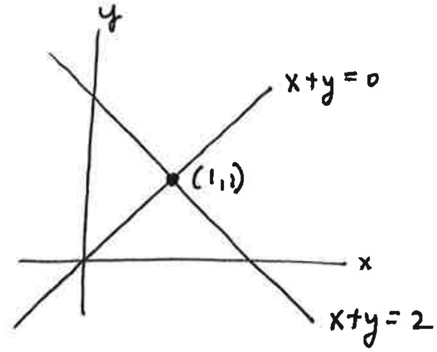
Diana Davis | Math 34 | January 22, 2018 | Class #1 ① §1.1–1.5

Mathematician spotlight: Diana Davis (my introduction).

Overview of course and syllabus:
- Come to class! 11:30 class will be videotaped.
- Homework: WebWork (due Mon) & written homework (due Fri).
- Exams: Friday 23 Feb, Friday 6 April, Exam week.
- Materials: 3-ring binder (for these notes), colored pencils/pens.
- Textbook: Susan Colley, Vector Calculus, 4^{th} edition.

Today: lines, planes and the cross product.



In linear algebra, you solved systems of linear equations.

Example. Find the point where two lines intersect:

$$\begin{cases} x+y=2 \\ x-y=0 \end{cases} \xrightarrow[\text{matrix}]{\text{in}} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{l} x=1 \\ y=1 \end{array}$$

$$\left[\text{Two variables}\right] - \left[\text{Two equations}\right] = \left[\begin{array}{c}\text{zero degrees} \\ \text{of freedom}\end{array}\right] \Rightarrow$$ solution set is empty or is isolated point(s).

In this case, solution is one point.

Question. What kind of object does $x-2y+3z=6$ describe?
- It's a linear equation (no x^2, no $\sin(y)$, etc.) so it's a line, plane, etc. (not curved)
- [THREE variables] − [ONE equation] = [Two deg. of freedom] ⇒ it's a plane!

- Give some examples of points on this plane: ______________________

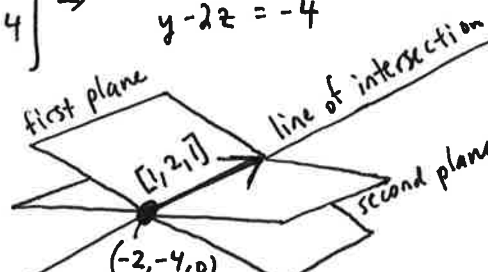
Question. How would we write the equation of a line in $\mathbb{R}^3$?

Multi-tasking. Let's find the line that is the intersection of two $\overbrace{\text{non-parallel}}^{\text{we'll explore this part soon.}}$ planes.

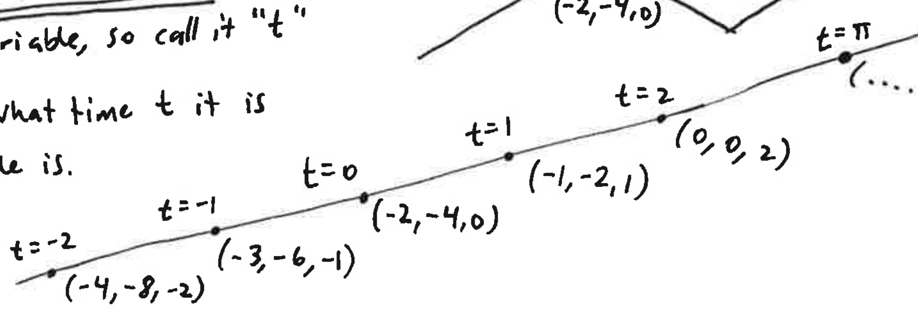
$$\begin{cases} x-2y+3z=6 & \leftarrow \text{first plane} \\ 3x-2y+z=2 & \leftarrow \text{second plane} \end{cases} \xrightarrow[\text{matrix}]{\text{in}} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 3 & -2 & 1 & 2 \end{array}\right] \xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -4 \end{array}\right] \Rightarrow \begin{array}{l} x \quad -z=-2 \\ y-2z=-4 \end{array}$$

$$\Rightarrow \begin{cases} x=-2+z \\ y=-4+2z \\ z=0+z \end{cases} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{\underline{\begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} t}}.$$

↖ z is our free variable, so call it "t"

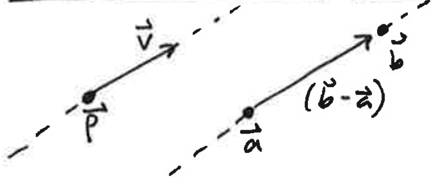


This is a parametric equation: what time t it is tells you where $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ your particle is.



<u>To define a line, you need:</u> • a point $\vec{p}$ and a direction $\vec{v}$: $\vec{r}(t) = \vec{p} + t\vec{v}$

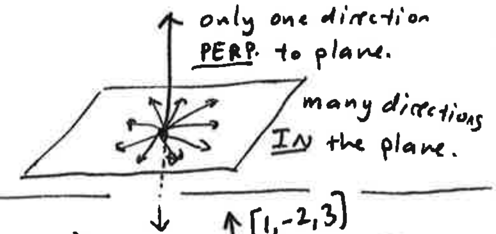
• two points $\vec{a}$ and $\vec{b}$: $\vec{r}(t) = \vec{a} + (\vec{b}-\vec{a})t$



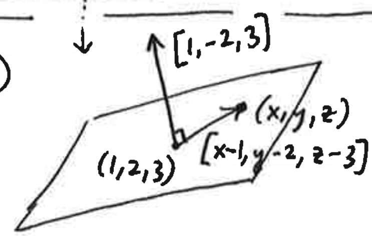
<u>To define a plane, you need:</u> • three points

• a point and a PERPENDICULAR! direction

⋮ more options



<u>Goal:</u> Write an equation for the plane containing the point $(1,2,3)$ that has $[1,-2,3]$ as its <u>normal vector</u>.



<u>Idea:</u> Notice that for any point (x,y,z) on the plane,

$$[1,-2,3] \bullet [x-1, y-2, z-3] = 0 \Rightarrow 1(x-1) - 2(y-2) + 3(z-3) = 0$$

$$\Rightarrow x - 1 - 2y + 4 + 3z - 9 = 0 \Rightarrow \underline{x - 2y + 3z = 6.}$$

In general: $[a,b,c] \bullet [x-x_0, y-y_0, z-z_0] = 0$

$$\Rightarrow \underbrace{ax + by + cz}_{\text{normal vector gives you coefficients}} = d = \underbrace{ax_0 + by_0 + cz_0.}$$

← to get constant, plug in a point.

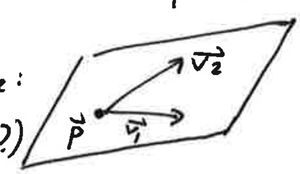
What if we had used $(0,0,2)$ as our point instead? $1(x-0) - 2(y-0) + 3(z-2) = \underline{x - 2y + 3z = 6.}$ as before.

Okay, but what if I'm not given the normal vector (perpendicular direction) to the plane?

What if I have: • three points in the plane

• a point on the plane and two directions in the plane:

(these amount to the same information, do you see why?)



<u>We need:</u> A method to take two (non-parallel) vectors in $\mathbb{R}^3$, and find a new vector that is perpendicular to the first two.

$$[x_1, y_1, z_1],\ [x_2, y_2, z_2] \downarrow [a,b,c]$$

Solve this: $\begin{cases} [a,b,c] \bullet [x_1, y_1, z_1] = 0 \\ [a,b,c] \bullet [x_2, y_2, z_2] = 0 \end{cases} \Rightarrow$ 3 variables a,b,c; 2 equations $\Rightarrow$ 1 degree of freedom, <u>length</u> of vector.

Skip to answer: $\underset{\text{vector perp. to both}}{[a,b,c]} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \vec{i}(y_1 z_2 - z_1 y_2) - \vec{j}(x_1 z_2 - z_1 x_2) + \vec{k}(x_1 y_2 - y_1 x_2)$

<u>Cross product!</u> determinant → $= [y_1 z_2 - z_1 x_2,\ x_1 z_2 - z_1 x_2,\ x_1 y_2 - y_1 x_2].$

<u>Example.</u> Given 3 points:

$(0,0,2)$, $(0,-3,0)$, $(6,0,0)$ → $[0,3,2]$, $[6,3,0]$

$$[0,3,2] \times [6,3,0] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 2 \\ 6 & 3 & 0 \end{vmatrix}$$

$= [-6, 12, -18]$ or reduce to $[1,-2,3]$. ☺

<u>Length</u> of cross product vector:

Area of parallelogram spanned by the two input vectors.

<u>Direction:</u> follows right hand rule.

fingers toward $\vec{v_1}$, curl toward $\vec{v_2}$ $\Rightarrow$ thumb points in direction $\vec{v_1} \times \vec{v_2}$.

Diana Davis   Math 34   January 24, 2018   Class # 2 ①

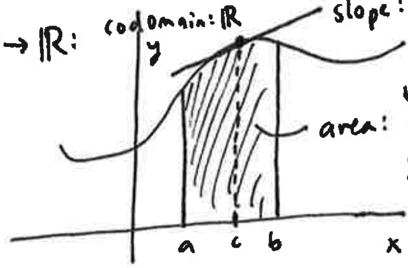
§ 2.1

Mathematician spotlight: John Urschel – PhD student at MIT, Penn St undergrad.

- spectral graph theory, numerical linear algebra, machine learning
- Baltimore Ravens, guard, 2014–2017.

You know: single-variable calculus.

$f: \mathbb{R} \to \mathbb{R}$:

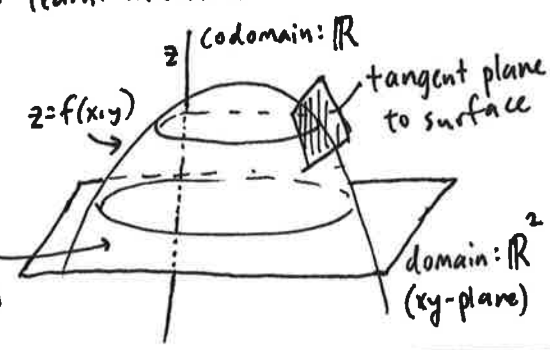


You want to learn: multivariable calculus.

$f: \mathbb{R}^m \to \mathbb{R}^n$

for example,

$f: \mathbb{R}^2 \to \mathbb{R}$:



We'll study functions of several variables. Here is some vocabulary to talk about them:

domain: all possible inputs

codomain: the set where outputs live

range: outputs that are actually achieved

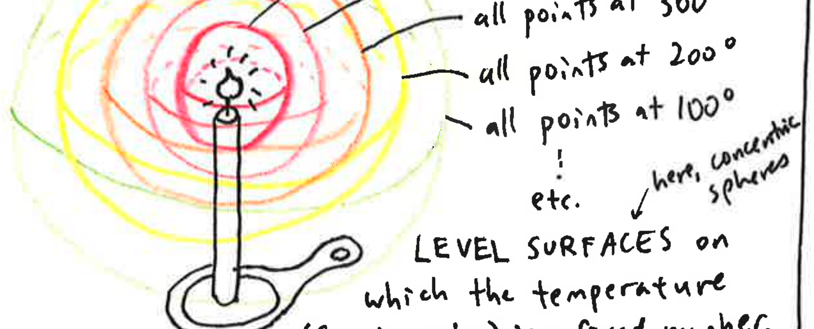
A function is onto if its range is the entire codomain.

A function is one-to-one if each element of the range comes from exactly one element in the domain, i.e. $f(x)=f(y) \Rightarrow x=y$.

function	domain	codomain	range	onto?	one-to-one?
$f(x)=x^2$	$\mathbb{R}$	$\mathbb{R}$	nonnegative real #s.	no, misses negatives	no, $f(-1)=f(1)$.
$f(x,y)=x+y+z$					
f(point in room) = temperature					
f(point on field) = wind direction					

To visualize multivariable functions, use "level curves" and "level surfaces."

Example. Let $f(x,y,z) =$ temperature at point (x,y,z) and suppose you have a candle in the cold air.



LEVEL SURFACES on which the temperature (function value) is a fixed number.

Example. Let $f(x,y,z) = x+y+z$.

What does this function look like?

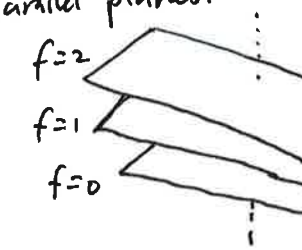
Let's try some fixed function values.

$f=0 \Rightarrow x+y+z=0$, plane through $(0,0,0)$ w/ $\vec{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$f=1 \Rightarrow x+y+z=1$, plane through $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ w/ $\vec{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$f=2 \Rightarrow x+y+z=2$, plane through $(0,0,2)$, etc. w/ $\vec{n} =$

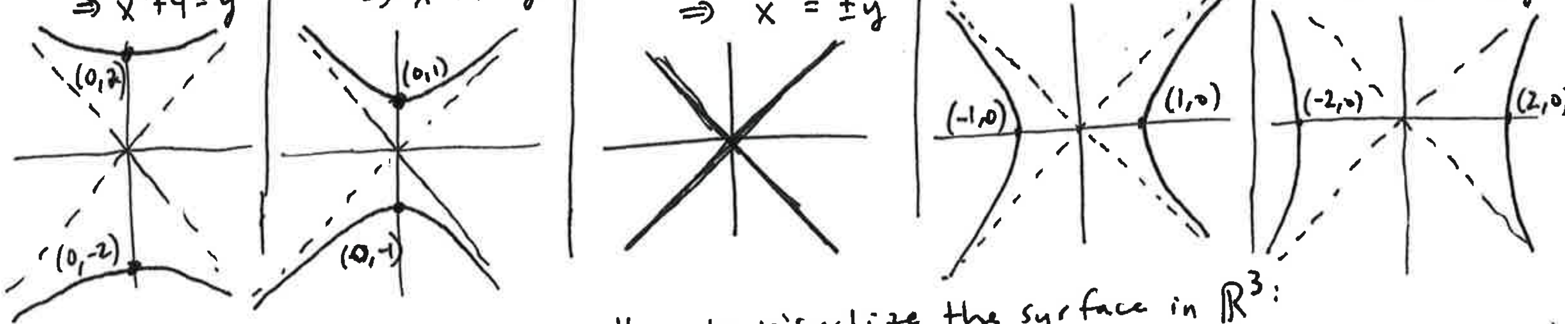
these are parallel planes!



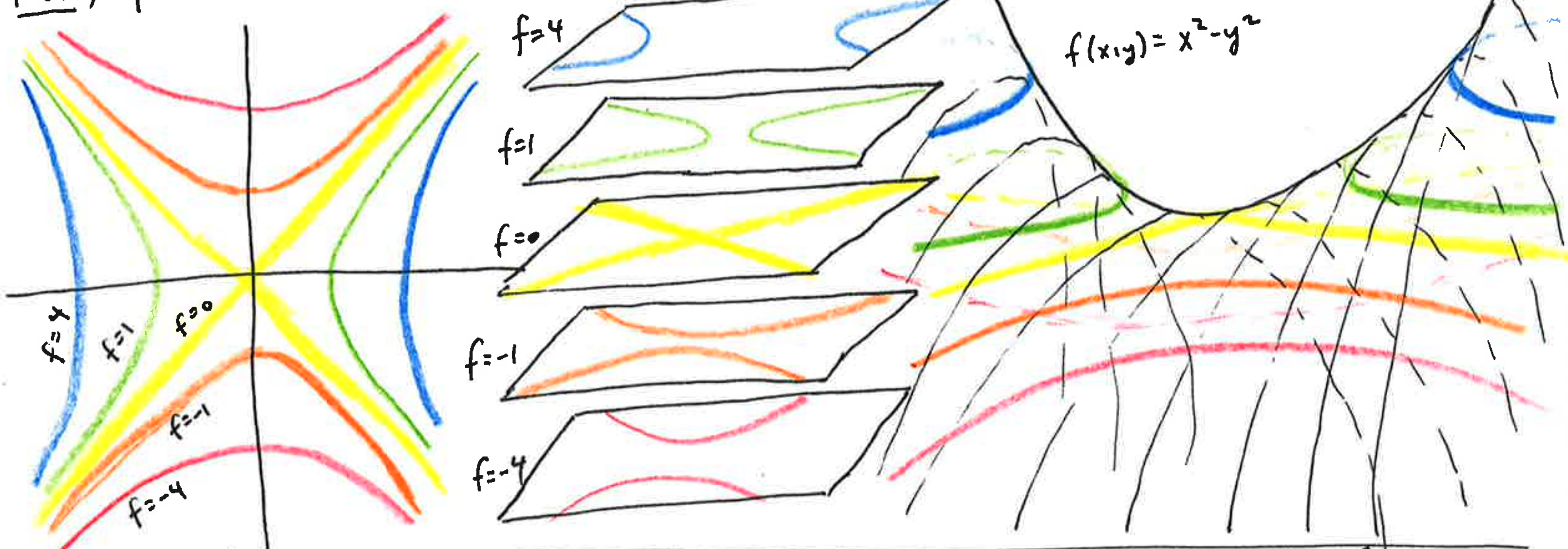
Those were level <u>surfaces</u> for functions $f:\mathbb{R}^3\to\mathbb{R}$ that we can't draw.
We can use level <u>curves</u> for functions $f:\mathbb{R}^2\to\mathbb{R}$ to understand and draw them.

<u>Example.</u> Sketch level curves at levels $-4,-1,0,1,4$ for $f(x,y)=x^2-y^2$.

$f=-4 \Rightarrow x^2-y^2=-4 \Rightarrow x^2+4=y^2$ | $f=-1 \Rightarrow x^2-y^2=-1 \Rightarrow x^2+1=y^2$ | $f=0 \Rightarrow x^2-y^2=0 \Rightarrow x^2=y^2 \Rightarrow x=\pm y$ | $f=1 \Rightarrow x^2-y^2=1 \Rightarrow x^2=1+y^2$ | $f=4 \Rightarrow x^2-y^2=4 \Rightarrow x^2=4+y^2$



<u>Now</u>, put the level curves together to visualize the surface in $\mathbb{R}^3$:

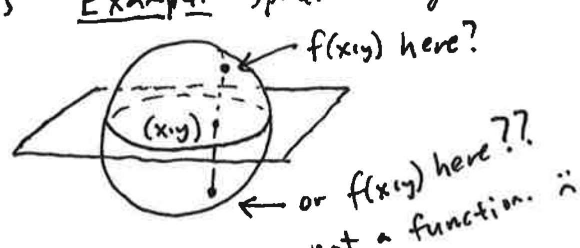


<u>Definition.</u> Let $f:\mathbb{R}^2\to\mathbb{R}$ be a $\underbrace{\text{scalar-valued}}_{}$ function of two variables. (codomain is $\mathbb{R}$; domain is $\mathbb{R}^2$)
Then the <u>level curve of f at height c</u> is the curve in $\mathbb{R}^2$ defined by $f(x,y)=c$, i.e. the set of points $\{(x,y)\in\mathbb{R}^2 : f(x,y)=c\}$.

Similarly, if $g:\mathbb{R}^3\to\mathbb{R}$ is a scalar-valued function of three variables, the <u>level surface of g at level c</u> is the surface $\{(x,y,z)\in\mathbb{R}^3 : g(x,y,z)=c\}$.

<u>Remark.</u> Some surfaces in $\mathbb{R}^3$ cannot be described as a function $f:\mathbb{R}^2\to\mathbb{R}$, i.e. $z=f(x,y)$, because they are not graphs of functions, because they fail the "vertical line test."

<u>Example.</u> Sphere $x^2+y^2+z^2=r^2$



Diana Davis Math 34 January 26, 2018 Class #3 ①
§1.7

Mathematician spotlight: Moon Duchin, associate professor at Tufts University.
- geometric group theory, geometric topology, dynamical systems.
- gerrymandering: using geometry to advance civil rights.

<u>Last time:</u> introduction to functions of two & three variables, $f(x,y)$ and $f(x,y,z)$.

<u>Today:</u> cylindrical & spherical coordinates.
- they work great for round surfaces like cylinders & spheres (!), cones,...
- you can easily describe some surfaces that are not graphs of functions, $z = f(x,y)$.

