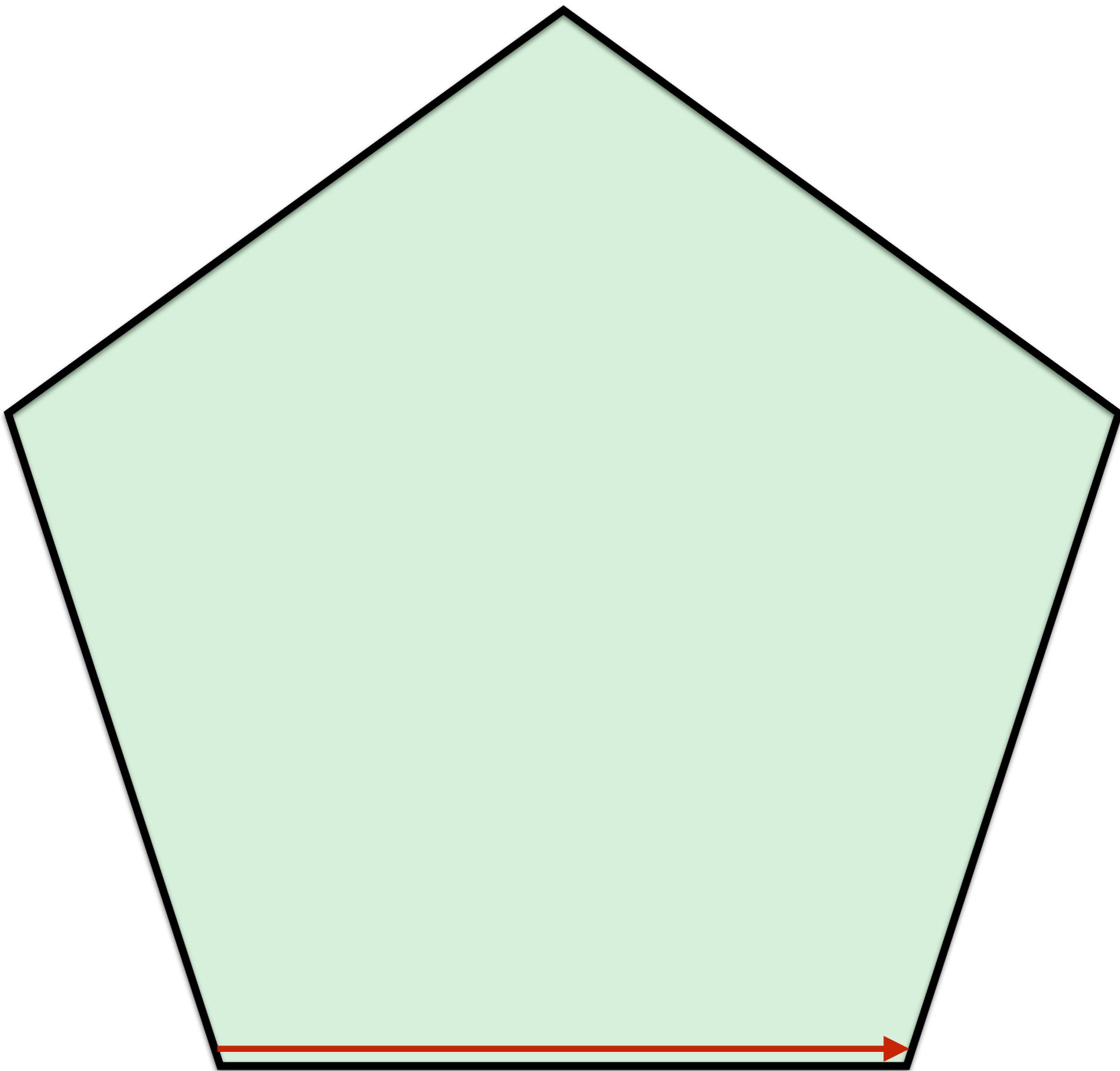
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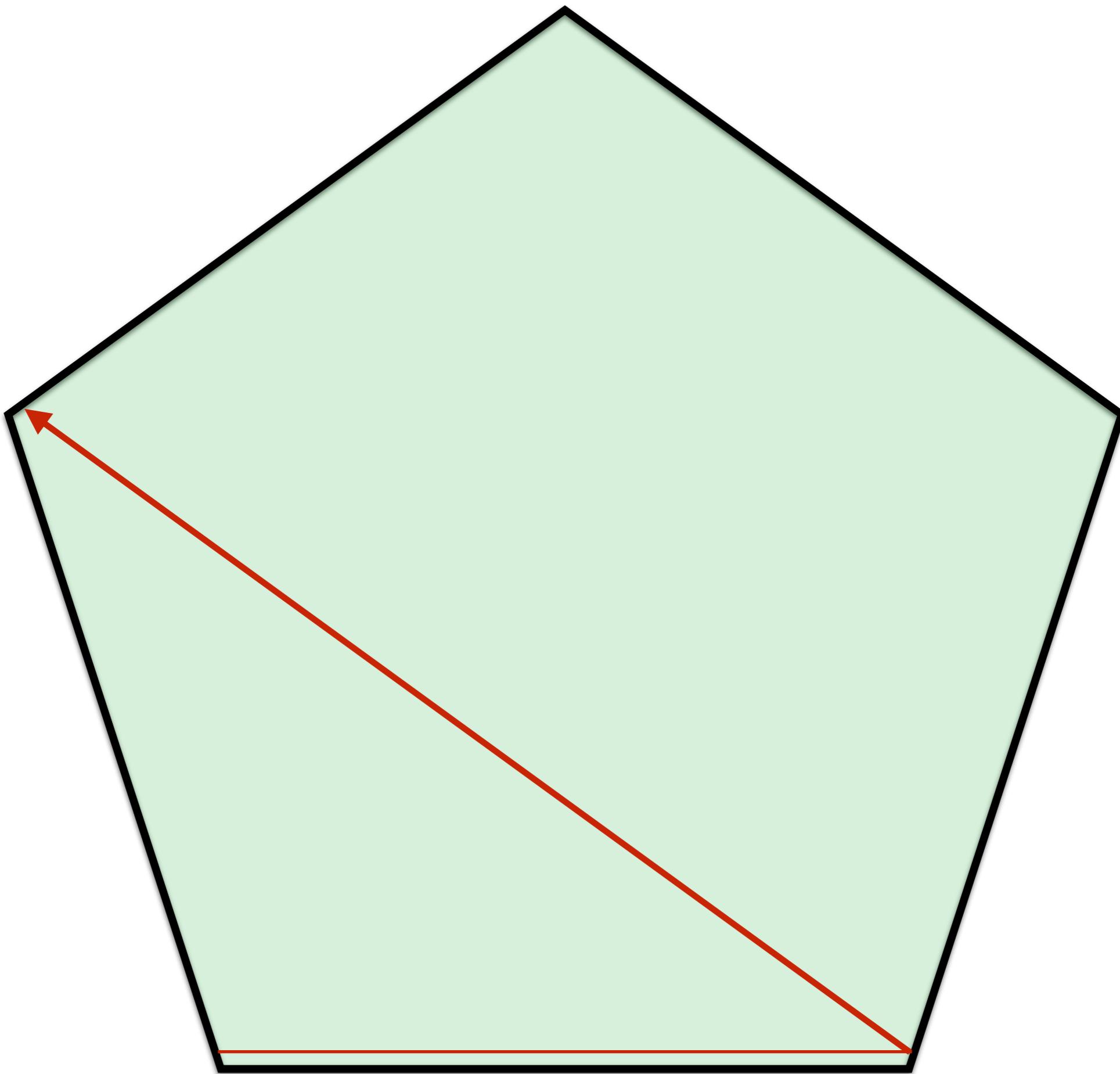


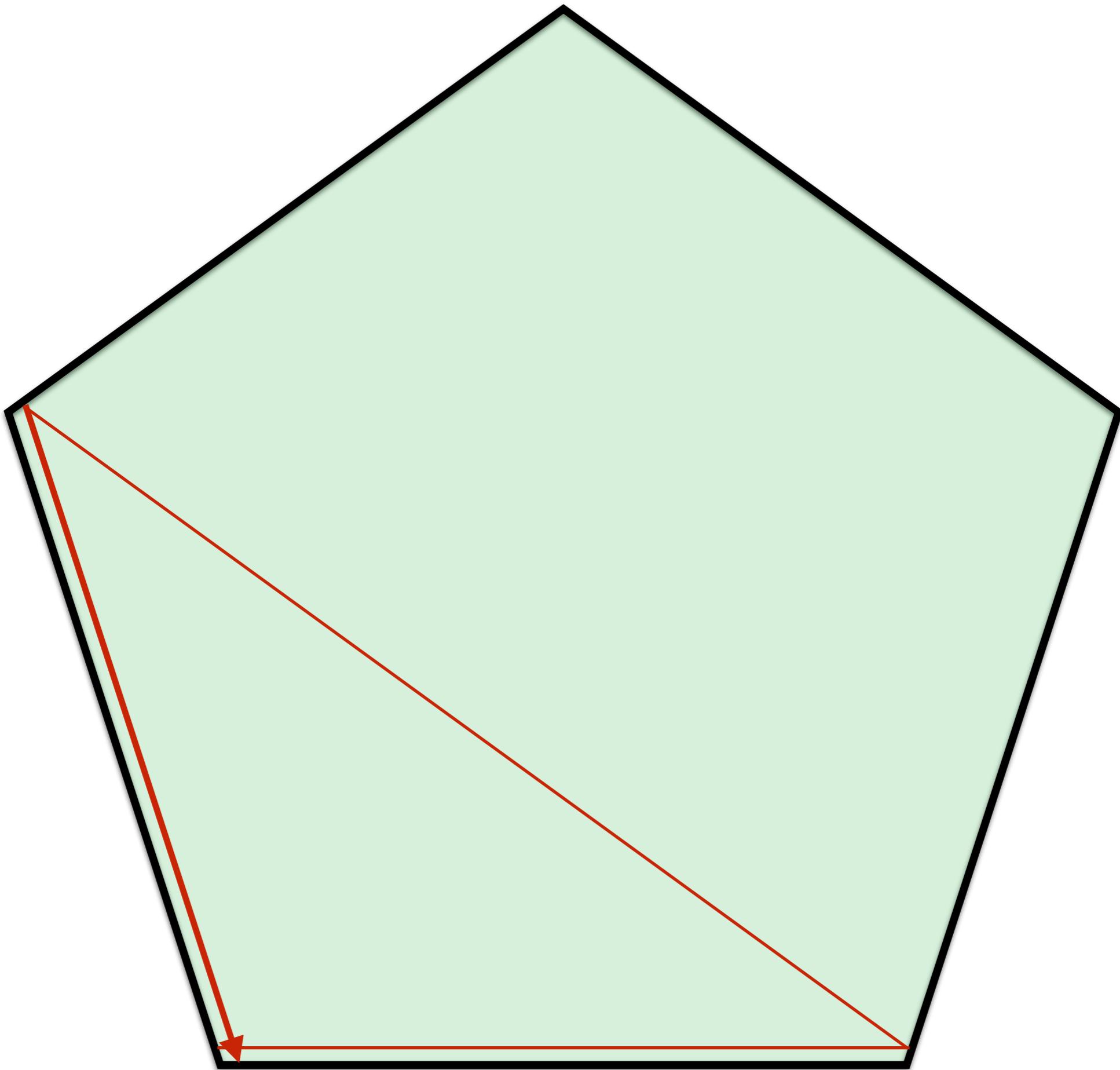
# BUDDIES

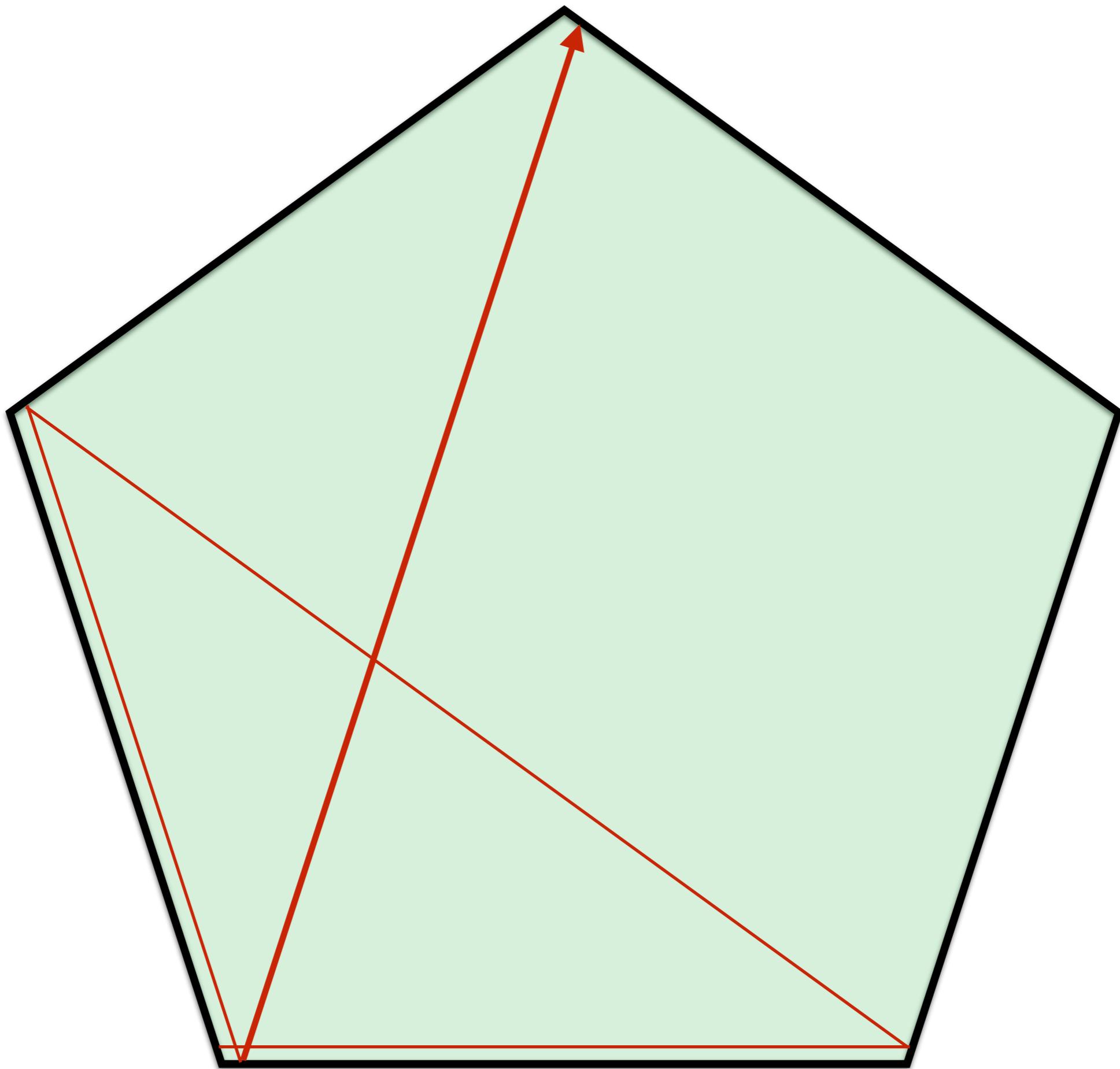
Joint with  
**Samuel Lelièvre**  
**Université  
Paris-Sud**

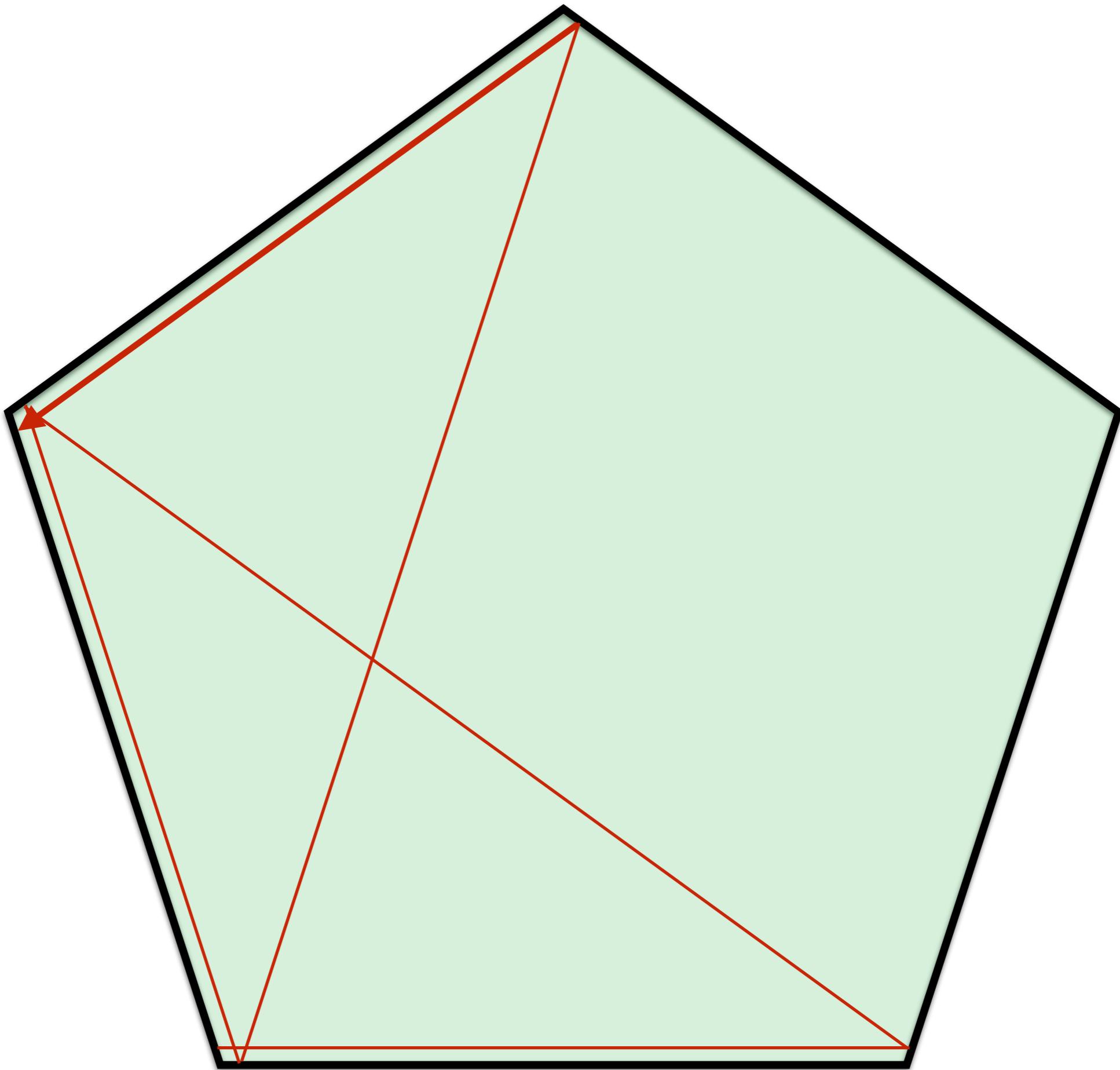


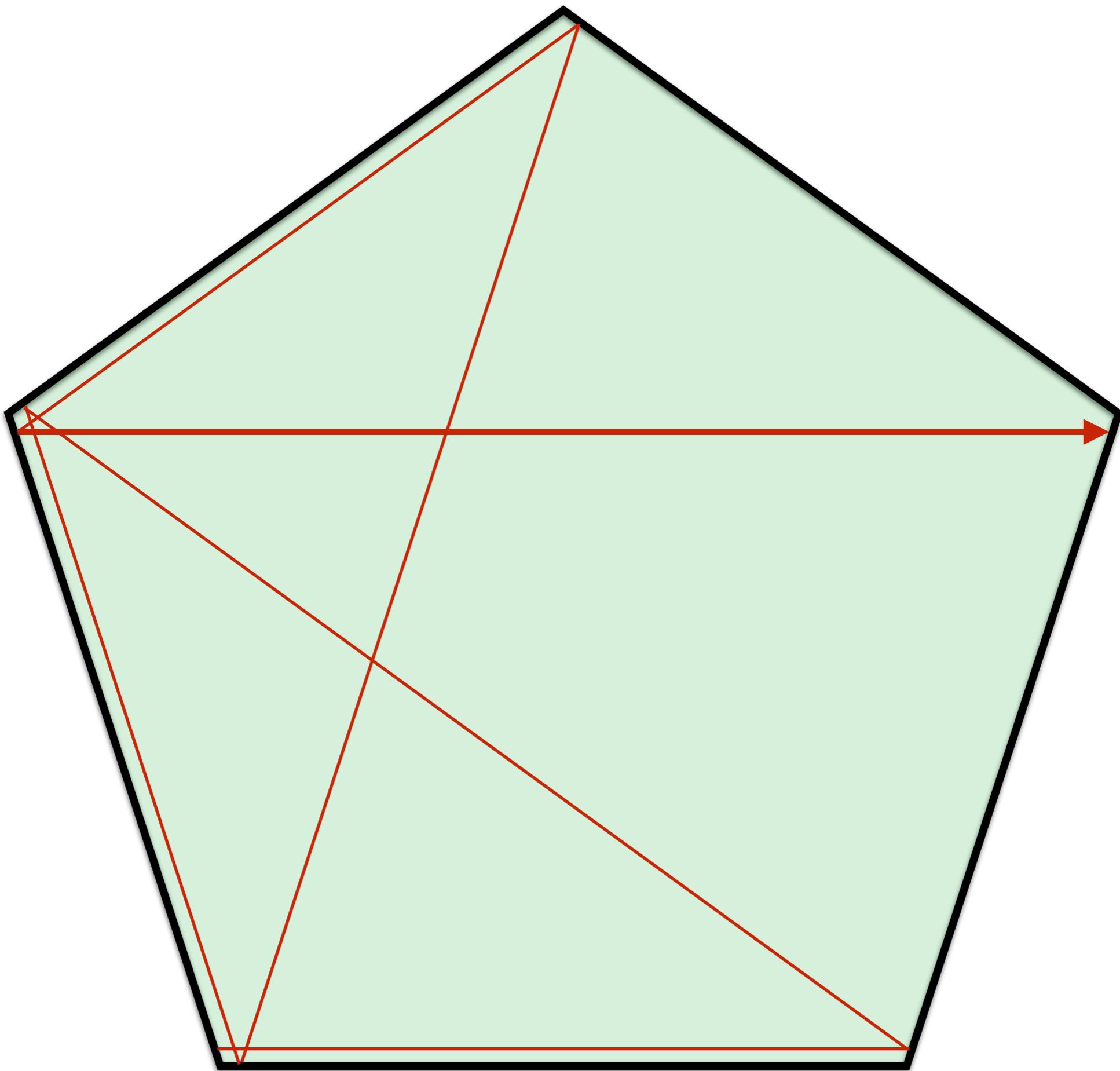


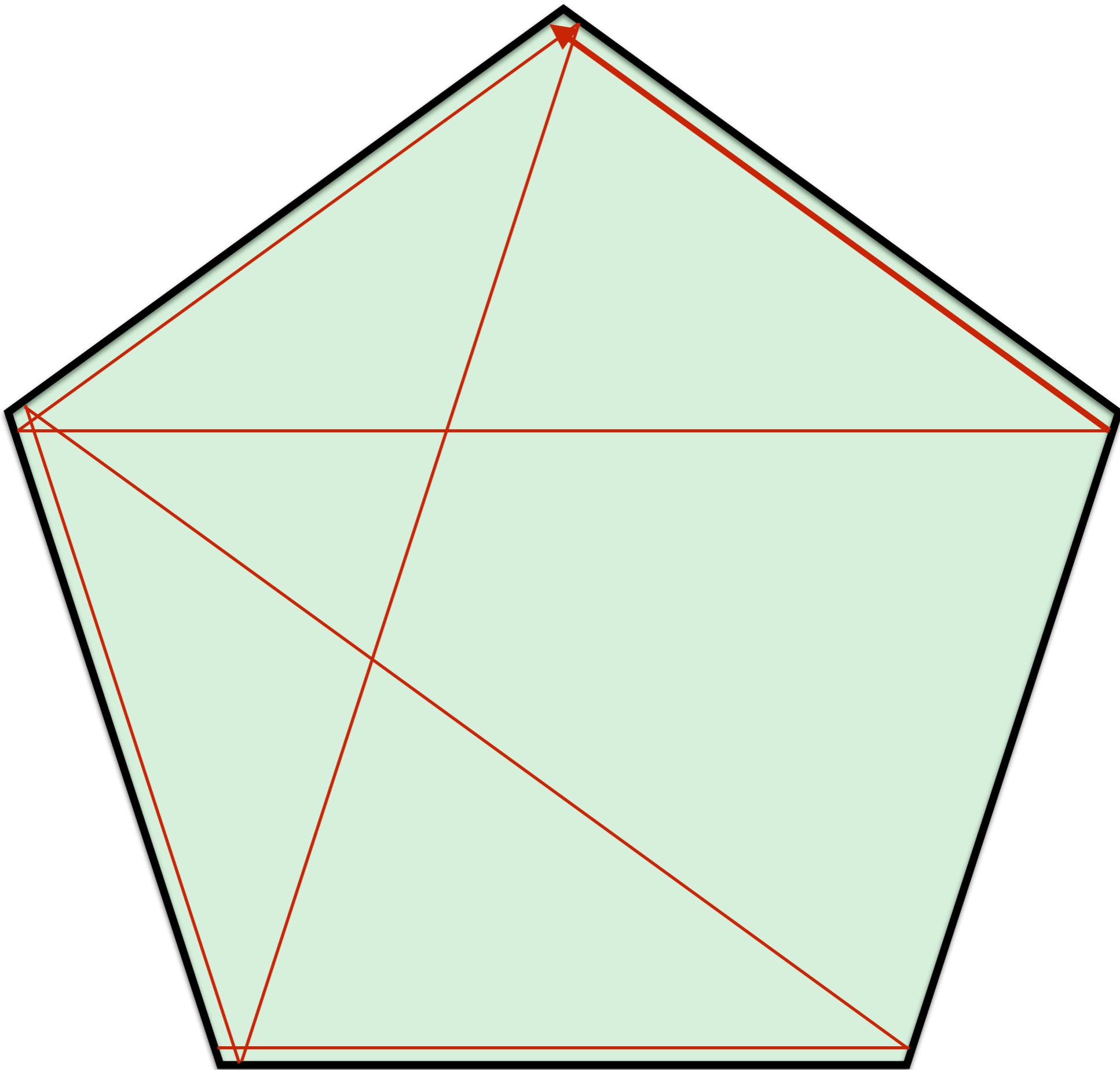


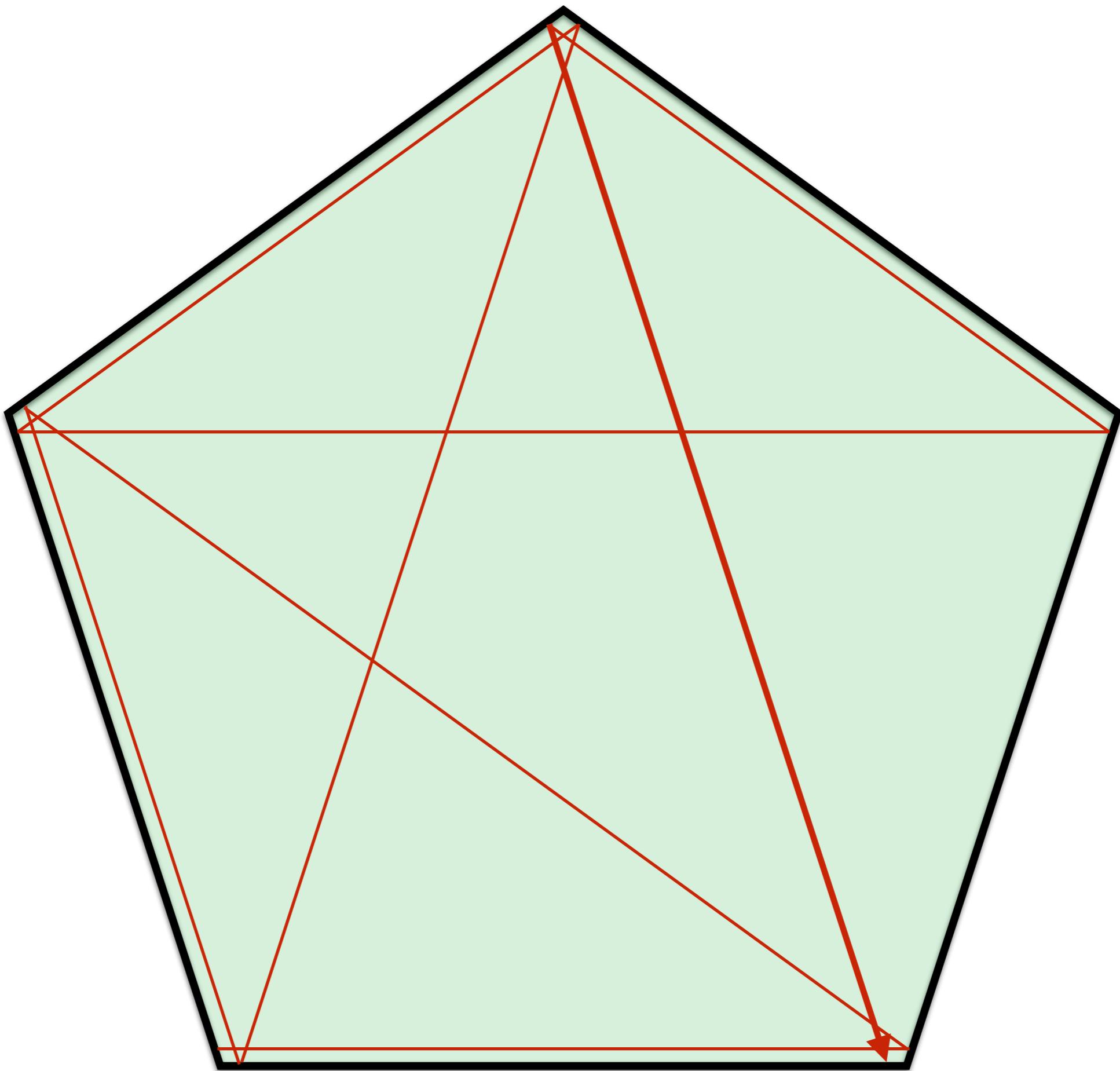


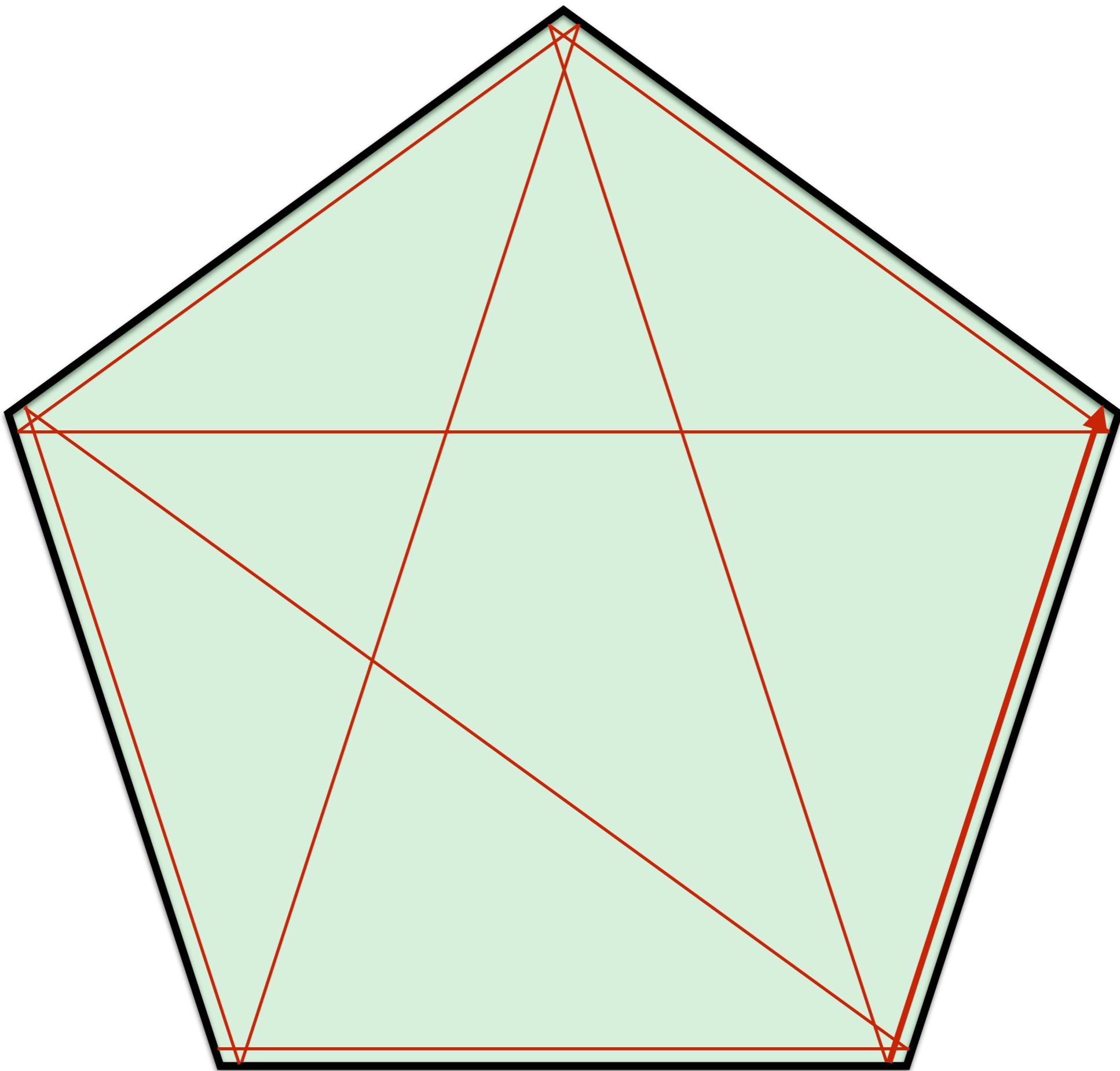


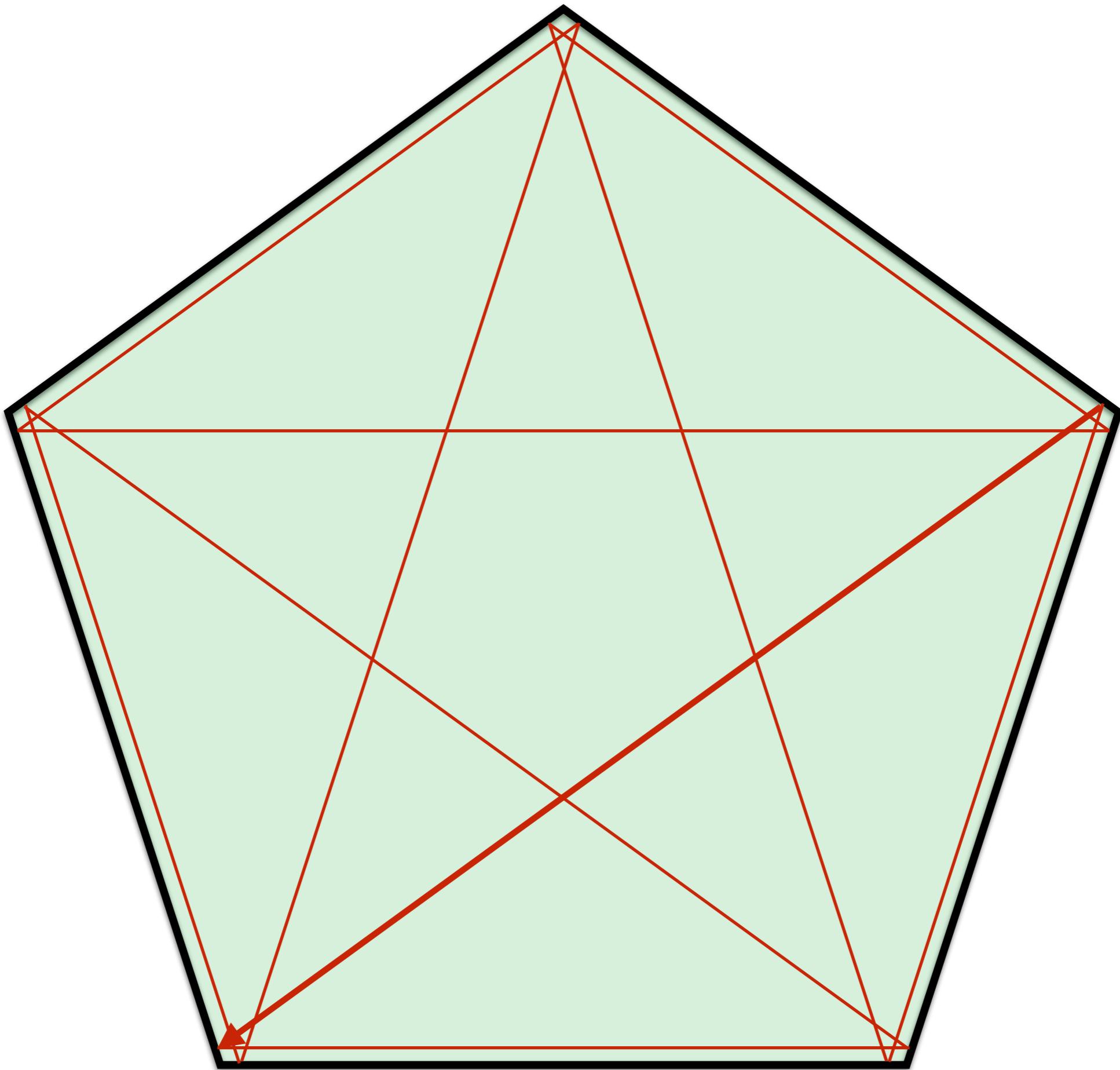


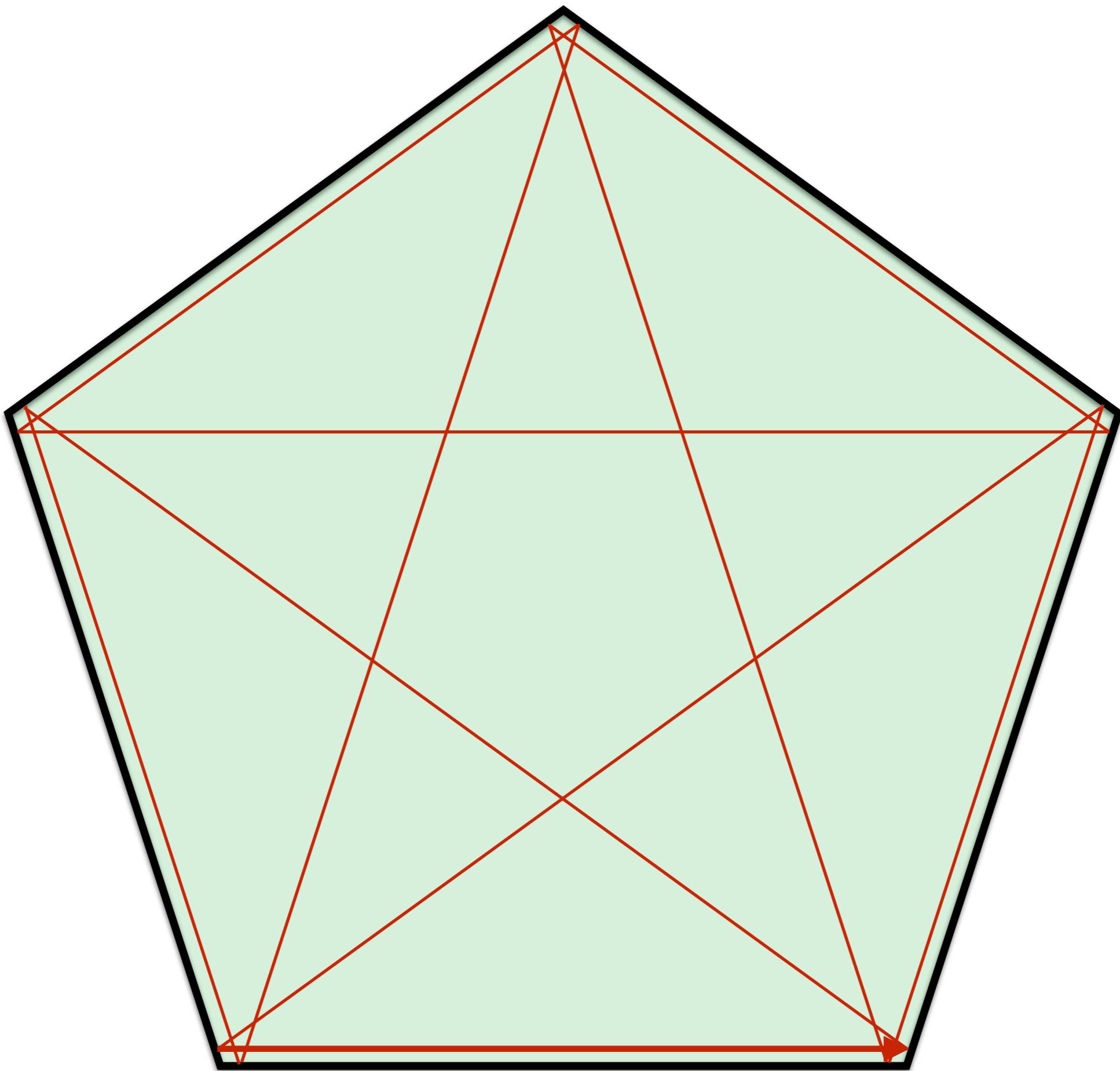


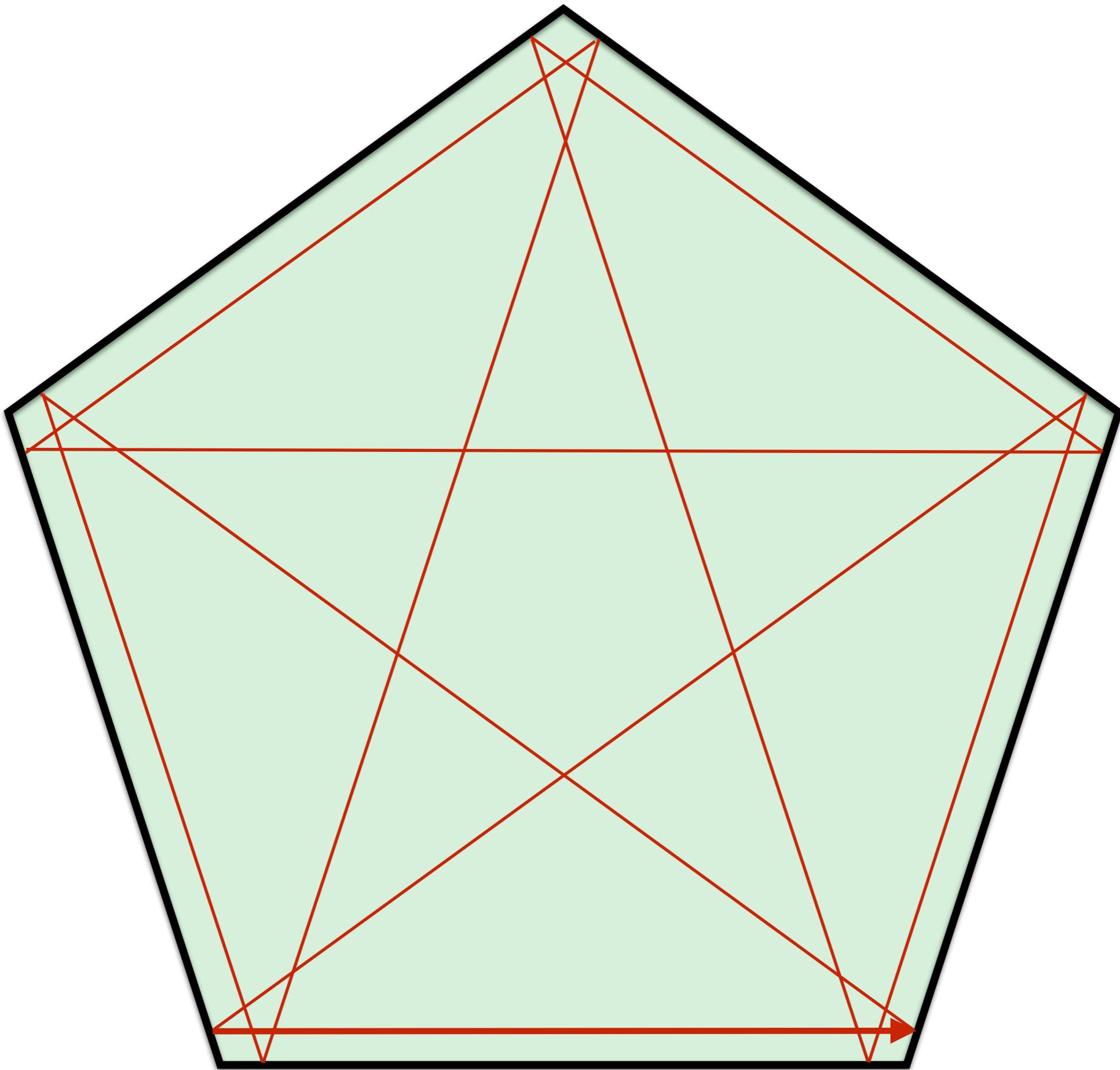


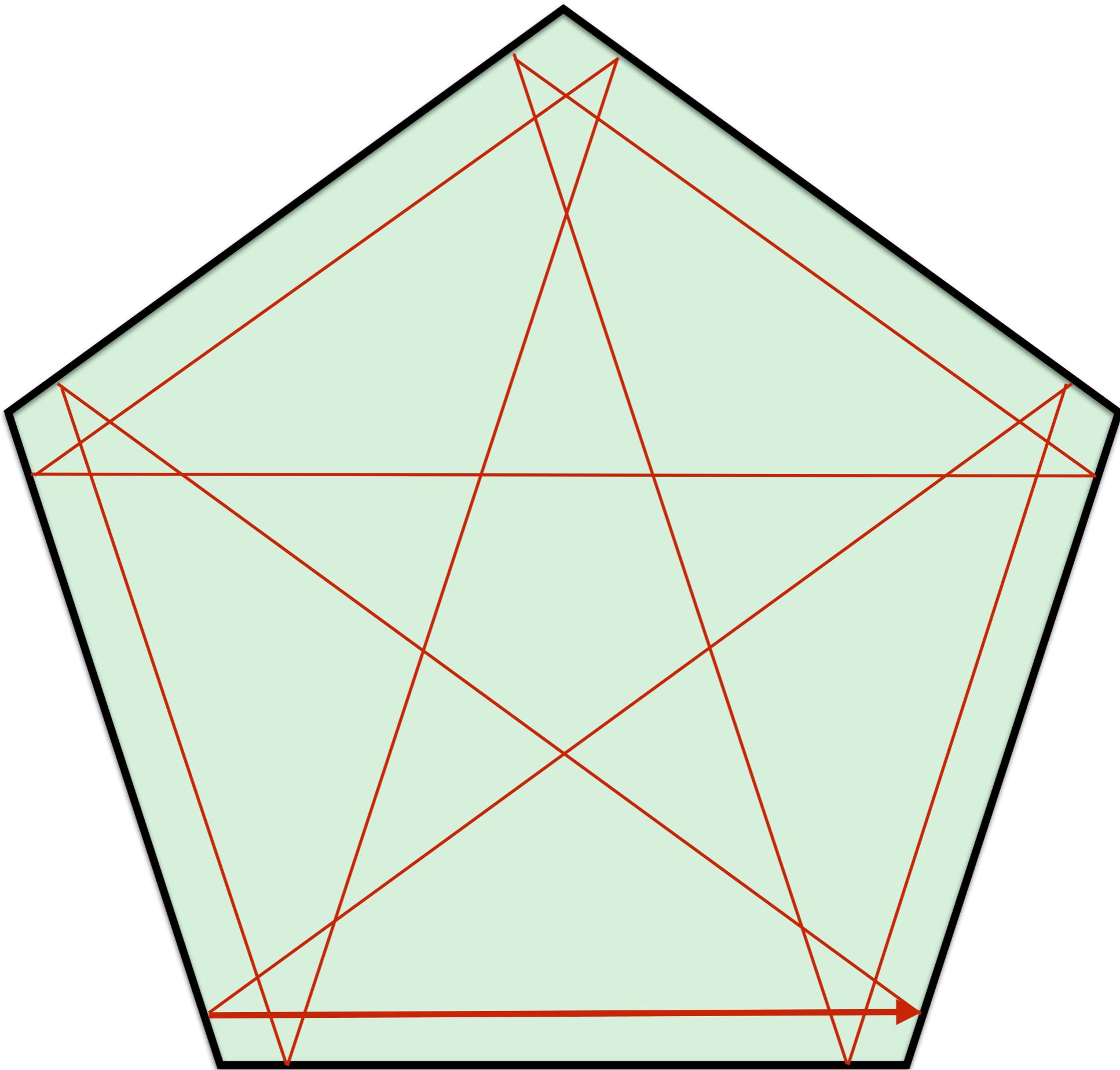


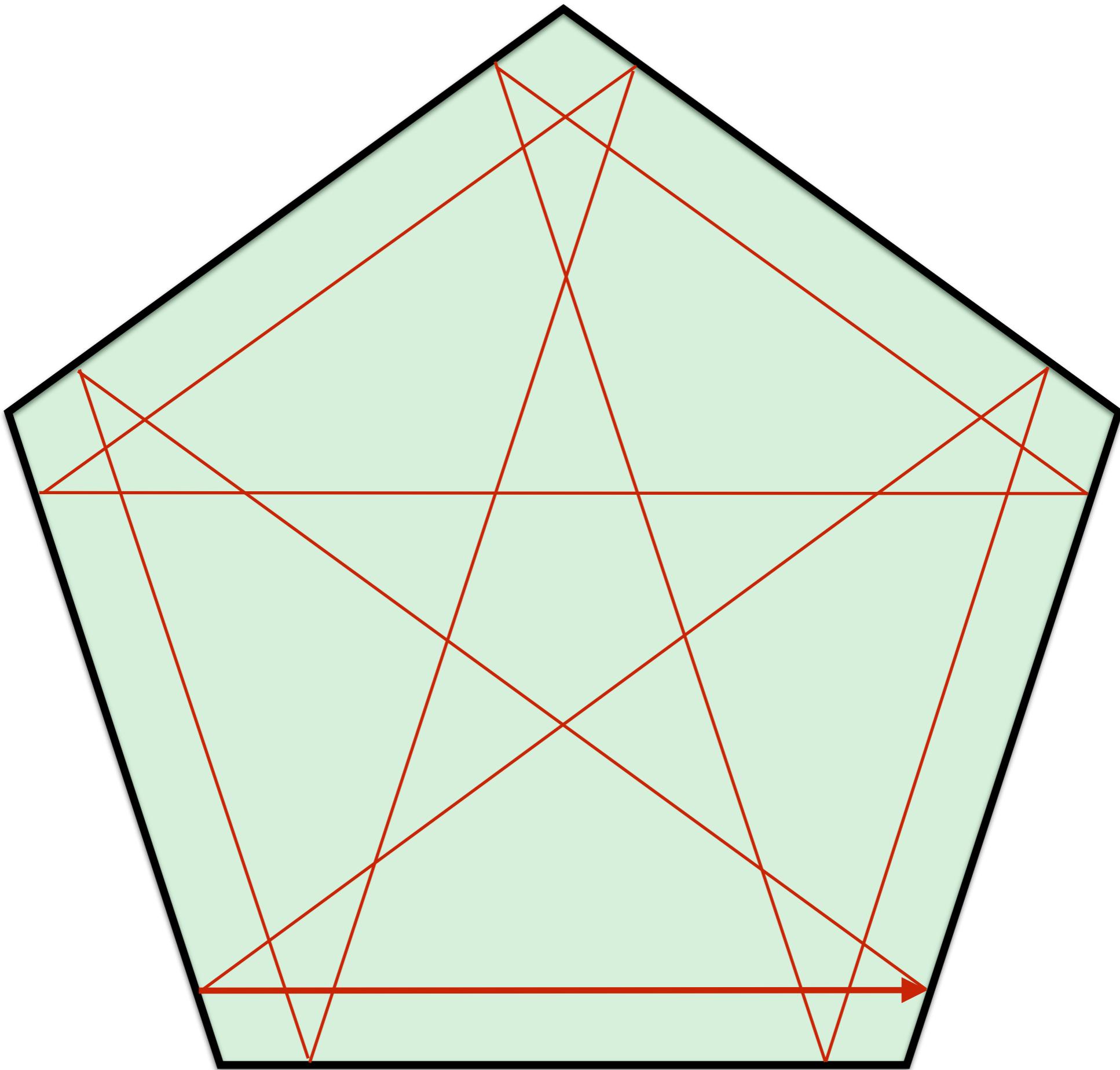


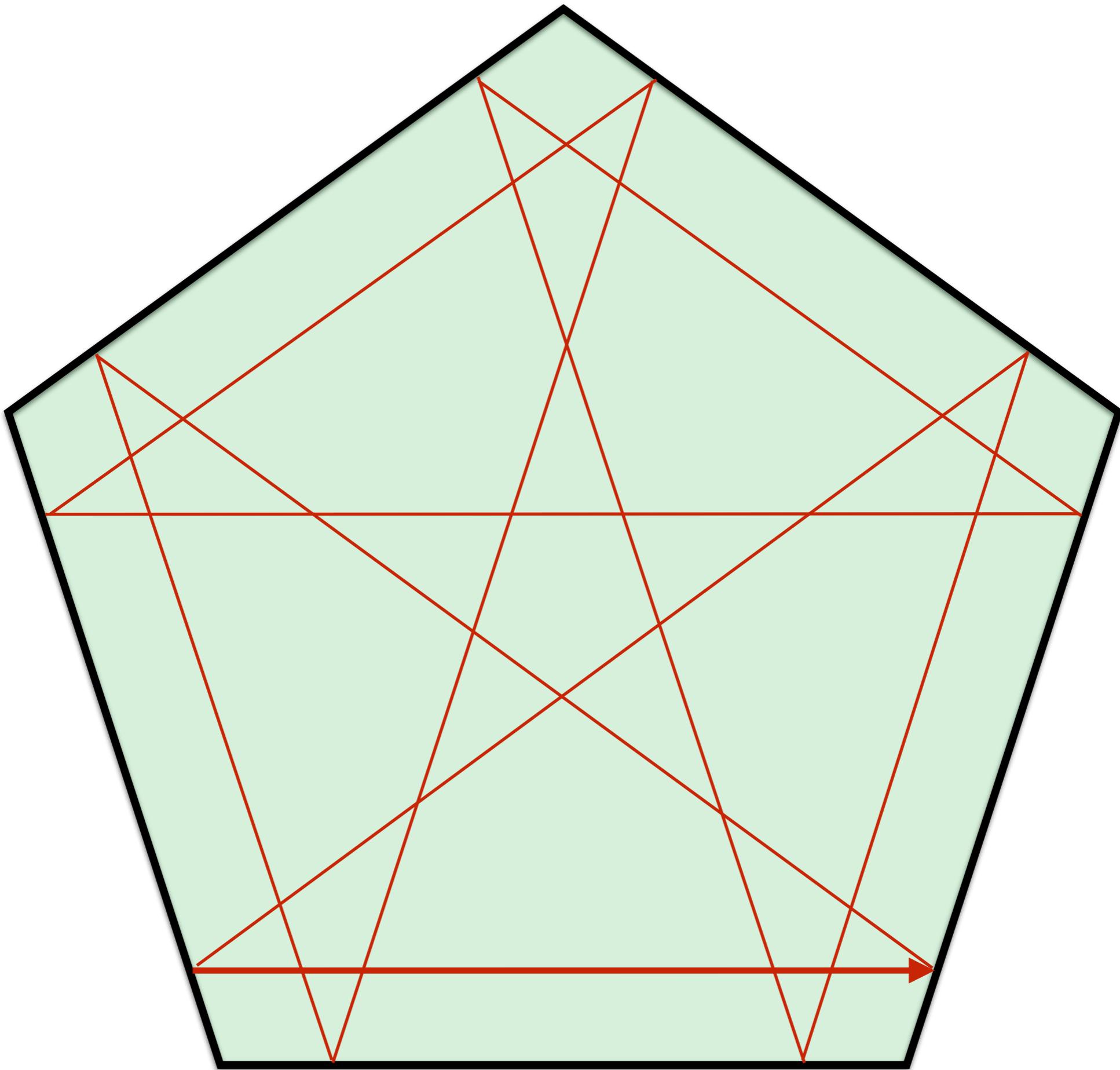


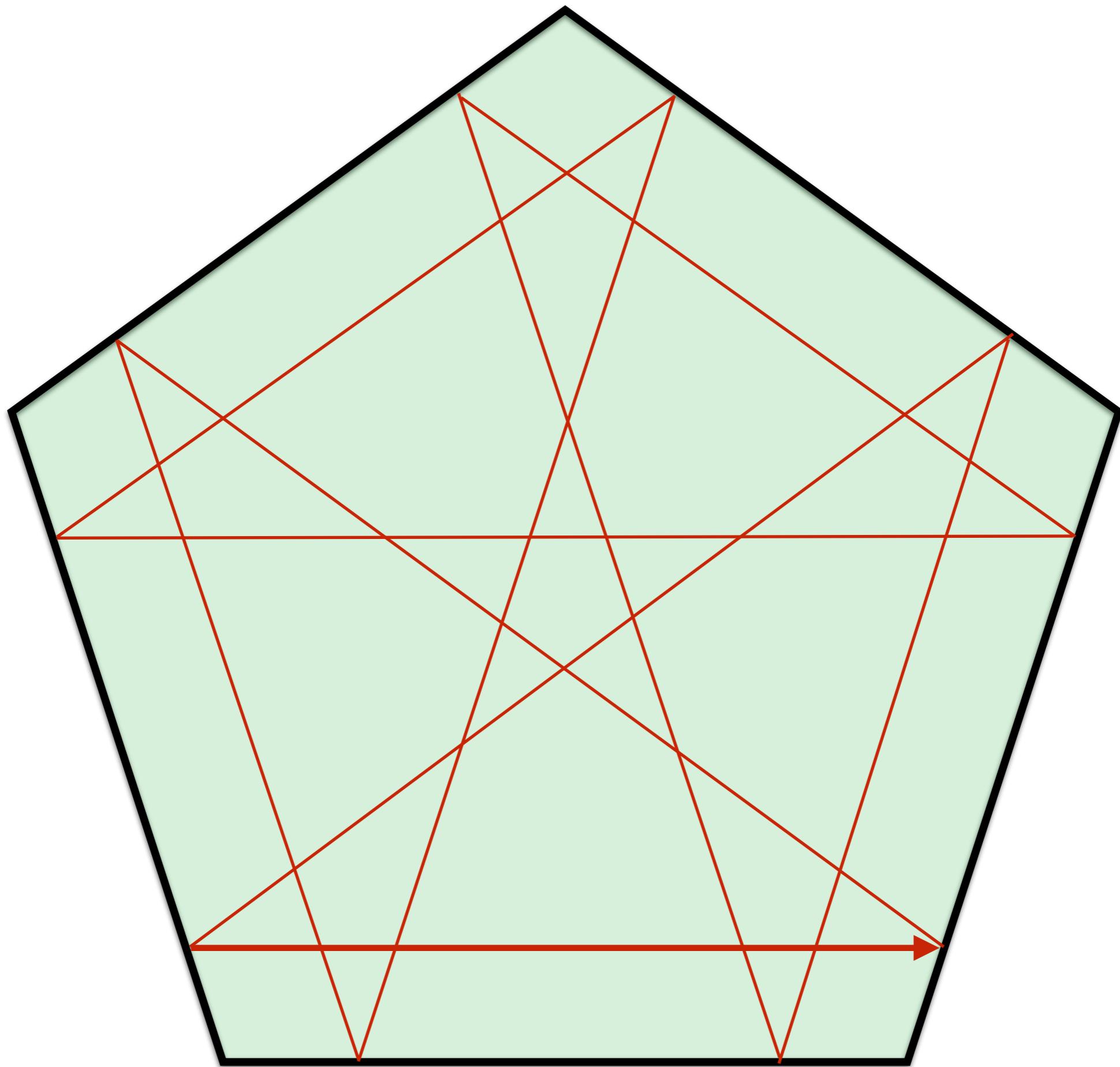


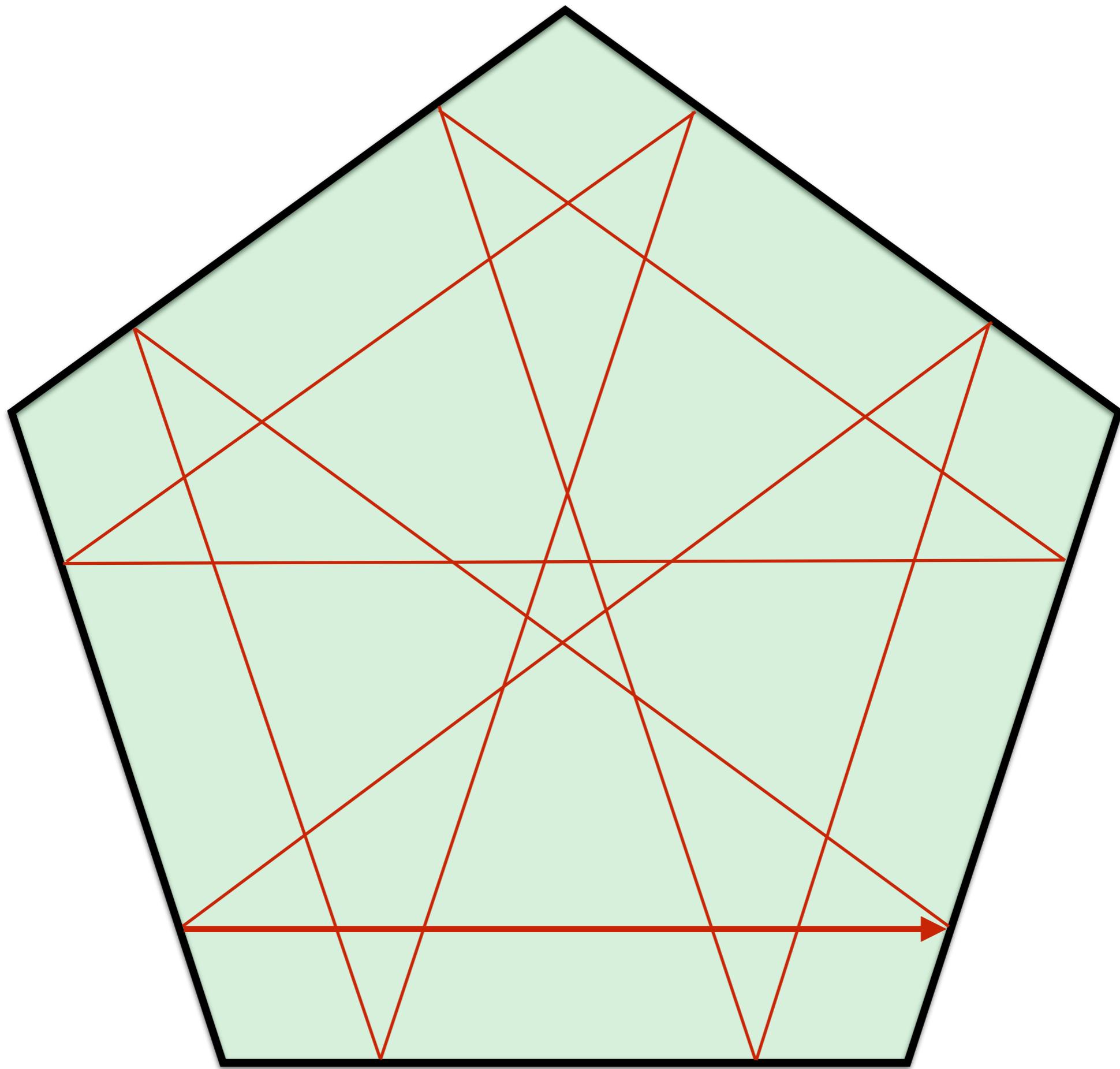


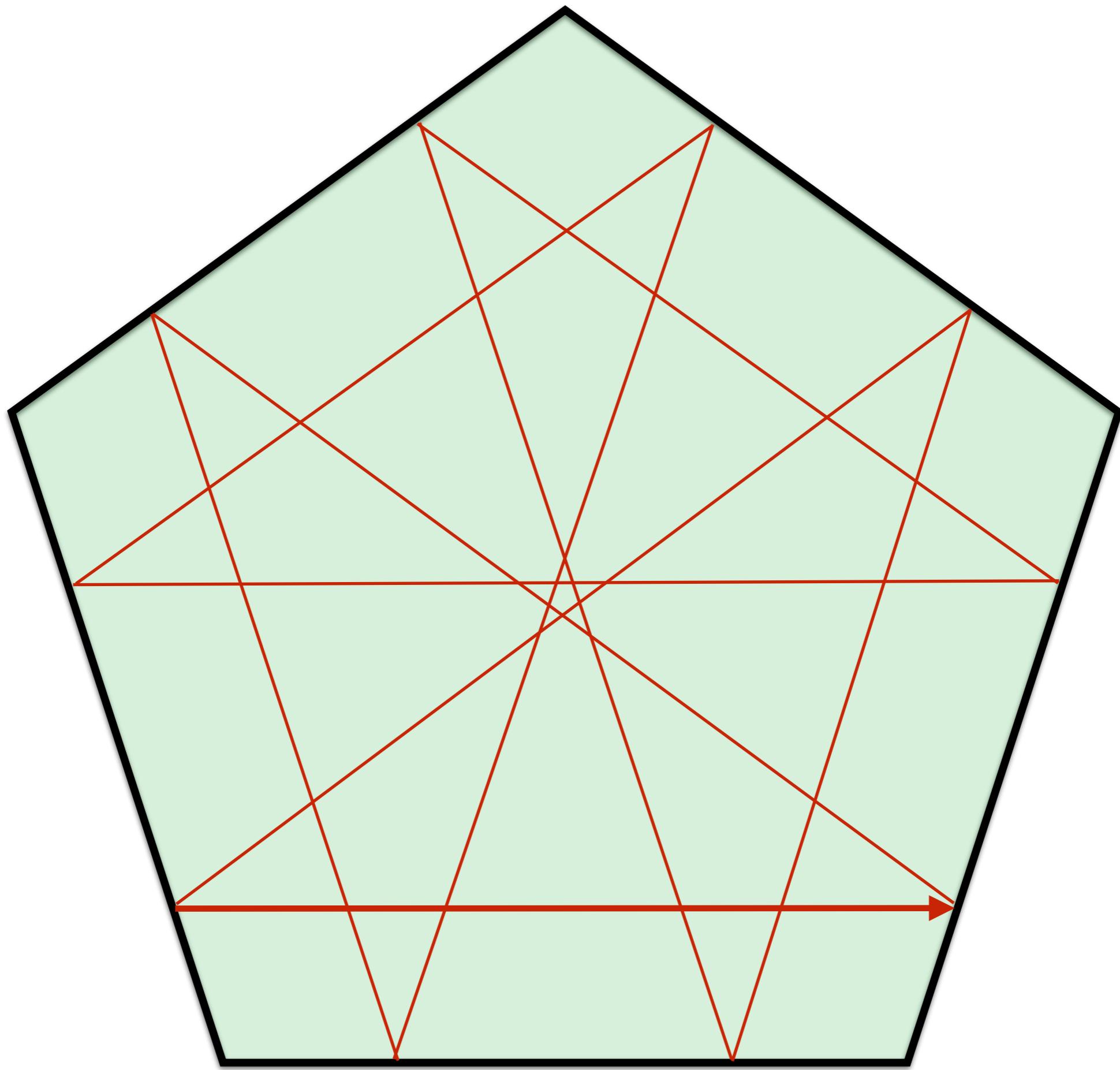


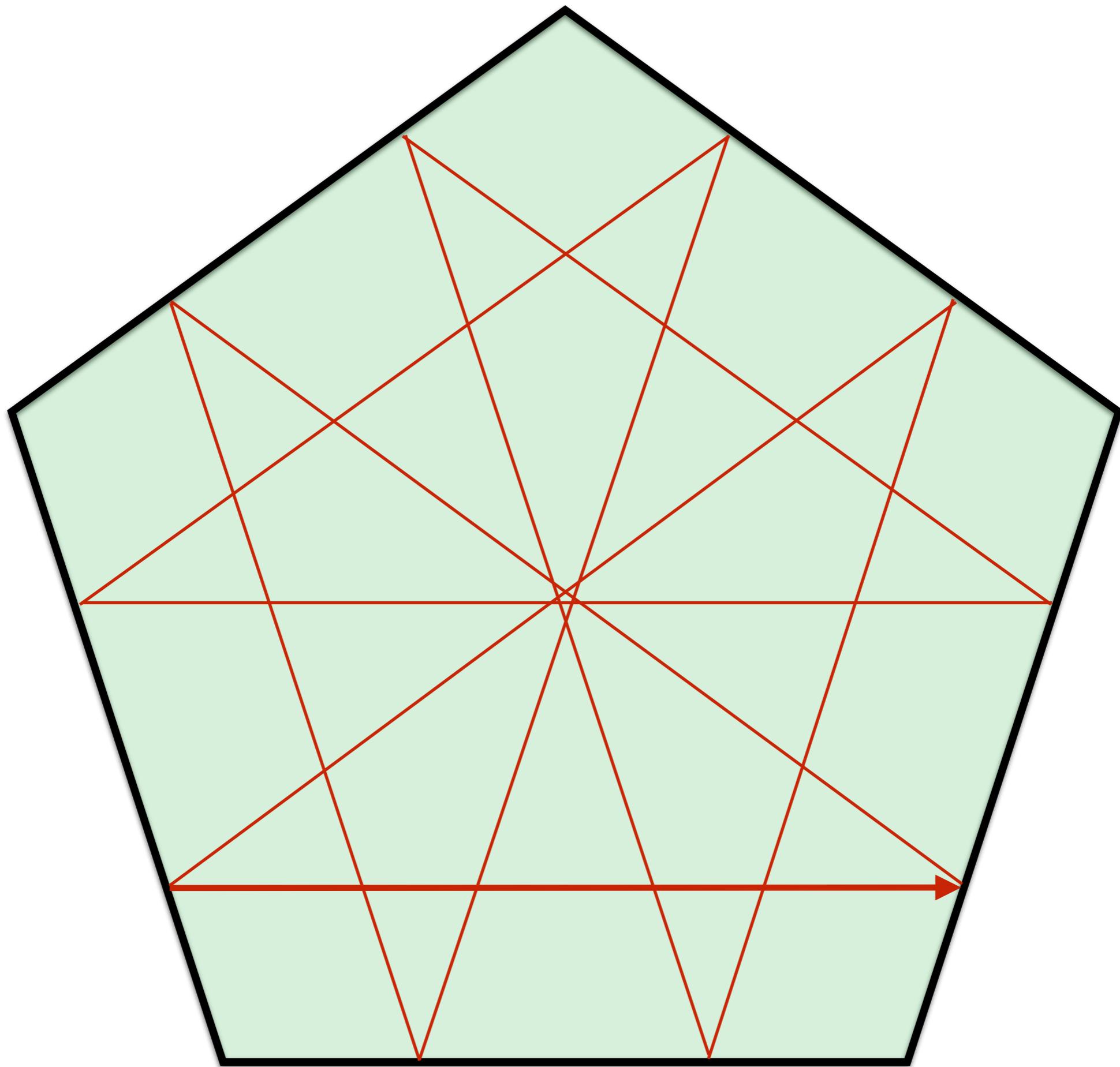


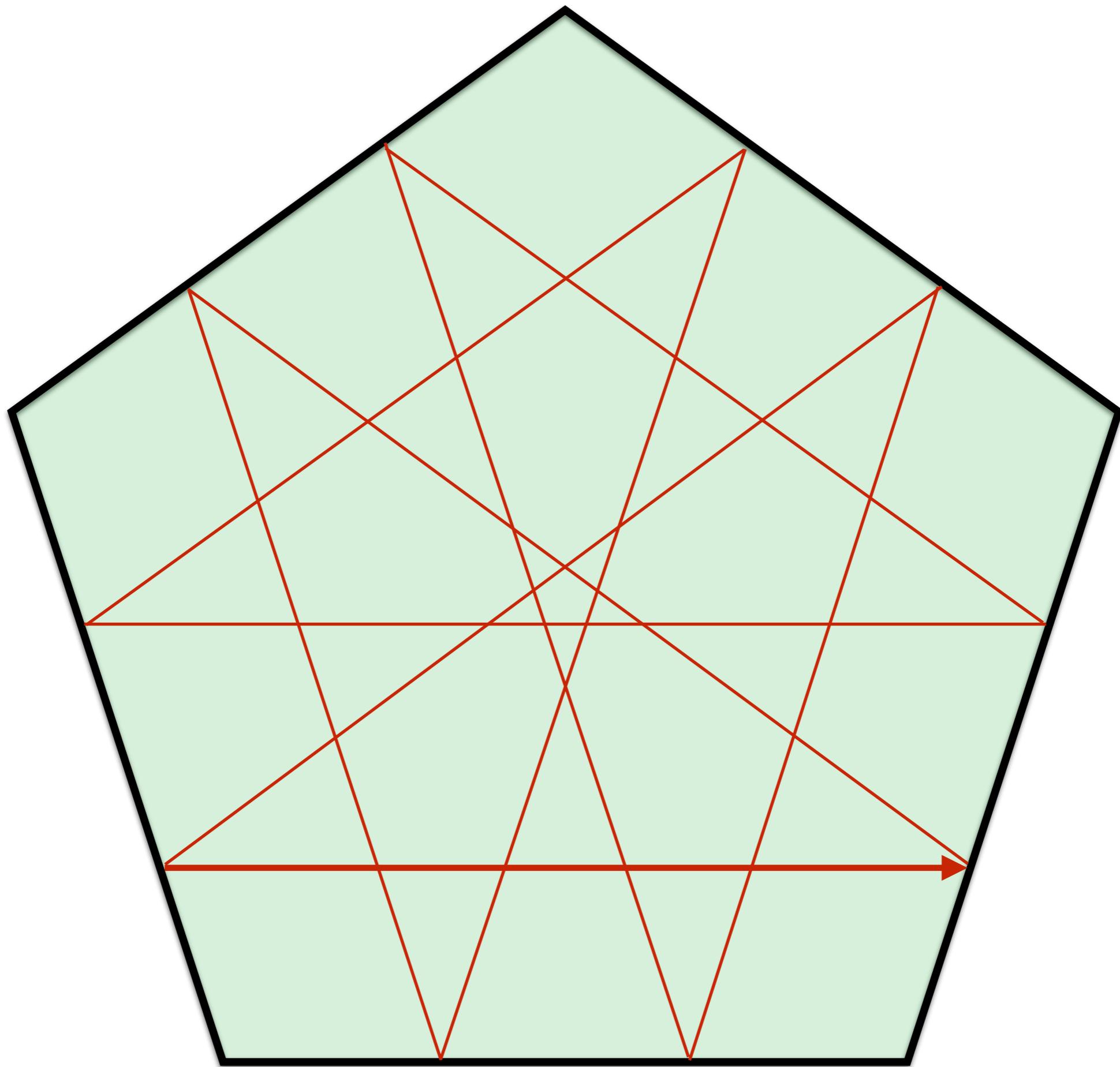


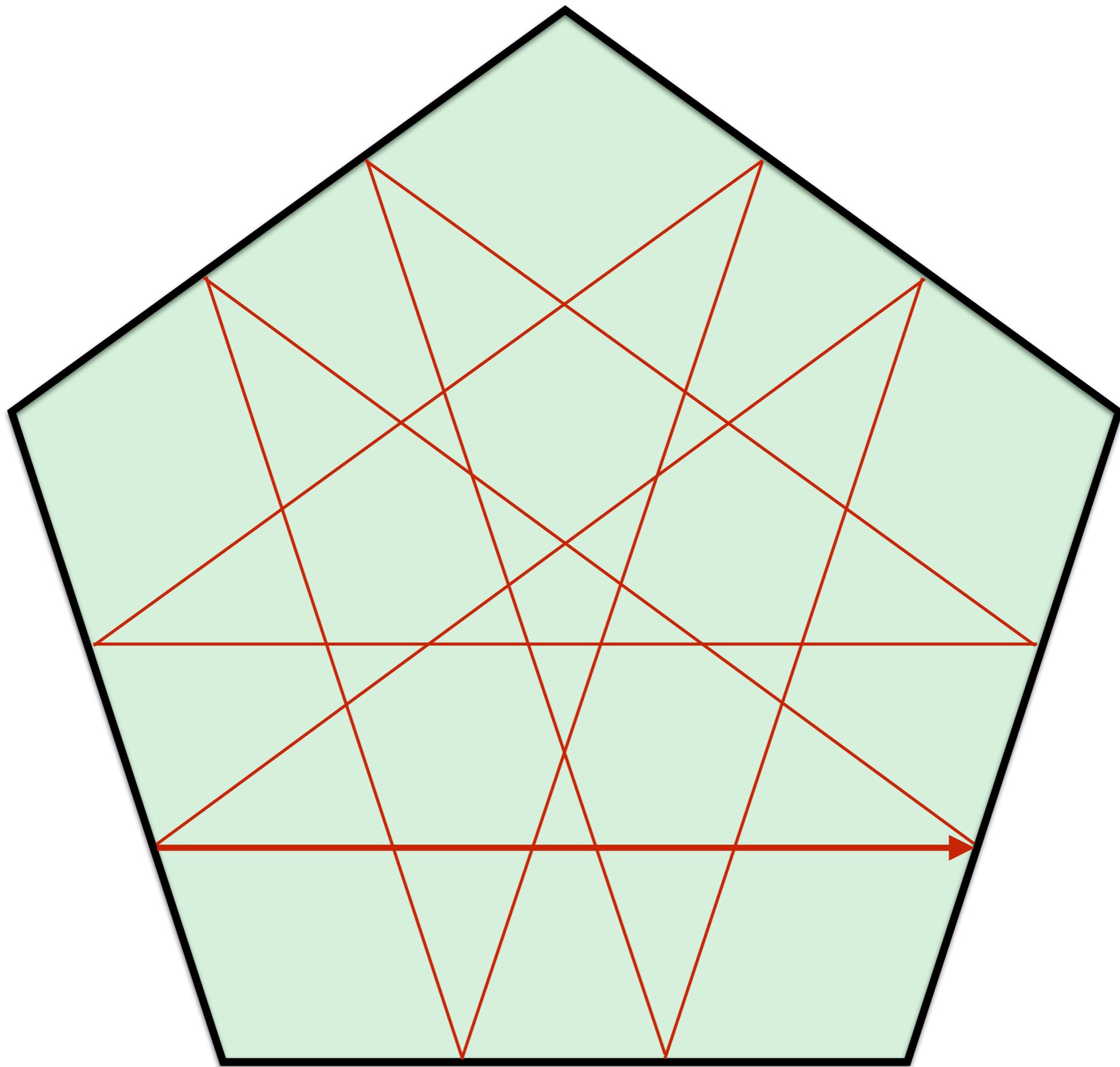


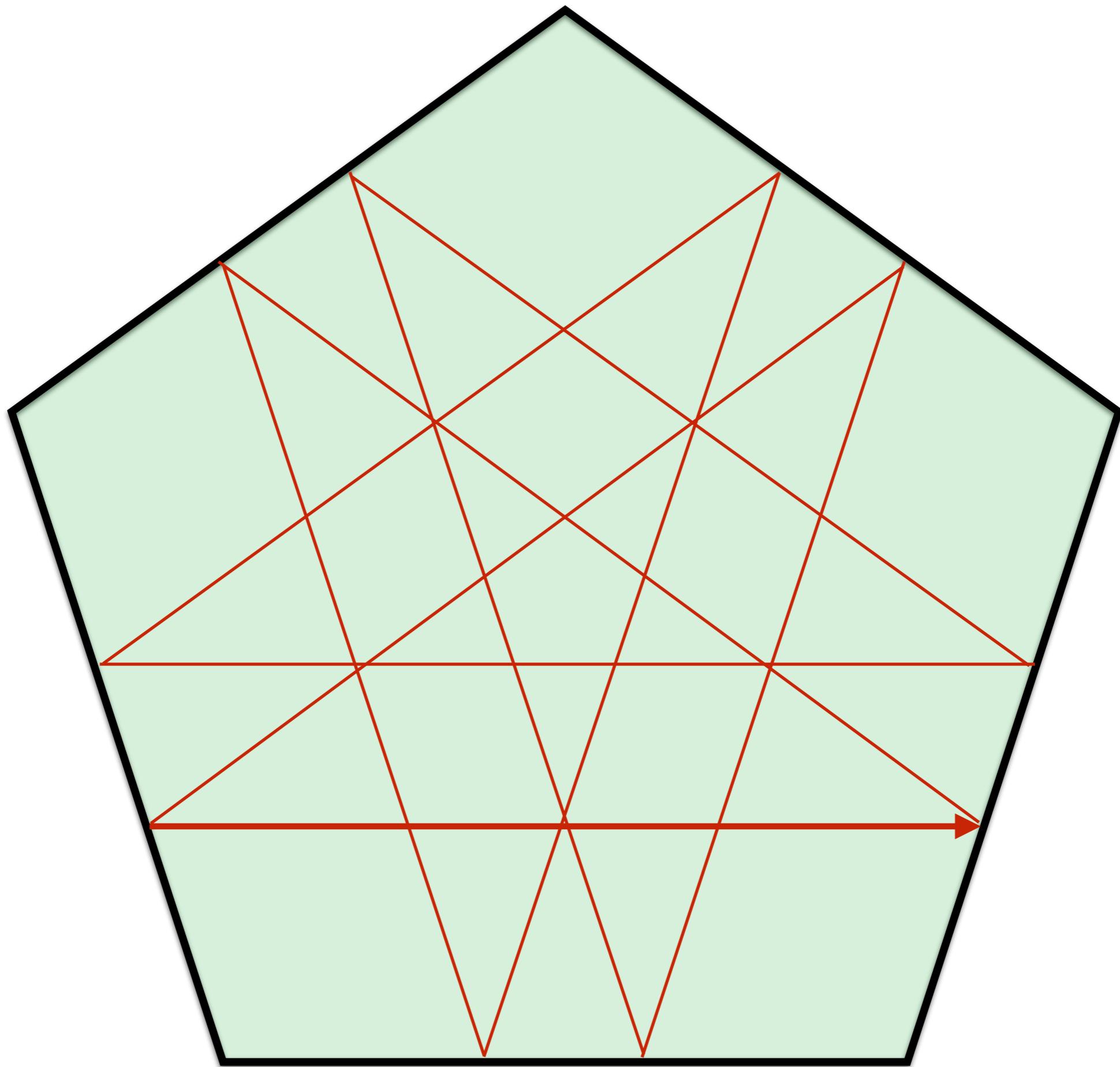


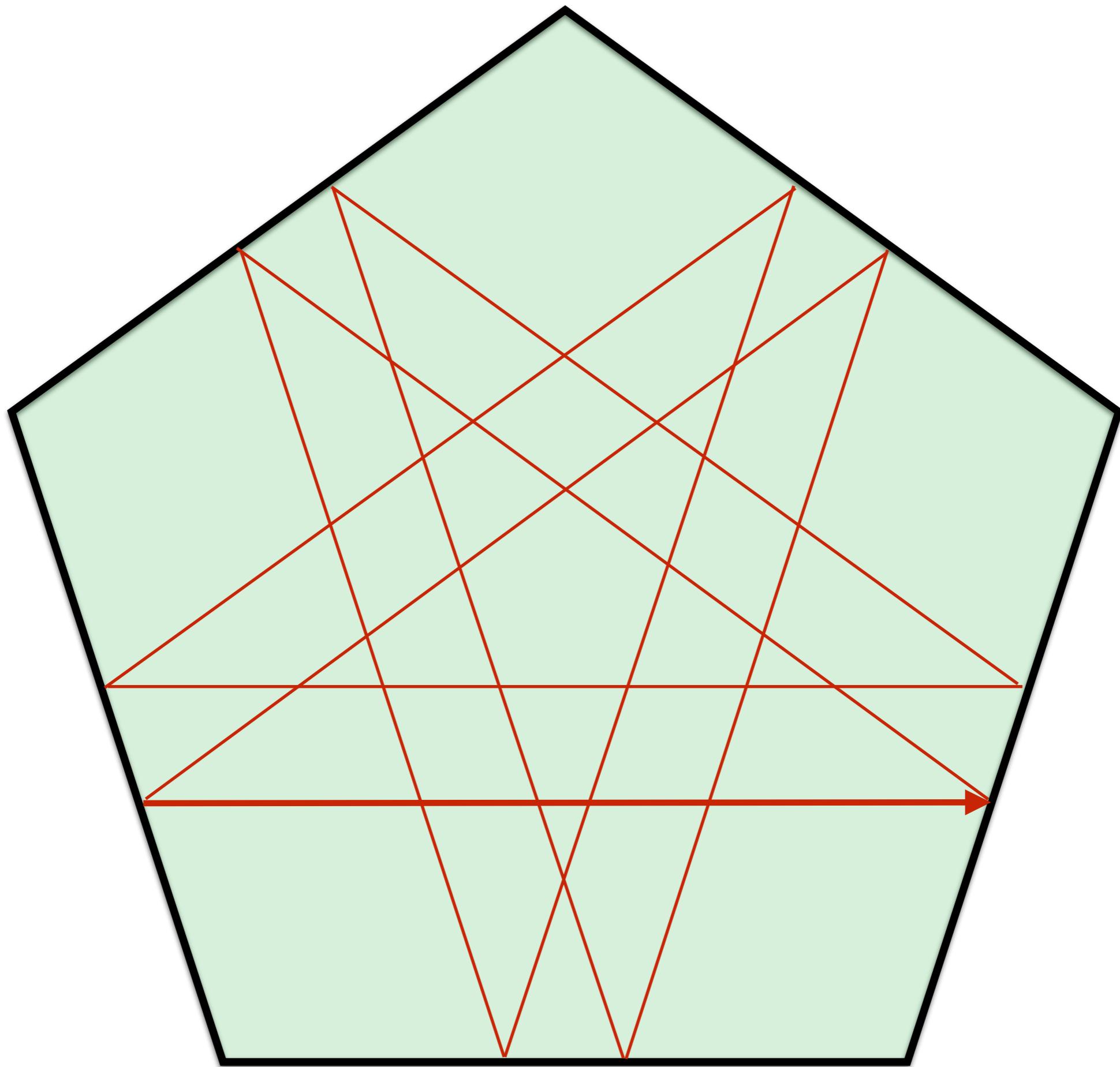


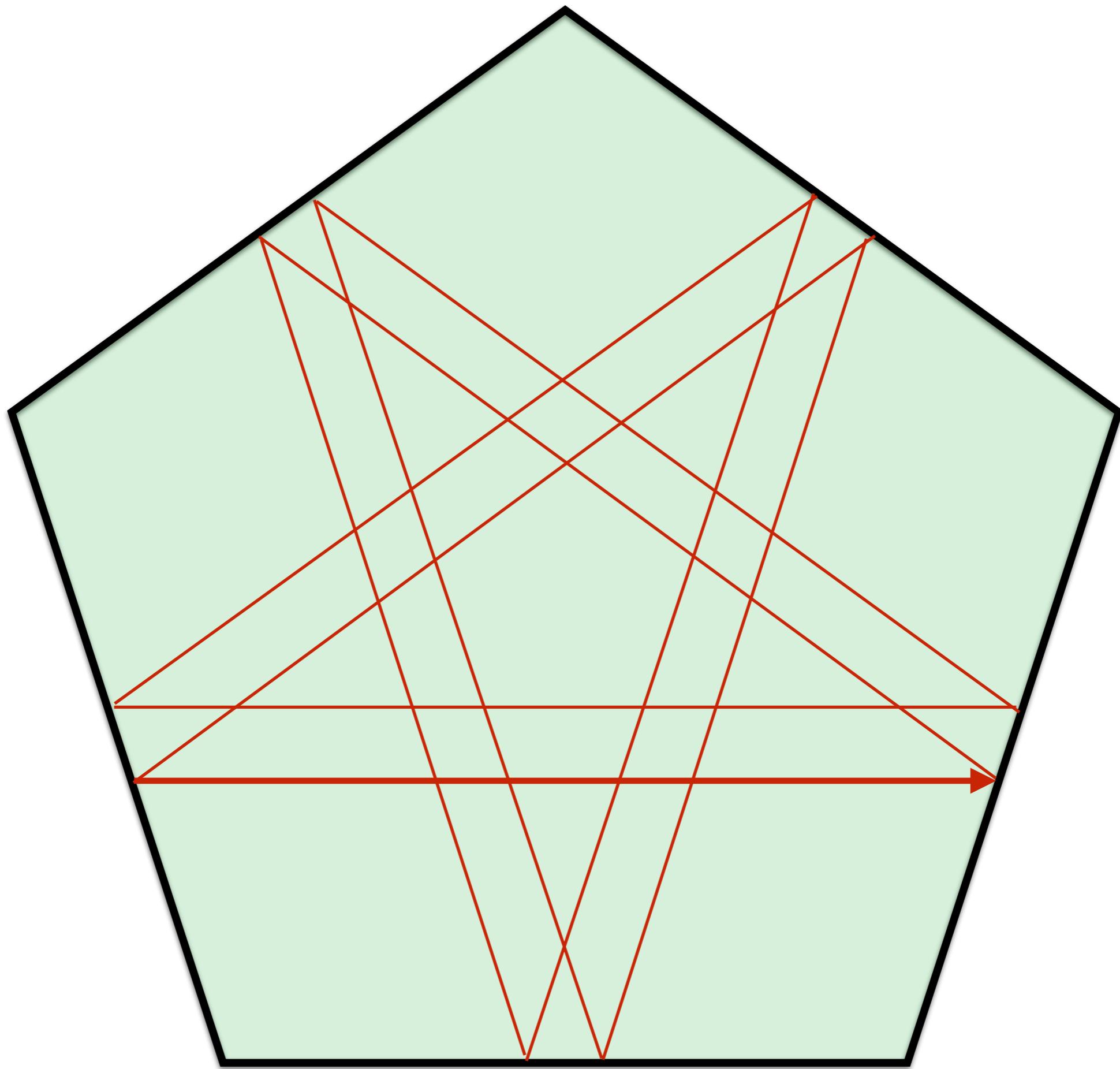


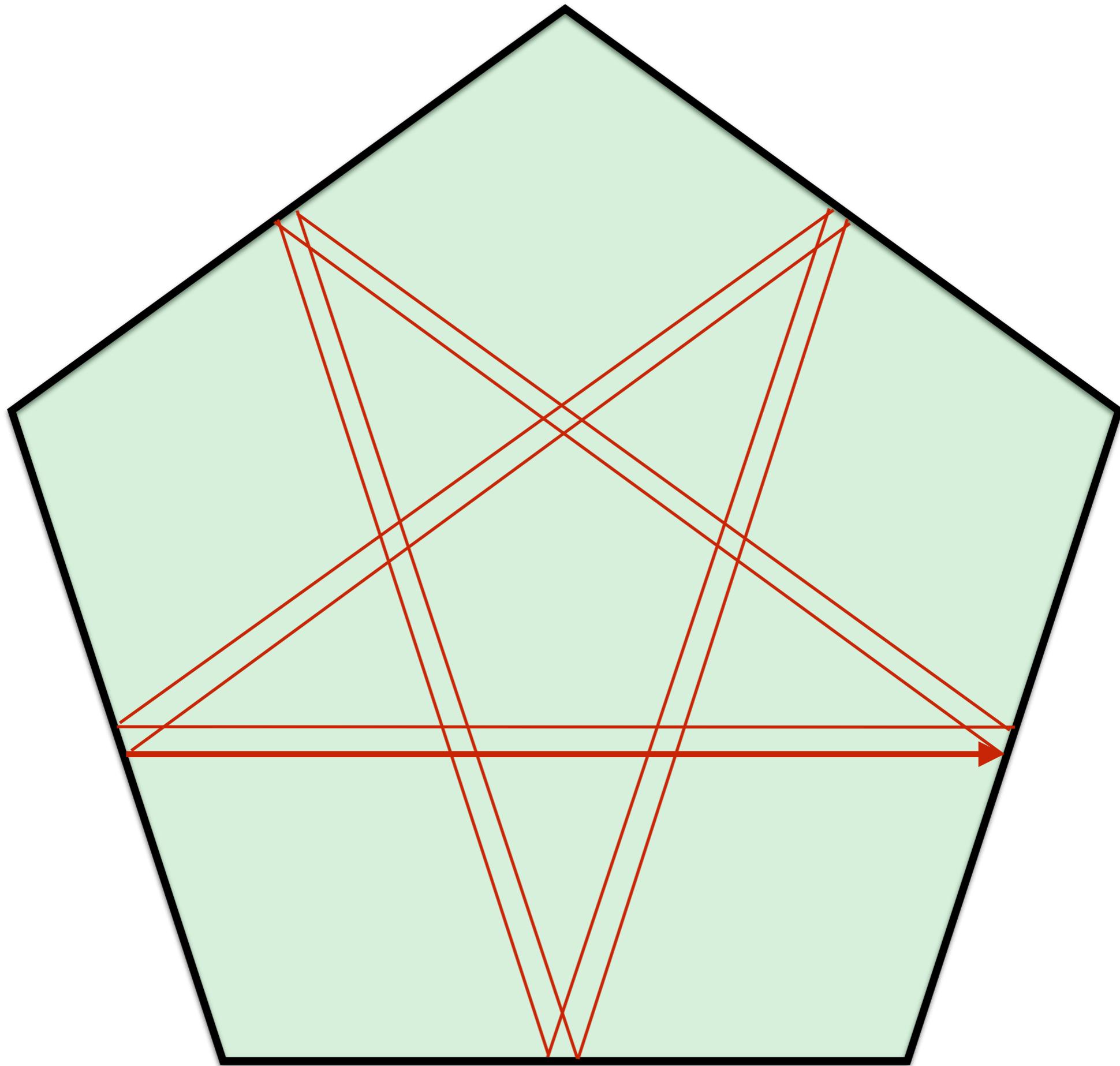


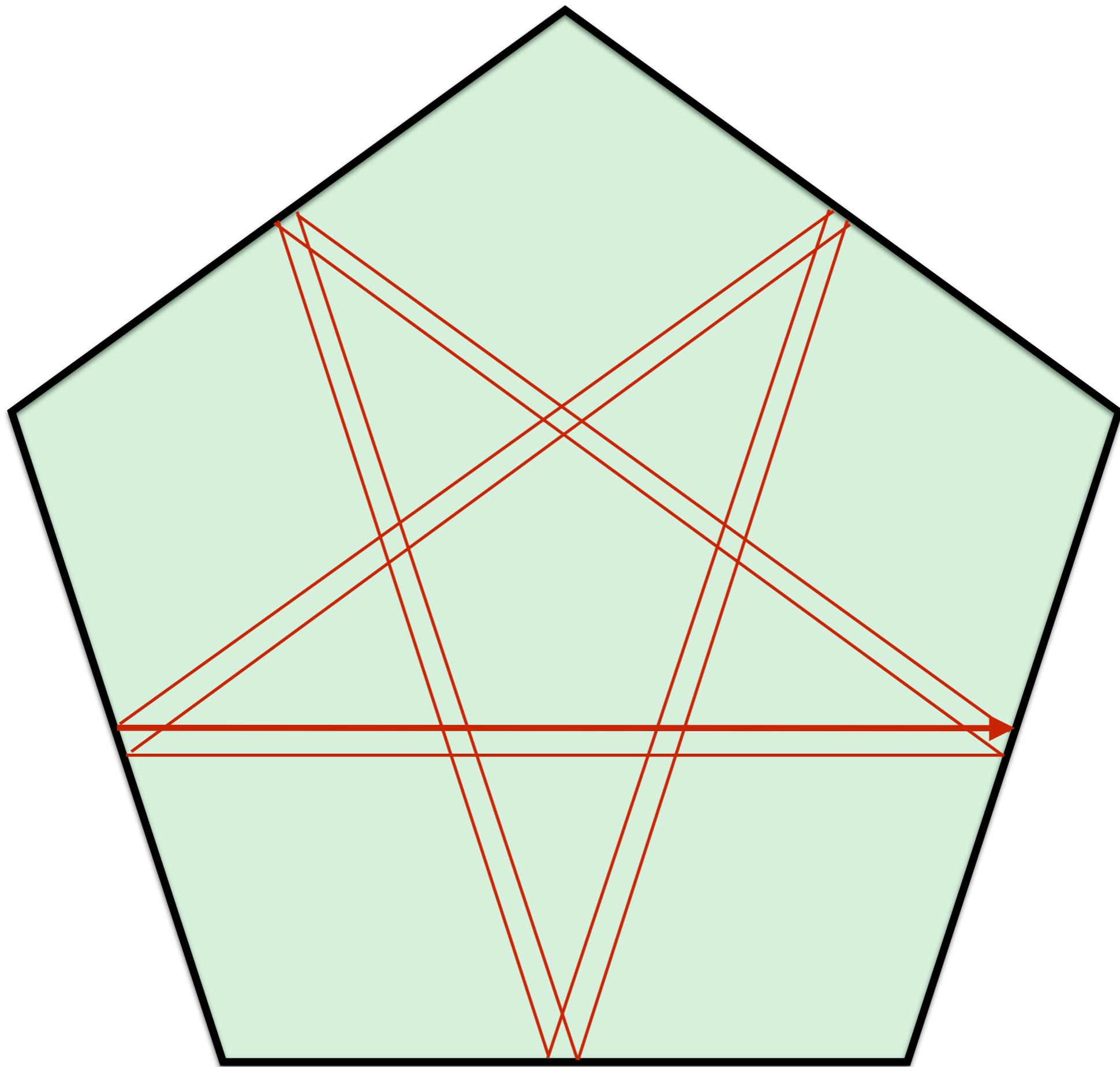


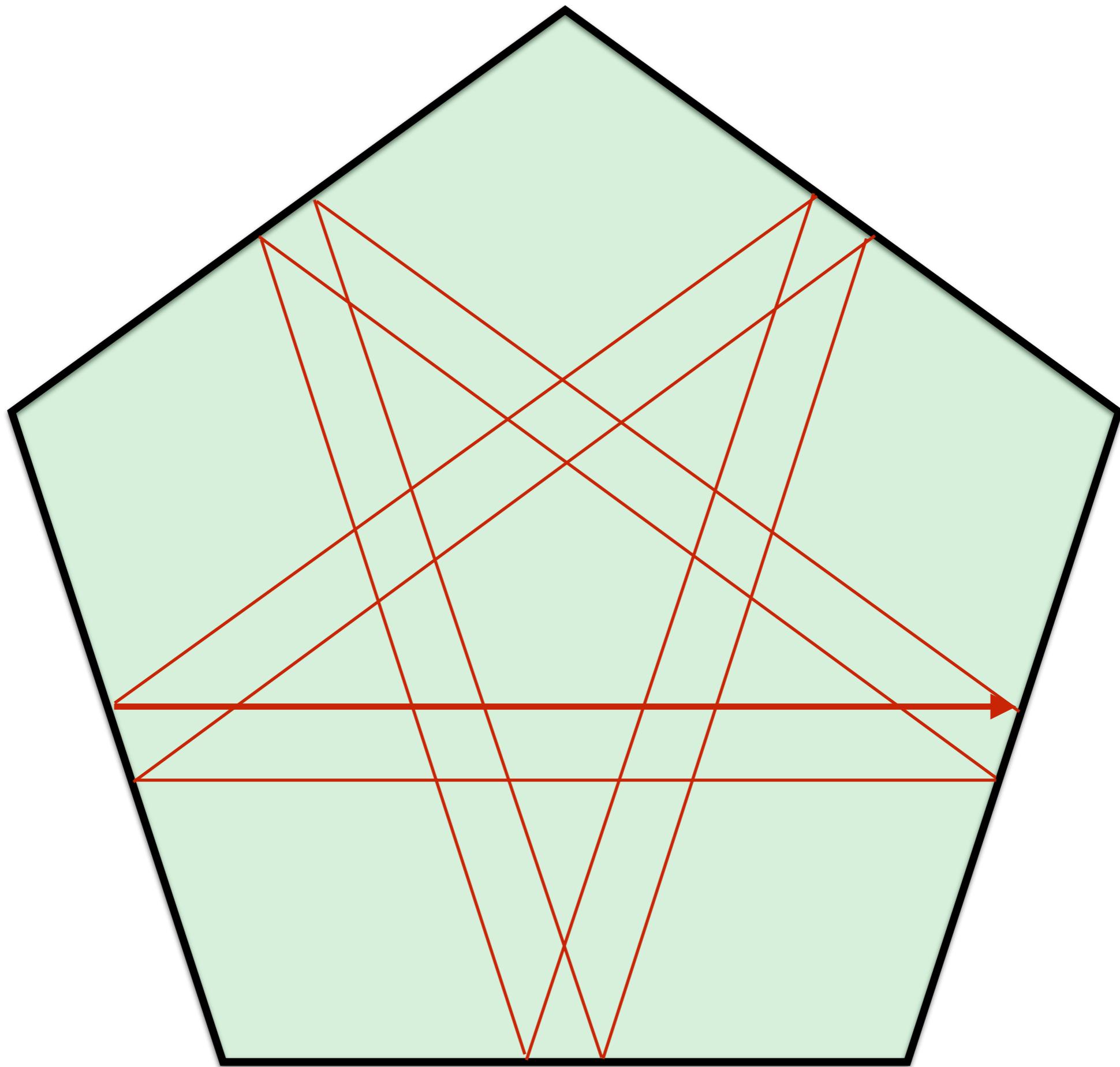


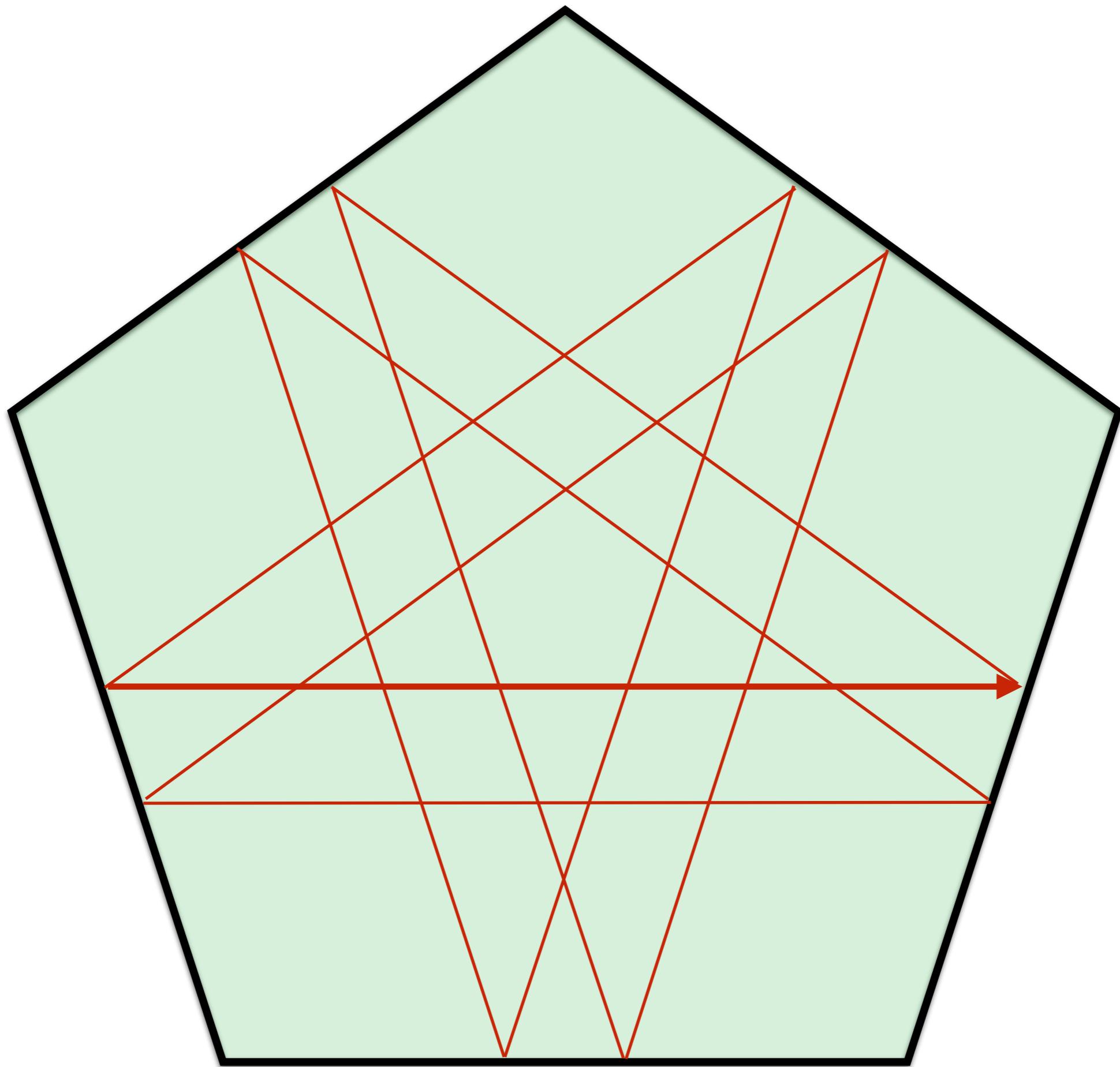


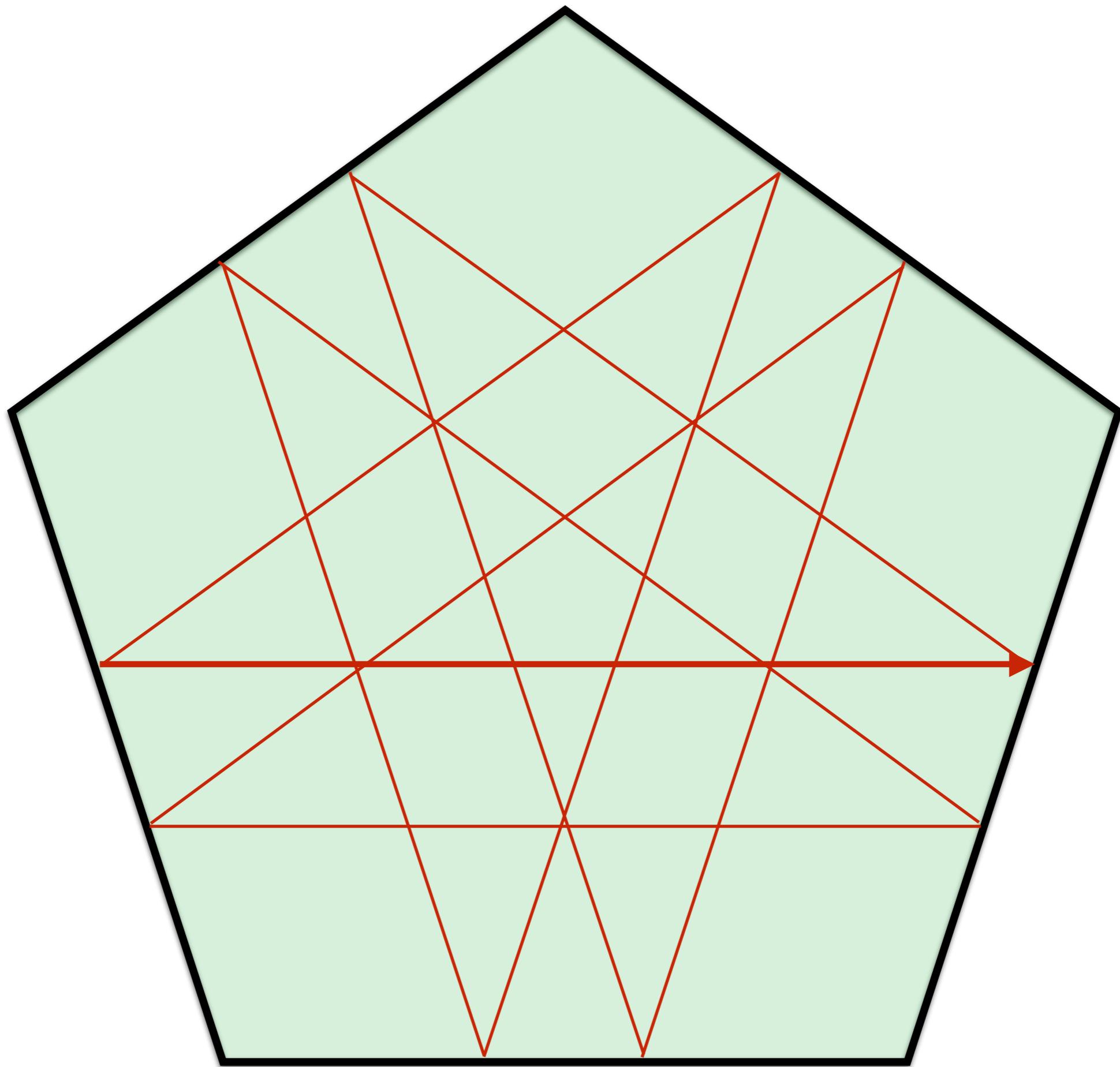


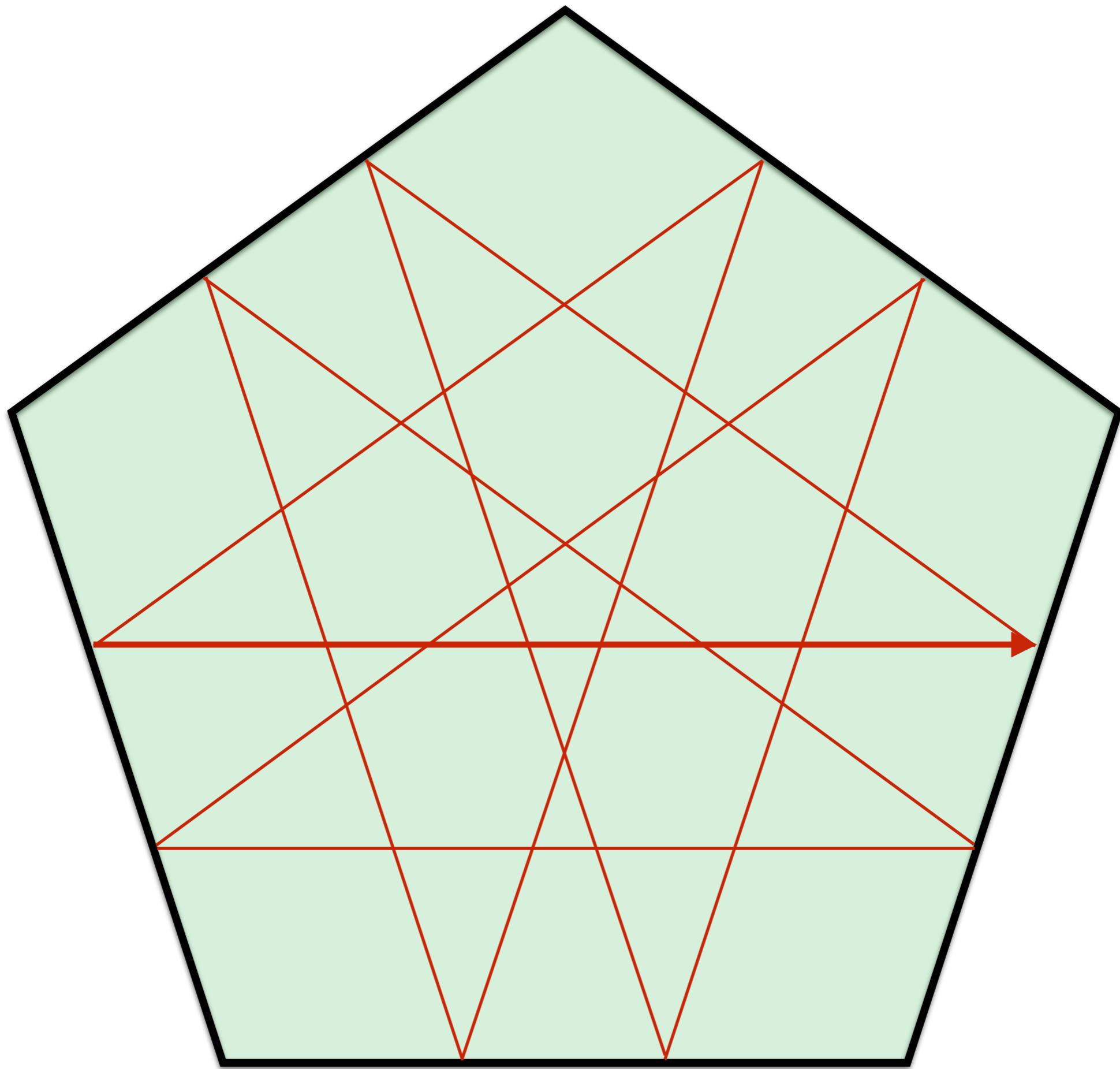


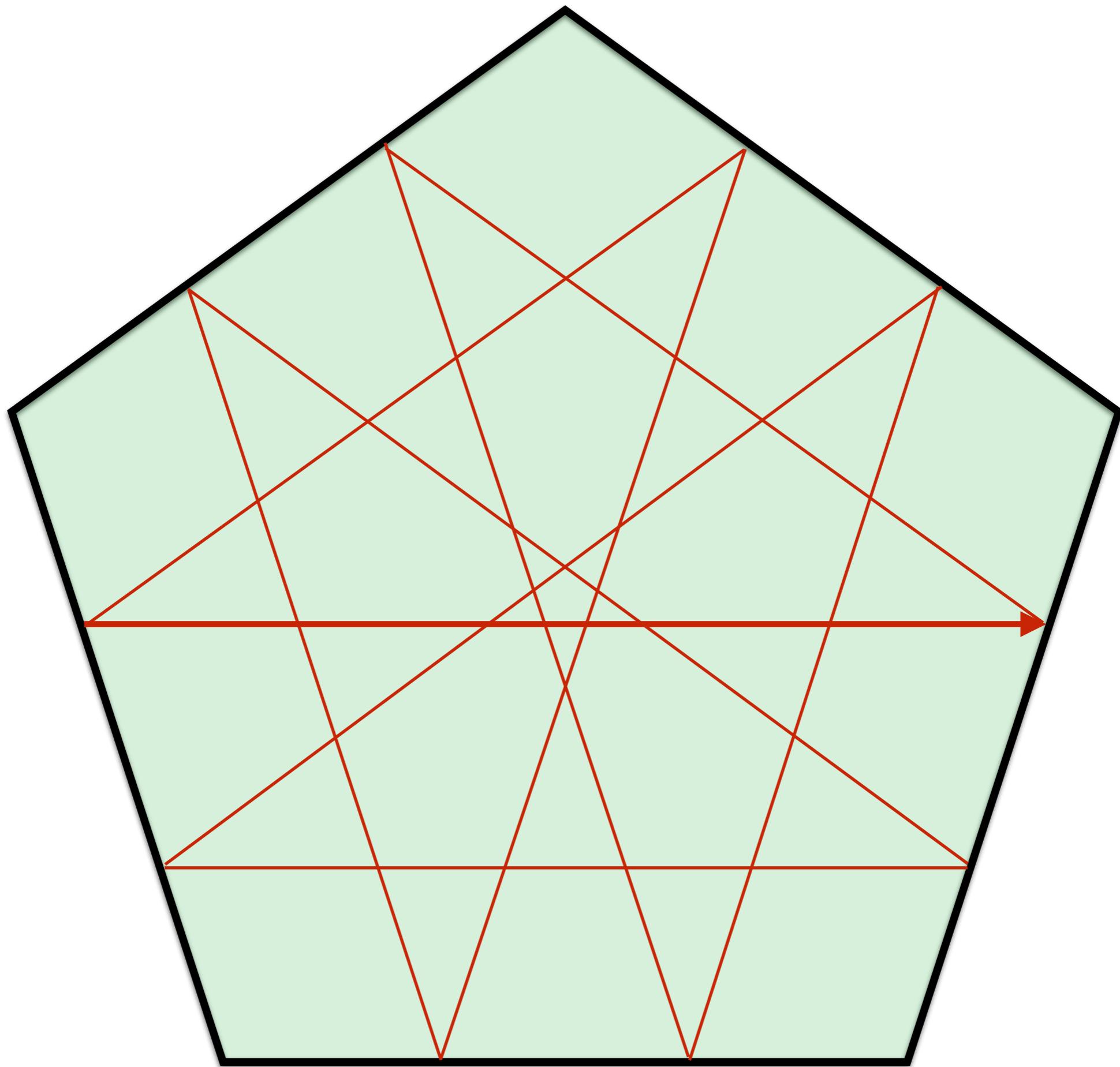


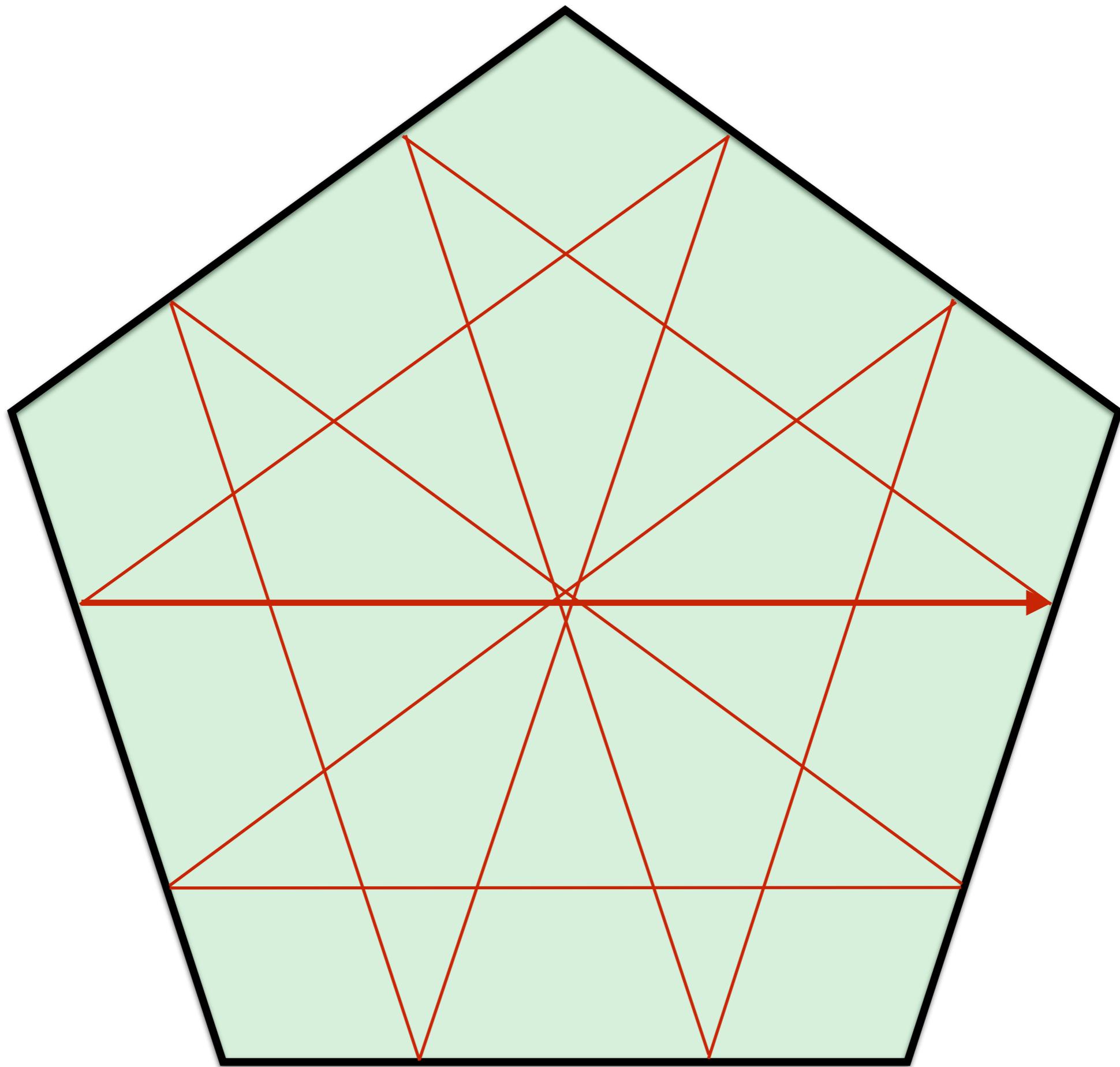


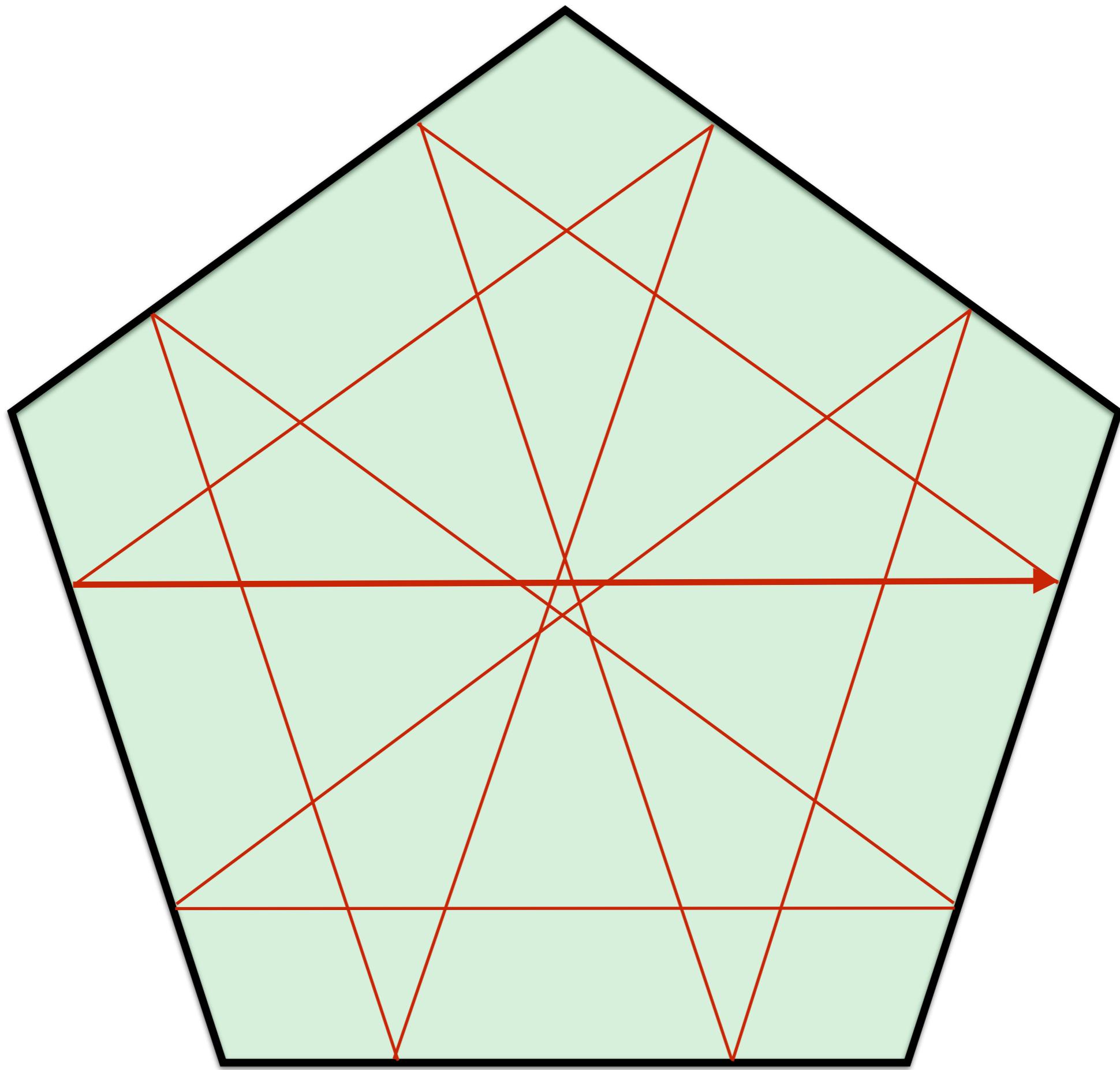


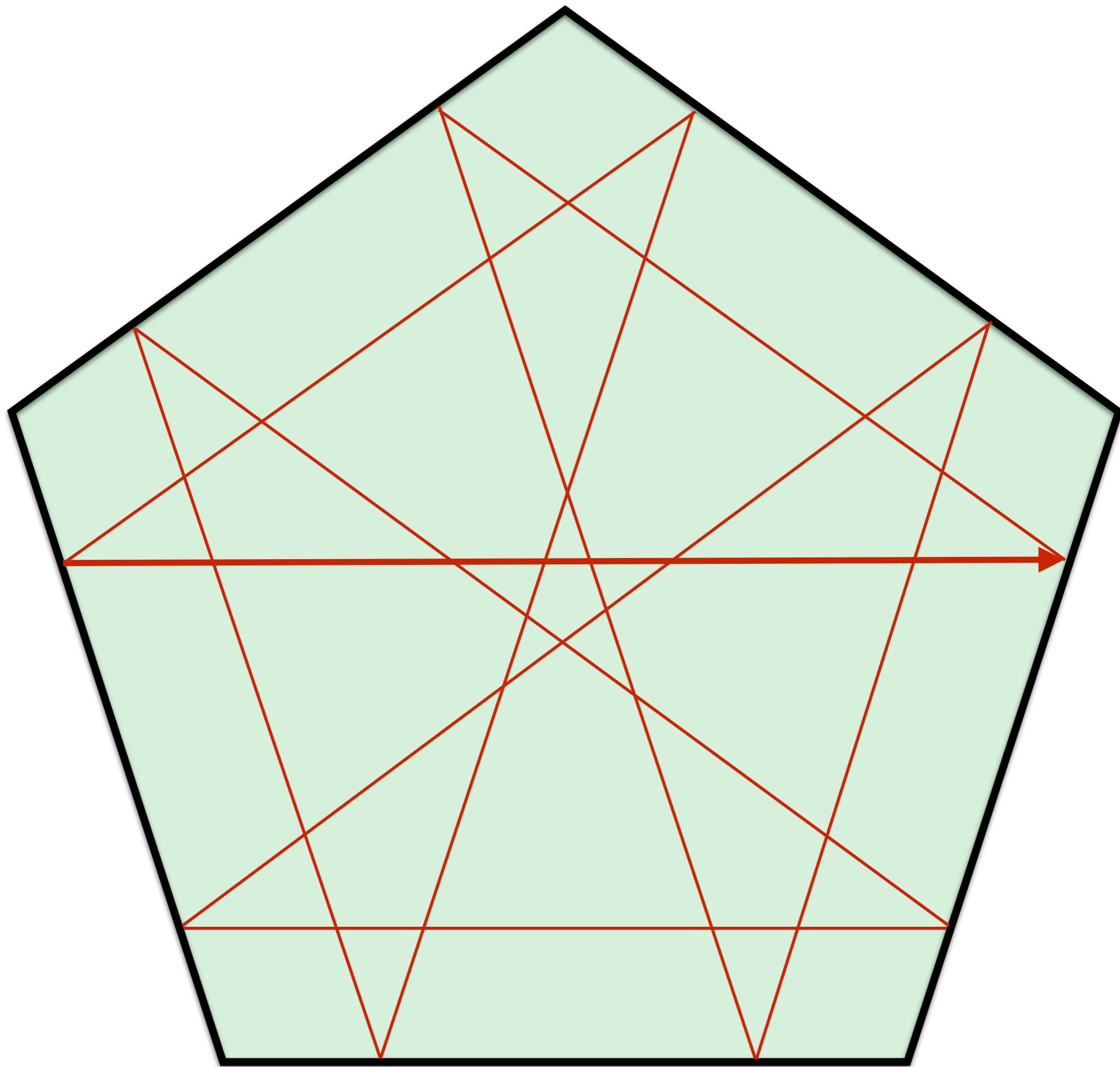


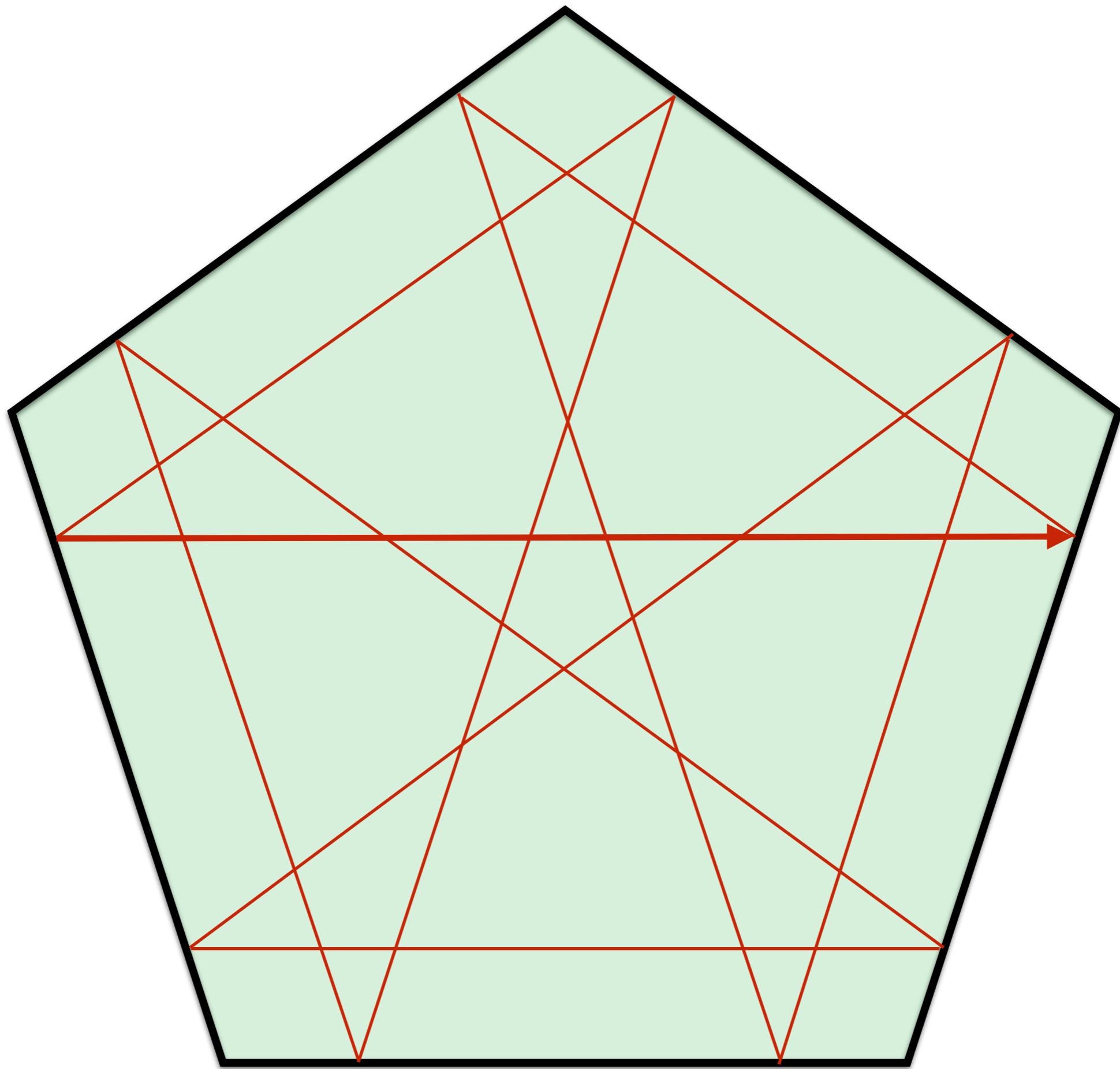


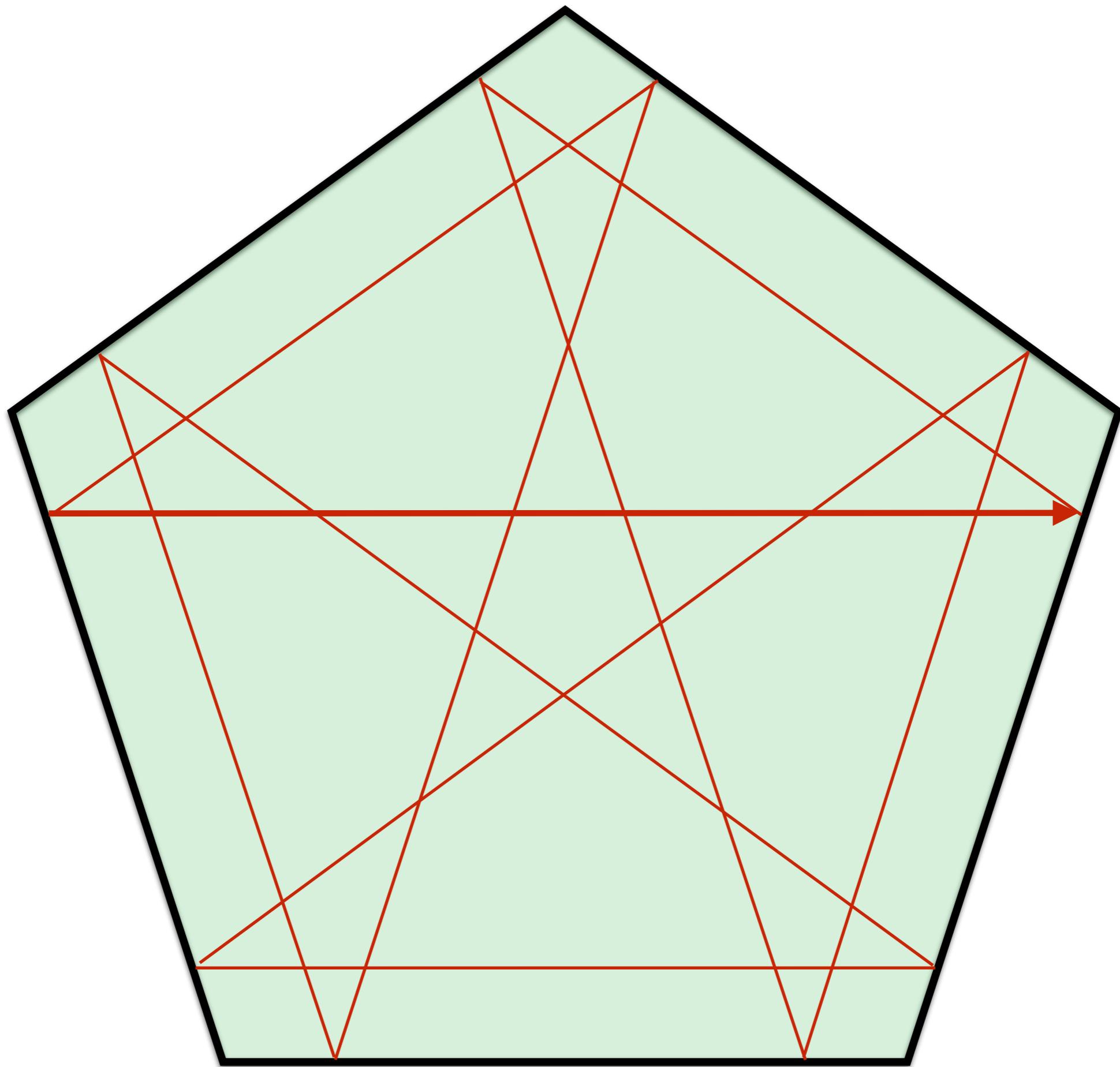


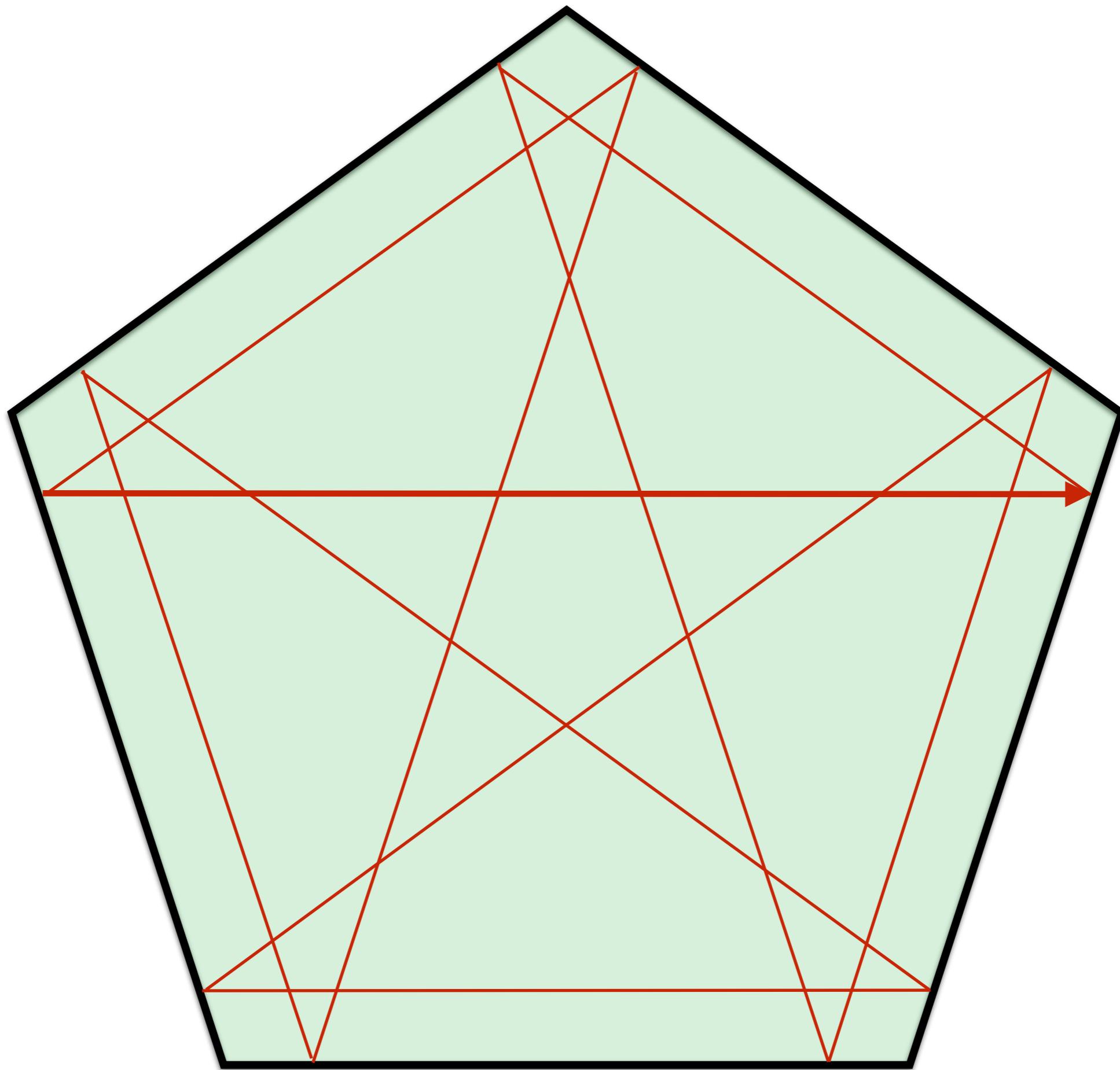


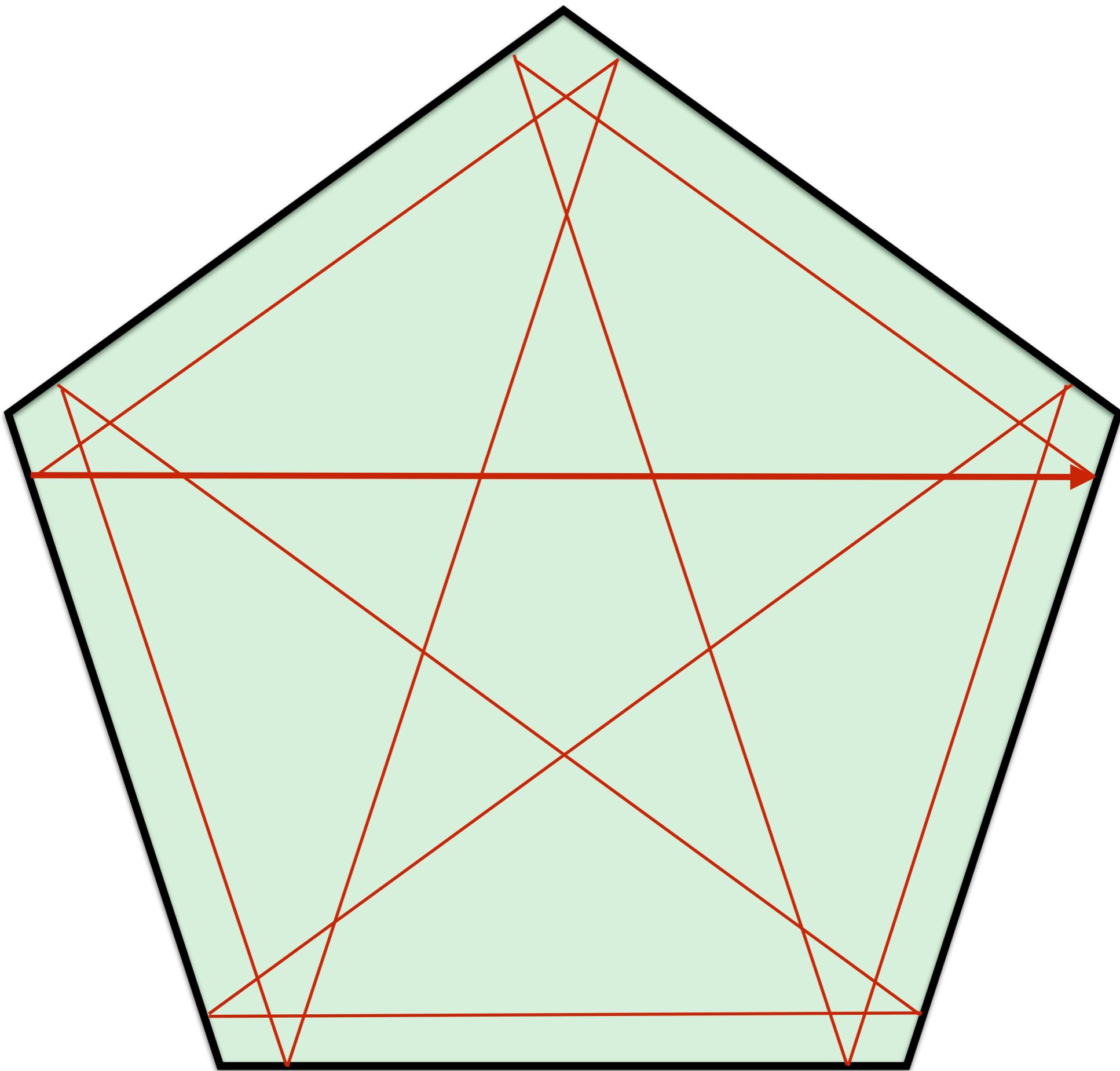


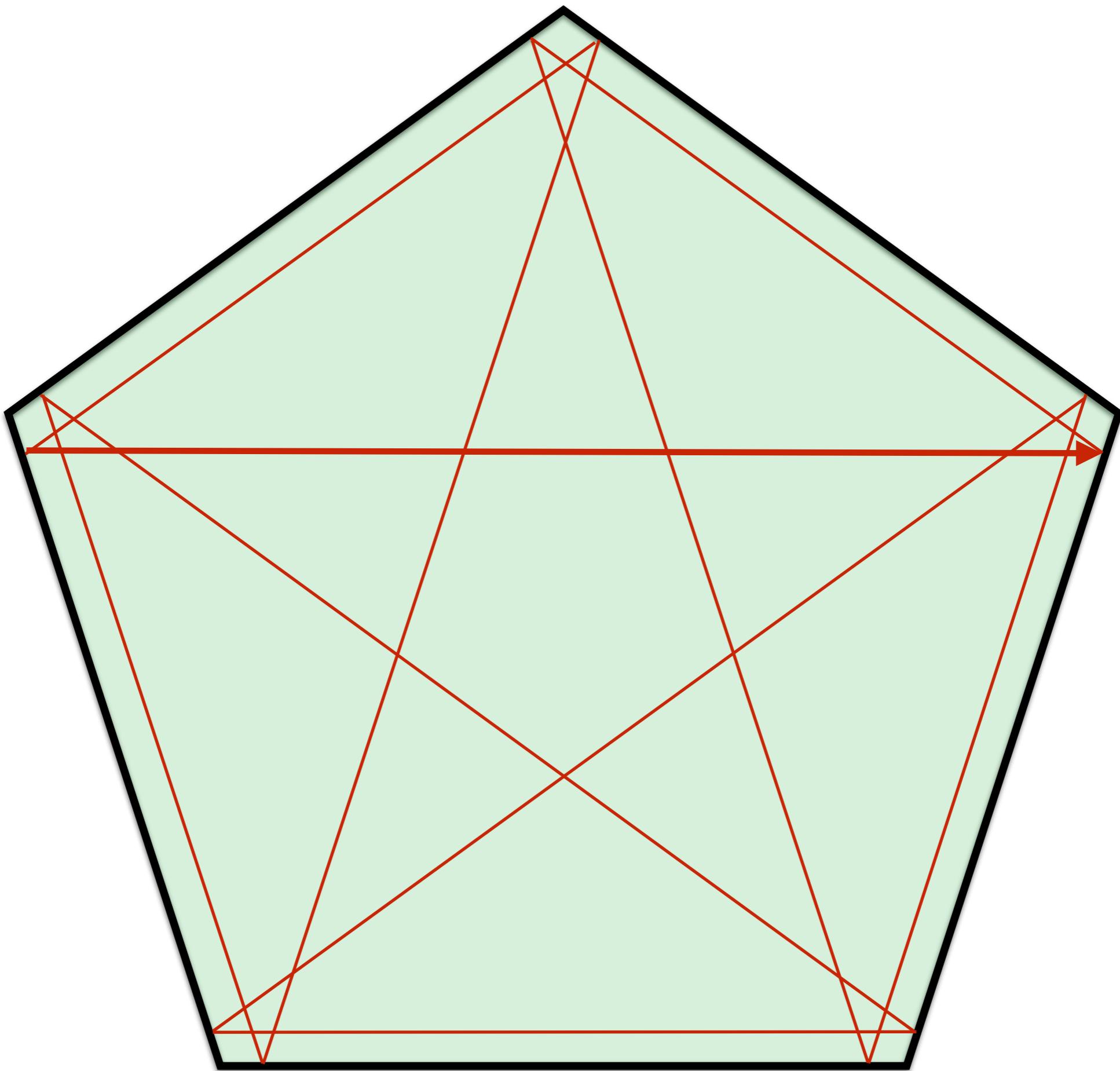


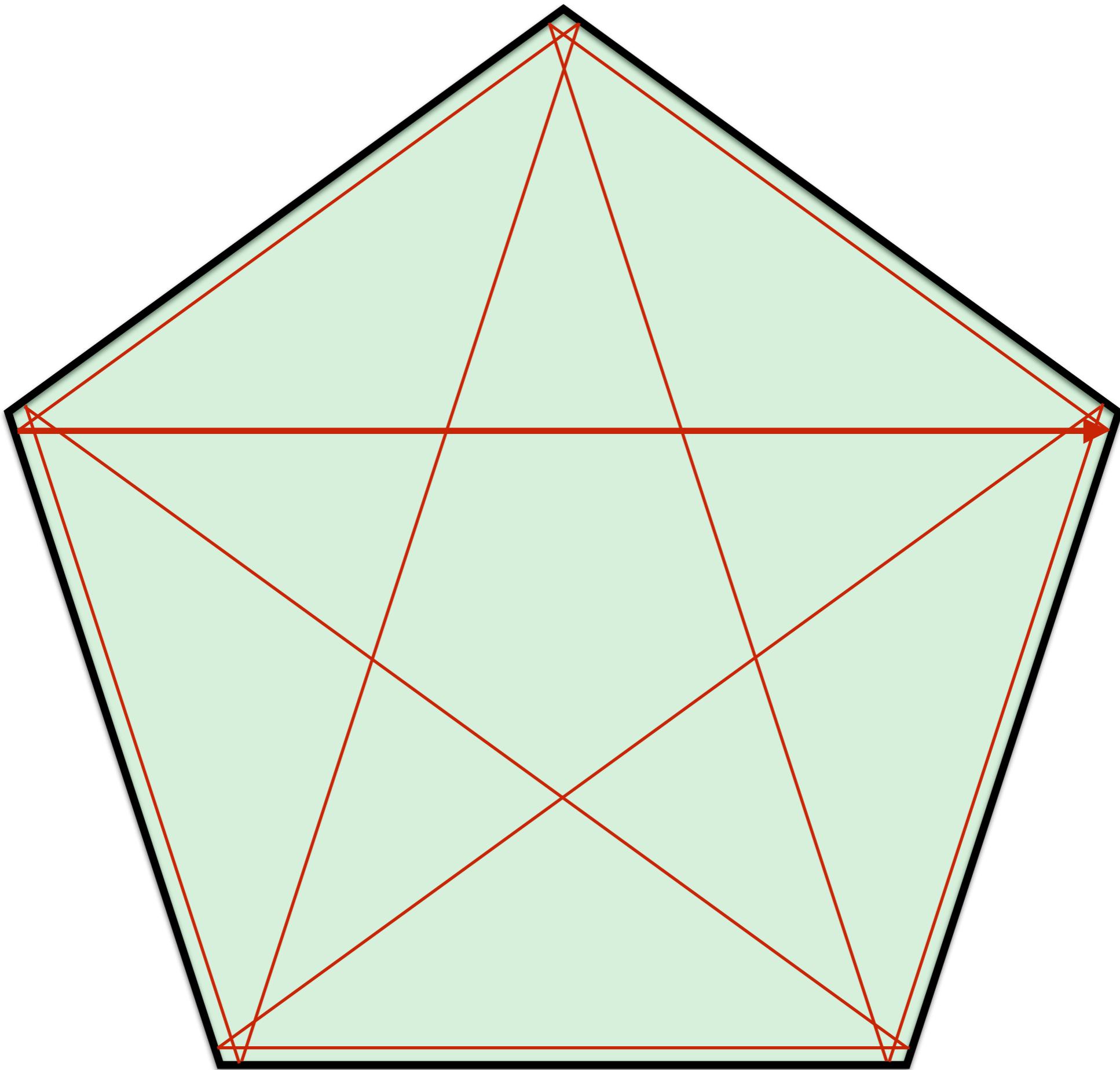


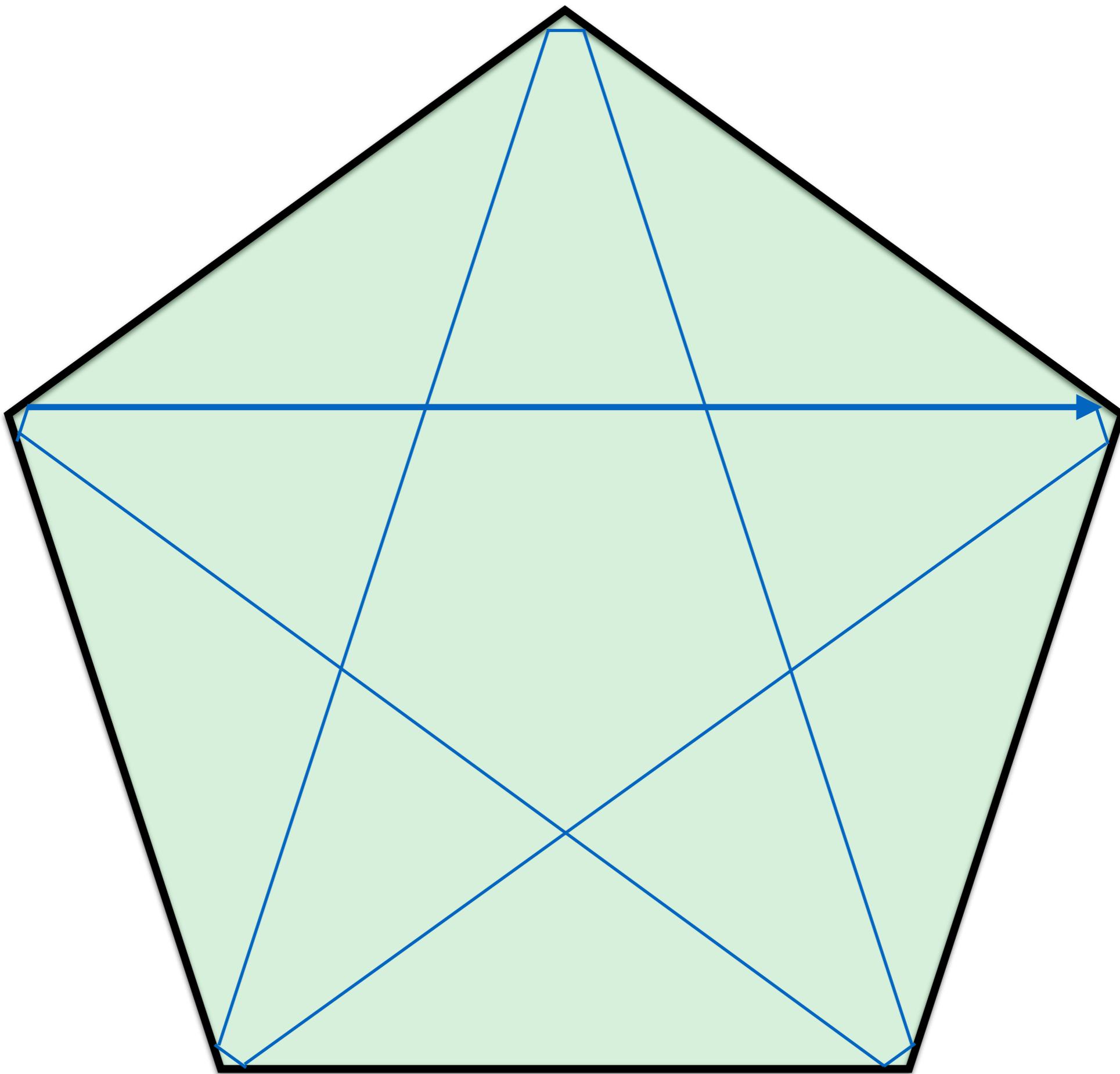


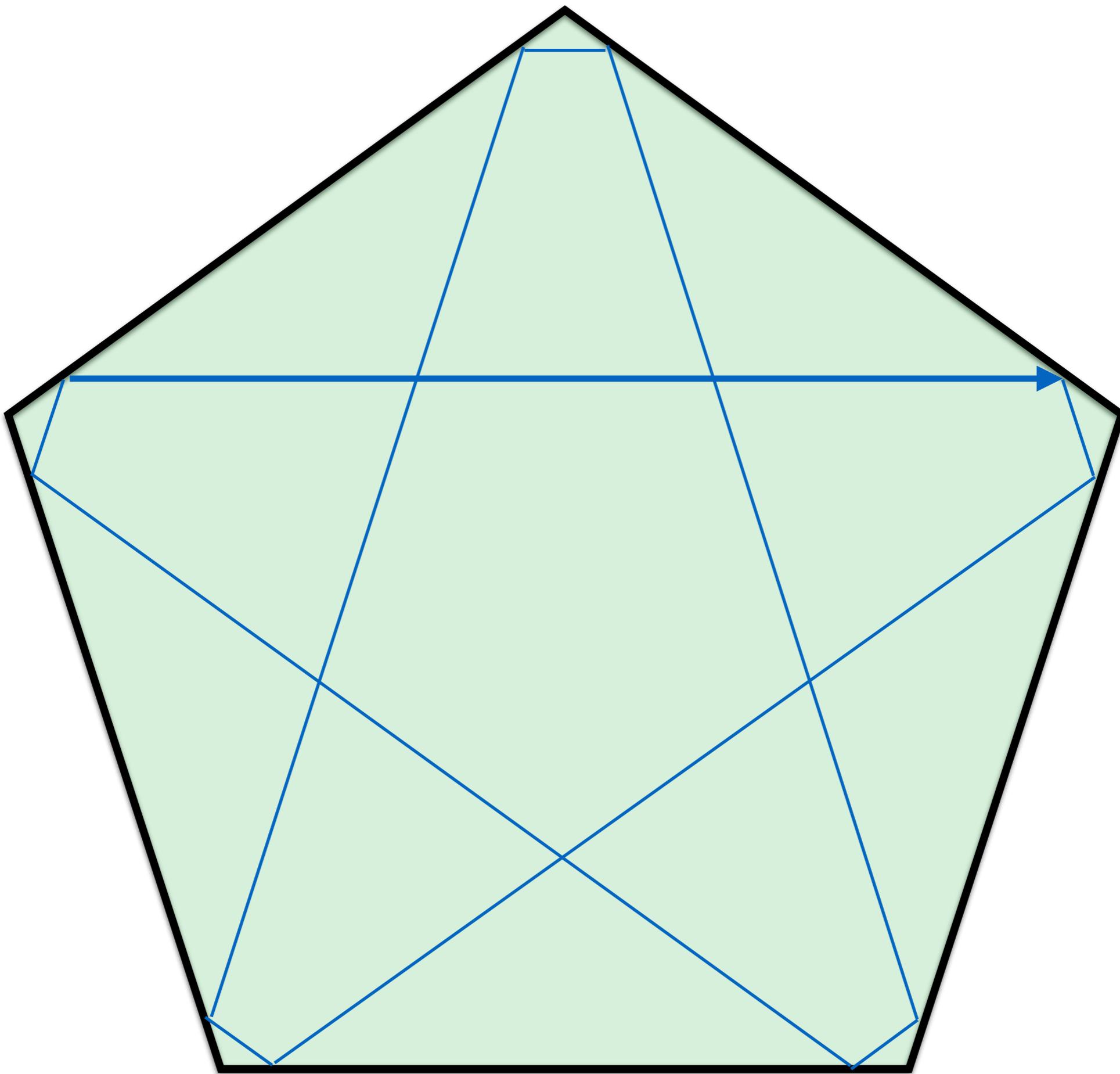


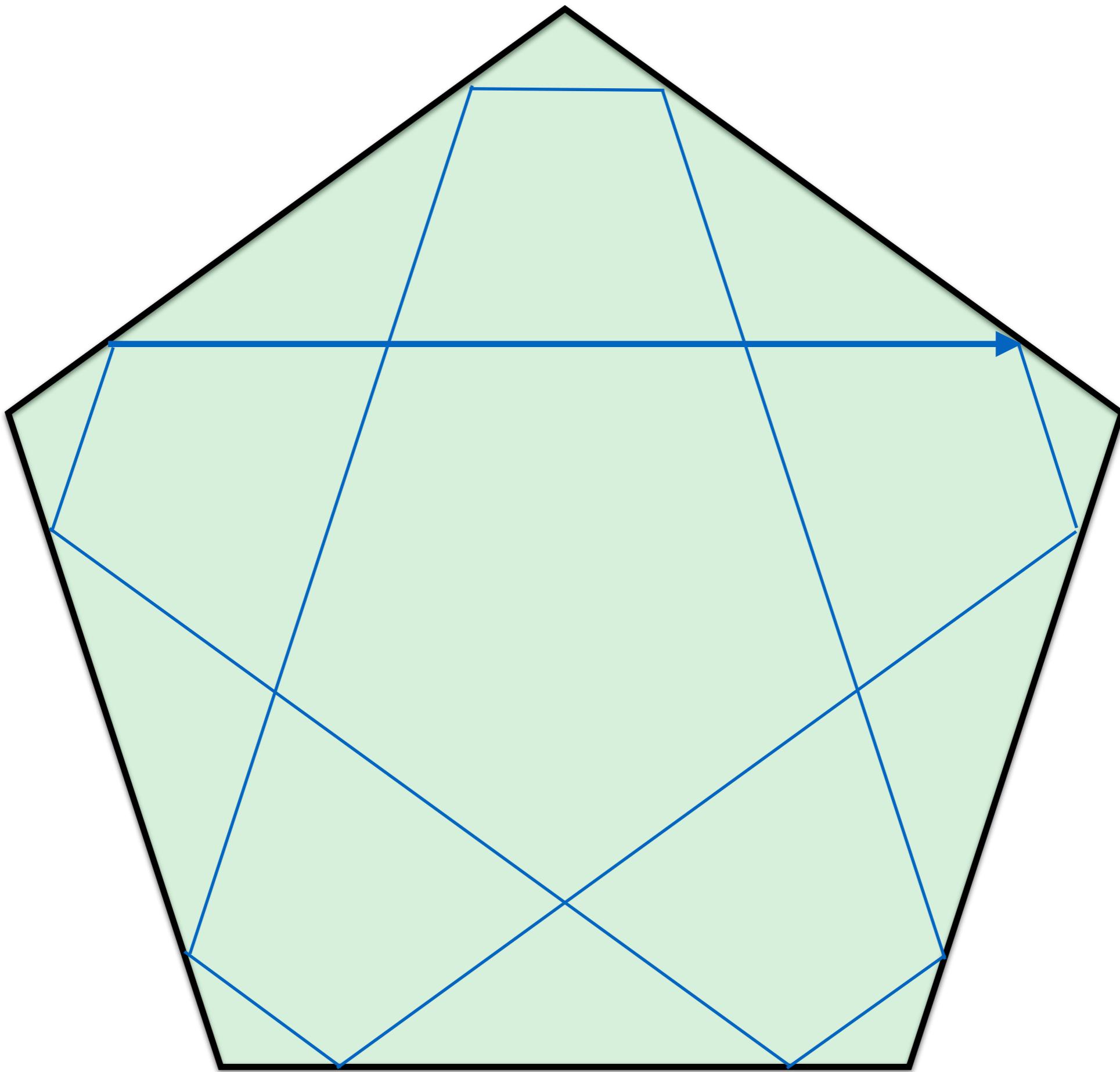


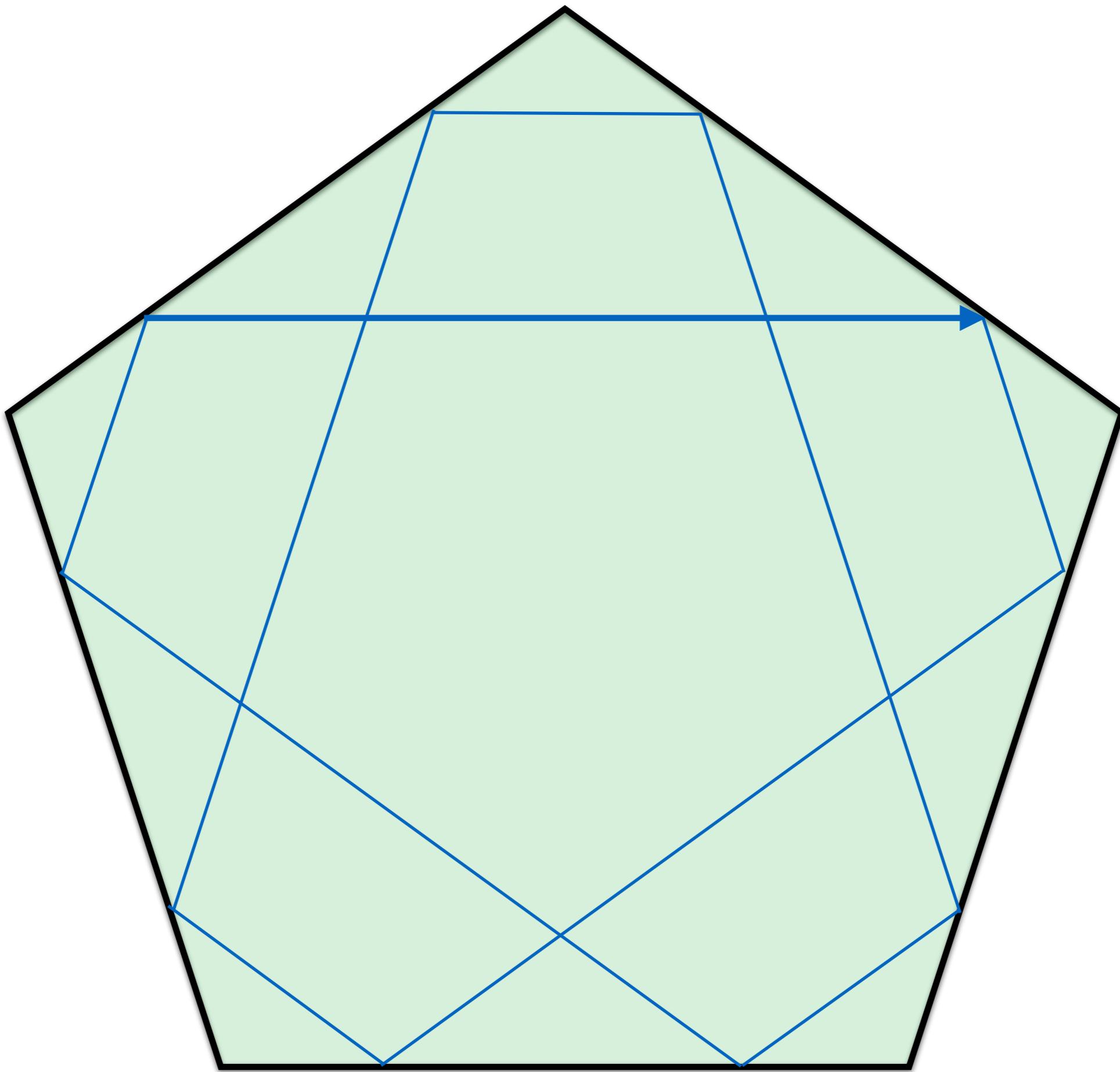


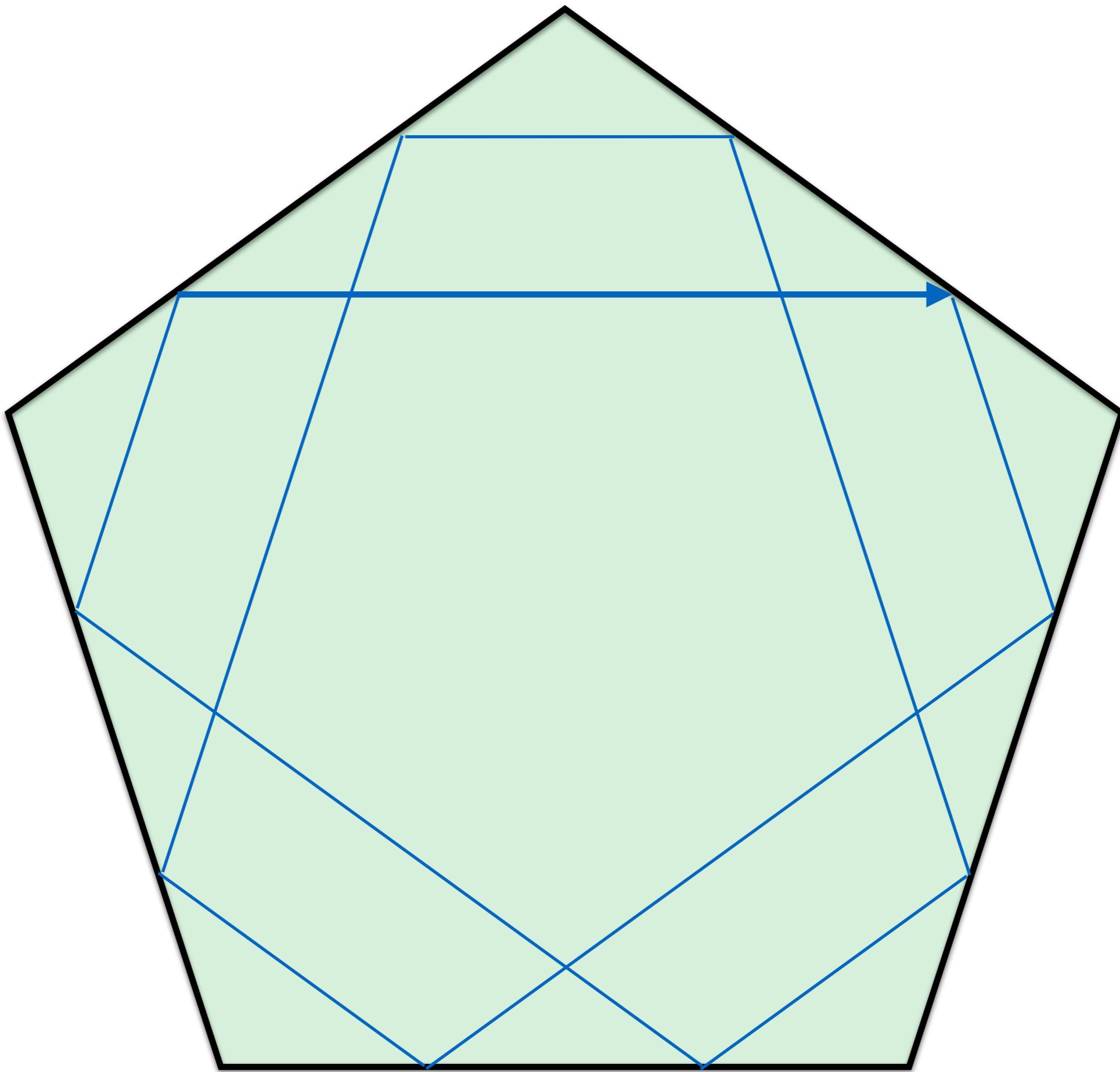


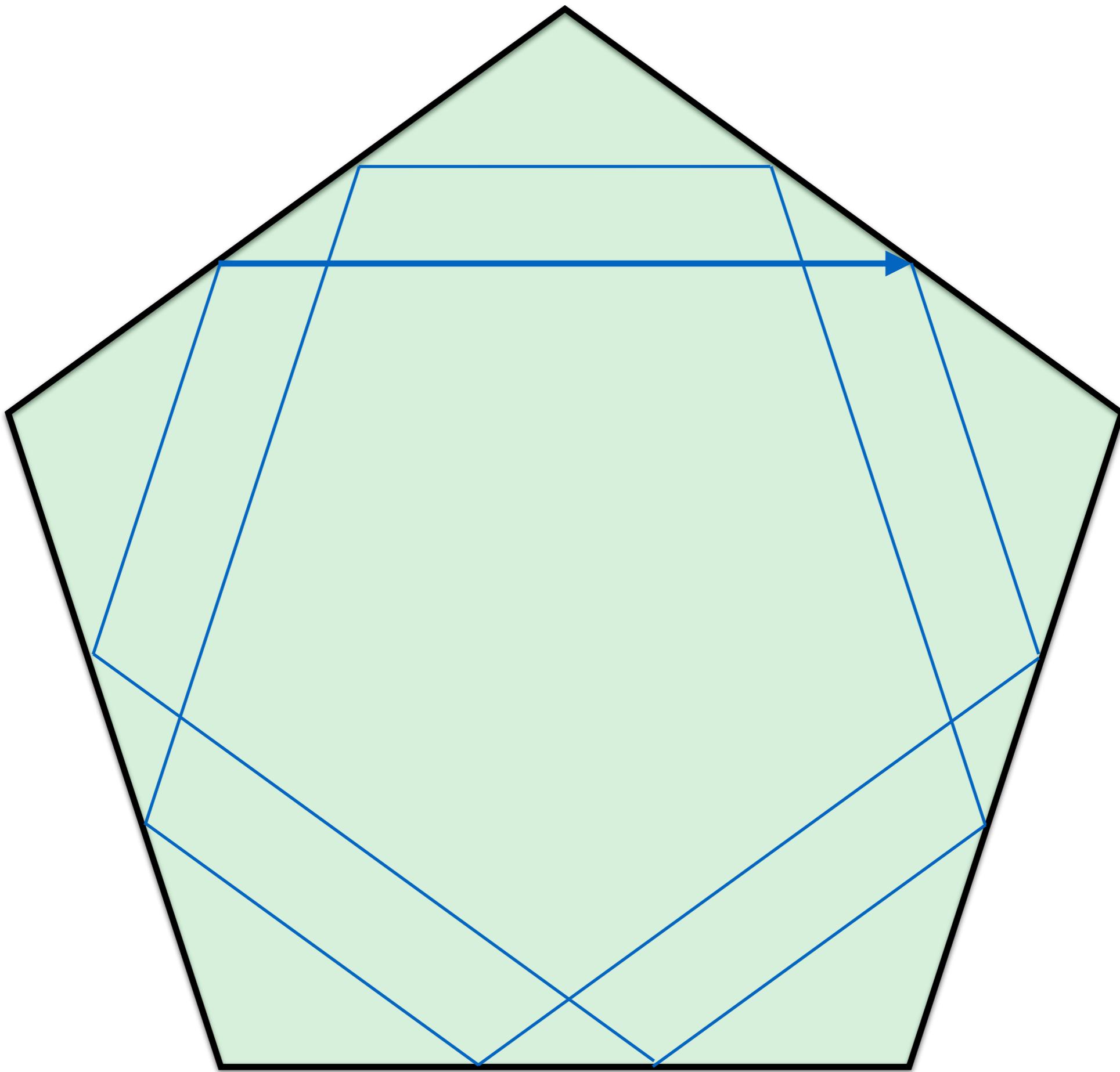


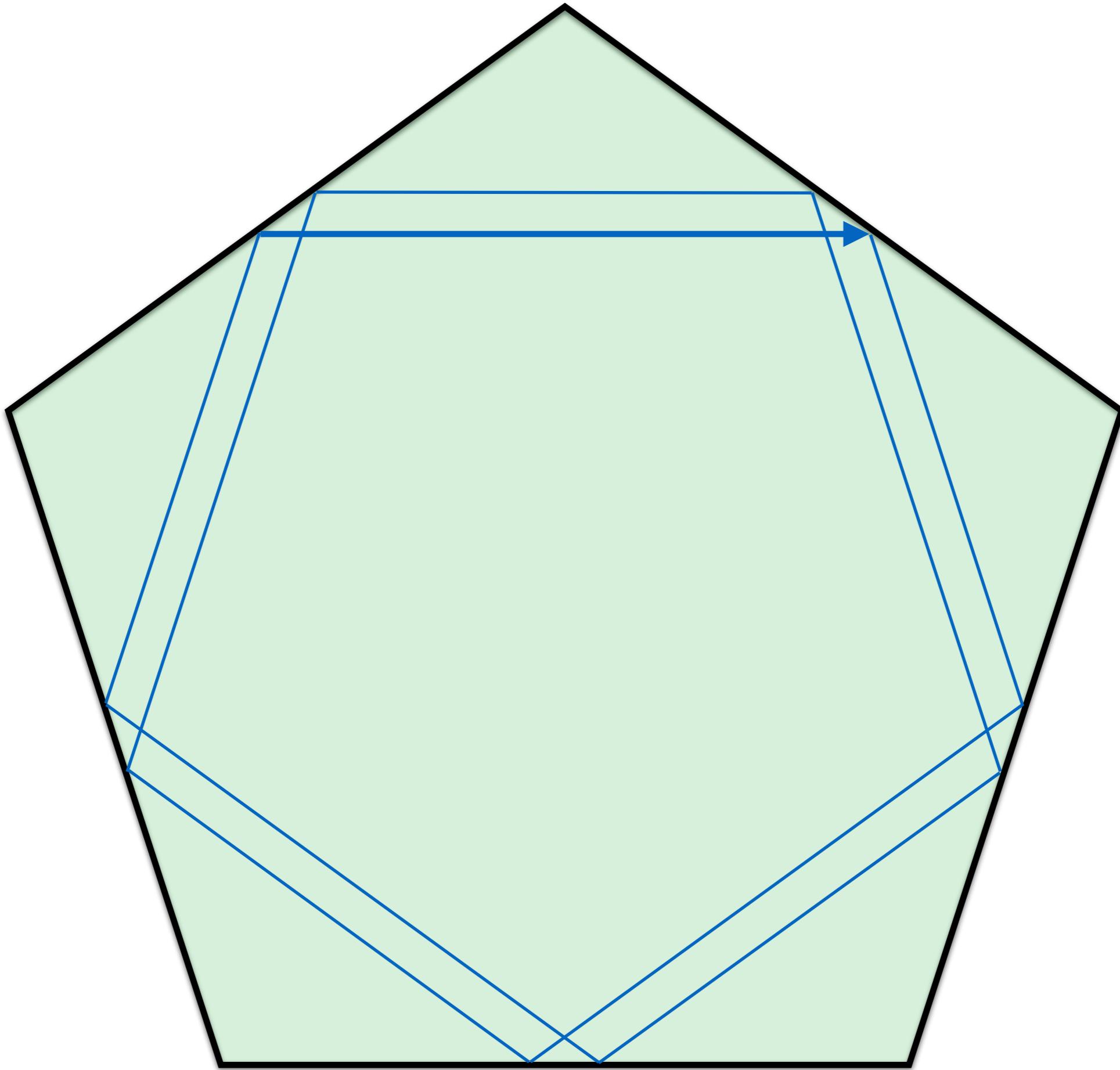


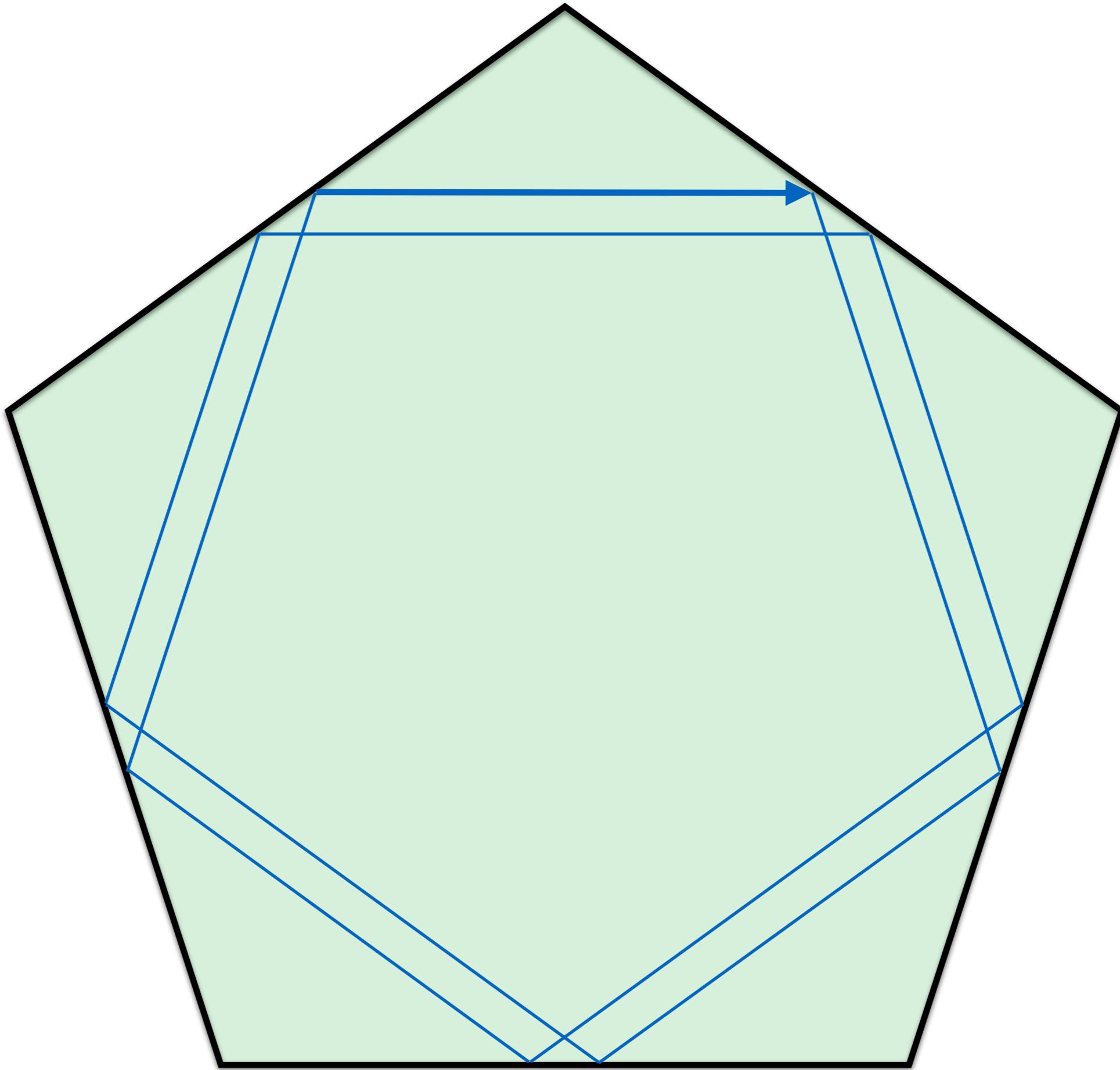


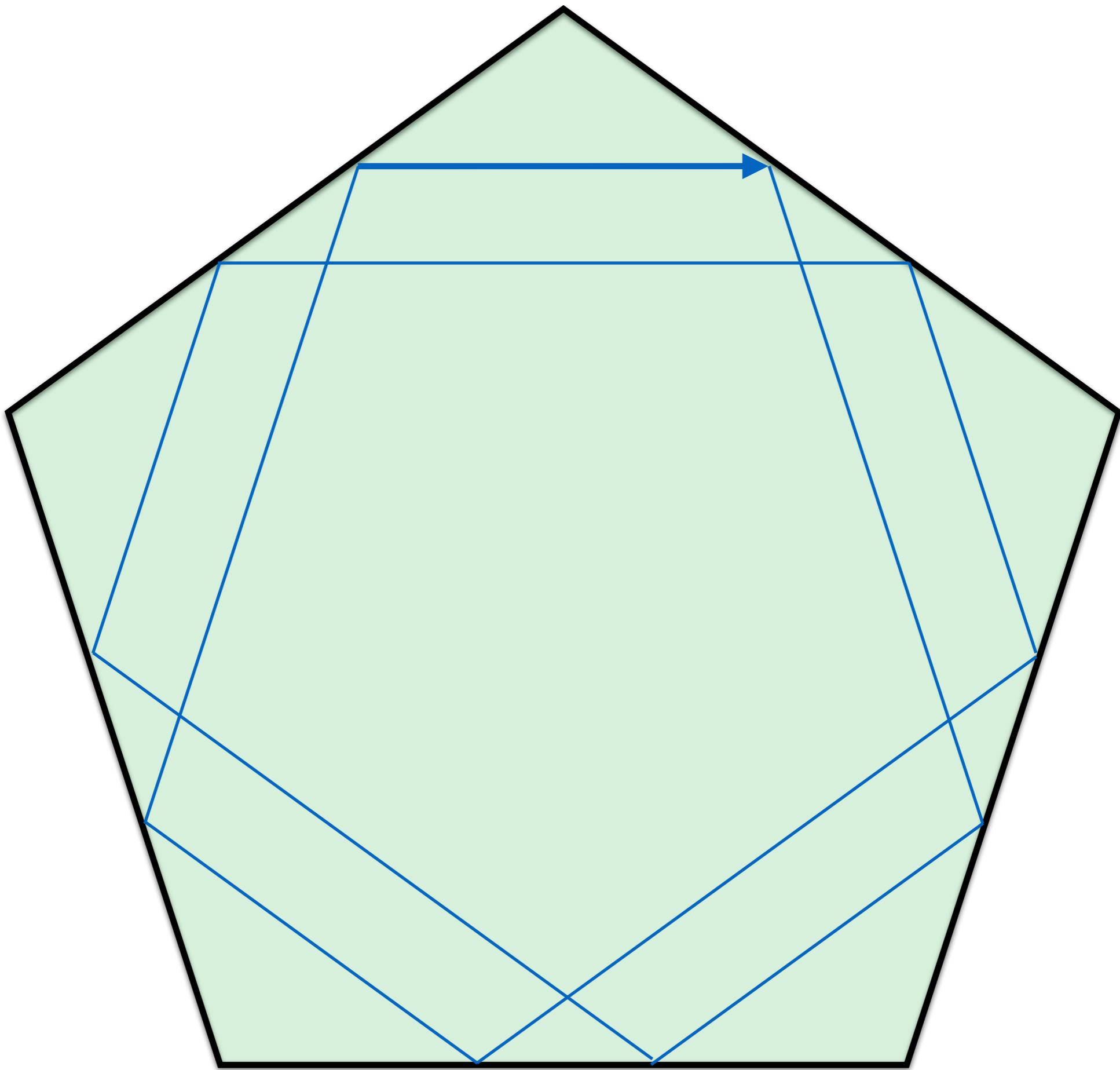


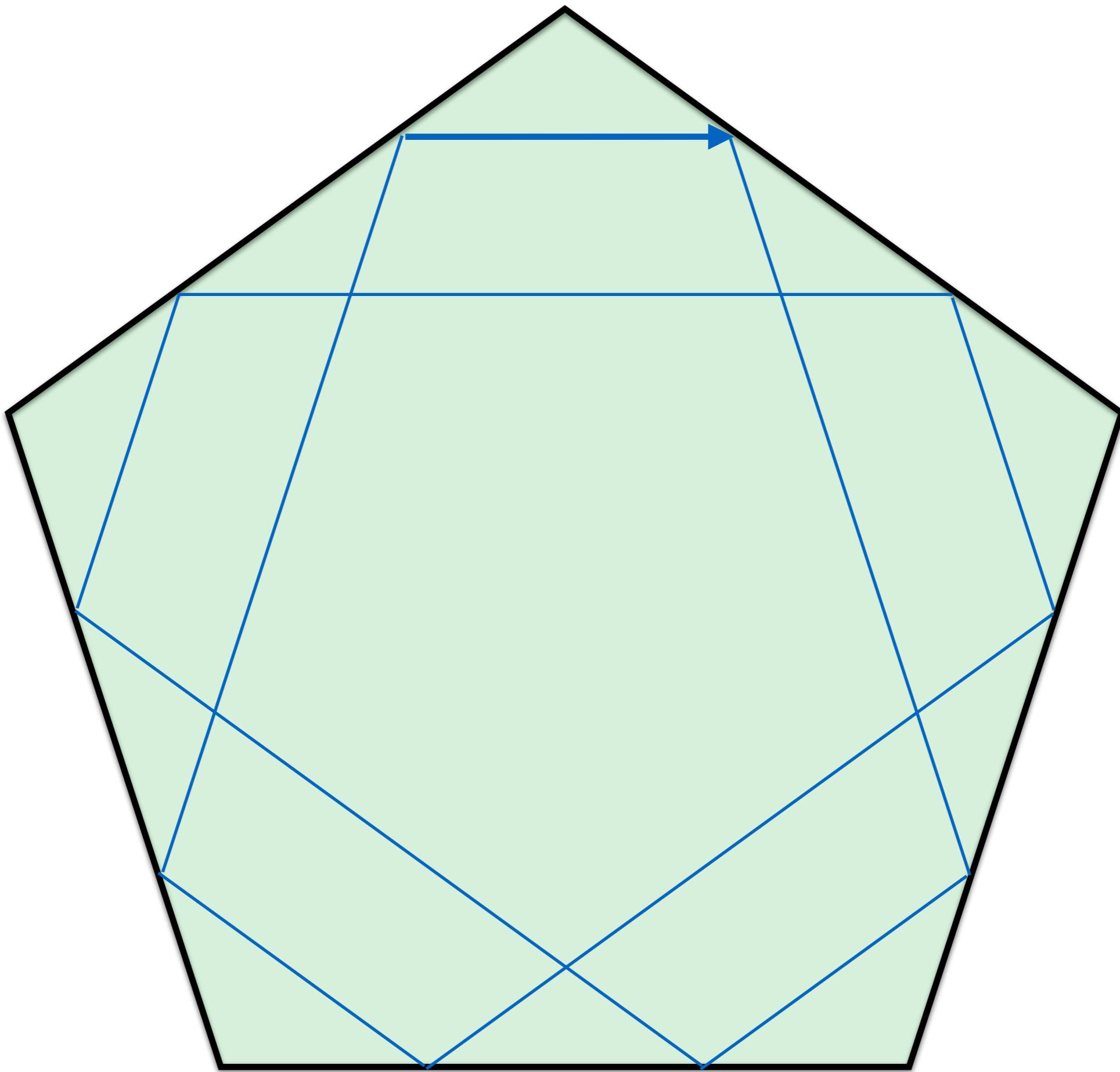


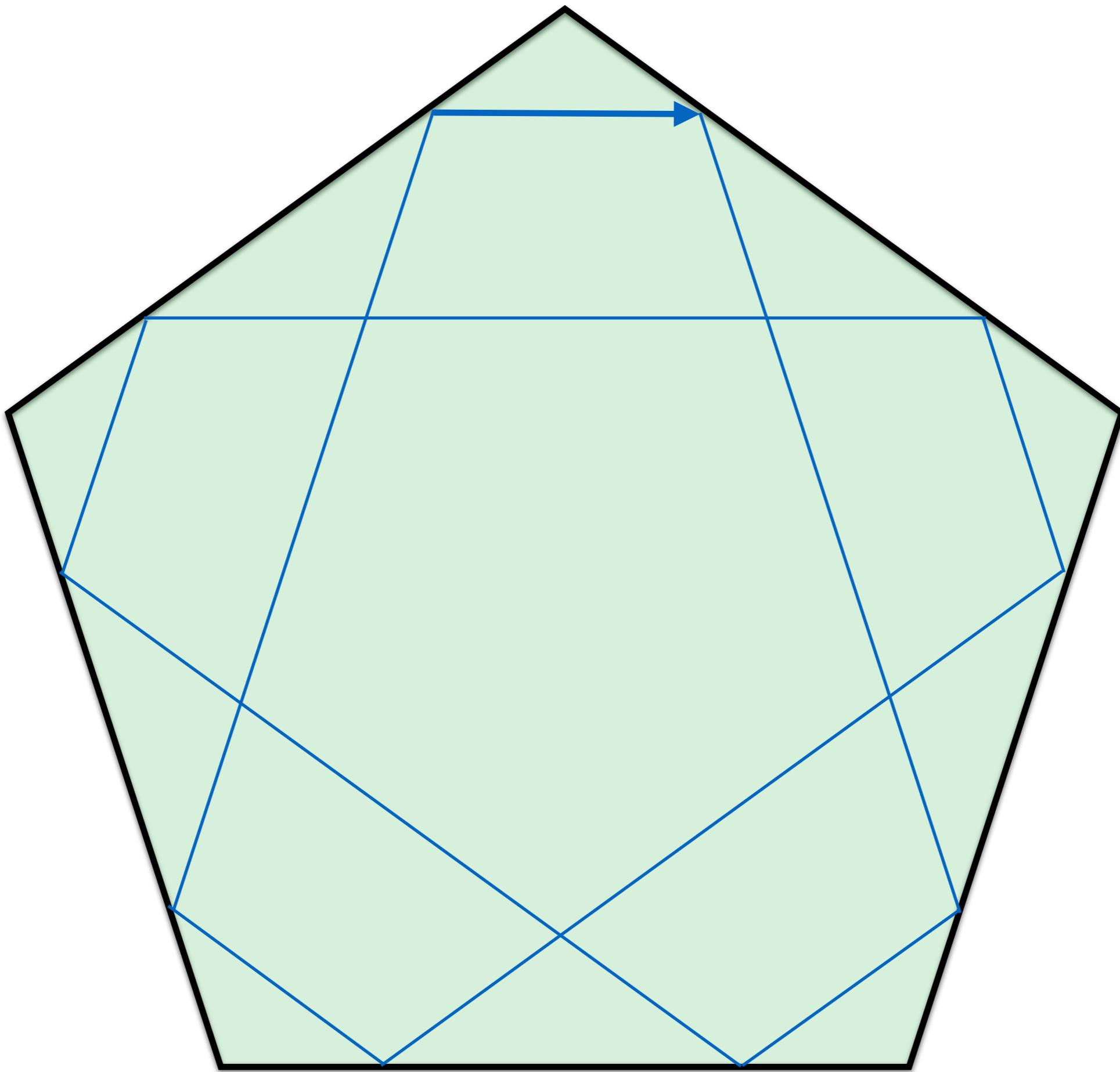