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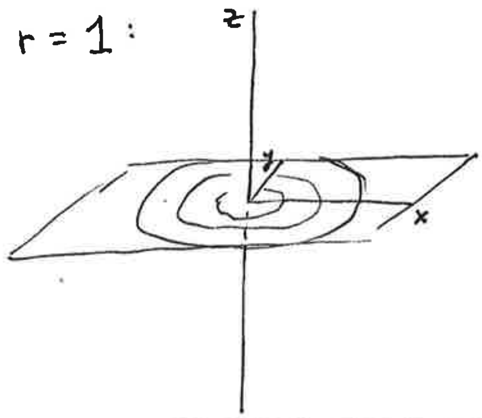
① Cylindrical coordinates (r, θ, z). "Polar with z."

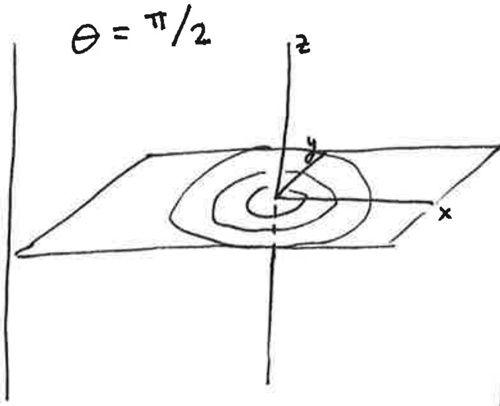
distance from z-axis
ccw angle from positive x-axis
height above xy-plane

OK, so we have 3 variables. If we fix one, and let the other two be anything, we get a surface. Let's see what we get.

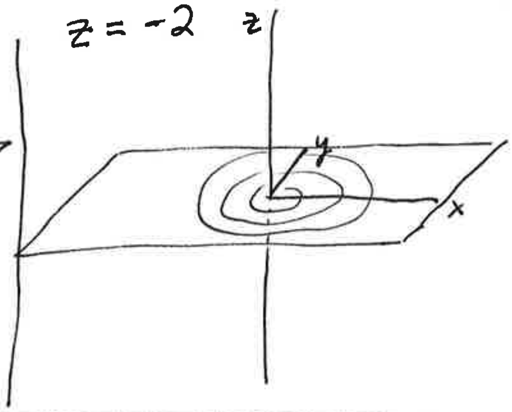
$r = 1$:



$\theta = \pi/2$

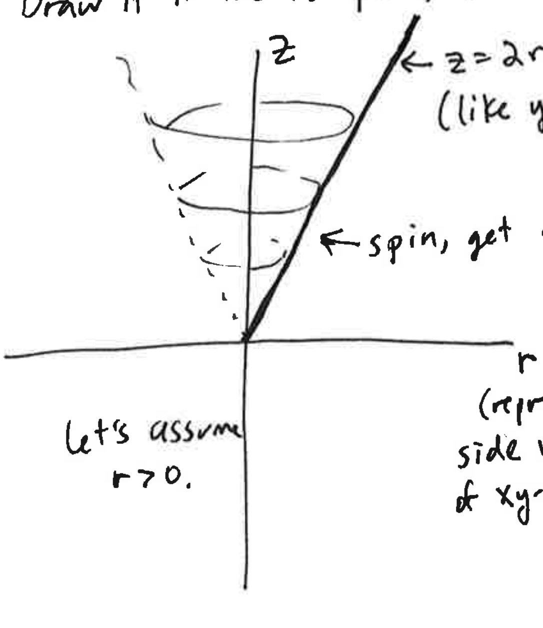


$z = -2$



<u>Example.</u> $z = 2r$.

Note that θ can be anything, so the surface has rotational symmetry. Draw it in the rz-plane, then spin!



← $z = 2r$ (like $y = 2x$)

← spin, get cone!

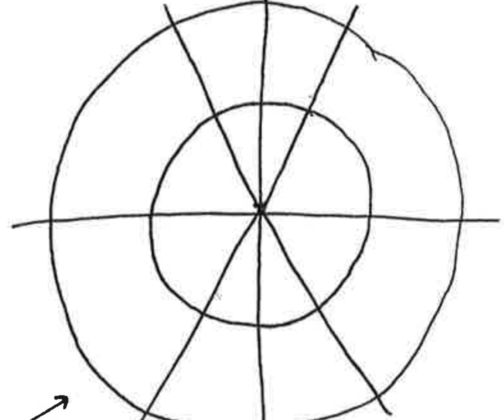
(represents side view of xy-plane)

Let's assume $r > 0$.

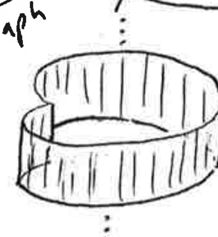
<u>Example.</u> $r = 1 + \cos\theta$.

Note that z can be anything, so it's a vertical "cylinder" (not necessarily circular) over some shape. Let's graph it in the (r,θ)-plane.

θ	$\cos\theta$	$1+\cos\theta$
0	1	
$\pi/3$	$1/2$	
$\pi/2$	0	
$2\pi/3$	$-1/2$	
π	-1	
$4\pi/3$	$-1/2$	
$3\pi/2$	0	
$5\pi/3$	$1/2$	
2π	1	



graph



cylinder over a cardioid

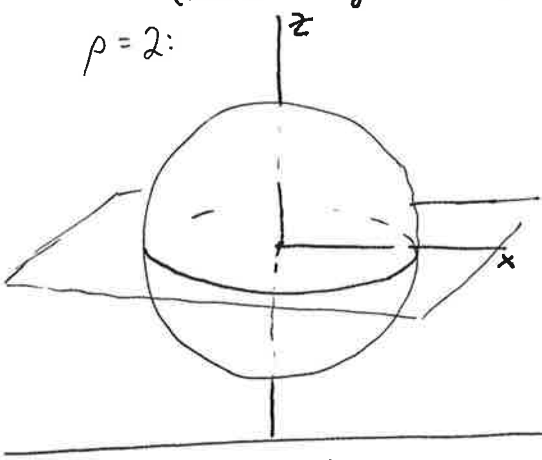
(2) Spherical coordinates (ρ, φ, θ)

distance (radius) from origin → ρ
angle down from positive z-axis → φ
angle ccw from positive x-axis (same as cylindrical). → θ

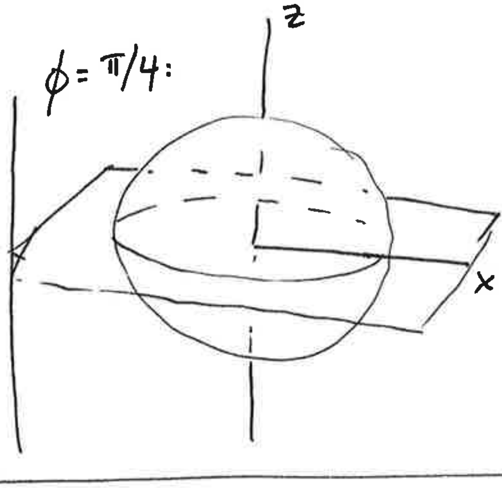
← Some books, and most physicists, use the order (ρ, θ, φ). For clarity, we will always write $(\rho, \varphi, \theta) = \ldots$ so there is no confusion.

We usually restrict to $\rho \geq 0$.

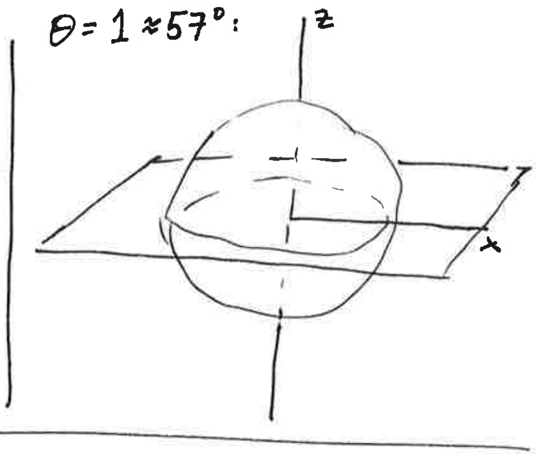
$\rho = 2$:



$\phi = \pi/4$:



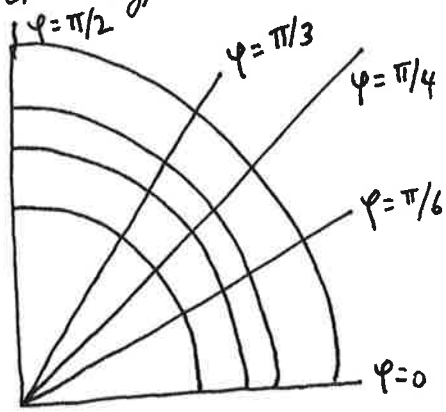
$\theta = 1 \approx 57°$:



<u>Example.</u> $\rho = 2a\cos\varphi$.

Note that θ is not in the equation, so θ can be anything, so the surface must be rotationally symmetric. Equivalently, cross sections for each value of θ look the same.

φ	$\cos\varphi$	$2a\cos\varphi$
0	1	$2a$
$\pi/6$	$\sqrt{3}/2$	$\sqrt{3}\,a$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}\,a$
$\pi/3$	$1/2$	a
$\pi/2$	0	0

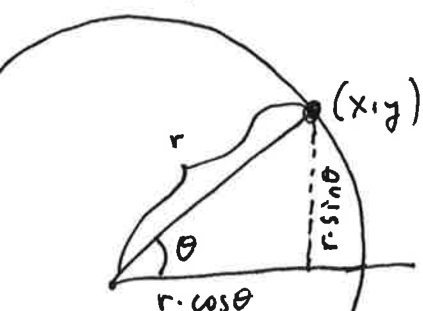


Sketch the surface in $\mathbb{R}^3$:

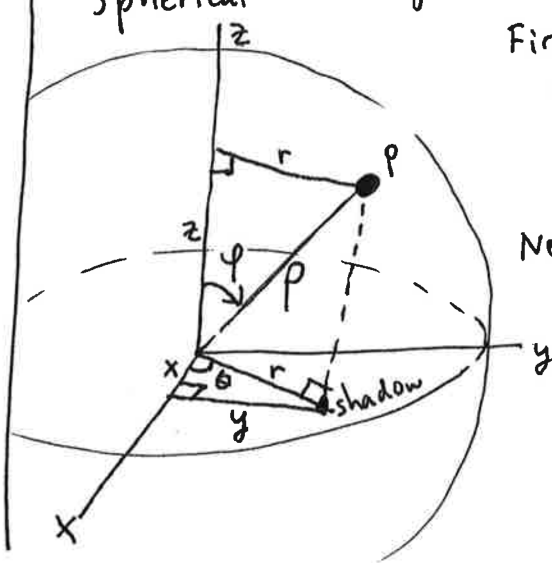
Going back and forth between cylindrical & spherical, to rectangular coordinates:

Cylindrical → rectangular

$$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \\ z = z \end{cases}$$



Spherical → rectangular



First step: use "shadow" of P in the xy-plane.

$x = r\cdot\cos\theta$, $y = r\cdot\sin\theta$ } as in polar & cylindrical.

Next step: use vertical triangle.

$z = \rho\cdot\cos\varphi$ ←
$r = \rho\cdot\sin\varphi$
$\Rightarrow x = \rho\cdot\sin\varphi\cdot\cos\theta$ ←
$\Rightarrow y = \rho\cdot\sin\varphi\cdot\sin\theta$. ←

☺

Mathematician spotlight: Ralph Gomez, Associate Professor, Swarthmore
- differential geometry

Last time: cylindrical & spherical coordinates; graphing surfaces
This time: "quadric surfaces," and how to figure out what they look like.

In 2D, consider curves of the form
$ax^2+bxy+cy^2+dx+ey+f=0.$
Such an equation describes one of the following, depending on a, b, c, d, e, f:

______ ______
______ ______
______

In 3D, consider surfaces of the form
$ax^2+by^2+cz^2+dxy+eyz+fxz+gx+hy+iz+j=0.$
What do these surfaces look like?
Idea: Set $x=$ constant, or $y=$ constant, etc.
This gives us a cross section (level curve) and reduces it to the 2D case of ellipses, parabolas, etc.!
Then use the cross sections to see what the surface is.

Example. $z=x^2-y^2$. We've seen this before! In color.
Let's look at cross-sections: let K be a constant.
① $z=K \Rightarrow K=x^2-y^2$: family of hyperbolas
② $x=K \Rightarrow z=K^2-y^2$: family of downward-facing parabolas
③ $y=K \Rightarrow z=x^2-K^2$: " " " upward - " "

Conclusion: horizontal cross-sections are hyperbolas
vertical cross-sections in one direction look like ∪
and in the perpendicular direction look like ∩.

Note! $\underbrace{\frac{z}{c}=\frac{y^2}{a^2}-\frac{x^2}{b^2}}$ is essentially the same as above, but stretched in each direction.
↳ because cross sections are hyperbolas & paraboloids, it's a "HYPERBOLIC PARABOLOID."

Example. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$
① $x=0 \Rightarrow$ ellipse in yz-plane
② $y=0 \Rightarrow$ " " xz-plane
③ $z=0 \Rightarrow$ " " xy-plane

what if $x=a$? ____________________

what if $x=2a$? ____________________

Cross sections are ellipses, a single point, or empty.

ELLIPSOID

Example. $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$
$x=0$: $z=y^2$ parabola
$y=0$: $z=x^2$ " "
$z=0$: point $(0,0,0)$
$z>0$: ellipse
$z<0$: empty
$x=K$: parabola shifted up
$y=K$: " " "

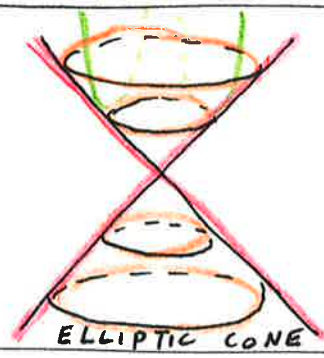
ELLIPTIC PARABOLOID

Now we will consider the family of surfaces of the form $\boxed{z^2 = x^2+y^2 \pm K.}$

First, suppose $K=0$.

$$z^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}$$

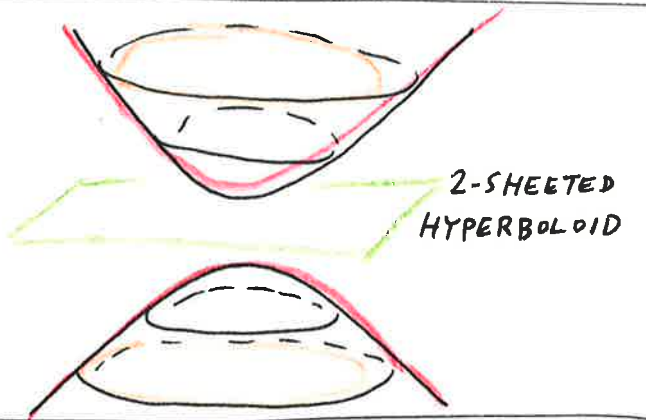
$x=0 \Rightarrow z^2=y^2 \Rightarrow z=\pm y$
$y=0 \Rightarrow z^2=x^2 \Rightarrow z=\pm x$ "
$z=0 \Rightarrow x=0, y=0 \Rightarrow (0,0,0)$
$z=k \Rightarrow k^2=\frac{x^2}{a^2}+\frac{y^2}{b^2} \Rightarrow$ ellipse
$x=k, y=k \Rightarrow$ upward parabolas



Now, suppose $K>0$ (added to x^2+y^2).

$$z^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}+K$$

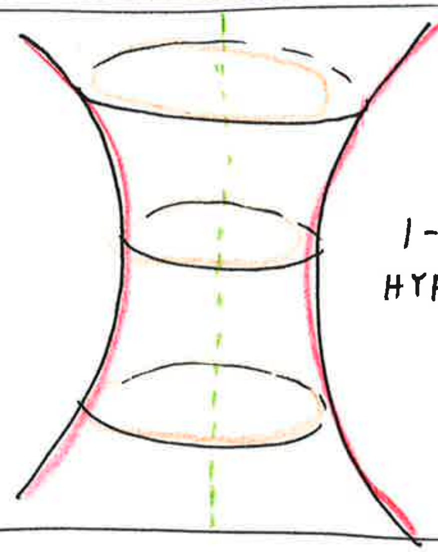
$x=0$: $z^2=y^2+K$, hyperbola
$y=0$: $z^2=x^2+K$, " "
$z=0$: EMPTY! tells you that the surface doesn't intersect the xy-plane! So 2 pieces!
$z^2>K$: $(+)=\frac{x^2}{a^2}+\frac{y^2}{b^2} \Rightarrow$ ellipse



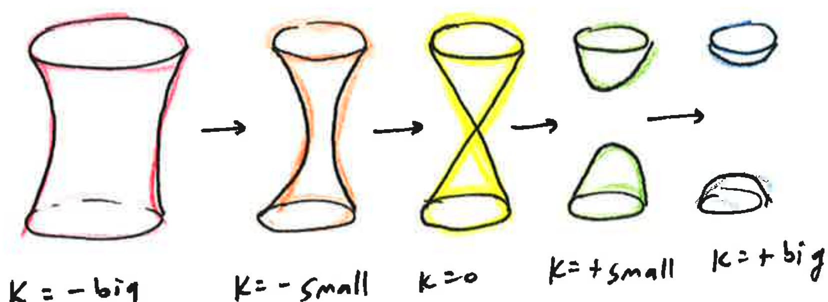
Now, suppose $K<0$ (subtracted from x^2+y^2).

$$z^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}-K$$

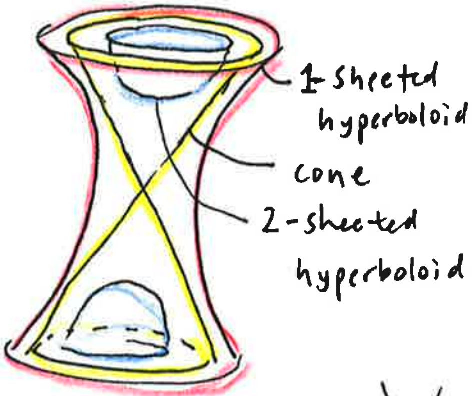
$x=0$: $z^2=y^2-K$, hyperbola
$y=0$: $z^2=x^2-K$, "
$z=0$: $K=\frac{x^2}{a^2}+\frac{y^2}{b^2}$, ellipse
$(x,y)=(0,0) \Rightarrow z^2=-K$ NO SOLUTION
tells you that the surface doesn't intersect the z-axis
$z=c \Rightarrow c^2+K=\frac{x^2}{a^2}+\frac{y^2}{b^2} \Rightarrow$ ellipse



Finally, think about the surfaces as K goes from negative, to 0, to positive, like a movie:



Or, think of them as nested:



Just as hyperbolas approach asymptote lines: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\pm 1 \to y=\pm\frac{b}{a}x$, Hyperboloids approach asymptote cones: $\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2} \cancel{\pm 1}$

Diana Davis — Math 34 — January 31, 2018 — Class #5 ① §2.2

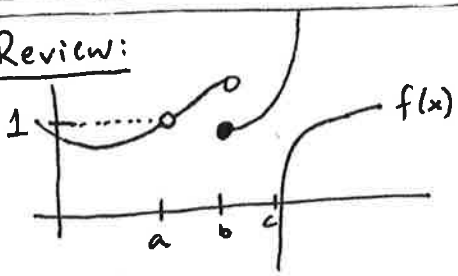
Mathematician spotlight: Emily Riehl, Assistant Professor, Johns Hopkins

· category theory, homotopy theory

Last time, we used cross-sections of a surface to figure out what it looks like.

This time, we'll think about what can "go wrong" when surfaces have vertical parts, etc.

Review:



$\lim_{x\to a} f(x) = 1$ (and it exists)

$\lim_{x\to b} f(x)$ does not exist (two different values) $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

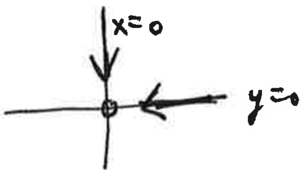
$\lim_{x\to c} f(x)$ does not exist (vertical asymptote)

For limits of multivariable functions, you must approach the point from all directions.

Example. $\lim_{(x,y)\to(1,2)} x^2+y^2 = 1^2+2^2 = 5$

If $f(x,y)$ is a product or sum of functions that are continuous at (a,b), just plug in (a,b) for the limit.