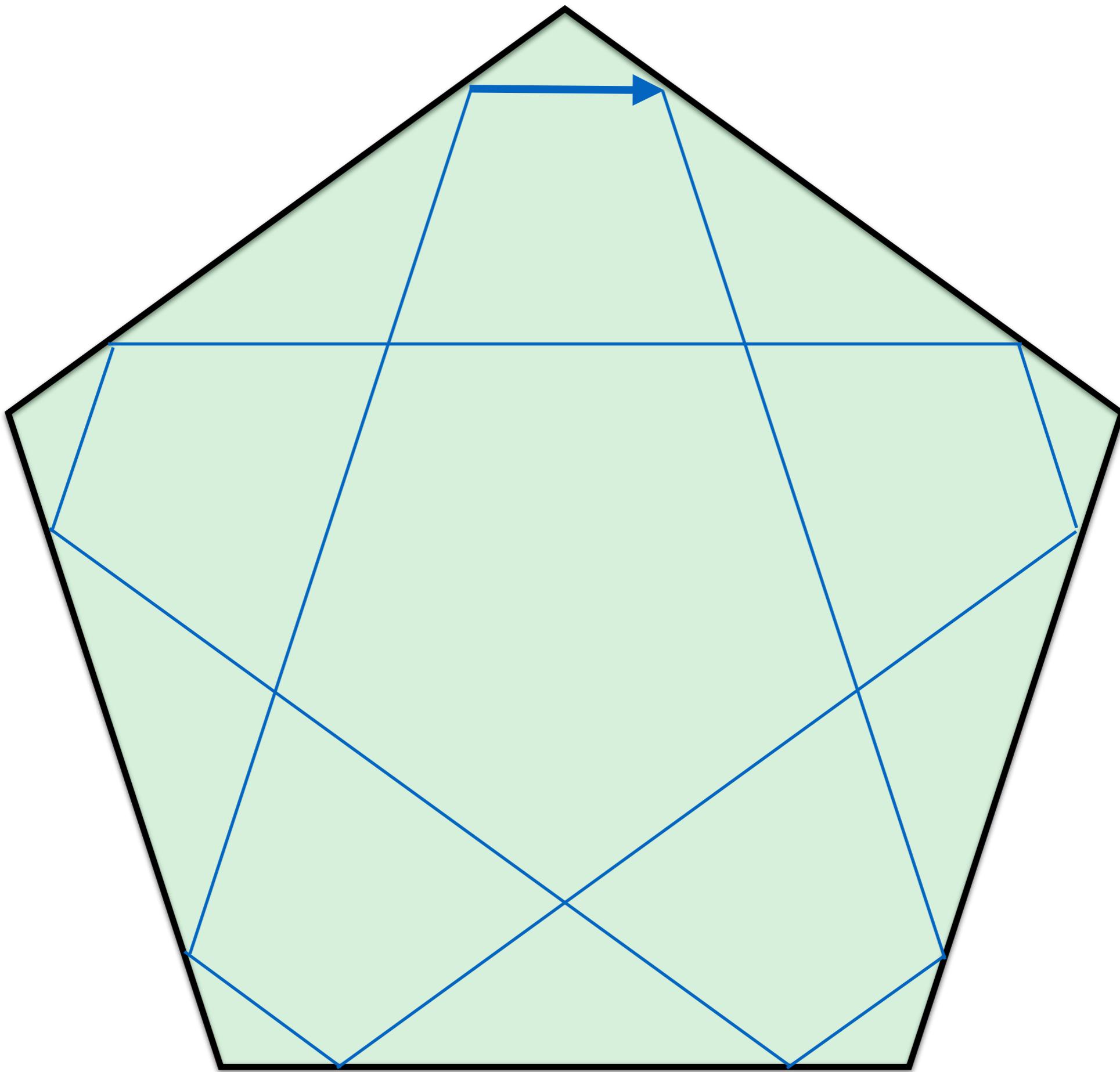


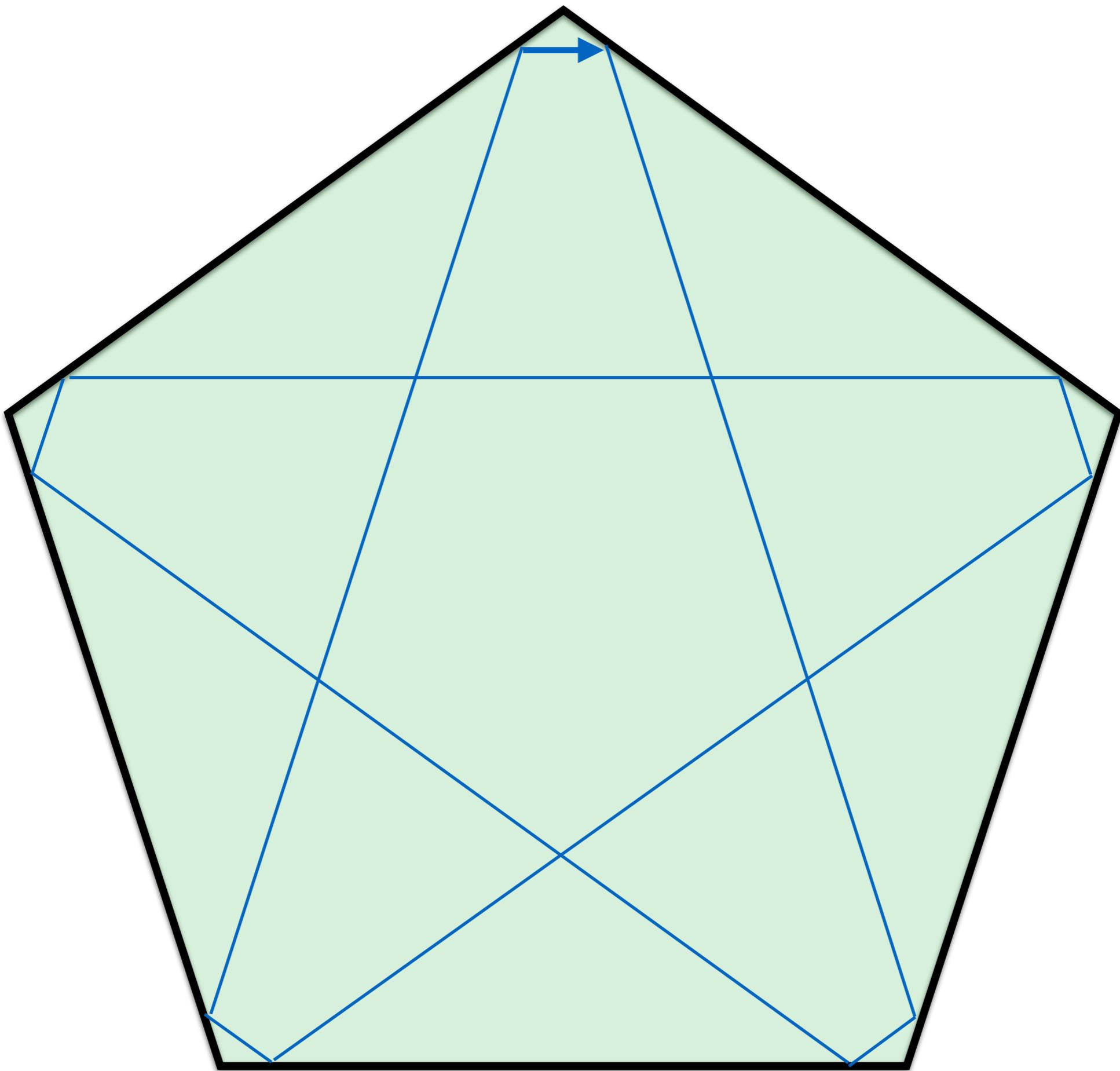


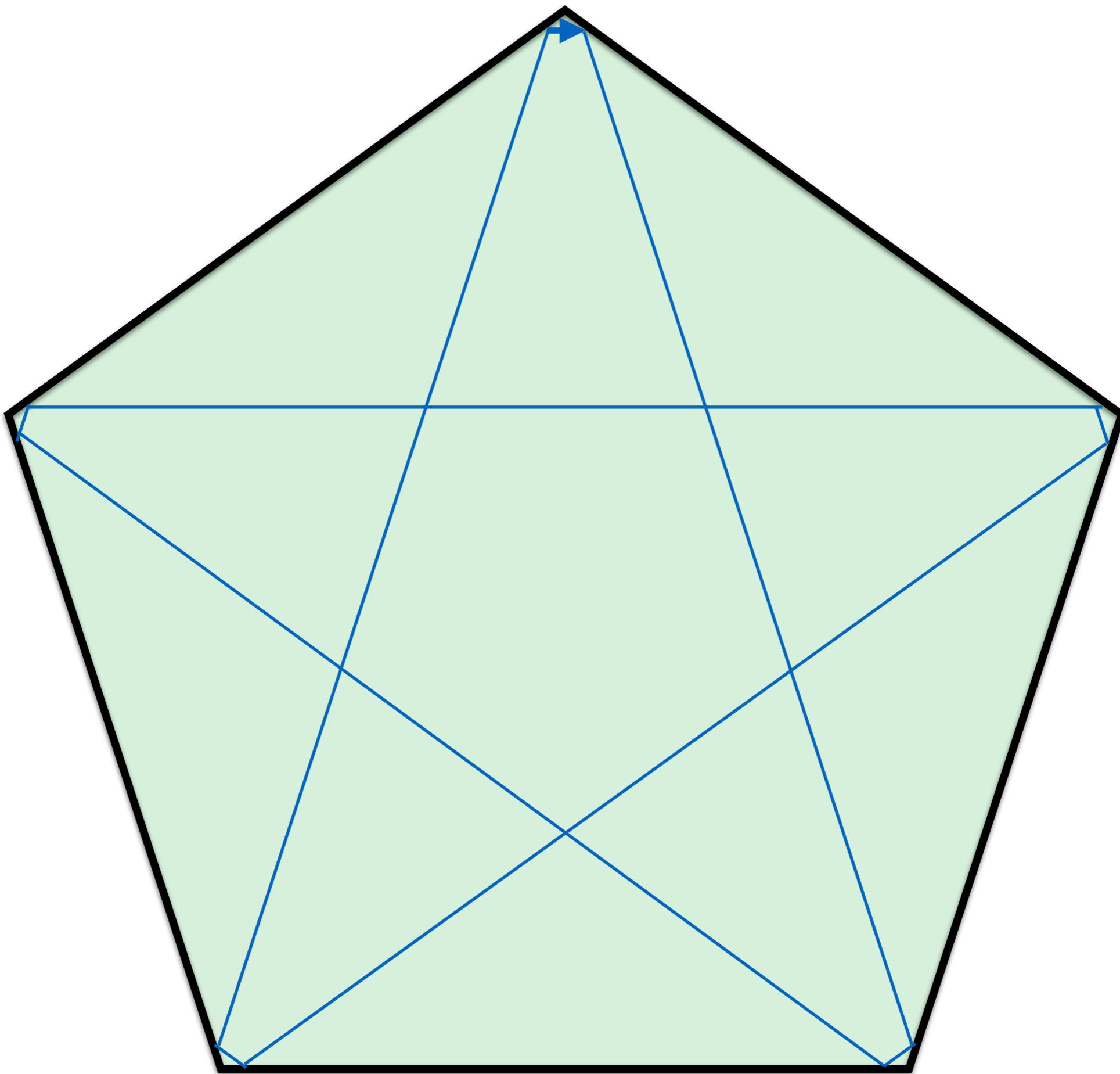








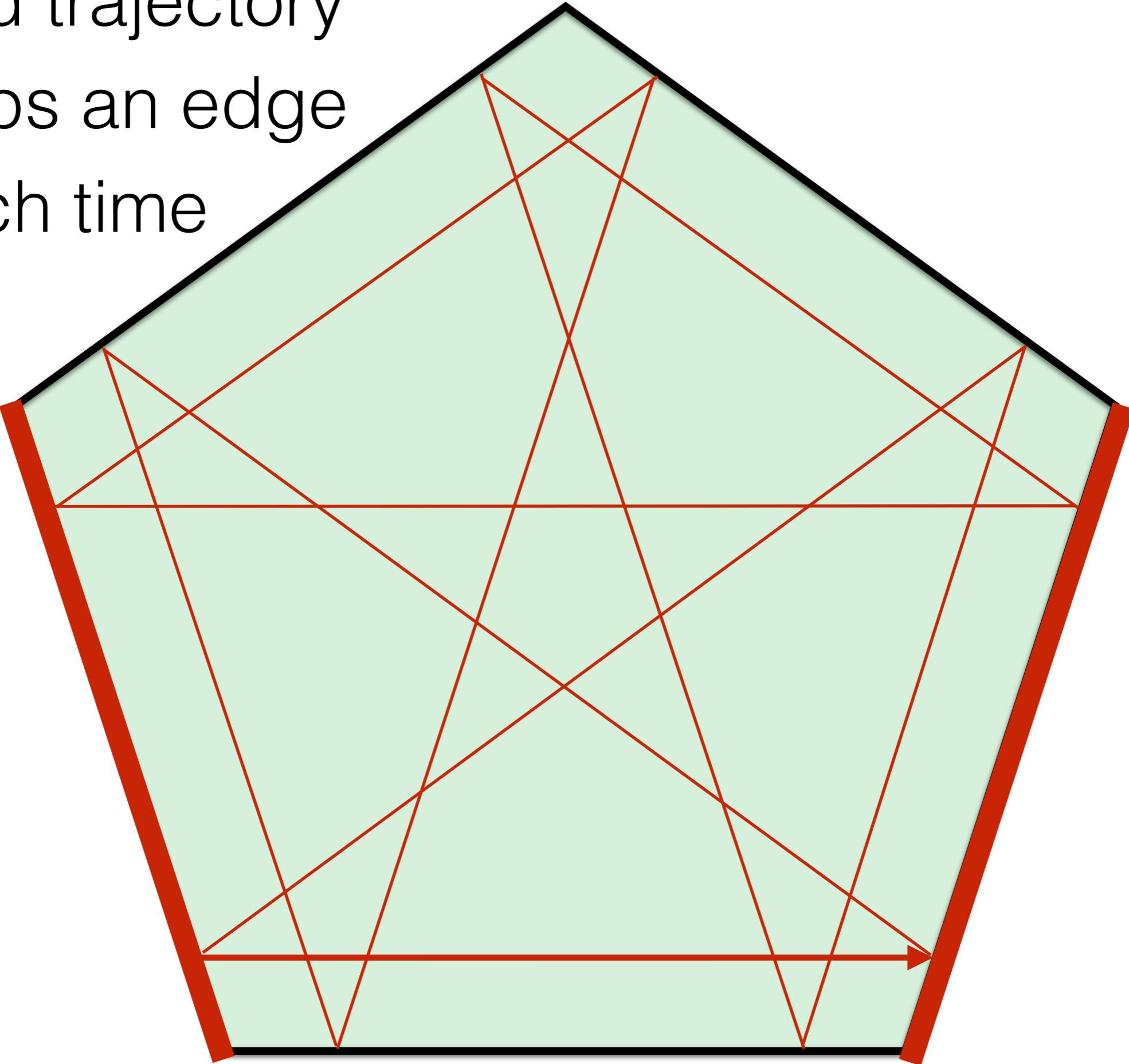




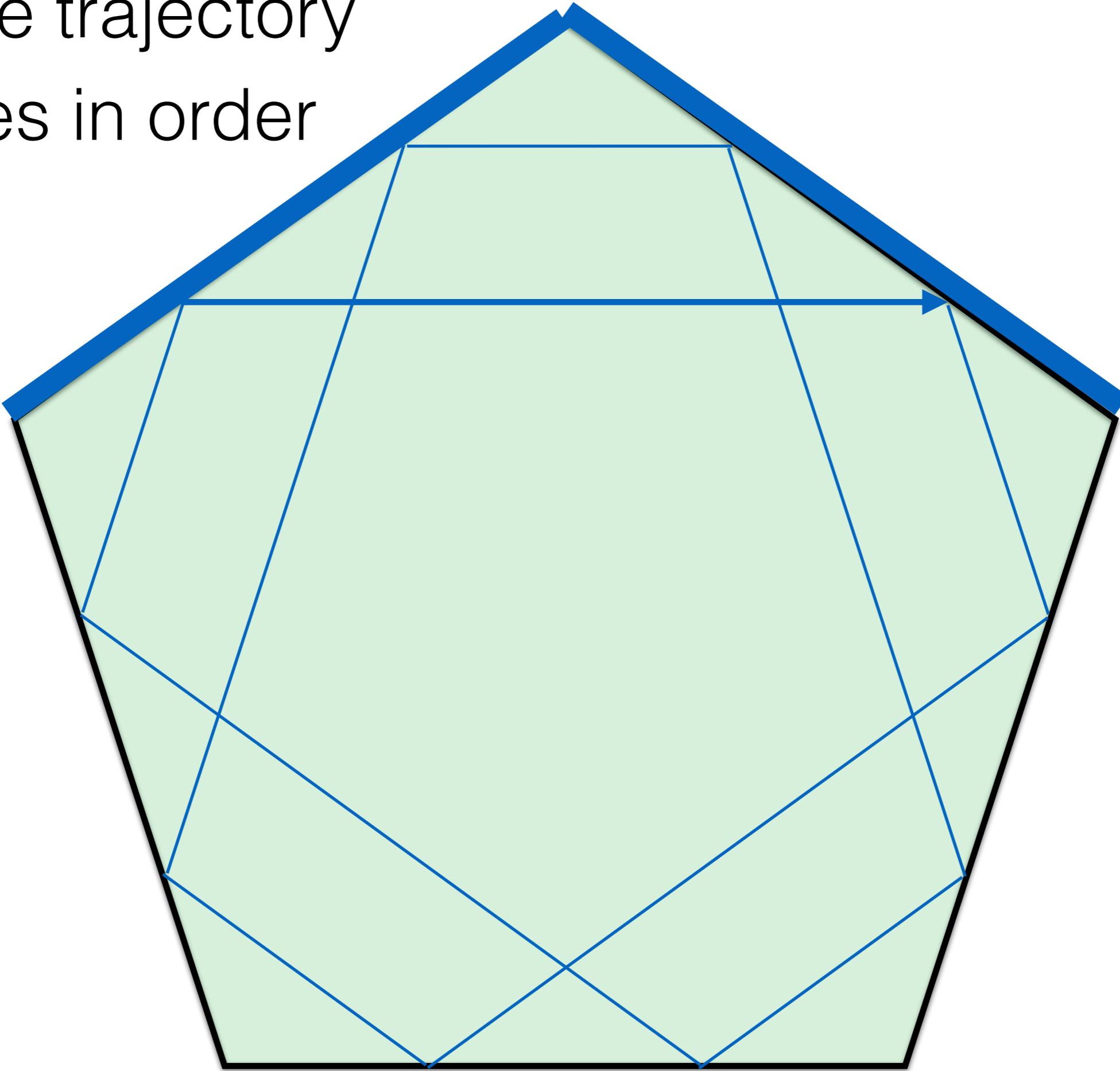
## Things to notice:

1. The red and blue trajectories hit the edges in different orders.
2. All the **red** trajectories are the same length.
3. All the **blue** trajectories are the same length.
4. The **red** trajectories are the golden ratio times as long as the **blue** trajectories.

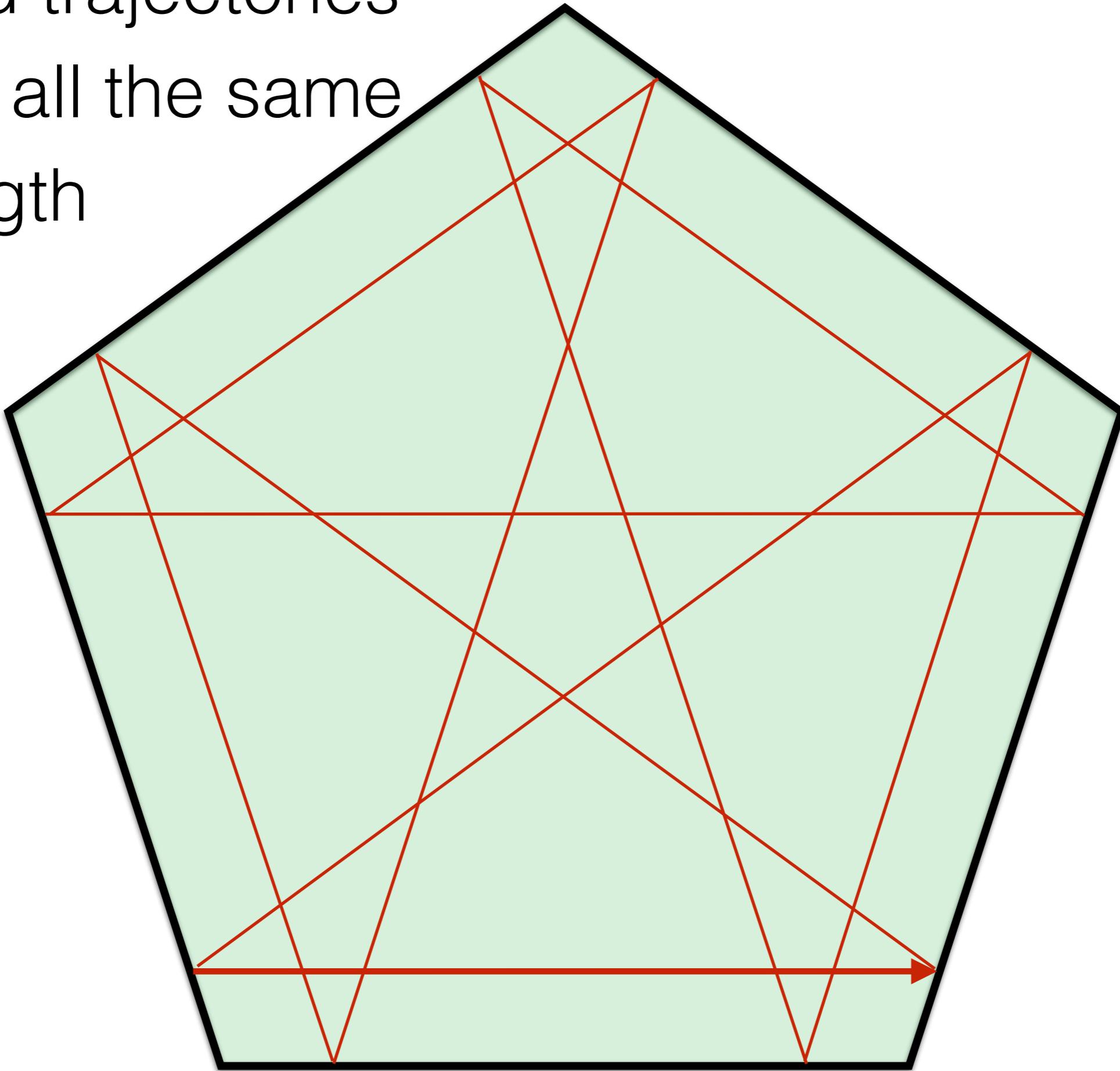
1. Red trajectory  
skips an edge  
each time



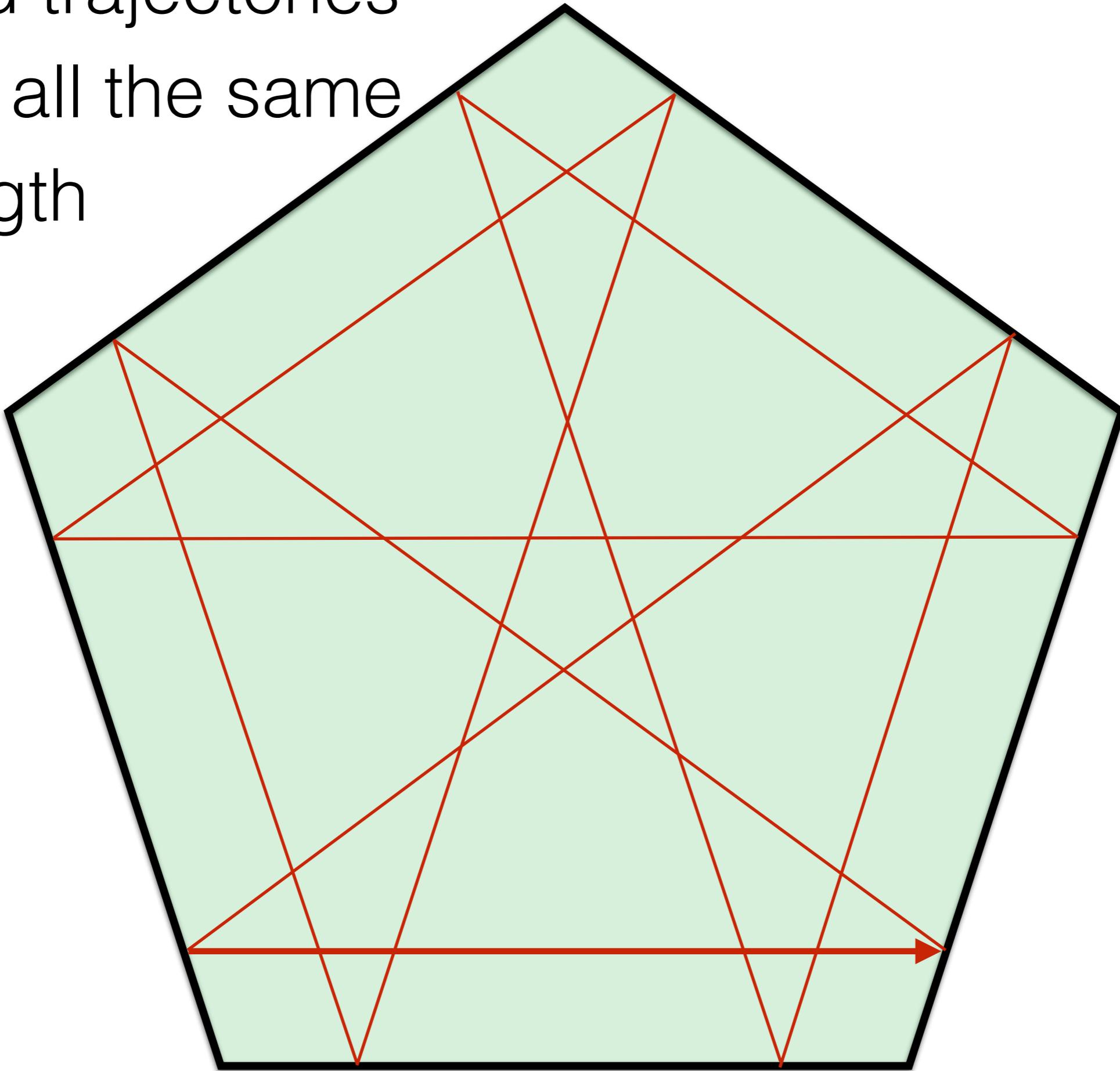
1. Blue trajectory  
goes in order



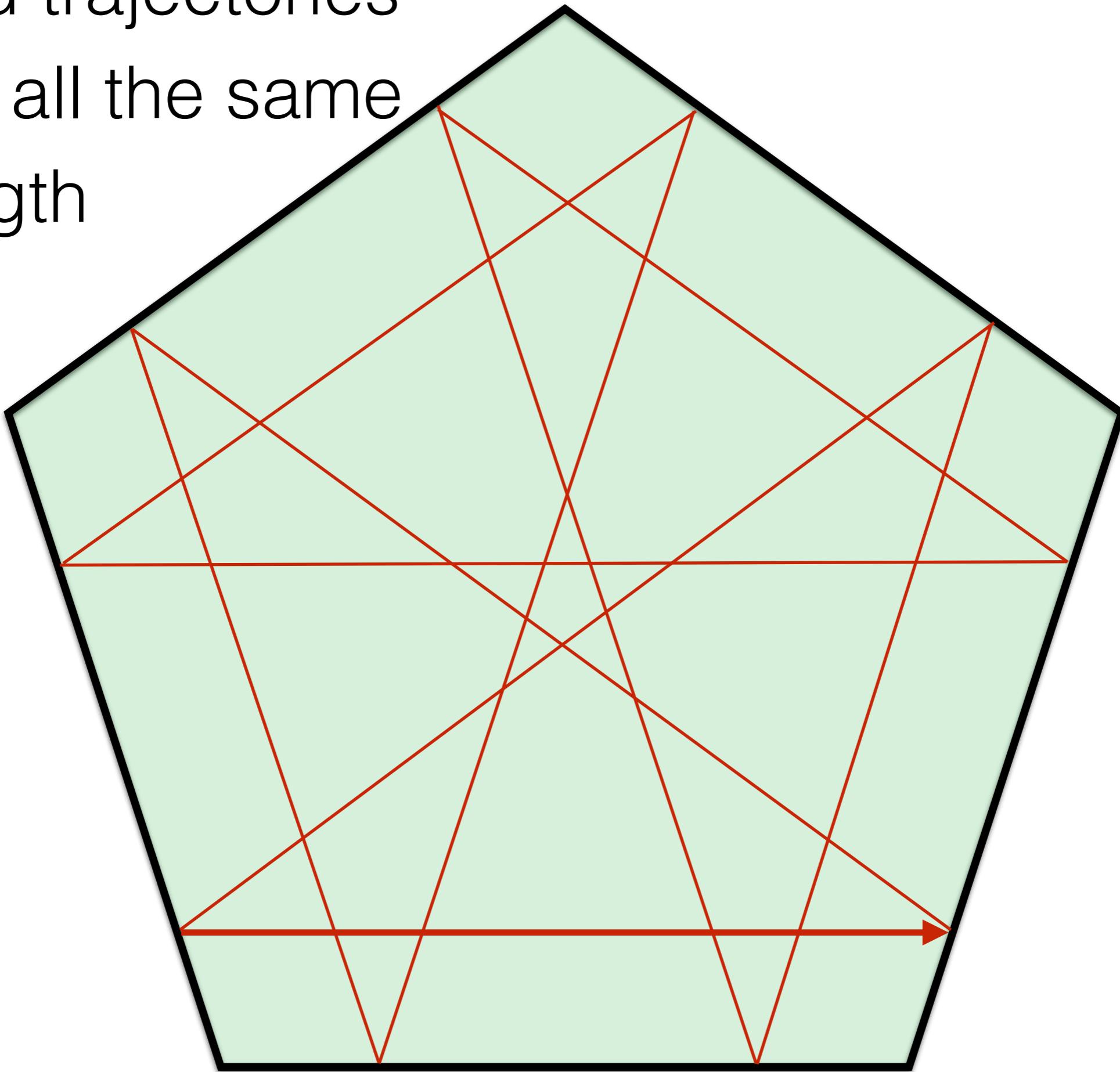
2. Red trajectories  
are all the same  
length



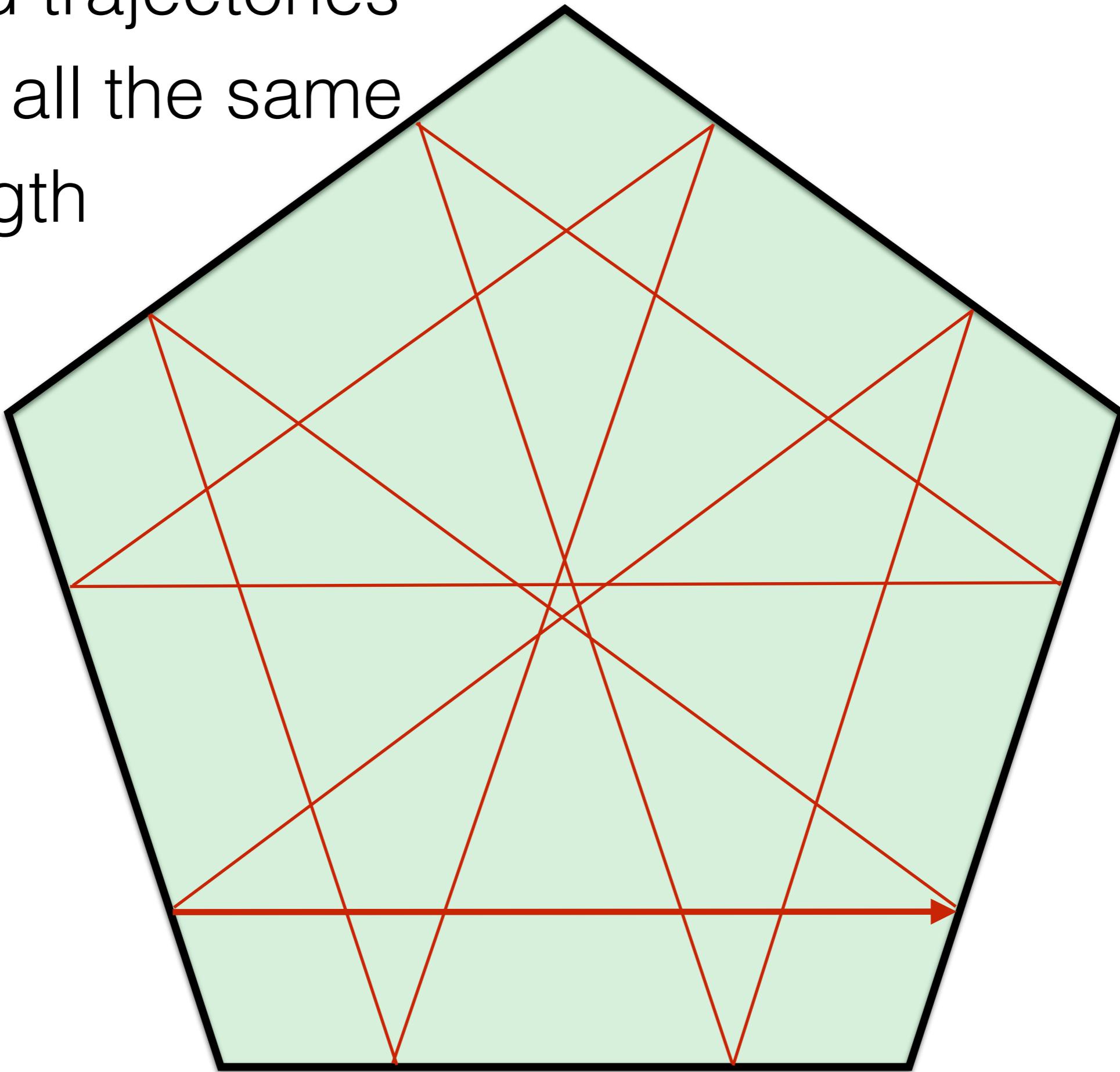
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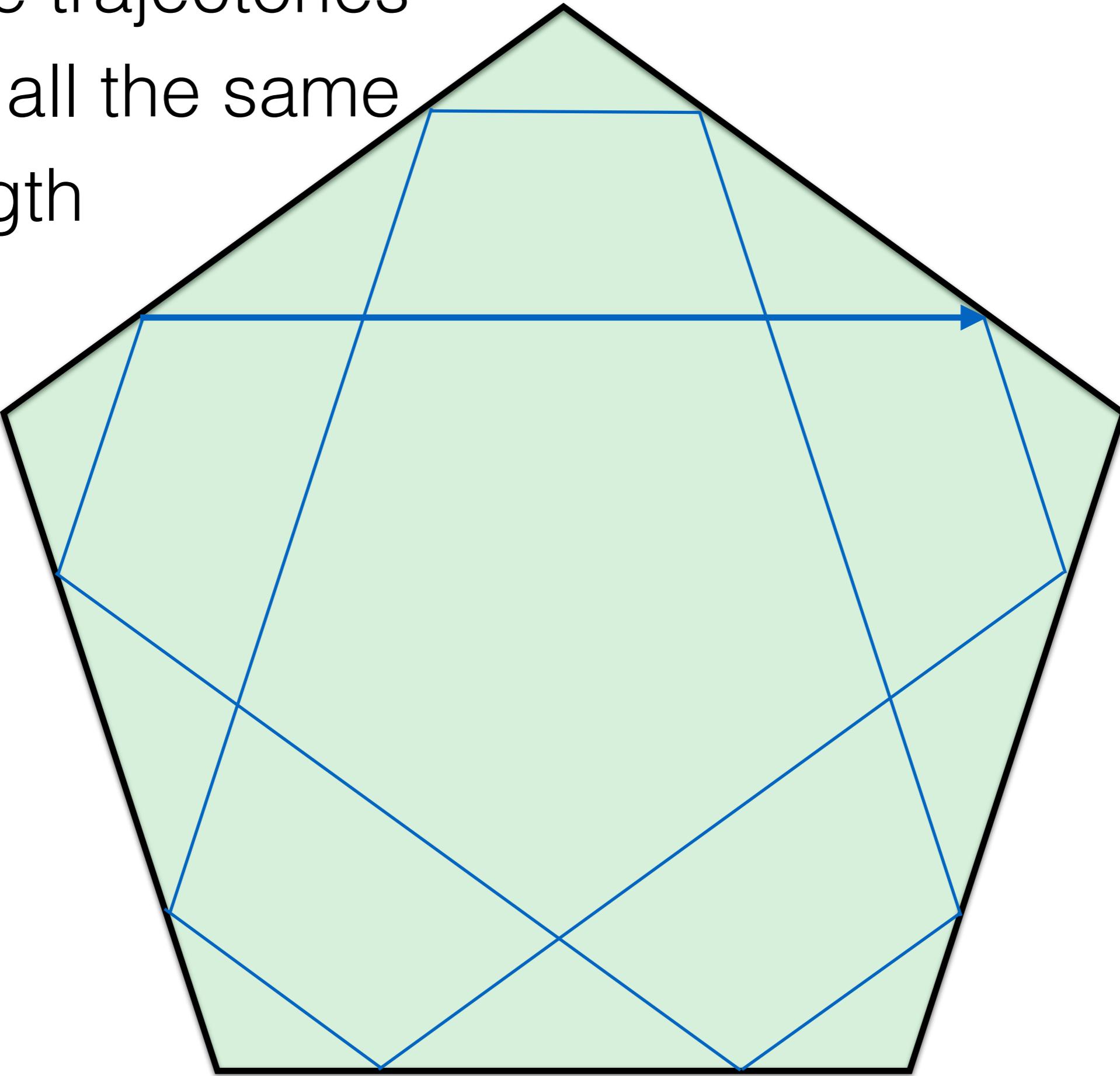
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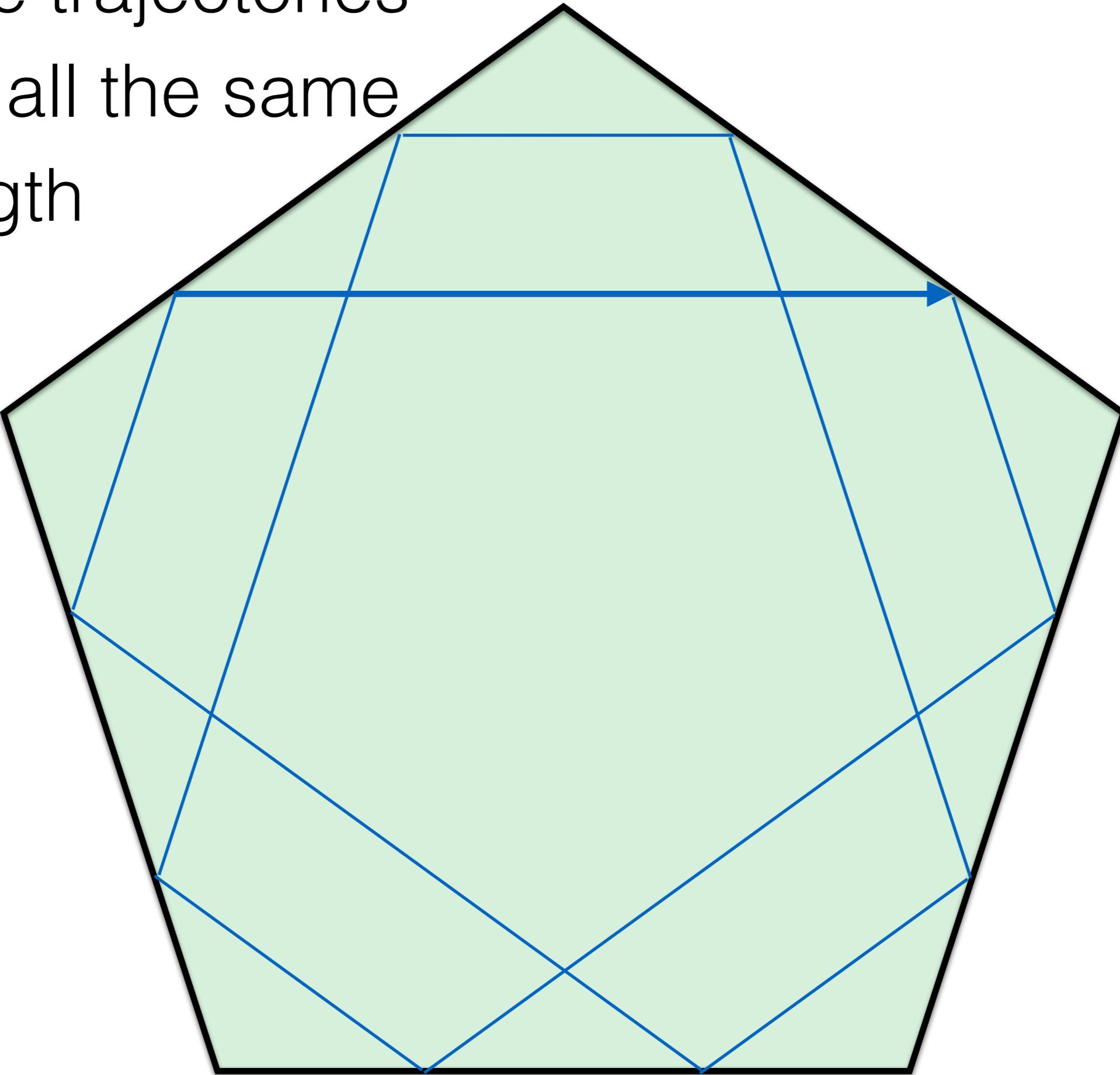
2. Red trajectories  
are all the same  
length



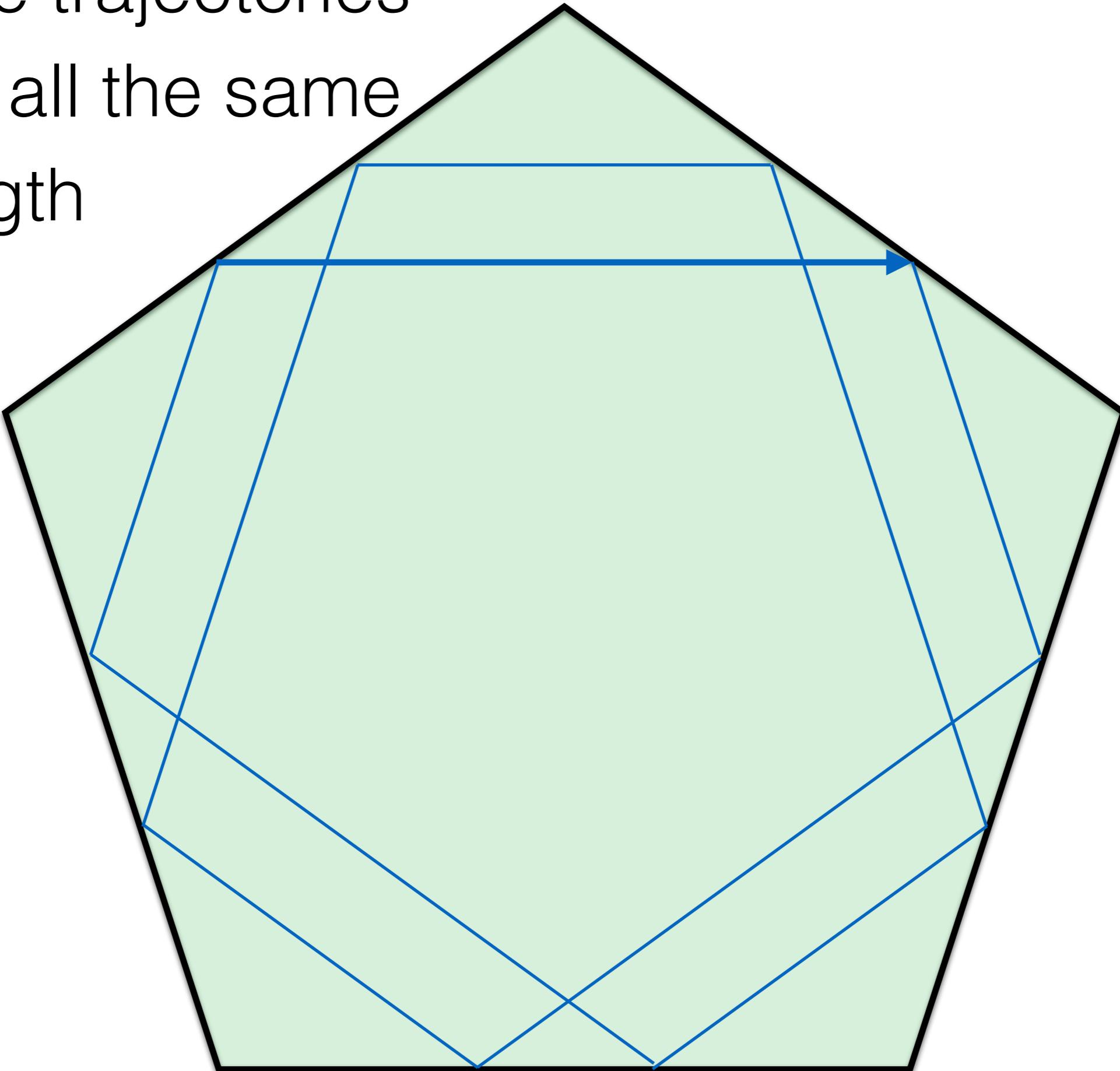
3. Blue trajectories  
are all the same  
length



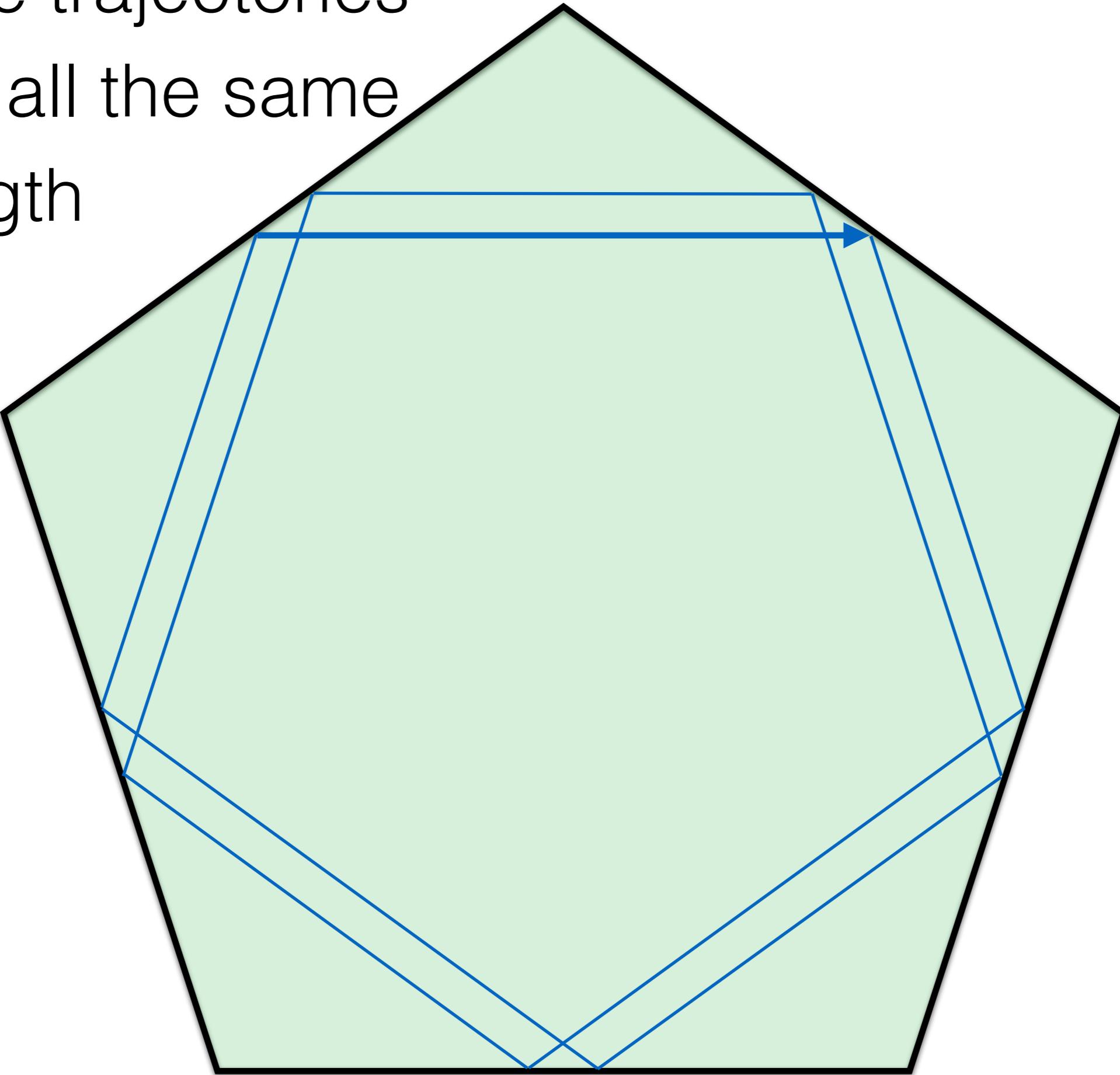
3. Blue trajectories  
are all the same  
length



3. Blue trajectories  
are all the same  
length

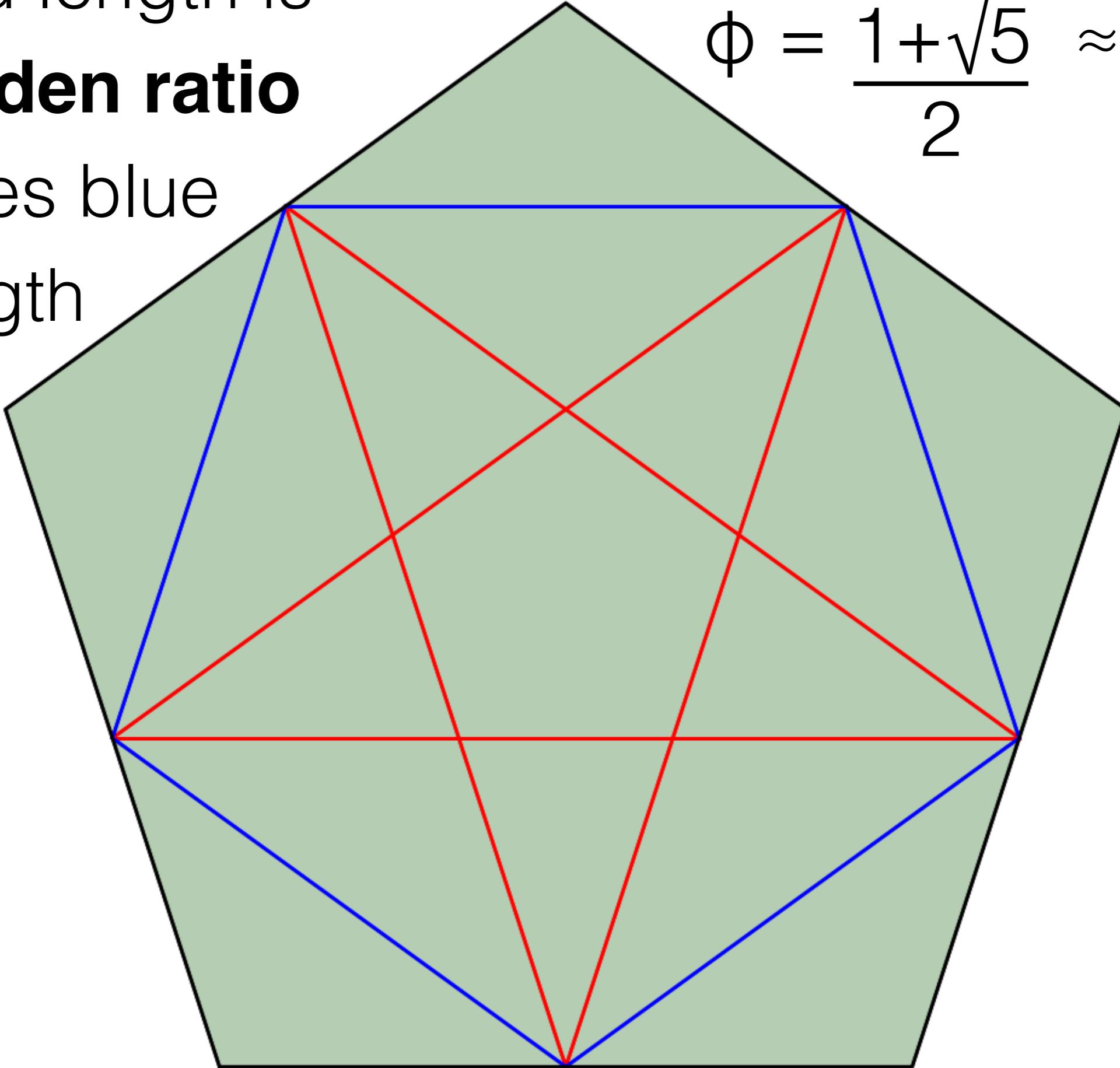


3. Blue trajectories  
are all the same  
length



4. Red length is  
**golden ratio**  
times blue  
length

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

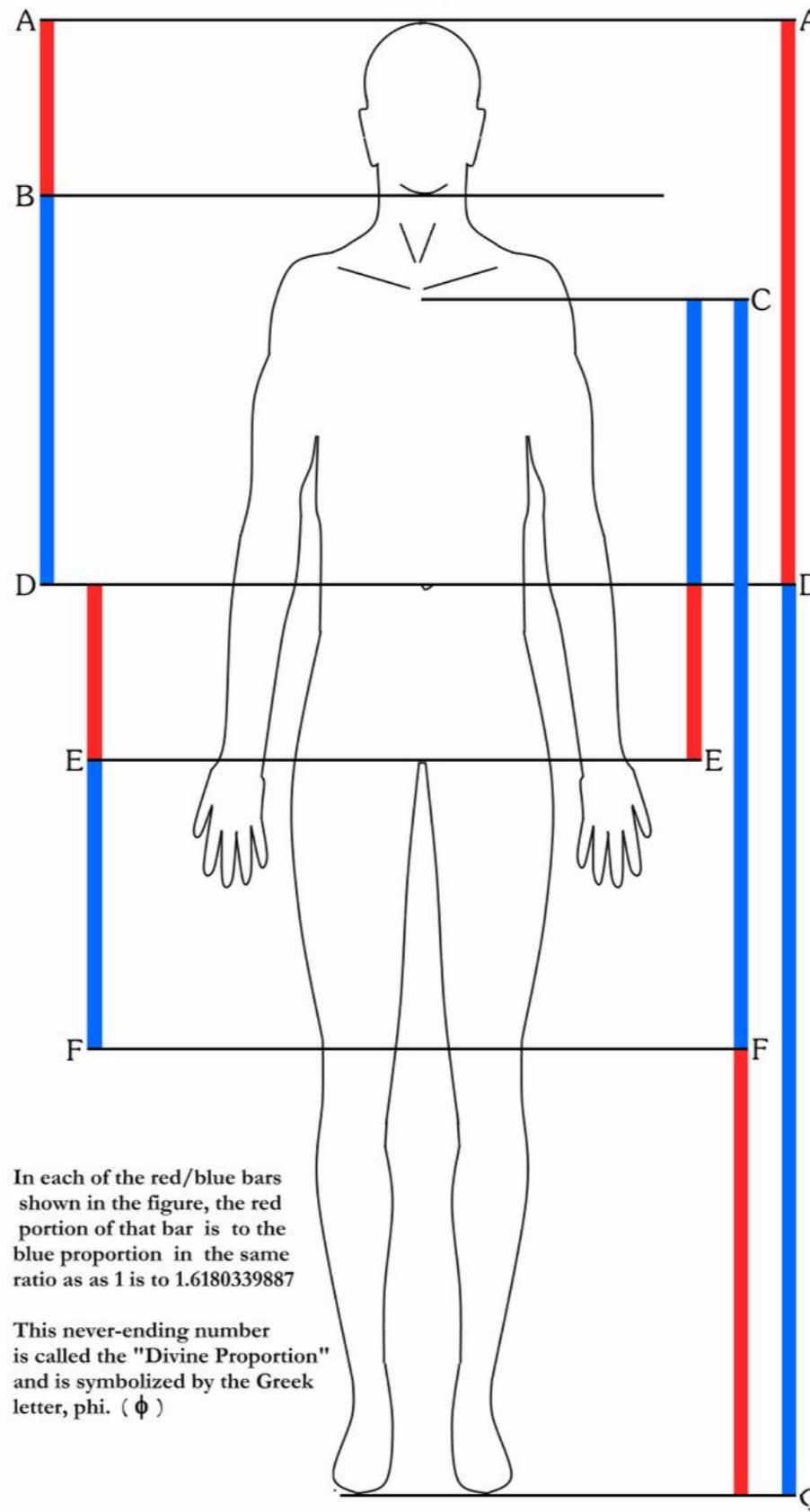


# THE GOLDEN RATIO

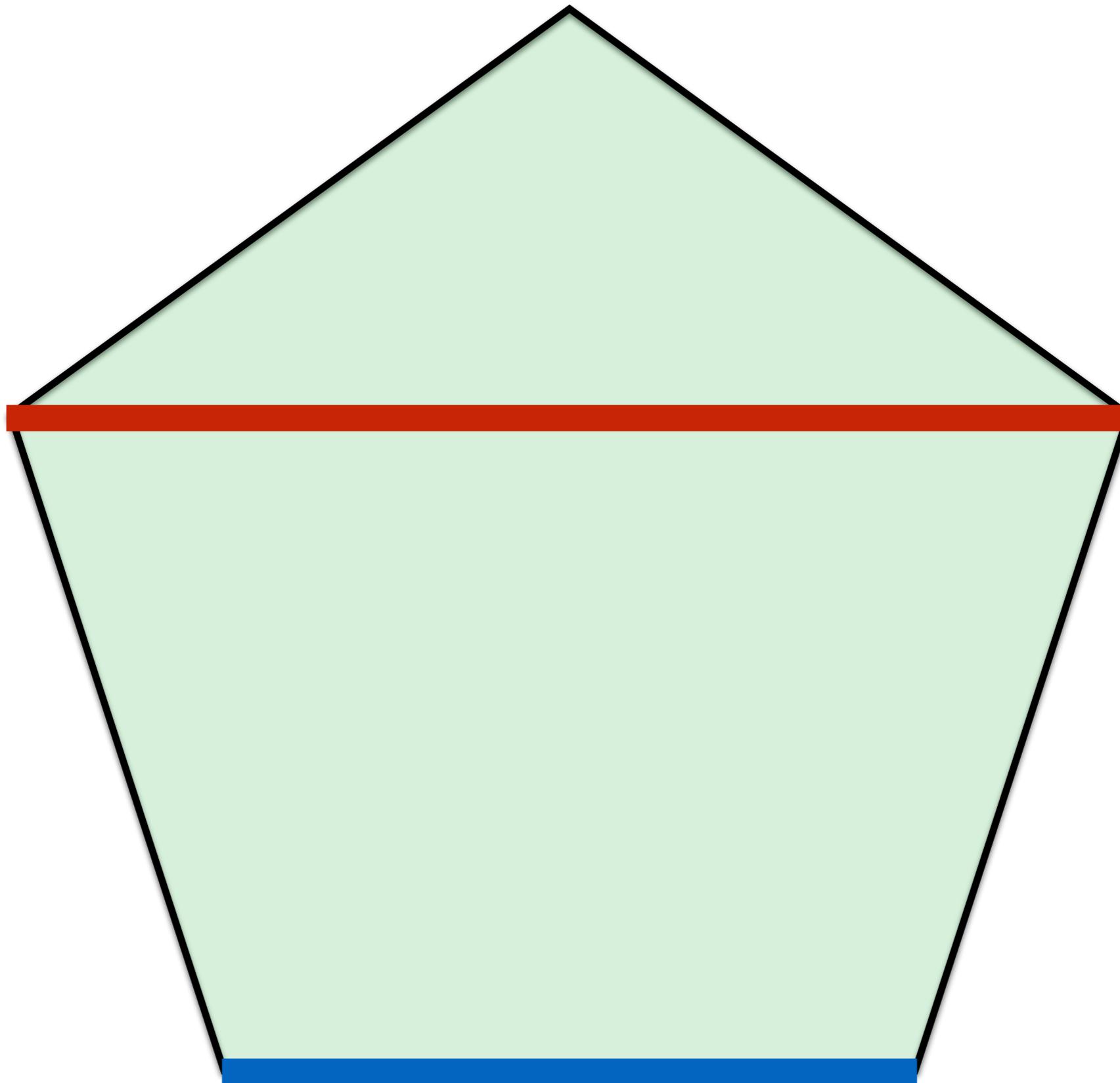


# THE GOLDEN RATIO

## DIVINE PROPORTIONS

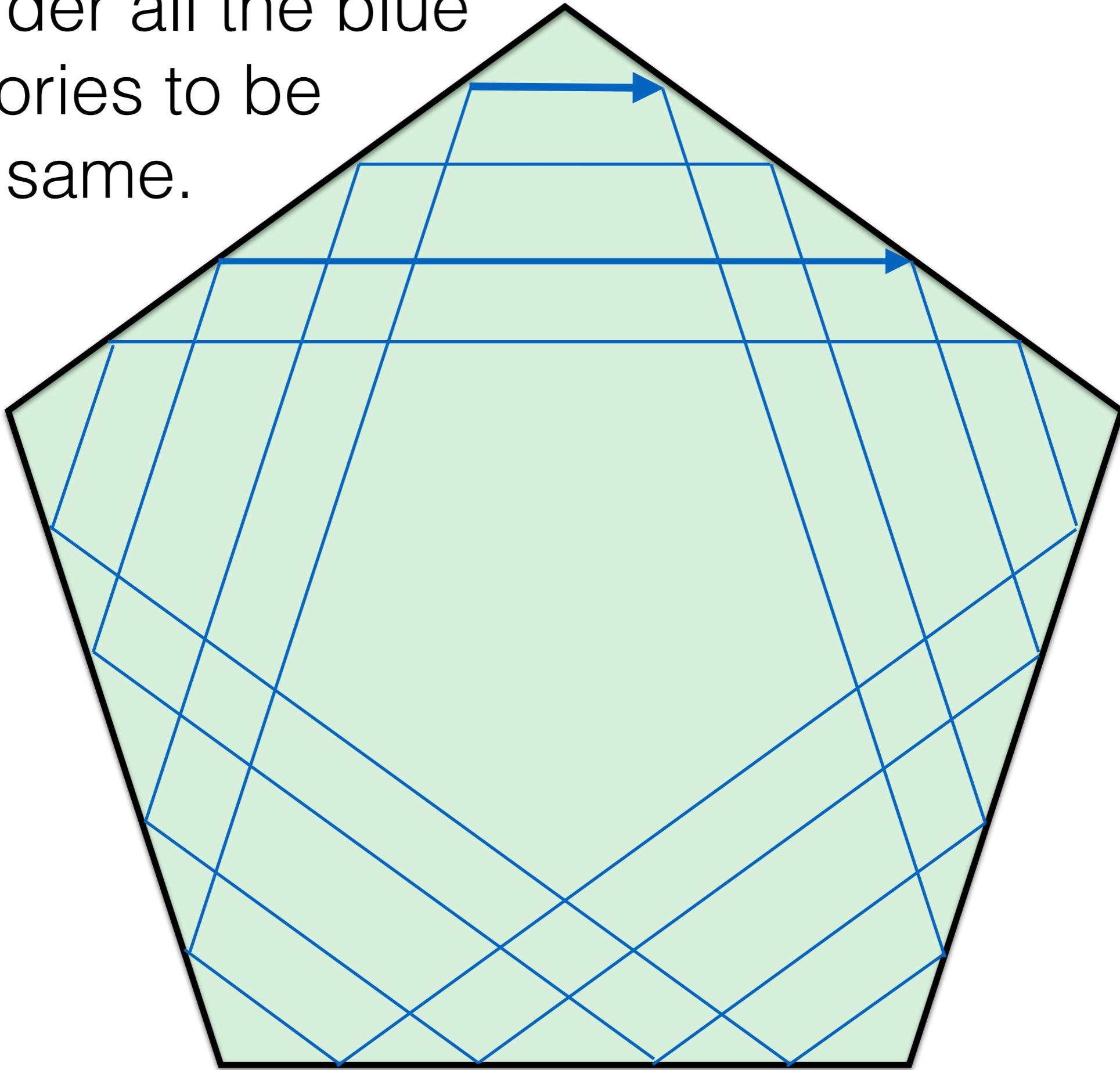


# THE GOLDEN RATIO

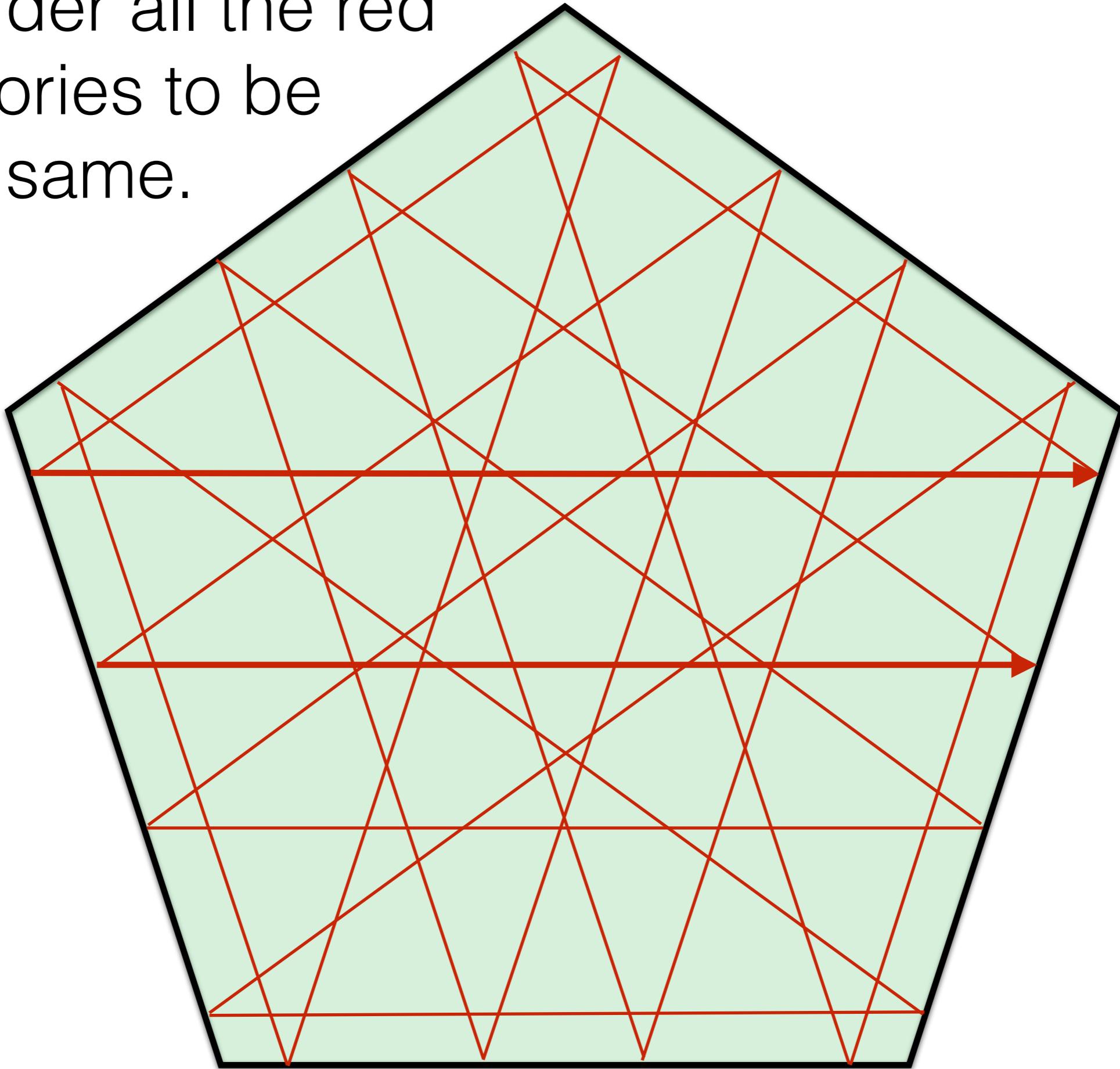


**BACK TO  
BUDDIES**

I consider all the blue  
trajectories to be  
all the same.

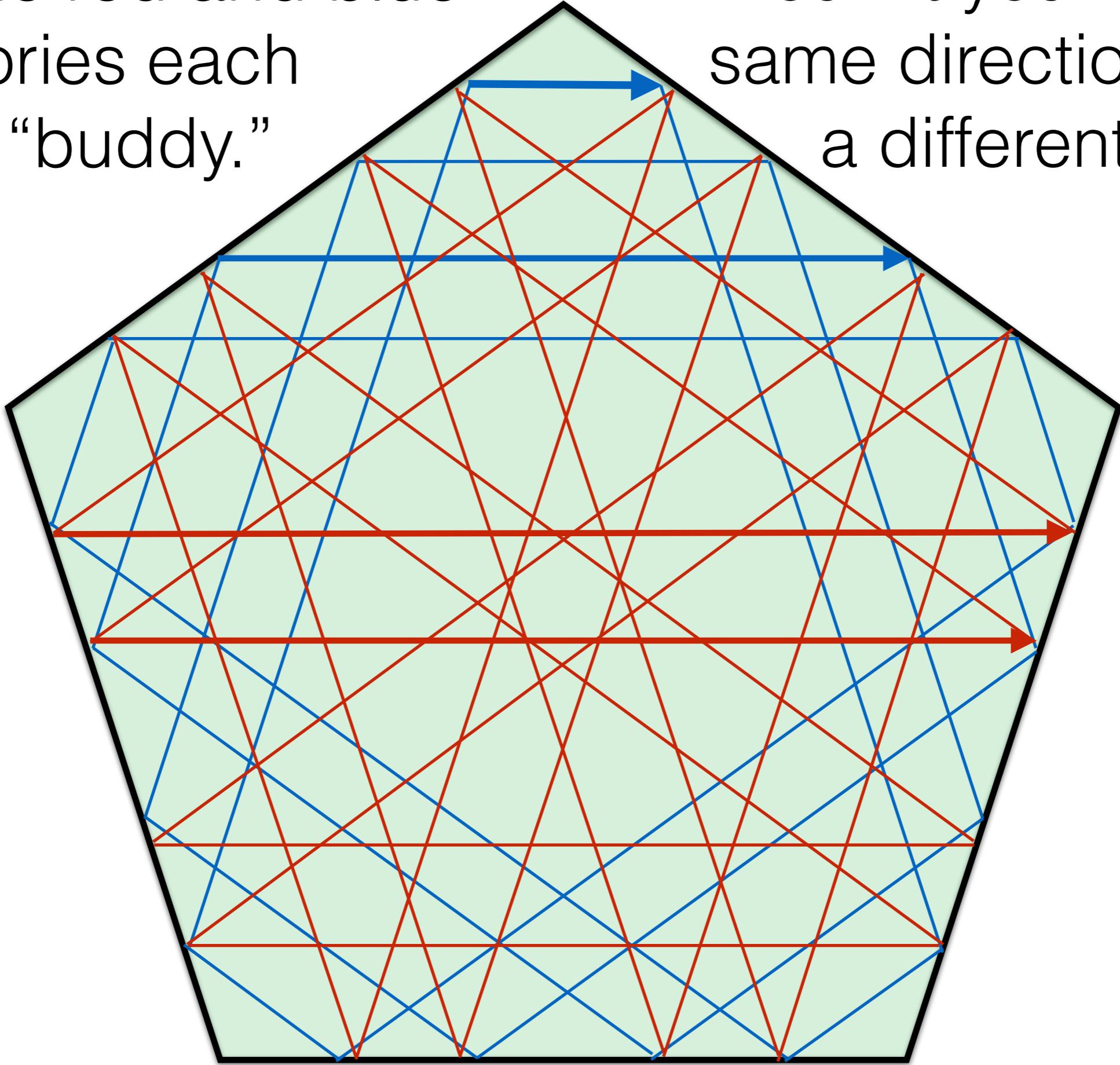


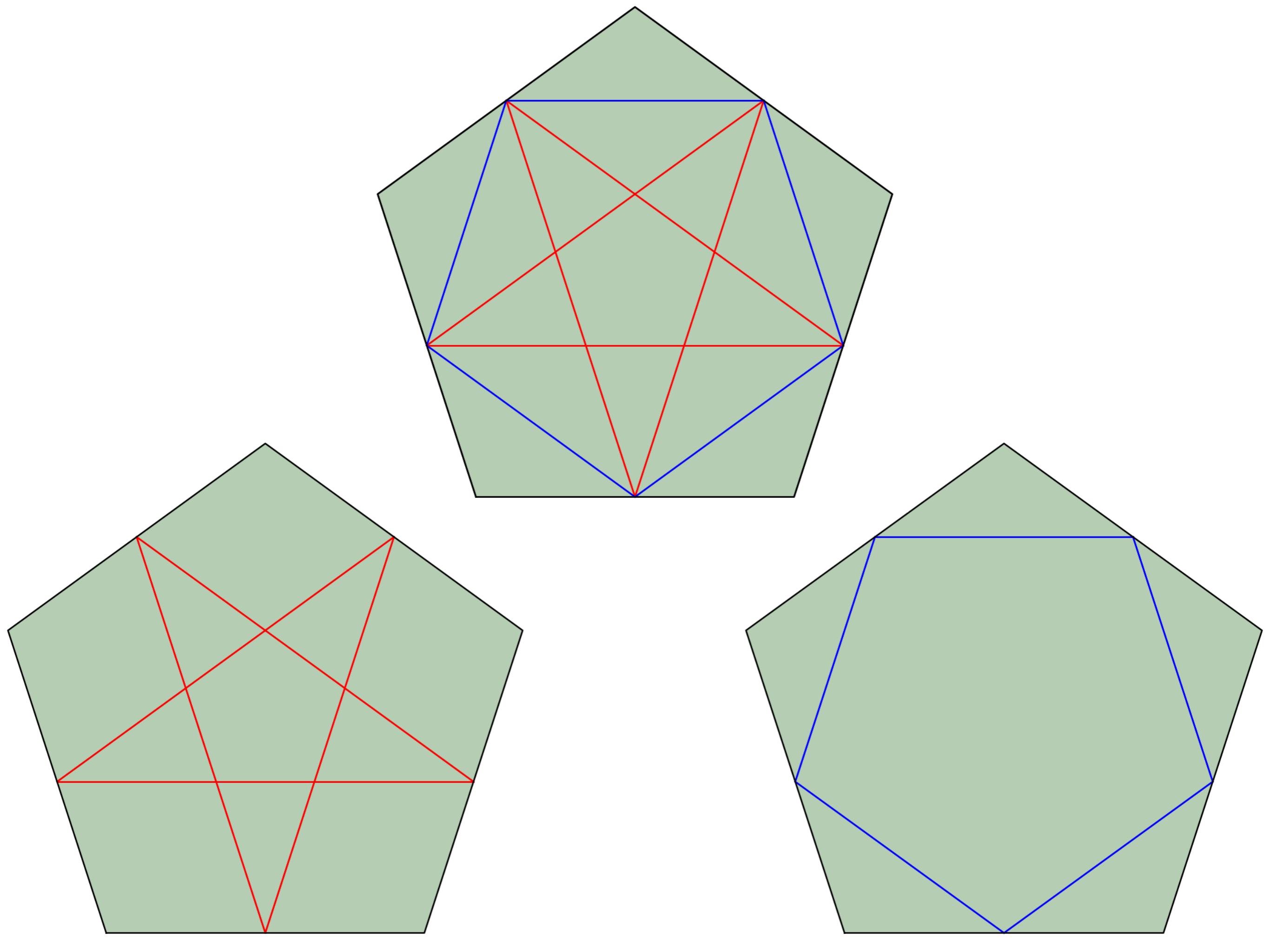
I consider all the red  
trajectories to be  
all the same.

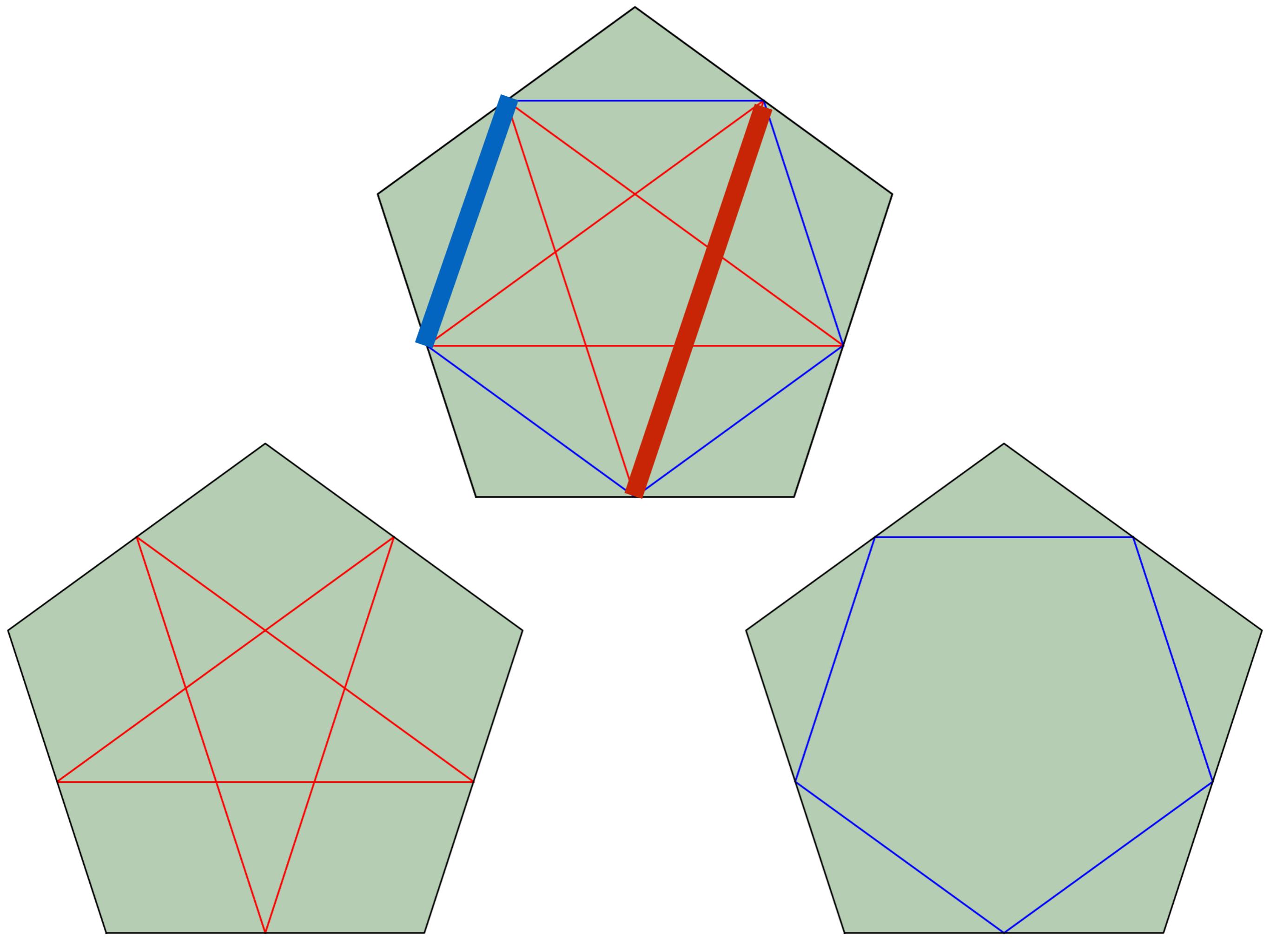


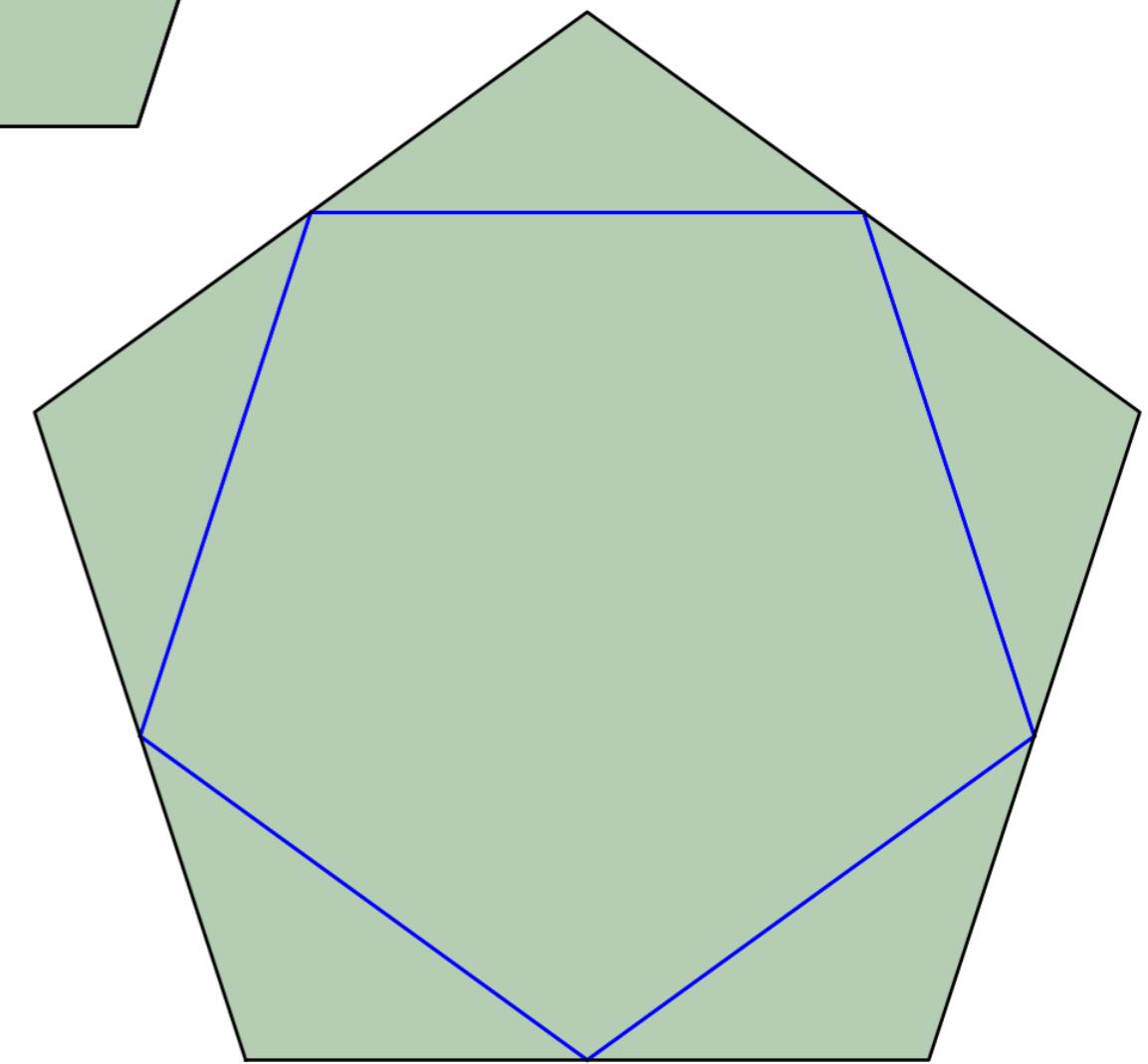
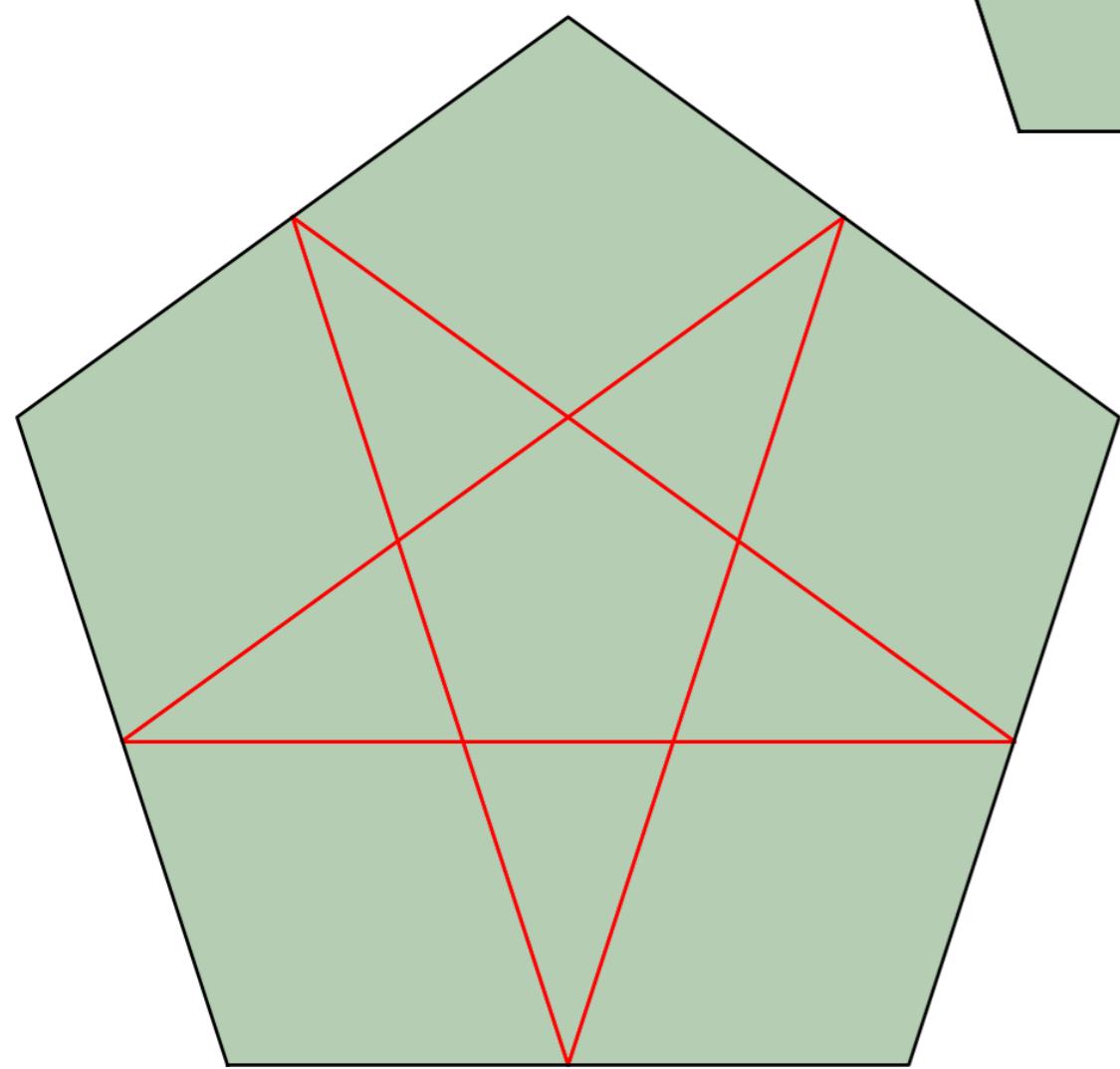
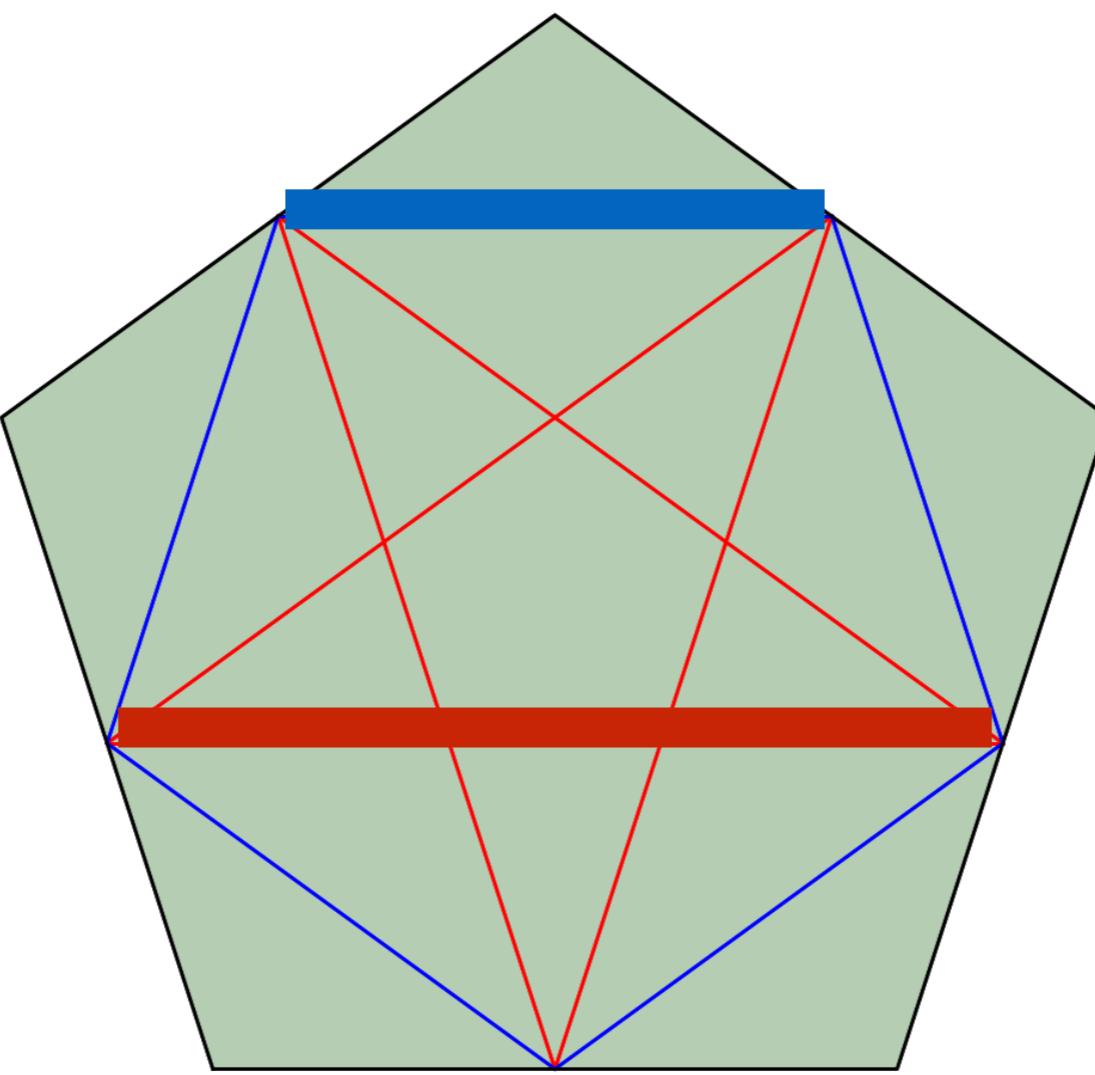
I call the red and blue  
trajectories each  
other's "buddy."

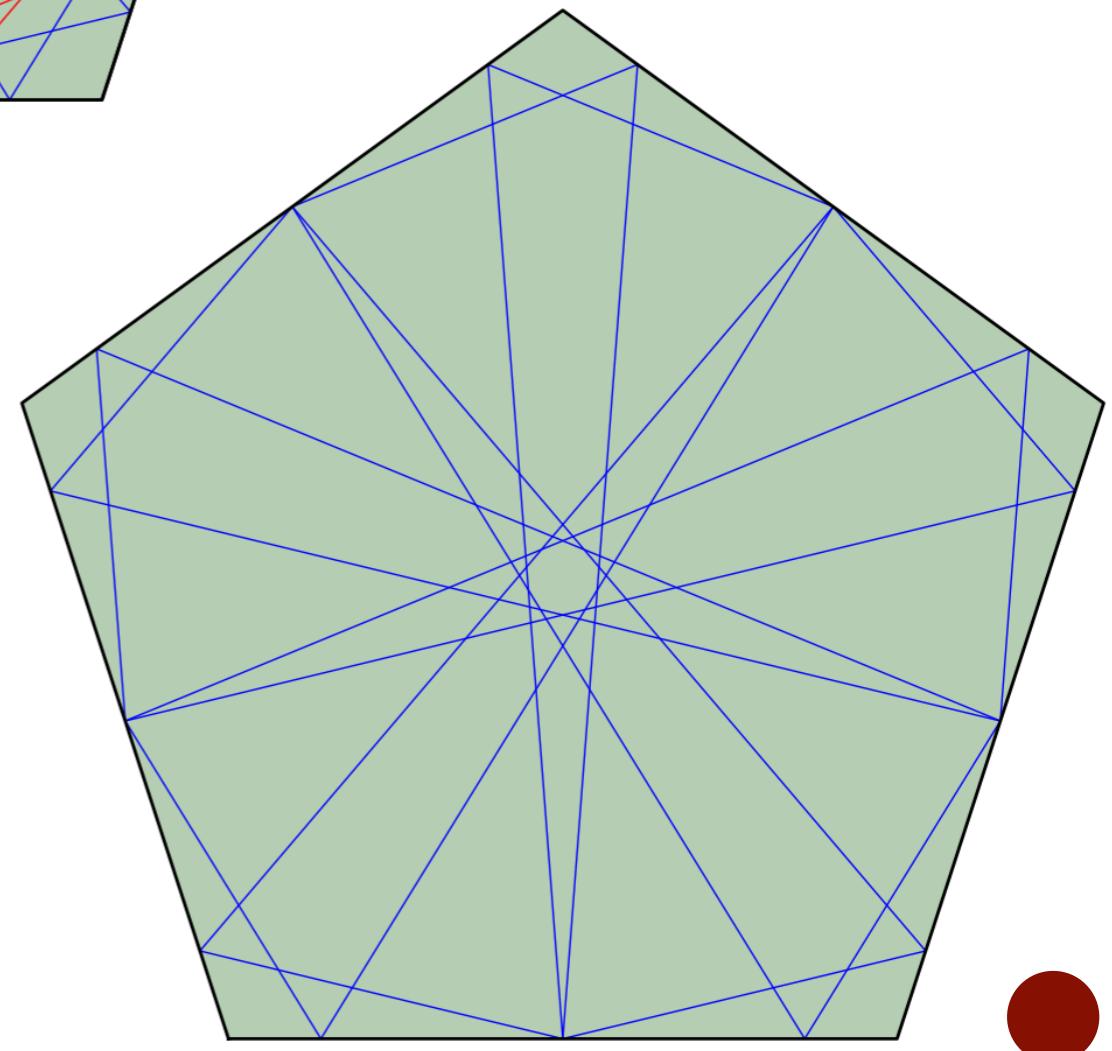
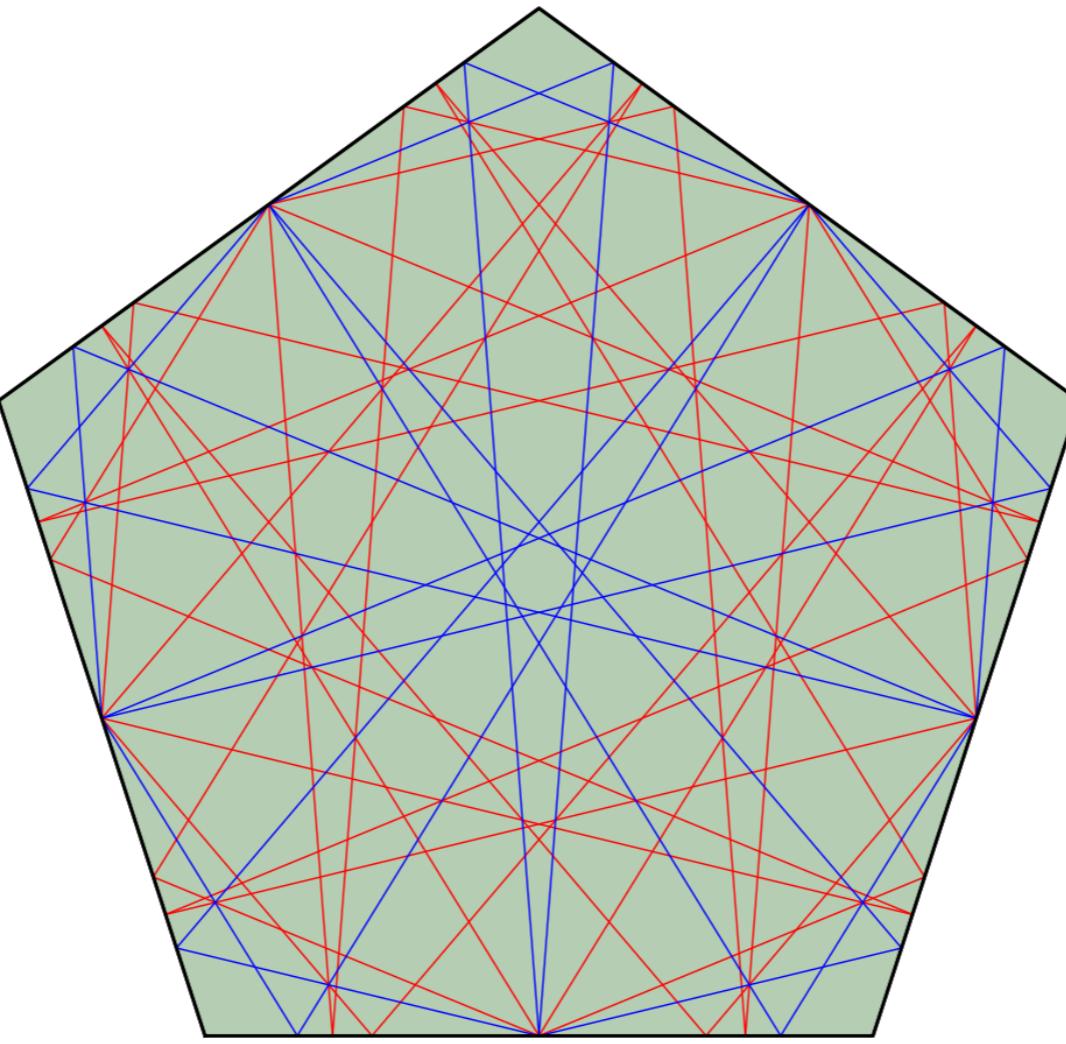
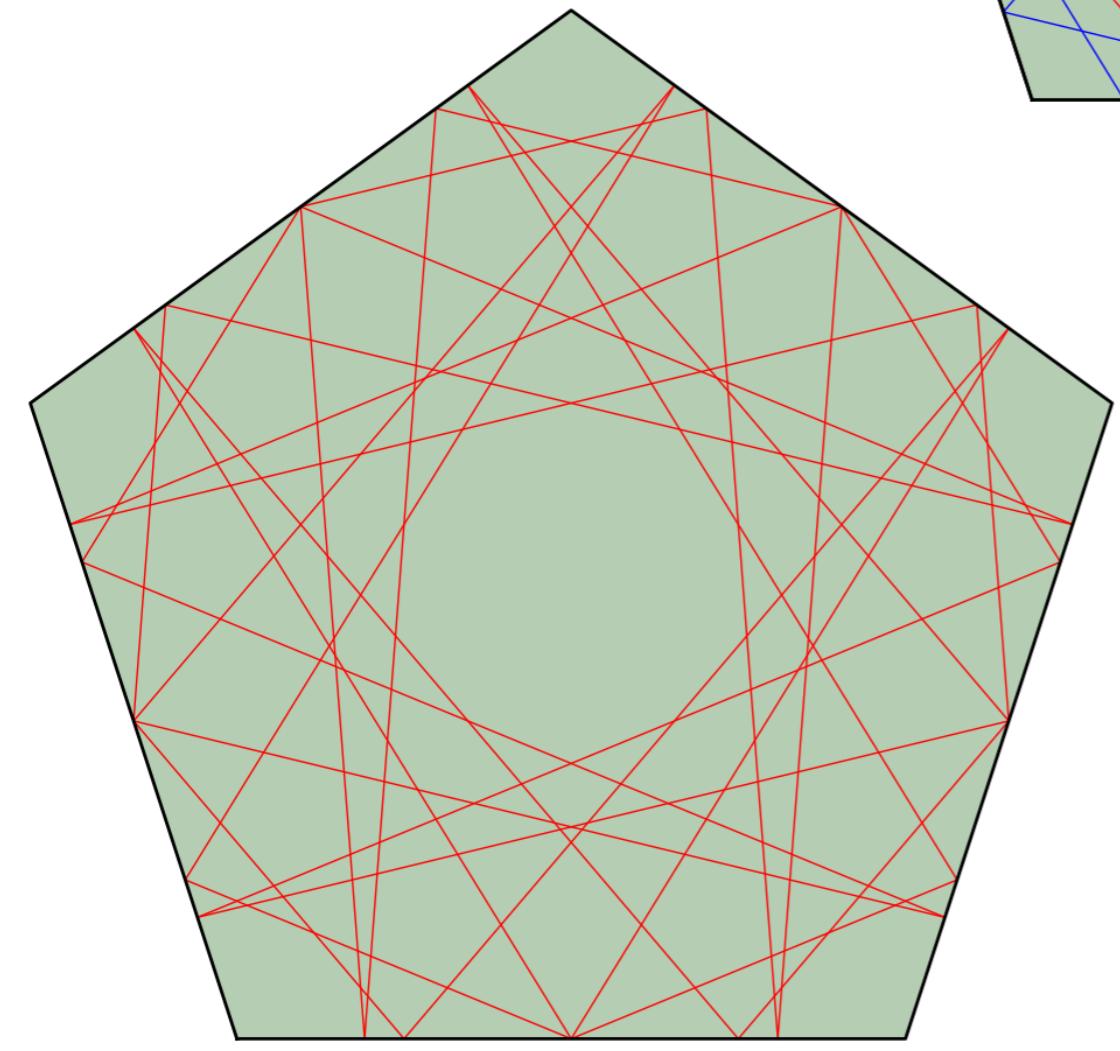
You hit your ball the  
same direction, from  
a different place.

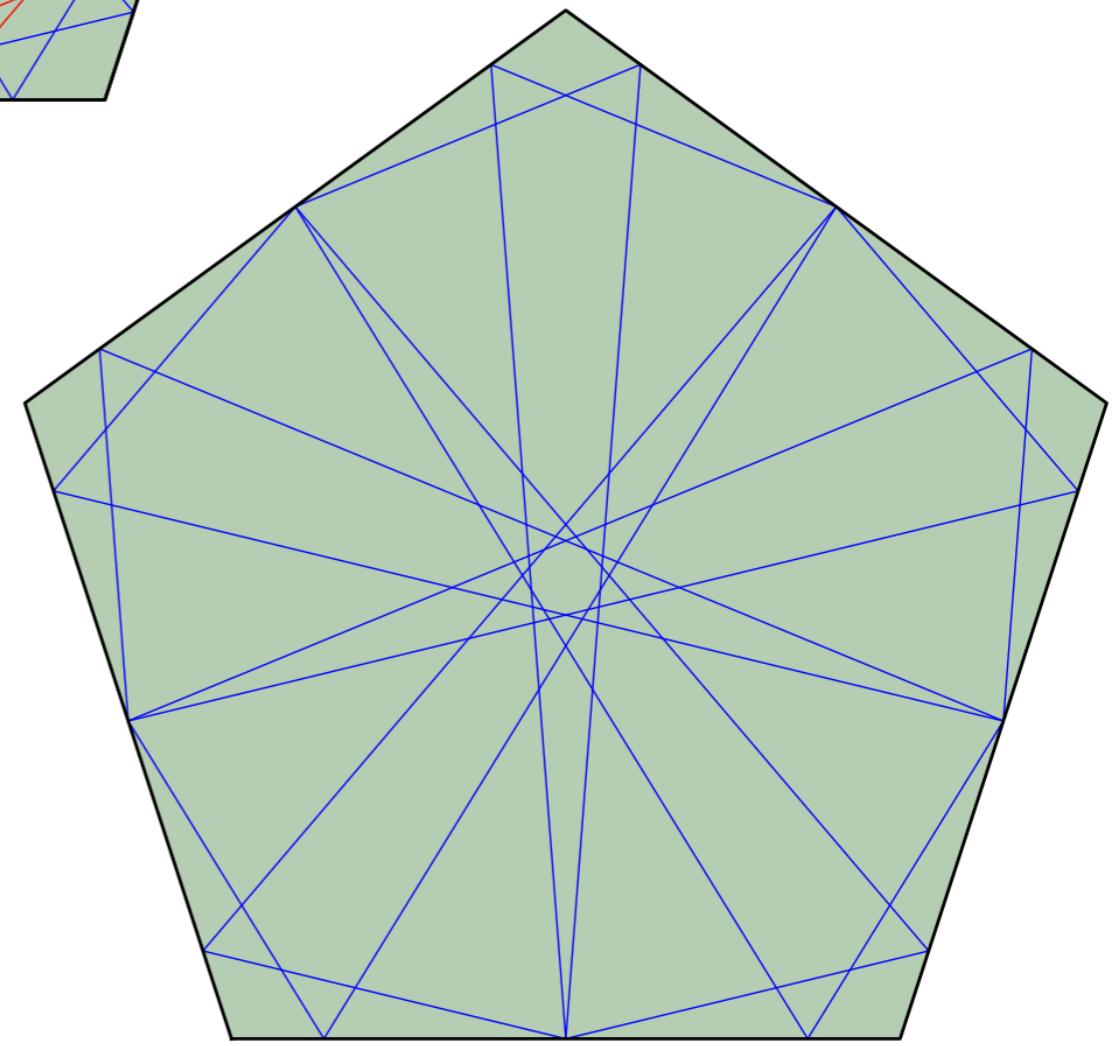
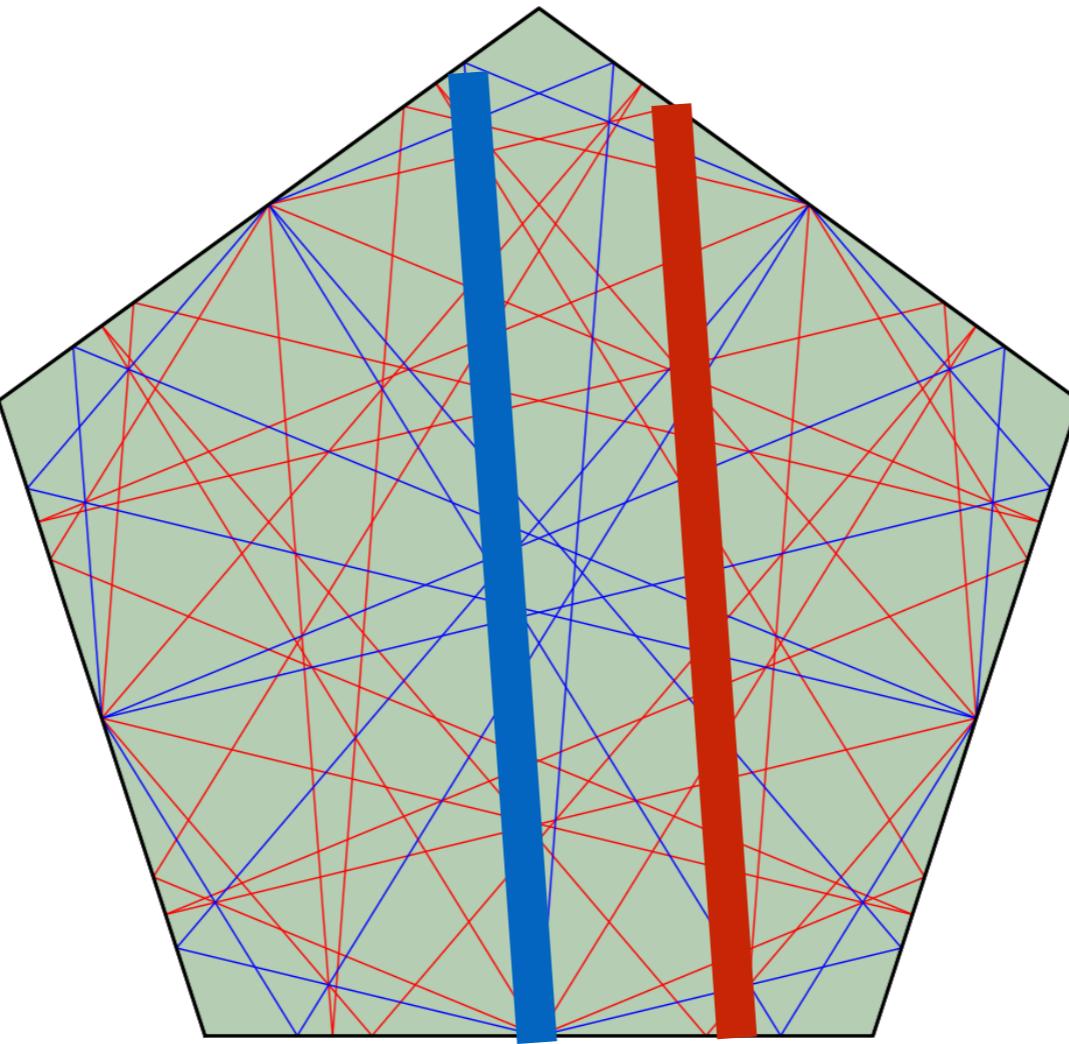
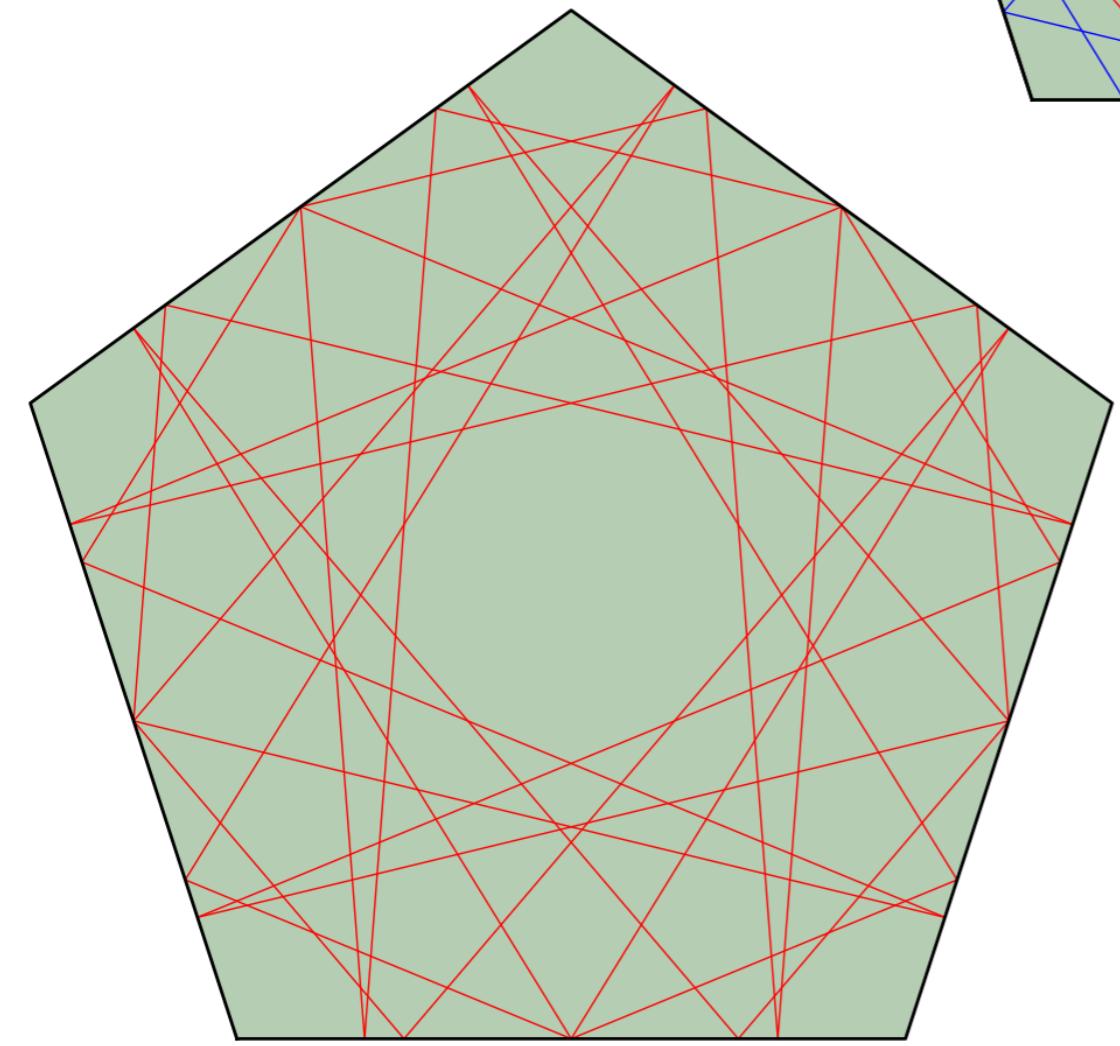


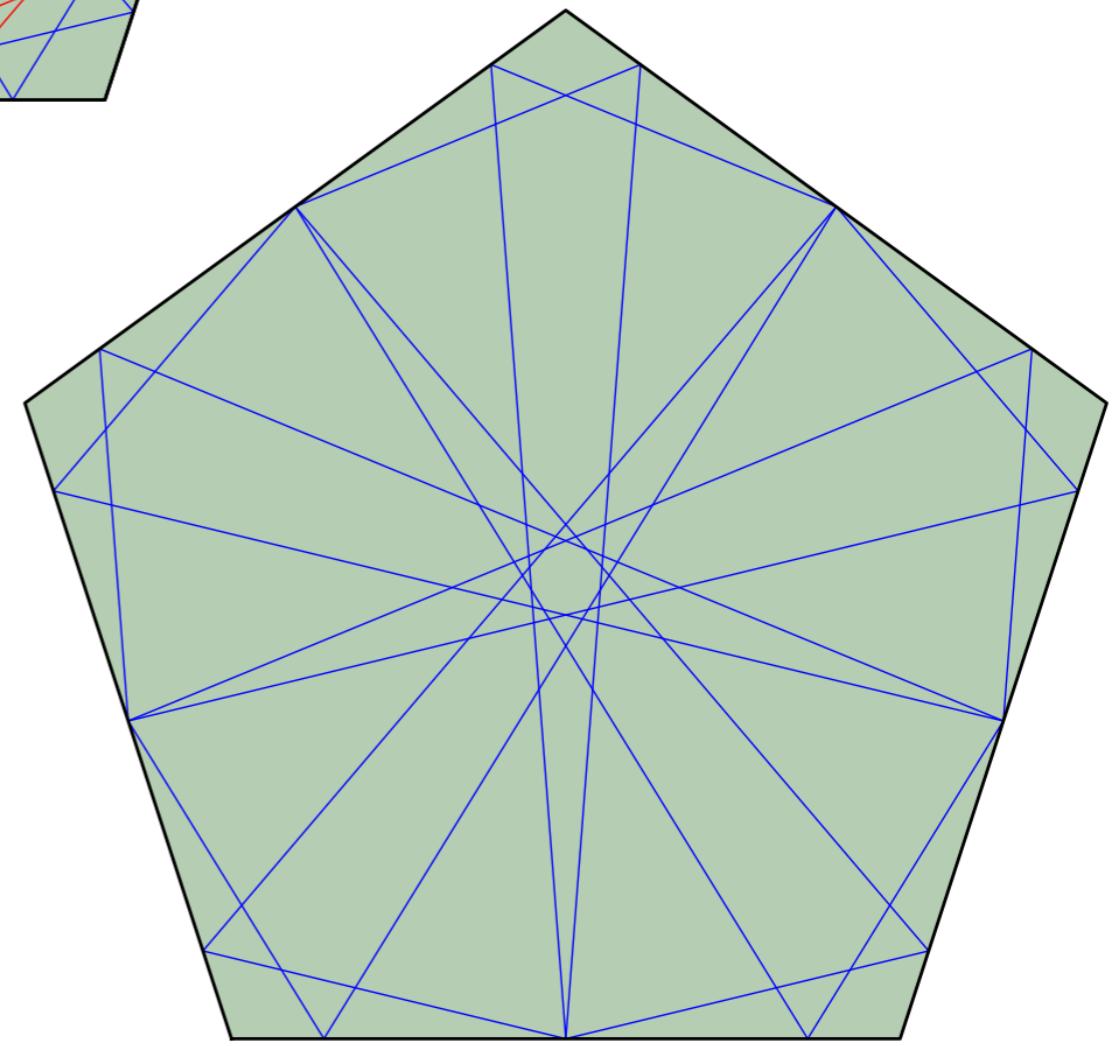
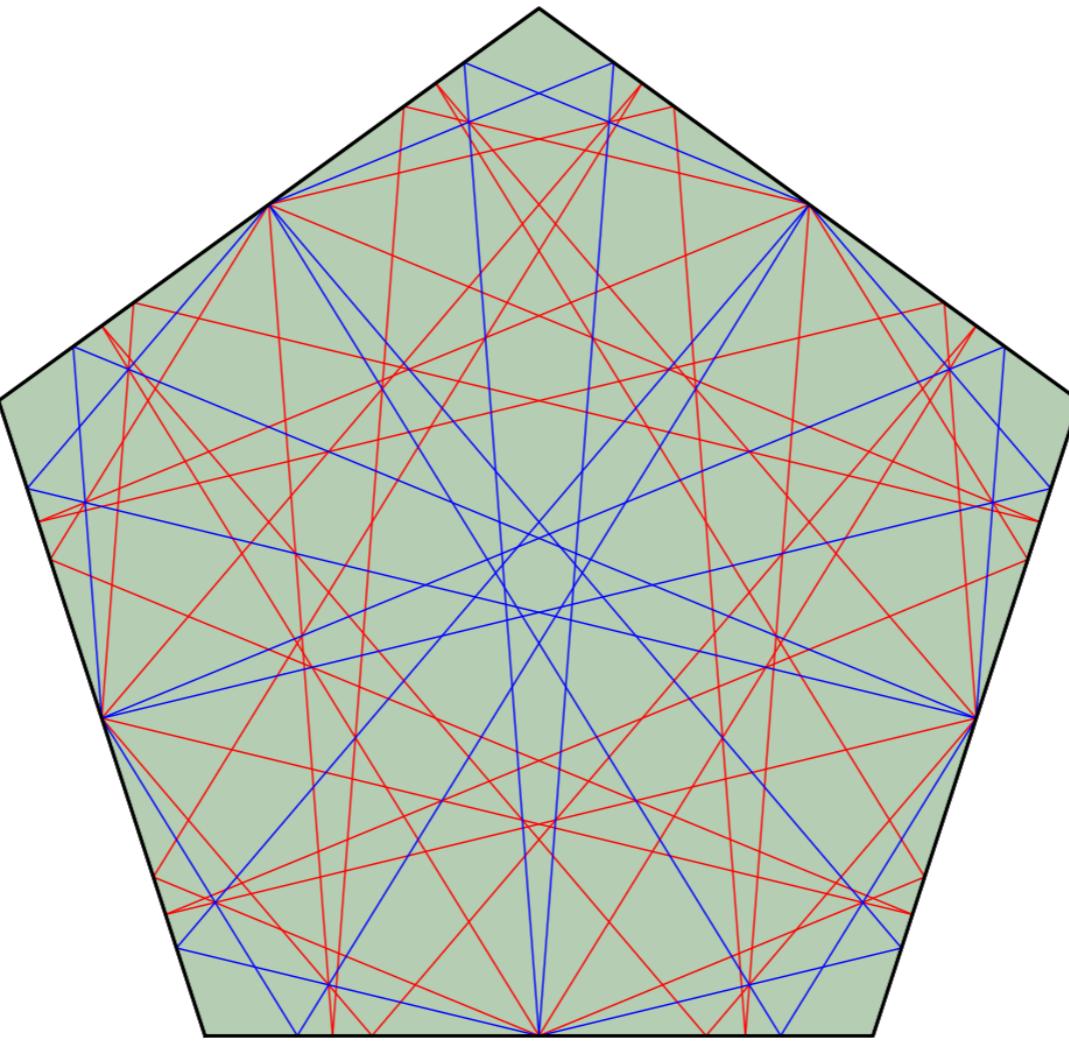
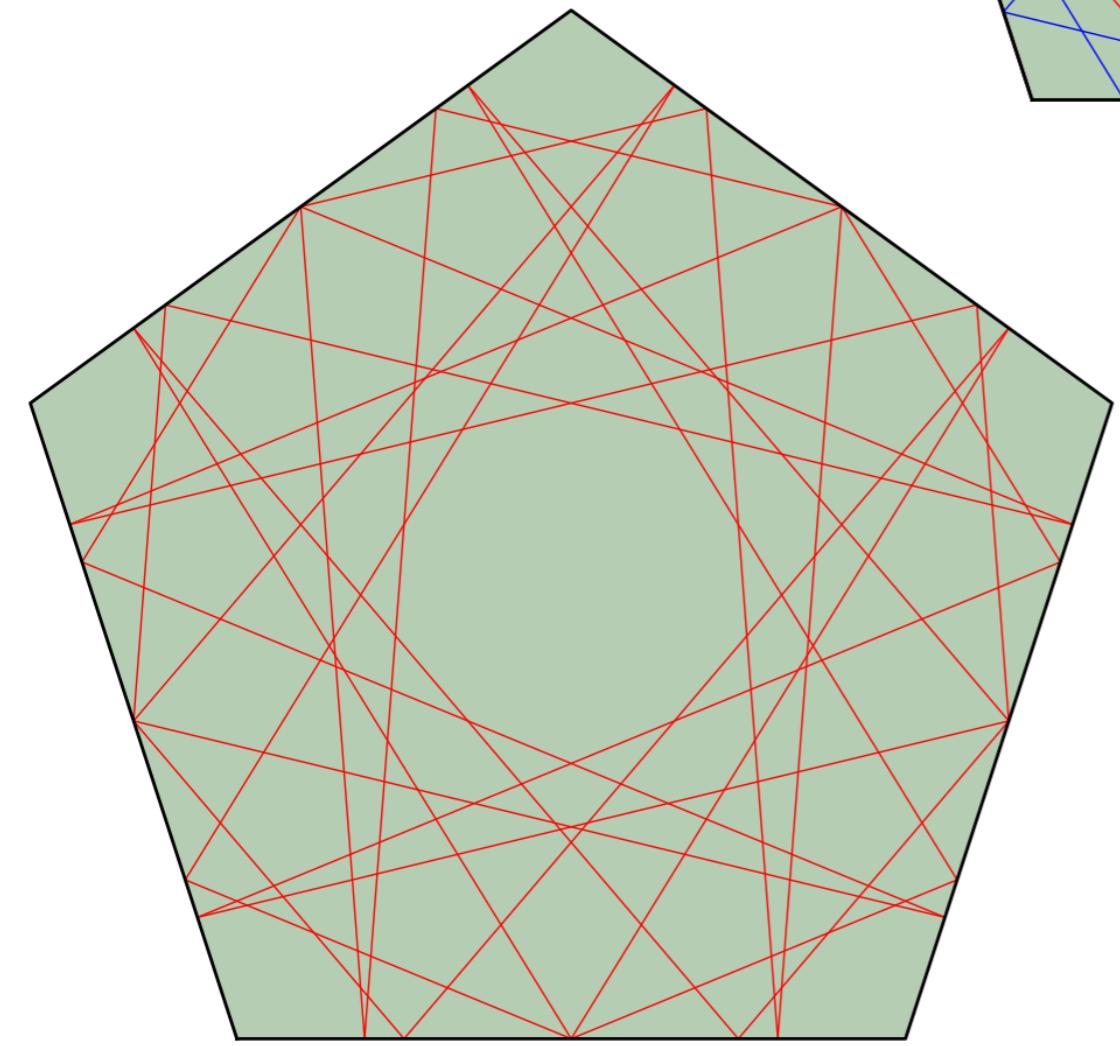


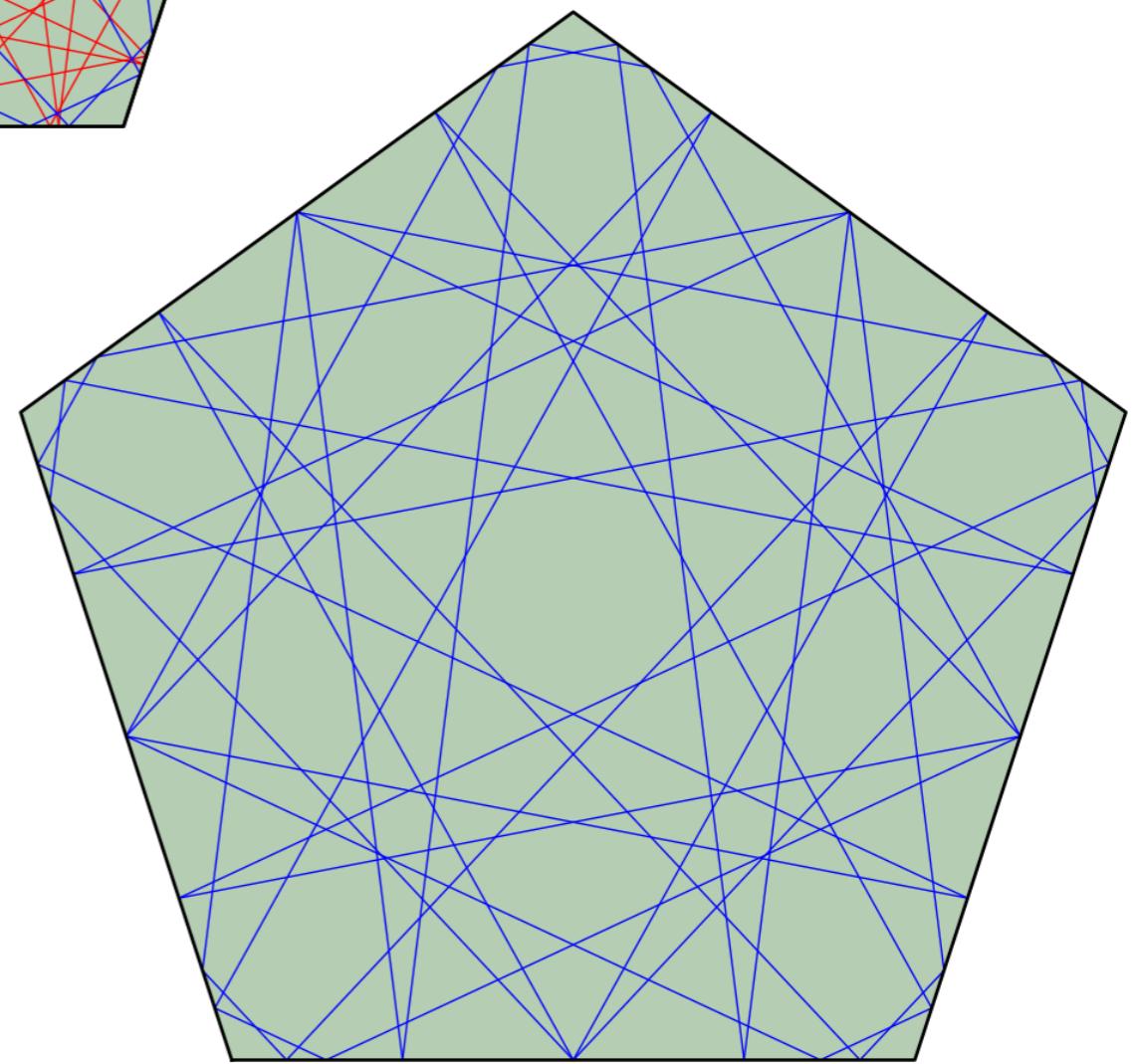
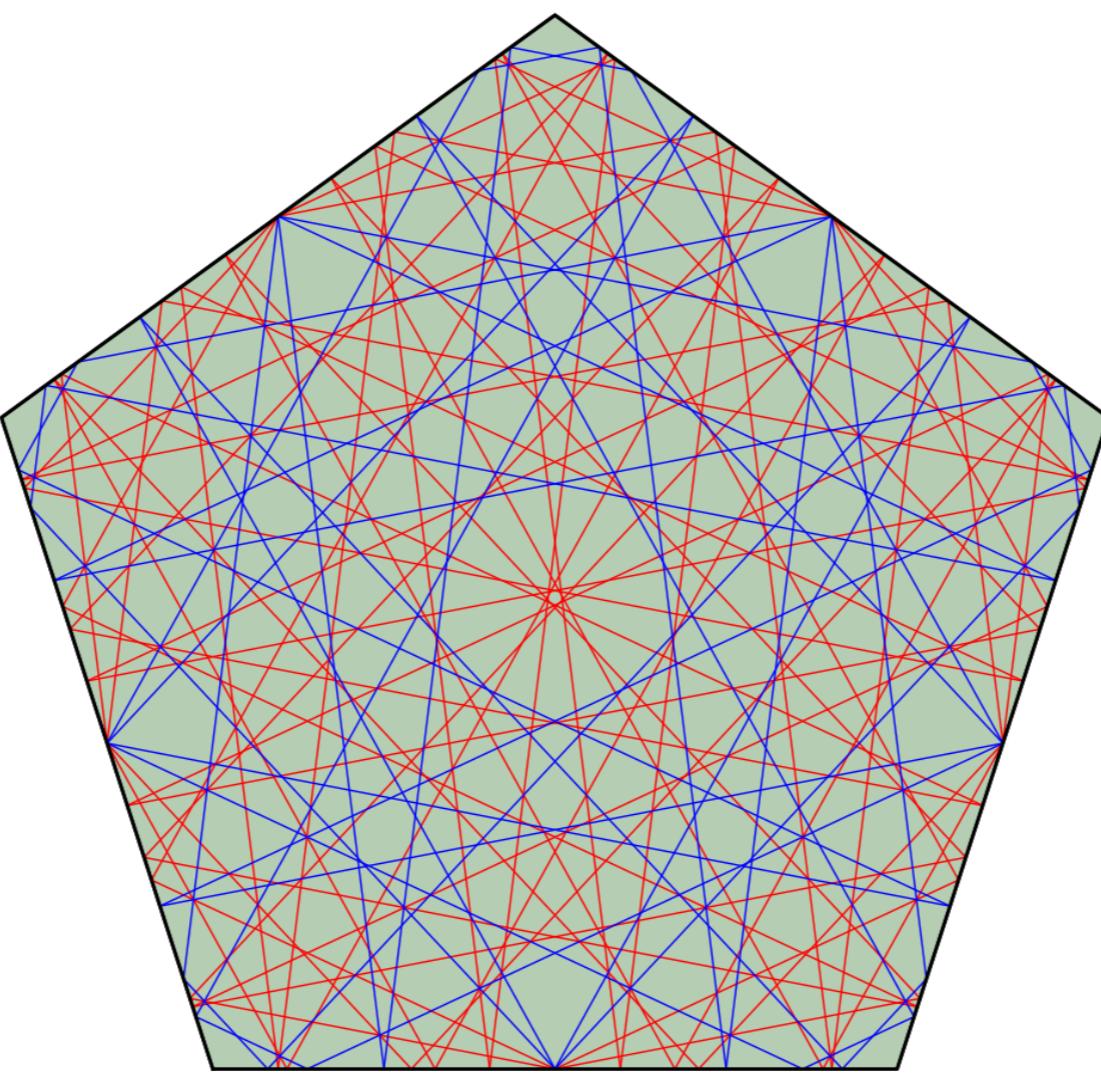
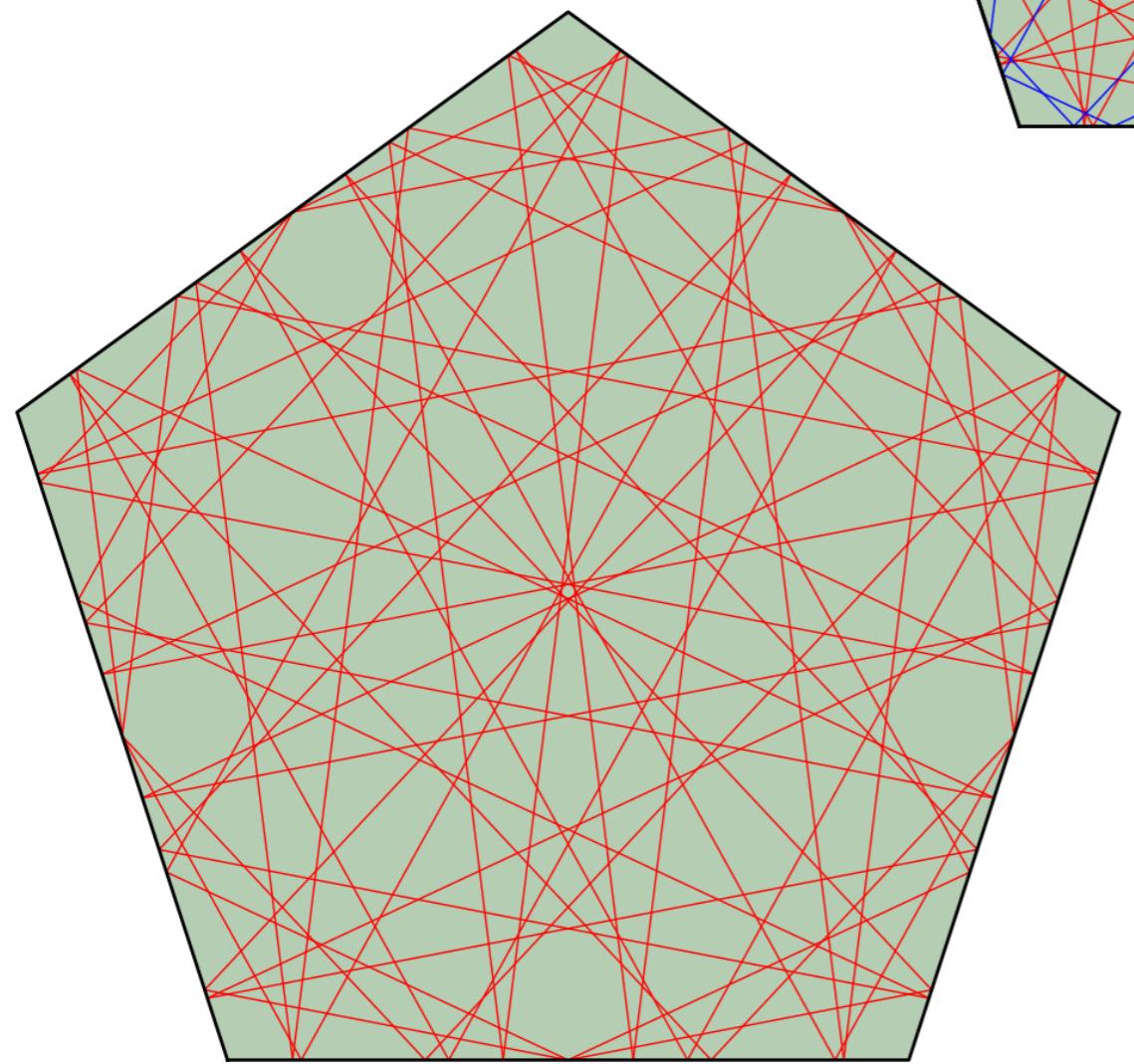


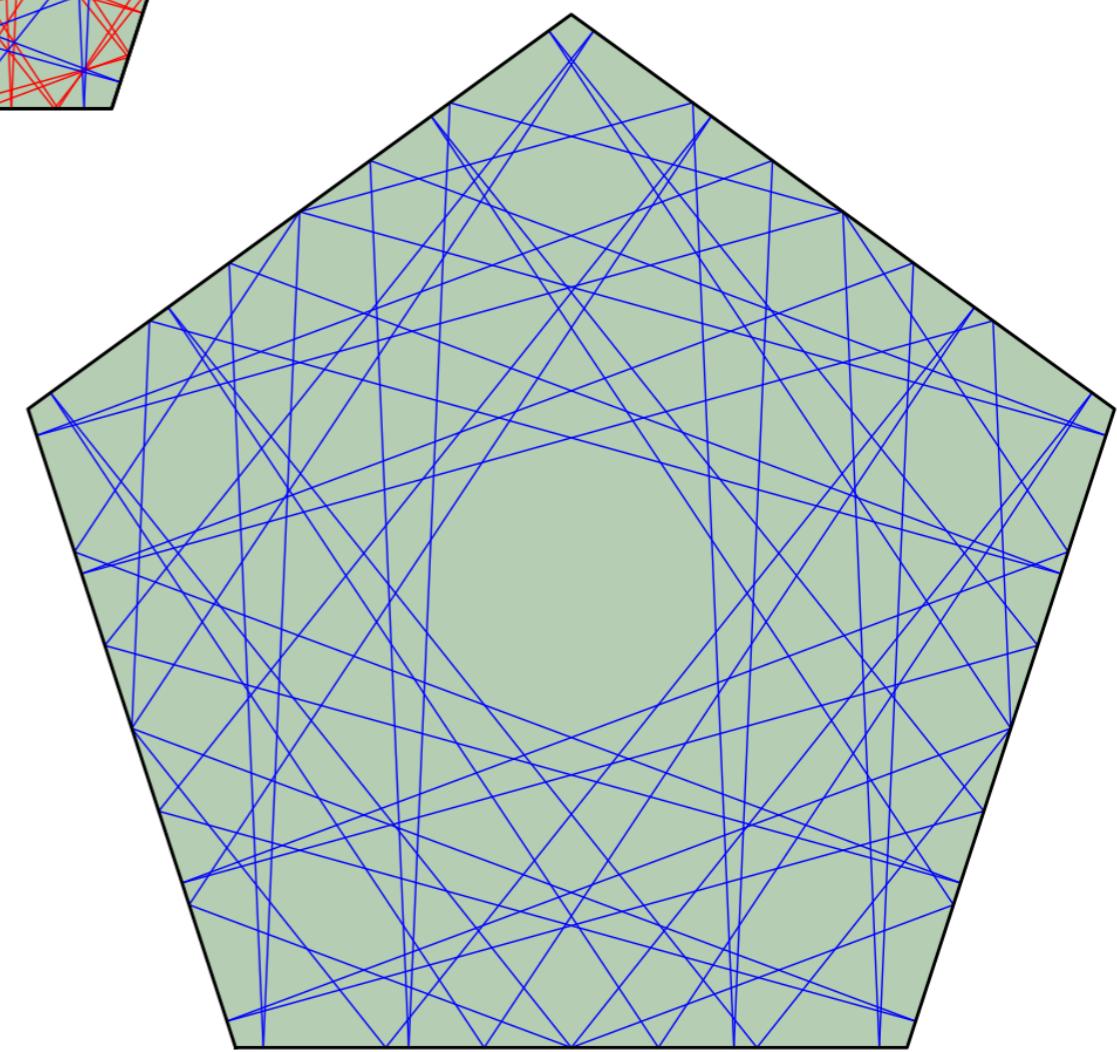
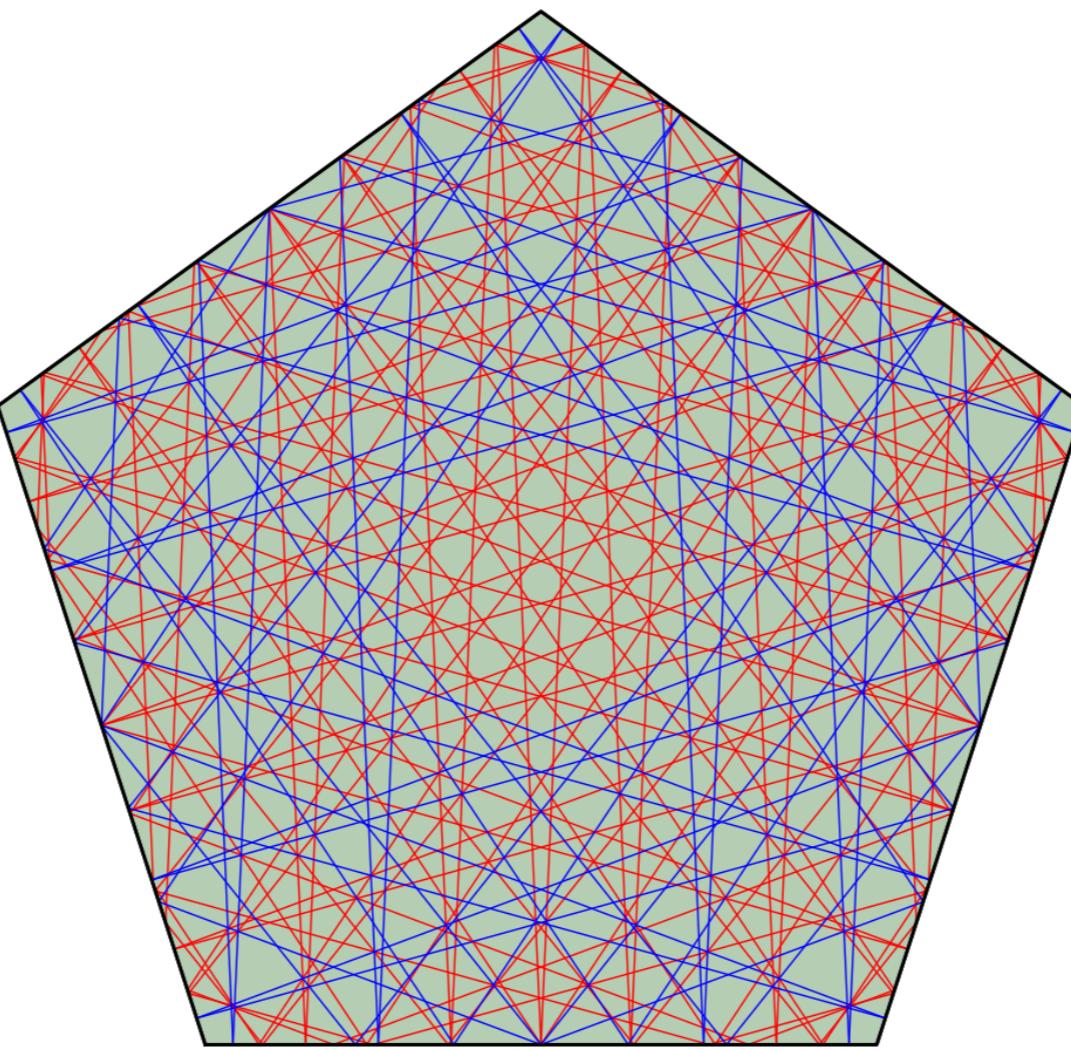
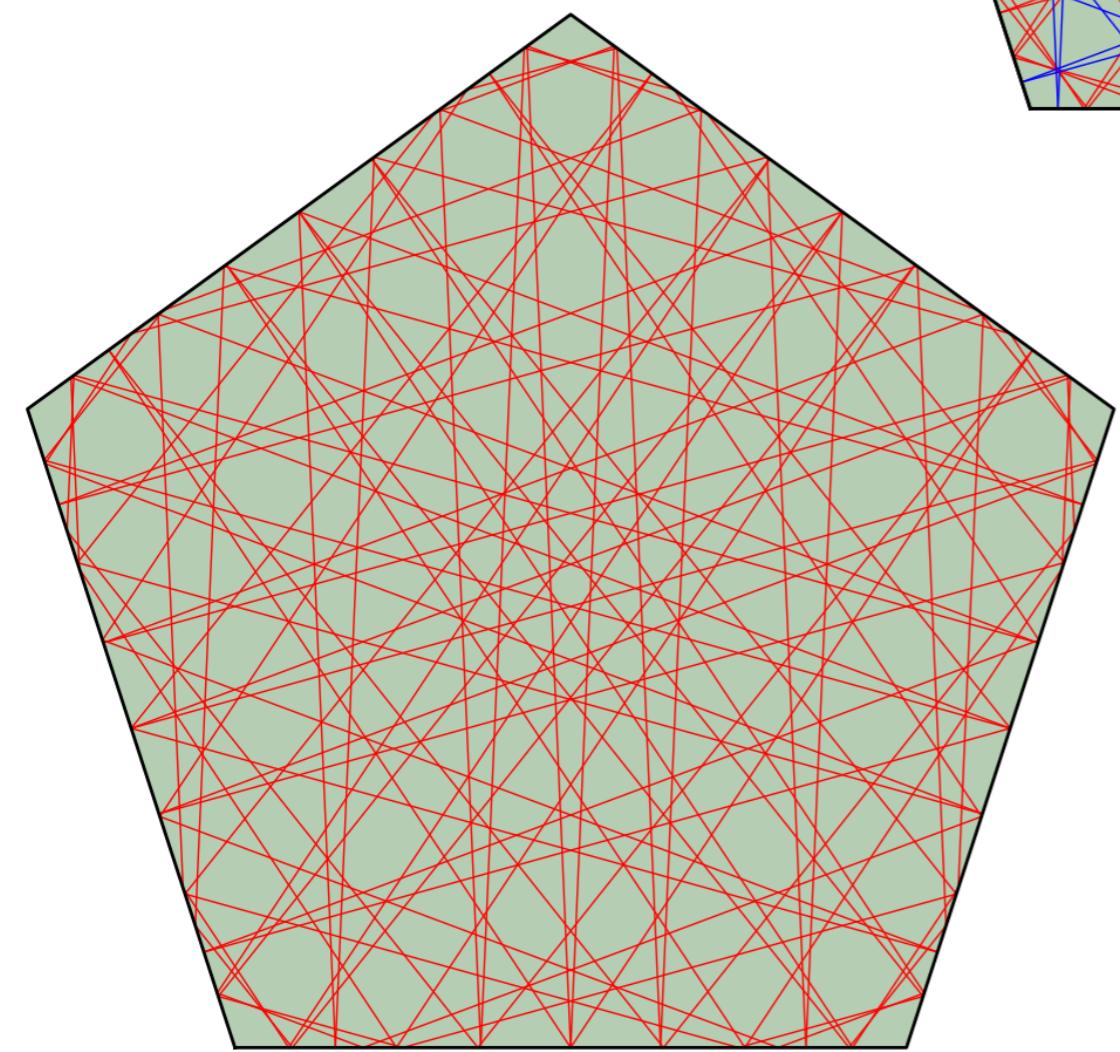


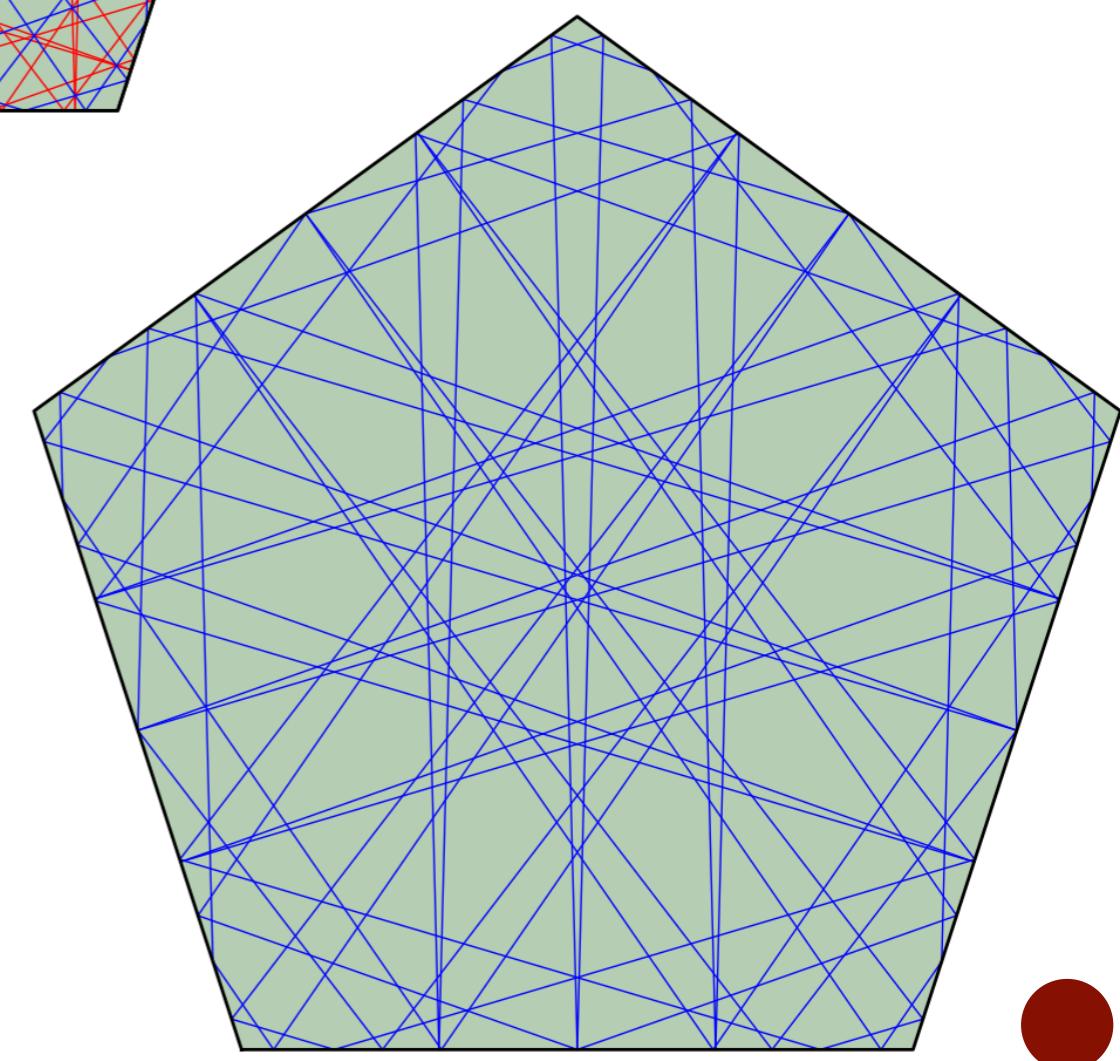
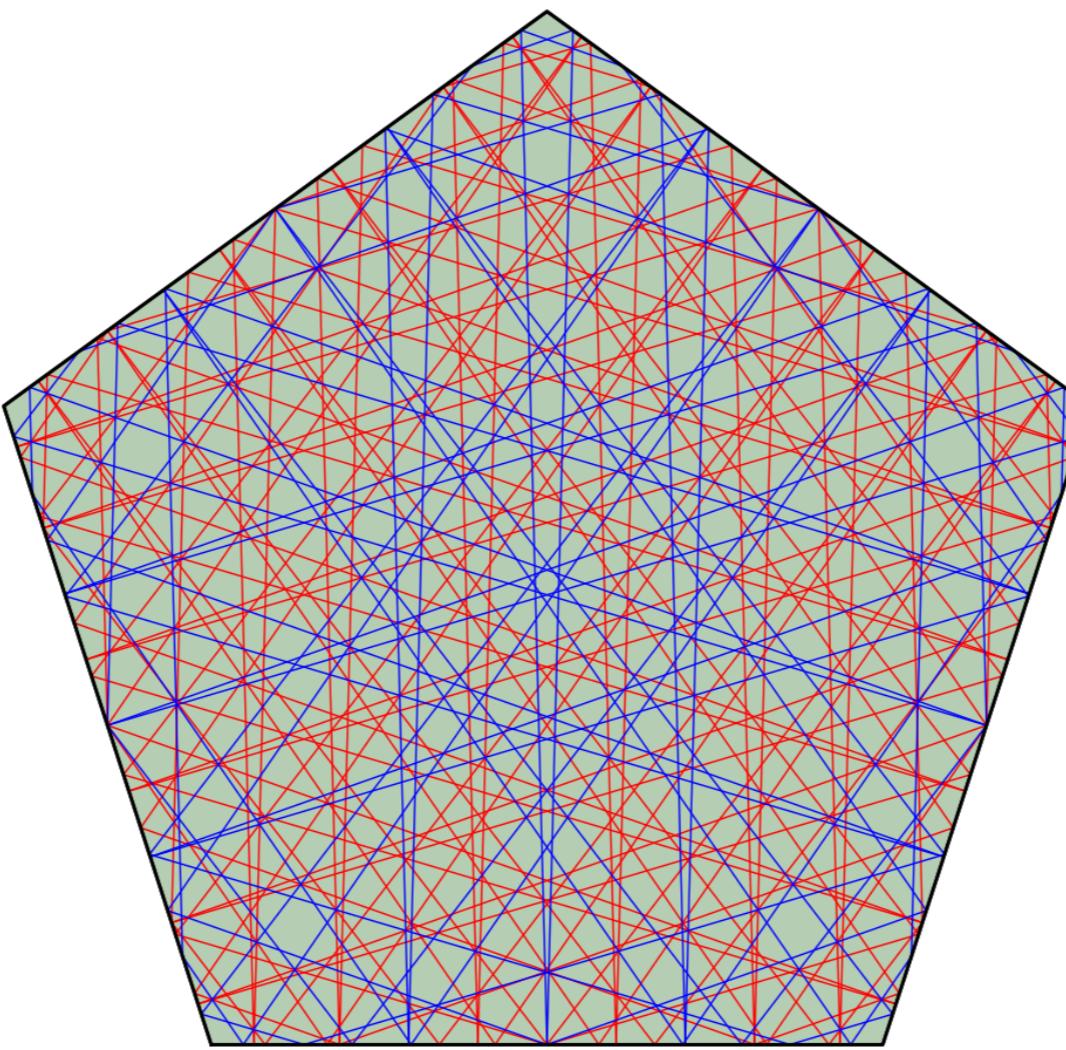
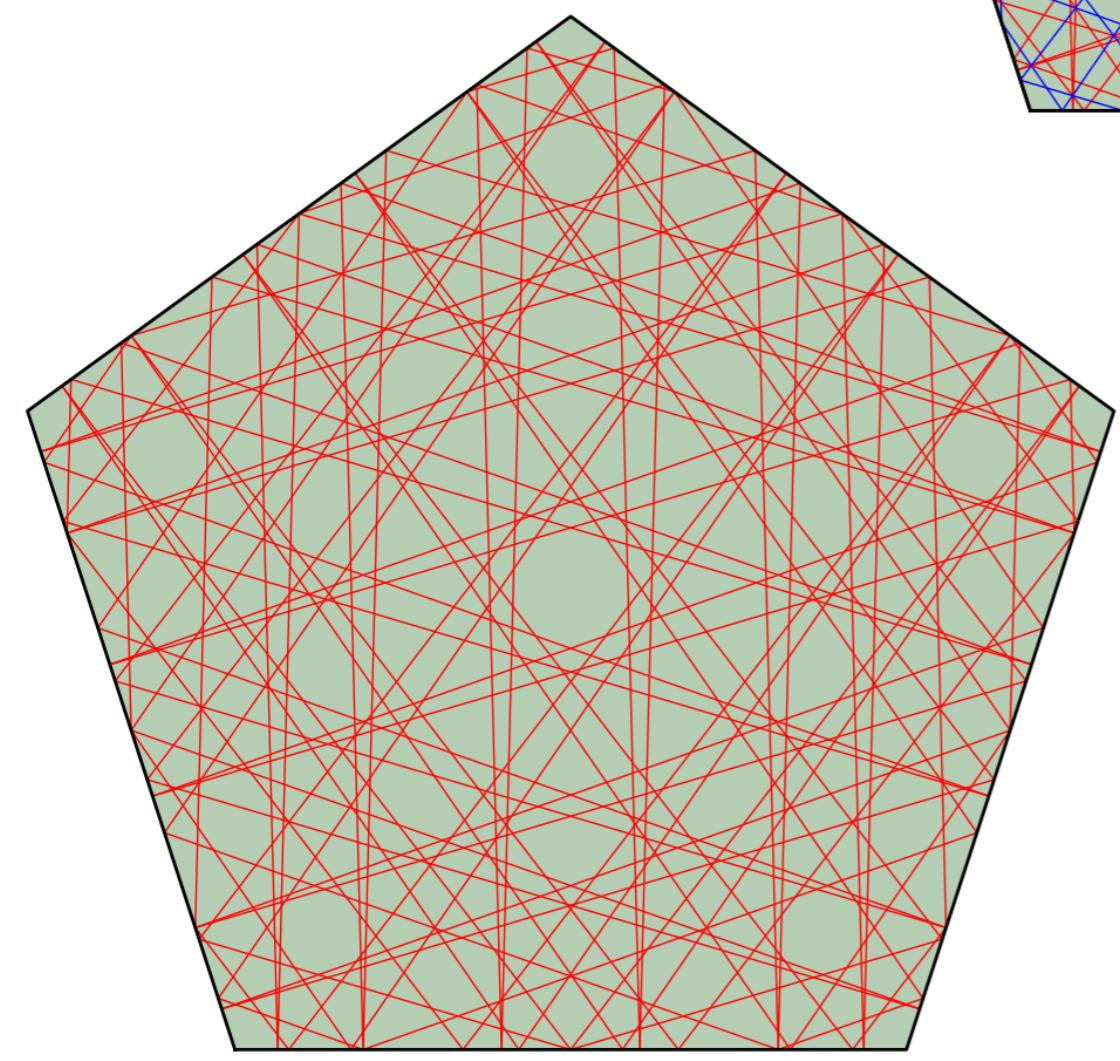


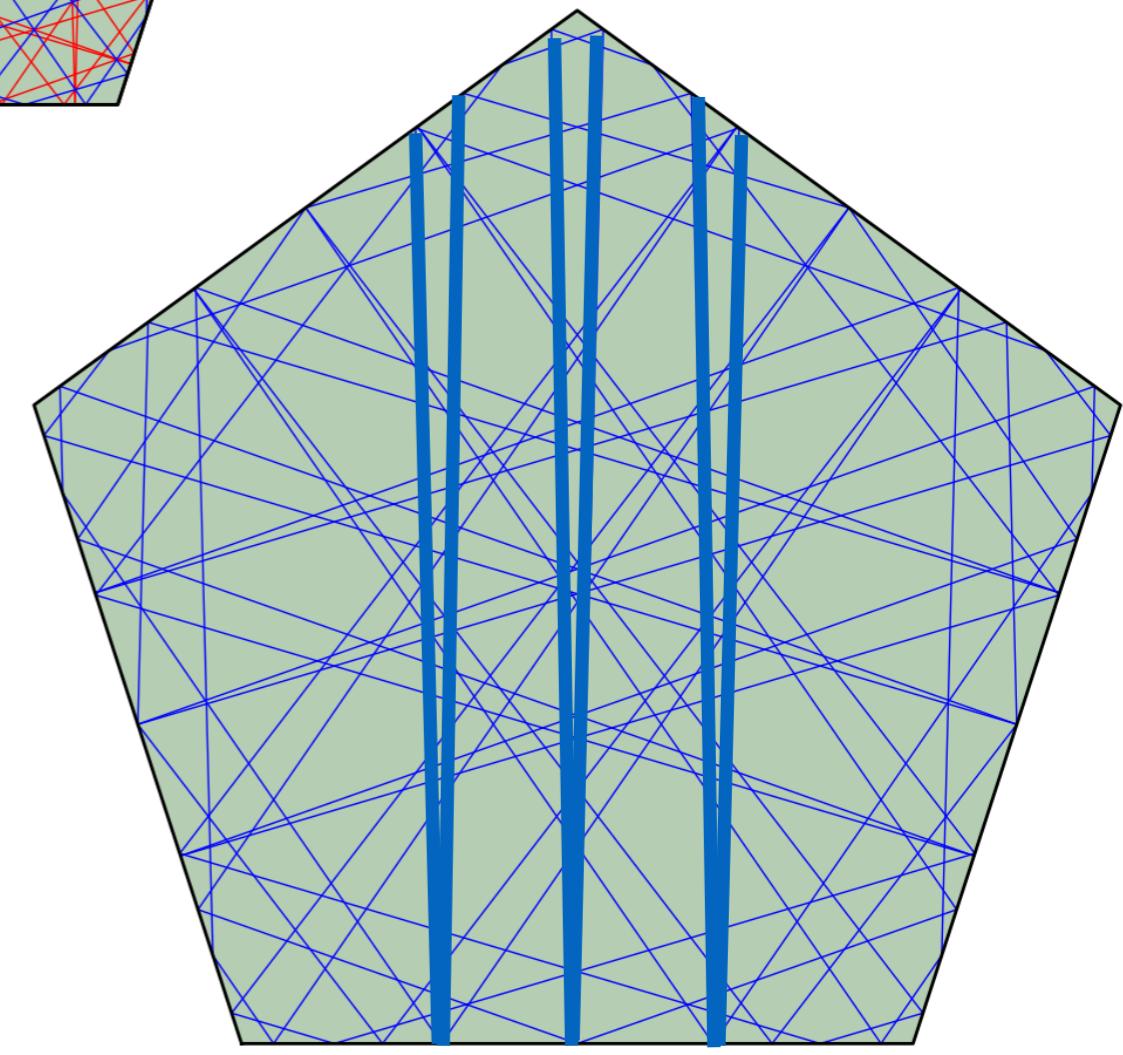
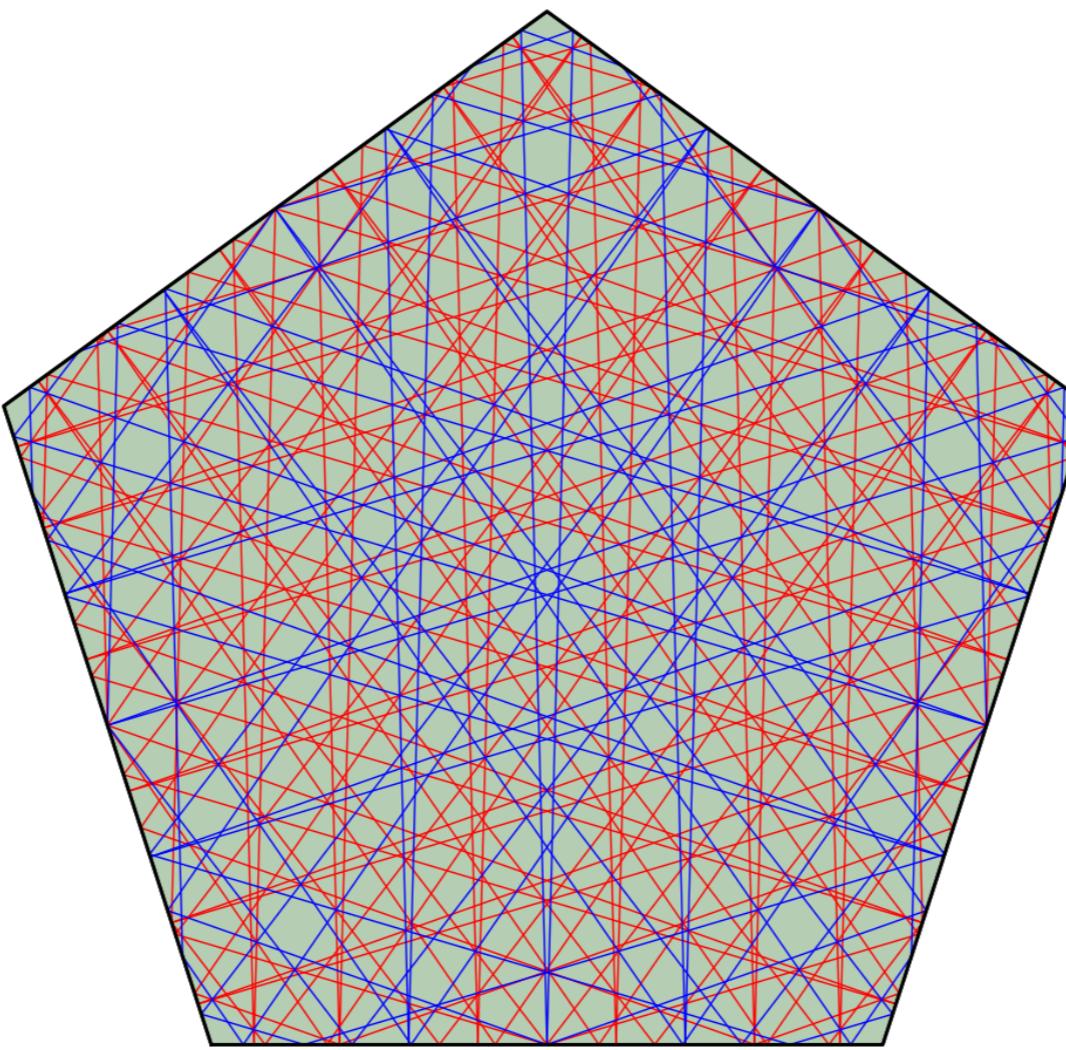
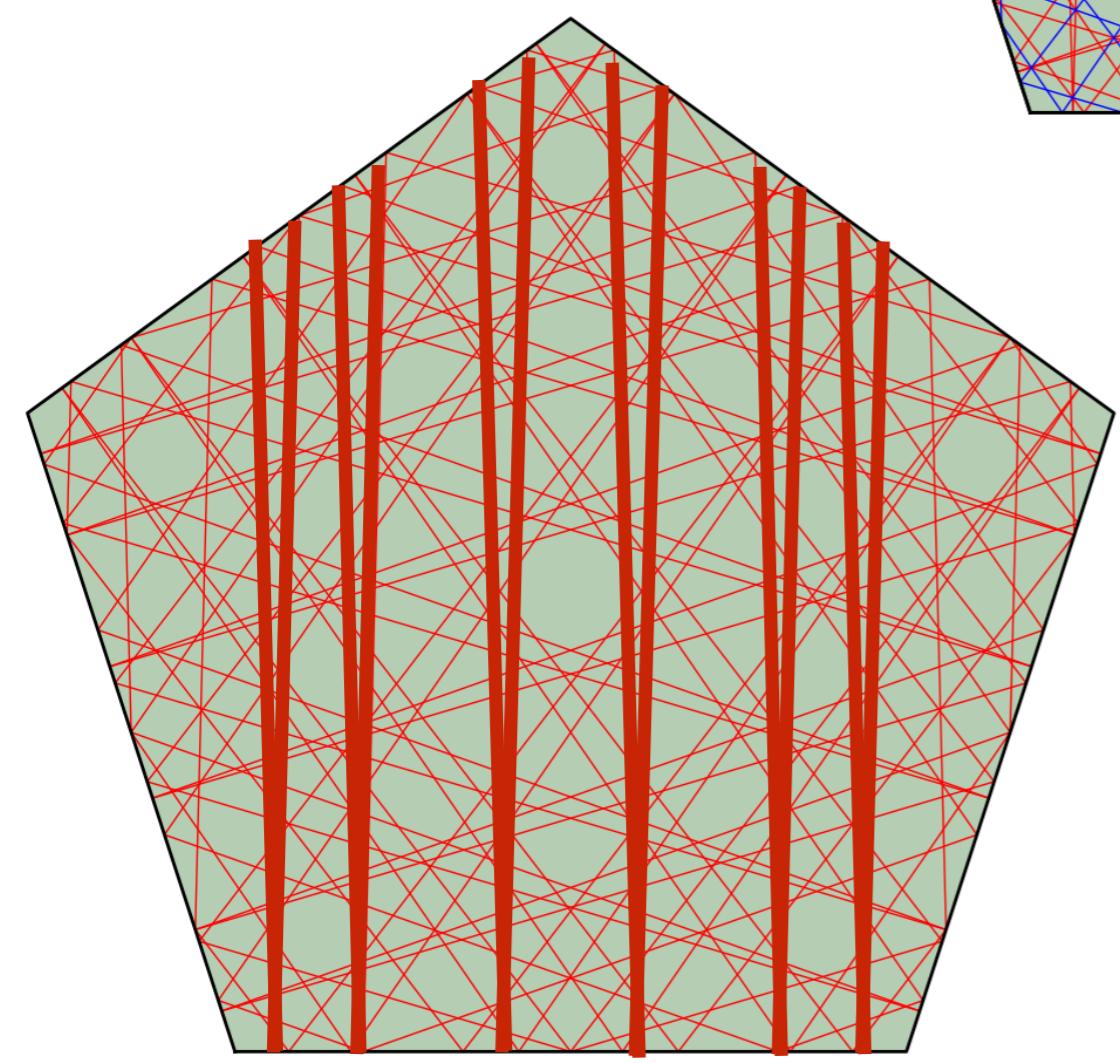