Example. $\lim_{(x,y)\to(0,0)} \frac{x+y}{2x+y}$

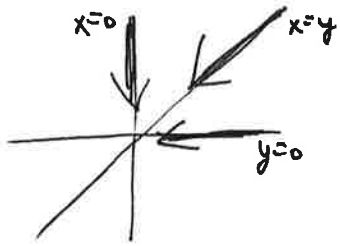


① Approach $(0,0)$ from the "east" $(y=0)$: $\lim_{(x,y)\to(0,0)} \frac{x+y}{2x+y} = \lim_{x\to 0} \frac{x+0}{2x+0} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}.$

② Approach $(0,0)$ from the "north" $(x=0)$: $\lim_{(x,y)\to(0,0)} \frac{x+y}{2x+y} = \lim_{y\to 0} \frac{0+y}{0+y} = \lim_{x\to 0}\frac{1}{1} = 1.$

The limit DOES NOT EXIST, because function value depends on the direction of approach.

Example. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$



① along $x=0$: $\lim_{y\to 0} \frac{0\cdot y}{0+y^2} = \lim_{y\to 0} \frac{0}{y^2} = \lim_{y\to 0} 0 = 0.$

② along $y=0$: $\lim_{x\to 0} \frac{x\cdot 0}{x^2+0} = \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0.$

③ along $y=x$: $\lim_{x\to 0} \frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$

again, limit DNE because value of f depends on direction of approach.

CLEVER TRICK: approach along all lines at once, using $y=mx$. (except $x=0$)

④ along $y=mx$: $\lim_{x\to 0} \frac{x\cdot mx}{x^2+(mx)^2} = \lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)} = \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$

limit DNE, value depends on slope (direction)

TAKEAWAY MESSAGE: If you think the limit does not exist, prove it by approaching on lines $y=mx$ and showing that the value depends on m.

Okay, now we know a method for showing that a limit <u>does not</u> exist.
How do we show that it does exist? Is checking all lines enough?
1. No, lines are not enough, because you could also approach along other curves.
2. To show the limit does exist, convert to polar (or spherical in $\mathbb{R}^3$).

<u>Example.</u> Here, the limit when approaching along all lines is 0, but the limit when approaching along a parabola is not! So the limit DNE.

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^4+y^2)^3} \quad \begin{pmatrix}\text{along}\\ y=mx:\end{pmatrix} : \lim_{x\to 0} \frac{x^4(mx)^4}{(x^4+(mx)^2)^3} = \lim_{x\to 0}\frac{x^8\cdot m^4}{x^6(x^2+m^2)^3} = \lim_{x\to 0} x^2\left(\frac{m^4}{(x^2+m^2)^3}\right) = \underline{\underline{0.}}$$

$$\begin{pmatrix}\text{along}\\ y=x^2:\end{pmatrix} : \lim_{x\to 0}\frac{x^4(x^2)^4}{(x^4+(x^2)^2)^3} = \lim_{x\to 0}\frac{x^{12}}{(2x^4)^3} = \lim_{x\to 0}\frac{x^{12}}{8x^{12}} = \lim_{x\to 0}\frac{1}{8} = \underline{\underline{\frac{1}{8}.}}$$

So the limit DNE, because it depends on the direction of approach.

<u>CLEVER TRICK:</u> (or, the only way to show that a limit <u>exists</u>): approach from all directions at once by converting to polar (r,θ) and do $\lim_{r\to 0}$.

<u>Example.</u>
$$\lim_{(x,y)\to(0,0)}\frac{x^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0}\frac{(r\cdot\cos\theta)^2}{\sqrt{r^2}} = \lim_{r\to 0}\frac{r^2\cdot\cos^2\theta}{r} = \lim_{r\to 0}\underbrace{r}_{\text{goes to }0}\cdot\underbrace{\cos\theta}_{\text{finite}} = \underline{\underline{0.}}$$

Also works in spherical coordinates: take $\lim_{\rho\to 0}$ to approach from all directions.

<u>Example.</u>
$$\lim_{(x,y,z)\to(0,0,0)}\frac{xz}{x^2+y^2+z^2} = \frac{(\rho\sin\varphi\cos\theta)(\rho\cos\varphi)}{\rho^2} = \lim_{\rho\to 0}\frac{\rho^2(\sin\varphi\cos\theta\cos\varphi)}{\rho^2}$$

$$= \lim_{\rho\to 0}\sin\varphi\cos\theta\cos\varphi \Rightarrow$$ limit DNE, because it depends on the direction of approach.

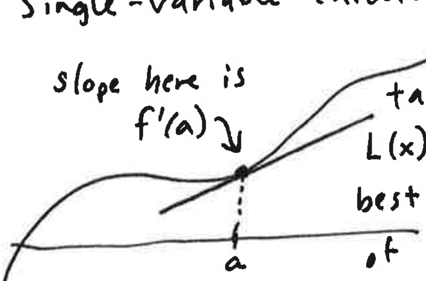
Good ways to make a computer draw graphs for you: (type in z = x^2 / (x^2 + y^2)^(1/2), for example)
- Google (type in z = x^2 / (x^2 + y^2)^(1/2), for example)
- WolframAlpha.com
- Grapher (comes standard on every Apple computer)
- Many free apps

Mathematician spotlight: Rich Schwartz, Brown University
- geometry, dynamical systems
- children's books explaining serious math (e.g. infinity)

Last time: multivariable limits.

This time: partial derivatives, like "slopes" in a given direction, and tangent planes.

Single-variable calculus:

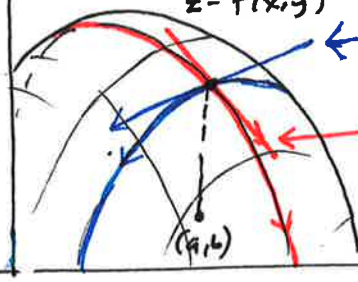


slope here is $f'(a)$

tangent line is $L(x)=f(a)+f'(a)(x-a)$, best linear approximation of $f(x)$ at $x=a$.

Multivariable calculus:

$z=f(x,y)$



slope in this direction is $f_x(a,b)$ or $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$.

slope in this direction is $f_y(a,b)$ or $\left.\frac{\partial f}{\partial y}\right|_{(a,b)}$

To find the "slope" or "tilt" of the surface at the desired point, we give the slope in the two axis-parallel directions:
- slope in positive x-direction is f_x
- " " y-direction is f_y.

We can use these to find the tangent plane to $z=f(x,y)$ at the point $(a,b,f(a,b))$, which is the best linear approximation of f at (a,b).

Example. Find the partial derivatives of $f(x,y)=5-x^2-2y^2$ at $(1,1)$.

① To find it in the x-direction, we can set $y=1$ and take the derivative with respect to x:

$$f(x,y)=5-x^2-2y^2$$
$$f(x,1)=5-x^2-2=3-x^2$$
$$\frac{d}{dx}f(x,1)=-2x$$
at $x=1$, $\frac{d}{dx}f(x,1)=-2=f_x(1,1)$.

② To find it in the y-direction, do the same for y, setting $x=1$: $f(1,y)=5-1-2y^2=4-2y^2$

$$\frac{d}{dy}f(1,y)=-4y$$
at $y=1$, $\frac{d}{dy}f(1,y)=-4=f_y(1,1)$.

In practice, we don't plug in a number for the other variable; we just treat it "as a constant": $f(x,y)=5-x^2-2y^2$

$$f_x(x,y)=-2x \qquad f_y(x,y)=-4y$$
$$f_x(1,1)=-2 \qquad f_y(1,1)=-4$$

Notation: $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}=f_x(a,b)$ mean the same thing.

$\frac{\partial f}{\partial x}$ is partial derivative, different from total derivative $\frac{df}{dx}$.


"del"

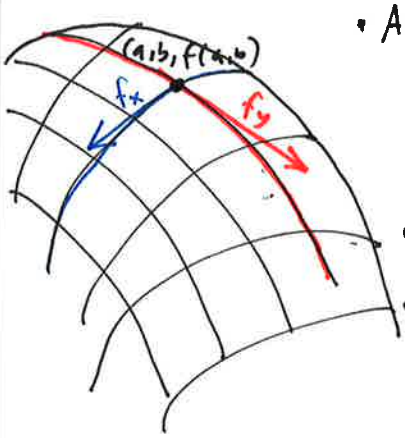
Practice taking a partial derivative. Note that chain rule, product rule, etc. still apply.

$g(x,y) = x^2 y + e^x \cdot \sin(xy)$.

$\frac{\partial g}{\partial x} =$ ______________________

$\frac{\partial g}{\partial y} = x^2 + e^x \cdot \cos(xy) \cdot x$.

Now we want to find the tangent plane to $z = f(x,y)$ at the point $(a, b, f(a,b))$.



- A vector tangent to the surface in the x-direction is given by $\begin{bmatrix} 1 \\ 0 \\ f_x(a,b) \end{bmatrix}$ ← y is not changing ← amount of rise, for run of 1 in the x-direction
- A vector tangent to the surface in the y-direction is given by $\begin{bmatrix} 0 \\ 1 \\ f_y(a,b) \end{bmatrix}$.
- To find the tangent plane to the surface at (a,b), we need:
  1. a point on the plane: ______________
  2. a normal vector to the plane: ______________

Find the normal vector:

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix} = \vec{i}\left(-f_x(a,b)\right) - \vec{j}\left(f_y(a,b)\right) + \vec{k}(1) = \begin{bmatrix} -f_x(a,b) \\ -f_y(a,b) \\ 1. \end{bmatrix}$$

The equation for the plane through (x_0, y_0, z_0) with $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$,

so " " " " $(a, b, f(a,b))$ " $\vec{n} = \begin{bmatrix} -f_x(a,b) \\ -f_y(a,b) \\ 1 \end{bmatrix}$ is given by

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + 1\,(z - f(a,b)) = 0.$$

rearranging the terms: $z = \underbrace{f(a,b)}_{\text{number}} + \underbrace{f_x(a,b)}_{\text{number}}(x - \overset{\uparrow}{a}) + \underbrace{f_y(a,b)}_{\text{number}}(x - b)$. ↖ number

(under a: number)

<u>Example.</u> Find the tangent plane to $z = 5 - x^2 - 2y^2$ at $(1,1,2)$.

We know: $f_x(1,1) = -2$ $\Rightarrow$ $z = 2 + (-2)(x-1) + (-4)(y-1)$.

$f_y(1,1) = -4$ $\qquad = 8 - 2x - 4y$.

This is the best linear approximation of $f(x,y) = 5 - x^2 - 2y^2$ at $(1,1)$:

- the value is the same
- the partial derivatives are the same.

$L(x,y) = 8 - 2x - 4y$

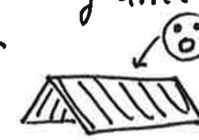
$\left.\begin{array}{l} L(1,1) = 2 \checkmark \\ L_x(1,1) = -2 \checkmark \\ L_y(1,1) = -4 \checkmark \end{array}\right\}$ same as for $f(x,y)$.

Diana Davis Math 34 February 5, 2018 class #7 ①
§2.3

Mathematician spotlight: Piper Harron, postdoc, University of Hawaii
- algebraic number theory (PhD at Princeton)
- intersectional radical feminism, anti-racism

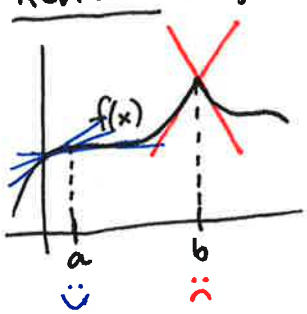
<u>Previously</u>: <u>Limits</u> - exploring one way in which functions can fail to be "nice".

for function values:
- function graph has a "hole" - e.g. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim\limits_{(x,y)\to(0,0)} x-y = \underline{\underline{0}}$. Can fill in, no problem.
- function graph has a "vertical part" - e.g. $\lim\limits_{(x,y)\to(0,0)} \frac{x+y}{2x+y}$ DNE - no way to fix ☹

<u>(Partial) Derivatives</u> - rate of change of function (slope) in x- or y-direction
- if function is "smooth", we know what to do. 
- if function has "creases," the derivative is not defined. 

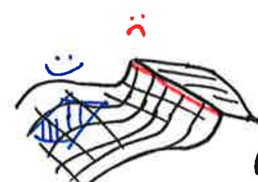
<u>Today:</u> Differentiability & non-differentiability of multivariable functions.

<u>Review:</u> single-variable calculus



f(x) is <u>differentiable</u> at p if:
- f has a well-defined tangent line at p, or (equivalently)
- $\lim\limits_{x\to p^-} f'(x) = \lim\limits_{x\to p^+} f'(x)$, or (equivalently)
- $\lim\limits_{x\to p} \frac{f(x) - (\text{tangent line at } p)}{x-p} = 0.$

<u>New:</u> multivariable calculus



f(x,y) is differentiable at (a,b) if f has a tangent plane at (a,b, f(a,b)) that is a good approximation of f, i.e.

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - (\text{tangent plane at } (a,b))}{\underbrace{\|(x,y)-(a,b)\|}_{\text{distance between points.}}} = \underline{\underline{0}}.$$

<u>Example:</u> The nice, everywhere-differentiable function from last time.
$f(x,y) = 5 - x^2 - y^2$ we found the tangent plane at (1, 1, 3) to be
$L(x,y) = 7 - 2x - 2y = z$.

> note: I am using $5-x^2-y^2$ instead of $5-x^2-2y^2$ so things work out cleaner.

To show that f(x,y) is differentiable at (1,1), we see if the tangent plane is a good approximation by taking the limit:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|(x,y)-(a,b)\|} = \lim_{(x,y)\to(1,1)} \frac{(5-x^2-y^2)-(7-2x-2y)}{\sqrt{(x-1)^2+(y-2)^2}} = \lim_{(x,y)\to(1,1)} \frac{-(x^2-2x+1+y^2-2y+1)}{\sqrt{(x-1)^2+(y-2)^2}}$$

$$= \lim_{(x,y)\to(1,1)} \frac{-((x-1)^2+(y-1)^2)}{\sqrt{(x-1)^2+(y-1)^2}} = \lim_{(x,y)\to(1,1)} \frac{-\left(\sqrt{(x-1)^2+(y-1)^2}\right)^{\cancel{2}}}{\cancel{\sqrt{(x-1)^2+(y-1)^2}}} = \lim_{(x,y)\to(1,1)} -\sqrt{(x-1)^2+(y-1)^2} = \underline{\underline{0}}.$$

So the plane is a good approximation, so f is diff'ble!

Diana Davis — Math 34 — February 5, 2018 — Class # 7 (2)

So, what does it look like when a function is not differentiable at a point?

Geometrically: a sharp cusp  or crease 

Analytically: the limit $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y) - (\text{tangent plane at }(a,b))}{\|(x,y)-(a,b)\|} \neq 0$ because no tangent plane approximates the function well there.

Example. $f(x,y) = \big||x|-|y|\big| - |x| - |y|$ will turn out to be non-differentiable at the origin (see cardboard model in class).

First, let's find a candidate tangent plane at $(0,0)$.
- taking the partial derivative with respect to x while treating y as constant seems complicated here, so let's use the definition: set $y=0$ and take the limit as $x\to 0$.

at $(0,0)$: $f_x(x,y) = \lim\limits_{x\to 0} f(x,0) = \lim\limits_{x\to 0} \big||x|-|0|\big| - |x| - |0| = \lim\limits_{x\to 0} \big||x|\big| - |x| = \lim\limits_{x\to 0} |x|-|x| = \lim\limits_{x\to 0} 0 = \underline{\underline{0}}.$

now let's do the same for y: set $x=0$ and take the limit as $y\to 0$.

at $(0,0)$: $f_y(x,y) = \lim\limits_{y\to 0} f(0,y) = \lim\limits_{y\to 0} \big||0|-|y|\big| - |0| - |y| = \lim\limits_{y\to 0} \big||y|\big| - |y| = \lim\limits_{y\to 0} |y|-|y| = \lim\limits_{y\to 0} 0 = \underline{\underline{0}}.$

and the function value:

$$f(0,0) = \big||0|-|0|\big| - |0| - |0| = 0.$$

So our candidate tangent plane equation is
$$\begin{aligned} z = L(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\ &= 0 + 0\,(x-0) + 0\,(y-0) \\ &= \underline{\underline{0}}. \end{aligned}$$

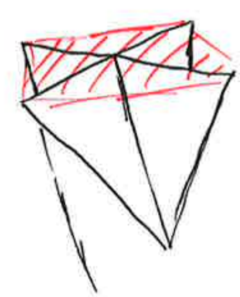
OK, now let's write down the limit that we expect to come out $\neq 0$:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - (\text{tangent plane at }(0,0))}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\big||x|-|y|\big| - |x| - |y|}{\|(x,y)\|}$$

For the limit to exist and be 0, it has to exist and be 0 from every direction of approach. Let's approach along the line $y=x$.

$$\left(\begin{matrix}\text{along}\\ y=x:\end{matrix}\right) \quad \lim_{x\to 0} \frac{\big||x|-|x|\big| - |x| - |x|}{\|(x,x)\|} = \lim_{x\to 0} \frac{-2|x|}{\sqrt{2}\,|x|} = \lim_{x\to 0} -\sqrt{2} = \underline{\underline{-\sqrt{2}.}} \quad \neq 0$$

So the candidate tangent plane does not approximate the function well at $(0,0)$, so the function is not differentiable at $(0,0)$.



Diana Davis Math 34 February 7, 2018 Class #8 ① §2.4

Mathematician spotlight: Ryan Hynd, University of Pennsylvania
- differential equations applied to inelastic collisions
- colloquium speaker here at Swarthmore yesterday.

Last time: A function is differentiable at a given point if its graph has a well-defined tangent plane there: no "sharp point" or "crease."

This time: Generalize, organize & compute the notion of a tangent plane, or best linear approximation, & differentiability, for $f: \mathbb{R}^m \to \mathbb{R}^n$ using "Jacobian matrix."

Recall: Our equation for the tangent plane (best linear approximation) to $z = f(x,y)$ at $(a, b, f(a,b))$ is $z = L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$.

Changing notation, if $\vec{a} = (a,b)$ and $\vec{x} = (x,y)$: $= f(\vec{a}) + f_x(\vec{a})(x-a) + f_y(\vec{a})(y-b)$.

we can express this as a dot product: $= f(\vec{a}) + \begin{bmatrix} f_x(\vec{a}) \\ f_y(\vec{a}) \end{bmatrix} \bullet \begin{bmatrix} x-a \\ y-b \end{bmatrix}$

now define the "gradient of f at (a,b)":
$$\boxed{\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} f_x(a,b) \\ f_y(a,b) \end{bmatrix} = \begin{bmatrix} f_x(\vec{a}) \\ f_y(\vec{b}) \end{bmatrix}}$$

$$z = L(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \bullet (\vec{x} - \vec{a}).$$

Note: this is similar to point-slope form for the tangent line to $y = f(x)$ at $x = a$: $y = f(a) + f'(a)(x-a)$.

Also, we can now express the definition of differentiability more precisely:

$f(\vec{x})$ is differentiable at $\vec{a}$ if $\displaystyle\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - [f(\vec{a}) + \nabla f(\vec{a}) \bullet (\vec{x} - \vec{a})]}{\|\vec{x} - \vec{a}\|} = 0.$
$(f: \mathbb{R}^n \to \mathbb{R})$

We can generalize this notion of "best linear approximation" to functions $f: \mathbb{R}^m \to \mathbb{R}^n$:

Now $f(\vec{x}) = f(x_1, \ldots, x_m) = \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{bmatrix} = \begin{bmatrix} f_1(x_1, \ldots, x_m) \\ f_2(x_1, \ldots, x_m) \\ \vdots \\ f_n(x_1, \ldots, x_m) \end{bmatrix}$. Instead of the column vector gradient ∇f, we now have the $n \times m$ Jacobian matrix Df:

$$Df = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_m \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_m \\ \vdots & \vdots & & \vdots \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \cdots & \partial f_n/\partial x_m \end{bmatrix}.$$

For $f: \mathbb{R}^2 \to \mathbb{R}$, this reduces to the gradient:
$$Df = \left[\partial f_1/\partial x_1 \quad \partial f_1/\partial x_2\right] = [f_x, f_y] = \nabla f.$$

Now the best linear approximation of f at $\vec{a}$ is given by

$$L(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a}) = \underbrace{\begin{bmatrix} f_1(\vec{a}) \\ f_2(\vec{a}) \\ \vdots \\ f_n(\vec{a}) \end{bmatrix}}_{\text{numbers}} + \begin{bmatrix} Df(\vec{a}) \\ (n \times m) \end{bmatrix} \begin{bmatrix} \vec{x} - \vec{a} \\ (m \times 1) \end{bmatrix} = \overbrace{\begin{bmatrix} L_1(\vec{x}) \\ L_2(\vec{x}) \\ \vdots \\ L_n(\vec{x}) \end{bmatrix}}^{\text{all linear terms}},$$

a linear approximation in each coordinate.