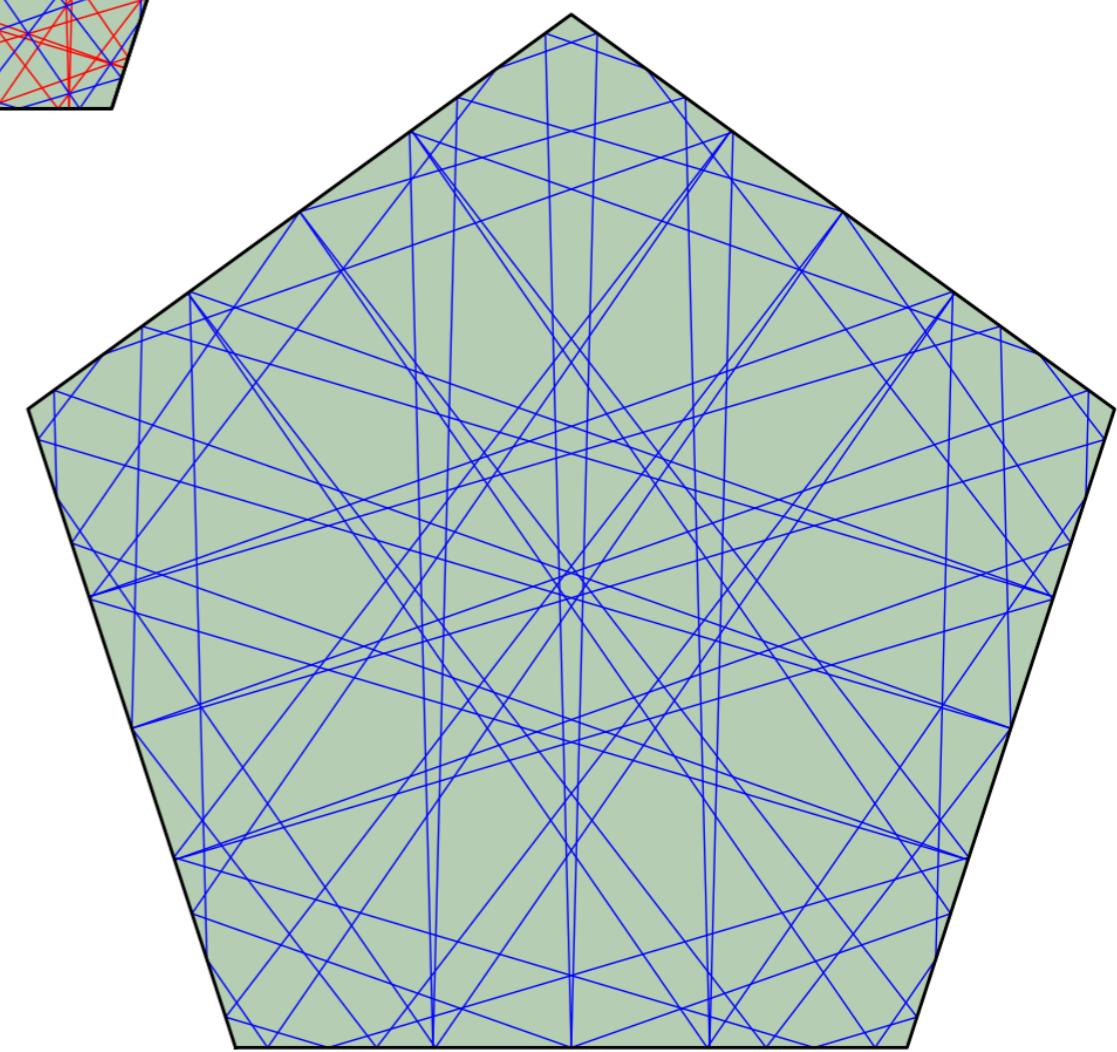
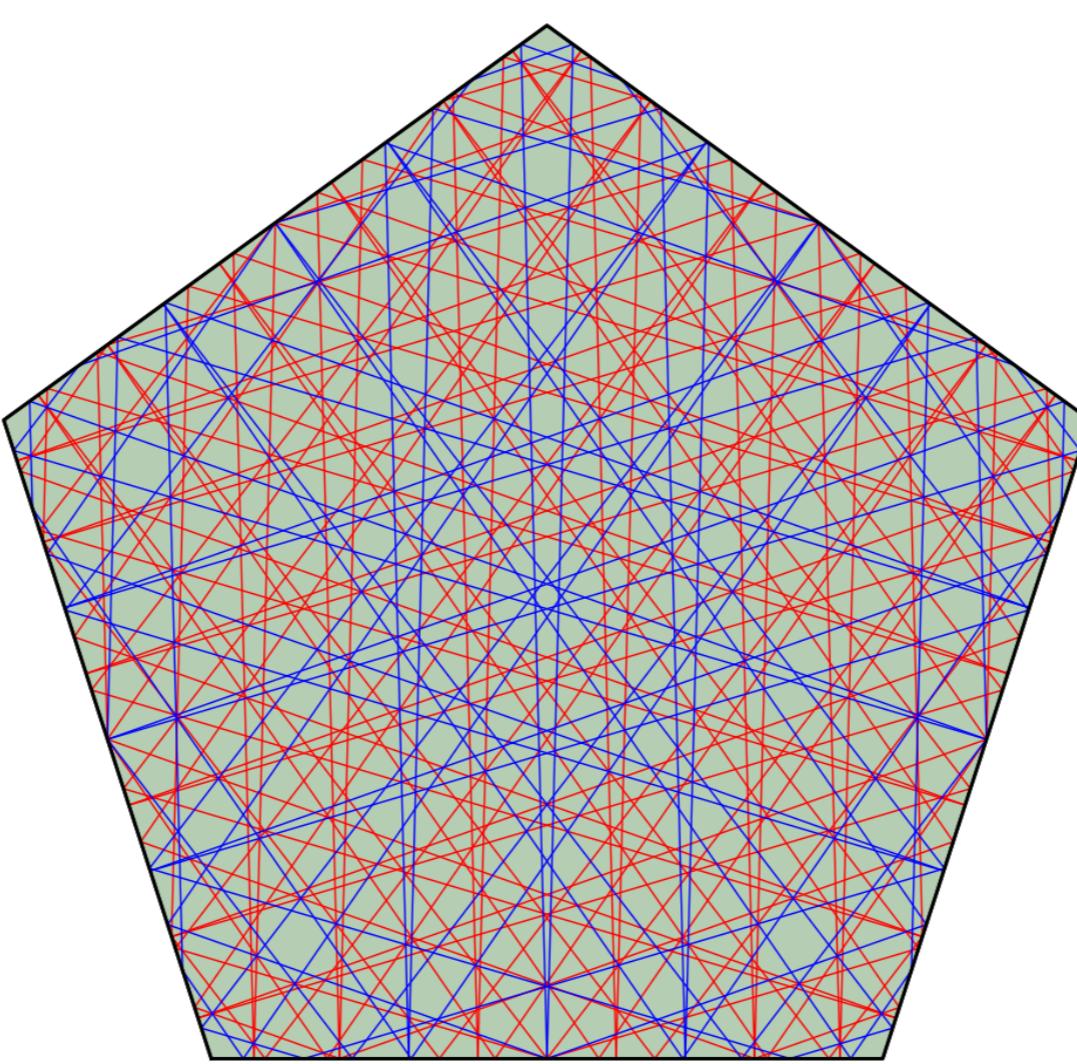
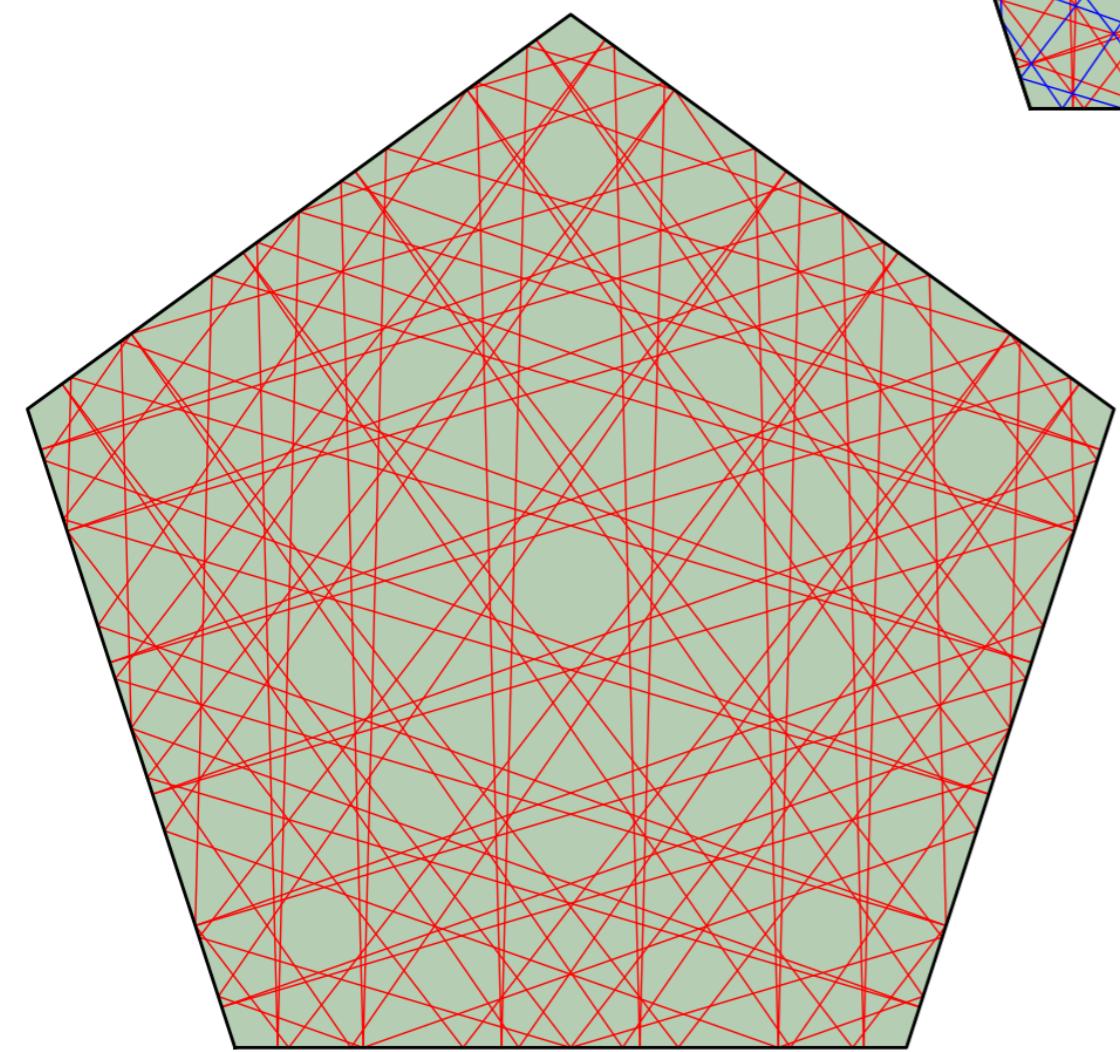


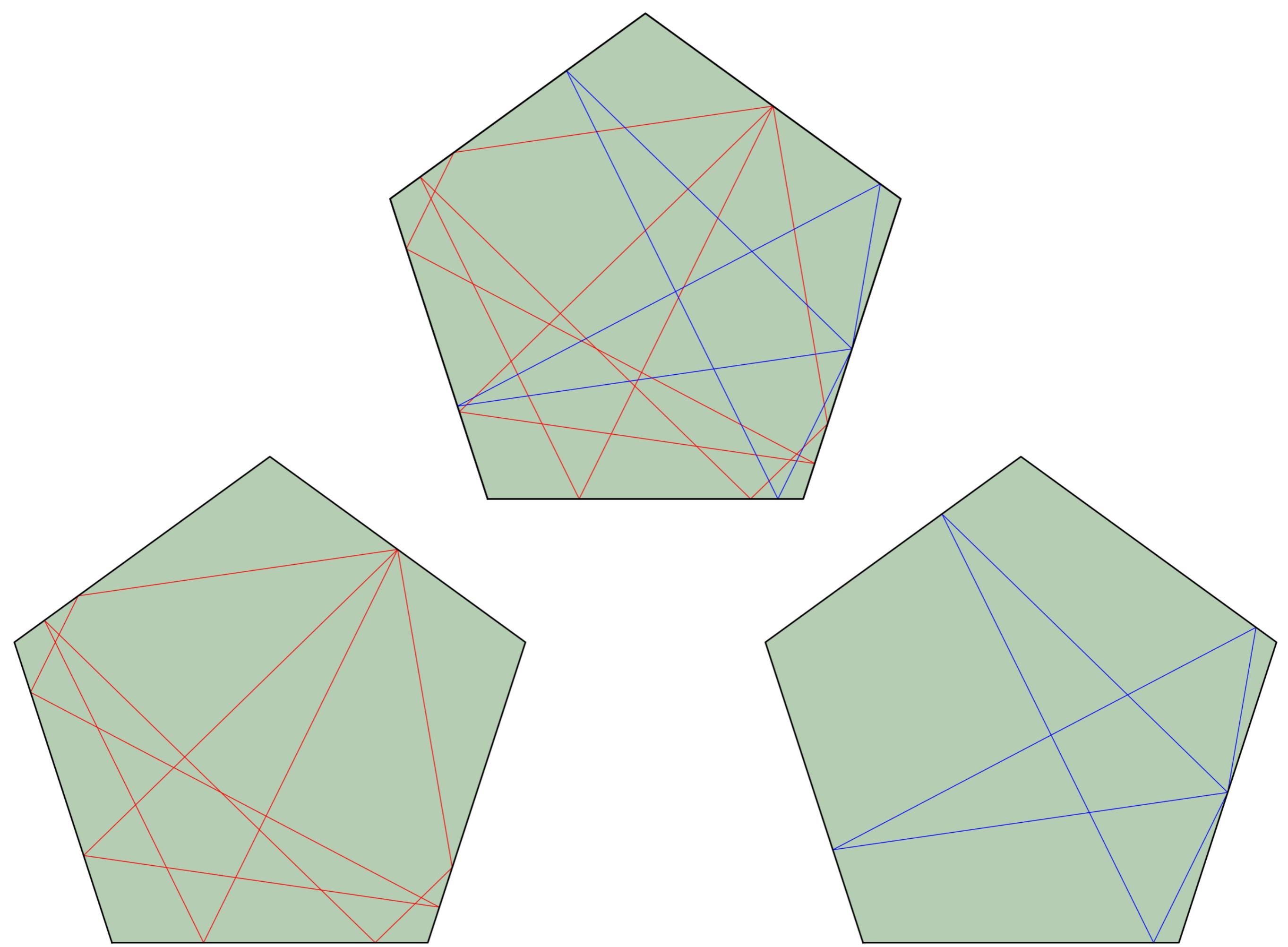


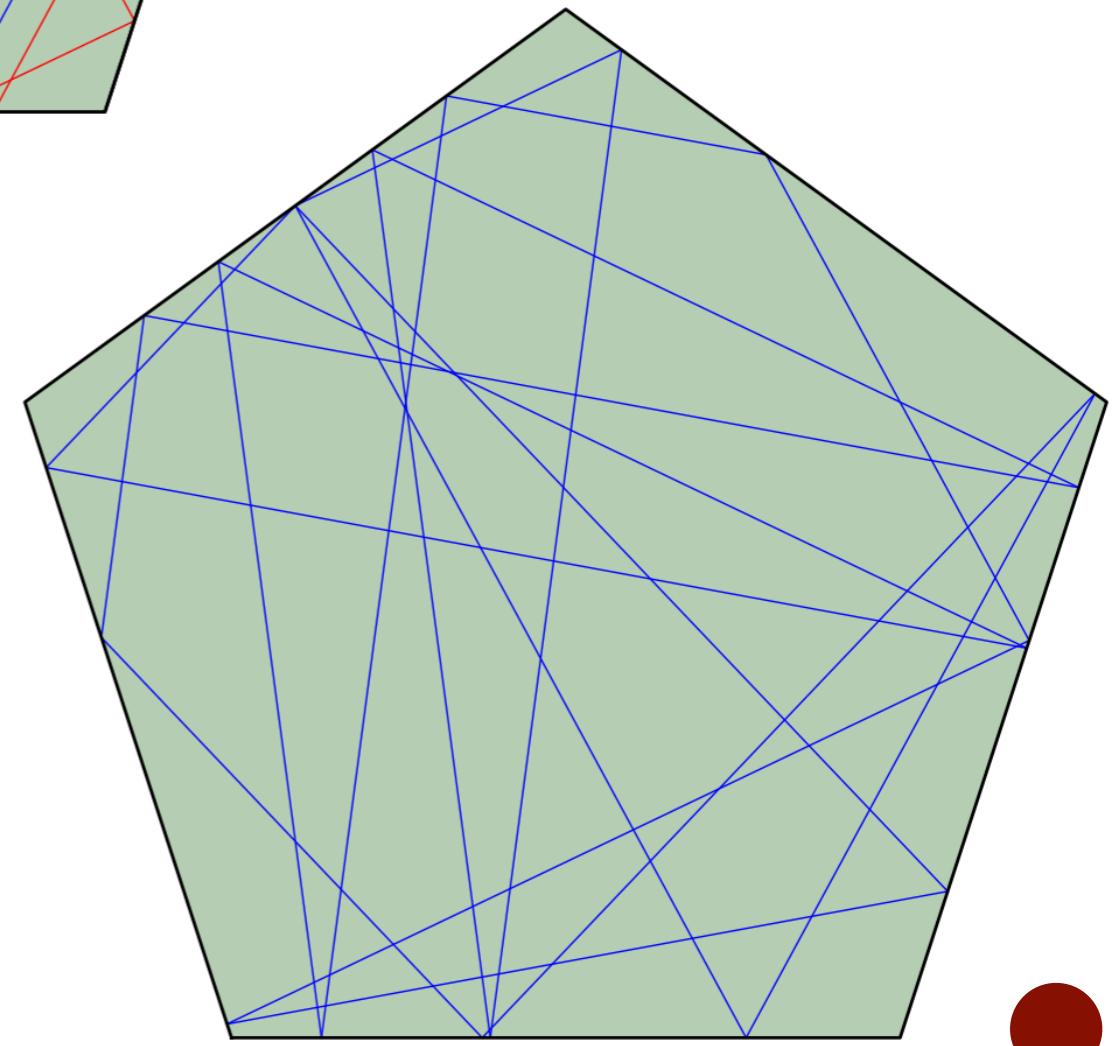
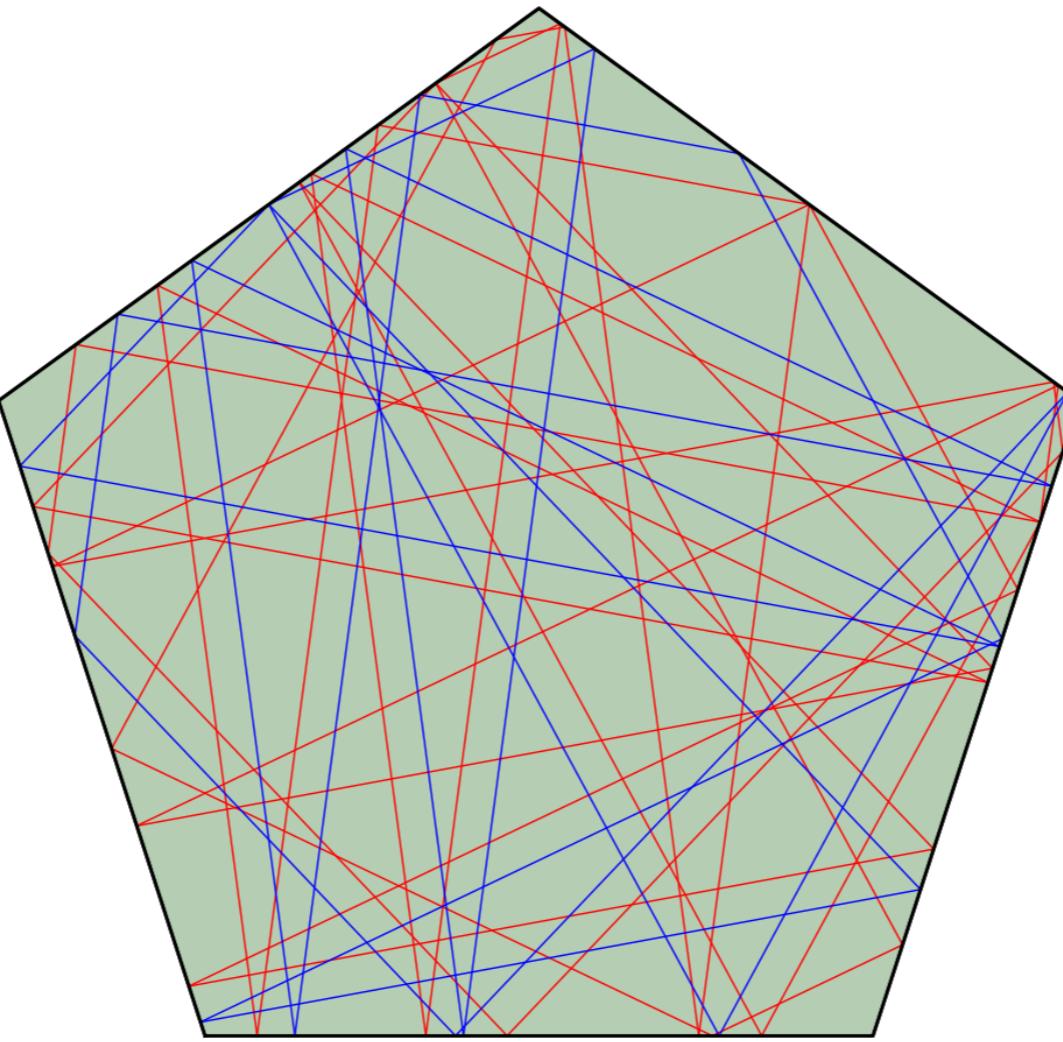
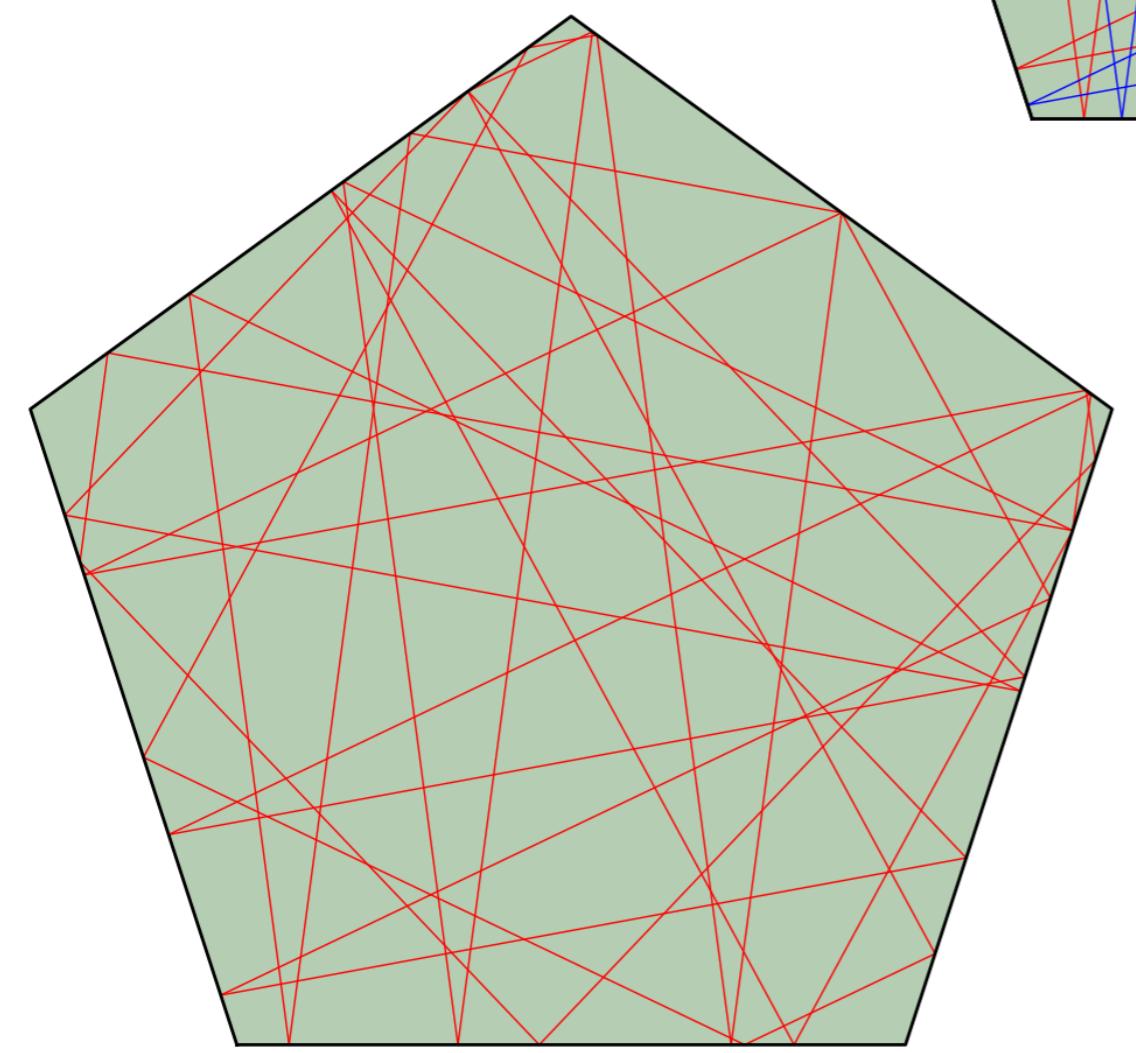


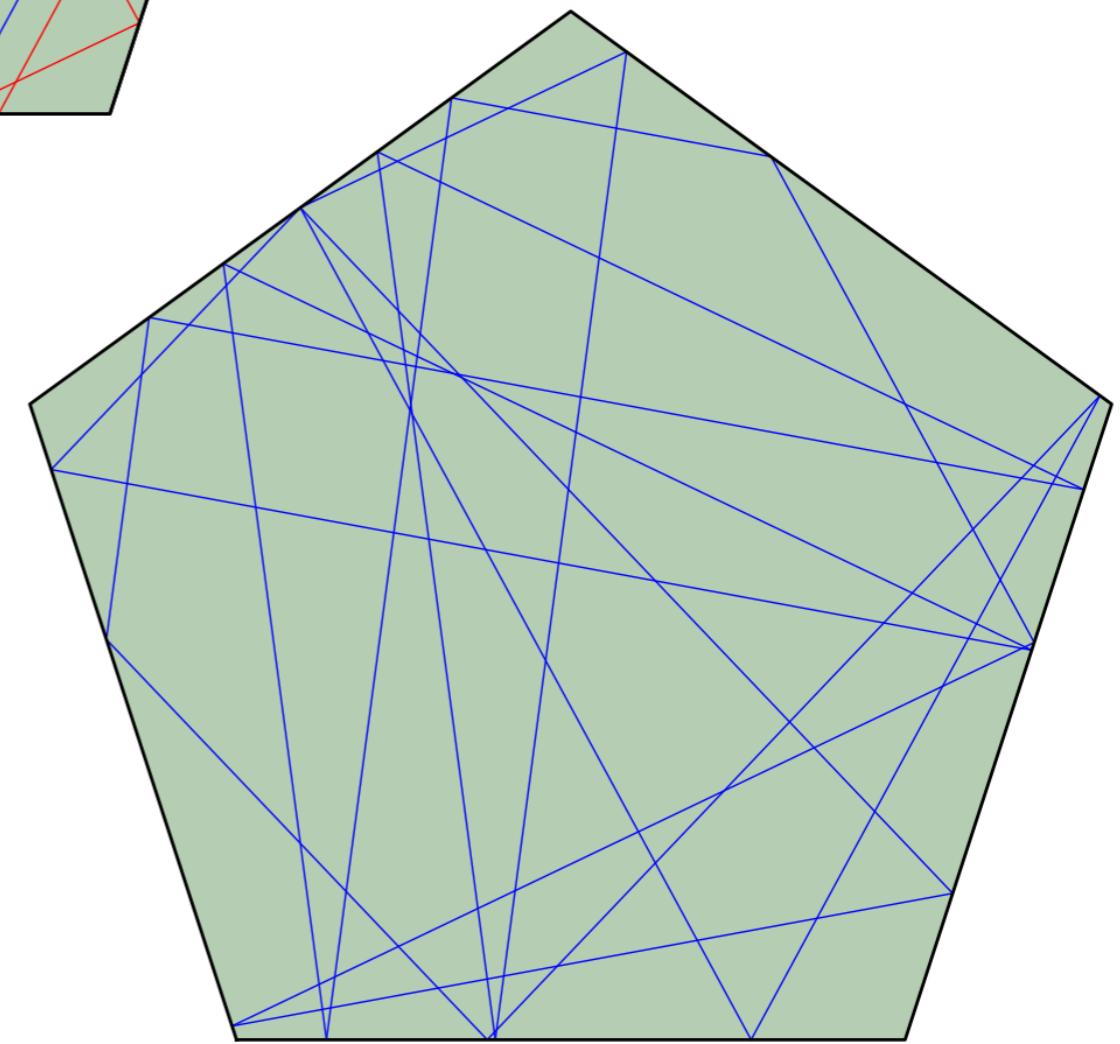
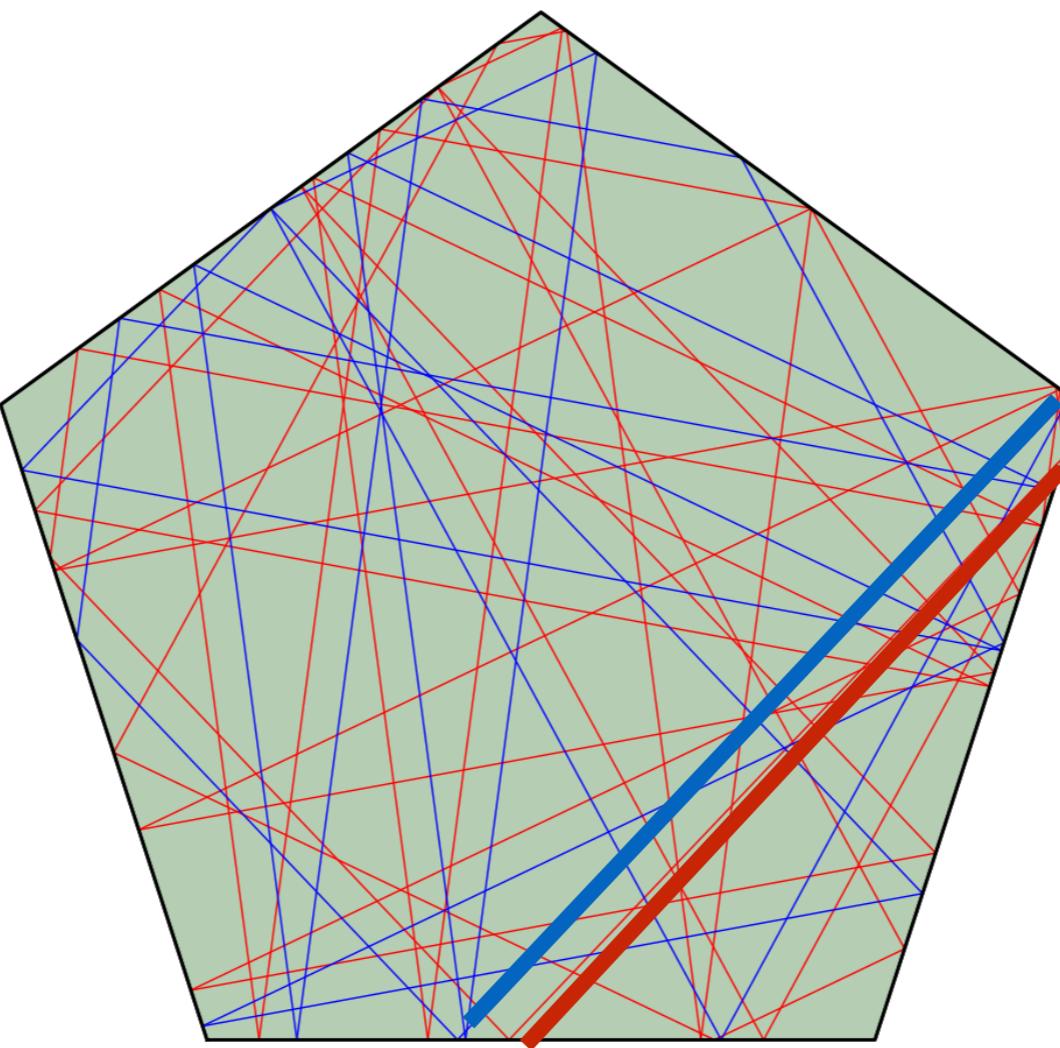
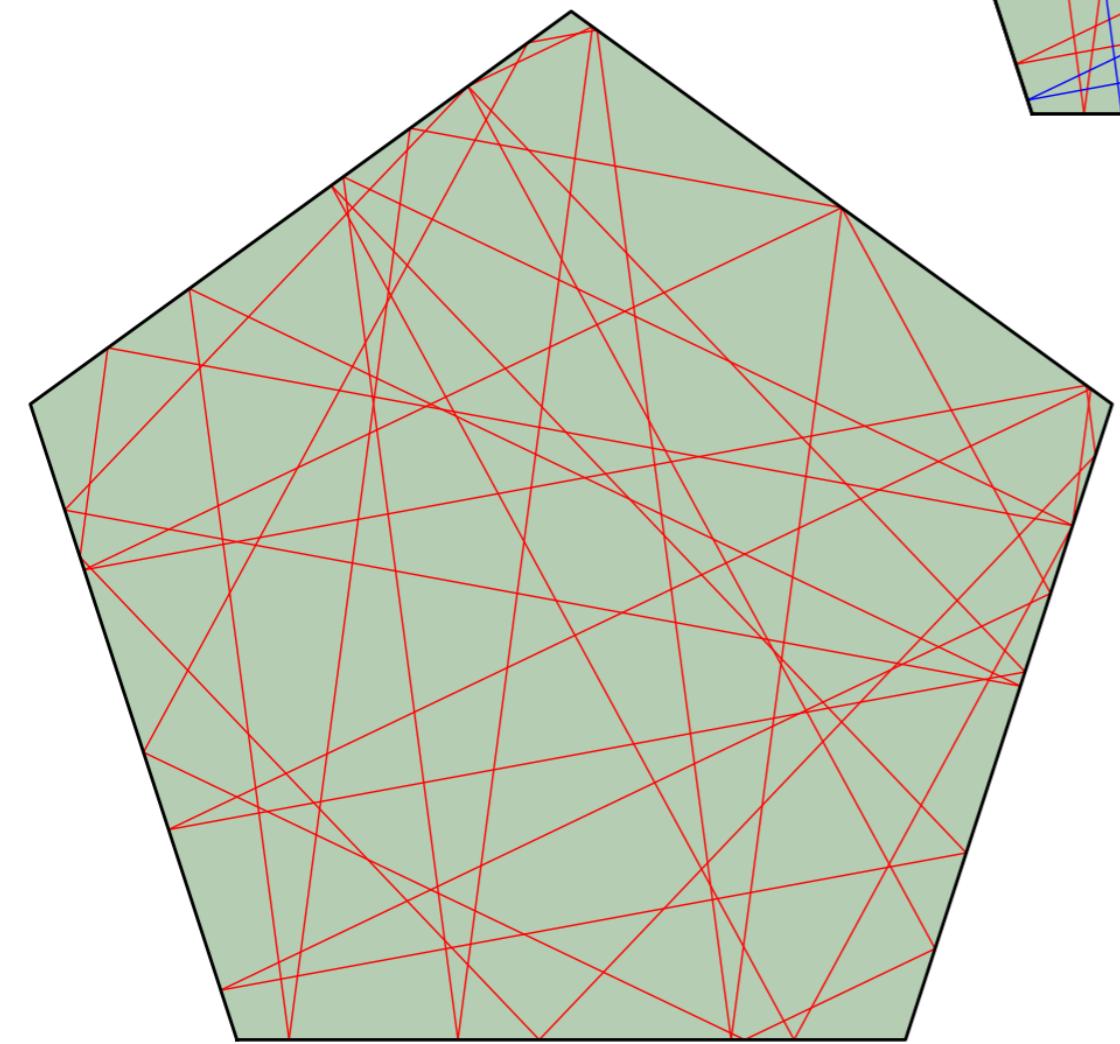


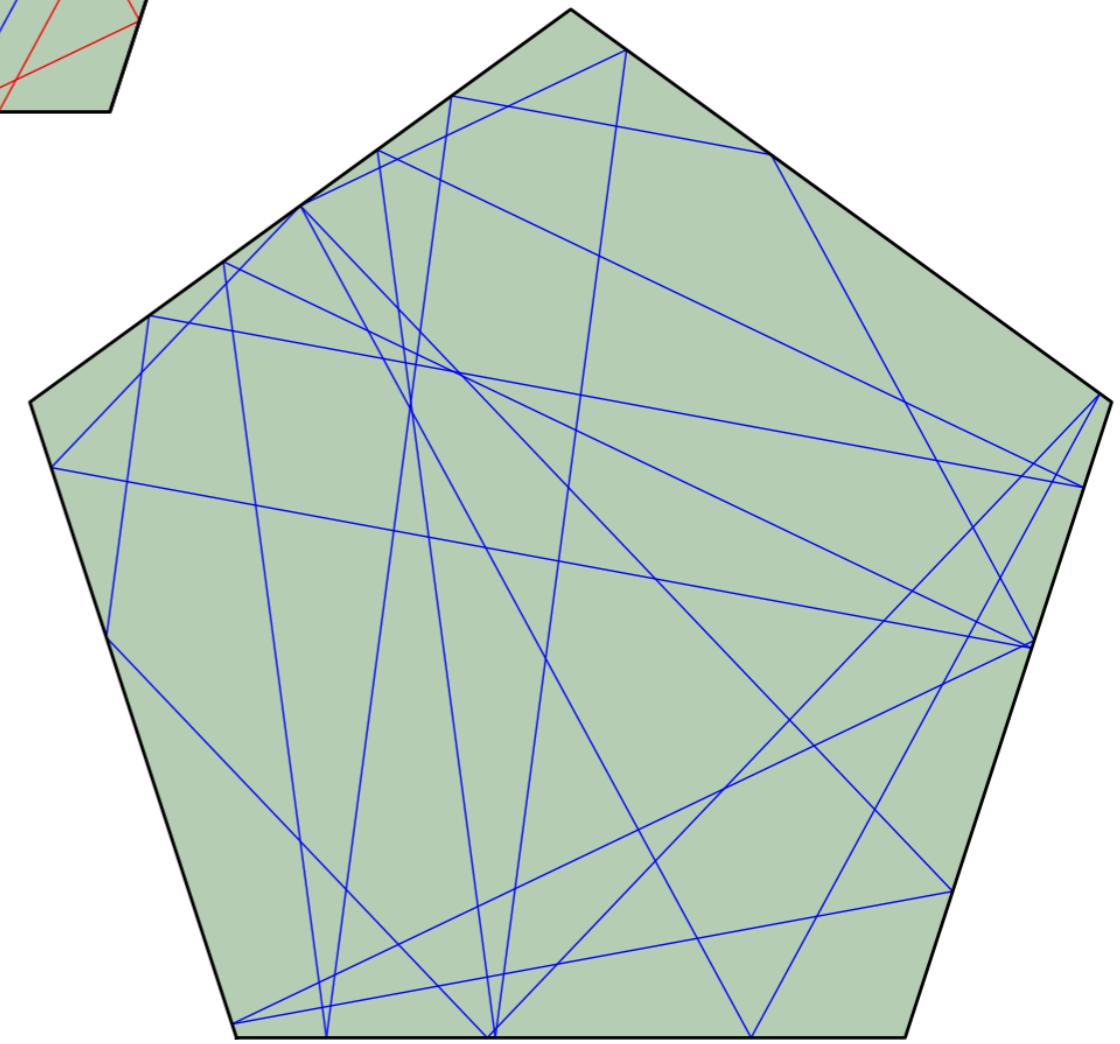
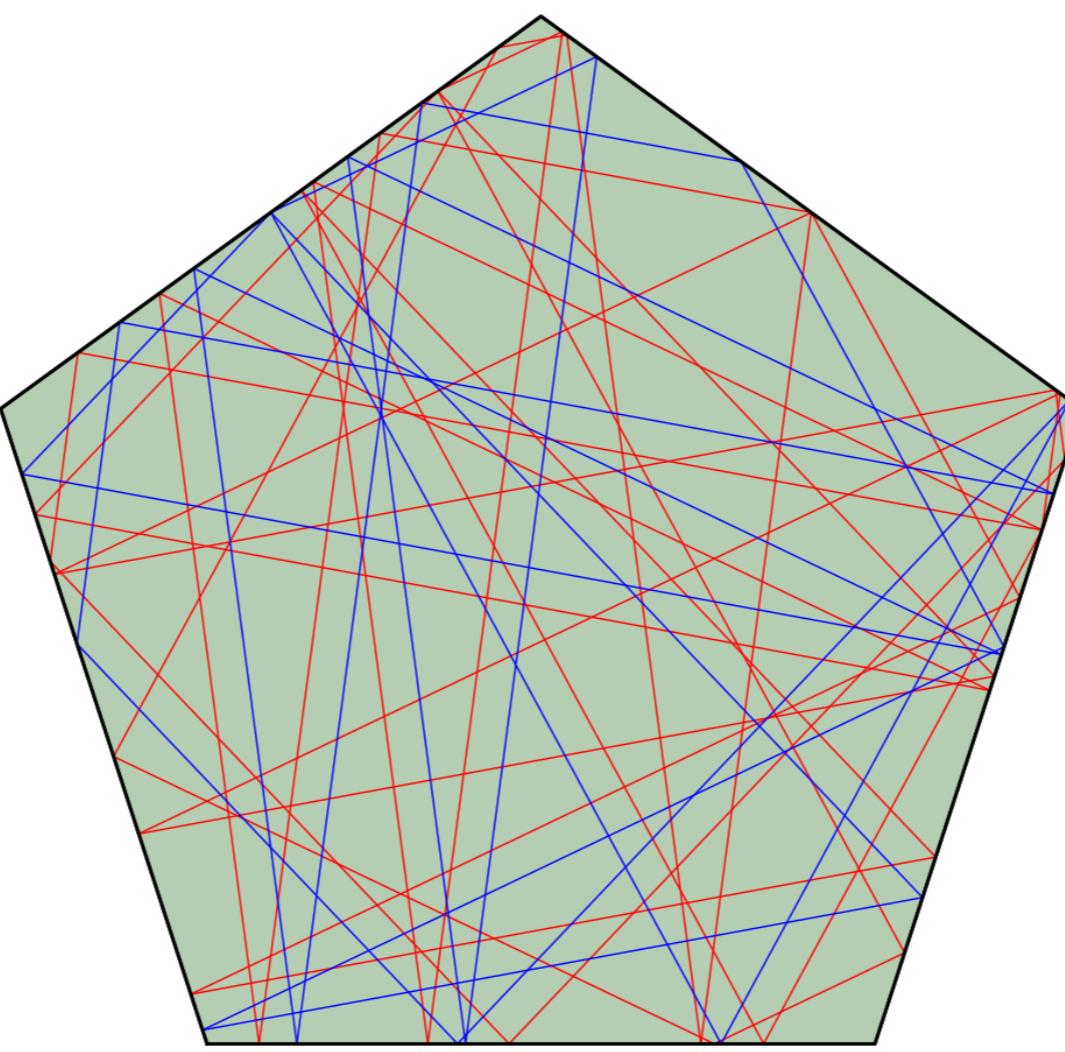
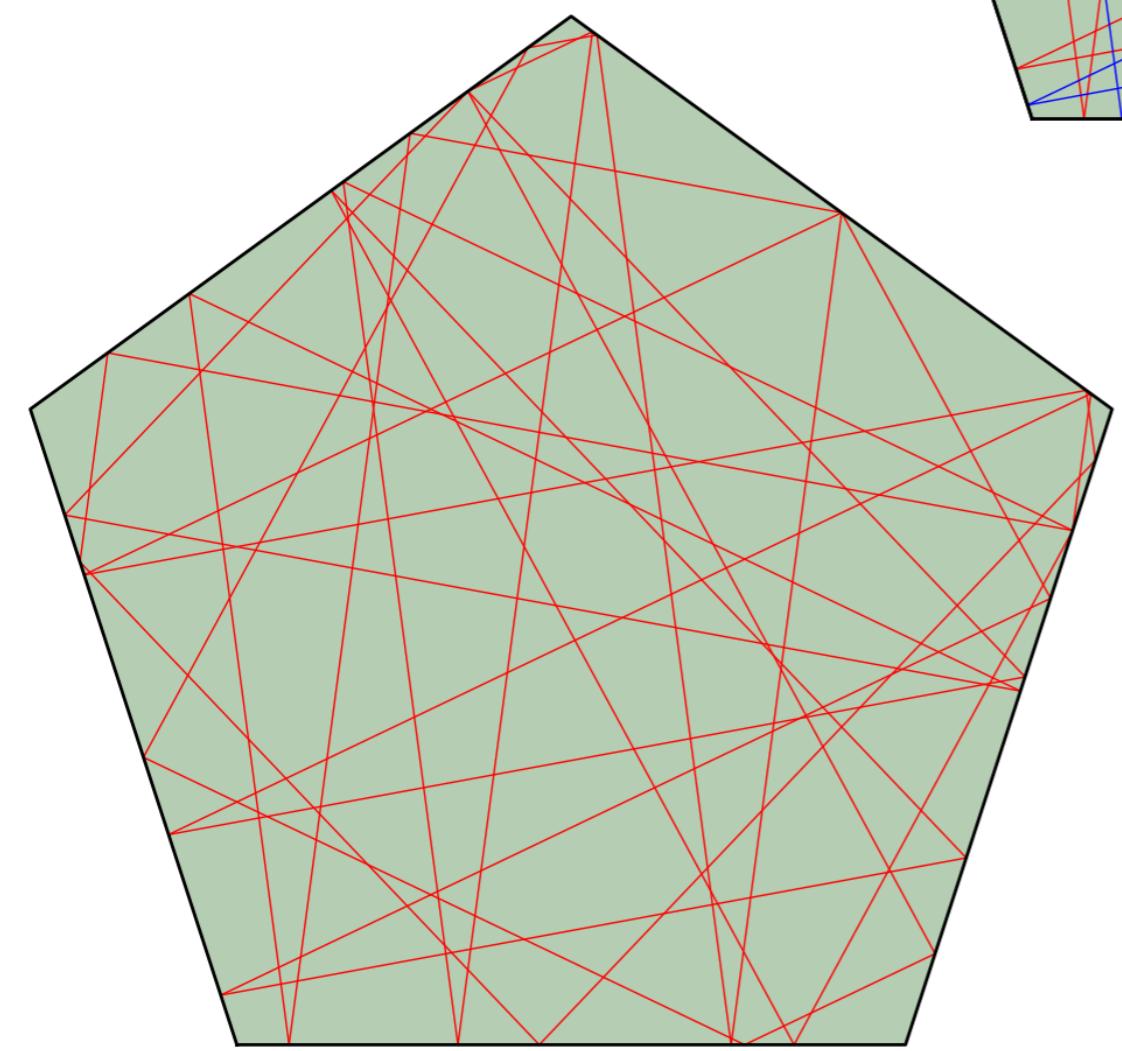




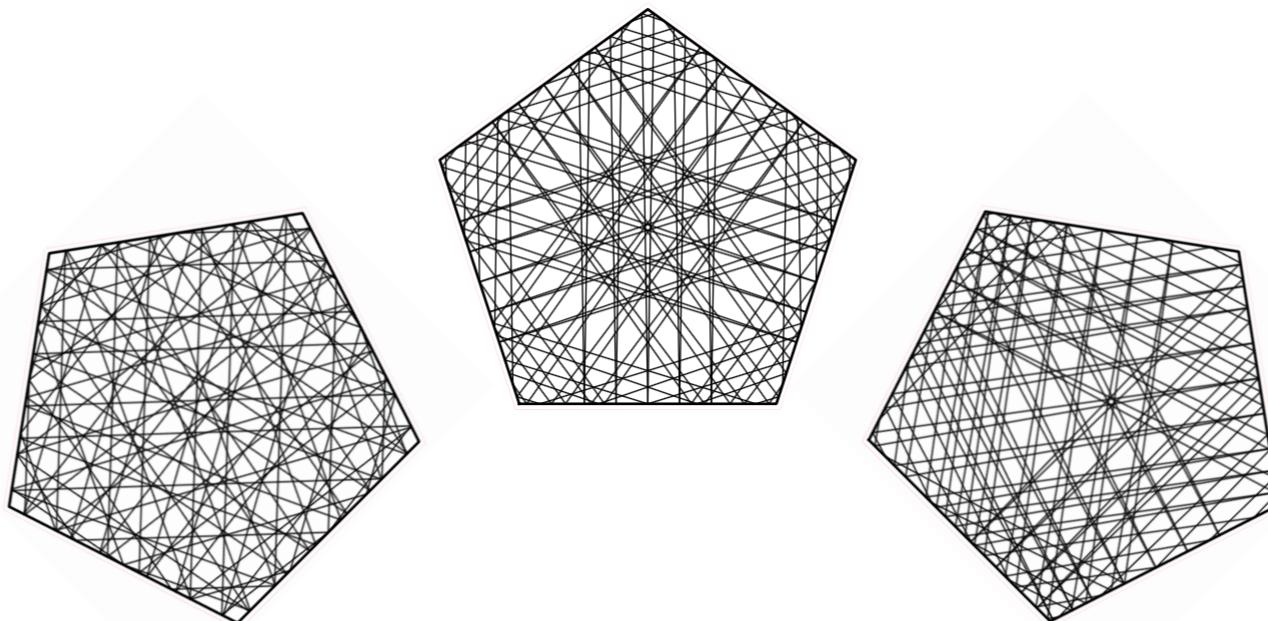




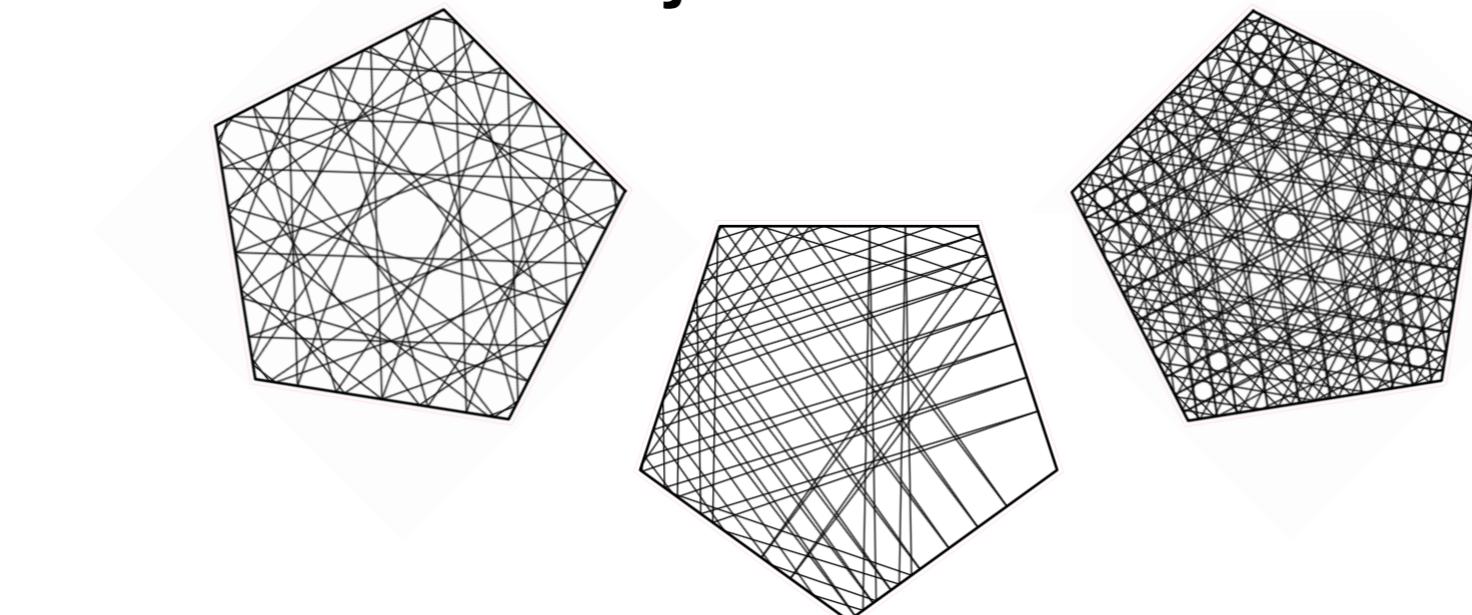


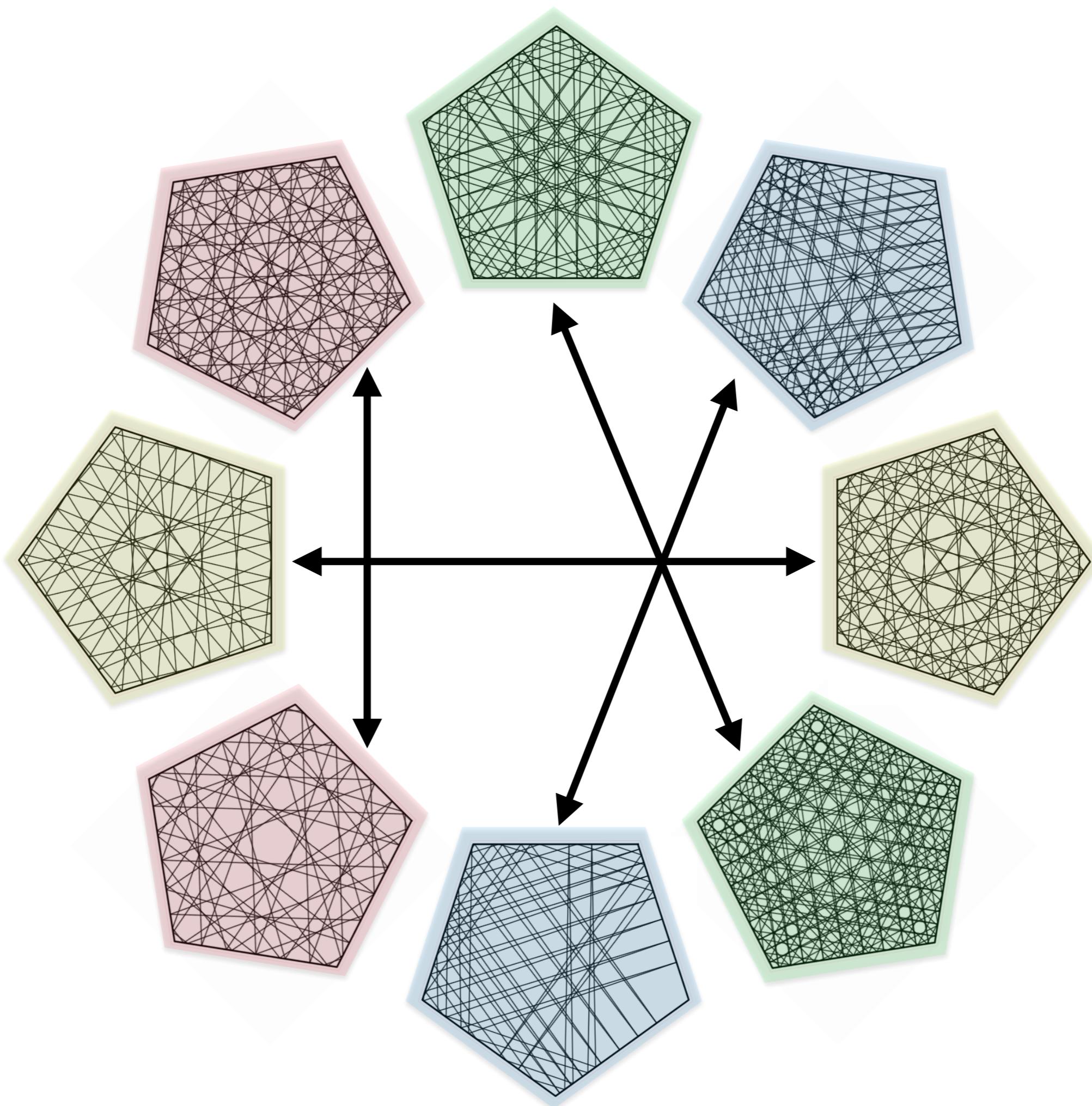


# MATCH THE BUDDIES!



Make 4 pairs  
of parallel  
trajectories.





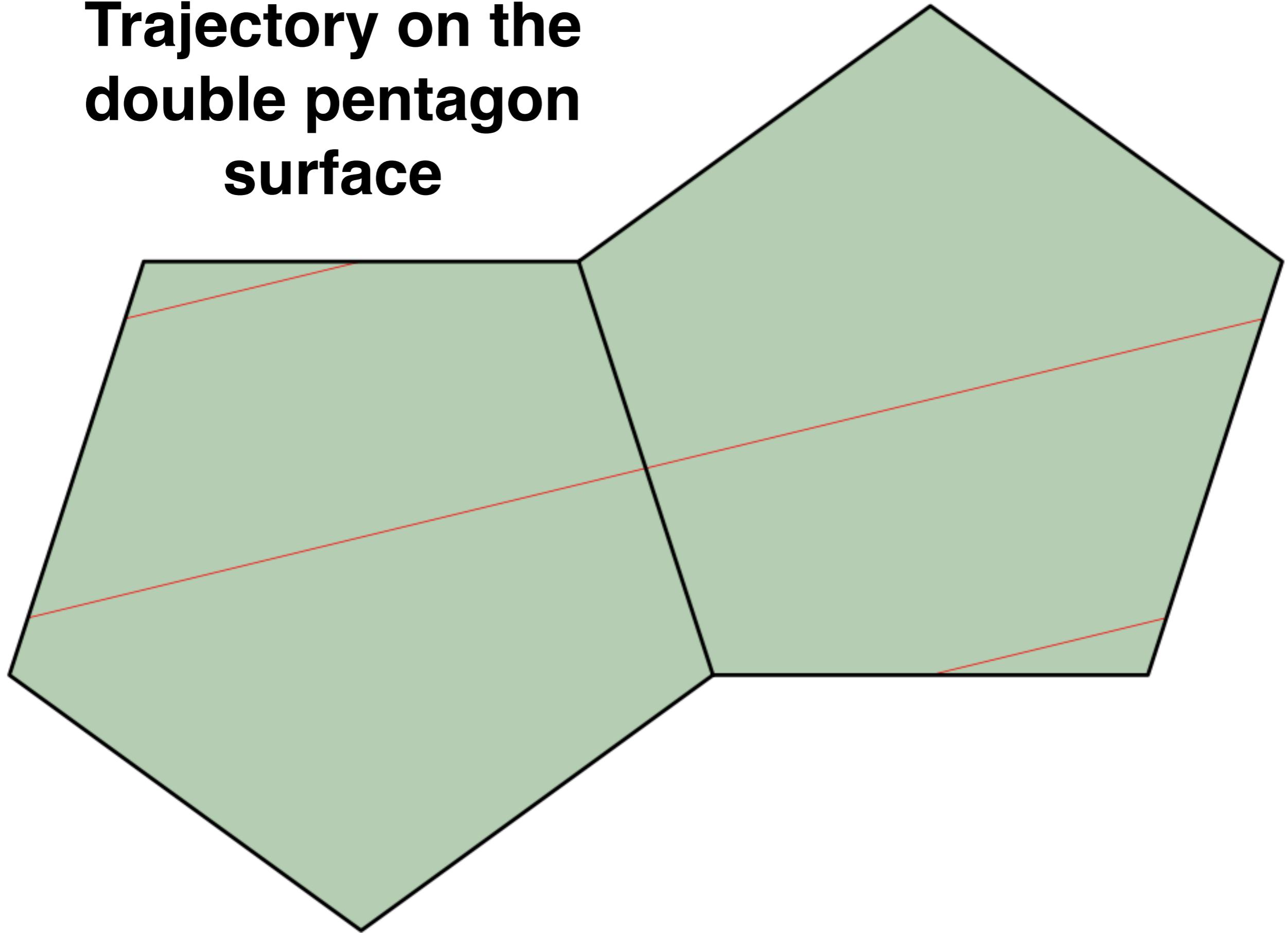
# FAMILIES

Joint with  
Barak Weiss

Tel Aviv  
University



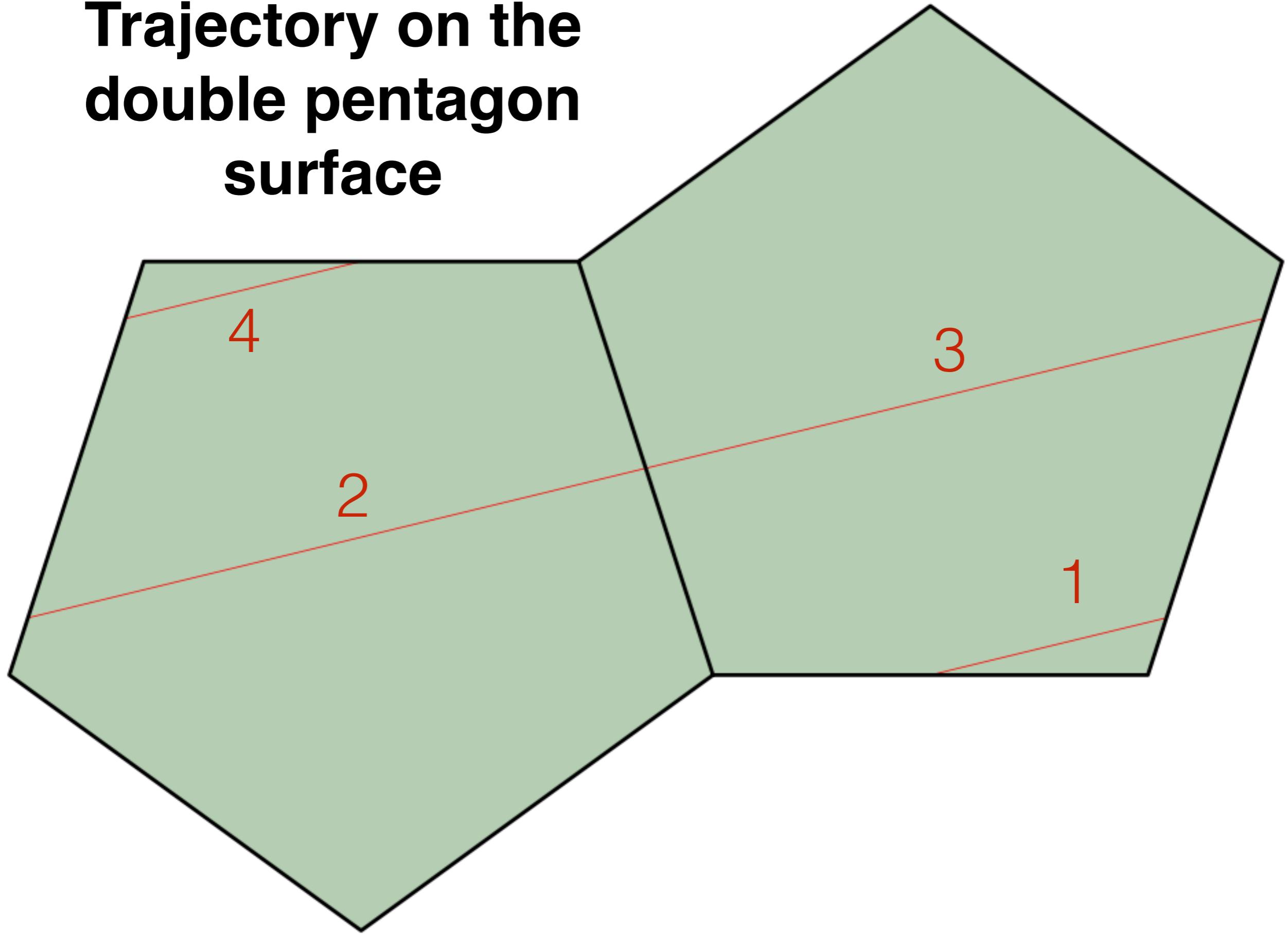
# Trajectory on the double pentagon surface



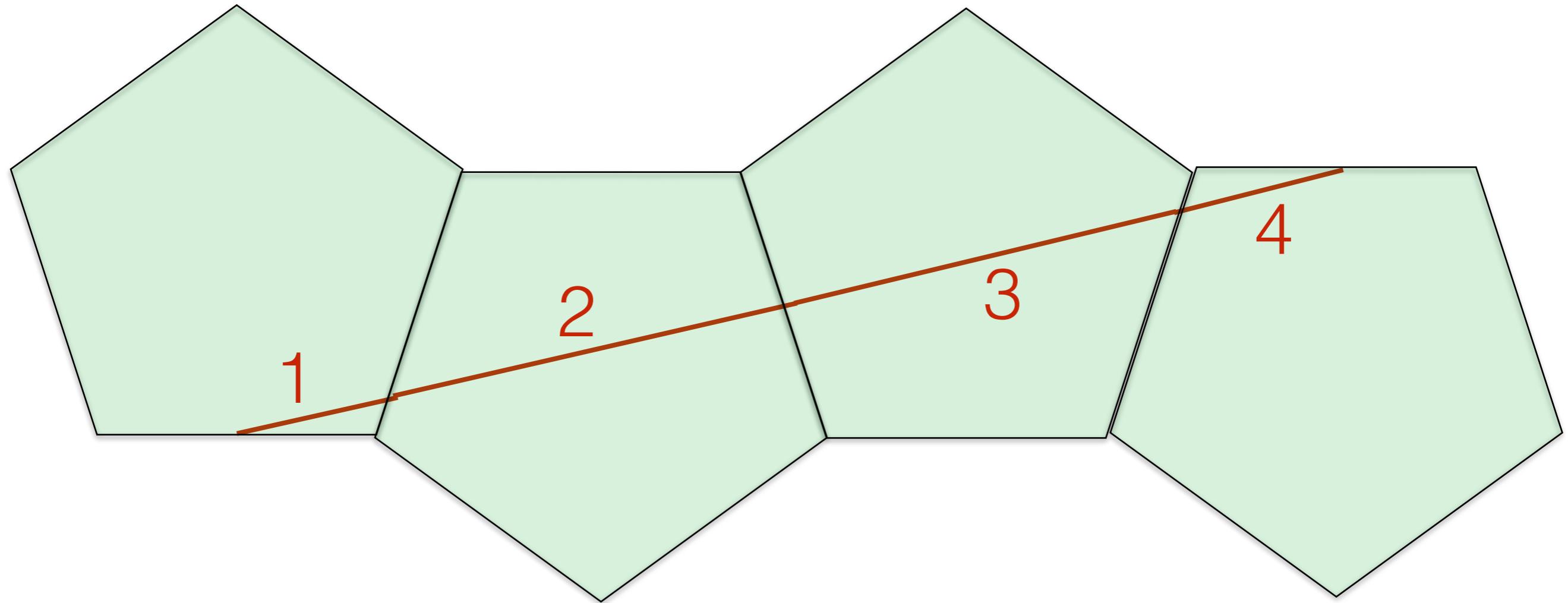
# Double pentagon surface



# Trajectory on the double pentagon surface

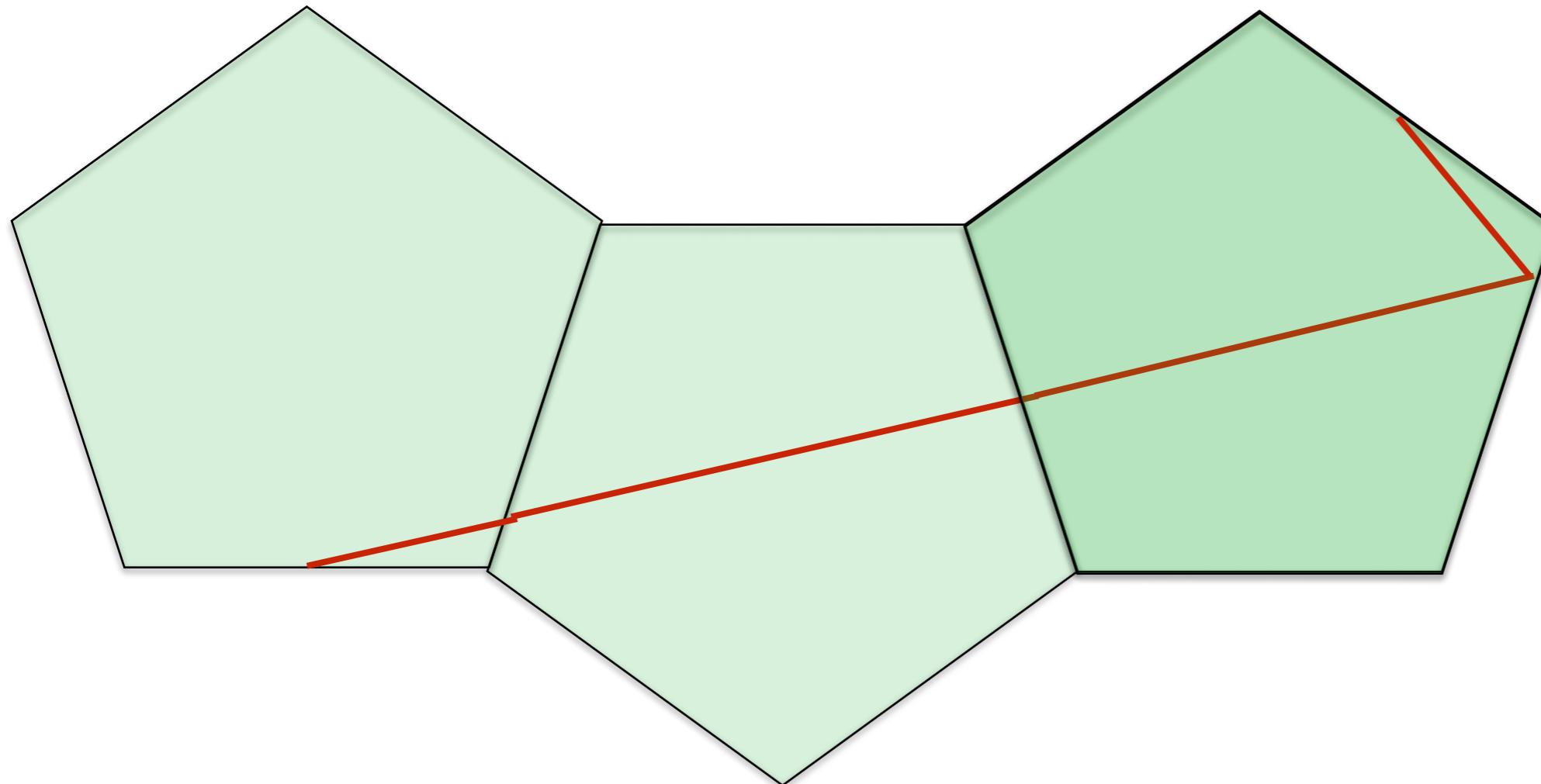


Start with a trajectory on the surface



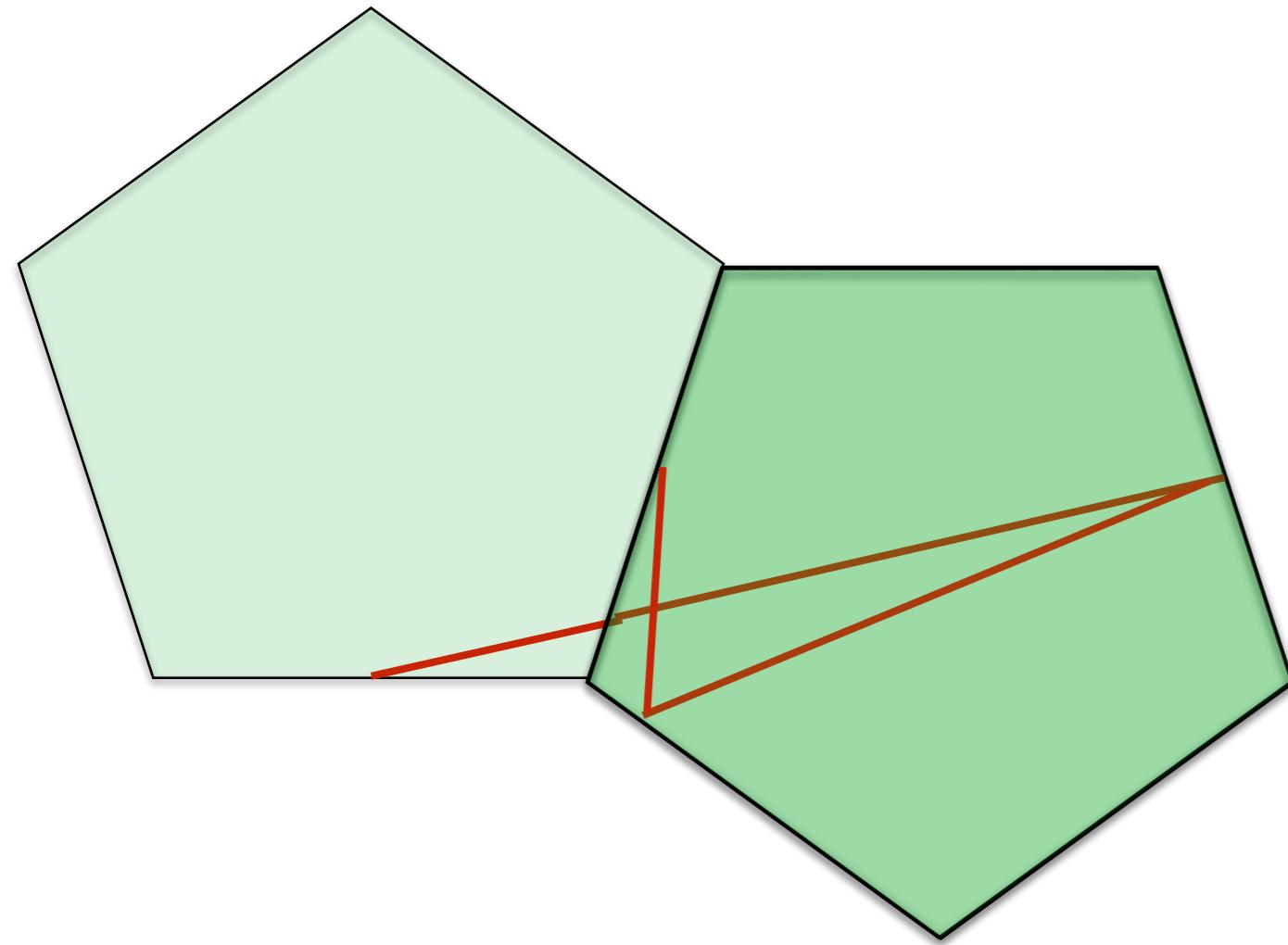
Fold each pentagon onto the previous one

Start with a trajectory on the surface



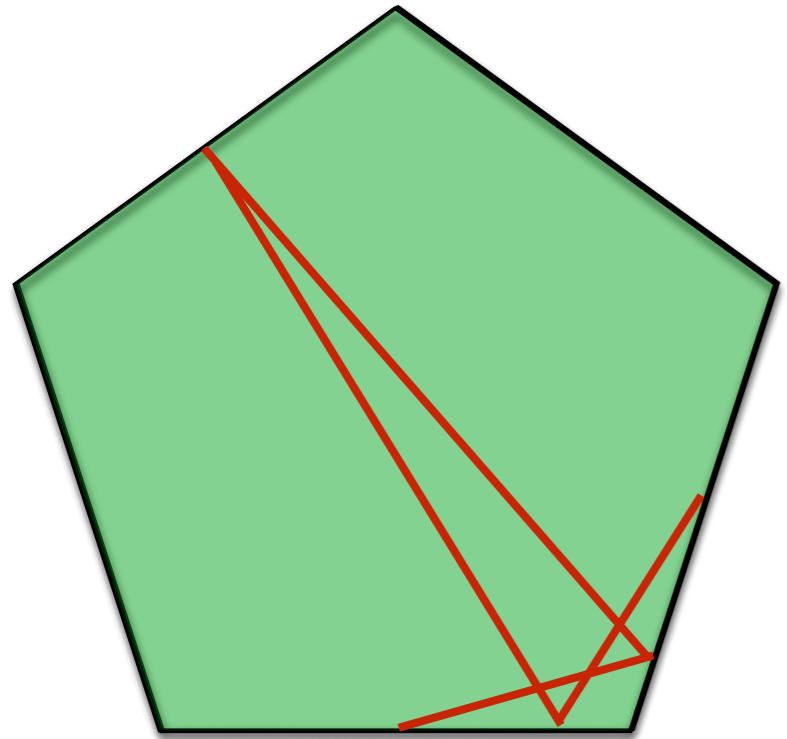
Fold each pentagon onto the previous one

Start with a trajectory on the surface



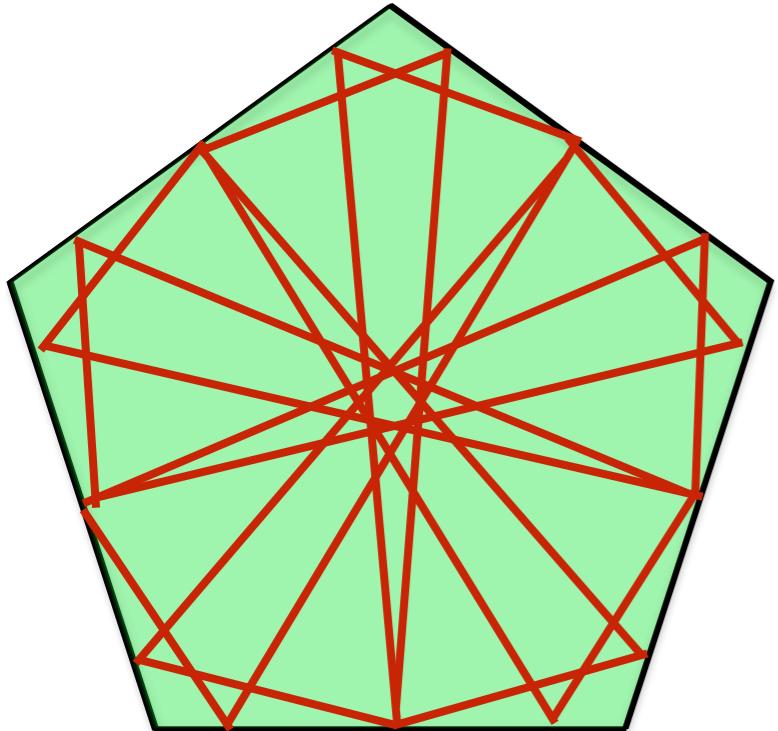
Fold each pentagon onto the previous one

Start with a trajectory on the surface



Fold each pentagon onto the previous one  
Get a pentagon billiard trajectory!

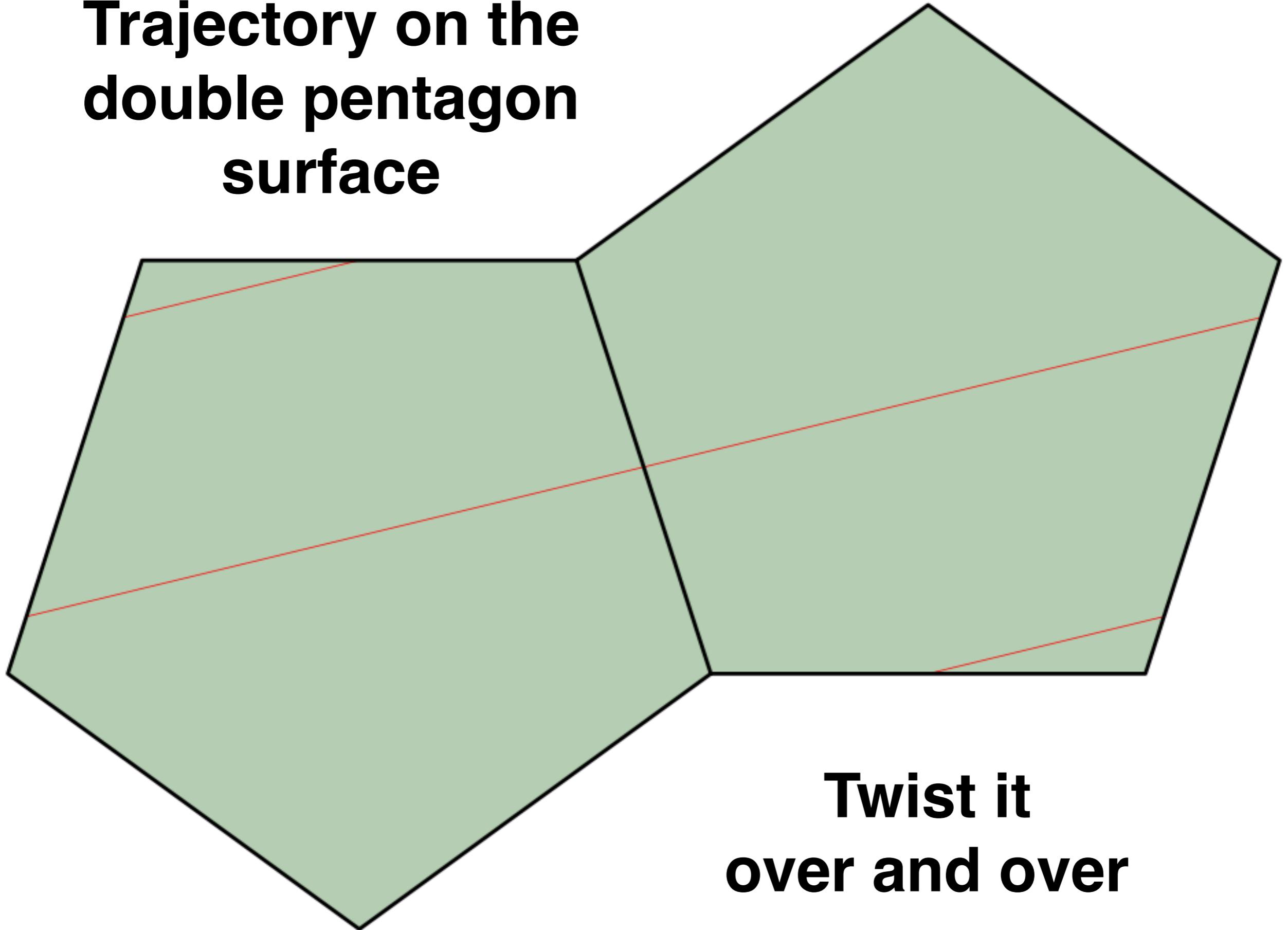
Start with a trajectory on the surface



Do it 5 times  
to get back  
where we started

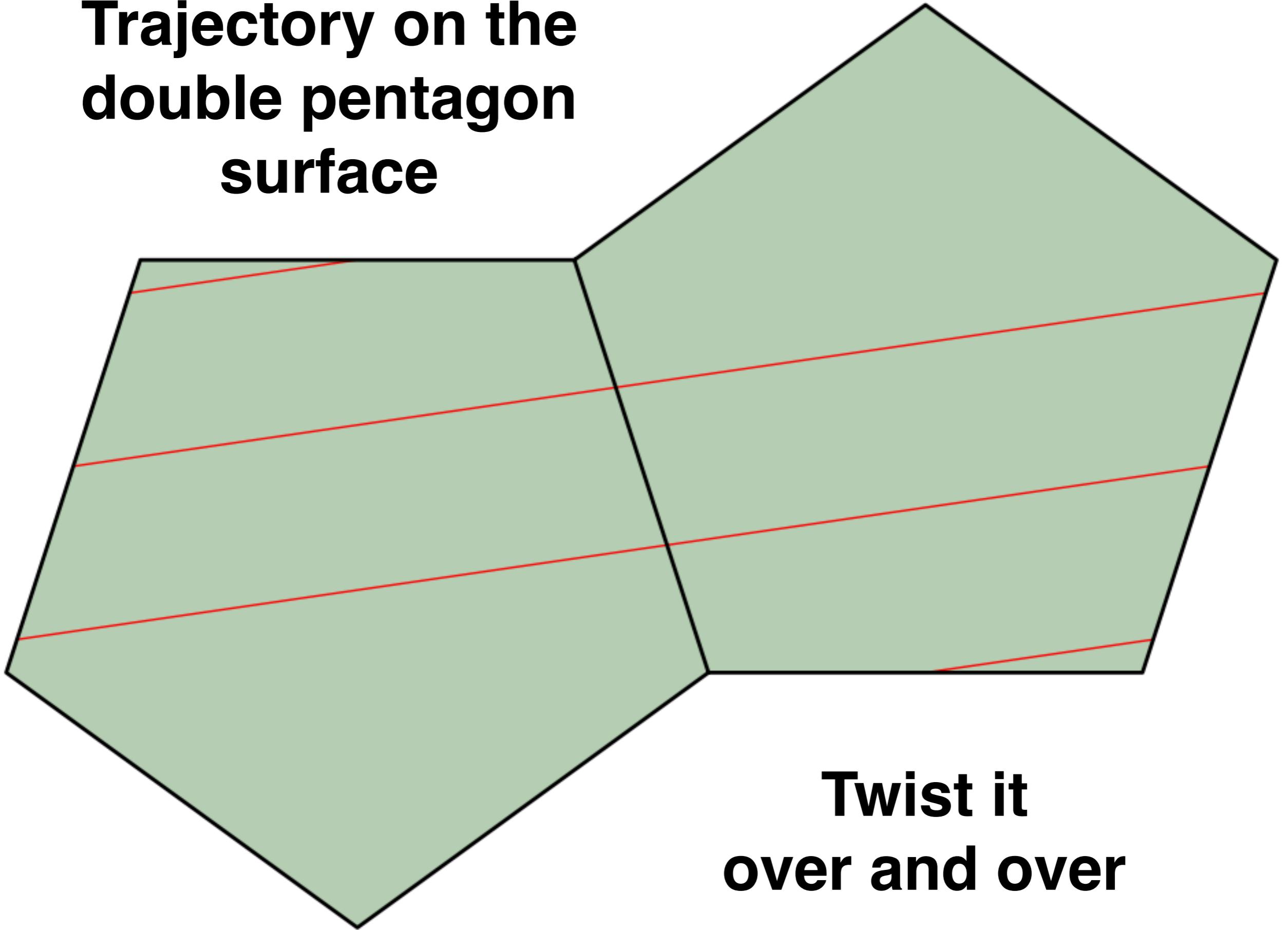
Fold each pentagon onto the previous one  
Get a pentagon billiard trajectory!