Definition. The Jacobian matrix of a function $f:\mathbb{R}^m \to \mathbb{R}^n$ is the $n \times m$ matrix Df of partial derivatives of f. If $f:\mathbb{R}^m \to \mathbb{R}$, the Jacobian matrix is a row vector, and is called the gradient of f, denoted by ∇f. Given a point $\vec{a}$, the function $L(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x}-\vec{a})$ provides the best linear approximation of f in the region near the point $\vec{a}$.

Example. Let $\underbrace{f:\mathbb{R}^3 \to \mathbb{R}^2}_{\text{think of it as the wind direction vector for any point in 3-space.}}$ be the function $f(x,y,z) = \begin{bmatrix} xy^2z + z\cdot e^y \\ z + \sin(xyz) \end{bmatrix} = \begin{bmatrix} f_1(x,y,z) \\ f_2(x,y,z) \end{bmatrix}$

Let's find the best linear approximation of f at the point $\vec{a} = (1,1,1)$:

1. Find the Jacobian matrix of f:

$$Df = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z \end{bmatrix} = \begin{bmatrix} y^2z & \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ & \_\_\_\_ \end{bmatrix} \Rightarrow Df(1,1,1) = \begin{bmatrix} 1 & 2+e & 1+e \\ \cos 1 & \cos 1 & 1+\cos 1 \end{bmatrix}$$

2. Find the linear approximation of f at $(1,1,1)$:

$$L(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x}-\vec{a}) = f(1,1,1) + Df(1,1,1)\begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+e \\ 1+\sin 1 \end{bmatrix} + \begin{bmatrix} 1 & 2+e & 1+e \\ \cos 1 & \cos 1 & 1+\cos 1 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} = \begin{bmatrix} -3-e+x+(2+e)y+(1+e)z \\ \sin 1 - 3\cos 1 + \cos 1(x+y) + (1+\cos 1)z \end{bmatrix}.$$

all linear terms! No polynomials, exponents, trig... just numbers & x, y, z.

We generalized derivatives to higher dimensions. Now we'll do higher-order derivatives:

Example. Let $f(x,y) = xy^2 + e^{xy}$.

Then $f_x(x,y) = \dfrac{\partial f}{\partial x} =$ ________

$f_y(x,y) = \dfrac{\partial f}{\partial y} =$ ________

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}(x,y) = y^2e^{xy}.$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}(x,y) = 2y + e^{xy} + xy\,e^{xy}.$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}(x,y) = 2y + e^{xy} + xy\,e^{xy}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}(x,y) = 2x + x^2e^{xy}.$$

Notice: $f_{xy} = f_{yx}$. This is always true:

Clairaut's Theorem: If $f(x_1,\ldots,x_m)$ has continuous 1^{st} and 2^{nd} partial derivatives, then the order of differentiation does not matter: $f_{x_ix_j} = f_{x_jx_i}$ for all i,j.

In fact, if f has continuous $1^{st}, 2^{nd}, \ldots, k^{th}$ partial derivatives, you can take them in any order: $f_{xyzxyz} = f_{xxyyzz} = f_{zxyxyz} = \ldots$ (any reordering of 2 x's, 2y's, 2z's).

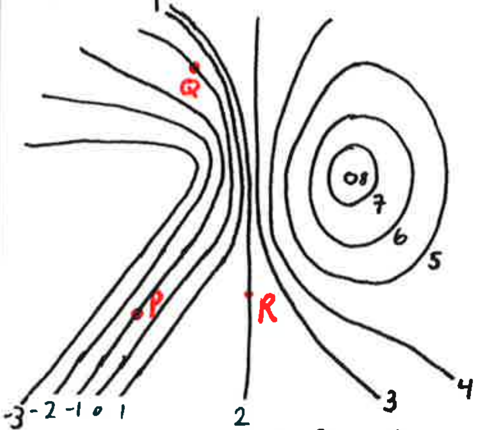
So, use the order that is easiest and makes things (terms) disappear early!

Mathematician spotlight: Sarah Koch, Associate Professor, University of Michigan
- complex dynamics, complex analysis
- comes with beautiful pictures – search for "Julia set" online

Last time: ① linear approximation of a function $f: \mathbb{R}^m \to \mathbb{R}^n$
② second partial derivatives f_{xx}, f_{xy}, f_{yy} (etc. for more variables).

This time: Explore those more, plus the Chain Rule.

What partial derivatives mean, geometrically: Fill in with ">", "<" or "=".



Level curves of $f(x,y)$

$f_x(R)$ ___ 0 slope if you walk in $+x$ direction

$f_y(P)$ ___ 0 slope if you walk in $+y$ direction

$f_{xx}(P)$ ___ 0 rate of change of slope in $+x$ direction

$f_{yy}(Q)$ ___ 0 rate of change of slope in $+y$ direction

$f_{xy}(R)$ ___ 0 how the slope of an eastward $(+x)$ path changes, if you move your path slightly to the north $(+y)$.

$f_y(R)$ ___ 0 slope if you walk in $+y$ direction

Example of linear approximation: Find the (x,y) coordinates of the following points:

① $(r,\theta) = (2, \frac{\pi}{3}) \Rightarrow (x,y) =$ ________ $(r,\theta) = (2.1, \frac{\pi}{3} - 0.1) \Rightarrow (x,y) =$ ________

Hmmm, need a calculator... or linear approximation!

Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
$$f(r,\theta) = \begin{bmatrix} r\cdot\cos\theta \\ r\cdot\sin\theta \end{bmatrix} = \begin{bmatrix} x(r,\theta) \\ y(r,\theta) \end{bmatrix}.$$
Then $Df = \begin{bmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$.

We want the linear approximation at $\vec{a} = (2, \frac{\pi}{3})$. So let's plug in:
$$L(r,\theta) = f(\vec{a}) + Df(\vec{a})(\vec{r} - \vec{a})$$
$$\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + \begin{bmatrix} 1/2 & -\sqrt{3} \\ \sqrt{3}/2 & 1 \end{bmatrix}\begin{bmatrix} r-2 \\ \theta - \frac{\pi}{3} \end{bmatrix}$$
$$\Rightarrow L(2.1, \tfrac{\pi}{3} - 0.1) = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + \begin{bmatrix} 1/2 & -\sqrt{3} \\ \sqrt{3}/2 & 1 \end{bmatrix}\begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} \approx \begin{bmatrix} 1.223 \\ 1.719 \end{bmatrix}$$
actual value is $\begin{bmatrix} 1.226 \\ 1.705 \end{bmatrix}$

wow, so close!

It seems like we can differentiate anything! How about functions of other functions?

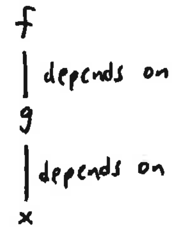
Example. Let $f(x,y) = x^2 y$ and suppose x and y are also functions:
$x(s,t) = st$
$y(s,t) = e^{st}$.

To find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$, we can just plug in:
$$f(s,t) = (st)^2 \cdot e^{st}$$
$$\Rightarrow \partial f/\partial s = \ldots$$

This looks like a tedious pain. Let's use the Chain Rule!

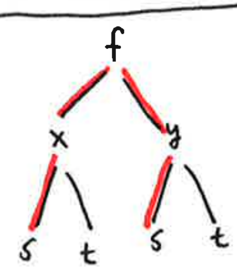
Diana Davis   Math 34   February 9, 2018   Class #9 ②

Recall: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\cdot g'(x)$. In Leibniz notation: $\frac{df}{dx} = \frac{df}{dg}\cdot\frac{dg}{dx}$.



For more variables: $\frac{\partial f}{\partial s} = \underbrace{\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}} + \underbrace{\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}}$

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$



Find the dependence of f on s from both contributions.

$f(x,y) = x^2y$ $x(s,t) = st$ $y(s,t) = e^{st}$

$\partial f/\partial x =$ $\partial x/\partial s =$ $\partial y/\partial s =$

$\partial f/\partial y =$ $\partial x/\partial t =$ $\partial y/\partial t =$

So now we can compute:

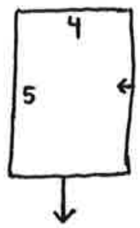
$\partial f/\partial s = (2xy)(t) + (x^2)(te^{st})$ ← not done; need in terms of s and t

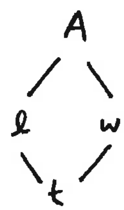
$= 2\cdot st\cdot e^{st}\cdot t + (st)^2\cdot t\cdot e^{st}$

$\partial f/\partial t = (2xy)(s) + (x^2)(se^{st})$

$= 2\cdot st\cdot e^{st}\cdot s + (st)^2\cdot s\cdot e^{st}$.

Example. A patch of moss at the Scott Arboretum is rectangular. In February 2018, its length was 5 meters and increasing by 1 meter per month, and its width was 4 meters and decreasing by ½ meter per month. At what rate was its area changing?





We know: $\ell = 5$ $w = 4$ $A = \ell w$

$d\ell/dt = 1$ $dw/dt = -1/2$ $\partial A/\partial \ell = w$ $\partial A/\partial w = \ell$

← ∂ for partial derivative: function depends on ℓ and w.

↑ d for total derivative: function only depends on t

We want: $\frac{dA}{dt}$ ← d for total derivative: function only depends on t

Chain Rule: $\frac{dA}{dt} = \frac{\partial A}{\partial \ell}\cdot\frac{d\ell}{dt} + \frac{\partial A}{\partial w}\cdot\frac{dw}{dt} = (w)(1) + (\ell)(-1/2) = 4\times 1 + 5\times\frac{-1}{2} = 4 - 2.5 = \underline{+1.5}$ square m per month.

What if we have more functions and more variables? Make a tree; follow the branches.

Example. $g(x,y,z) = xy^2z^3$

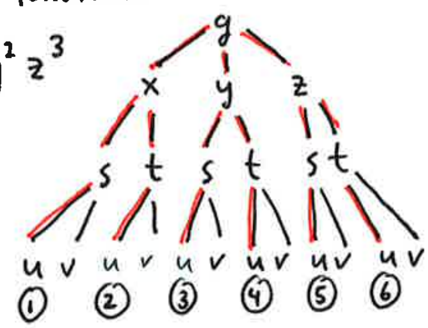
$x(s,t) = st$

$y(s,t) = e^{st}$

$z(s,t) = s+t$

$s(u,v) = u+v$

$t(u,v) = 2u-v$



$\frac{\partial g}{\partial u} = \underset{①}{\frac{\partial g}{\partial x}\frac{\partial x}{\partial s}\frac{\partial s}{\partial u}} + \underset{②}{\frac{\partial g}{\partial x}\frac{\partial x}{\partial t}\frac{\partial t}{\partial u}} + \cdots$ (4 more terms)

These functions ↗ are all scalar-valued: $g:\mathbb{R}^3\to\mathbb{R}$ and the rest are from $\mathbb{R}^2\to\mathbb{R}$.

What about vector-valued functions from $\mathbb{R}^m\to\mathbb{R}^n$? Use matrices, not scalars:

If $f:\mathbb{R}^m\to\mathbb{R}^p$ and $g:\mathbb{R}^n\to\mathbb{R}^m$, then the Jacobian $D(f\circ g)$ for $f\circ g:\mathbb{R}^n\to\mathbb{R}^p$ is:

$$D(f\circ g) = \underset{p\times m}{\begin{bmatrix} Df \end{bmatrix}}\underset{m\times n}{\begin{bmatrix} Dg \end{bmatrix}} = \underset{p\times n}{\begin{bmatrix} Df\cdot Dg \end{bmatrix}}.$$

Matrix multiplication instead of scalar multiplication.

Same idea, higher dimensions.

Mathematician spotlight: David Rockoff, University of Arizona
- differential item functioning - fairness of tests
- uses randomization, simulations & data

Last time: chain rule for multivariable functions

This time: vector-valued chain rule example, plus directional derivatives!

Review: Suppose $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$. Then $\frac{d}{dx} f(g(x)) = \underbrace{\frac{d}{dx}(f\circ g)(x)}_{\text{function composition notation}} = f'(g(x))\cdot g'(x)$
$= \frac{df}{dg}\cdot\frac{dg}{dx}$.

New: Suppose $f:\mathbb{R}^m\to\mathbb{R}^p$, $g:\mathbb{R}^n\to\mathbb{R}^m$.

Then the Jacobian of $f\circ g:\mathbb{R}^n\to\mathbb{R}^p$ is

$$D(f\circ g) = \begin{bmatrix} Df \\ (p\times m)\end{bmatrix}\begin{bmatrix} Dg \\ (m\times n)\end{bmatrix} = \begin{bmatrix} Df\cdot Dg \\ (p\times n).\end{bmatrix}$$

Let's do an example!

Example. Suppose $f(x,y,z) = \begin{bmatrix} xyz \\ x+y\end{bmatrix} = \begin{bmatrix} f_1(x,y,z) \\ f_2(x,y,z)\end{bmatrix}$ and $x(s,t)=st$, $y(s,t)=s+t$, $z(s,t)=s^2-t^2$, so $g(s,t)=\begin{bmatrix} st \\ s+t \\ s^2-t^2\end{bmatrix}$.

Consider $f\circ g = f(g):\mathbb{R}^2\to\mathbb{R}^2$, because $\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2$.

Then $D(f\circ g) = Df\cdot Dg = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z\end{bmatrix}\begin{bmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \\ \partial z/\partial s & \partial z/\partial t\end{bmatrix} = \underbrace{\begin{bmatrix} yz & xz & xy \\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix} t & s \\ 1 & 1 \\ 2s & -2t\end{bmatrix}}_{\text{now multiply out and put everything in terms of } s \text{ and } t.}$.

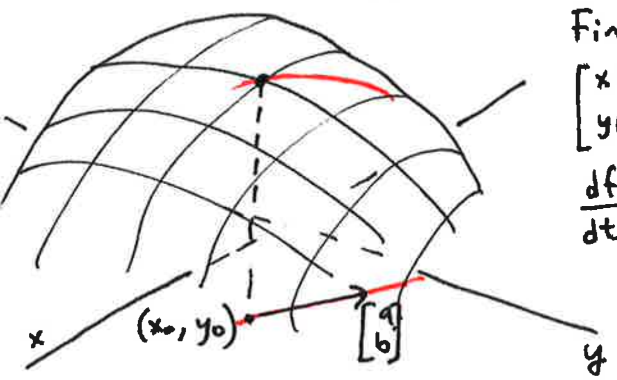
If you want to find this Jacobian matrix at a point $(s,t)=(1,1)$, compute $(x,y,z)=(1,2,0)$ and plug in:

$$D(f\circ g)(\vec{a}) = Df(g(\vec{a}))\cdot Dg(\vec{a})$$

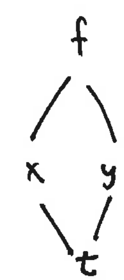
$$\left.\begin{aligned} D(f\circ g)(1,1) &= Df(g(1,1))\cdot Dg(1,1) \\ &= Df(1,2,0)\cdot Dg(1,1)\end{aligned}\right\} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & -2\end{pmatrix} = \underline{\underline{\begin{pmatrix} 4 & -4 \\ 2 & 2\end{pmatrix}}}. = \begin{pmatrix} \partial f_1/\partial s & \partial f_1/\partial t \\ \partial f_2/\partial s & \partial f_2/\partial t\end{pmatrix}.$$

Today: Directional derivatives! We know the "slope" of $z=f(x,y)$ in the positive x-direction is f_x, and in pos y-direction is f_y, but what about the slope in other directions?

$z=f(x,y)$



Find "slope" at (x_0,y_0) in direction $\begin{bmatrix} a \\ b\end{bmatrix}$: This line is $\begin{bmatrix} x(t) \\ y(t)\end{bmatrix} = \begin{bmatrix} x_0+at \\ y_0+bt\end{bmatrix}$ and we want $\frac{d}{dt} f(x(t),y(t))$.



$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = \frac{\partial f}{\partial x}\cdot a + \frac{\partial f}{\partial y}\cdot b$$

$$= \left[\partial f/\partial x,\ \partial f/\partial y\right]\cdot\begin{bmatrix} a \\ b\end{bmatrix}. \Rightarrow \boxed{D_{\vec{u}} f = \nabla f\cdot\vec{u}}$$

Directional derivative of f in direction $\vec{u}=\begin{bmatrix} a \\ b\end{bmatrix}$.

Note that the "slope" should not depend on the length of the direction vector $\vec{u}$ that we choose, so we always take $\vec{u}$ to be a unit vector:

> The directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of the unit vector $\vec{u}$ is $D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0) \bullet \vec{u}$.

<u>Example.</u> Let $f(x,y) = x^2 - y^2$. Find the directional derivative of f at $(3,1)$ in the direction $\begin{bmatrix}1\\2\end{bmatrix}$.

① Find $\nabla f = \begin{bmatrix}2x\\-2y\end{bmatrix}$ ② Find $\nabla f(3,1) = \begin{bmatrix}6\\-2\end{bmatrix}$ ③ Find a unit direction vector: $\vec{u} = \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$

④ Put them together! $D_{\vec{u}} f(3,1) = \nabla f(3,1) \bullet \vec{u} = \begin{bmatrix}6\\-2\end{bmatrix} \bullet \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix} = 6/\sqrt{5} - 4/\sqrt{5} = \underline{\underline{2/\sqrt{5}}}.$

⑤ What does it mean? ____________________

<u>Pondering the geometry:</u> $D_u f(x_0,y_0) = \nabla f(x_0,y_0) \bullet \vec{u} = \|\nabla f(x_0,y_0)\| \cdot \|\vec{u}\| \cdot \cos\theta$ ← where θ is the angle between $\nabla f(x_0,y_0)$ and $\vec{u}$

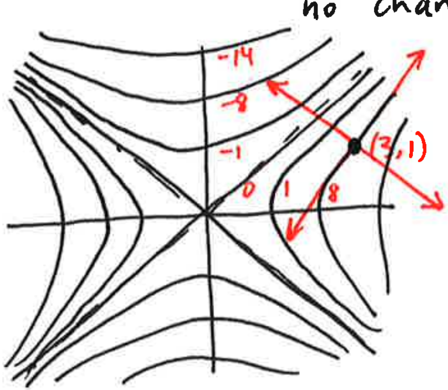
$= \|\nabla f(x_0,y_0)\| \cdot \cos\theta$ ← since $\|\vec{u}\| = 1$

- When is $D_{\vec{u}} f$ the largest positive number? → when $\underbrace{\cos\theta = 1}$, $D_{\vec{u}} f$ is $\|\nabla f\|$, the largest
  ↳ happens when $\theta = 0$, i.e. ∇f and $\vec{u}$ in same direction
  ⇒ gradient points in direction of steepest ascent.

- When is $D_{\vec{u}} f$ the largest negative number? → when $\underbrace{\cos\theta = -1}$, $D_{\vec{u}} f$ is $-\|\nabla f(x_0,y_0)\|$
  ↳ happens when $\theta = \pi$, i.e. ∇f and $\vec{u}$ in opposite directions
  ⇒ direction of steepest descent is $-\nabla f(x_0,y_0)$.

- When is $D_u f = 0$, i.e. which direction(s) can you go to stay on the same level? → ① when $\cos\theta = 0$, so when $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, so when ∇f and $\vec{u}$ are perpendicular
  → ② when you go along a level curve.

<u>Example.</u> For $f(x,y) = x^2 - y^2$, find directions of steepest ascent, steepest descent, and no change, and the directional derivatives (slopes) in those directions.



- Steepest ascent: $\nabla f(3,1) = \begin{bmatrix}6\\-2\end{bmatrix}$ and slope is $\|\nabla f(3,1)\| = \sqrt{6^2 + (-2)^2} = \sqrt{40}$.
- steepest descent: ____________________
- no change: ____________________

Diana Davis Math 34 February 14, 2018 Class #11 ① §2.6

Mathematician spotlight: Ron Buckmire ♥ Dean Elzinga

Happy Valentine's Day!

Ron: NSF program officer for financially supporting student math.

Dean: Opera singer (20 years), now senior machine learning engineer.

Last time: The gradient of $f(x,y)$ is $\nabla f(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$ or $\nabla f(x,y,z) = \begin{bmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{bmatrix}$ etc.

The directional derivative of f at (x_0, y_0) in the direction of unit vector $\vec{u}$ is

$$D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0) \bullet \vec{u}.$$

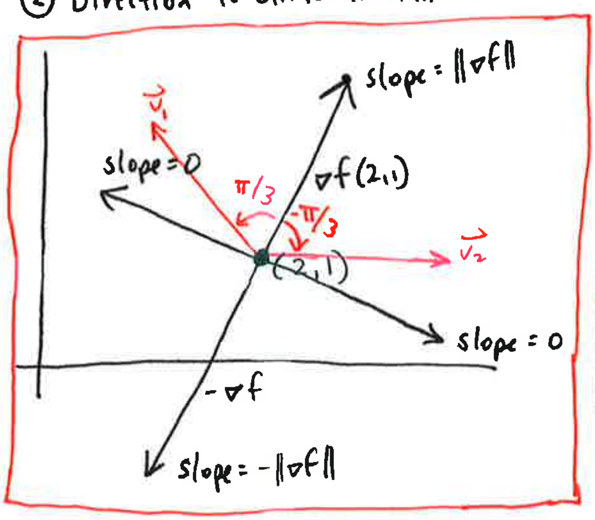
Today: The gradient is perpendicular to the level curves/surfaces, which gives us a new way to find the tangent plane to a surface, even one that isn't the graph of a surface (e.g. sphere).

Example. Let $f(x,y) = xy$. Find the direction of steepest ascent at $(2,1)$.
Then find the direction(s) you could go to climb at half this steepness.

① Direction of steepest ascent is ______________ so let's compute:

$\nabla f(x,y) = [\quad] \Rightarrow \nabla f(2,1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The rate of increase, or slope, in this direction, is ______.

② Direction to climb at half the maximum steepness: find direction $\vec{u}$ so that



$$\underbrace{D_{\vec{u}} f(2,1)}_{\text{same}} = \underbrace{\nabla f(2,1) \bullet \vec{u}}_{\text{same}} = \underbrace{\|\nabla f(2,1)\| \|\vec{u}\| \cos\theta}_{\text{same}} = \|\nabla f(2,1)\| \cos\theta$$

want $\frac{1}{2}$ maximum steepness: $= \frac{1}{2}\|\nabla f(2,1)\|$.

$\Rightarrow \cos\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$. So, rotate:

rotate by $\pi/3$:

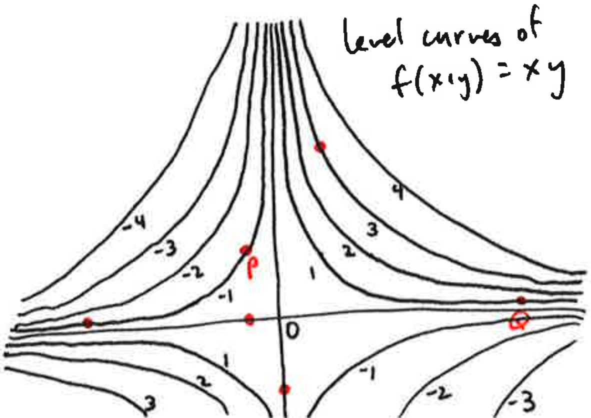
$$\begin{bmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \frac{1-2\sqrt{3}}{2\sqrt{5}} \\ \frac{2+\sqrt{3}}{2\sqrt{5}} \end{bmatrix} = \vec{v}_1$$

divide by $\sqrt{5}$ to make it a unit vector

rotate by $-\pi/3$:

$$\begin{bmatrix} \cos-\pi/3 & -\sin-\pi/3 \\ \sin-\pi/3 & \cos-\pi/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \frac{1+2\sqrt{3}}{2\sqrt{5}} \\ \frac{2-\sqrt{3}}{2\sqrt{5}} \end{bmatrix} = \vec{v}_2$$

Use level curves to sketch the gradient vector at some points. (these: •)



- $\nabla f(x,y)$ should be perpendicular to level curve
- point in direction of ascent
- be appropriately scaled.

which vector has greater magnitude, $\nabla f(P)$ or $\nabla f(Q)$? ______

why? ______________________

Diana Davis   Math 34   February 14, 2018   Class #11 ②

Why is the gradient perpendicular to the level sets (level curves, level surfaces)?

① ∇f points in the direction of steepest increase, and $-\nabla f$ points in the direction of steepest decrease, so the direction "between," perpendicular to both, has no change.

② The level curve of a function $f(x,y)$ at level k has equation $f(x,y)=k$.

Suppose the level curve has equation $\vec{r}(t)=\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$, so $f(x(t),y(t))=k$.

Differentiate both sides with respect to t:
$$\frac{d}{dt}\Big(f(x(t),y(t))\Big)=\frac{d}{dt}(k)$$
$$\Rightarrow \frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}=0$$
$$\Rightarrow \left[\partial f/\partial x,\ \partial f/\partial y\right]\cdot\left[dx/dt,\ dy/dt\right]=0 \Rightarrow \nabla f\cdot\begin{bmatrix}x'(t)\\y'(t)\end{bmatrix}=0 \Rightarrow \nabla f\cdot\vec{r}\,'(t)=0$$

tangent vector to the level curve!

<u>Example.</u> Consider the implicitly-defined curve $e^{xy}-xy=1$. Find the tangent line at $(0,1)$.

We need: ① a point on the line: ____

② the tangent vector to the curve. Set $F(x,y)=e^{xy}-xy$.

Our curve is the level set $F(x,y)=$ ____. So $\nabla F(0,1)$ is perpendicular to our desired tangent vector:

$$\nabla F(x,y)=\begin{bmatrix}\quad\end{bmatrix}\Rightarrow \nabla F(0,1)=\begin{bmatrix}\quad\end{bmatrix}\Rightarrow \text{tangent vector is }\begin{bmatrix}\quad\end{bmatrix}\text{ so }\begin{bmatrix}x(t)\\y(t)\end{bmatrix}=\begin{bmatrix}\quad\end{bmatrix}+\begin{bmatrix}\quad\end{bmatrix}t=\begin{bmatrix}\quad\end{bmatrix}.$$

<u>Example.</u> Find an equation for the tangent plane to $x^2+y^2+z^2=3$ at $(1,1,1)$.

The sphere is a level surface of $g(x,y,z)=x^2+y^2+z^2$ at level ____.

For a tangent plane, we need ① a point $(1,1,1)$ and ② a normal vector,

$\nabla g(1,1,1)$: $\nabla g(x,y,z)=\begin{bmatrix}2x\\2y\\2z\end{bmatrix}\Rightarrow \nabla g(1,1,1)=\begin{bmatrix}2\\2\\2\end{bmatrix}\Rightarrow$ equation is
$$\underline{\underline{2(x-1)+2(y-1)+2(z-1)=0.}}$$

<u>Example.</u> On the surface $xyz=8$, which points have a tangent plane parallel to $x+2y+4z=100$? $\rightarrow$ find points whose tangent vector to surface is multiple of $\begin{bmatrix}1\\2\\4\end{bmatrix}$.

View this surface as a level surface of $g(x,y,z)=$ ______ at level ____.

$$\nabla h(x,y,z)=\begin{bmatrix}yz\\xz\\xy\end{bmatrix}=\lambda\begin{bmatrix}1\\2\\4\end{bmatrix}\Rightarrow\begin{cases}yz=\lambda\\xz=2\lambda\\xy=4\lambda\end{cases}$$

gradient can be any (nonzero) multiple of $\begin{bmatrix}1\\2\\4\end{bmatrix}$

<u>notice:</u> ① x,y,z cannot be zero since $xyz=8$

② Solve to get $x^3=64\Rightarrow x=4$
$\Rightarrow y=2,\ z=1$

So $(4,2,1)$ is the unique (only) point whose tangent plane is parallel to $x+2y+4z=100$. ☺

Diana Davis Math 34 16 February 2018 Class #12 ①
§4.2

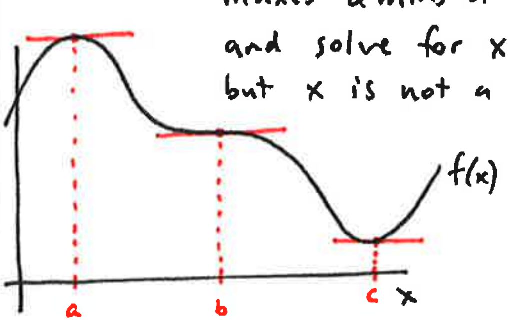
Midterm 1 a week from today in class - covering through class 11 / §2.6 / gradients.

Mathematician spotlight: Radia Perlman - math undergrad, CS Ph.D. (MIT)
- invented Spanning Tree Protocol - crucial to Internet ('70s)
- founding expert on network & security protocols

Last time: Using gradients to find · direction of greatest increase/decrease
· tangent lines & tangent planes to implicitly-defined $\begin{cases}\text{curves}\\\text{surfaces.}\end{cases}$

This time: Optimization in multivariable situations! Finding extrema (max/min) of a function!

Review: In single-variable calculus, to find the maxes & mins of $f(x)$, you set $f'(x)=0$ and solve for x. Sometimes $f'(x)=0$ but x is not a max or min, as at b.



You also need to check the endpoints if you're optimizing on a closed, bounded set.

New: In multivariable calculus, to find the maxes & mins of $f(x,y)$, you set $f_x=0$ and $f_y=0$ and solve for x and y. As in single-variable calculus, this sometimes picks up points that are not maxes or mins, and we have to check the boundary if optimizing over a closed, bounded set.

To find maxes & mins of a function, we find all the critical points, where $\begin{cases} f_x(x,y)=0 \\ f_y(x,y)=0\end{cases} \Leftrightarrow \nabla f(x,y)=\vec{0}$, and then classify them as a max, min or saddle.

Notice that at each critical point, the tangent plane is horizontal.

Example. Find all local extrema of $f(x,y)= 4x+6y-x^2-y^2-12$.

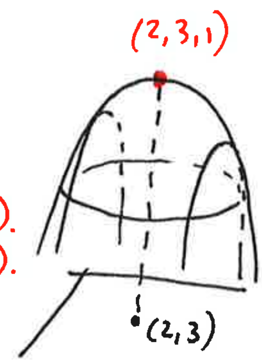
$\nabla f = \begin{bmatrix} 4-2x \\ 6-2y\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix} \Rightarrow \begin{cases} x=2 \\ y=3\end{cases}$ is the only critical point of f.

Is $(2,3)$ a max, min or saddle?

Hmm. $f(x,y) = -(x^2-4x+4)-(y^2-6x+9)+1$

$= \underbrace{-(x-2)^2-(y-3)^2+1}$

a downward-opening paraboloid whose maximum is at $(x,y)=(2,3)$. or $(x,y,z)=(2,3,1)$.



So $(2,3)$ gives a maximum value for f.

→ Being able to complete the square was very lucky! What do we do in general??

Isn't there a more systematic way to do this? Maybe a multivariable second derivative test? YES!

<u>Review:</u> second derivative test from single-variable calculus.

$f(x)$ - value of function - position

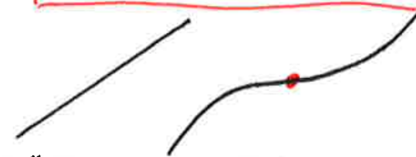
$f'(x)$ - rate of change of function - slope - velocity

$f''(x)$ - rate of change of slope - concavity - acceleration

If $f'(a)=0$, then
- $f''(a)>0 \Rightarrow a$ is a ______
- $f''(a)<0 \Rightarrow a$ is a ______
- $f''(a)=0 \Rightarrow$ ______

$f''(x)>0$ when the curve is concave-up

$f''(x)<0$ when the curve is concave-down



$f''(x)=0$ when the curve is linear, or instantaneously flat (infl. pt)

The multivariable second derivative test uses the "Hessian" matrix of partial derivatives:

$$Hf = \begin{bmatrix} f_{xx}(x,y) & f_{yx}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{bmatrix}$$

<u>Here's how it goes:</u> Set $\nabla f = \vec{0}$ and solve for critical points.

Let $\vec{a}$ be one of the critical points. Let's classify it.

$Hf(\vec{a})$ is a 2×2 matrix of numbers. It has 2 eigenvalues: λ_1, λ_2.

- if $\lambda_1, \lambda_2 > 0$, this corresponds to a local shape like a paraboloid opening up , so $\vec{a}$ is a local <u>min</u> for f.
- if $\lambda_1, \lambda_2 < 0$, this corresponds to a local shape like a paraboloid opening down , so $\vec{a}$ is a local <u>max</u> for f.
- if one is positive and one is negative, this corresponds to a shape like one parabola opening up and the other opening down, , like a hyperbolic paraboloid, so $\vec{a}$ is a saddle point.

<u>Example.</u> $f(x,y) = 4x + 6y - x^2 - y^2 - 12$ has one critical point, $\vec{a} = (2,3)$.

$$Hf(2,3) = \begin{bmatrix} \_ & \_ \\ \_ & \_ \end{bmatrix} \Rightarrow \lambda_1 = -2,\ \lambda_2 = -2,$$

so $(2,3)$ is a local <u>maximum</u> for f.

<u>Example.</u> The same works in higher dimensions, with criteria "all positive," "all negative" and "some pos, some neg."

$f(x,y,z) = xy + xz + 2yz - 1/x$

$$\nabla f(x,y,z) = \begin{bmatrix} y+z+1/x^2 \\ x+2z \\ x+2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \text{one critical point, } (1, -1/2, -1/2).$$

$$Hf = \begin{bmatrix} 2/x^3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \Rightarrow Hf(1,-1/2,-1/2) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

eigenvalues are $-2, -2, 2$

$\Rightarrow (1, -1/2, -1/2)$ is a <u>saddle point.</u>

Diana Davis   Math 34   19 February 2018   Class #13 ①
§4.2

Exam in class on Friday - through gradients / class 11 / §2.6

Mathematician spotlight: Kathryn Lindsey, Boston College (Williams math undergrad)
- dynamical systems, complex dynamics
- showed that you can get a Julia set in any shape, incl. cat.

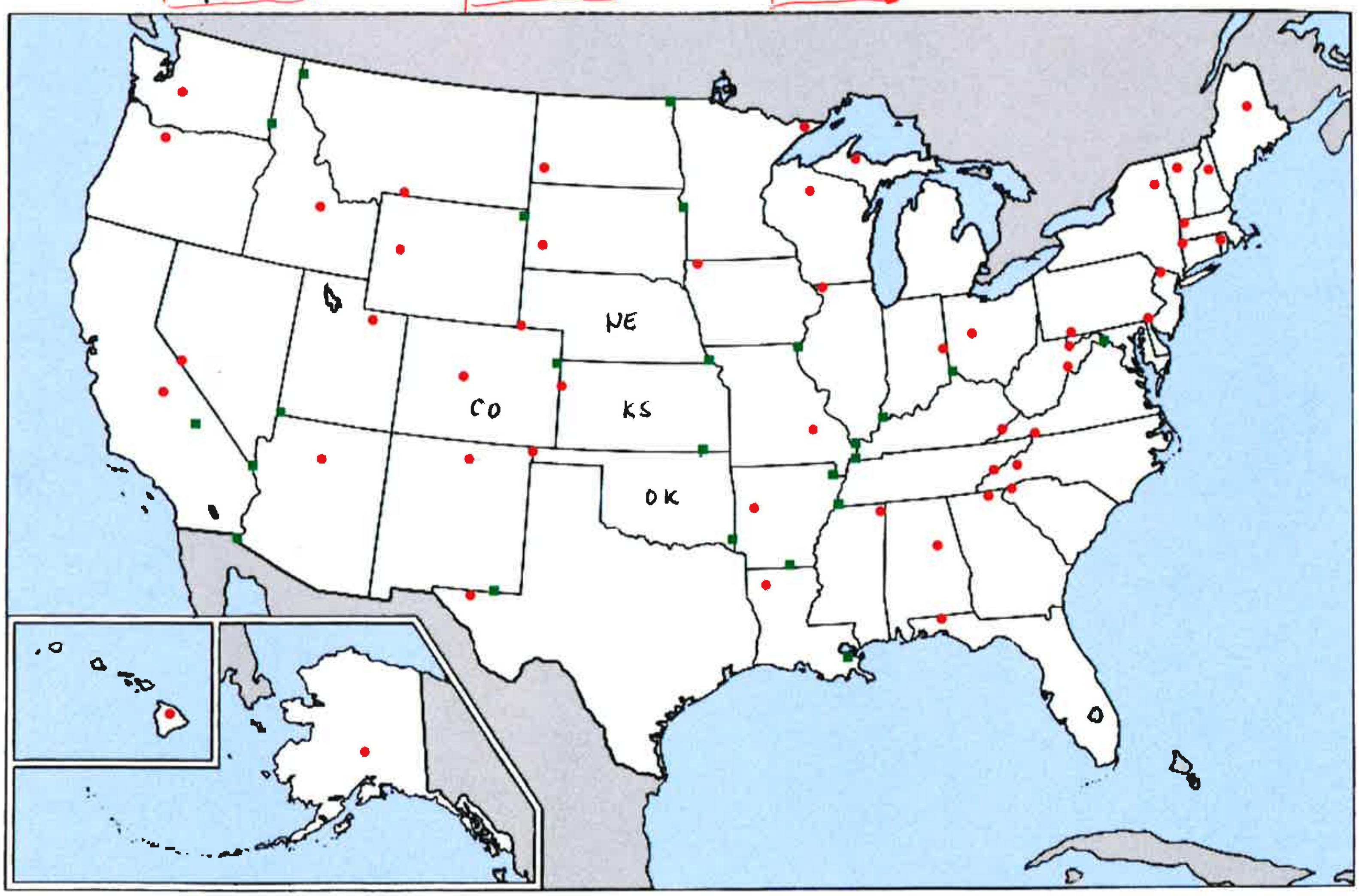
Last time: Find & classify critical points (where $\nabla f = 0$) of multiv. fcns using eigenvalues of Hf.

Today: Find absolute max & min of a function on a constrained (closed, bounded) region.

The map below marks the highest (red) & lowest (green) point of each U.S. state.

Choose 10 states, and for each one, say whether the highest point is:

at an interior point   along an edge   at a corner   somewhere else ?



Record your results & observations!

-
-
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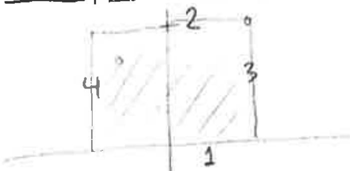
-

Starting from the high point of Colorado (CO), sketch in plausible level curves for elevation (topo lines) that result in the high point locations for NE, KS and OK.

Example. Find the absolute extrema of $f(x,y) = x^2 + xy + y^2 - 6y$ over the rectangle $-3 \le x \le 3$, $0 \le y \le 5$:



① Find critical points of f that are inside the region.

$$\nabla f = \begin{bmatrix} 2x+y \\ x+2y-6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} y=-2x \\ x+2y-6=0 \end{matrix} \Rightarrow x + 2(-2x) - 6 = 0 \Rightarrow -3x = 6 \Rightarrow x=-2 \Rightarrow y = 4.$$

so $(-2,4)$ is the only C.P.
notice that it is inside the rectangle.

② Check the four boundary components for critical points along the region.

1: $y=0$ so $f(x,0) = x^2$
$f'(x,0) = 2x = 0 \Rightarrow x = 0 \Rightarrow (0,0)$

notice that these are on the boundary of the rectangle.