# **Trajectory on the double pentagon surface**



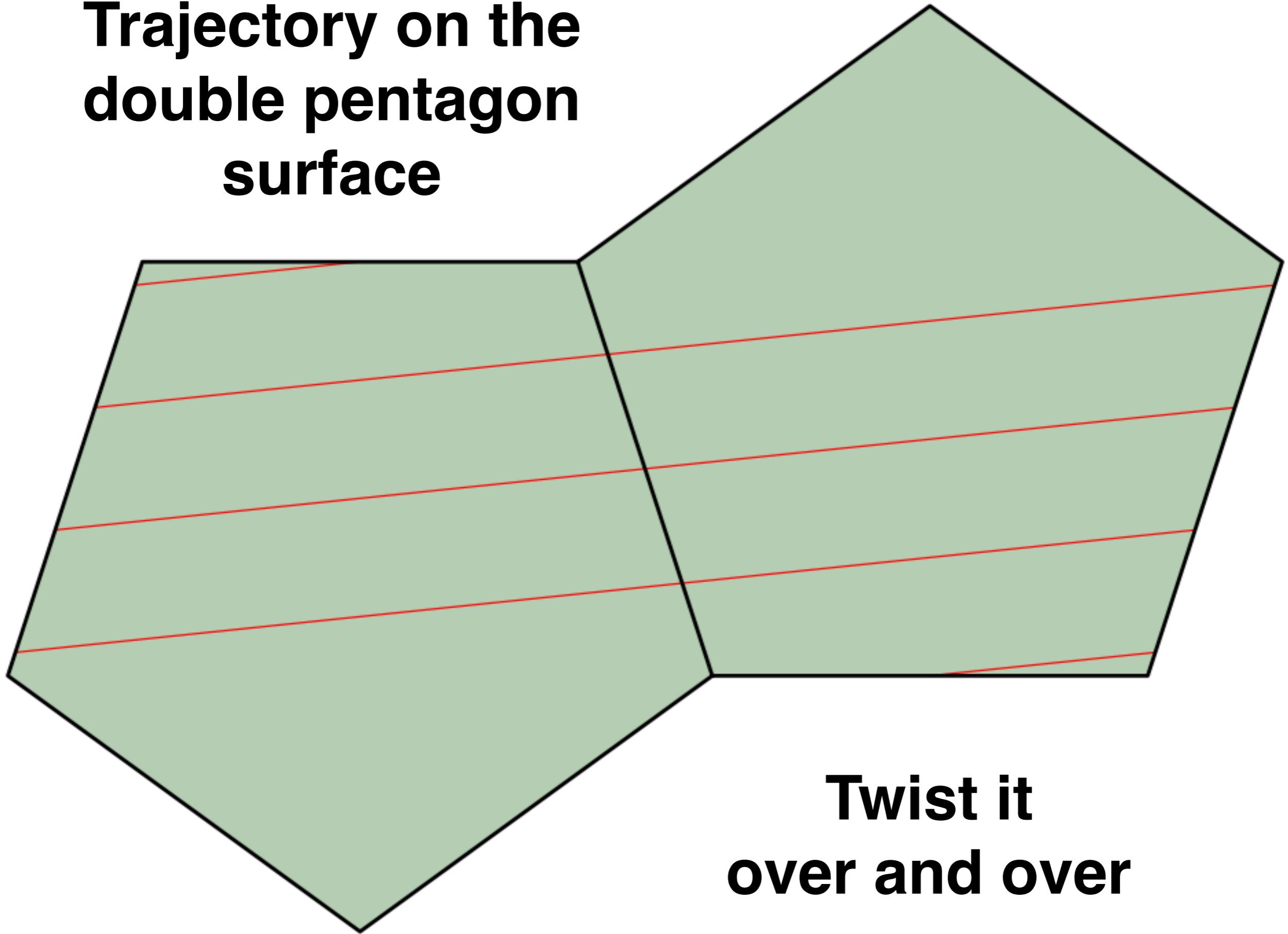
**Twist it  
over and over**

# **Trajectory on the double pentagon surface**



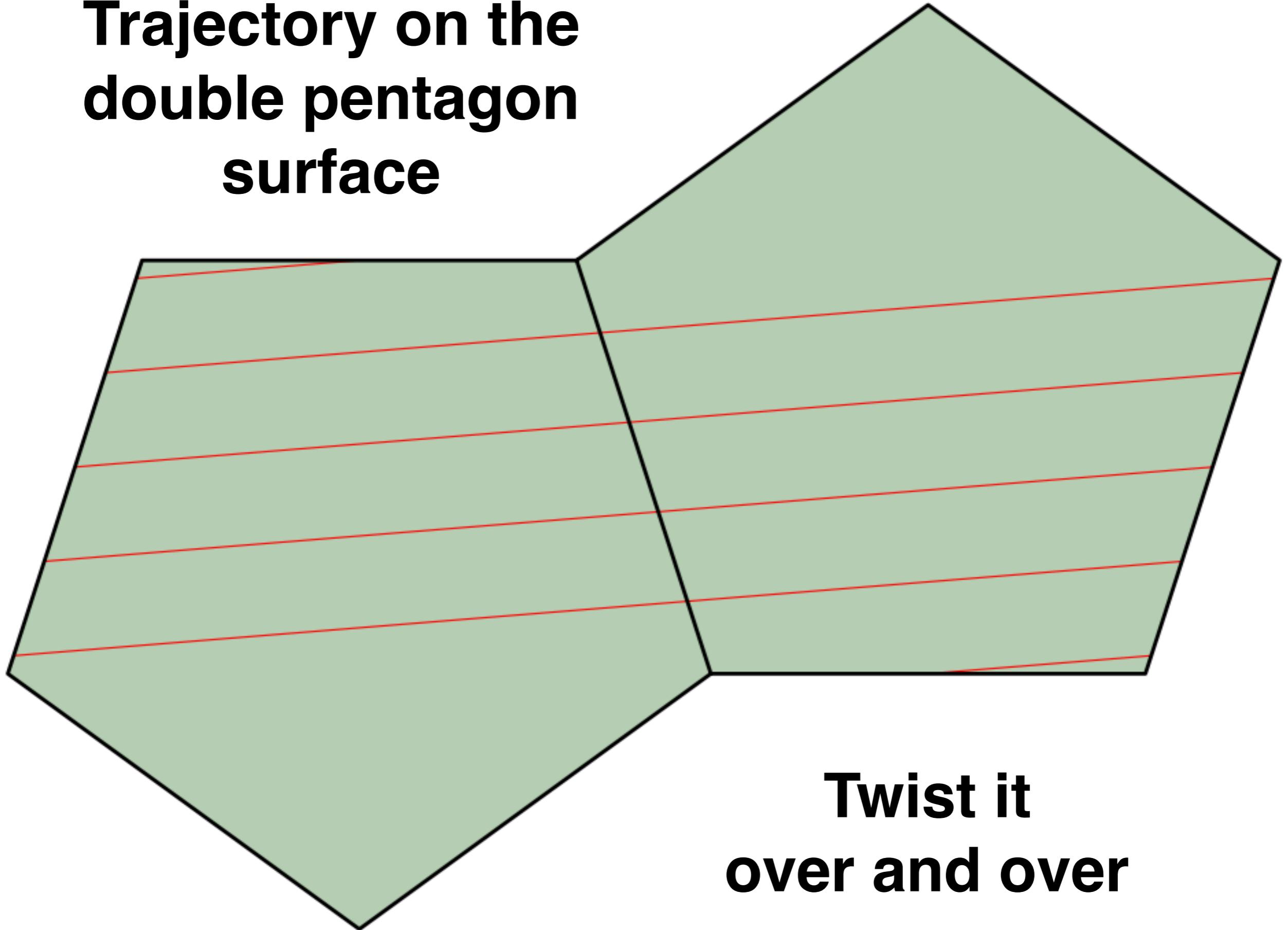
**Twist it  
over and over**

# **Trajectory on the double pentagon surface**



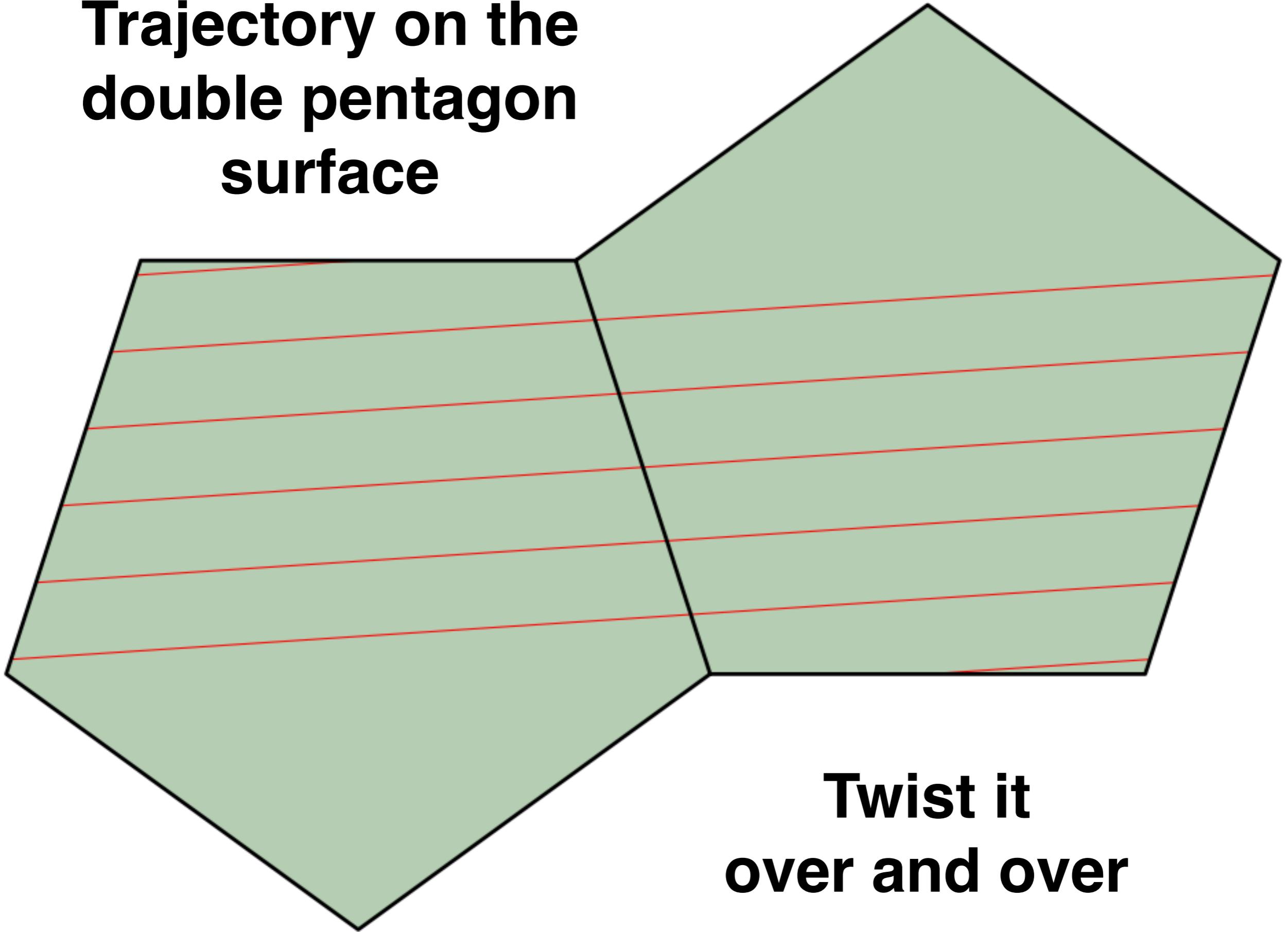
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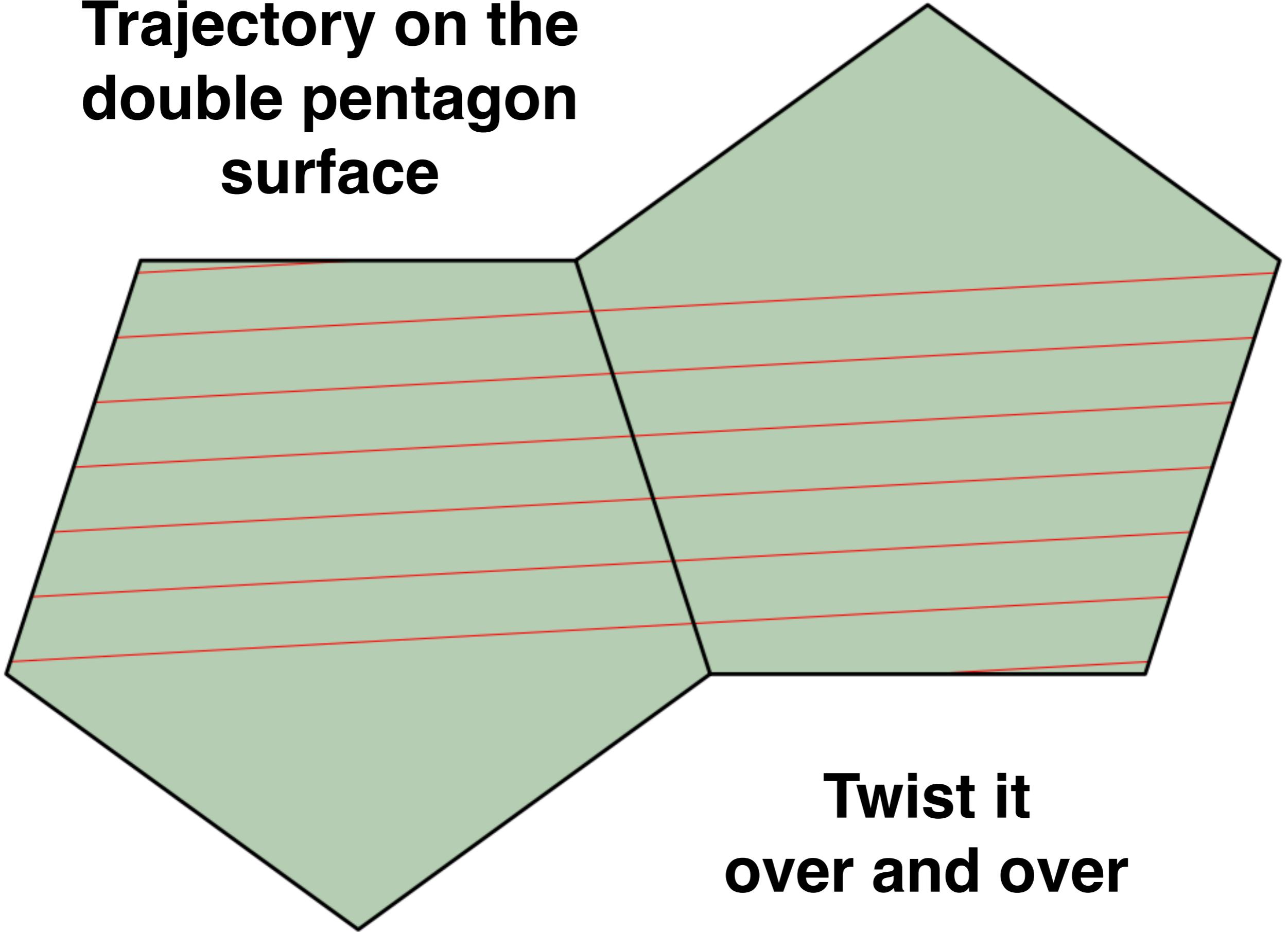
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# **Trajectory on the double pentagon surface**



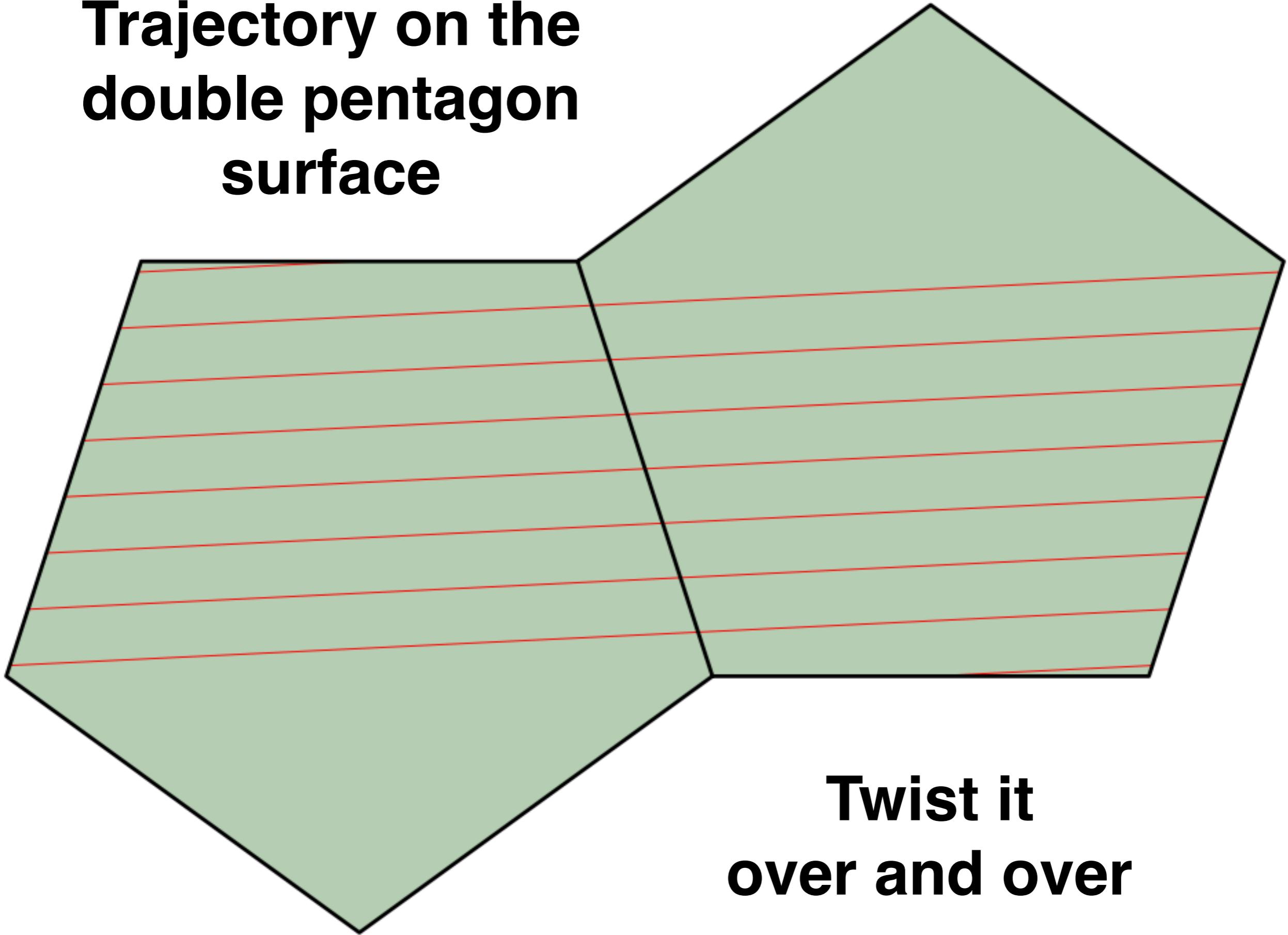
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# **Trajectory on the double pentagon surface**



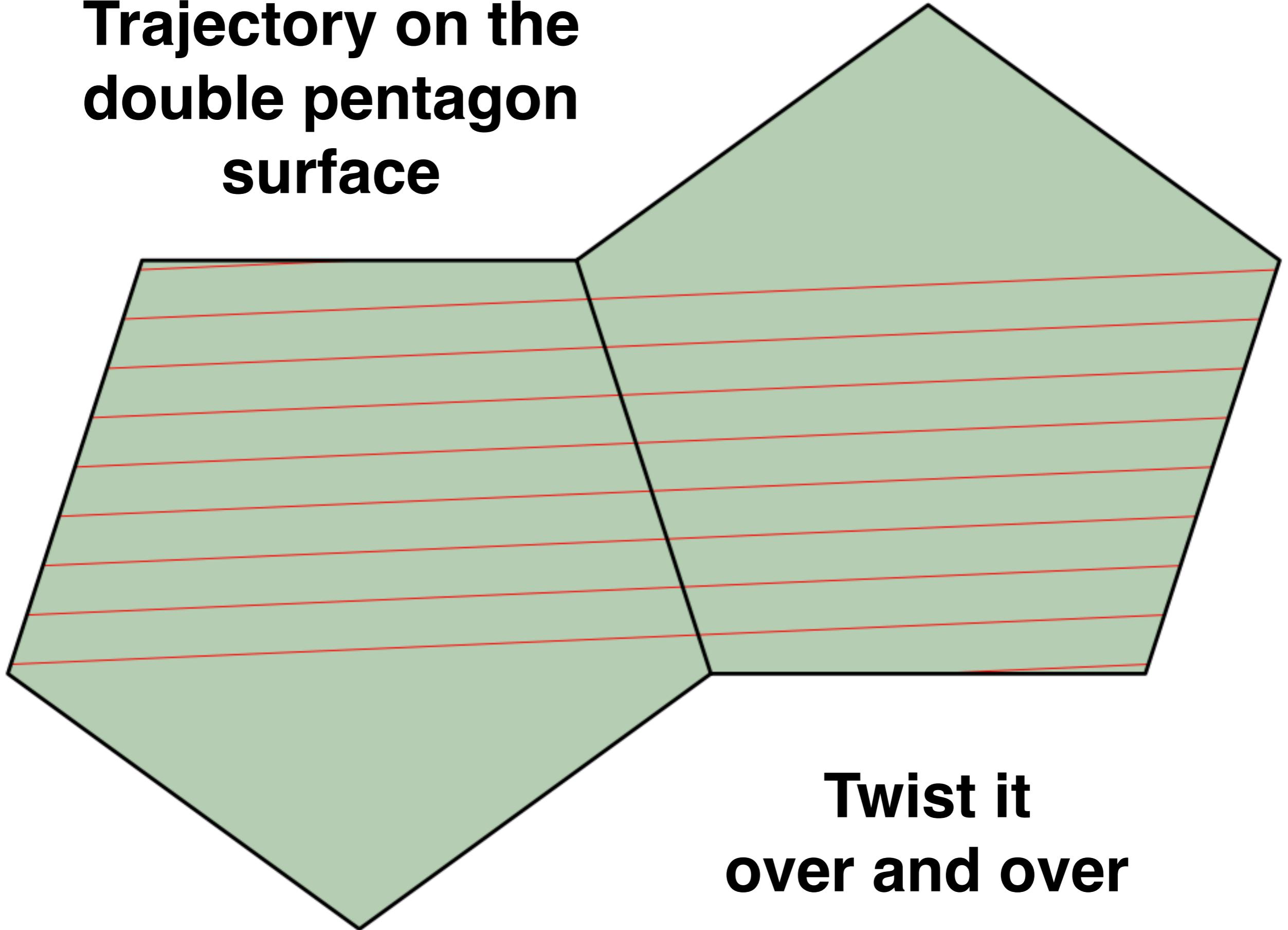
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# **Trajectory on the double pentagon surface**



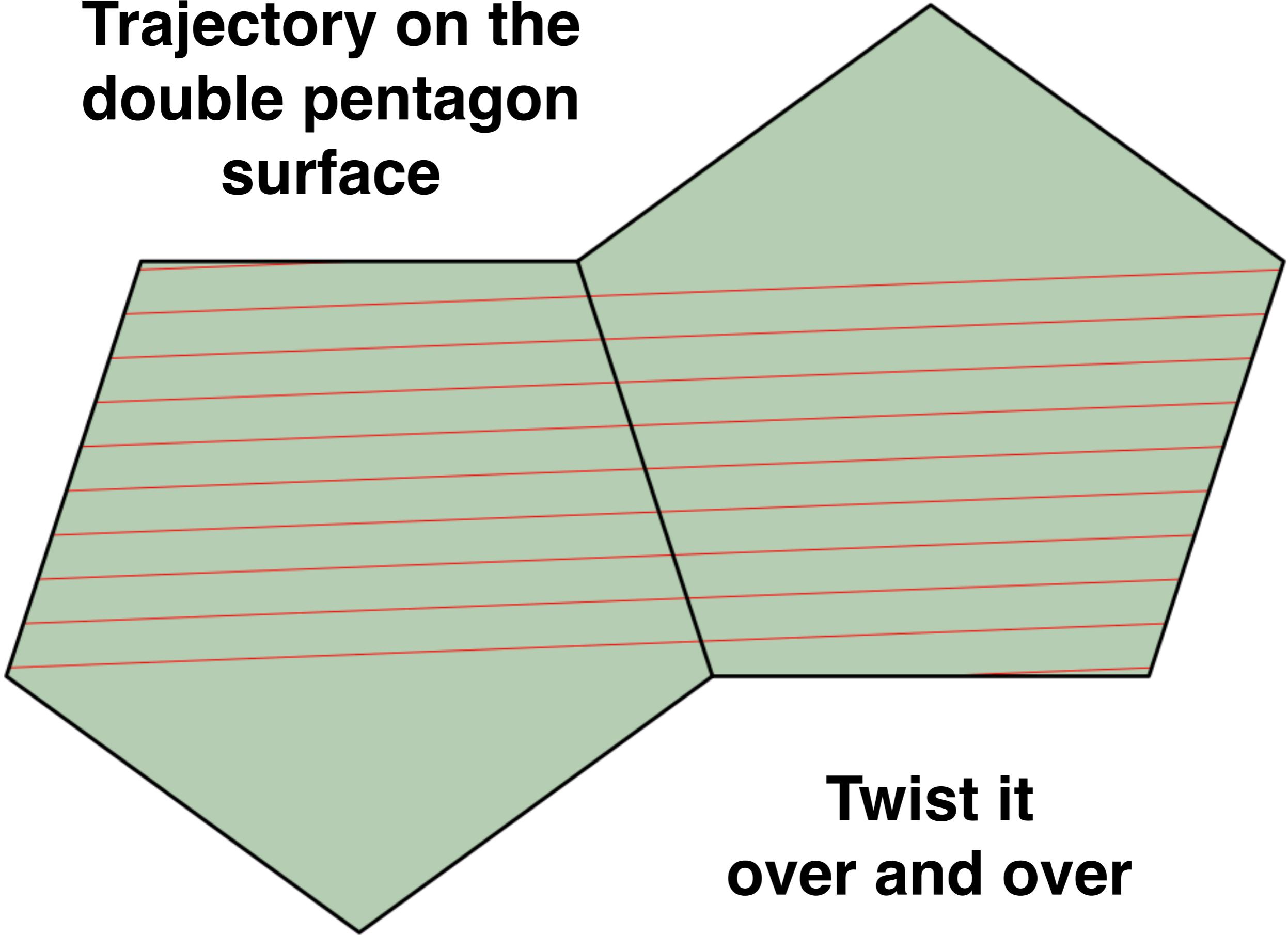
**Twist it  
over and over**

# **Trajectory on the double pentagon surface**



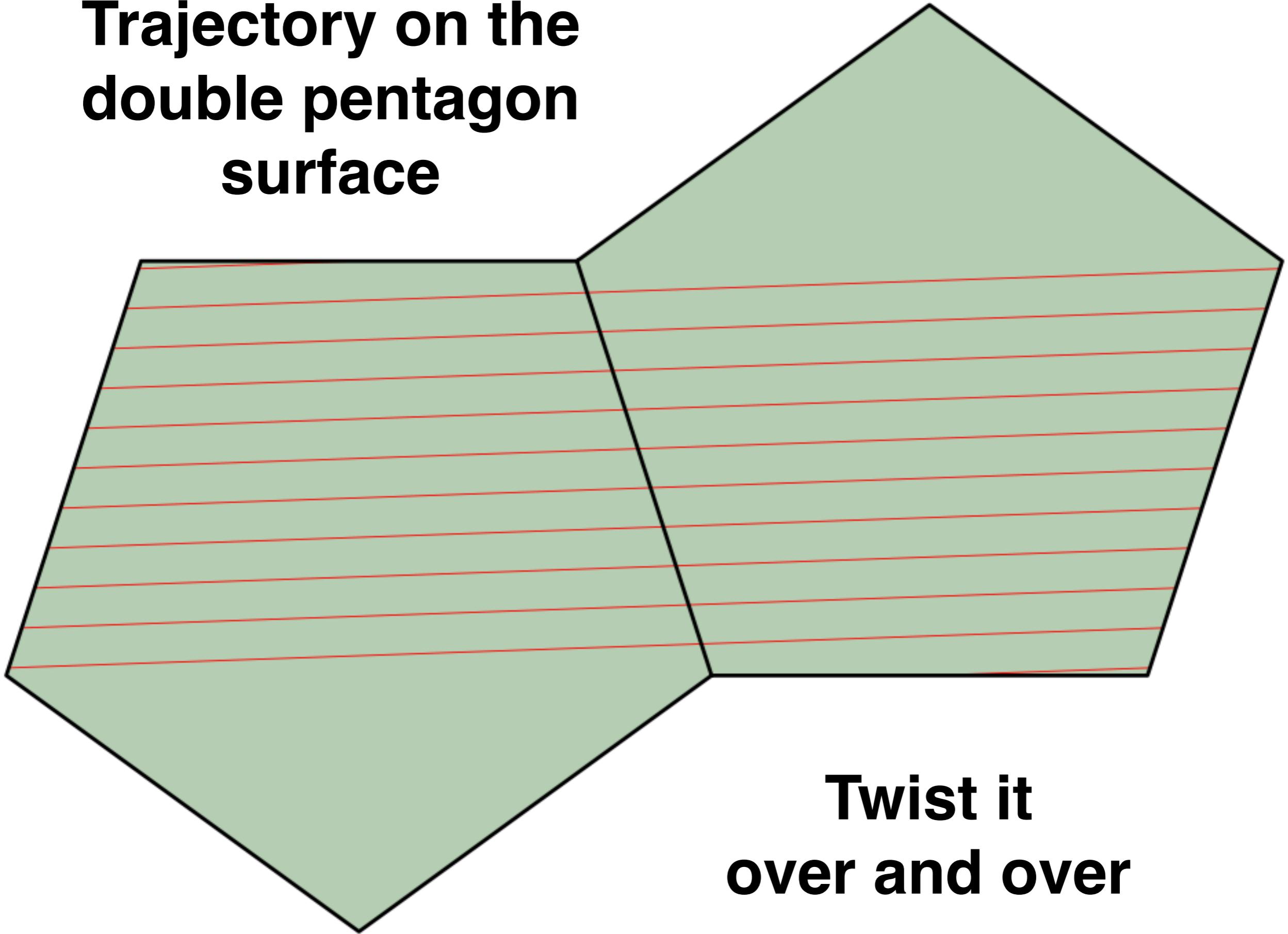
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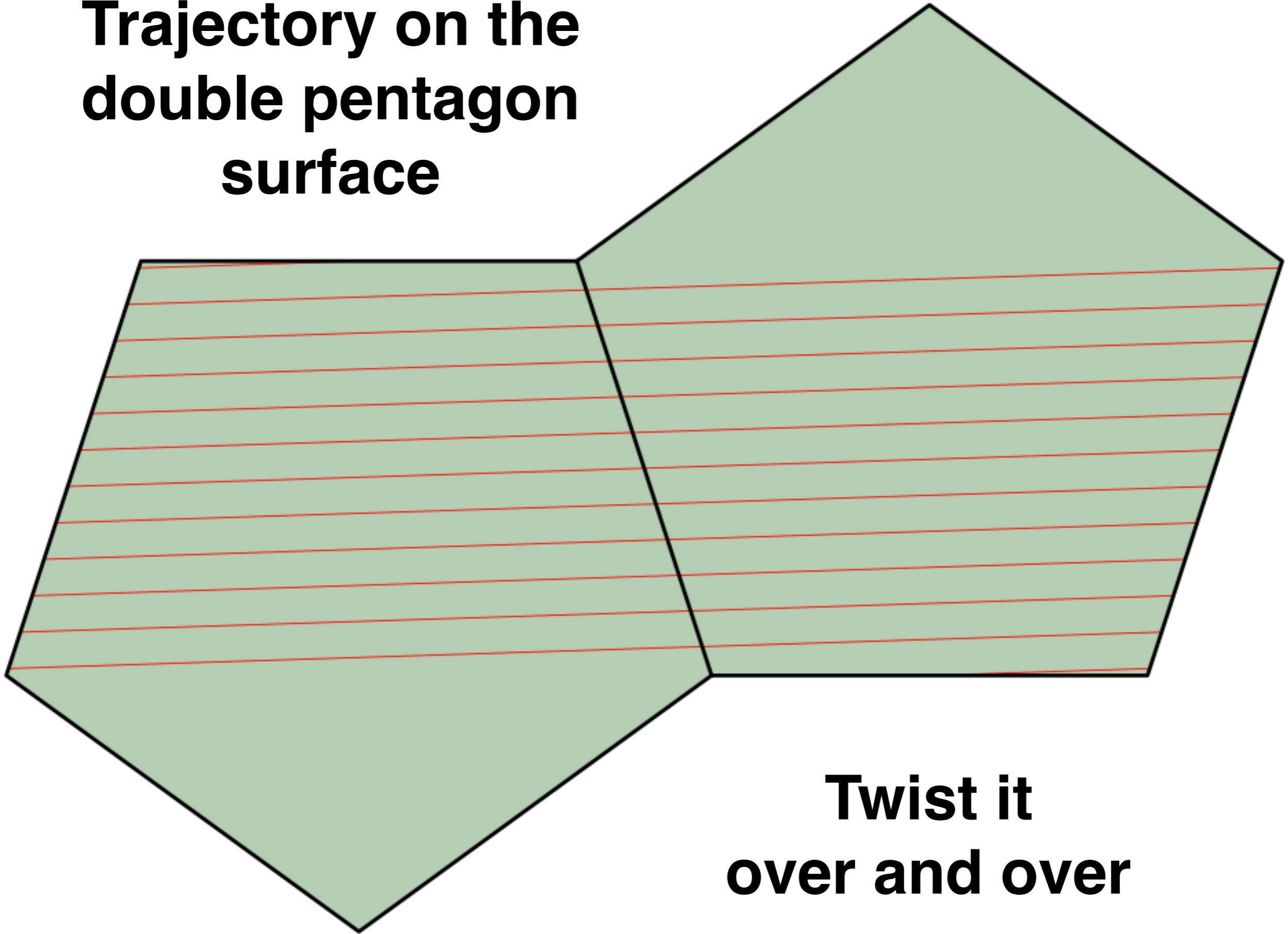
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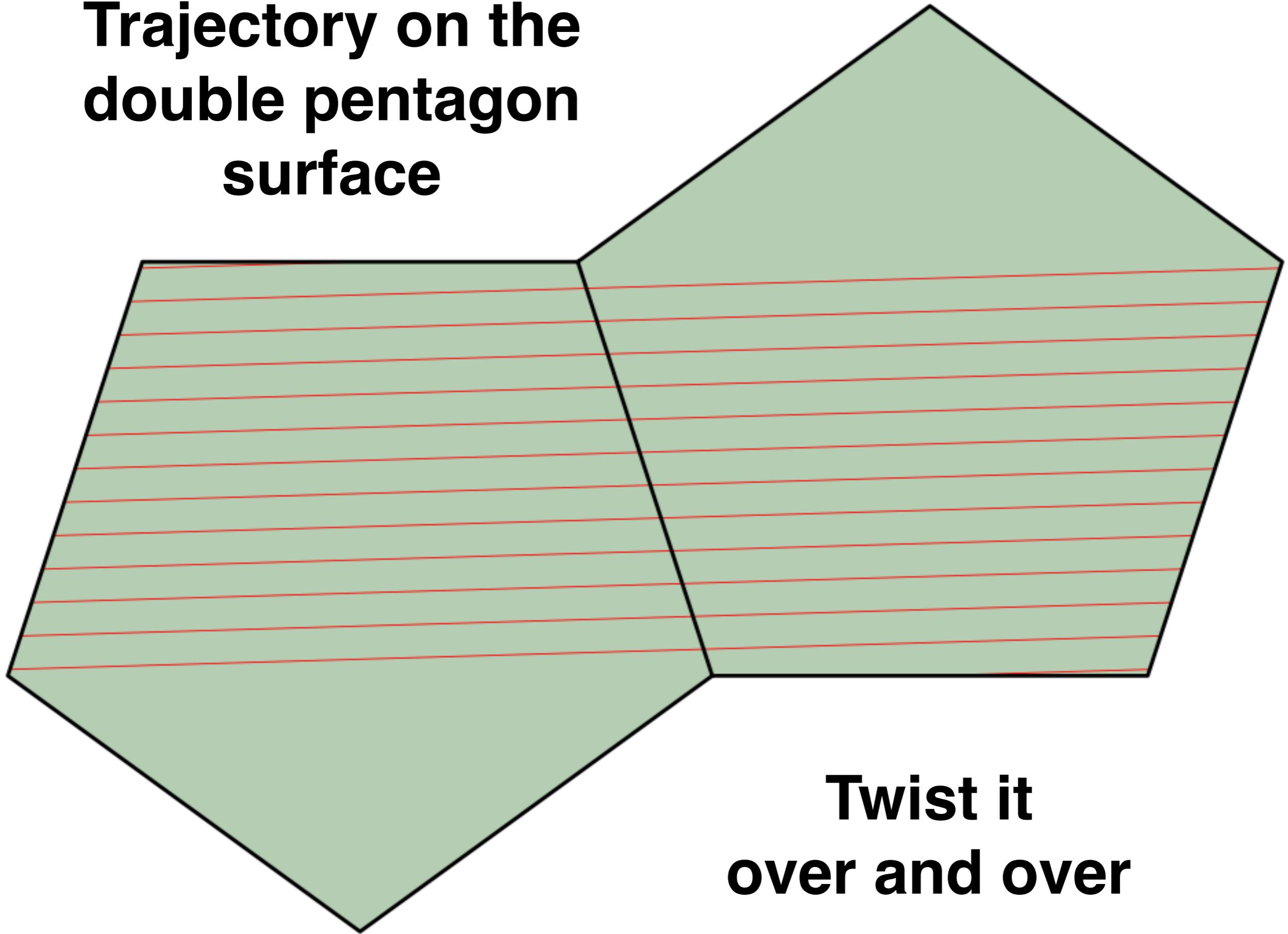
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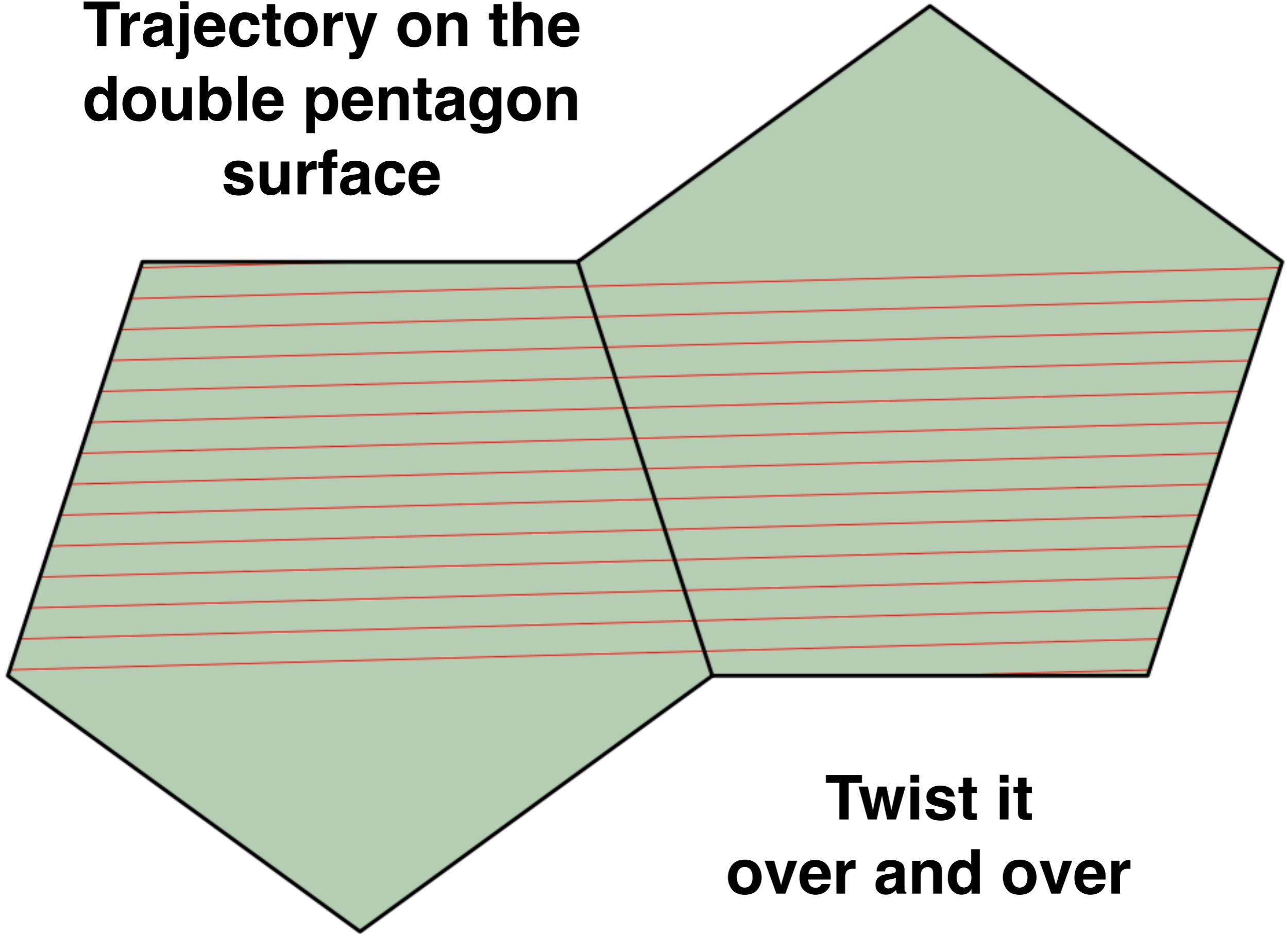
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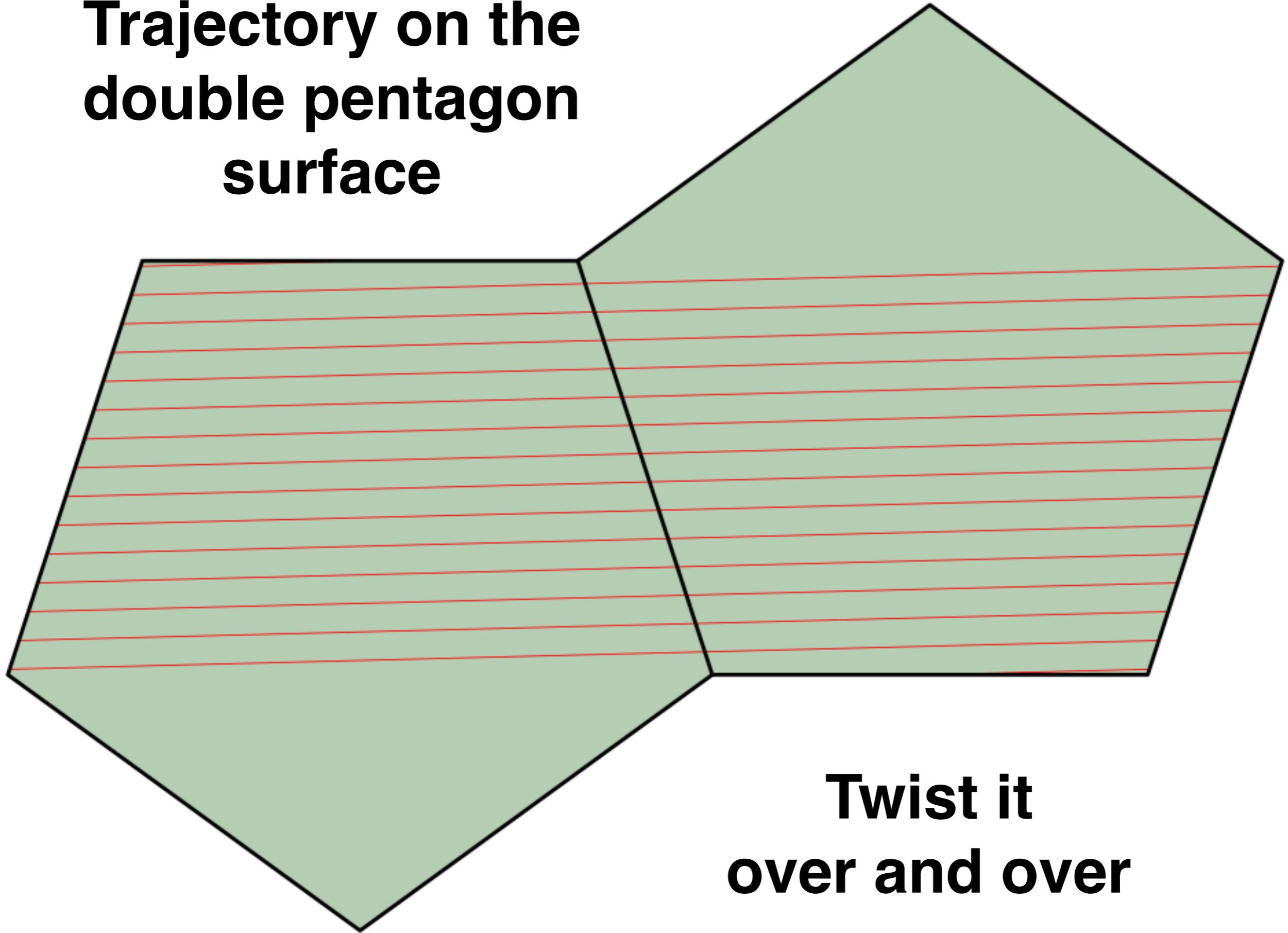
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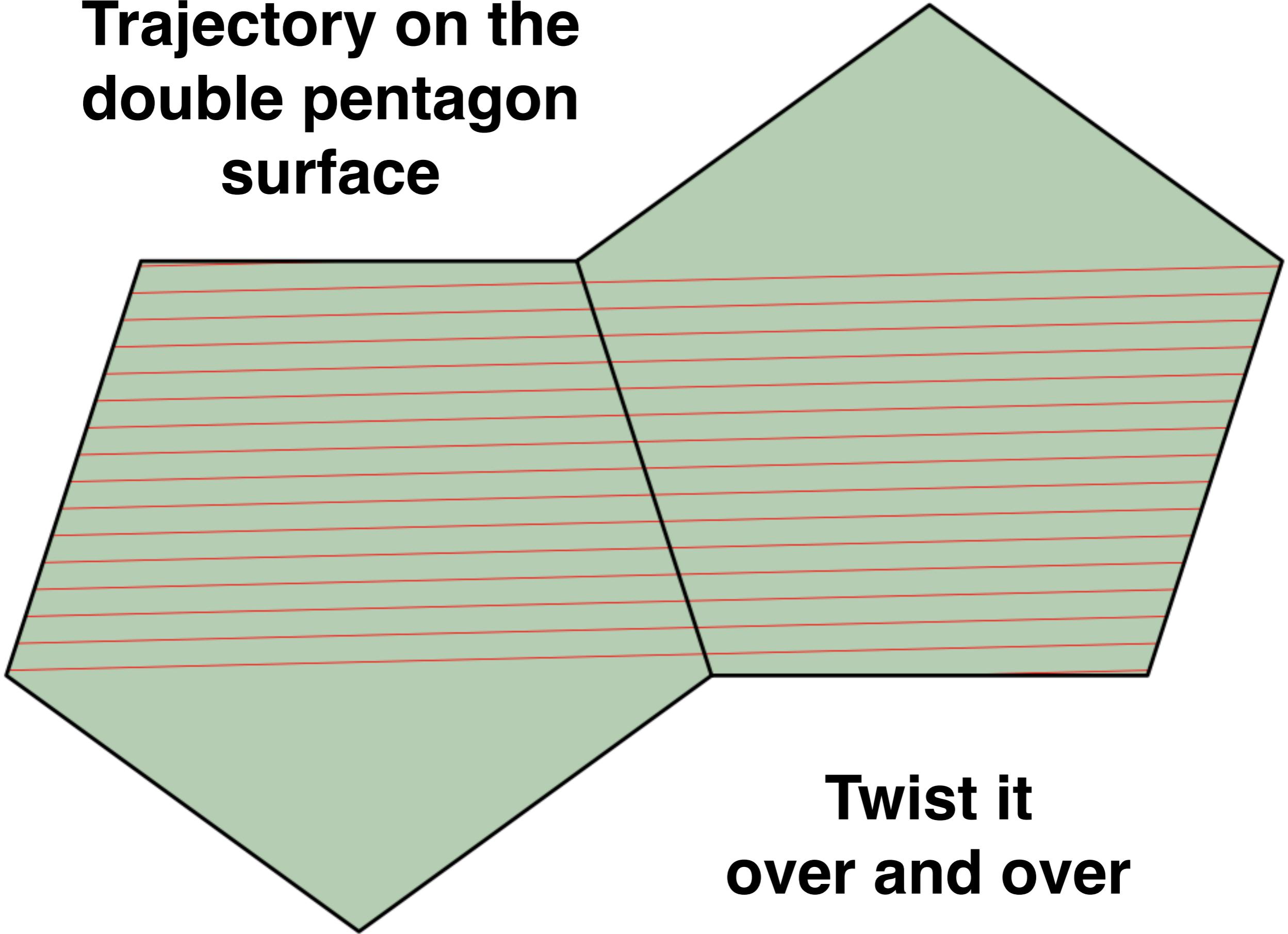
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# **Trajectory on the double pentagon surface**



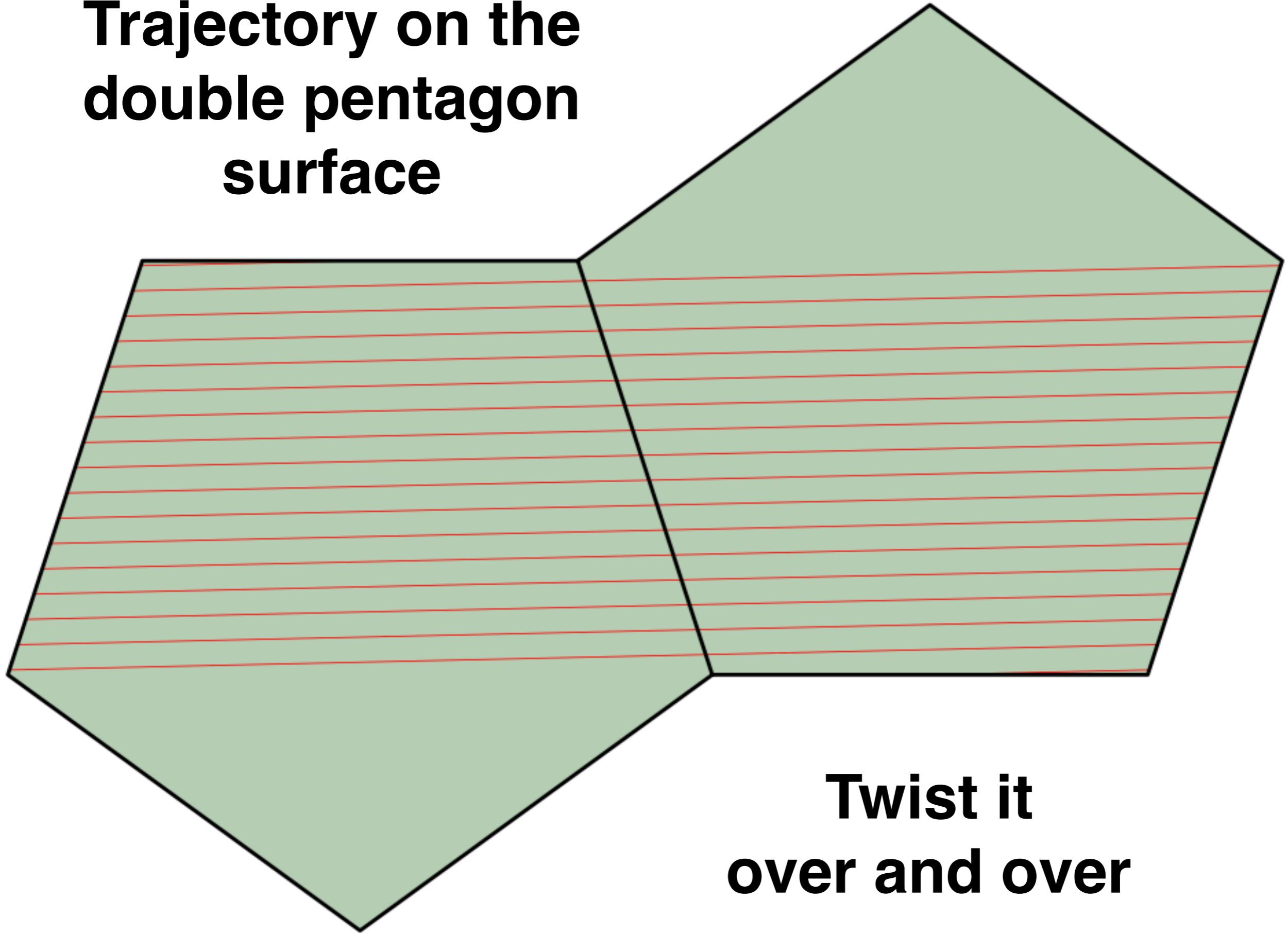
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# **Trajectory on the double pentagon surface**



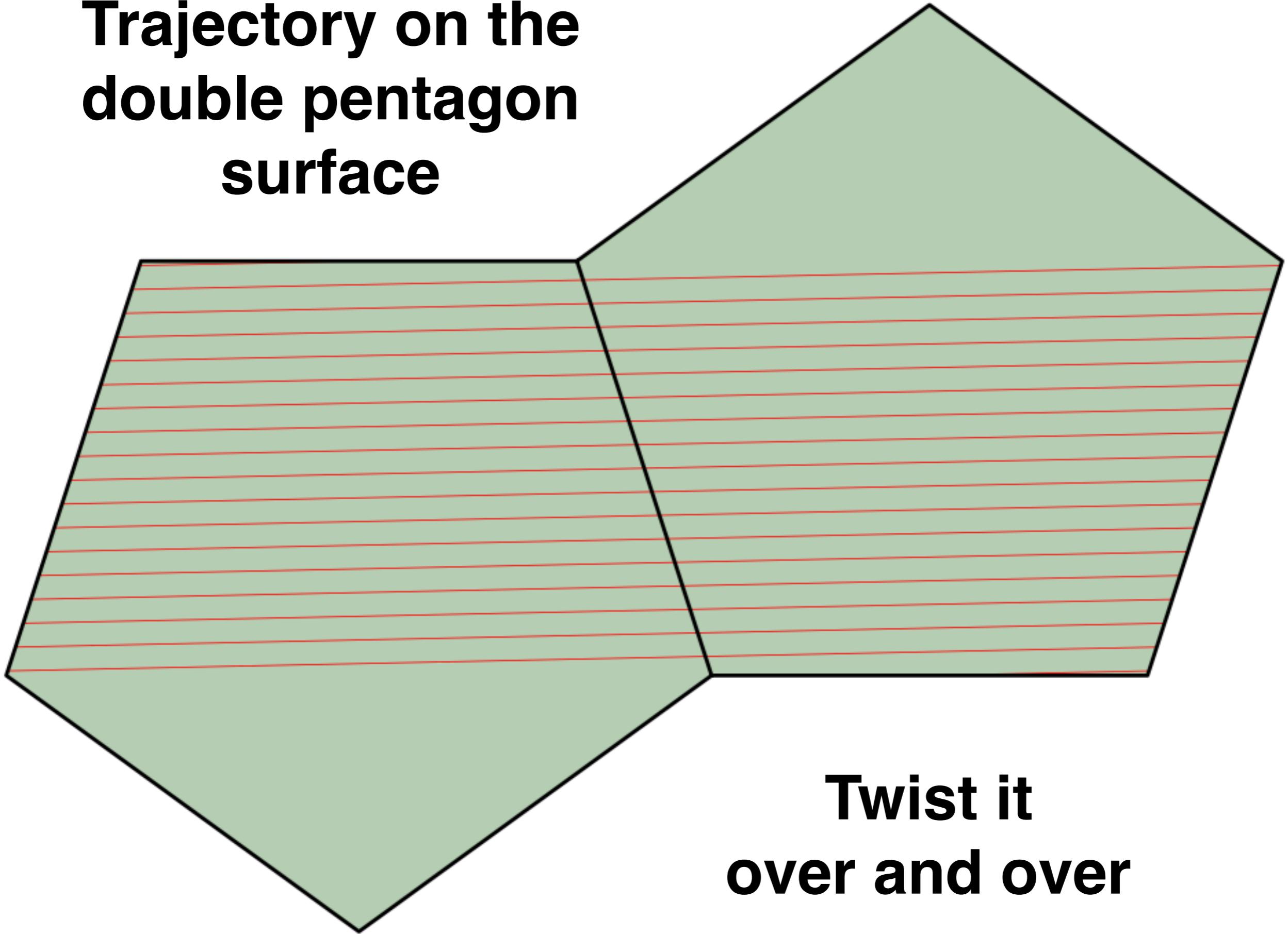
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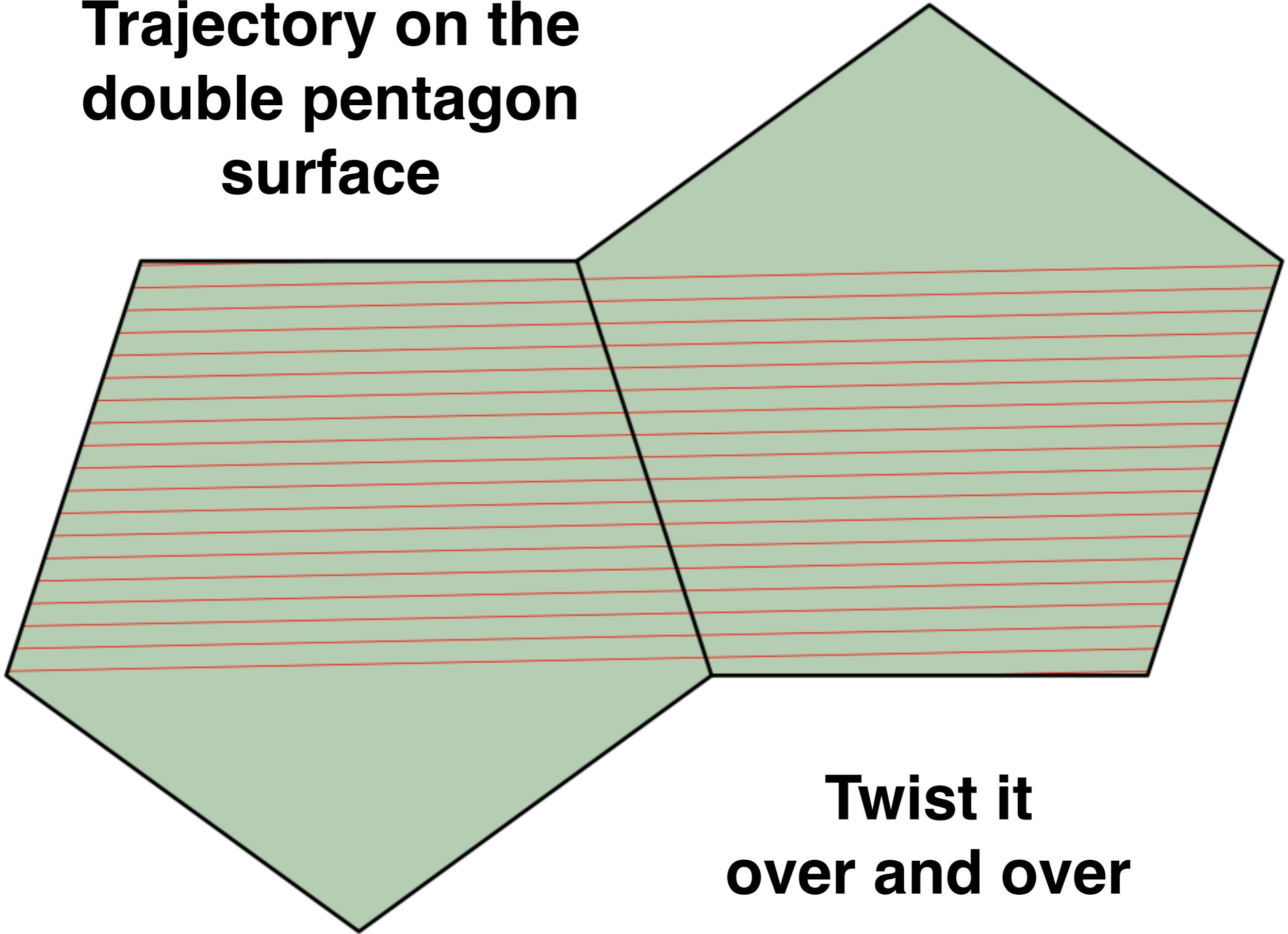
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# **Trajectory on the double pentagon surface**



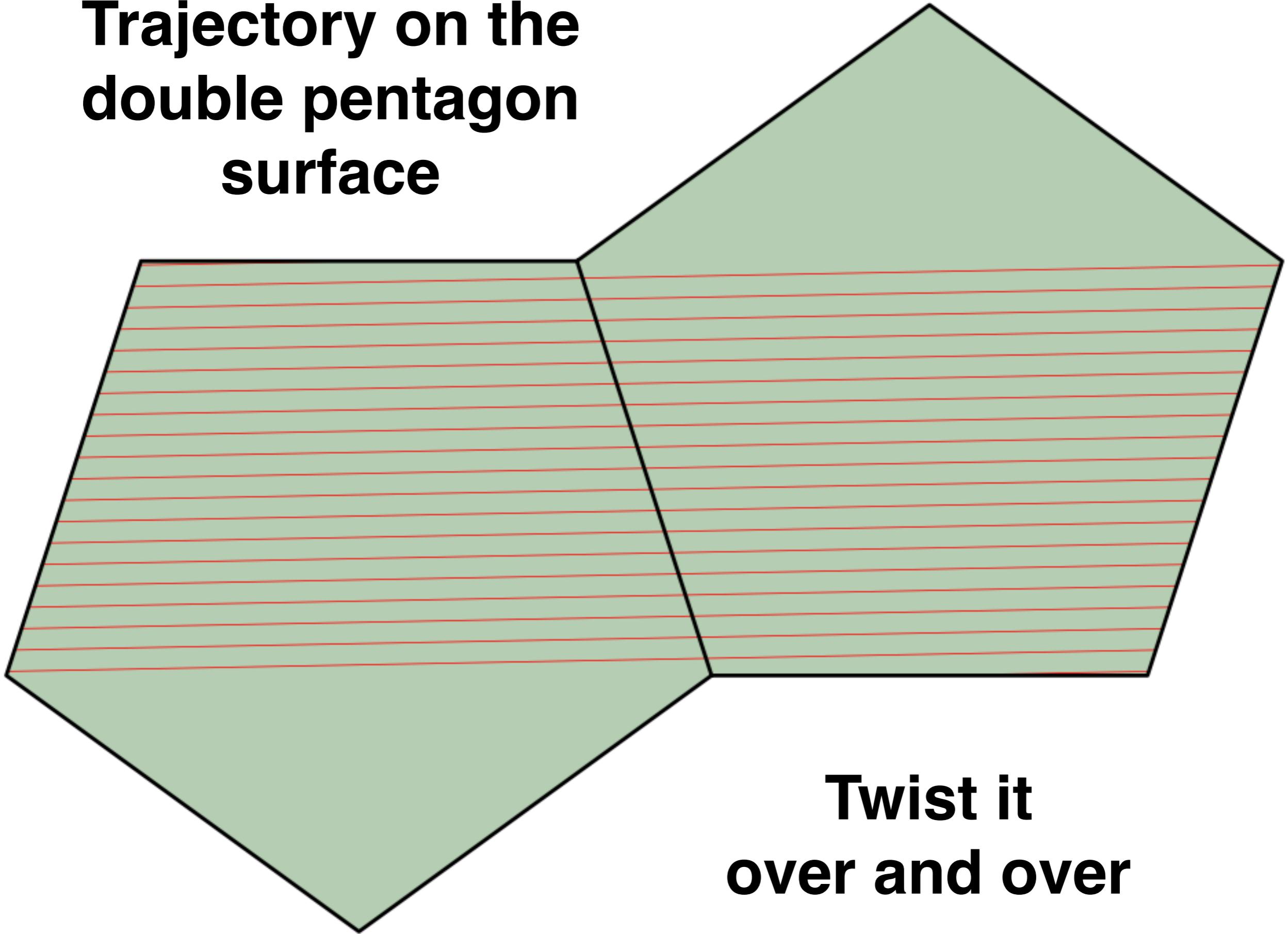
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# **Trajectory on the double pentagon surface**



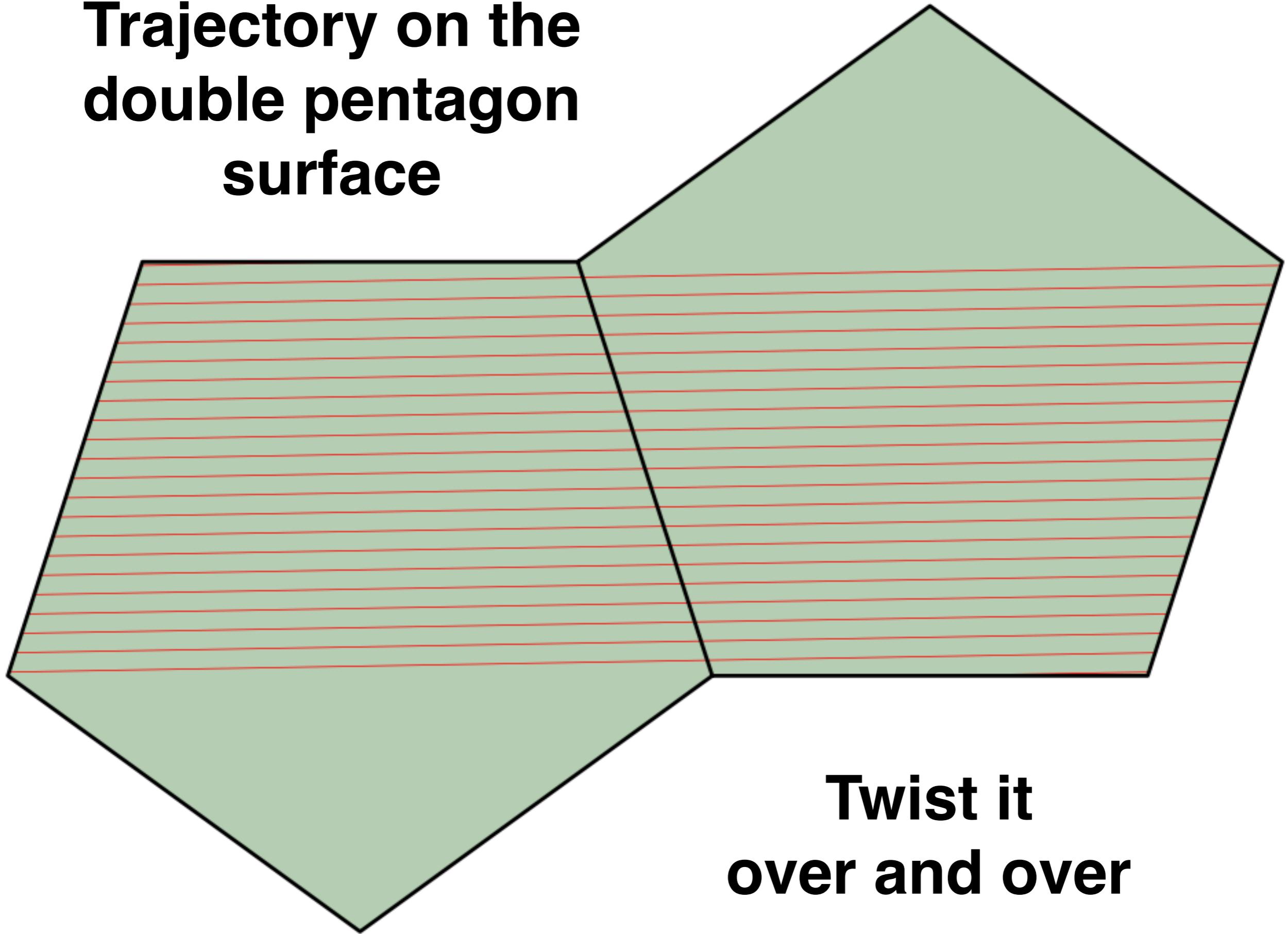
**Twist it  
over and over**

# **Trajectory on the double pentagon surface**



**Twist it  
over and over**

# **Trajectory on the double pentagon surface**

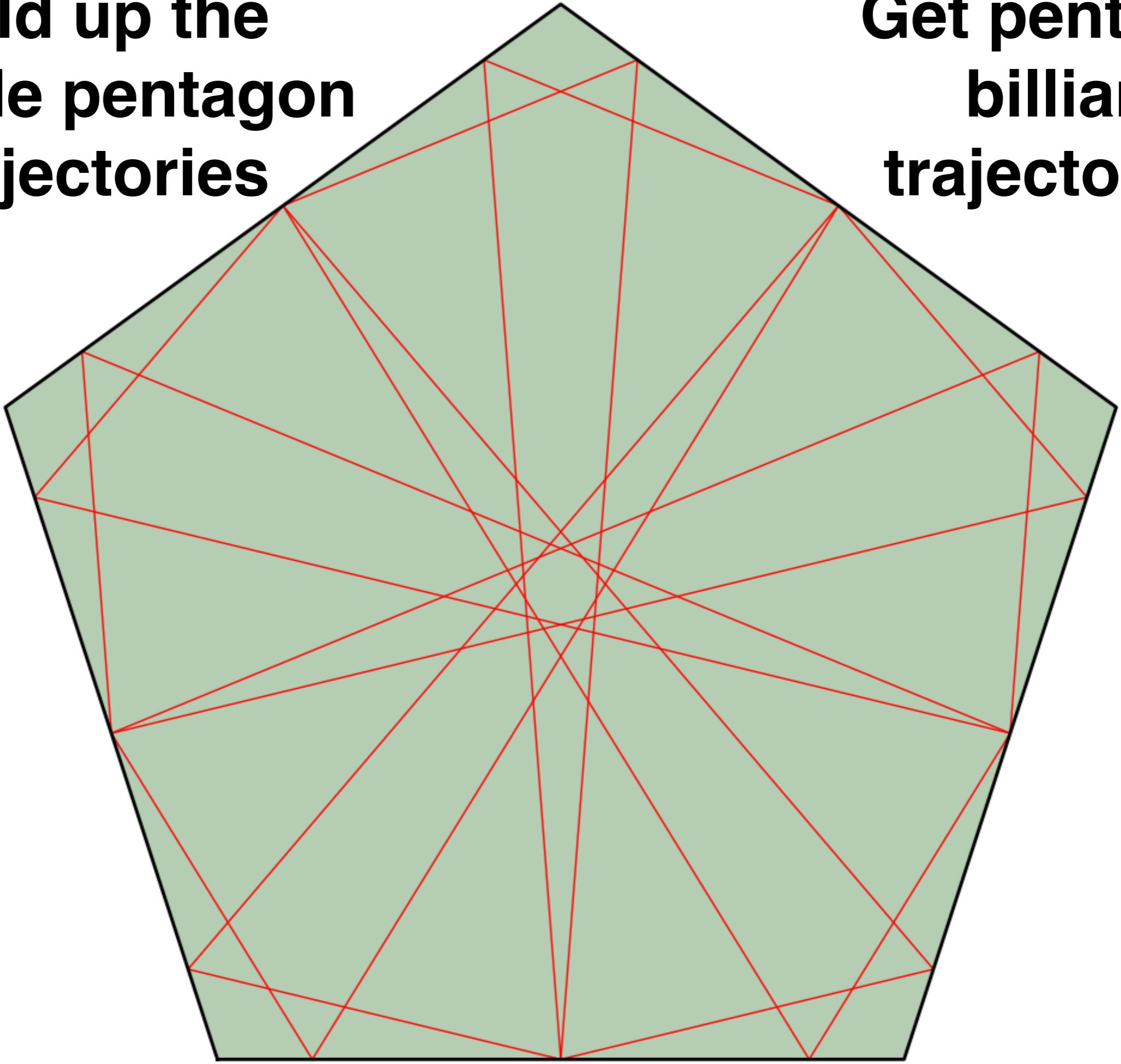


**Twist it  
over and over**



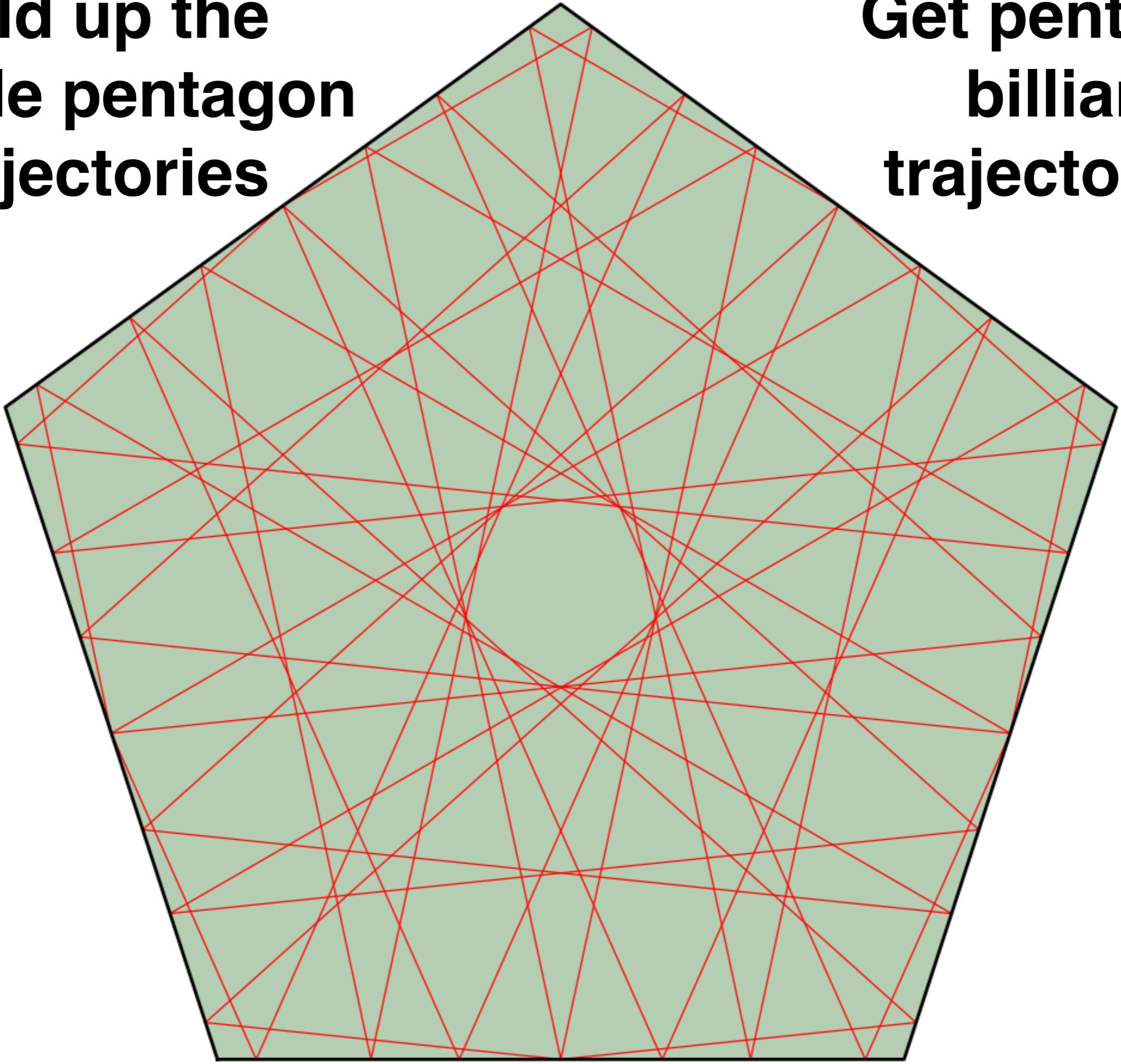
**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**