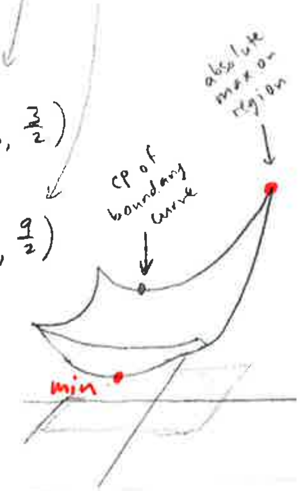
2: $y=5$ so $f(x,5) = x^2 + 5x + 25 - 30$
$f'(x,5) = 2x+5 = 0 \Rightarrow x = -\frac{5}{2} \Rightarrow (-\frac{5}{2}, 5)$

3: $x=3$ so $f(3,y) = 9 + 3y + y^2 - 6y$
$f'(3,y) = 3 + 2y - 6 = 2y - 3 = 0 \Rightarrow y = \frac{3}{2} \Rightarrow (3, \frac{3}{2})$

4: $x=-3$ so $f(-3,y) = 9 - 3y + y^2 - 6y$
$f'(-3,y) = -3 + 2y - 6 = 2y - 9 = 0 \Rightarrow y = \frac{9}{2} \Rightarrow (-3, \frac{9}{2})$

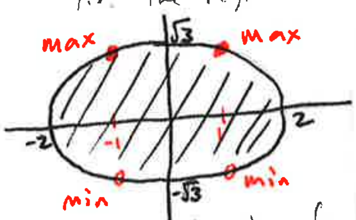
③ check the four corners, in case the extreme value is not a critical point of the boundary curve:

List of candidates	value of f there	
$(-2,4)$	-12	← min; interior
$(0,0)$	0	edges
$(-\frac{5}{2},5)$	-11.25	
$(3,\frac{3}{2})$	6.75	
$(-3,\frac{9}{2})$	-11.25	
$(3,0)$	9	corners
$(3,5)$	19	← max
$(-3,5)$	-11	
$(-3,0)$	9	



So the max value of f is 19, attained at $(3,5)$, and the min value of f is -12, at $(-2,4)$.

Example. Find the absolute extrema of $f(x,y) = x^2 y$ over the region $3x^2 + 4y^2 \le 12$, i.e. the region inside (and including the boundary) of the ellipse $3x^2 + 4y^2 = 12$.



① Find critical points of f that are inside the region.

$$\nabla f = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = 0 \Rightarrow \text{the entire } y\text{-axis is critical points of } f.$$

② Find critical points of the function restricted to the boundary:

Method 1: $3x^2 + 4y^2 = 12 \Rightarrow x^2 = \frac{12 - 4y^2}{3} = 4 - \frac{4}{3}y^2$

So on the boundary, $f(x,y) = x^2 y$ is just
$f(y) = (4 - \frac{4}{3}y^2)y = 4y - \frac{4}{3}y^3$
$f'(y) = 4 - 4y^2 = 0 \Rightarrow y = \pm 1 \Rightarrow 3x^2 + 4 = 12$
$\Rightarrow x^2 = \frac{8}{3}$
$\Rightarrow x = \pm\sqrt{8/3}$

Method 2: Parameterize the boundary:

$x = 2\cos t$, $y = \sqrt{3}\sin t$ $\Rightarrow f(x,y) = f(t) = (2\cos t)^2 \cdot \sqrt{3}\sin t = 4\sqrt{3}\cos^2 t \sin t$

$\Rightarrow f'(t) = 4\sqrt{3}(-2\cos t \sin^2 t + \cos^3 t) = 0$
$\Rightarrow -2\cos t \sin^2 t + \cos^3 t = 0$
$\Rightarrow -2\cos t(1 - \cos^2 t) + \cos^3 t = 0$
$\Rightarrow -2\cos t + 2\cos^3 t + \cos^3 t = 0$
$\Rightarrow \cos t(-2 + 3\cos^2 t) = 0$ $3\cos^2 t = 2$
$\Rightarrow \underbrace{\cos t = 0}_{\Rightarrow x = 0,\ y = \pm\sqrt{3}}$ or $\underbrace{\cos t = \sqrt{2/3}}$ $\Rightarrow \sin t = \pm\sqrt{1 - (\sqrt{2/3})^2} = \pm\sqrt{\frac{1}{3}}$

(ALREADY HAVE)

$\Rightarrow x = 2\sqrt{2/3} = \sqrt{8/3}$, $y = \sqrt{3} \cdot \pm\sqrt{\frac{1}{3}} = \pm 1$ } as in Method 1

③ check corners: no corners!

List of candidates	value of f there	
y-axis $(0,y)$	0	
$(\sqrt{8/3}, 1)$	$8/3$	← max
$(-\sqrt{8/3}, 1)$	$8/3$	← max
$(\sqrt{8/3}, -1)$	$-8/3$	← min
$(-\sqrt{8/3}, -1)$	$-8/3$	← min

↳ So the absolute max of f is $\sqrt{8/3}$, attained at $(1, \sqrt{8/3})$ and $(-1, \sqrt{8/3})$, and the absolute min of f is $-\sqrt{8/3}$, attained at $(1, -\sqrt{8/3})$ and $(-1, -\sqrt{8/3})$.

Diana Davis Math 34 21 February 2018 Class #14 ① §4.3

<u>Today:</u> office hours 1:00-2:30, 3:00-4:15; review 8pm Sci 101

<u>Friday:</u> Midterm 1 in class

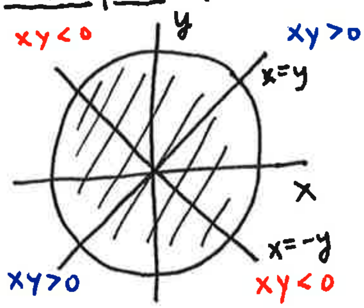
<u>Mathematician Spotlight:</u> Kwadwo Antwi-Fordjour, Earlham College

- modeling head re-growth of hydra using differential equations
- speaking TODAY 4:30 pm in Sci 181 (VAP job candidate)

<u>Last time:</u> To find absolute extrema of a function on a bounded region, you have to check for critical points on interior, critical pts of boundary curve(s), and corners.

<u>Today & Monday:</u> Absolute extrema <u>on</u> a constraint curve using <u>Lagrange multipliers.</u>

<u>Example.</u> (review using strategy from last time) Find absolute extrema of $f(x,y)=xy$ over the closed unit disk $x^2+y^2\le 1$.



① Find critical points on interior by setting $\nabla f=\vec{0}$:

$$\nabla f=\begin{bmatrix} y\\ x\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow (0,0) \text{ is the only critical point.}$$

② Find critical points of f on the boundary curve:

boundary curve is unit circle: $\begin{cases} x=\cos\theta \\ y=\sin\theta\end{cases}$ so on the boundary, $f(x,y)=xy$

$\Rightarrow f(\theta)=\cos\theta\cdot\sin\theta$

$f'(\theta)=-\sin^2\theta+\cos^2\theta=0 \Rightarrow \cos^2\theta=\sin^2\theta$

$\Rightarrow x^2=y^2.$

$\Rightarrow x=\pm y$

$\Rightarrow \left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right).$

Now check value of f at these points:

$f(0,0)=0$ ← neither a max nor a min

$f\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)=f\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)=\frac{1}{2}$ ← two equal maxima

$f\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)=f\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)=-\frac{1}{2}$ ← two equal minima

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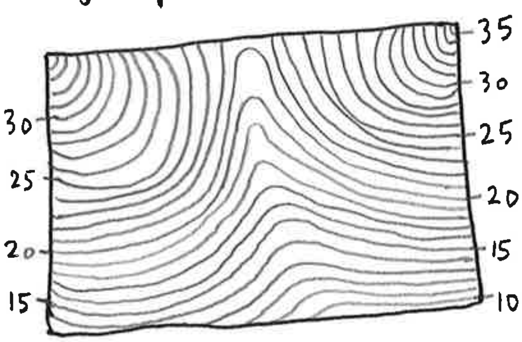
Lagrange's great idea: a new strategy for finding critical points of f on the boundary.



← This is a topographical (elevation) map of a piece of land.

Draw a lumpy blob outline. This is the fence that encloses your herd of sheep.

Find the points of highest & lowest elevation of the fence. How did you find them? ____________

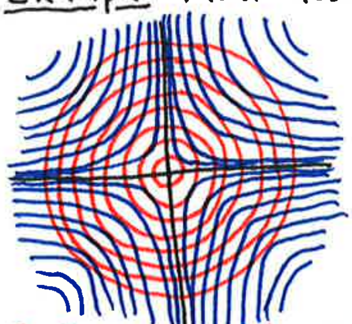
<u>Lagrange's idea:</u> At a critical point of the function $f(x,y)$ on the constraint $g(x,y)=k$,

- the boundary curve g is tangent to a level curve of f
- ∇f and ∇g point in the same direction: they are multiples of each other.

So, to optimize $f(x,y)$ or $f(x,y,z)$ subject to constraint $g(x,y)=k$ or $g(x,y,z)=k$, solve the Lagrange multipliers equation $\nabla f(x,y)=\lambda\cdot\nabla g(x,y).$

← lambda: some constant

Example. Find absolute extrema of $f(x,y)=xy$ on the unit circle. (same as before)



We wish to maximize/minimize the function $f(x,y)=$ ______

subject to the constraint $g(x,y)=$ ______ $=$ ___.

By the Lagrange multipliers equation,

$$\nabla f=\lambda\cdot\nabla g \Rightarrow \begin{bmatrix} y \\ x \end{bmatrix}=\lambda\begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow \underbrace{\begin{matrix} y=\lambda\cdot 2x \\ x=\lambda\cdot 2y \\ x^2+y^2=1. \end{matrix}}_{\text{3 eqns, 3 variables}}$$

Usual strategy: find a way to eliminate λ, because we don't care what it is.

Idea: multiply 1st equation by y, 2nd equation by x, so RHS will be equal.

$$\left.\begin{matrix} y^2=\lambda\cdot 2xy \\ x^2=\lambda\cdot 2xy \end{matrix}\quad \begin{matrix} \Rightarrow y^2=x^2. \\ \Rightarrow y=\pm x. \end{matrix}\right\}$$

now, this tells us that the places where the constraint curves of the form $x^2+y^2=r^2$ are tangent to the level curves of the function $g(x,y)=k$ occur along the lines $y=\pm x$.

Can you see this in the picture above?

Finally, plug this into the constraint equation to solve for points:

$y=x$ and $x^2+y^2=1$ $\Rightarrow$ $(\sqrt{2}/2, \sqrt{2}/2)$ and $(-\sqrt{2}/2, -\sqrt{2}/2)$ maxes as before.

$y=-x$ and $x^2+y^2=1$ $\Rightarrow$ $(\sqrt{2}/2, -\sqrt{2}/2)$ and $(-\sqrt{2}/2, \sqrt{2}/2)$ mins

---

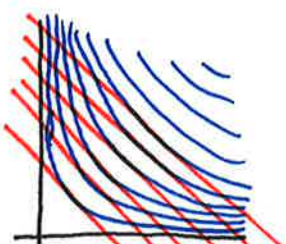
Example. Find the largest possible product of three numbers whose sum is 100.

→ not possible because ______

How about: Find the largest possible product of three positive numbers whose sum is 100.

Let the three numbers be x,y,z. We seek to maximize $f(x,y,z)=$ ______

under the constraint $g(x,y,z)=$ ______ $=$ ___.



By the Lagrange multipliers equation, $\nabla f=\lambda\cdot\nabla g$, so

$$\begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}=\lambda\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \underbrace{\begin{matrix} yz=\lambda \\ xz=\lambda \\ xy=\lambda \\ x+y+z=100 \end{matrix}}_{\text{4 eqns, 4 variables}}$$

$yz=xz$: $\frac{y\cancel{z}}{\cancel{z}}=\frac{x\cancel{z}}{\cancel{z}} \Rightarrow x=y$. we can divide by z because ______

$xz=xy$: $\frac{\cancel{x}z}{\cancel{x}}=\frac{\cancel{x}y}{\cancel{x}} \Rightarrow z=y$.

together, $x=y=z$.

This is where the constraint & level curves are tangent.

Now plug into our constraint: $x+y+z=100 \Rightarrow x+x+x=100 \Rightarrow x=\frac{100}{3}$. $\Rightarrow$ numbers are all $100/3$.

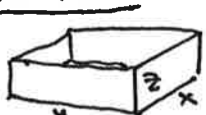
How do we know it's a max, not a min? $f(\frac{100}{3},\frac{100}{3},\frac{100}{3})\approx 37037$

$f(33,33,34)=37026$ ← function value at nearby point is lower, so it's a max!

Example. Find the largest volume of a rectangular box whose length, width, height sum to 100.

→ Did that!

How about: Find the largest volume of an open-top rectangular box made of 12 ft^3 of cardboard.



We wish to maximize $f(x,y,z)=$ ______

subject to constraint $g(x,y,z)=$ ______ $=$ ___.

so $\nabla f=\lambda\nabla g \Rightarrow \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}=\lambda\begin{bmatrix} y+2z \\ x+2z \\ 2x+2y \end{bmatrix}$

... etc.

Diana Davis — Math 34 — 26 February 2018 — Class #15 ① §4.3

Mathematician spotlight: Nsoki Mamie Mavinga - Associate* Professor, Swarthmore
- * Just promoted to Associate (so, tenured now) on Saturday!
- Studies nonlinear differential equations

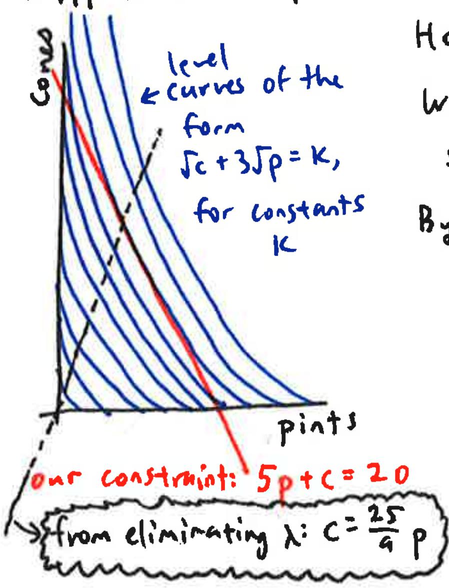
Last time: Introduction to Lagrange multipliers, for optimizing a function under a constraint.

Today: Fun applications, optimized using Lagrange multipliers!

Example from economics: Suppose that the amount of happiness (h) that you derive from consumption of ice cream pints (p) and cones (c) is described by the equation

$h(p,c) = \sqrt{c} + 3\sqrt{p}$ (an additional pint makes you happier than an additional cone).

Suppose that pints are \$5, cones are \$1, and you have \$20 to spend on them.



How many pints and cones should you buy, to maximize happiness???

We wish to maximize $h(p,c) =$ ________

subject to the constraint $g(p,c) =$ ________ $=$ ____.

By the Lagrange multipliers equation, $\nabla h = \lambda \nabla g$, so

$$\begin{bmatrix} \frac{1}{2}c^{-1/2} \\ \frac{3}{2}p^{-3/2} \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 5 \end{bmatrix} \Rightarrow \left.\begin{array}{l} \frac{1}{2\sqrt{c}} = \lambda \\ \frac{3}{2\sqrt{p}} = 5\lambda \end{array}\right\} \Rightarrow \frac{1}{2\sqrt{c}} = \frac{3}{10\sqrt{p}} \Rightarrow 10\sqrt{p} = 6\sqrt{c}$$

$$\Rightarrow 100p = 36c$$

$$\Rightarrow 25p = 9c$$

$$\underbrace{5p + c = 20}_{\text{3 eqns, 3 vars}}$$

$c = \frac{25}{9}p \approx 3p$ ← this is the happiness-maximizing ratio of pints to cones. Now plug into our personal budget constraint.

budget: $5p + c = 20$

$c = 20 - 5p$

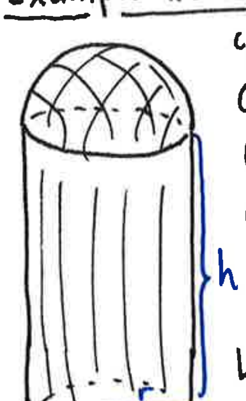
Lagrange multipliers: $25p = 9c = 9(20-5p) \Rightarrow 70p = 180 \Rightarrow p = \frac{18}{7} \approx \boxed{2.6 \text{ pints}}$

$\Rightarrow c = \frac{50}{7} \approx \boxed{7.1 \text{ cones.}}$

Is this actually a max, or perhaps a rogue min? $h(2.6, 7.1) = 7.5$ happiness

try a nearby point satisfying constraint: $h(2,10) = 7.4$ happiness ⇒ yay, a max indeed!

Example from construction: Suppose you are building a silo to store grain, which will be a cylinder with a hemispherical top, as shown. You have 600π square feet of corrugated aluminum to construct the sides and top (the floor is pre-existing). What dimensions should you make it to maximize the volume of storage, which is the volume of the cylindrical part (not the hemispherical part)?



circ of circle $= 2\pi r$
vol of cyl $= \pi r^2 h$
S.A. of sphere $= 4\pi r^2$

We wish to maximize $f(r,h) =$ ________

under the constraint $g(r,h) =$ ________ $=$ ______

Lagrange multipliers equation: $\nabla f = \lambda \nabla g$

$$\Rightarrow \begin{bmatrix} 2\pi r h \\ \pi r^2 \end{bmatrix} = \lambda \begin{bmatrix} 2\pi h + 4\pi r \\ 2\pi r \end{bmatrix} \Rightarrow \begin{array}{l} \cancel{2\pi} rh = \lambda(\cancel{2\pi}h + \cancel{4\pi}^{2} r) \\ \cancel{\pi} r^2 = \lambda \cdot 2\cancel{\pi} r \end{array} \Rightarrow \left.\begin{array}{l} rh = \lambda(h+2r) \\ r^2 = \lambda \cdot 2r \\ 2\pi rh + 2\pi r^2 = 600\pi \end{array}\right\}$$

From the Lagrange multipliers equation:

① $rh = \lambda(h+2r)$

② $r^2 = \lambda \cdot 2r$ ← we know $r \neq 0$ at a maximum, since that would give zero volume, so solve: $\lambda = \frac{r^2}{2r} = \frac{r}{2}$. Plug into ①: $rh = \frac{r}{2}(h+2r)$

$2rh = r(h+2r)$

$2rh = rh + 2r^2$

$rh = 2r^2$ again, $r \neq 0$, so

$h = 2r$ ← volume-maximizing proportion!

Now plug this volume-maximizing proportion $h=2r$ into constraint $\cancel{2\pi} rh + \cancel{2\pi} r^2 = \cancel{600}\,\cancel{\pi}$ 300

simplify to $rh + r^2 = 300$

$r(2r) + r^2 = 300$

$3r^2 = 300 \Rightarrow \boxed{r = 10 \Rightarrow h = 20.}$

Check that this is indeed a max: volume $(r=10, h=20) = \pi \cdot 10^2 \cdot 20 = 2000\pi\ ft^3$ ← max! :)

compute at a nearby point on constraint: volume$(r=15, h=5) = \pi \cdot 15^2 \cdot 5 = 1125\pi\ ft^3$

Suppose that we allowed the hemispherical top to also be filled with grain. What would be the volume-maximizing radius and height then?

we wish to maximize $f(r,h) = \underbrace{\pi r^2 h}_{\text{cylinder}} + \underbrace{\frac{2}{3}\pi r^3}_{\text{hemisphere}}$

subject to constraint $g(r,h) = 2\pi rh + 2\pi r^2 = 600\pi$.

$\nabla f = \lambda \nabla g \Rightarrow \begin{bmatrix} 2\pi rh + 2\pi r^2 \\ \pi r^2 \end{bmatrix} = \lambda \begin{bmatrix} 2\pi h + 4\pi r \\ 2\pi r \end{bmatrix} \Rightarrow$ $\cancel{2\pi} rh + \cancel{2\pi} r^2 = \lambda(\cancel{2\pi} h + \cancel{4\pi}^{2} r)$

$\cancel{\pi} r^2 = \lambda(2\cancel{\pi} r)$

$\Rightarrow rh + r^2 = \lambda(h+2r)$ ①

$r^2 = \lambda(2r) \longrightarrow$ again, $r \neq 0$ so $\lambda = \frac{r^2}{2r} = \frac{r}{2}$ $\Rightarrow rh + r^2 = \frac{r}{2}(h+2r)$ ②

$2rh + 2r^2 = r(h+2r)$

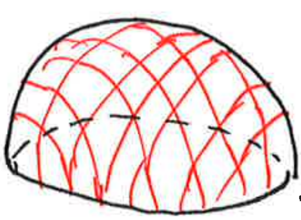
$2rh + \cancel{2r^2} = rh + \cancel{2r^2}$

$rh = 0 \Rightarrow \underbrace{r=0}$ or $\boxed{h=0.}$?!

not possible because then volume $=0$.

So Lagrange multipliers says volume is maximized when $h=0$.

Is this a mistake? How could this be?



$h=0$

← Oh, right! A hemisphere is volume-maximizing for given surface area when the base is "free."

This is why soap bubbles, solving the reverse problem (minimizing surface area for fixed volume of air) make hemispheres on a soapy surface.

Application to animal husbandry: Suppose that you wish to make a rectangular pen along the side of a building, that encloses $32\ m^2$ of area with minimum fencing. How to do it?

minimize $f(x,y) = 2x + y$

subject to $g(x,y) = xy = 32$

$\nabla f = \lambda \nabla g \Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} y \\ x \end{bmatrix} \Rightarrow \left.\begin{matrix} 2 = \lambda y \\ 1 = \lambda x \\ xy = 32 \end{matrix}\right\}$ 3 eqns 3 vars

Mult. ① by x: $2x = \lambda y x$

mult. ② by y: $1y = \lambda xy$

$\Rightarrow 2x = \lambda xy = 1y \Rightarrow \boxed{2x = y}$ fence-minimizing proportions

Plug into constraint: $xy = 32$

$x(2x) = 32 \Rightarrow 2x^2 = 32 \Rightarrow \boxed{x = 4 \Rightarrow y = 8.}$

Again, check nearby to ensure max, not min.

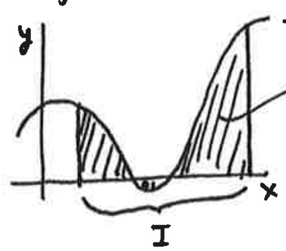
Mathematician spotlight: Erik Demaine - Professor, MIT (electrical eng. & C.S.)
- PhD thesis proved the "Fold and One Cut Theorem"
- child prodigy; works with father Martin; creates art

Theme of course: Take ideas from single-variable calculus, generalize to multivariable calculus.

So far, we've done this with: functions, limits, derivatives

Today, we begin: integrals!

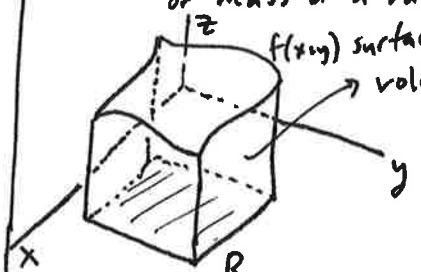
Single-variable integration: area under a curve



Area $= \int_I f(x)\,dx$

where area below the x-axis is taken to be negative

multivariable integration: volume under a surface or mass of a varying-density material



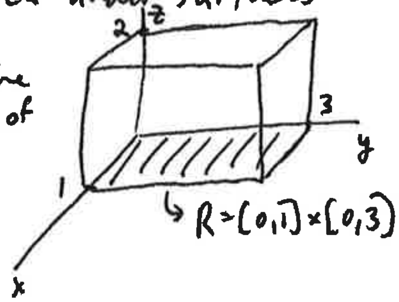
volume $= \iint_R f(x,y)\,dA$

where volume below the xy-plane is taken to be negative

Examples: finding area under surfaces

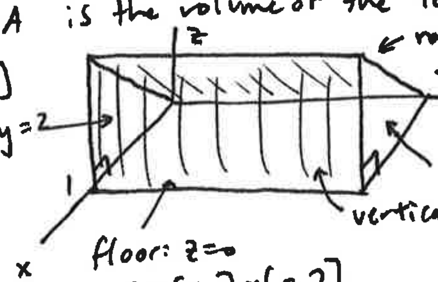
$\iint_{[0,1]\times[0,3]} 2\,dA$ ← this is the volume of

↑ x between 0 and 1; ↖ y between 0 and 3



So the volume of this box $= \iint_{[0,1]\times[0,3]} 2\,dA =$ ______.

$\iint_{[0,1]\times[0,2]} x\,dA$ is the volume of the "lean-to shed"



The volume is (area of Δ side) × (length) $= \frac{1}{2} \times 2 = 1$

or half of (rectangular box) $= \frac{1}{2} \times 1 \times 1 \times 2 = 1$.

so $\iint_{[0,1]\times[0,2]} x\,dA = 1$.

It is elegant to use geometry to find volumes, but it is often not possible; we have to use calculus to compute it algebraically. We use a double integral: first do the inner, then the outer.

$$\iint_{[0,1]\times[0,2]} x\,dA = \int_{y=0}^{2}\int_{x=0}^{1} x\,dx\,dy = \int_{y=0}^{y=2}\underbrace{\left(\int_{x=0}^{x=1} x\,dx\right)}_{\text{do this first}}dy = \int_{y=0}^{y=2}\left(\frac{x^2}{2}\Big|_{x=0}^{x=1}\right)dy = \int_{y=0}^{y=2}\left(\frac{1}{2}\right)dy = \frac{1}{2}y\Big|_{y=0}^{y=2} = 1-0 = 1.$$

or we could integrate first with respect to y:

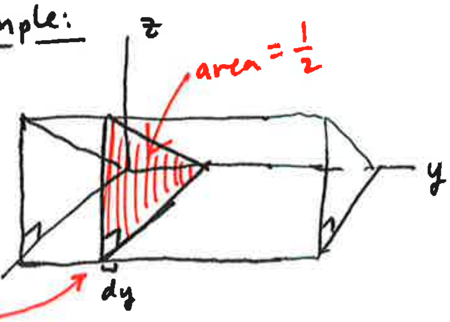
$$\iint_{[0,1]\times[0,2]} x\,dA = \int_{x=0}^{x=1}\int_{y=0}^{y=2} x\,dy\,dx = \int_{x=0}^{x=1}\underbrace{\left(\int_{y=0}^{y=2} x\,dy\right)}_{\text{do this first}}dx = \underbrace{\int_{x=0}^{x=1}\left(xy\Big|_{y=0}^{y=2}\right)dx}_{x\text{ is a constant with respect to }y} = \int_{x=0}^{x=1}(2x)\,dx = x^2\Big|_{x=0}^{x=1} = 1-0 = 1.$$

What do the inner and outer integrals <u>mean</u>?

- the inner integral gives the area of a slice of the region.
- the outer integral gives each slice a tiny thickness and sums up the volumes.

<u>For our shed example:</u>

$$\int_{y=0}^{y=2} \underbrace{\left(\int_{x=0}^{x=1} x\,dx\right)}_{\text{area of slice of volume for a fixed } y} dy:$$

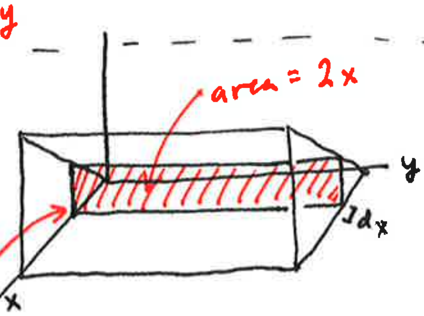


volume of solid
= sum of (area of slice) × (tiny thickness dy)

$$= \int_{y=0}^{y=2} \frac{1}{2}\,dy = \frac{1}{2}y\Big|_{y=0}^{y=2} = \frac{1}{2}\cdot 2 = 1.$$

In this case, the area of each slice is the same, $\frac{1}{2}$. However, the area may also depend on the outer variable of integration:

$$\int_{x=0}^{x=1} \underbrace{\left(\int_{y=0}^{y=1} x\,dy\right)}_{\text{area of slice of volume for a fixed } x} dx$$



volume of solid
= sum of (area of slice) × (tiny thickness dx)

$$= \int_{x=0}^{x=1} 2x\,dx = x^2\Big|_{x=0}^{x=1} = \underline{\underline{1}}.$$

If our region of integration is of the form $[a,b]\times[c,d]$, we can integrate in either order, as in the example above. It always works out the same in both ways:

<u>Fubini's Theorem:</u> If $f(x,y)$ is a sufficiently well-behaved function (as ours will always be), then
$$\iint_{[a,b]\times[c,d]} f(x,y)\,dA = \int_{y=c}^{y=d}\int_{x=a}^{x=b} f(x,y)\,dx\,dy = \int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dy\,dx.$$

> For a double integral over a rectangular region, you can change the order of integration.

<u>Why would we want to change it?</u> To turn an impossible integral into another order, which may be possible.

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=1} y\cdot\underbrace{\sin(x^2)\,dx}\,dy = \int_{x=0}^{x=1}\int_{y=-1}^{y=1} y\cdot\sin(x^2)\,dy\,dx = \int_{x=0}^{x=1}\left(\frac{y^2}{2}\sin(x^2)\Big|_{y=-1}^{y=1}\right)dx = \int_{x=0}^{x=1}\left(\frac{1}{2}\sin(x^2)-\frac{1}{2}\sin(x^2)\right)dx$$

$\sin(x^2)$ has no antiderivative with respect to x, so we are stuck. change the order!

$$= \int_{x=0}^{x=1} 0\,dx = \underline{\underline{0}}.$$

So the other order was not only possible, it was in the end quite easy.

Diana Davis Math 34 2 March 2018 Class #17 ①
§5.2

Mathematician spotlight: Autumn Kent, Associate Professor, Univ. of Wisconsin

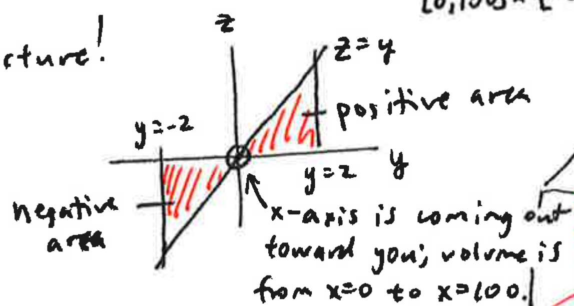
- Studies geometry, topology, dynamical systems, knot theory
- Knot theory works on answering the (HARD!) question of when two given knots are the same.

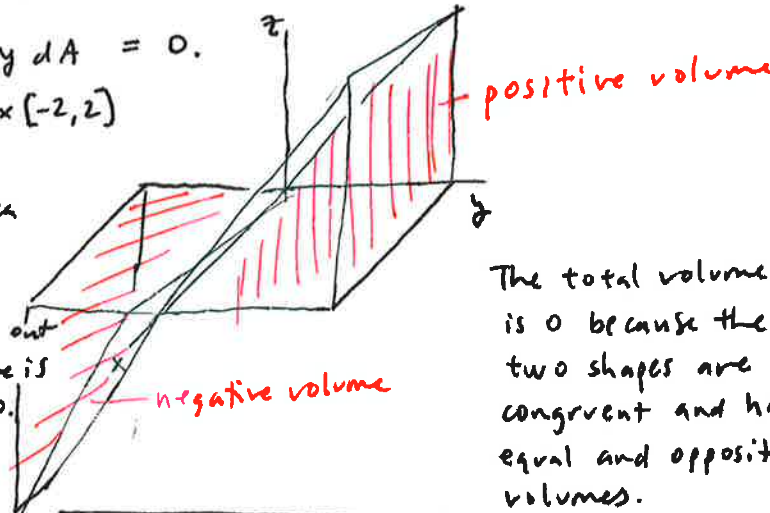
Last time: We introduced double integrals, computed them geometrically & algebraically, and stated Fubini's Theorem: ______

Today: More examples, plus the Riemann sum definition of double integrals.

Example: Justify geometrically why $\iint_{[0,100]\times[-2,2]} y \, dA = 0.$

Draw a picture!





The total volume is 0 because the two shapes are congruent and have equal and opposite volumes.

Example. Find the volume below the surface $z = xe^{xy}$ above $[1,2]\times[1,3]$ in the xy-plane.

This volume is given by $\iint_{[1,2]\times[1,3]} x e^{xy} \, dA$. Let's compute it in both orders:

$$\int_{x=1}^{x=2}\left(\int_{y=1}^{y=2} x e^{xy}\, dy\right) dx = \int_{x=1}^{x=2}\left(e^{xy}\Big|_{y=1}^{y=3}\right) dx = \int_{x=1}^{x=2}\left(e^{3x}-e^{x}\right)dx = \frac{1}{3}e^{3x}-e^{x}\Big|_{x=1}^{x=2} = \left(\frac{1}{3}e^{6}-e^{2}\right)-\left(\frac{1}{3}e^{3}-e\right)$$

$$= \underline{\underline{\frac{1}{3}e^{6}-\frac{1}{3}e^{3}-e^{2}+e.}}$$

x is a constant with respect to y

Now the other order:

$$\int_{y=1}^{y=3}\underbrace{\left(\int_{x=1}^{x=2} x e^{xy}\, dx\right)}_{\text{need integration by parts!}} dy = \int_{y=1}^{y=3}\left(\frac{x}{y}e^{xy}-\frac{1}{y^2}e^{xy}\Big|_{x=1}^{x=2}\right)dy = \int_{y=1}^{y=3}\left(\frac{2}{y}e^{2y}-\frac{1}{y^2}e^{2y}-\frac{1}{y}e^{y}+\frac{1}{y^2}e^{y}\right)dy$$

→ we cannot solve this, even with integration by parts. Conclusion: other order was easier!

$\boxed{\int u\, dv = uv - \int v\, du}$

Let $u=x$, $dv = e^{xy}\,dx$

$\Rightarrow du = dx$, $v = \frac{1}{y}e^{xy}$

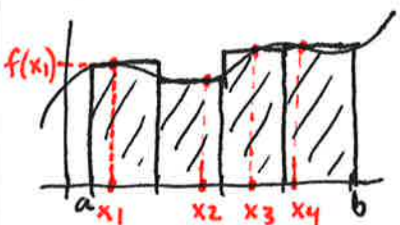
then

$$\Rightarrow \int x e^{xy}\,dx = \frac{x}{y}e^{xy} - \int \frac{1}{y}e^{xy}\,dx = \underbrace{\frac{x}{y}e^{xy} - \frac{1}{y^2}e^{xy}}_{\text{use this above}}$$

OK, now that we understand what double integrals do to find the volume under a surface, let's back up and define them more rigorously.

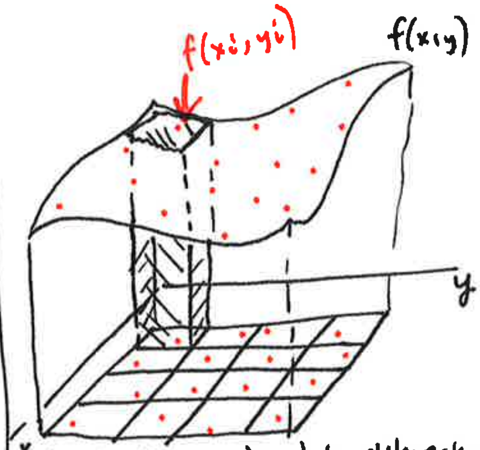
Diana Davis Math 34 2 March 2018 Class #17 ②

Single-variable Riemann sum:



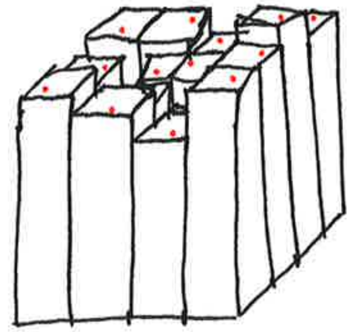
To compute $\int_a^b f(x)\,dx$, break the interval into subintervals, choose an x-value x_i in each, find each $f(x_i)$, and add up all the areas $f(x_i)\Delta x$ to get an area estimate. As $\Delta x \to 0$, rectangle area $\to \int_a^b f(x)\,dx$.

multivariable Riemann sum:

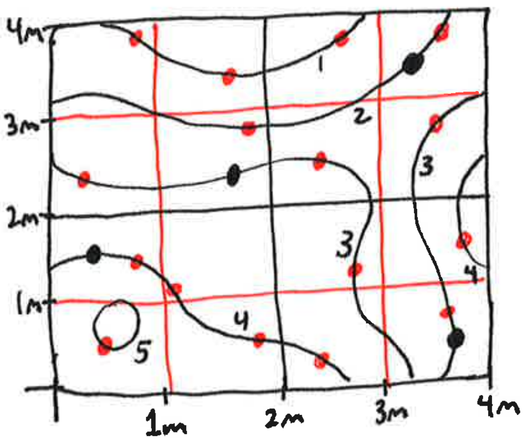


Break the region into subrectangles and choose an (x_i, y_i) point in each. Add up the volumes $f(x_i, y_i)\Delta x \Delta y$ of boxes to get volume.

total volume of boxes approximates volume under surface. Becomes more accurate as rects become thinner.



Example. The map shows level curves for elevation above bedrock of granite at a new quarry site. Estimate the total volume of granite in the quarry.



① First, let's break the region into four 2x2 squares, and choose a sample point in each (black).

Riemann sum = $\sum f(\text{sample point}) \times (\text{area of rectangle})$
$= 3\times4 + 2\times4 + 4\times4 + 3\times4 = 48\text{m}^3$.

② Now let's break the region into sixteen 1x1 squares, and choose a sample point in each (red)

Riemann sum = $\sum f(\text{sample point}) \times (\text{area of rectangle})$
$= 1\cdot1 + 1\cdot1 + 1\cdot1 + 2\cdot1 + 3\cdot1 + 2\cdot1 + 3\cdot1 + 3\cdot1 + 4\cdot1 + 4\cdot1 + 3\cdot1 + 4\cdot1 + 5\cdot1 + 4\cdot1 + 4\cdot1 + 3\cdot1 = 47\text{m}^3$.

With finer rectangles and finer level curves, we would get even better approximations.

Integrability: Integrals are defined as limits of Riemann sums. What if the limit doesn't exist??

A function is integrable over a rectangle R if the limit of Riemann sums used to define the double integral exists.

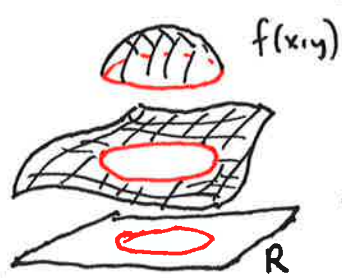
Example: let $f(x,y) = \begin{cases} \frac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

For any rectangle R containing (0,0), f is not integrable over R, because f(sample point) can be as large as you wish, so the limit does not exist.

Theorem: If f is bounded, and the set of points where f is not continuous has zero area, then f is integrable over such a region.

We usually consider continuous functions that are bounded over our region, so we won't worry much about integrability.

Example:



f is bounded over R, and f is only discontinuous on the circle, which is a curve, which has area 0.
$\Rightarrow$ f is integrable on R.

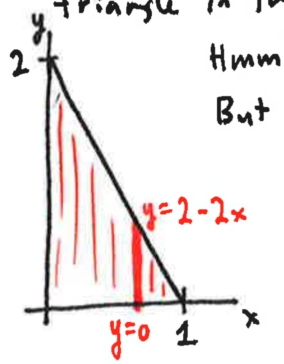
§5.2

Mathematician spotlight: Rodrigo Treviño, Assistant Professor, Univ. of Maryland
- ergodic theory, dynamical systems, geometry, math physics
- ergodic theory studies how "well mixed" a system becomes

Past two classes: Double integrals over rectangular regions; Riemann sums

Today: Double integrals over general regions - circles, triangles, anything you want!

Example: Find the volume of the solid under the surface $f(x,y)=2xy$, above the triangle in the xy-plane bounded by the x and y axes and the line $y=2-2x$.



Hmm! If we were integrating over the entire rectangle, it would be $\int_{x=0}^{x=1}\int_{y=0}^{y=1}(2-2x)\,dy\,dx$

But y doesn't go all the way to 2 - only up to $2-2x$.