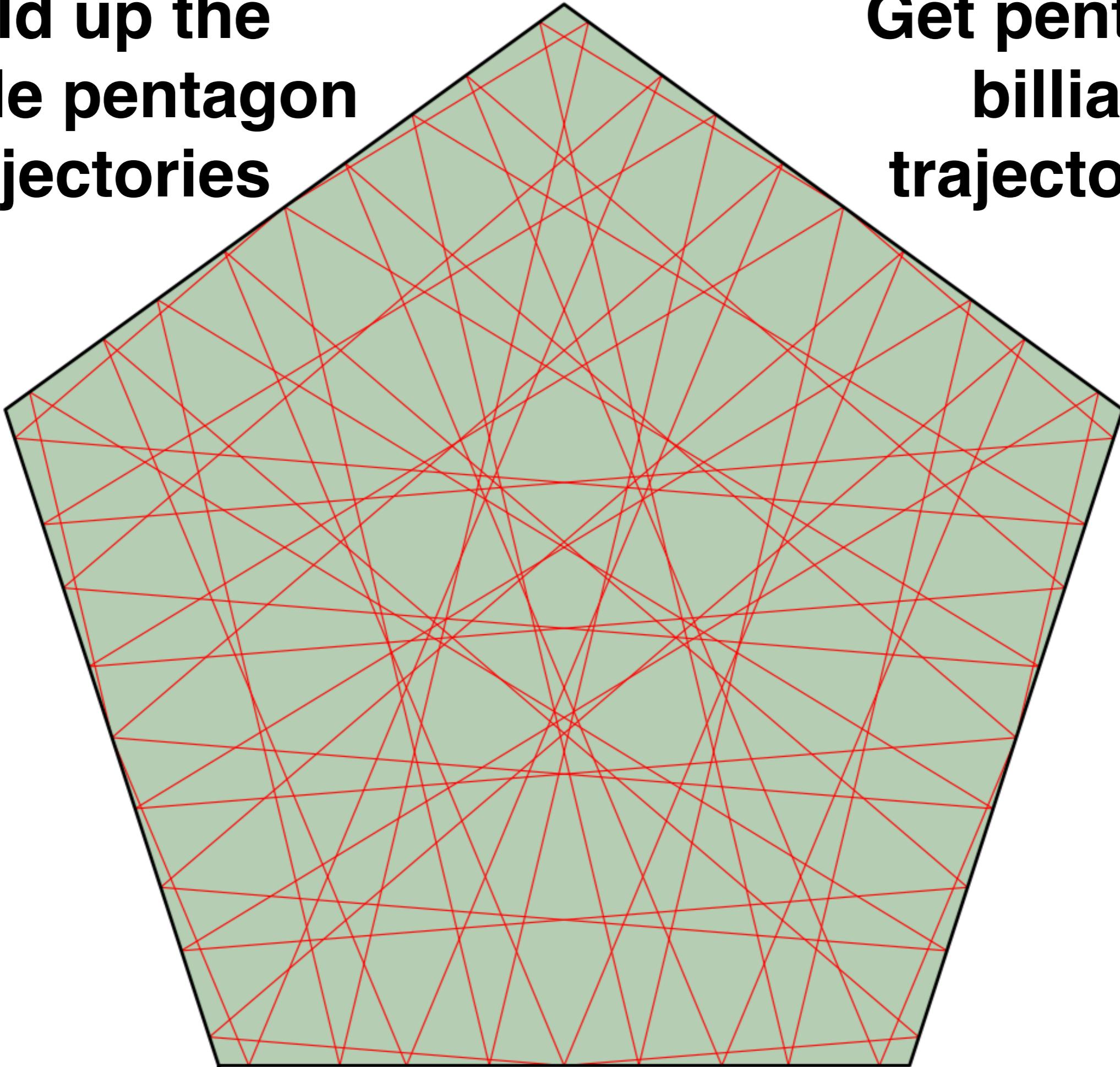
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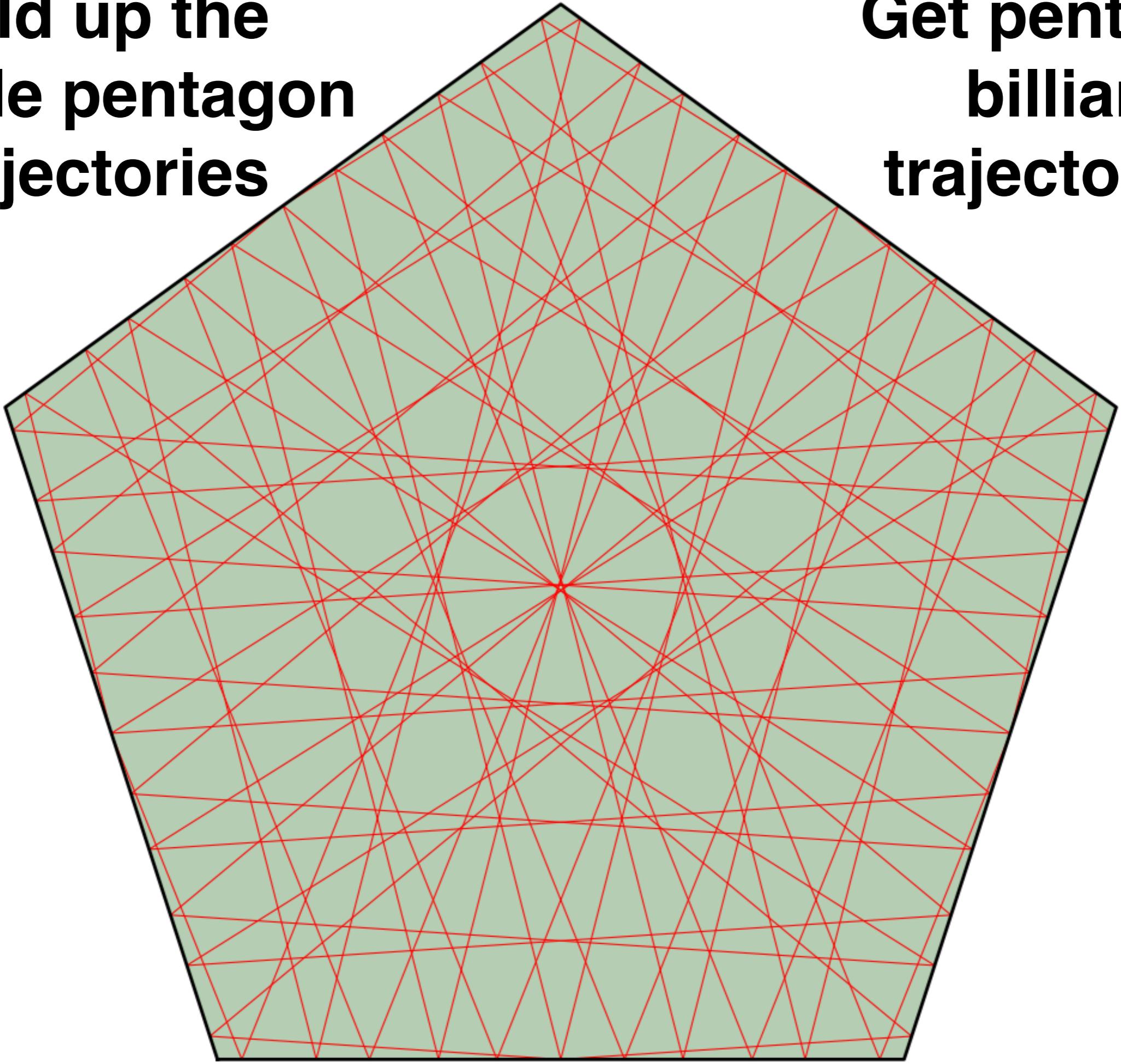
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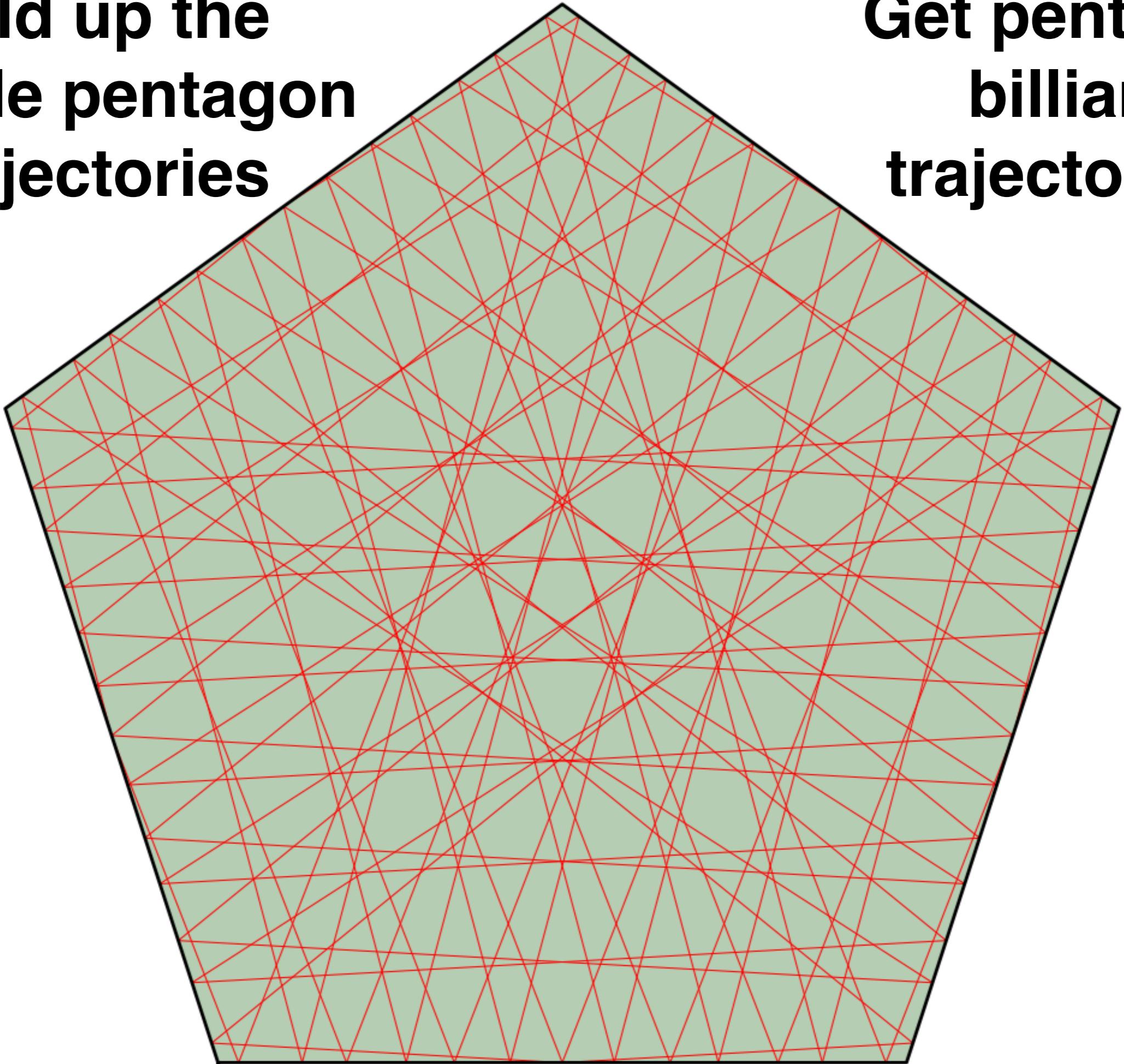
**Fold up the  
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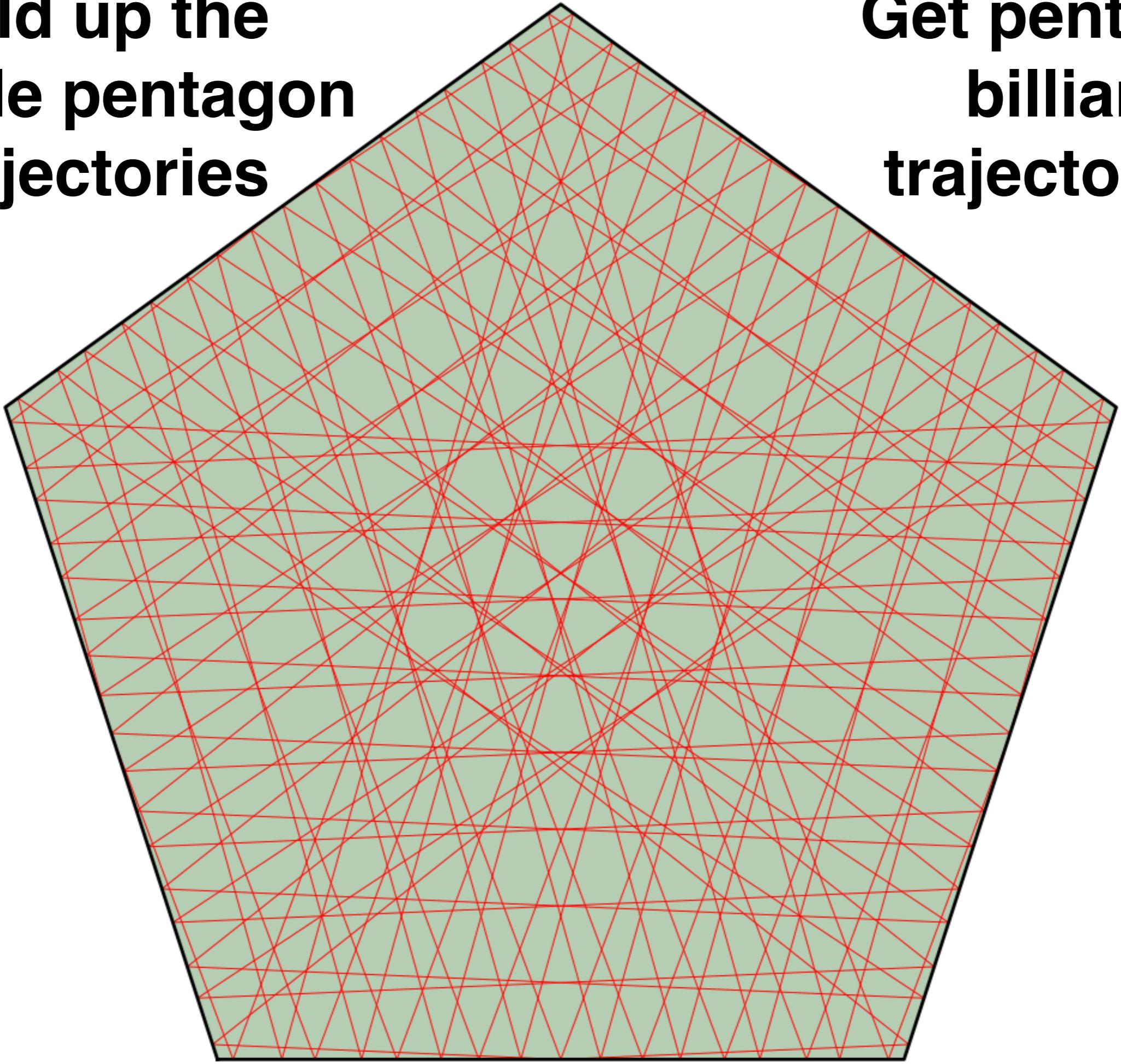
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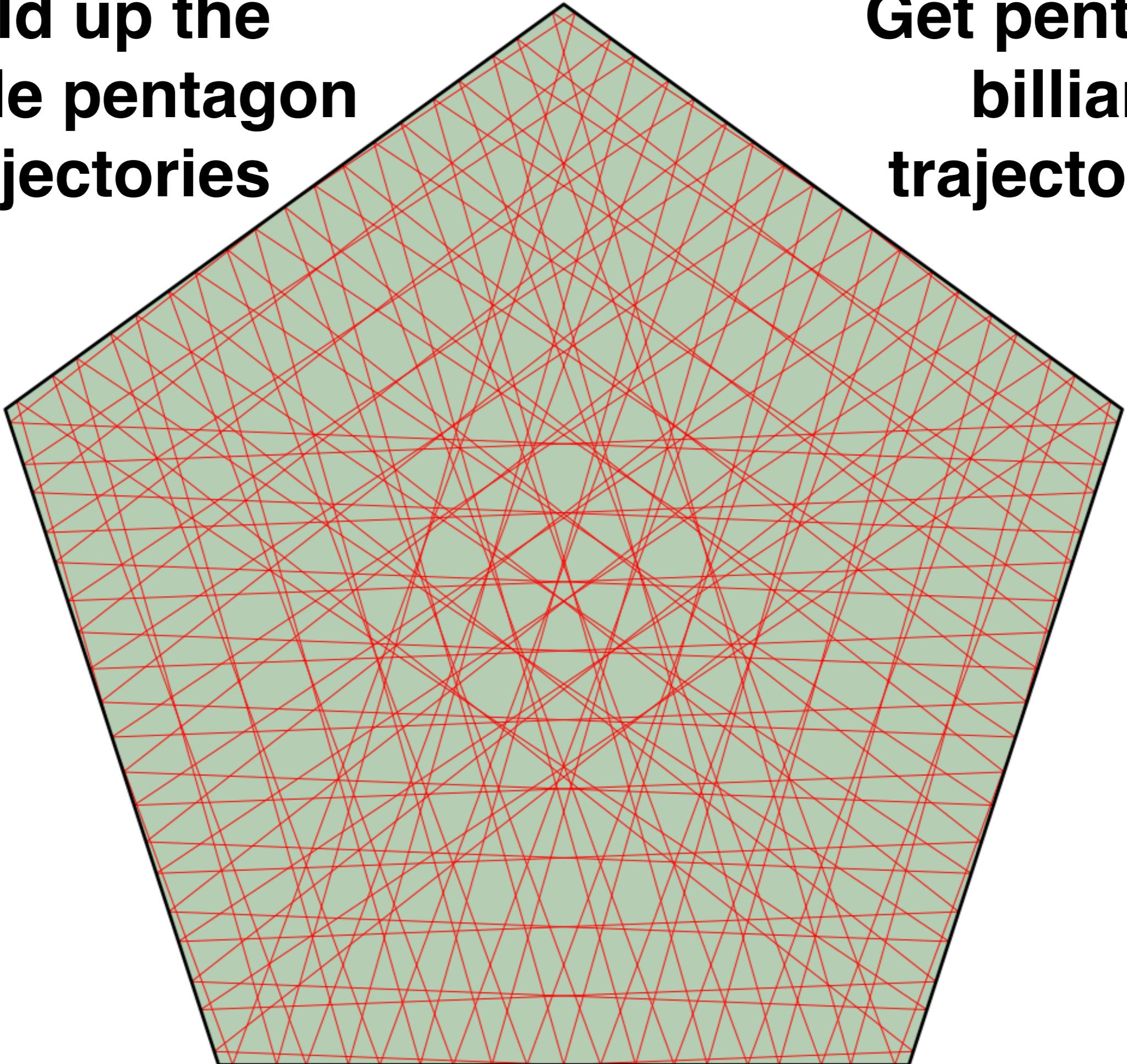
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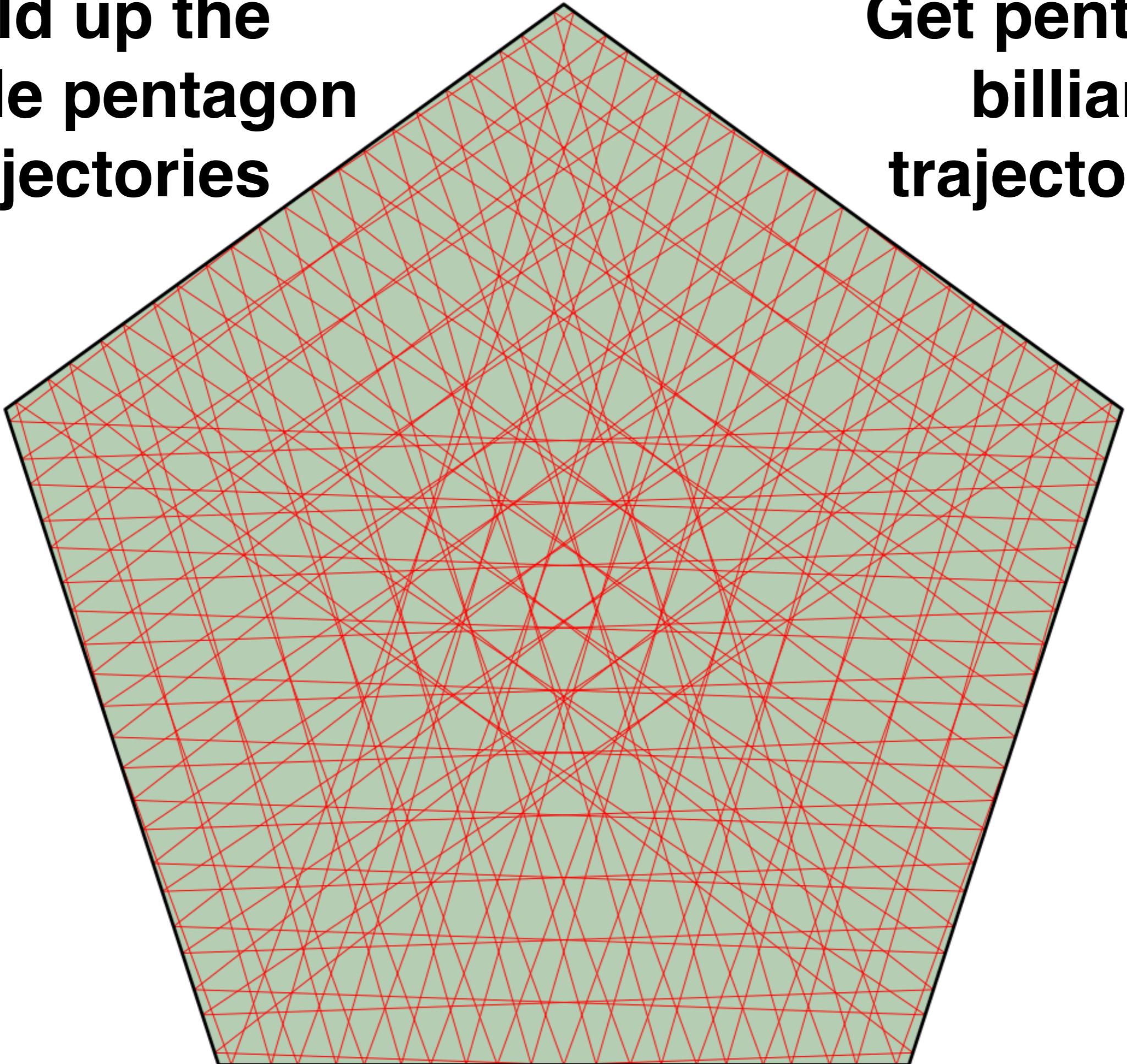
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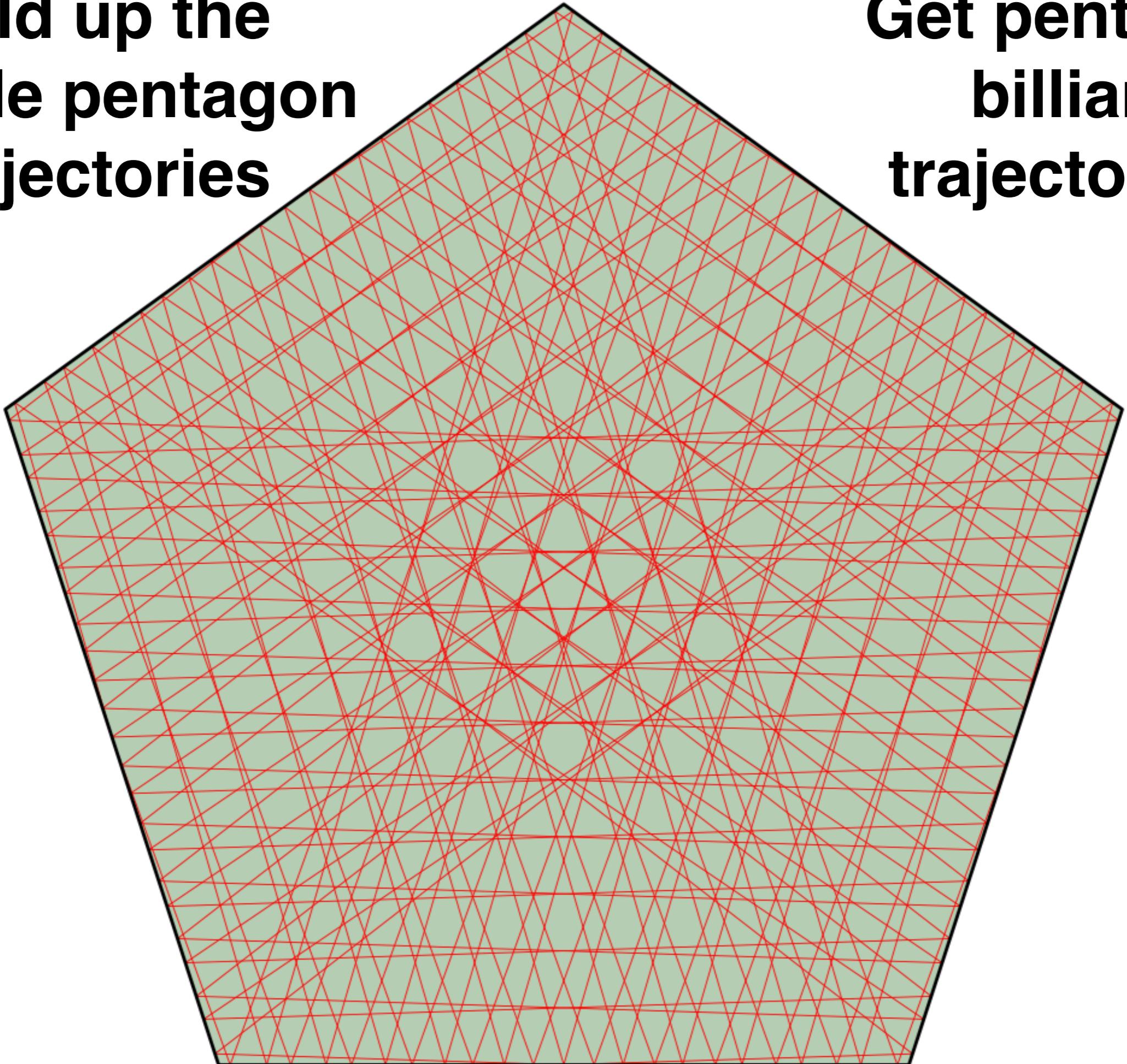
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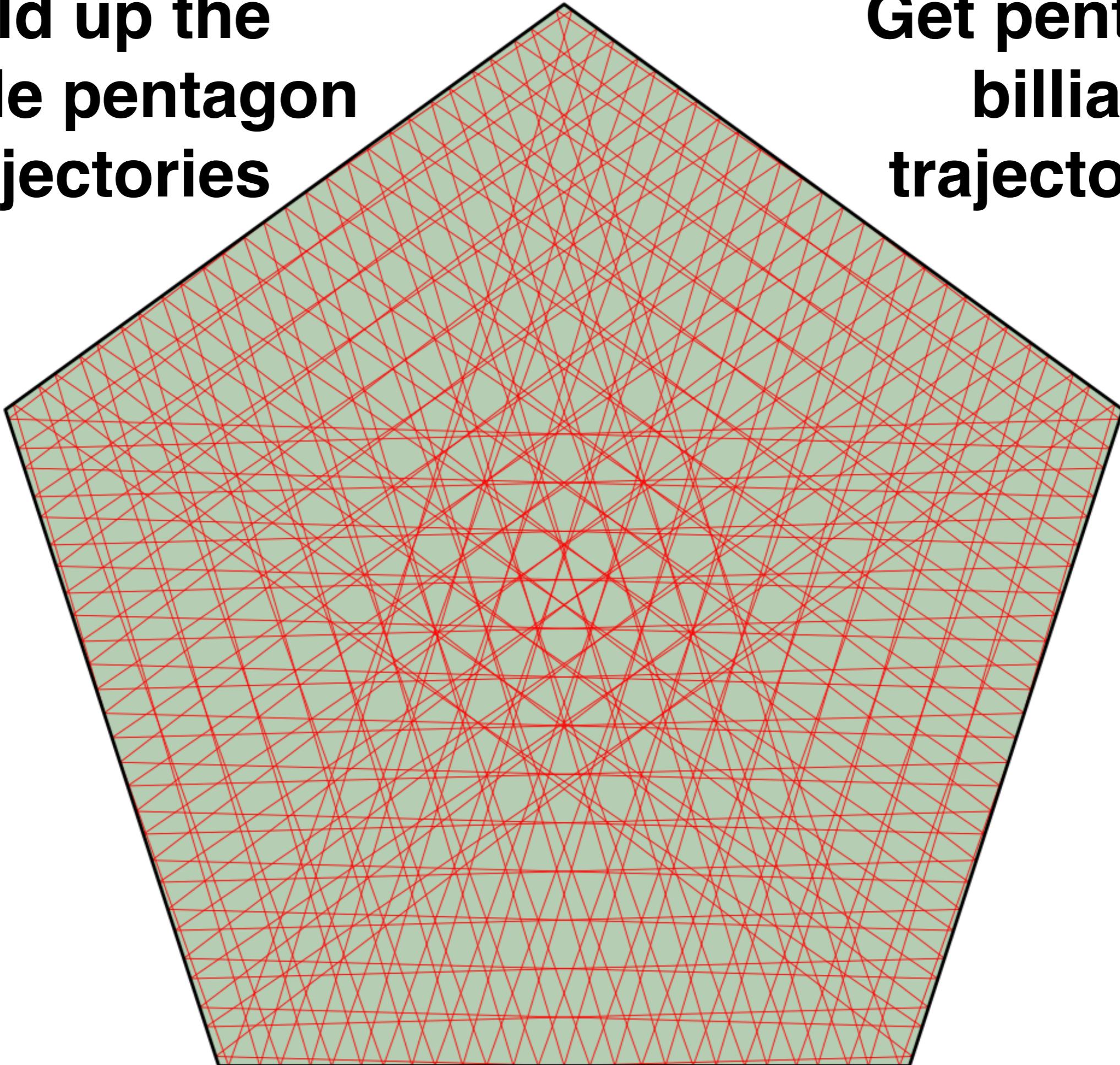
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billiard  
trajectories!**



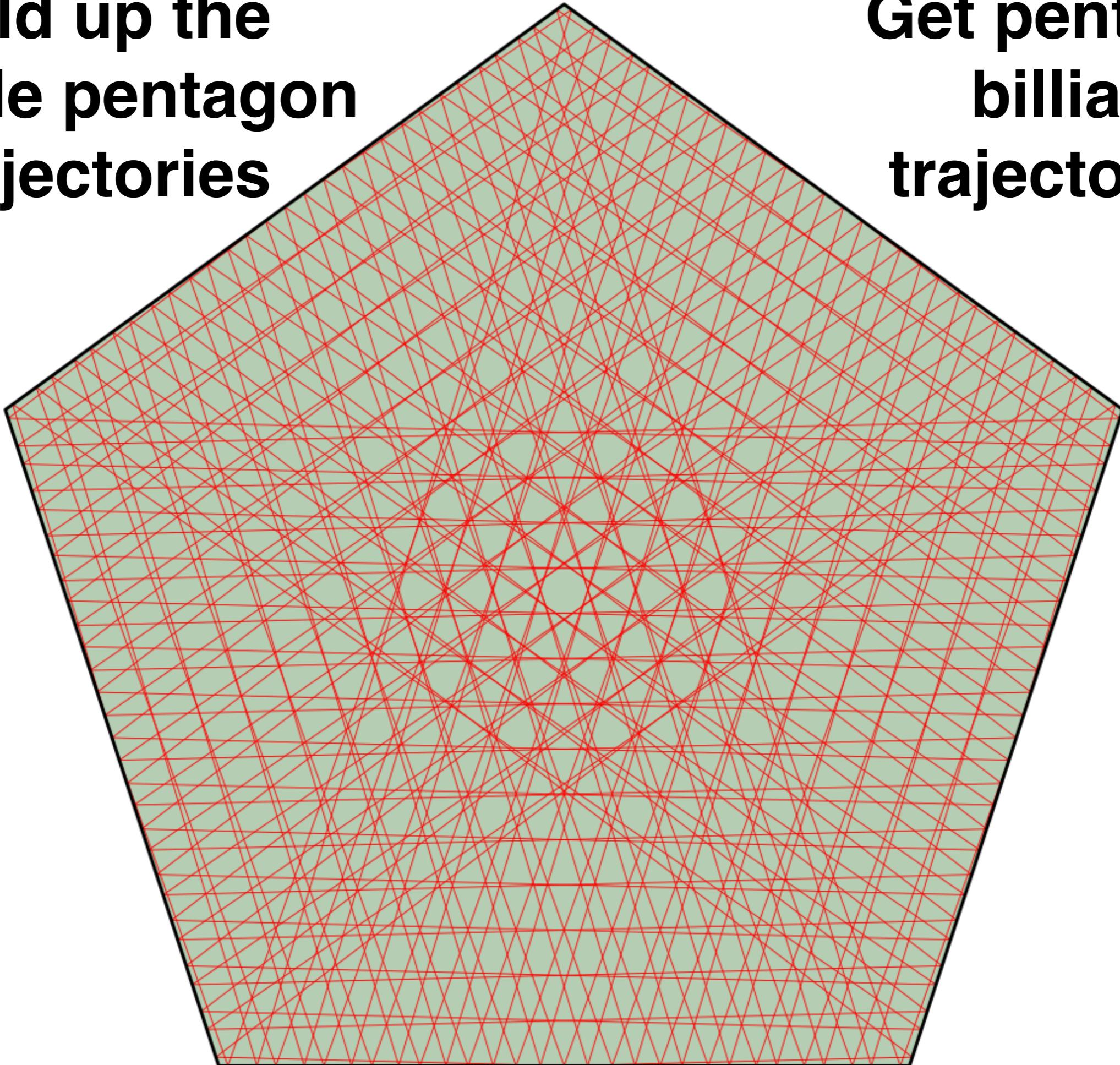
**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**



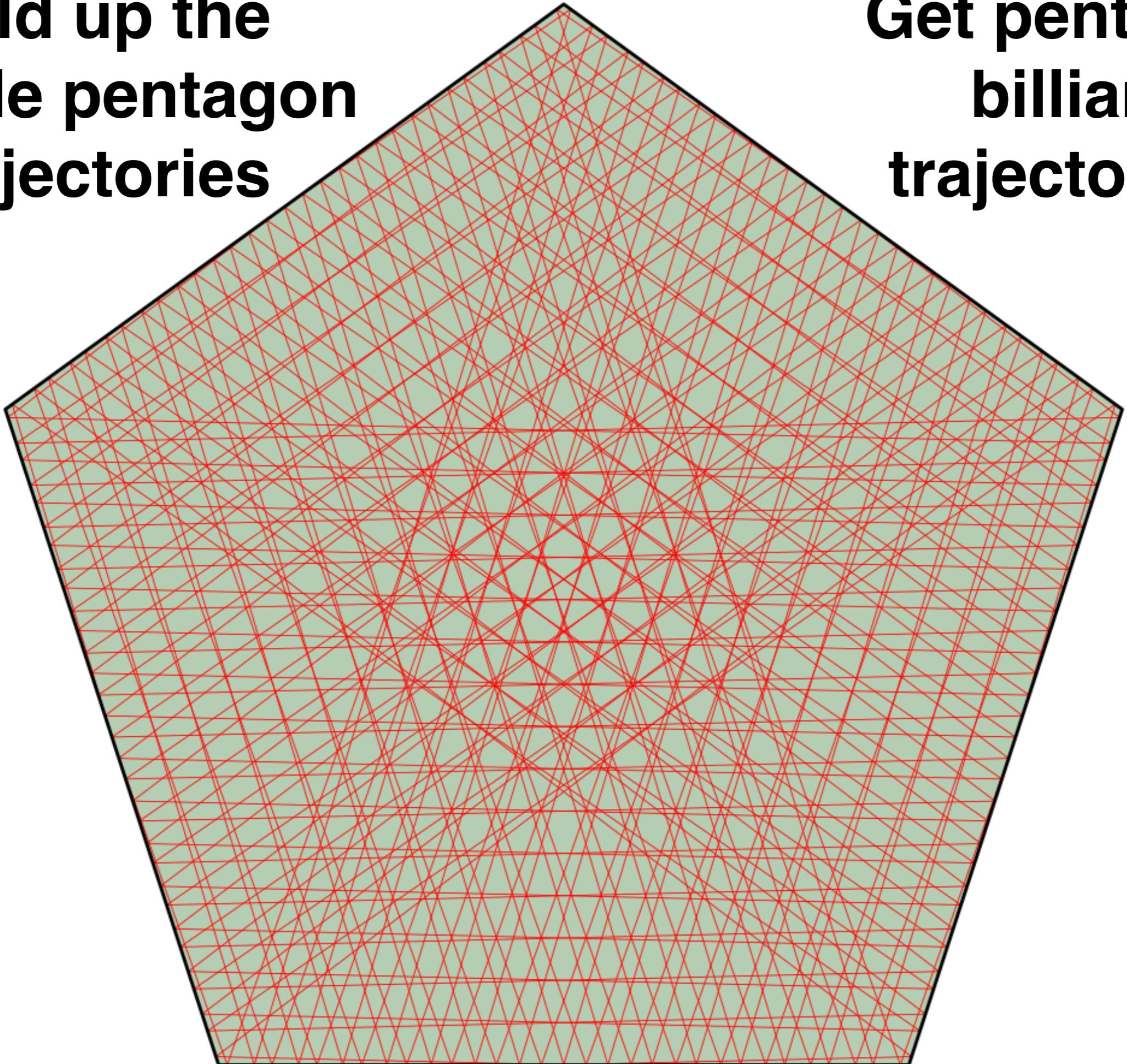
**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**



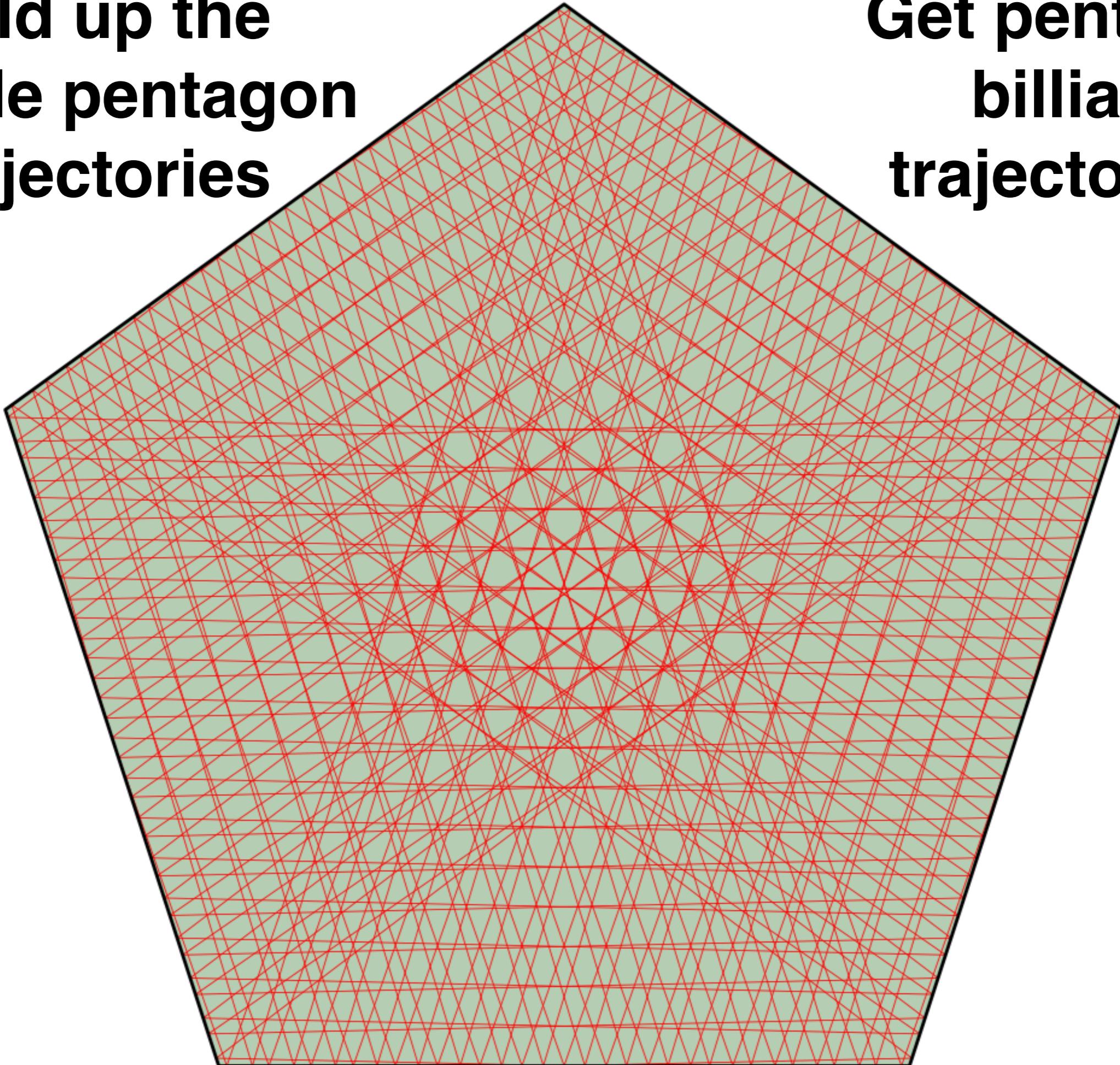
**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**



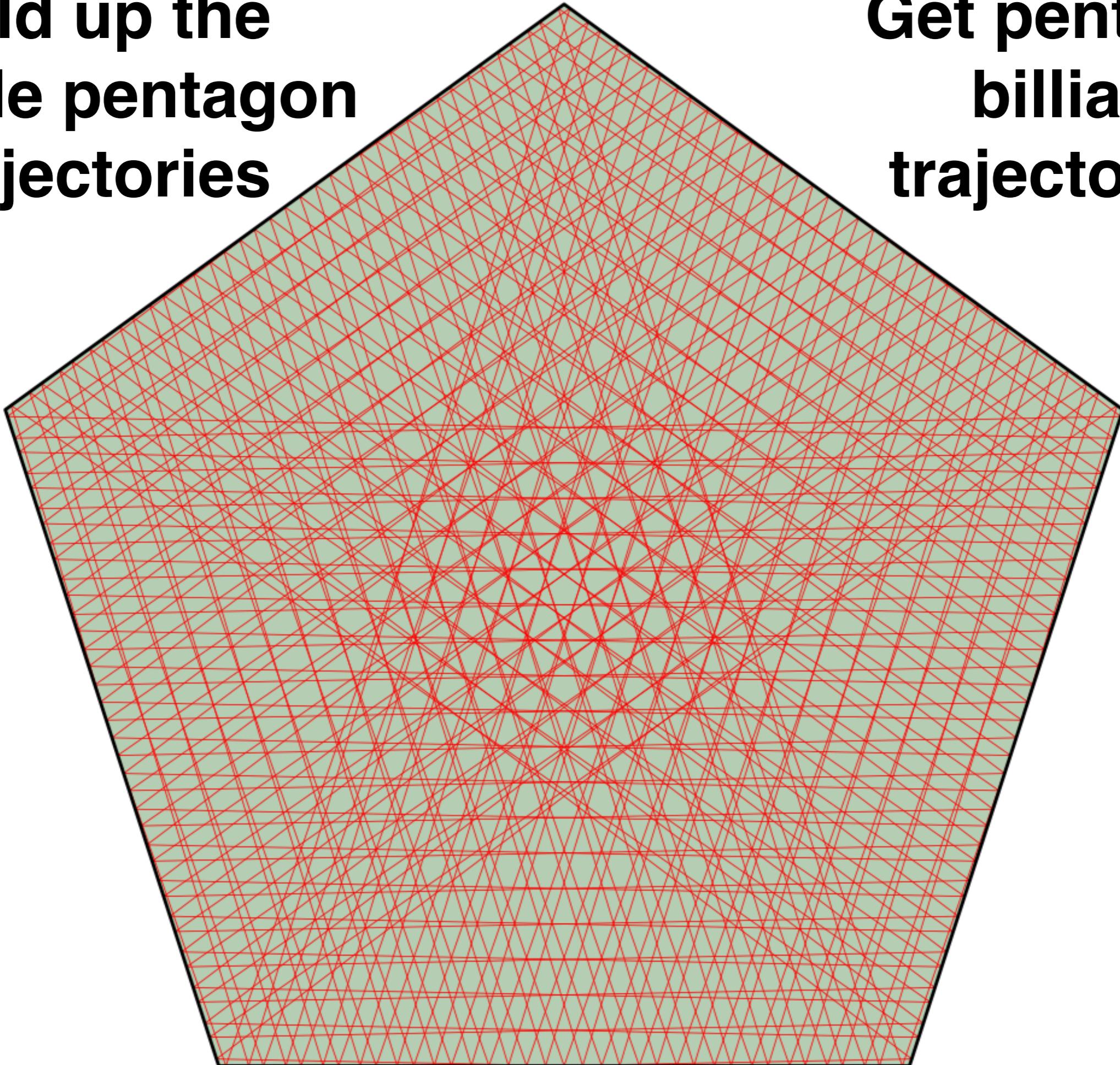
**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**



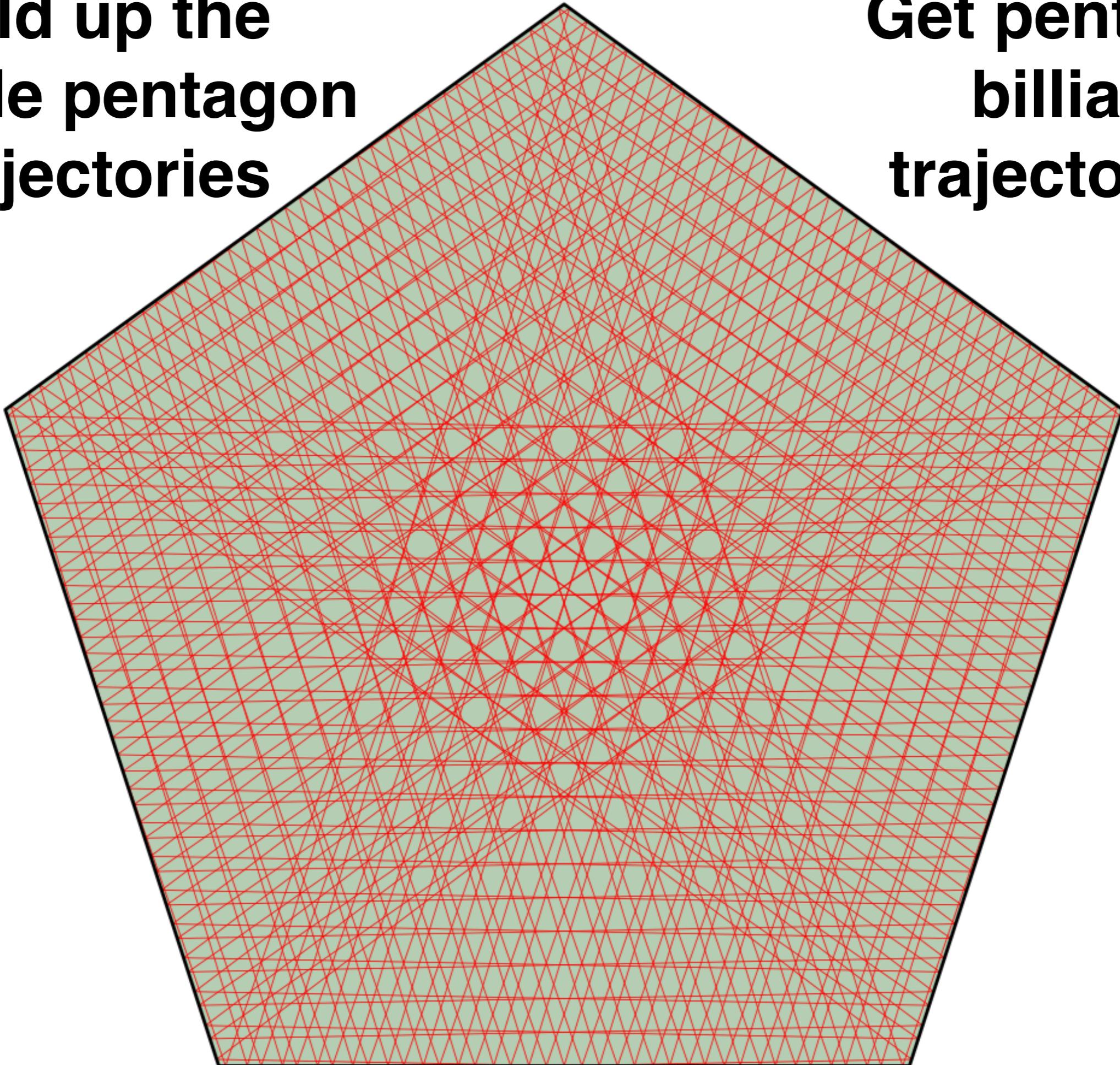
**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**



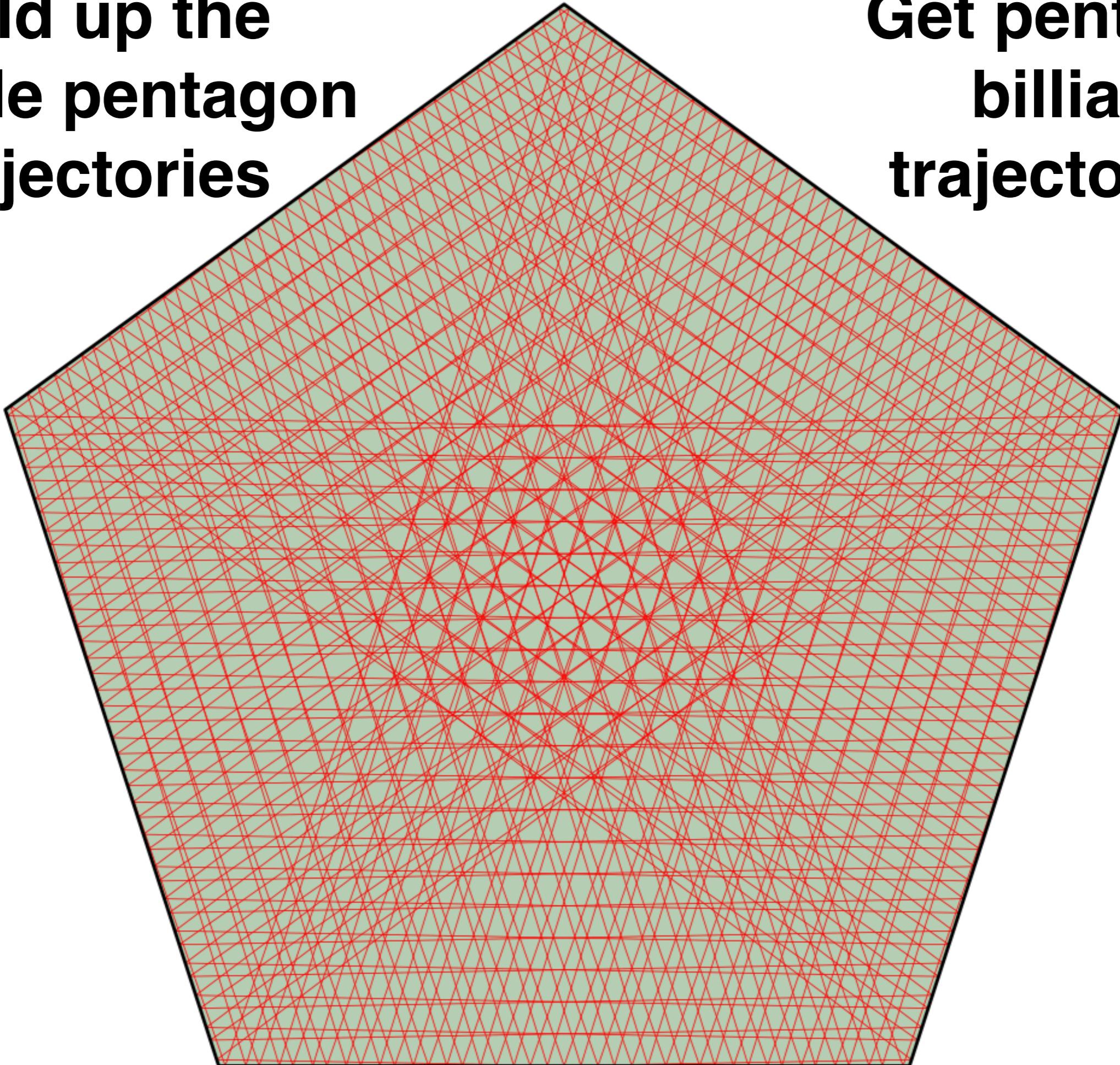
**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**



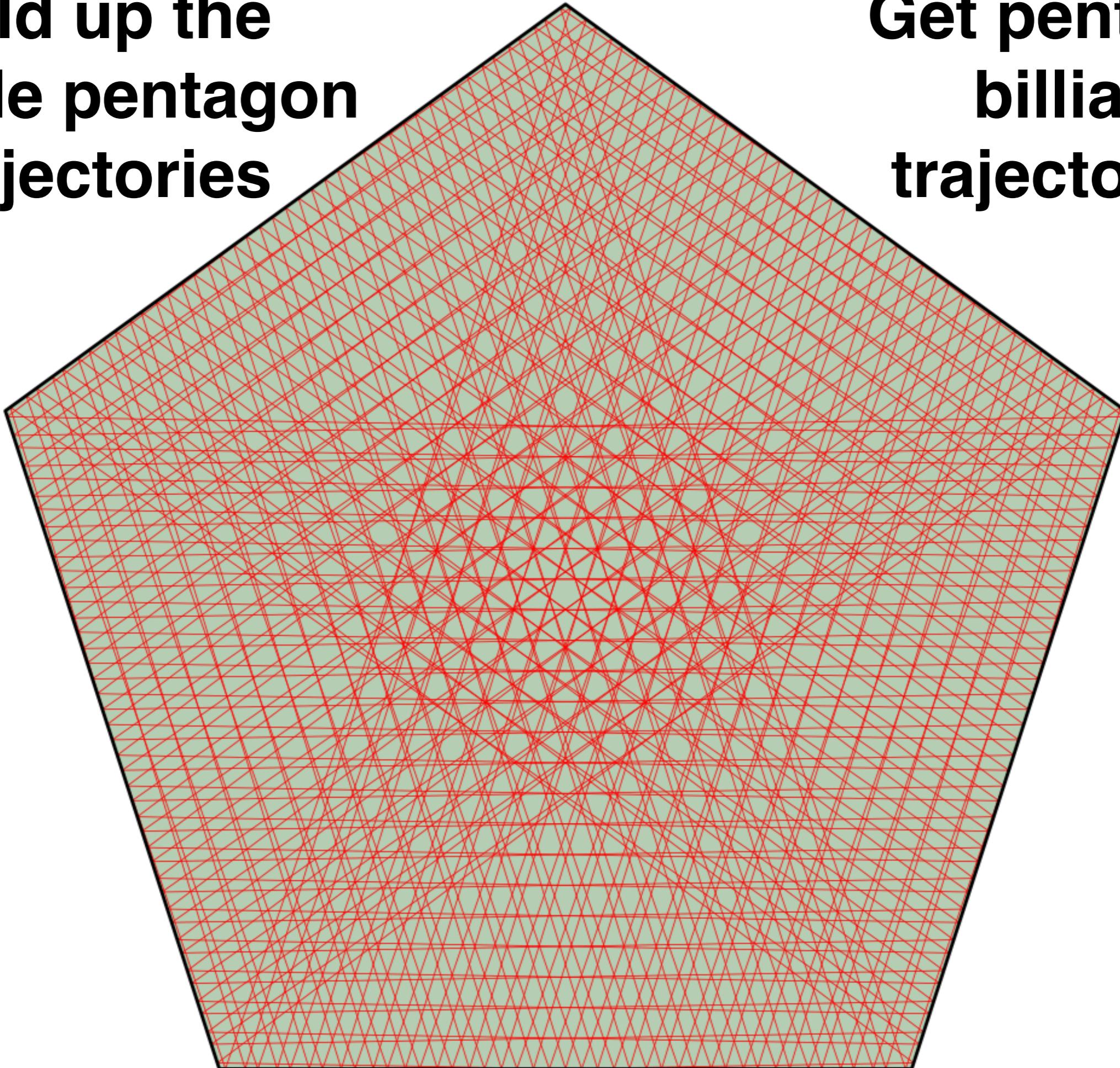
**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**



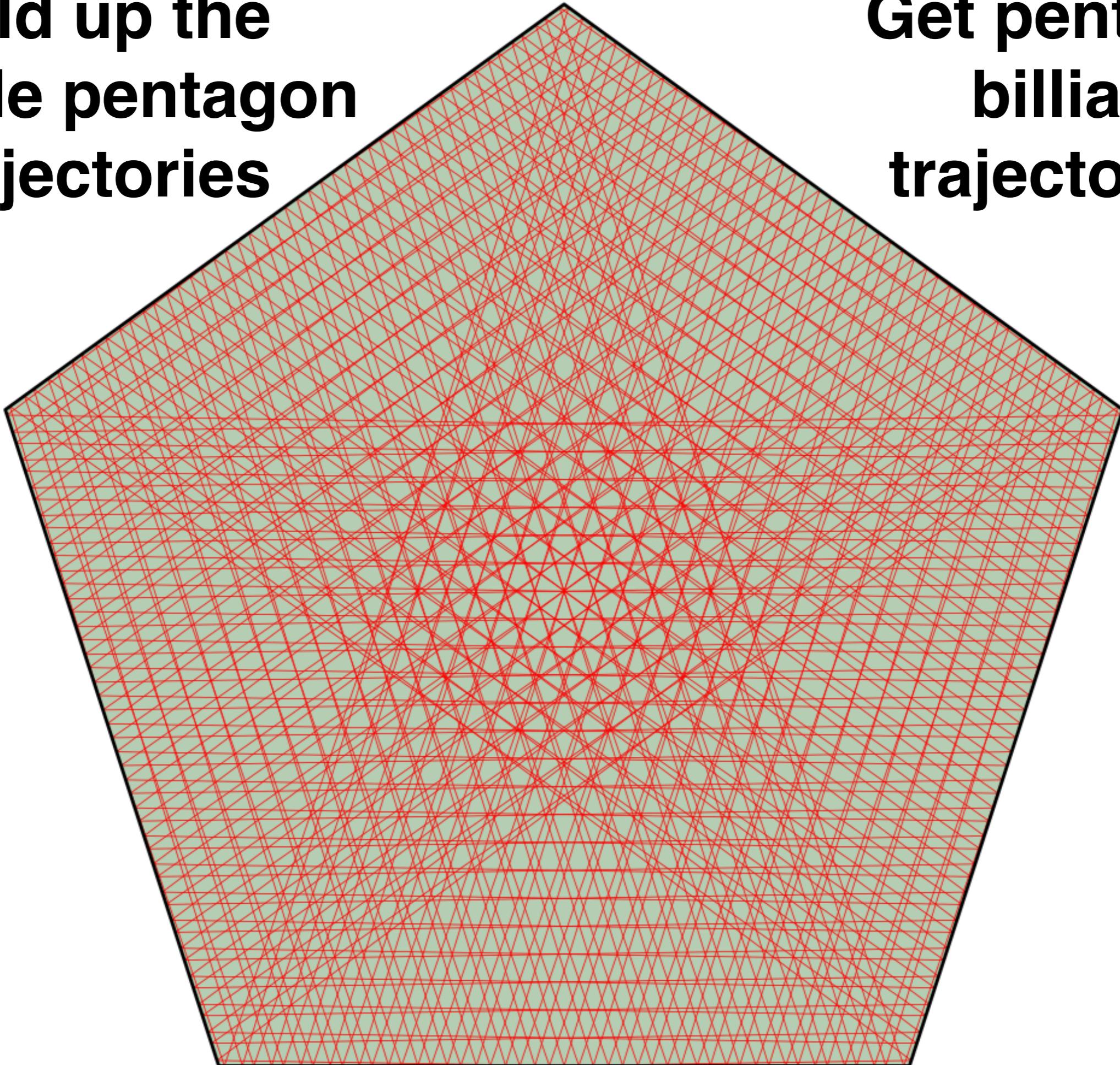
**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**



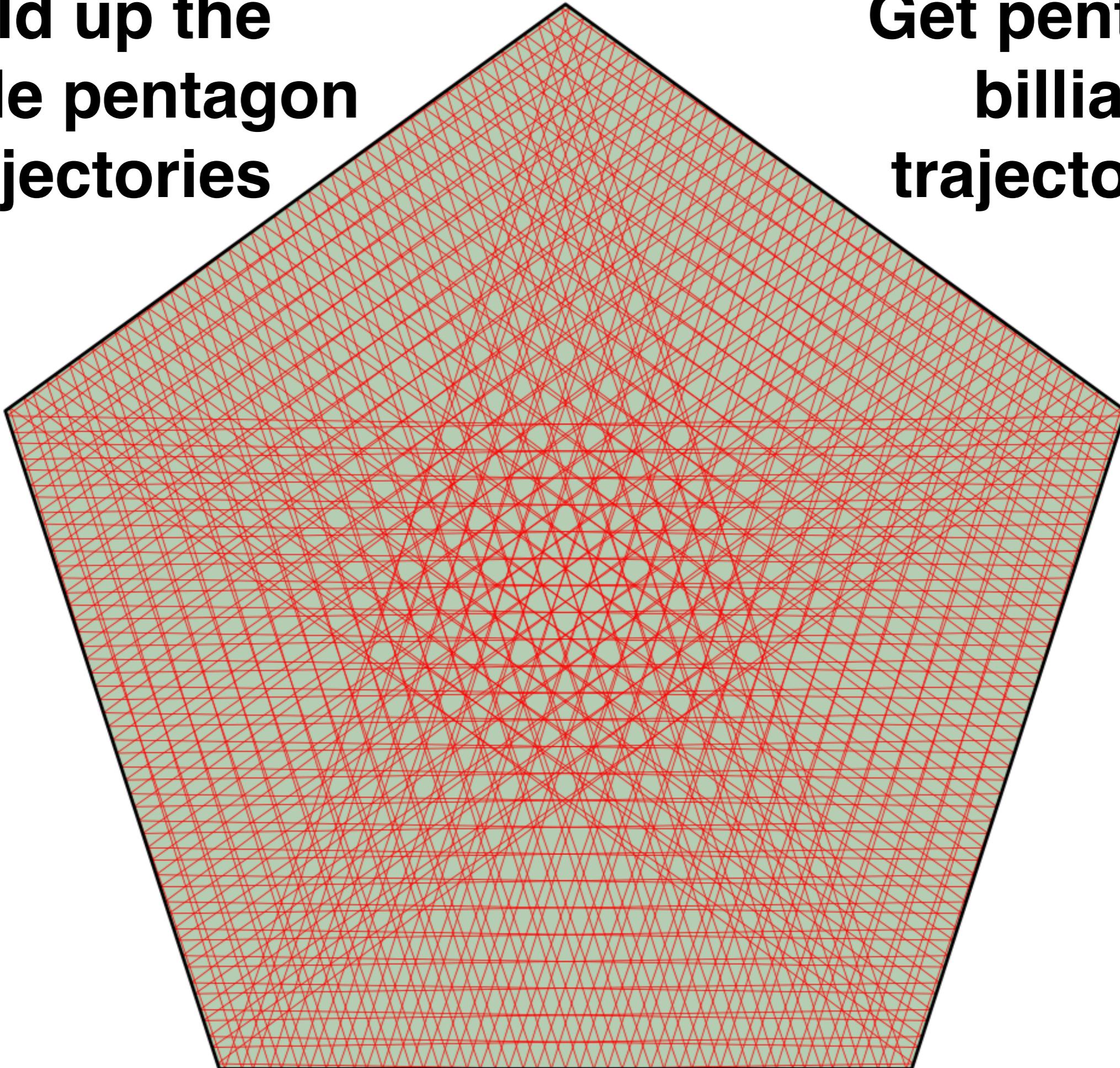
**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**



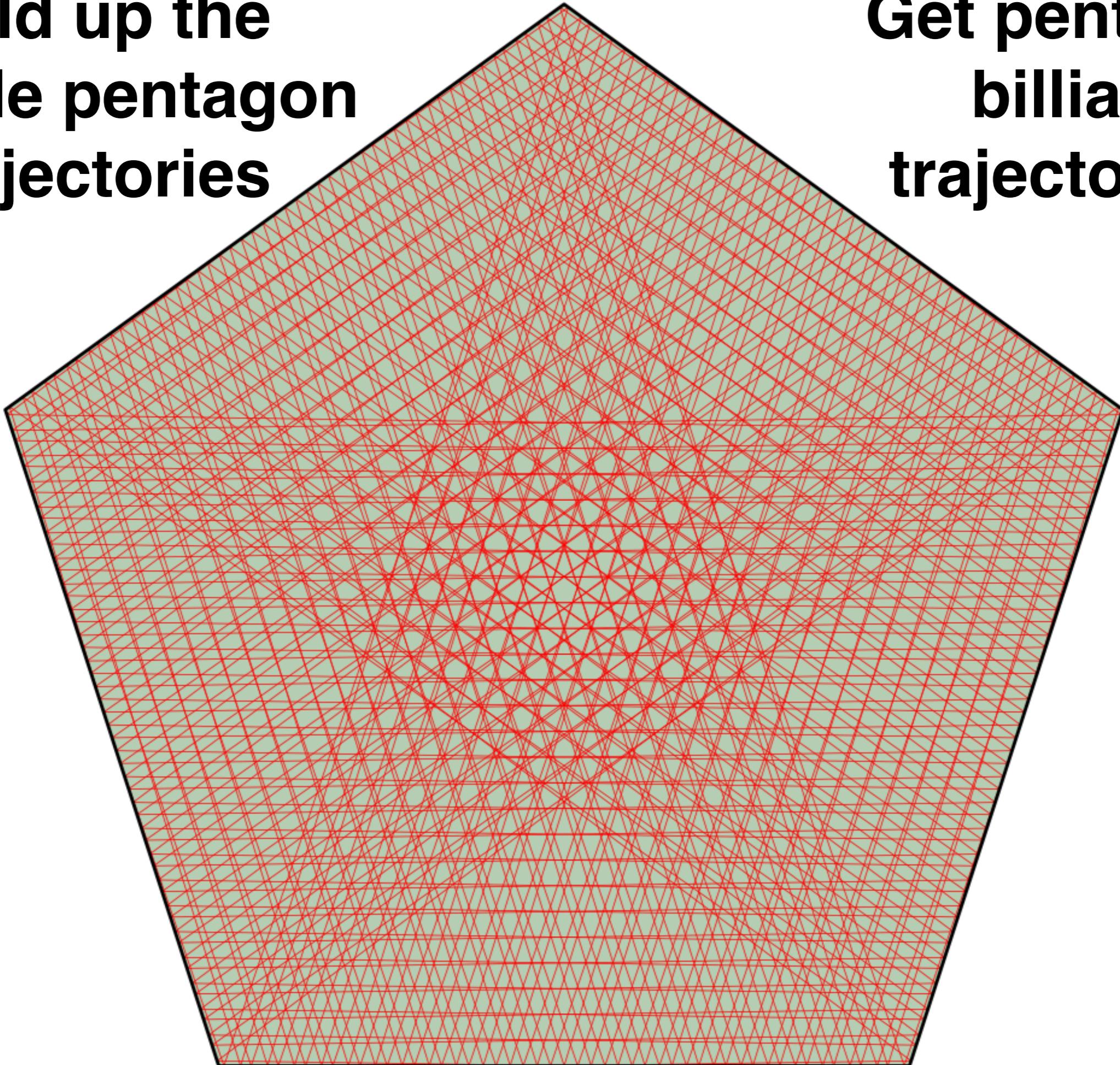
**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**



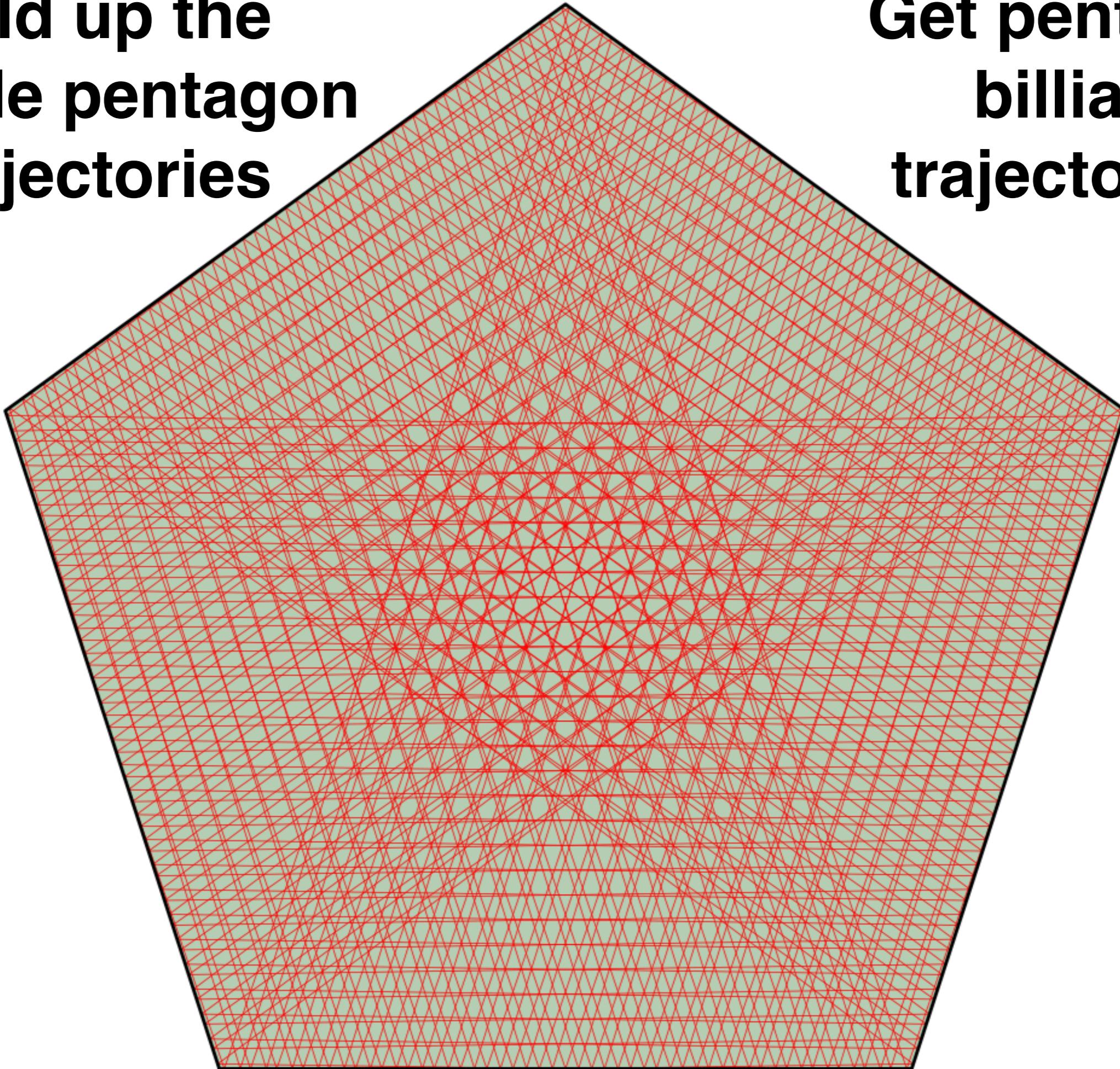
**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**



**Fold up the  
double pentagon  
trajectories**

**Get pentagon  
billiard  
trajectories!**

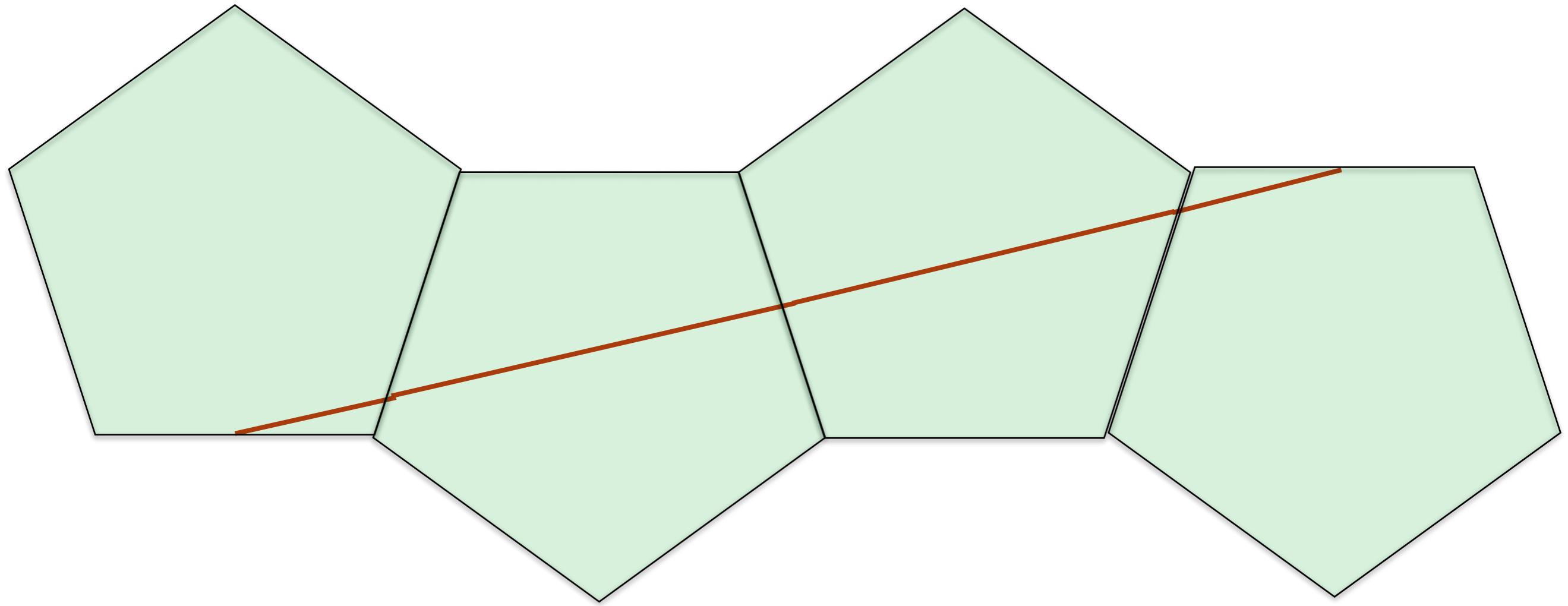


# **SYMMETRY**

**Joint with  
Samuel Lelièvre**

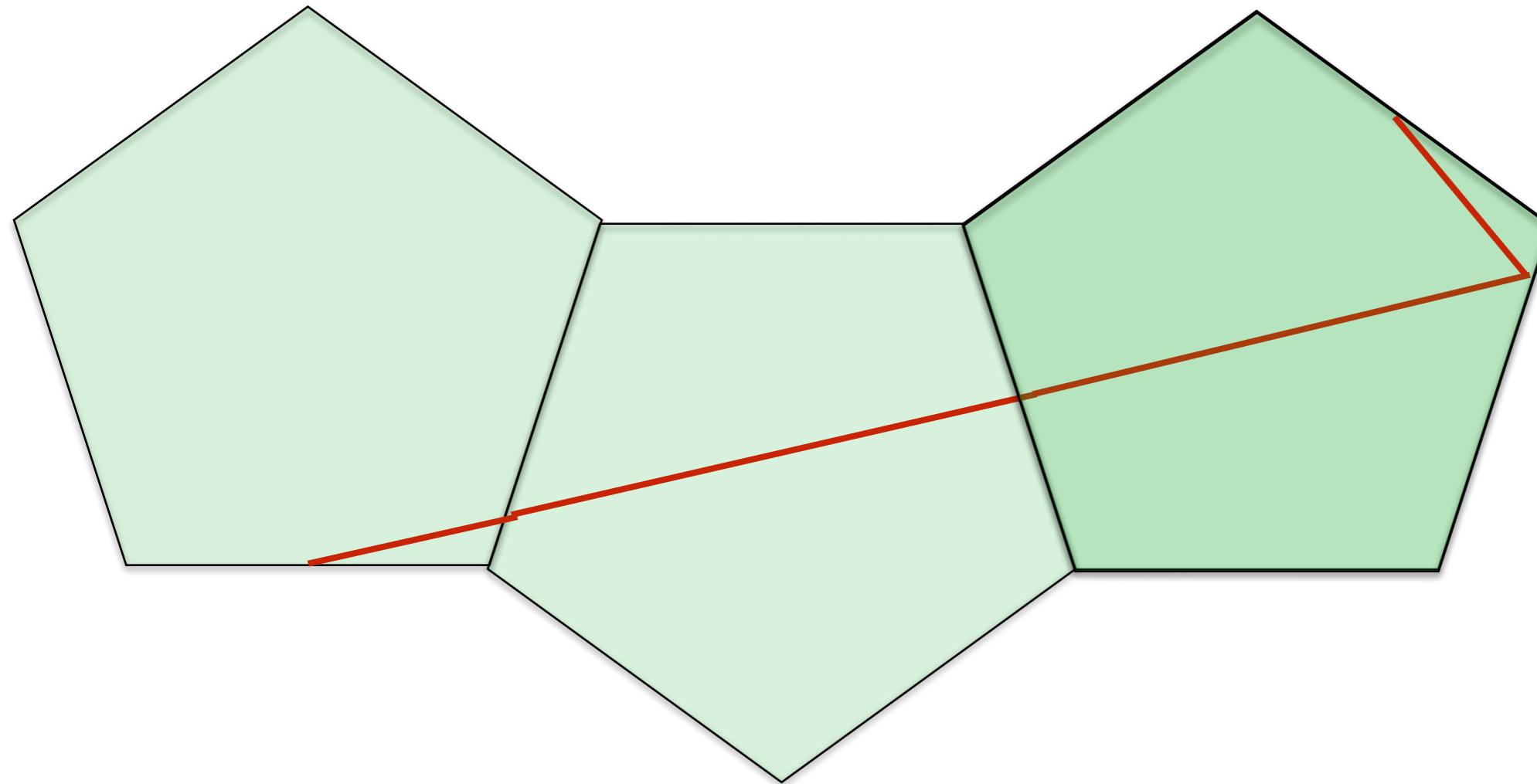


Recall: start with a trajectory on the surface



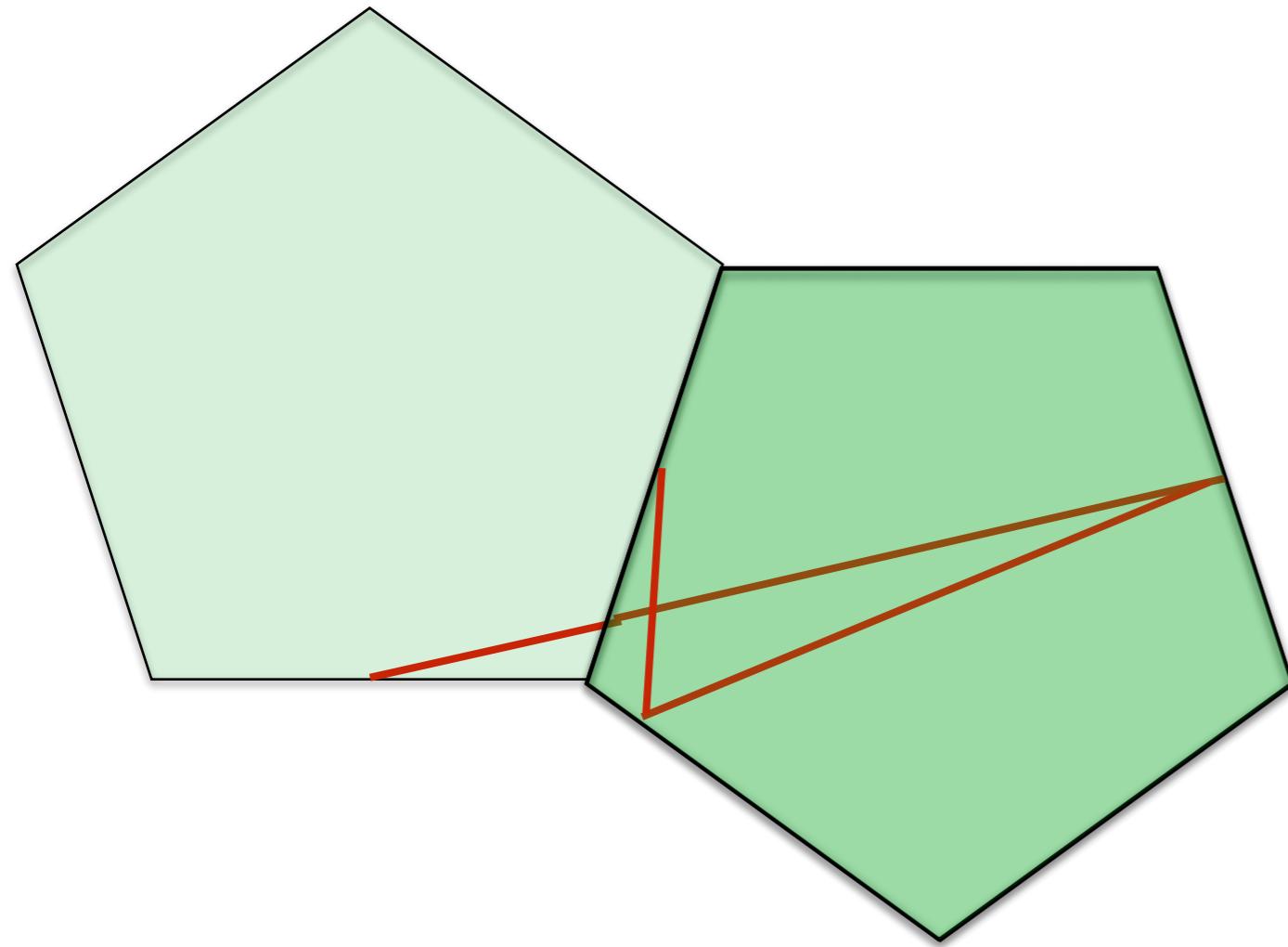
Fold each pentagon onto the previous one

Recall: start with a trajectory on the surface



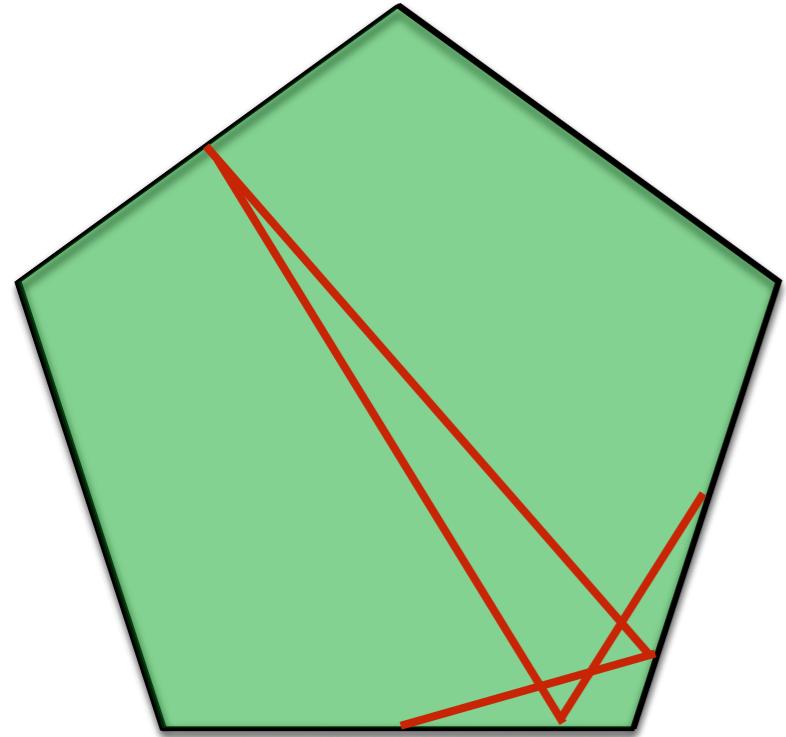
Fold each pentagon onto the previous one

Recall: start with a trajectory on the surface



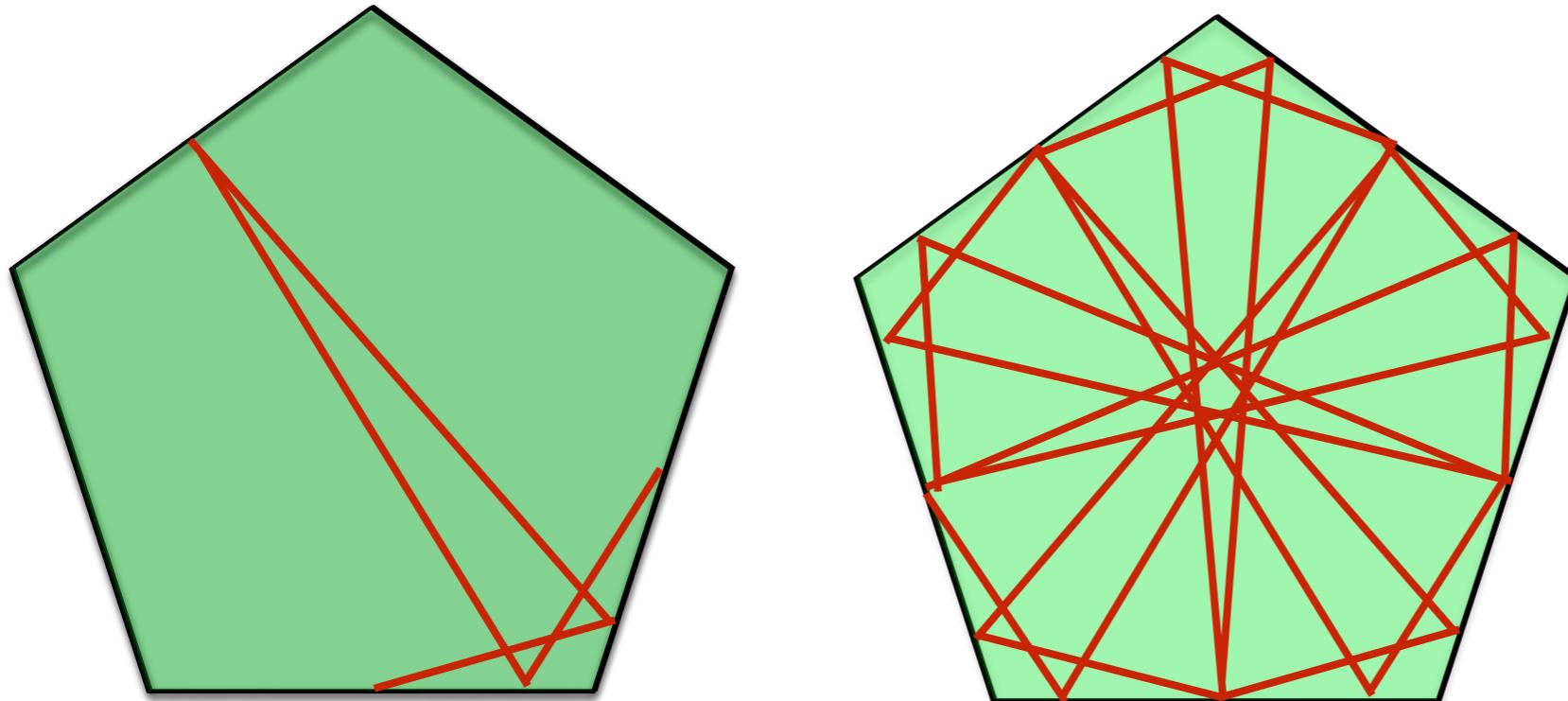
Fold each pentagon onto the previous one

Recall: start with a trajectory on the surface



Fold each pentagon onto the previous one  
Get INCOMPLETE billiard trajectory.

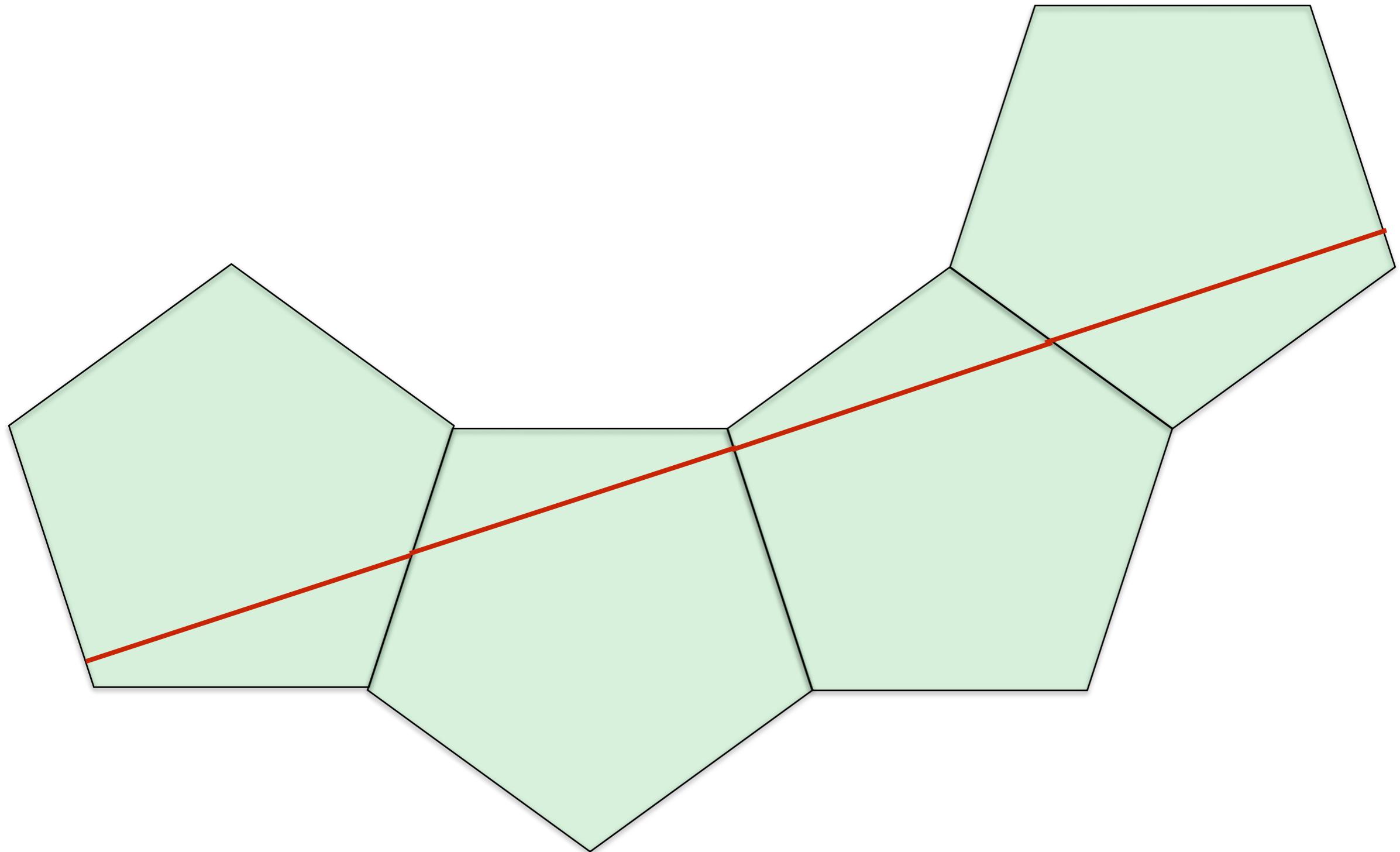
Recall: start with a trajectory on the surface



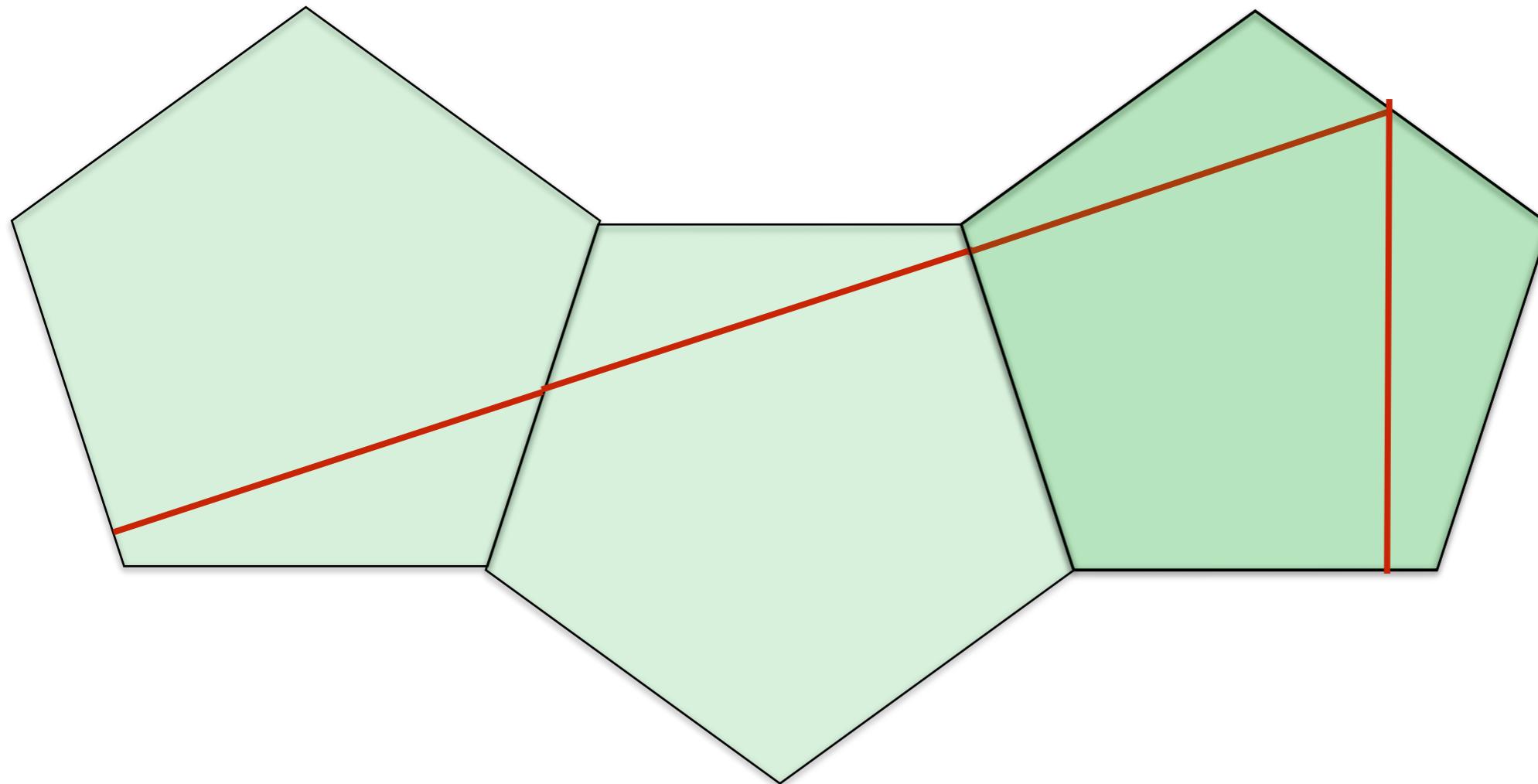
Have to do it 5  
times to get  
back  
where we  
started

Fold each pentagon onto the previous one  
Get INCOMPLETE billiard trajectory.

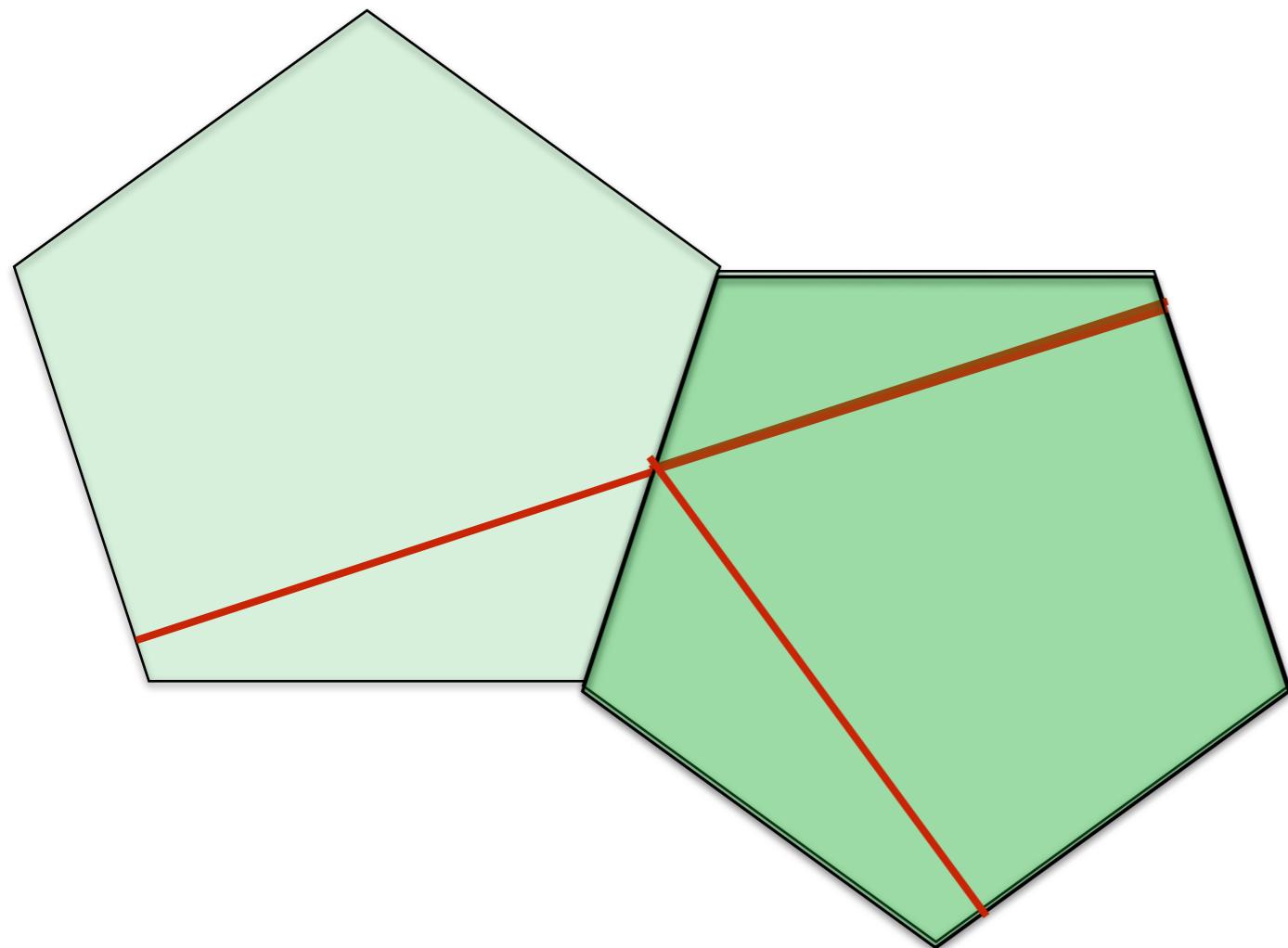
# Start with a different trajectory



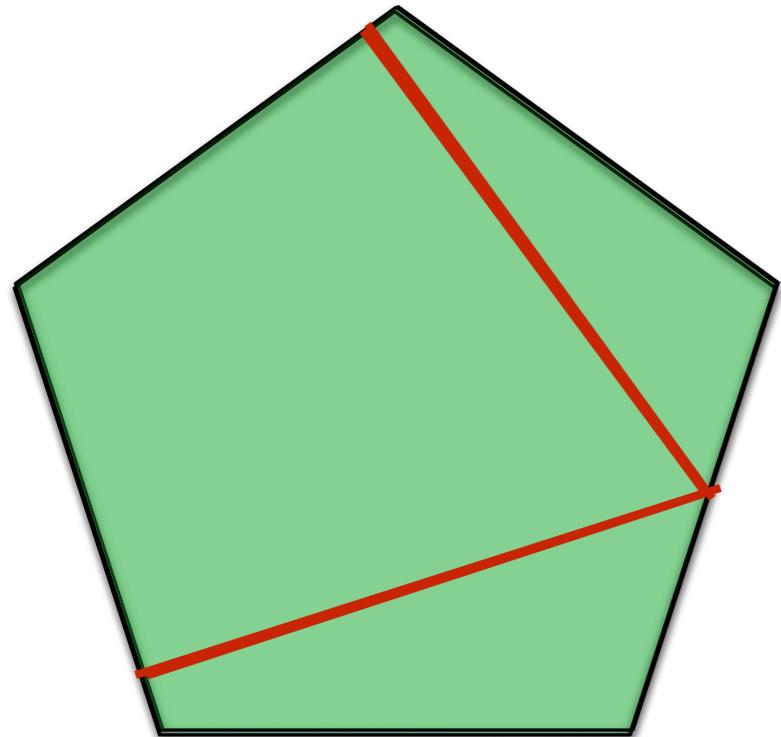
# Start with a different trajectory



# Start with a different trajectory



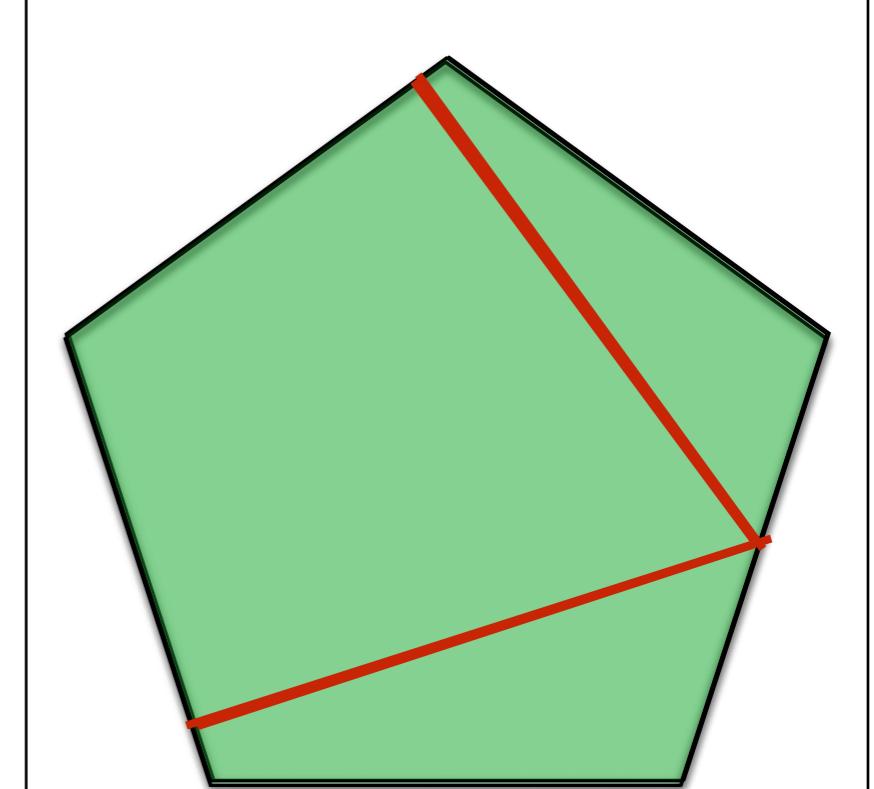
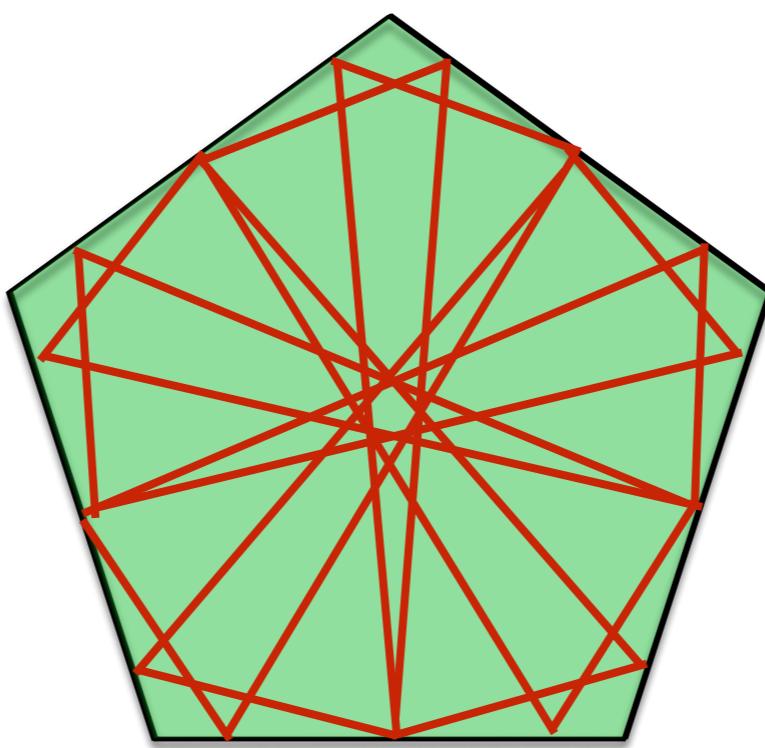
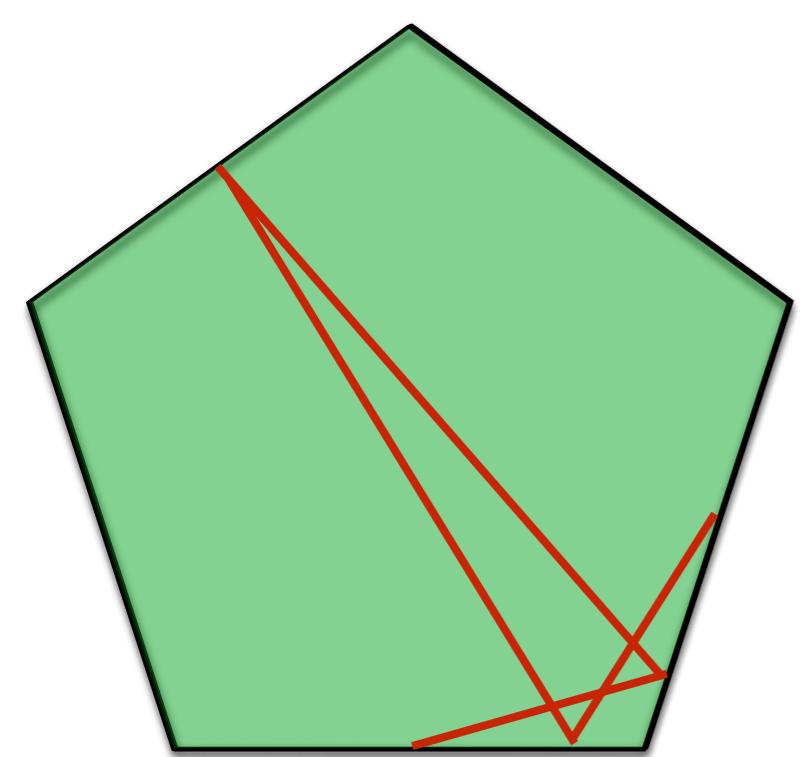
Start with a different trajectory

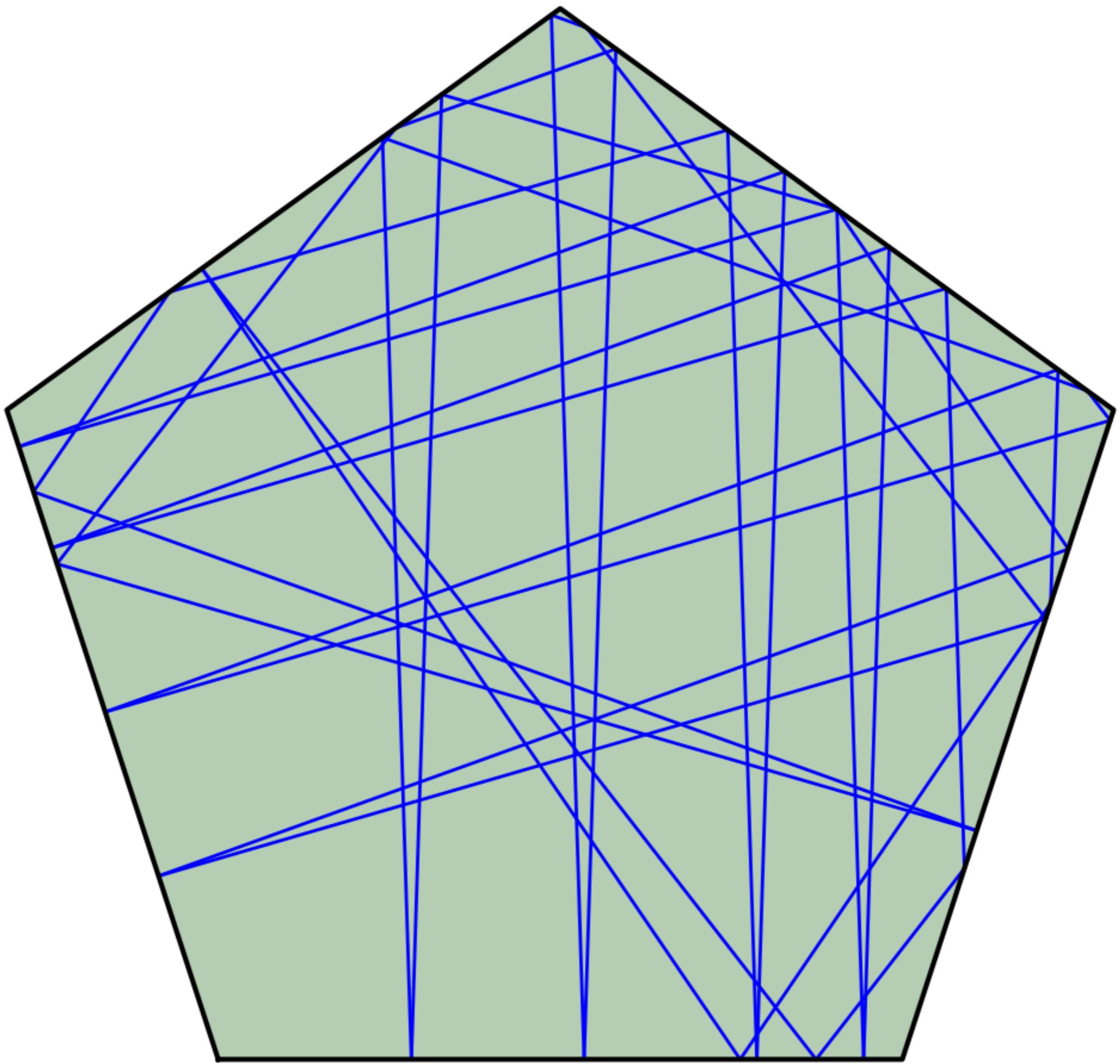


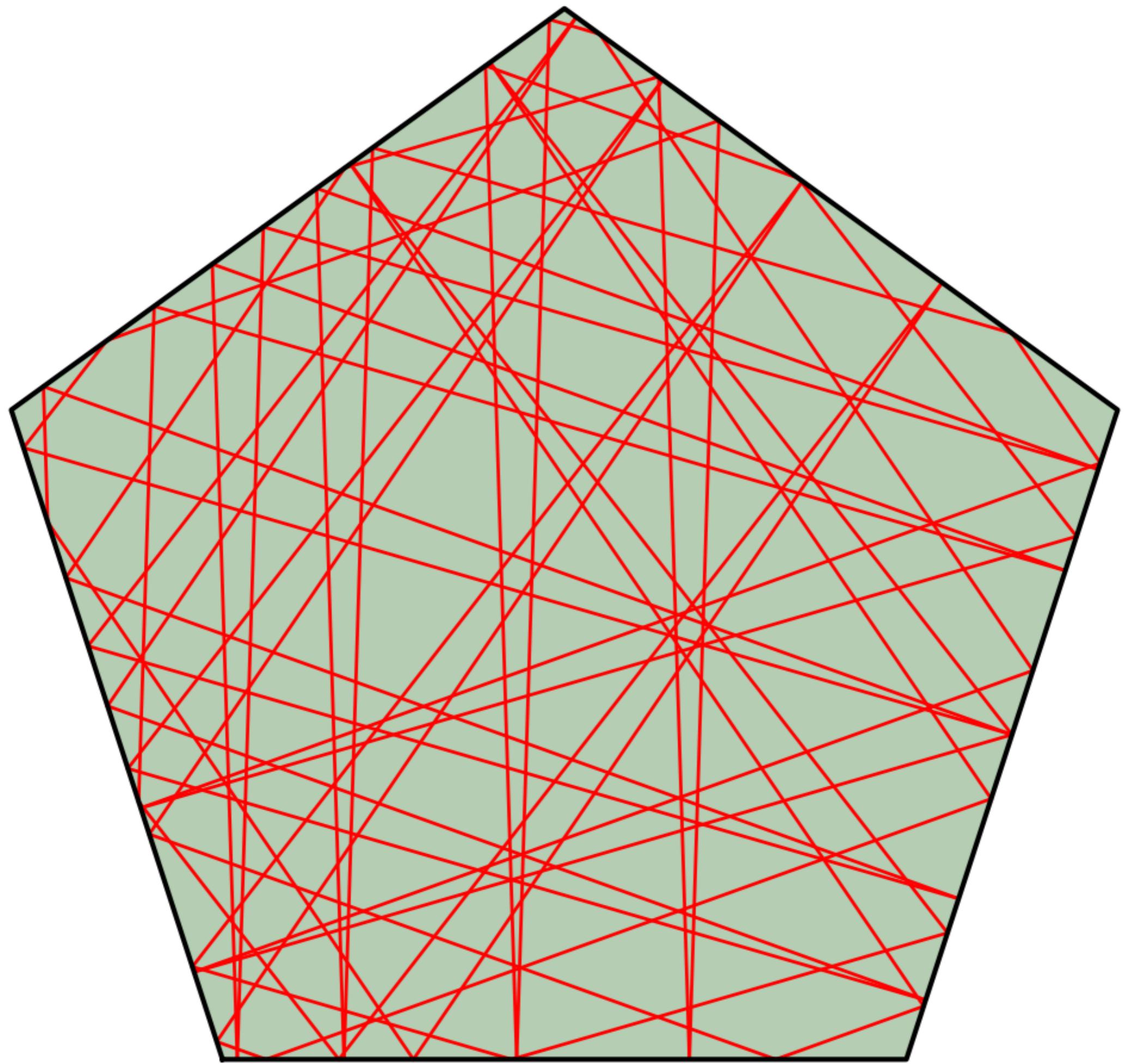
Get a COMPLETE billiard trajectory!

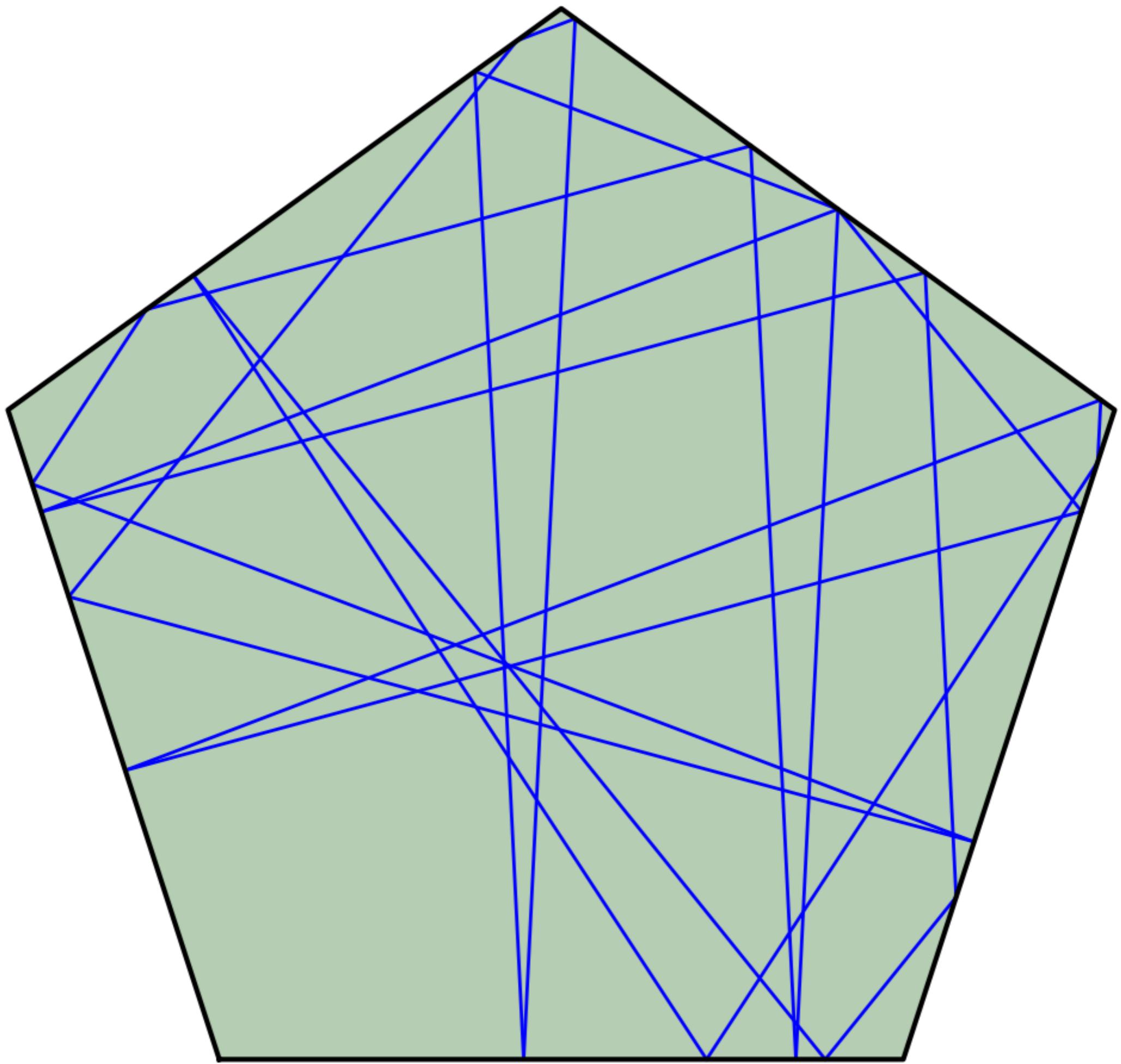
Paths needing 5  
**copies** to complete  
**have** symmetry.

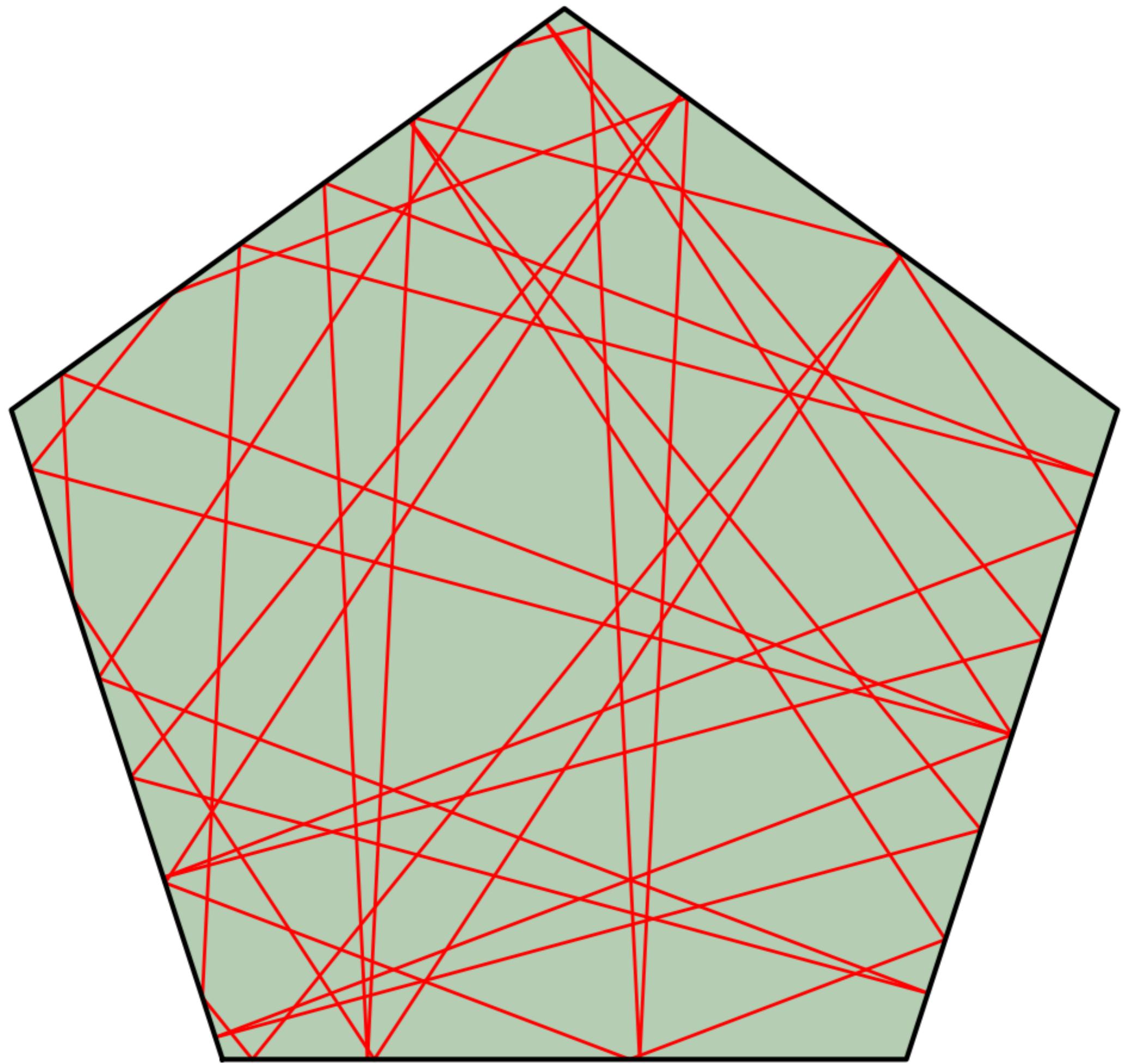
Paths that are  
**complete** when folded  
**don't** have symmetry.

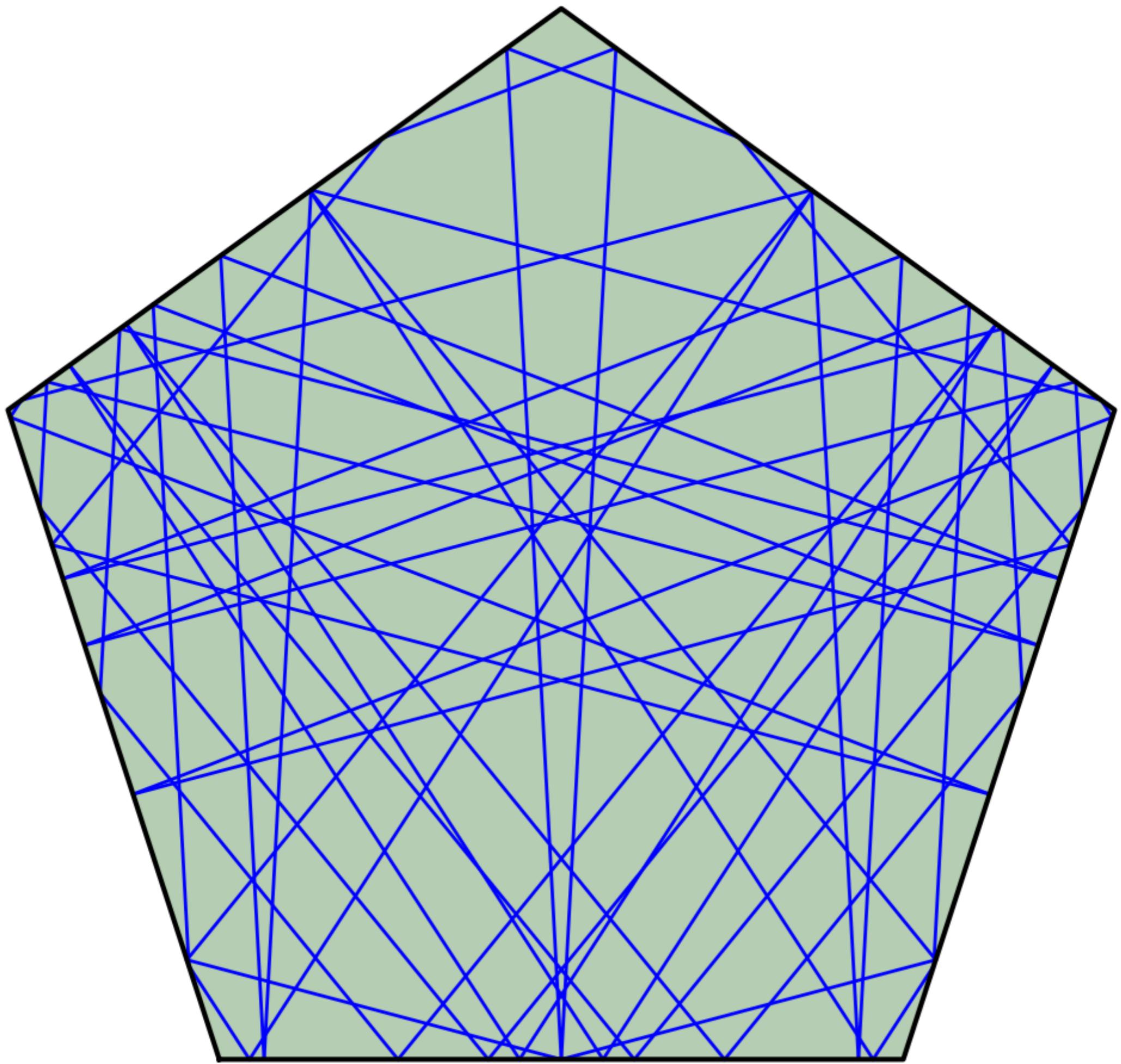


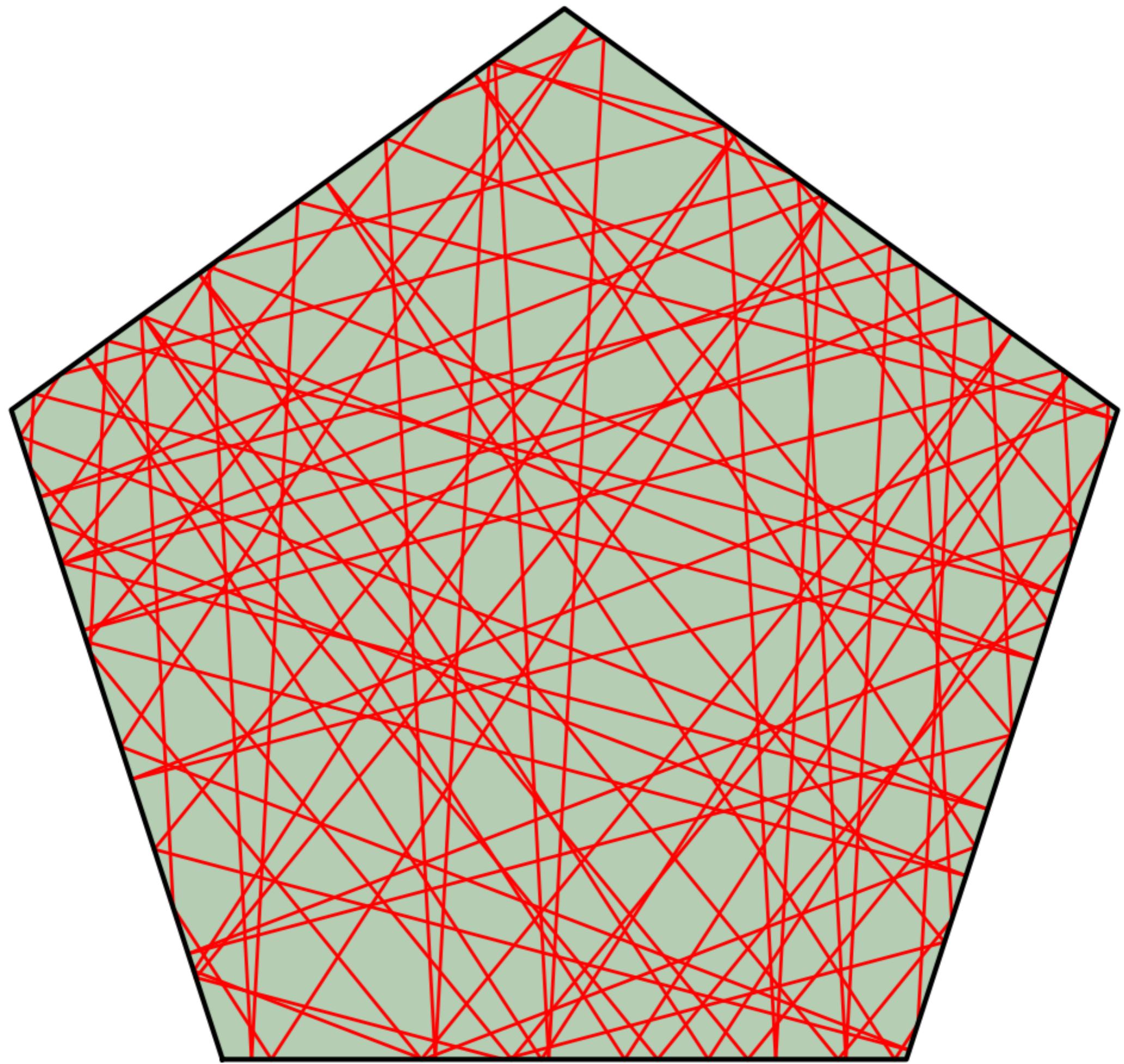


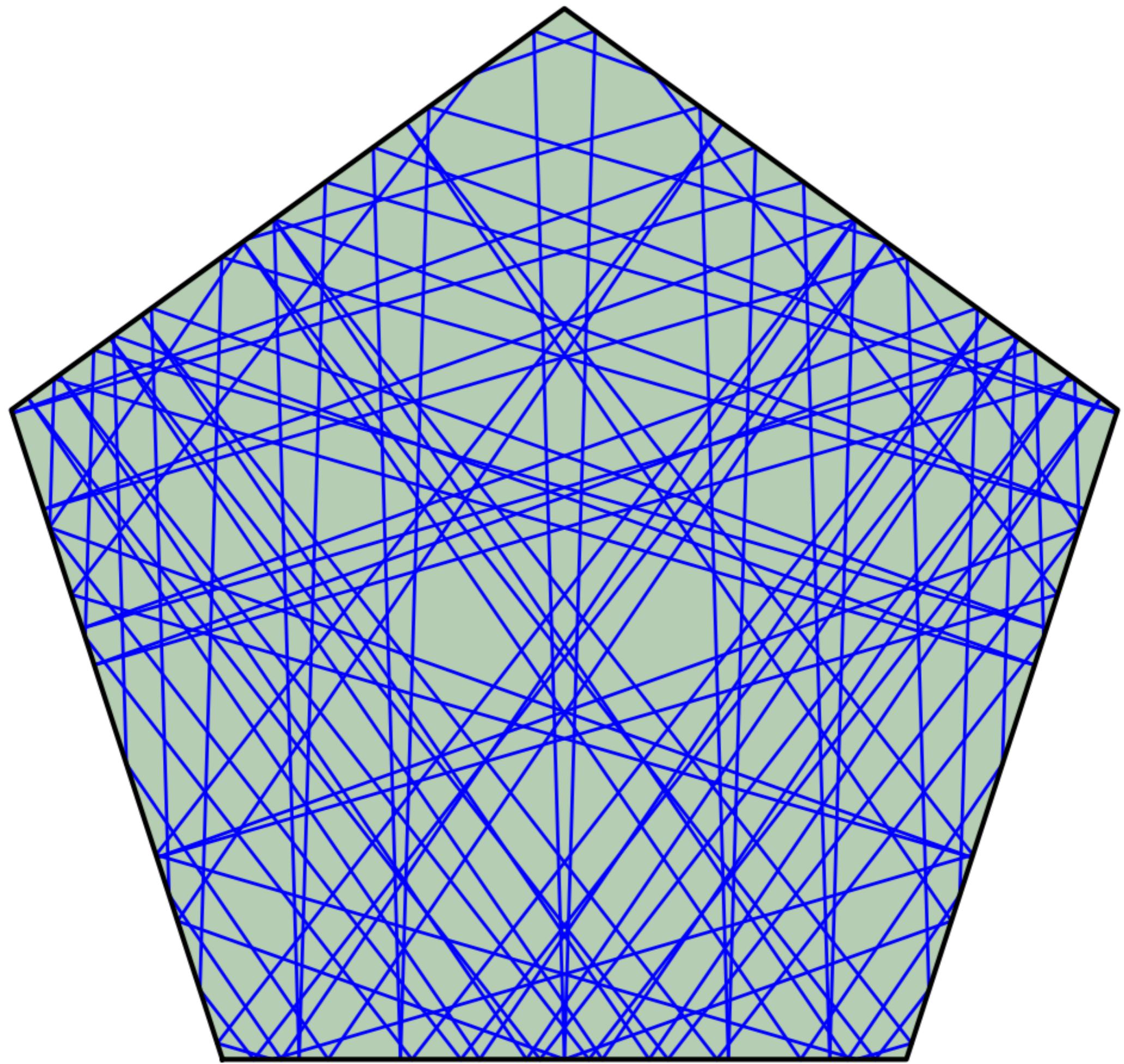


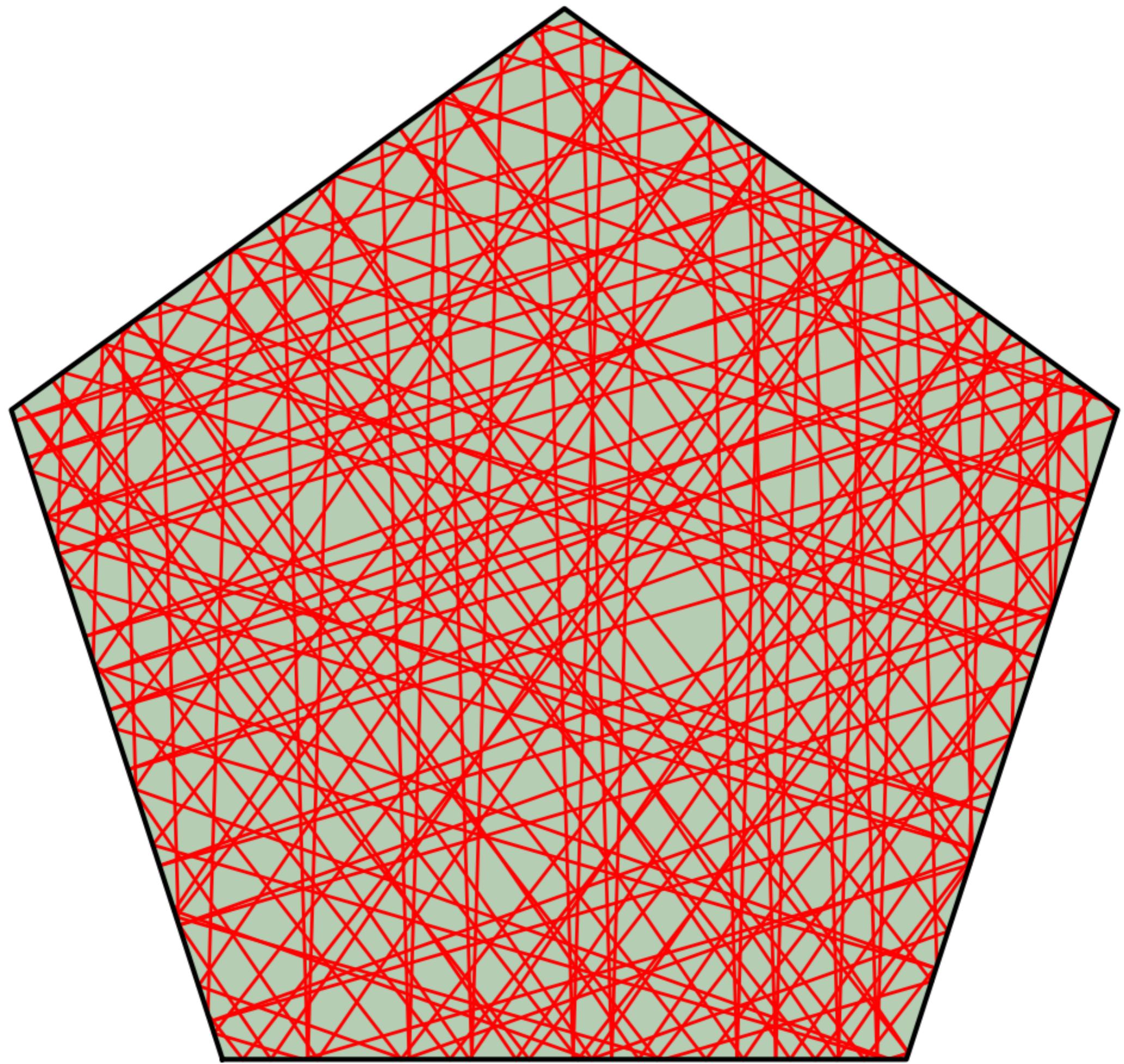


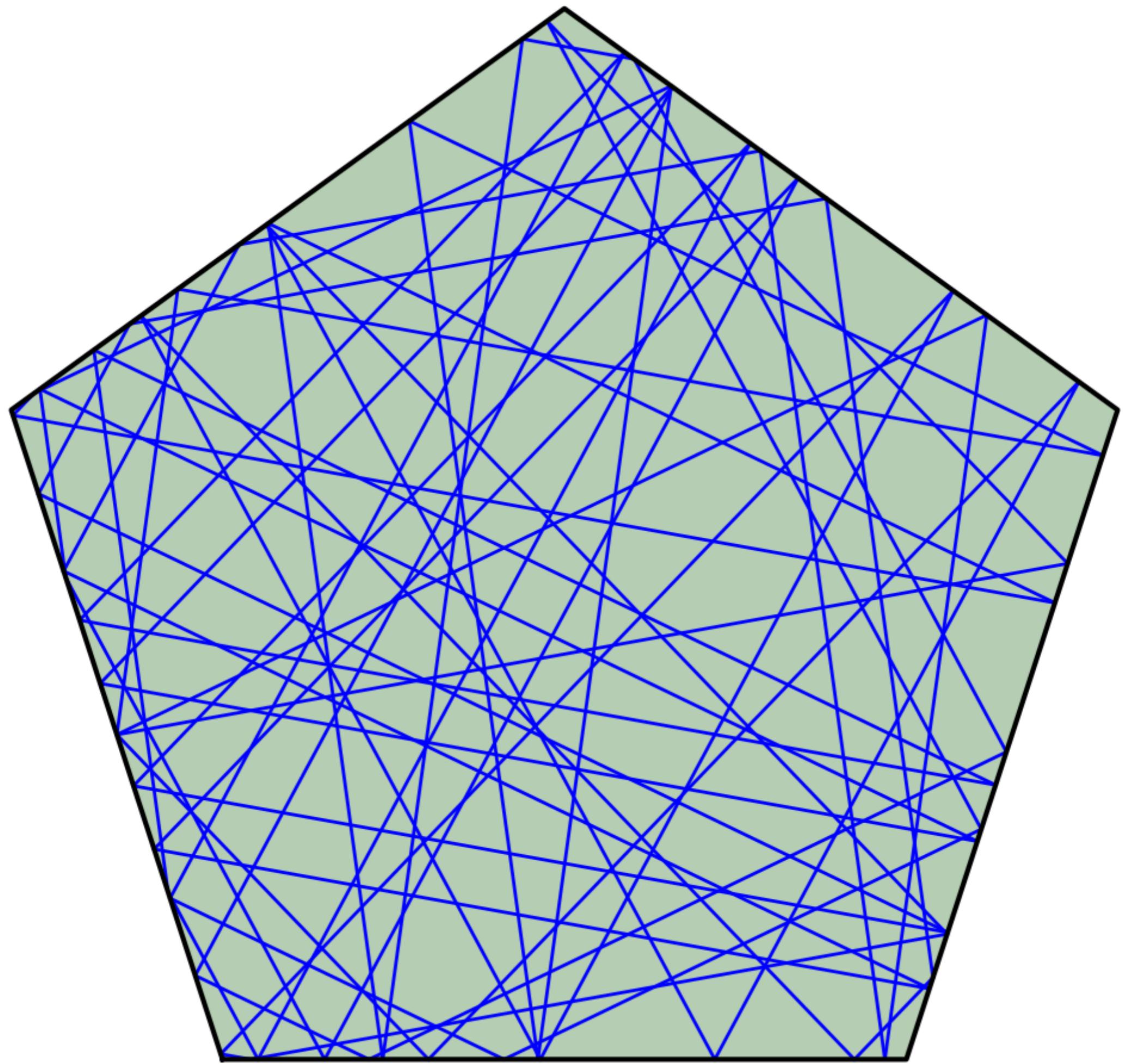


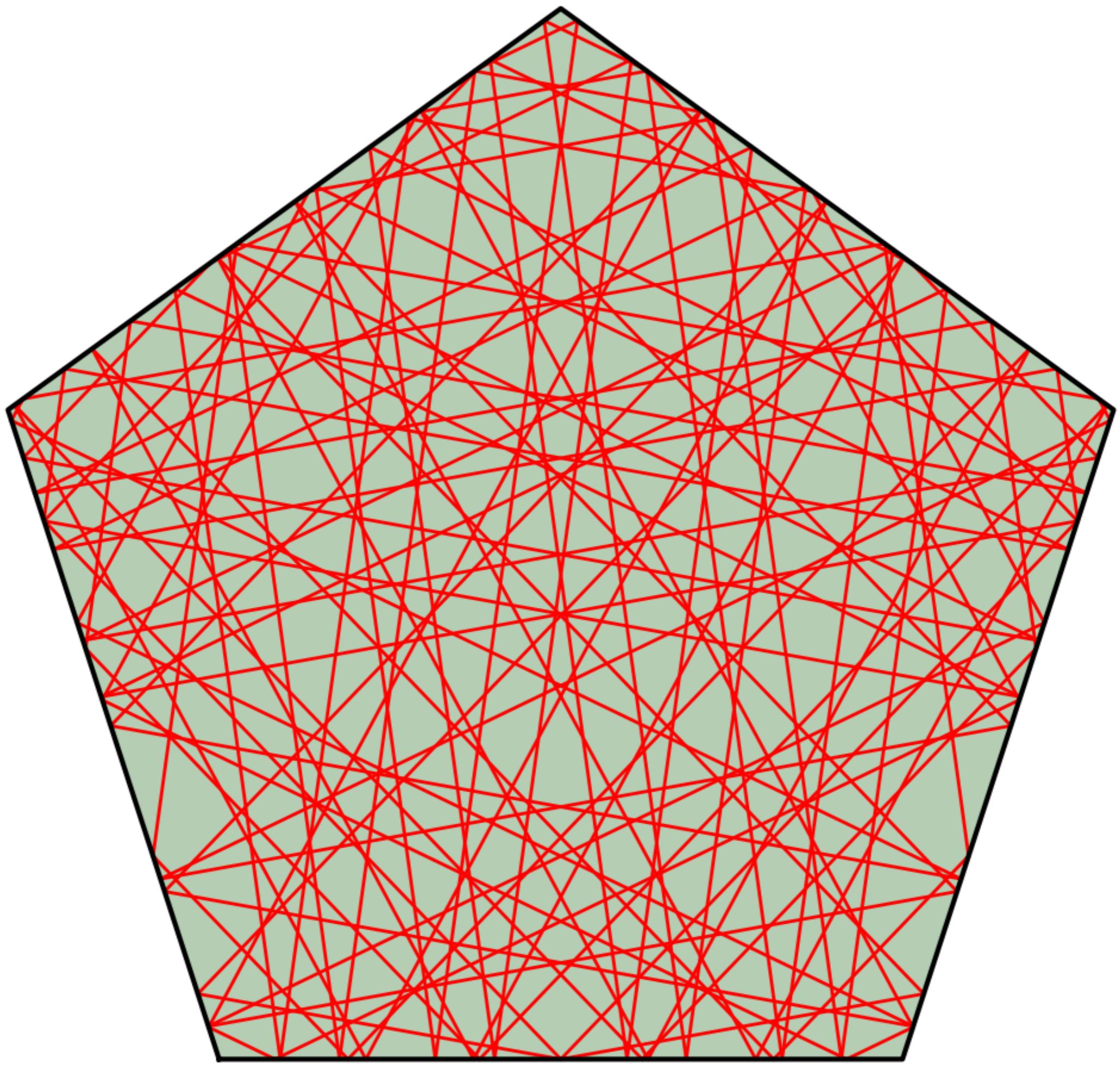


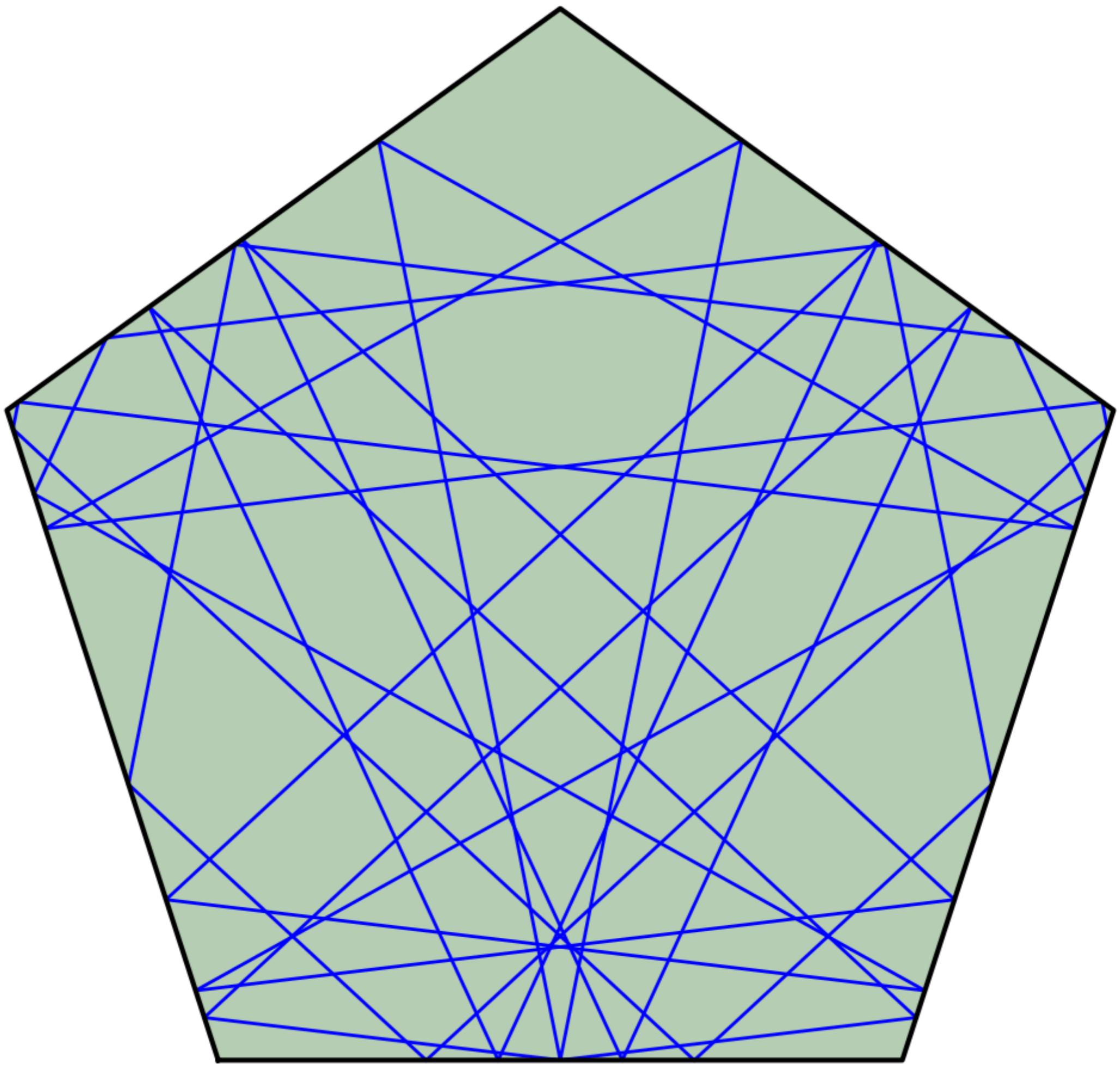


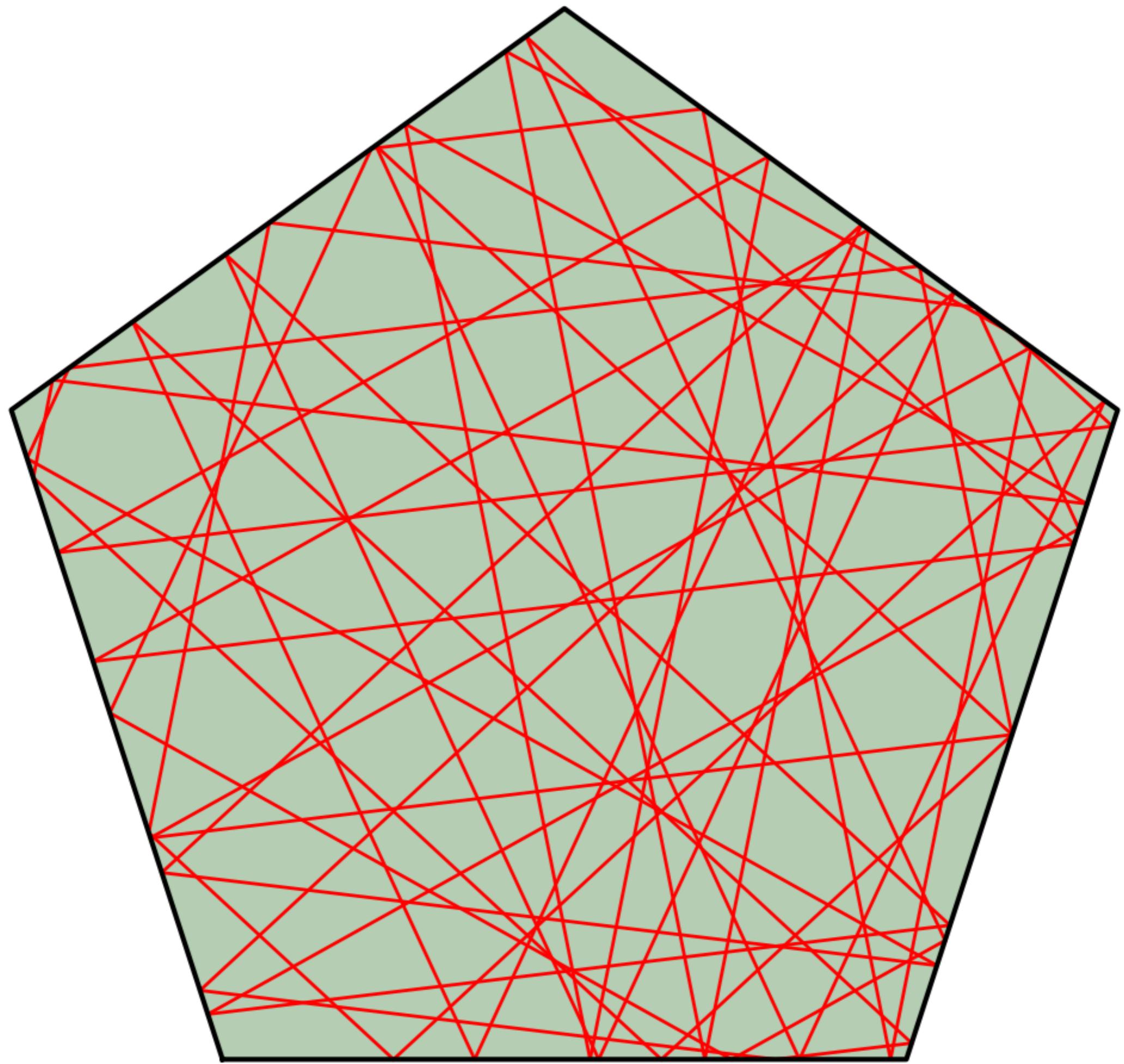


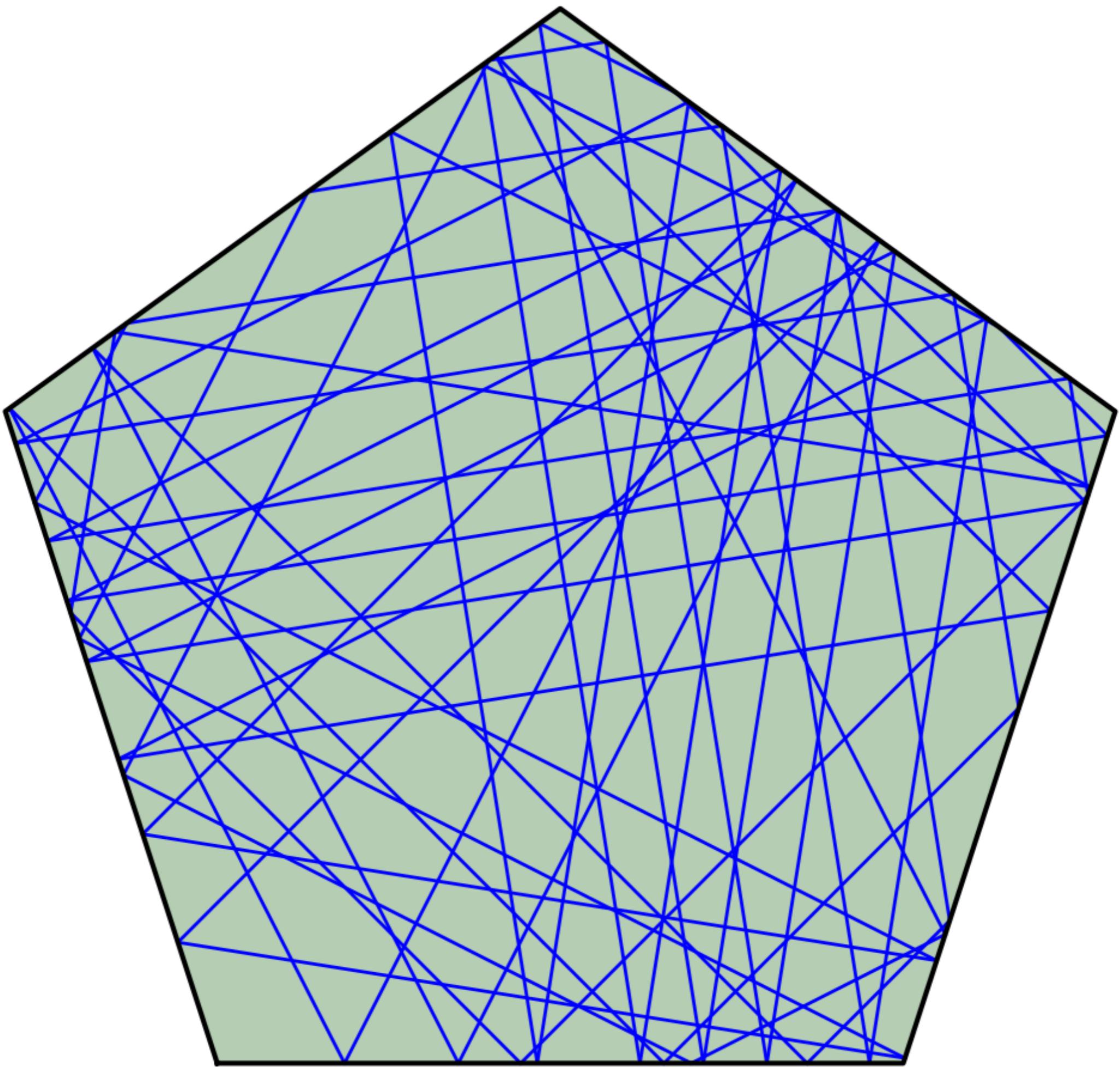


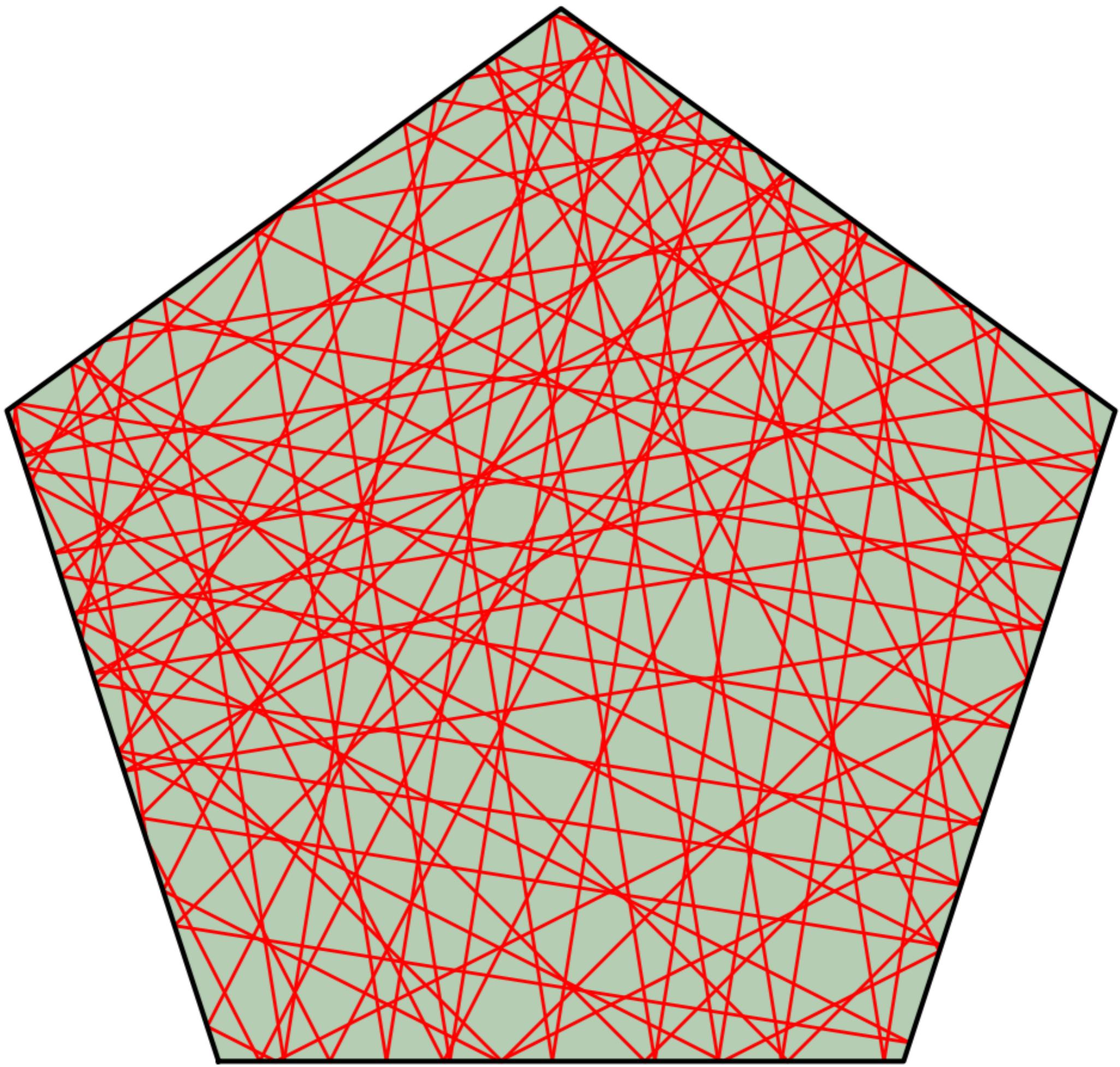


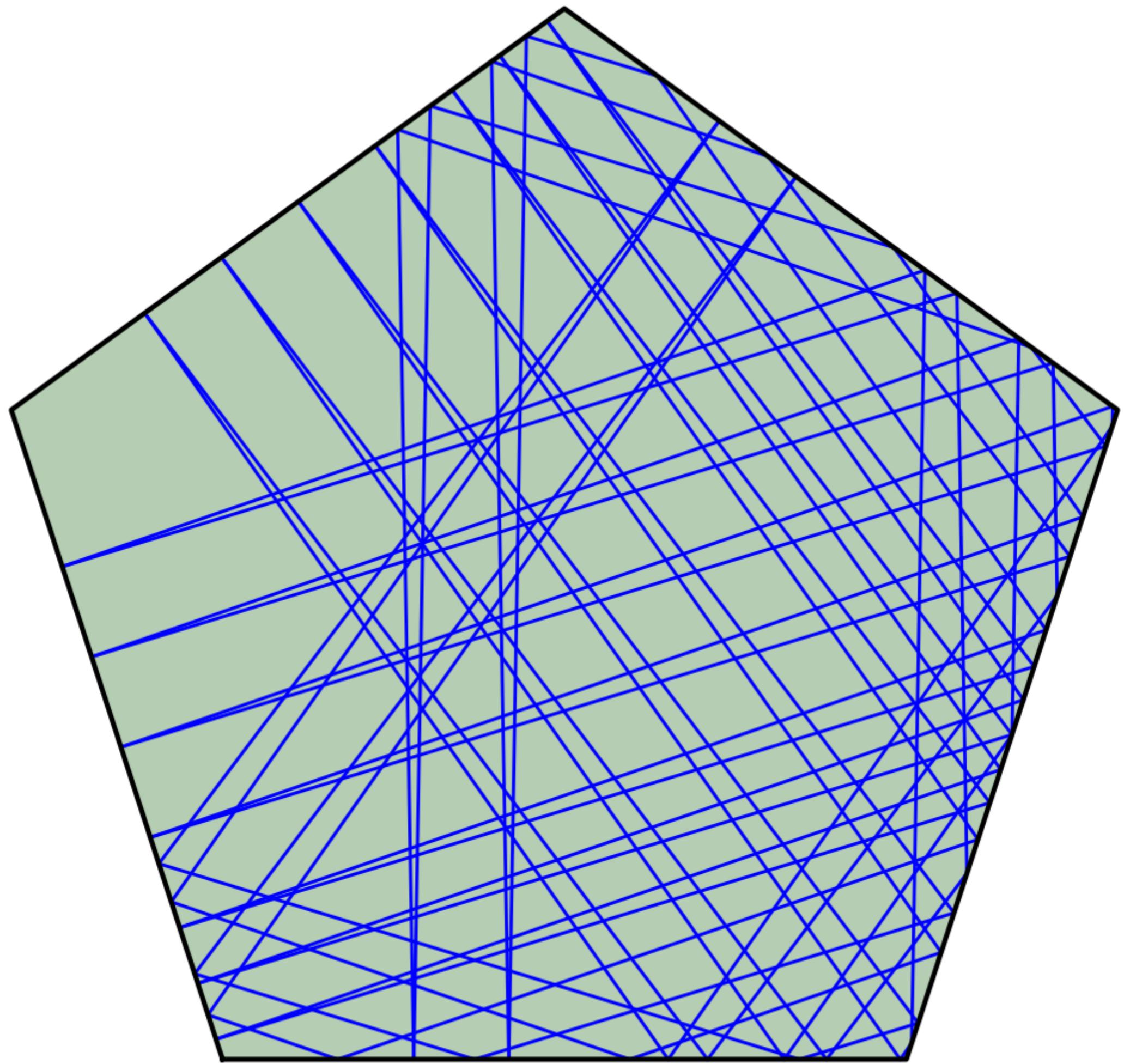


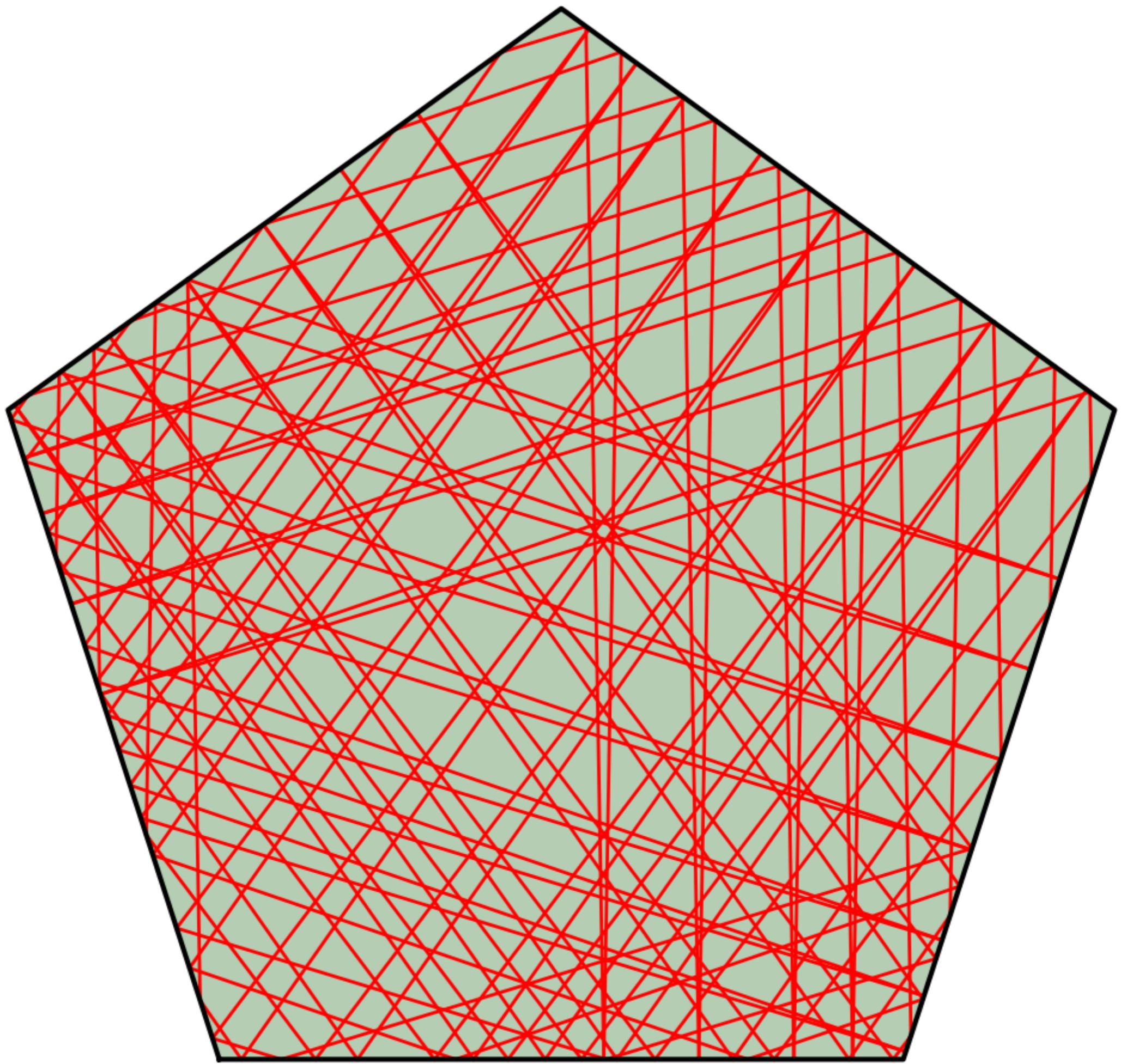


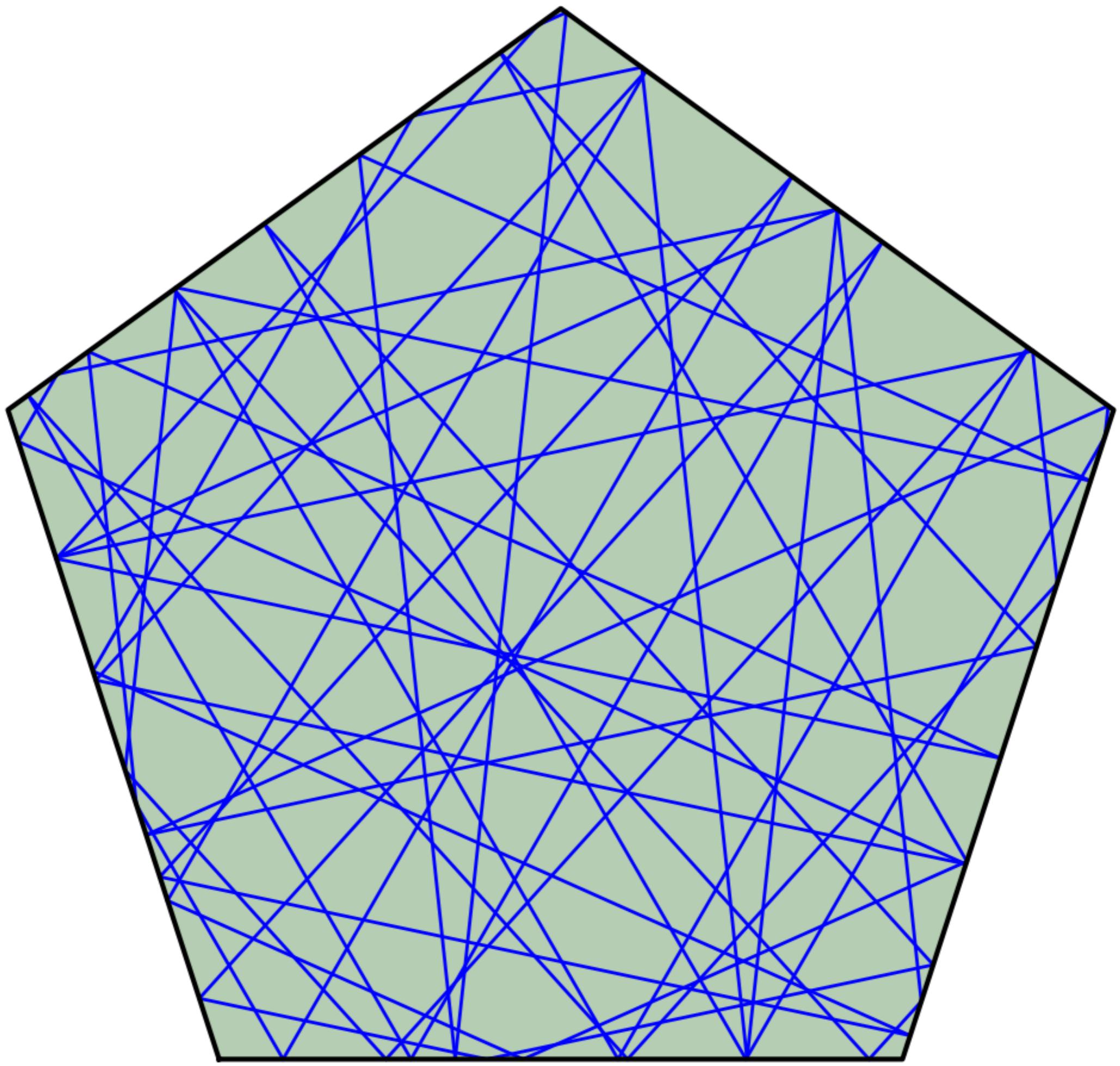


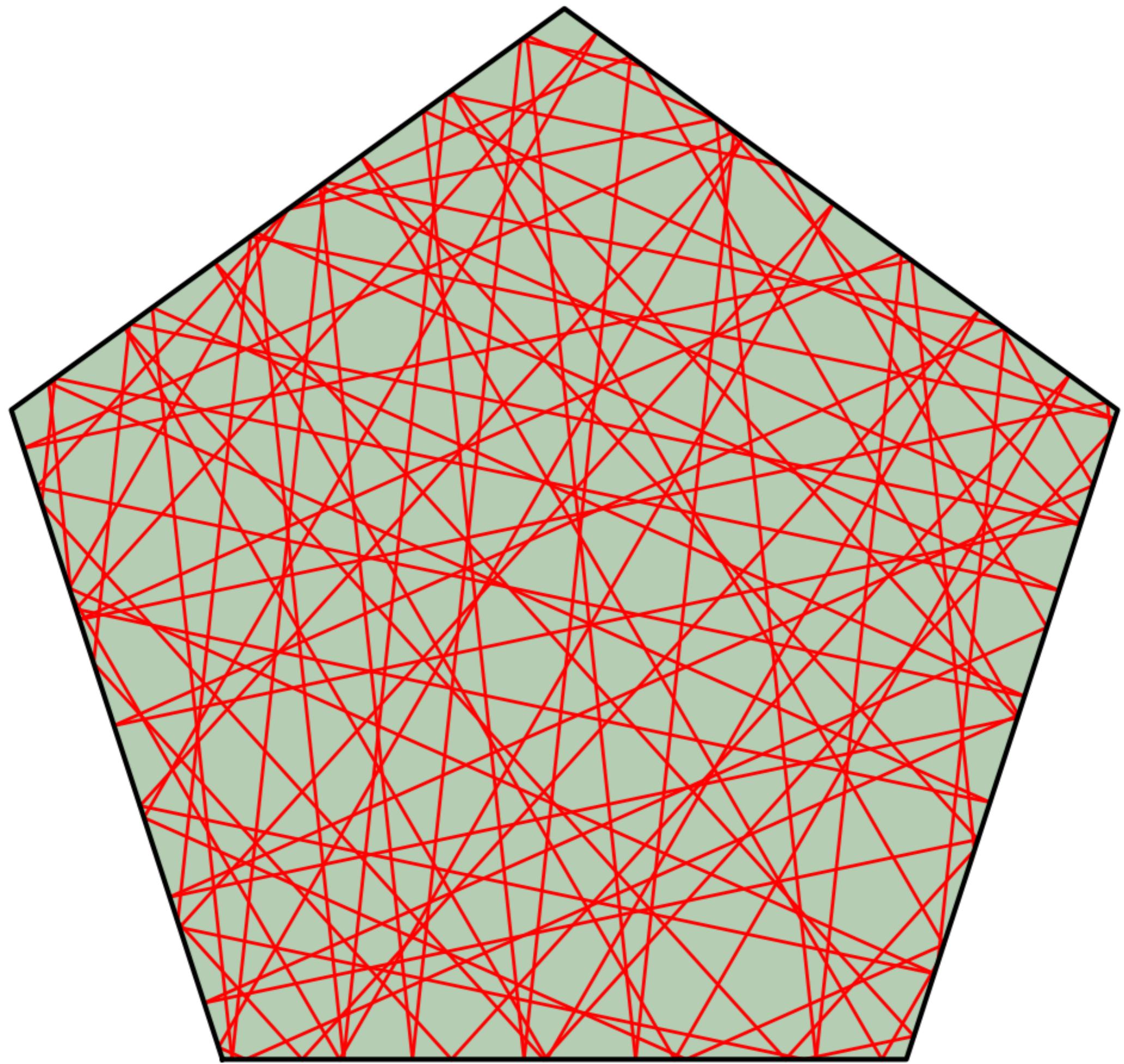


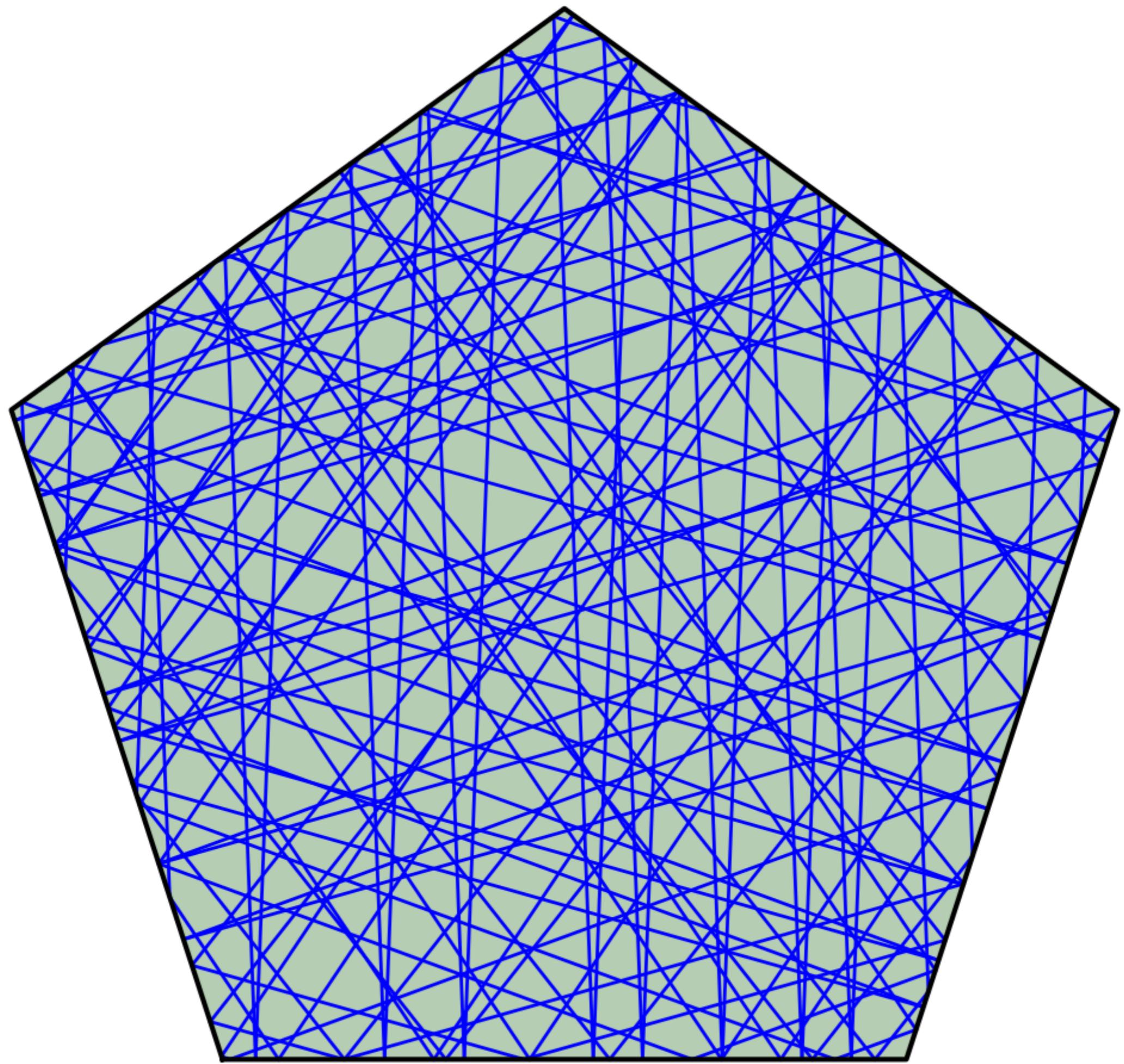


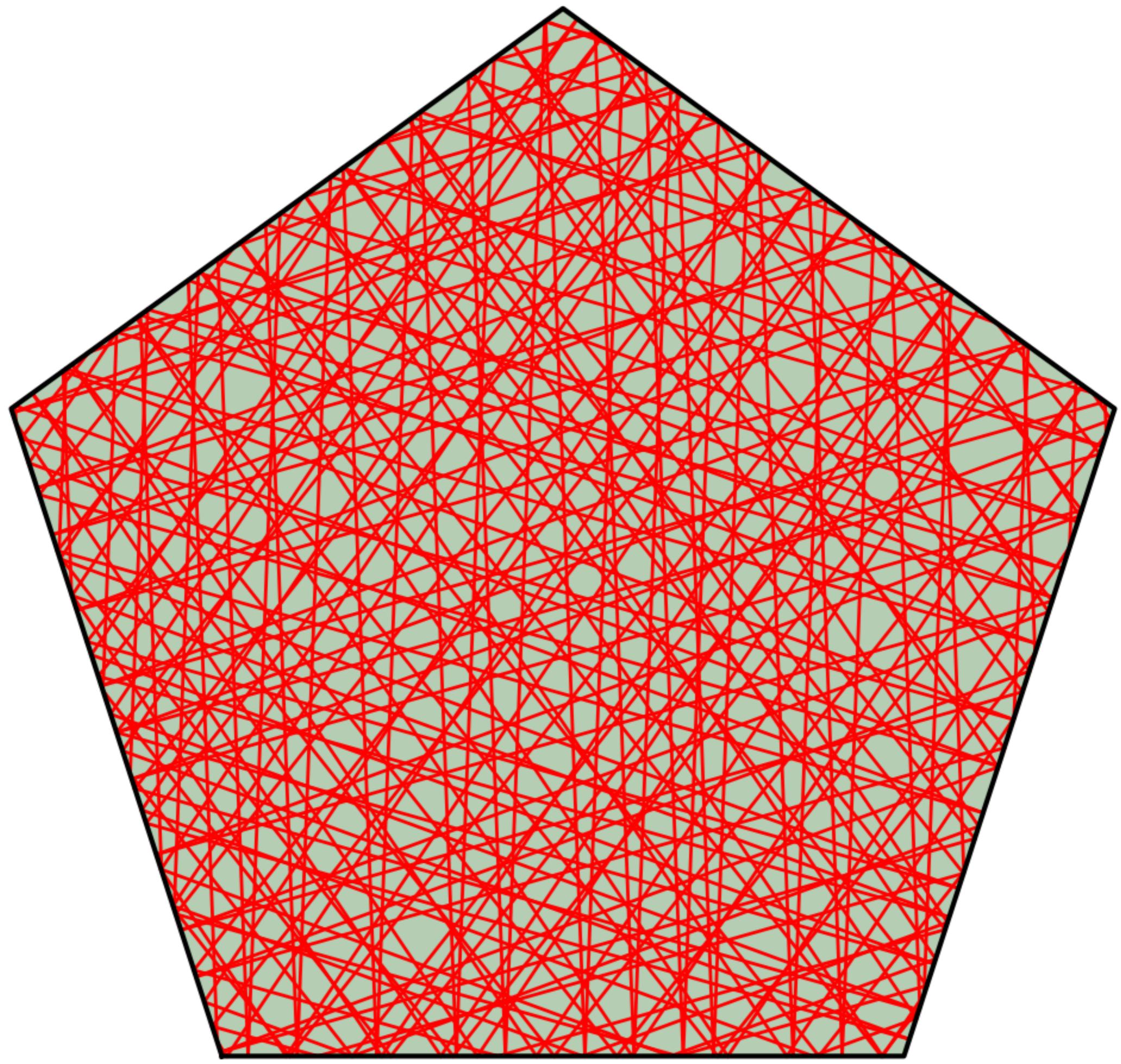












# FAB LAB

# Swarthmore's FabLab

Laser cutter

Two 3D printers



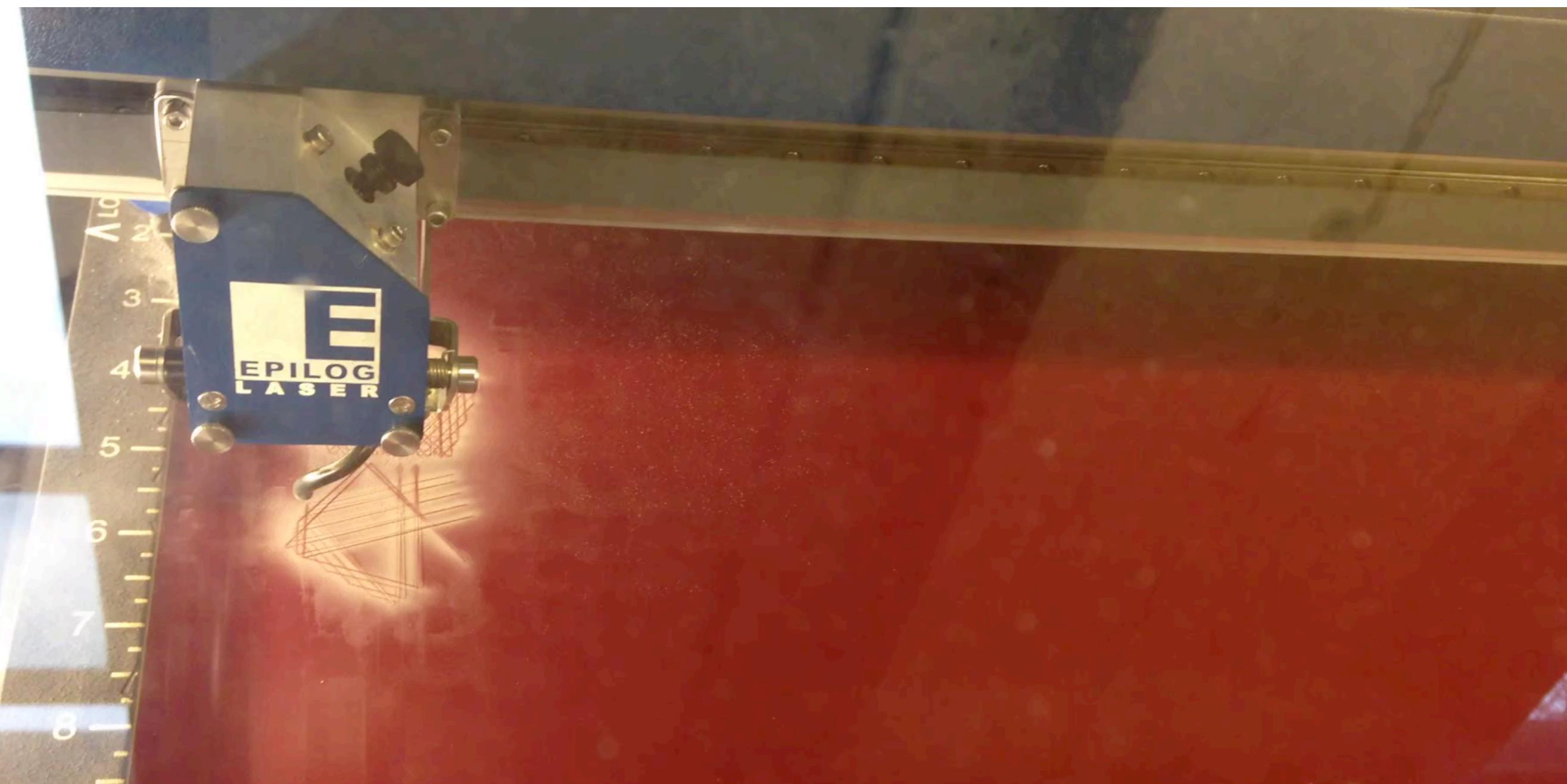
# Three steps to making pentagons:

1. Engrave (vector) the trajectory
2. Flip them over by hand
3. Engrave (raster) the back



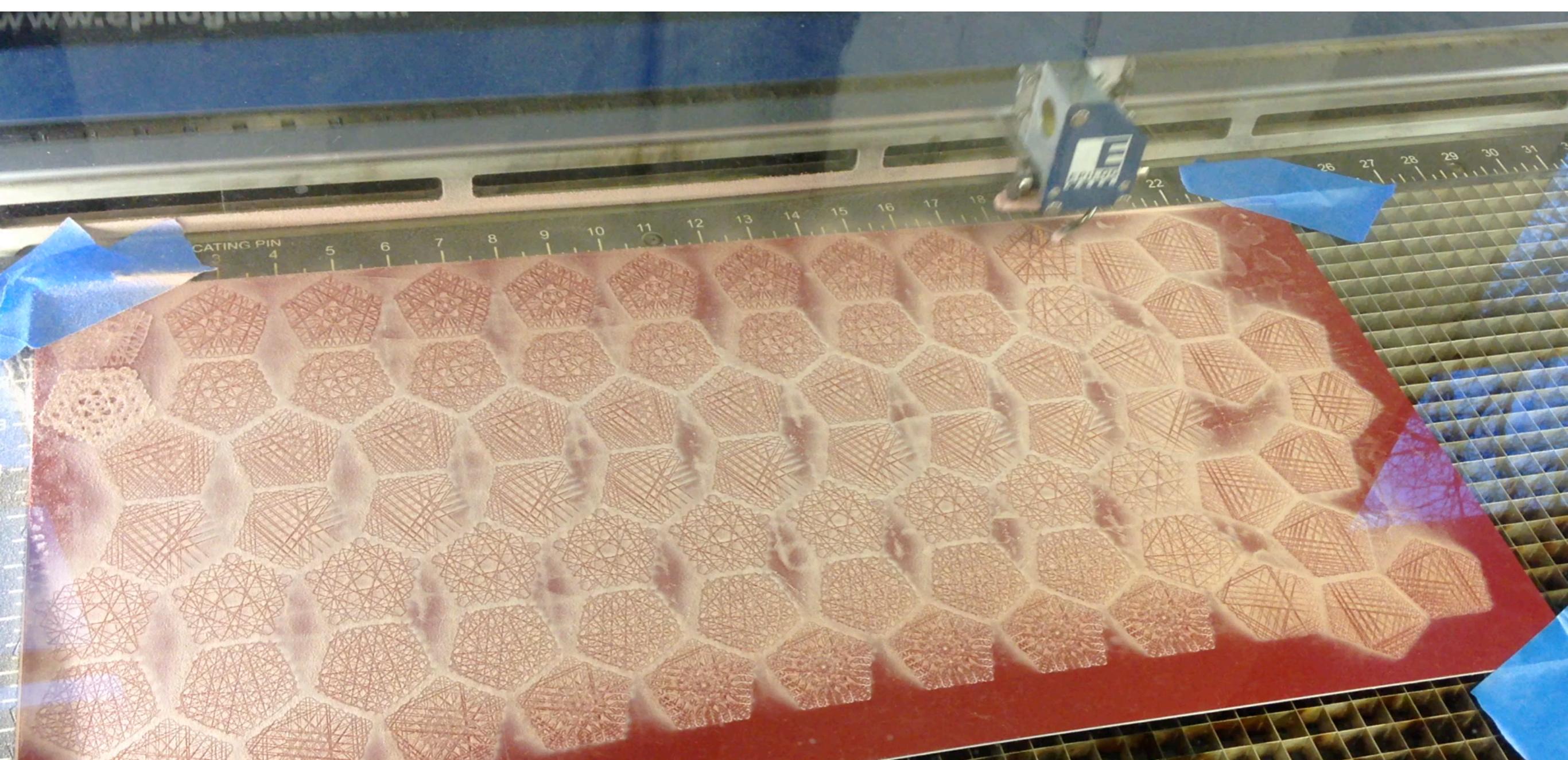
Three steps to making pentagons:

- 1. Engrave (vector) the trajectory**
2. Flip them over by hand
3. Engrave (raster) the back



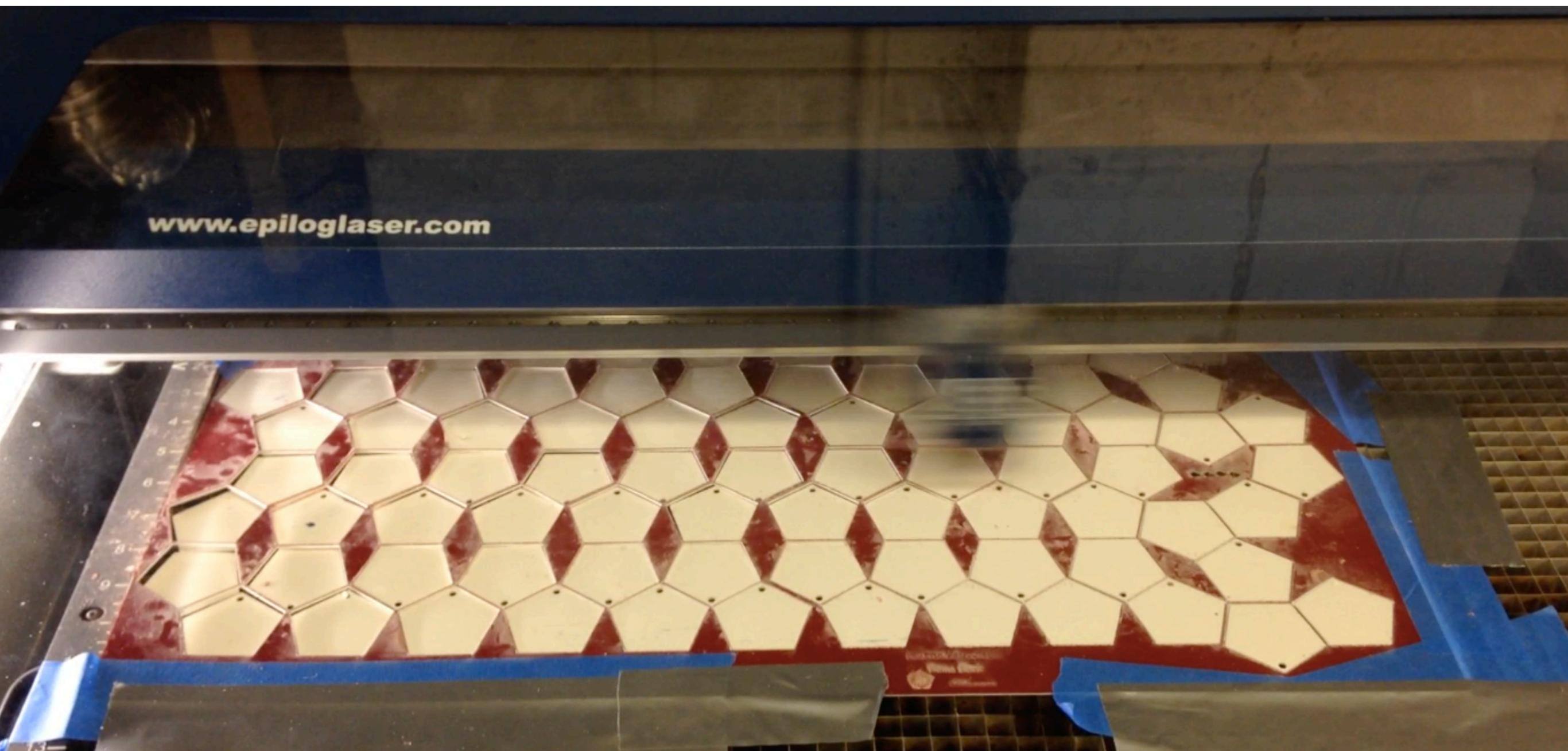
Three steps to making pentagons:

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Three steps to making pentagons:

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