So we compute:

$$\int_{x=0}^{x=1}\int_{y=0}^{y=2-2x} 2xy\,dy\,dx = \int_{x=0}^{x=1}\left(xy^2\Big|_{y=0}^{y=2-2x}\right)dx = \int_{x=0}^{x=1} x(2-2x)^2\,dx$$

$$= \int_{x=0}^{x=1}(4x-8x^2+4x^3)\,dx = 2x^2-\frac{8}{3}x^3+x^4\Big|_{x=0}^{x=1} = 2-\frac{8}{3}+1 = \frac{1}{3}.$$

For a region D in the xy-plane and a function $f(x,y)$,

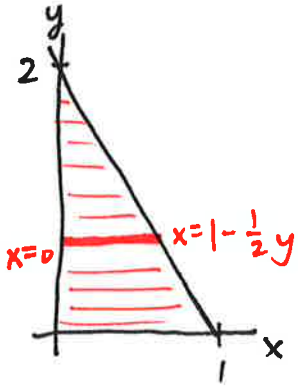
$$\iint_D f(x,y)\,dA = \int_{\text{min value of outer variable}}^{\text{max value of outer variable}} \int_{\text{lower bound of inner variable in terms of outer variable}}^{\text{upper bound of inner variable in terms of outer variable}} f(x,y)\,d(\text{inner})\,d(\text{outer}).$$

outer bounds are ALWAYS CONSTANTS

inner bounds can ONLY depend on the OUTER variable

Example. Re-do the example above, in the other order of integration ($dx\,dy$).

To find the upper bound for x in terms of y, we write down the equation of the line ($y=2-2x$) and solve for x: $x=1-\frac{1}{2}y$.



$$\iint_D 2xy\,dA = \int_{y=0}^{y=2}\int_{x=0}^{x=1-\frac{1}{2}y} 2xy\,dx\,dy = \int_{y=0}^{y=2}\left(x^2y\Big|_{x=0}^{x=1-\frac{1}{2}y}\right)dy$$

$$= \int_{y=0}^{y=2}\left(1-\tfrac{1}{2}y\right)^2\cdot y\,dy = \int_{y=0}^{y=2}\left(y-y^2+\tfrac{1}{4}y^3\right)dy = \frac{1}{2}y^2-\frac{1}{3}y^3+\frac{1}{16}y^4\Big|_{y=0}^{y=2}$$

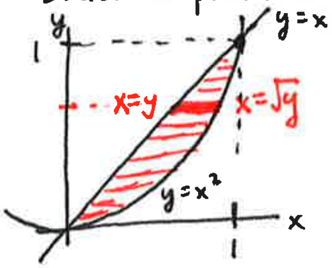
$$= 2-\frac{8}{3}+1 = 1/3.$$

$y=2-2x$
$2x=2-y$
$x=1-\frac{1}{2}y$

Same as above! ☺

Example. Compute $\iint_D (2y-x)\,dA$, where D is the domain in the first quadrant between $y=x$ and $y=x^2$.

Draw a picture:



① with x as the inner variable and y as the outer:

$$\int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}}(2y-x)\,dx\,dy = \int_{y=0}^{y=1}\left(2xy-\tfrac{1}{2}x^2\right)\Big|_{x=y}^{x=\sqrt{y}}dy = \int_{y=0}^{y=1}\left(2y\sqrt{y}-\tfrac{1}{2}(\sqrt{y})^2\right)-\left(2y\cdot y-\tfrac{1}{2}y^2\right)dy$$

$$=\int_{y=0}^{y=1}\left(2y^{3/2}-\tfrac{1}{2}y-\tfrac{3}{2}y^2\right)dy = \tfrac{4}{5}y^{5/2}-\tfrac{1}{4}y^3-\tfrac{1}{2}y^2\Big|_{y=0}^{y=1} = \tfrac{4}{5}-\tfrac{1}{4}-\tfrac{1}{2} = \tfrac{1}{20}.$$

$\frac{16}{20}-\frac{5}{20}-\frac{10}{20}$

② with y as the inner variable and x as the outer:

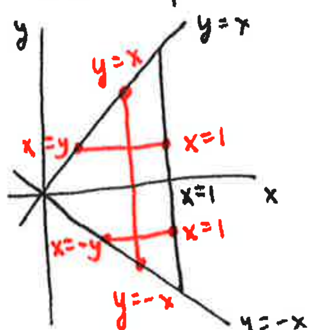
$$\int_{x=0}^{x=1}\int_{y=x^2}^{y=x}(2y-x)\,dy\,dx = \int_{x=0}^{x=1}\left(y^2-xy\Big|_{y=x^2}^{y=x}\right)dx = \int_{x=0}^{x=1}(x^2-x^2)-(x^4-x^3)\,dx = \int_{x=0}^{x=1}(x^3-x^4)\,dx = \frac{x^4}{4}-\frac{x^5}{5}\Big|_0^1 = \frac{1}{4}-\frac{1}{5} = \frac{1}{20}.$$

Same again! yay.

Example. Compute $\iint_D dA$, where D is the triangle formed by the lines $y=x$, $y=-x$, and $x=1$.

this is integrating $\iint_D 1\,dA$, which gives the area of the region D.

Draw a picture:



⓪ Geometry: area $= \frac{1}{2}\text{base}\cdot\text{height} = \frac{1}{2}\cdot 2\cdot 1 = 1.$

① using vertical segments: $\displaystyle\int_{x=0}^{x=1}\int_{y=-x}^{y=x}dy\,dx = \int_{x=0}^{x=1}(x--x)\,dx = \int_{x=0}^{x=1}2x\,dx = x^2\Big|_{x=0}^{x=1} = 1.$

② using horizontal segments: we have to break the region into two pieces!

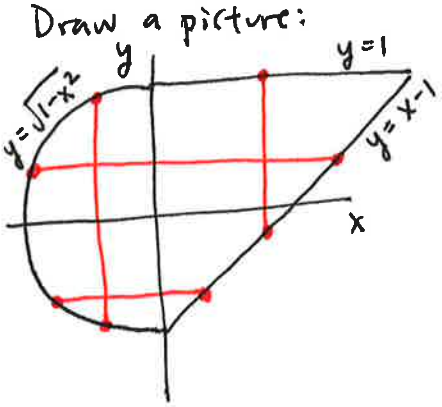
$$\underbrace{\int_{y=-1}^{y=0}\int_{x=-y}^{x=1}dx\,dy}_{\text{BOTTOM}} + \underbrace{\int_{y=0}^{y=1}\int_{x=y}^{x=1}dx\,dy}_{\text{TOP}} = \int_{y=-1}^{y=0}(1+y)\,dy + \int_{y=0}^{y=1}(1-y)\,dy$$

$$=\left(y+\tfrac{1}{2}y^2\Big|_{y=-1}^{y=0}\right)+\left(y-\tfrac{1}{2}y^2\Big|_{y=0}^{y=1}\right)$$

$$=(0+0)-\left(-1+\tfrac{1}{2}\right)+\left(1-\tfrac{1}{2}\right)-(0-0) = 1.$$

Example. Set up integrals in both orders for $\iint_D f(x,y)\,dA$, where D is the domain consisting of the left half of the unit disk, plus the triangle between $y=x-1$ and $y=1$.

Draw a picture:



① using horizontal segments: Solve for x in terms of y:
$x^2+y^2=1 \Rightarrow x^2=1-y^2 \Rightarrow x=\pm\sqrt{1-y^2}$
$y=x-1 \Rightarrow x=y+1$

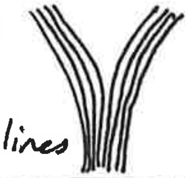
so $\displaystyle\int_{y=-1}^{y=1}\int_{x=-\sqrt{1-y^2}}^{x=y+1} f(x,y)\,dA.$

② using vertical segments, we need two pieces:

$$\underbrace{\int_{x=-1}^{x=0}\int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} f(x,y)\,dA}_{\text{LEFT (semi circle)}} + \underbrace{\int_{x=0}^{x=2}\int_{y=x-1}^{y=1} f(x,y)\,dA}_{\text{RIGHT (triangle)}}.$$

Diana Davis — Math 34 — 7 March 2018 — Class #19 ①

<u>Mathematician spotlight</u>: Eriko Hironaka, Professor Emerita, Florida State University
- studies dynamical systems
- also "train tracks" – like a doodle of many non-crossing lines

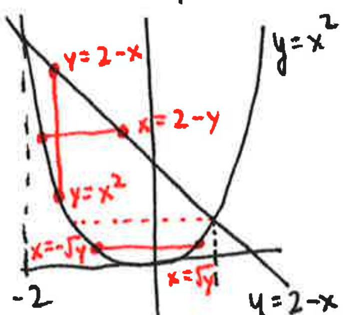


<u>Last time</u>, we explored how to integrate over a general region in the xy-plane.

<u>Today</u>, we'll discover how to change the order of integration: first, draw a picture to figure out what your region is, and then switch the order.

<u>Example.</u> Find the area of the region between $y=x^2$ and $y=2-x$.

Draw a picture:



intersection points:
$x^2 = 2-x$
$x^2+x-2=0$
$(x+2)(x-1)=0$
$\Rightarrow x=-2$ or $x=1$
$\Rightarrow (-2,4)$ and $(1,1)$.

① Using vertical slices:

$$\iint_D dA = \int_{x=-2}^{x=1}\int_{y=x^2}^{y=2-x} dy\,dx = \int_{x=-2}^{x=1}(2-x-x^2)\,dx = 2x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\Big|_{x=-2}^{x=1}$$
$$=\left(2-\tfrac{1}{2}-\tfrac{1}{3}\right)-\left(-4-2+\tfrac{8}{3}\right) = 2-\tfrac{1}{2}-\tfrac{1}{3}+4+2-\tfrac{8}{3} = 4\tfrac{1}{2}.$$

② using horizontal slices: need 2 pieces.

$$\iint_D dA = \underbrace{\int_{y=0}^{y=1}\int_{x=-\sqrt{y}}^{x=+\sqrt{y}} dx\,dy}_{\text{BOTTOM}} + \underbrace{\int_{y=1}^{y=4}\int_{x=-\sqrt{y}}^{x=2-y} dx\,dy}_{\text{TOP}} = \dots = \frac{4}{3}+\frac{19}{6} = 4\tfrac{1}{2}.$$

When would I use double integrals in real life? To compute probabilities!

<u>Example.</u> Your checkout line at the store has an average wait of 10 minutes, and your friend's of 5 minutes. What is the prob. that you check out first?

① Reality check: we expect an answer between ____ and ____.

② Let $x=$ amount of time until you check out
$y=$ " " " friend checks out.

We want: prob. $x<y$.

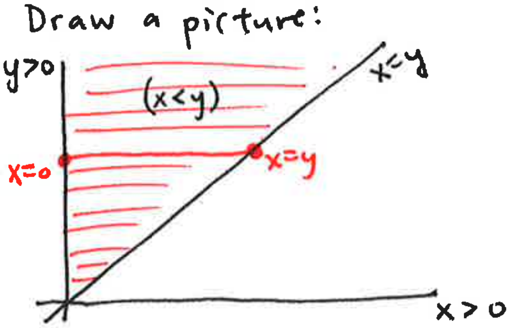
For one person, $P_\tau(t) = \begin{cases}\frac{1}{\tau}e^{-t/\tau} & \text{if } t\geq 0\\ 0 & \text{if } t<0\end{cases}$ ← the probability that you check out at t minutes, in a line whose average wait is τ minutes.

For two people, $p(x,y) = P_{10}(x)\cdot P_5(y)$ is the probability of the person in the 10-minute line checking out in x minutes, and the person in the 5-minute line checking out in y minutes.

We want: $\iint_{x<y} p(x,y)\,dA.$

$$= \int_{y=0}^{y=\infty}\int_{x=0}^{x=y}\frac{1}{10}e^{-x/10}\cdot\frac{1}{5}e^{-y/5}\,dx\,dy = \int_{y=0}^{y=\infty} -e^{-x/10}\cdot\frac{1}{5}e^{-y/5}\Big|_{x=0}^{x=y}dy$$

Draw a picture:



$$= \int_{y=0}^{y=\infty} -\frac{1}{5}e^{-y/5}\left(e^{-y/10}-1\right)dy = \int_{y=0}^{y=\infty}\left(-\frac{1}{5}e^{-3y/10}+\frac{1}{5}e^{-y/5}\right)dy = \frac{2}{3}e^{-3y/10}+e^{-y/5}\Big|_{y=0}^{y\to\infty}$$

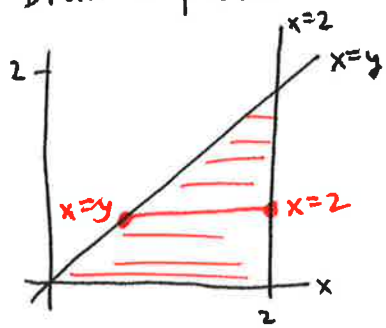
$$= (0-0)-\left(\frac{2}{3}-1\right) = \tfrac{1}{3}.$$

So there is a 1/3 probability that you'll check out before your friend.

Now let's change the order of integration.

Example. Compute $\displaystyle\int_{y=0}^{y=2}\int_{x=y}^{x=2} e^{x^2}\,dx\,dy.$ ← it is impossible to find an antiderivative for e^{x^2} with respect to x. Our only hope is to change the order.

Draw a picture:



Rewrite in the other order, using vertical slices:

$$=\int_{x=0}^{x=2}\int_{y=0}^{y=x} e^{x^2}\,dy\,dx = \int_{x=0}^{x=2}\left(y\cdot e^{x^2}\Big|_{y=0}^{y=x}\right)dx = \int_{x=0}^{x=2} x\cdot e^{x^2}\,dx = \frac{1}{2}e^{x^2}\Big|_{x=0}^{x=2}$$
$$=\underline{\frac{1}{2}e^4-\frac{1}{2}} = \underline{\frac{1}{2}(e^4-1)}.$$

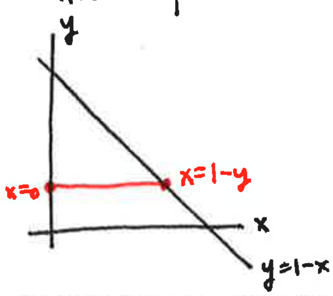
Note: we were lucky here, that changing the order yielded a computable integral. It happened that our region and our integrand played nicely together.

When changing the order of integration, use the limits of integration to sketch the region, and use the region to determine the new limits of integration.

A "good order" of integration generally depends on the function and on the region.

Example. Compute $\displaystyle\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\cos(1-y)^2\,dy\,dx.$ ← it is impossible to find an antiderivative for $\cos(1-y)^2$ w.r.t. y, so we must change the order.

Draw a picture:

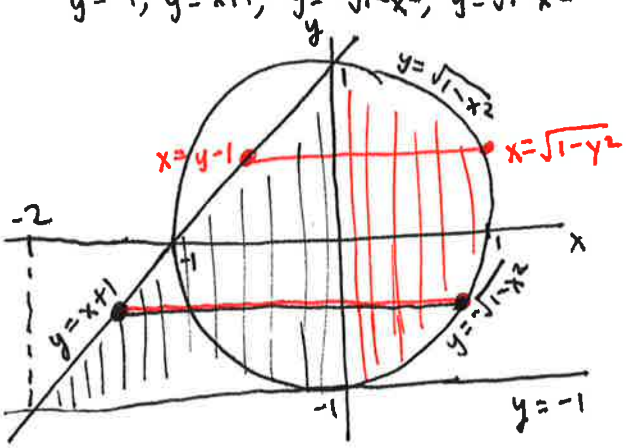


$$=\int_{y=0}^{y=1}\int_{x=0}^{x=1-y}\cos(1-y)^2\,dx\,dy = \int_{y=0}^{y=1}\left(\cos(1-y)^2\cdot(1-y)\right)dy = -\frac{1}{2}\sin(1-y)^2\Big|_{y=0}^{y=1} = -\frac{1}{2}(\sin 0-\sin 1)$$
$$=\underline{\frac{1}{2}\sin 1}.$$

Example. Consolidate into a single integral $\displaystyle\underbrace{\int_{x=-2}^{x=0}\int_{y=-1}^{y=x+1} f(x,y)\,dy\,dx}_{\text{FIRST}} + \underbrace{\int_{x=0}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx}_{\text{SECOND}}.$

First, draw all the bounding curves:

$y=-1,\ y=x+1,\ y=-\sqrt{1-x^2},\ y=\sqrt{1-x^2}$



$$=\int_{y=-1}^{y=1}\int_{x=y-1}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

Now, shade the region corresponding to each integral: FIRST, SECOND.

Finally, draw a horizontal slice and solve for the bounds.

Mathematician spotlight: Yitang (Tom) Zhang, Professor, Univ. of California Santa Barbara
- after PhD, didn't get math job; worked in restaurants & motels
- lecturer at UNH from 1999-2014, teaching calculus (etc)
- proved an extremely big theorem on gaps between prime numbers

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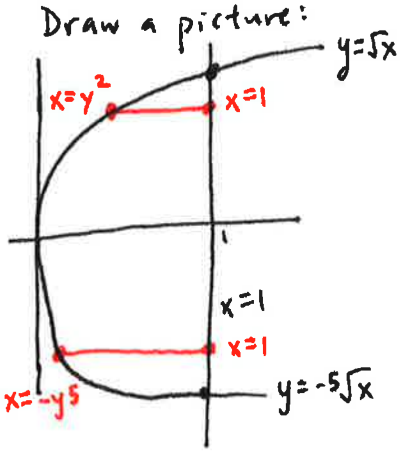
Last time: changing the order of integration for double integrals. How? Sketch the region!

Today: triple integrals! How? Visualize the surface in 3D, then set up the bounds.

Example. This is a toughie that will require all of the tools we have learned so far.

Compute $\int_0^1 \int_{-5\sqrt{x}}^{\sqrt{x}} \sin(y^3)\, dy\, dx$. ← $\sin(y^3)$ does not have an antiderivative with respect to y, so we have to change the order of integration.

Draw a picture:



$$= \int_{y=-1}^{y=0}\int_{x=-y^5}^{x=1} \sin(y^3)\,dx\,dy + \int_{y=0}^{y=1}\int_{x=y^2}^{x=1} \sin(y^3)\,dx\,dy.$$

BOTTOM TOP

$$= \int_{y=-1}^{y=0} \left(\sin(y^3) + y^5\sin(y^3)\right) dy + \int_{y=0}^{y=1}\left(\sin(y^3) - y^2\sin(y^3)\right) dy$$

$$= \underbrace{\int_{y=-1}^{y=0} \sin(y^3)\,dy}_{1} + \underbrace{\int_{y=-1}^{y=0} y^5\sin(y^3)\,dy}_{2} + \underbrace{\int_{y=0}^{y=1}\sin(y^3)\,dy}_{3} - \underbrace{\int_{y=0}^{y=1} y^2\sin(y^3)\,dy}_{4}$$

$$= \underbrace{\int_{y=-1}^{y=1} \sin(y^3)\,dy}_{1+3} + \underbrace{\int_{y=-1}^{y=0} y^5\sin(y^3)\,dy}_{2} - \underbrace{\int_{y=0}^{y=1} y^2\sin(y^3)\,dy}_{4} = \underbrace{0}_{1+3} \underbrace{- \frac{1}{3}y^3\cos(y^3)\Big|_{y=-1}^{y=0} + \frac{1}{3}\int_{y=-1}^{y=0}\cos(y^3)\cdot 3y^2\,dy}_{2} + 4$$

To compute 2, use Integration by Parts:

$u = y^3$ $\quad v = -\frac{1}{3}\cos(y^3)$

$du = 3y^2\,dy$ $\quad dv = y^2\sin(y^3)\,dy$

$\int u\,dv = uv - \int v\,du \Rightarrow \int y^5\sin(y^3) = -\frac{1}{3}y^3\cos(y^3) \cancel{-}\int \cancel{-}\frac{1}{3}\cos(y^3)\cdot 3y^2\,dy$

$$= -\frac{1}{3}\cdot 0\cdot\cos 0 + \frac{1}{3}\cdot(-1)\cos(-1) + \frac{1}{3}\sin(y^3)\Big|_{y=-1}^{y=0} + \frac{1}{3}\cos(y^3)\Big|_{y=0}^{y=1}$$

$$= \cancel{\frac{1}{3}\cos(-1)} - \frac{1}{3}\sin(-1) + \cancel{\frac{1}{3}\cos(1)} - \frac{1}{3}\cos(0)$$

$$= \frac{1}{3}\sin(1) - \frac{1}{3} = \frac{1}{3}(\sin 1 - 1).$$

Triple integrals! We compute $\iiint_R f(x,y,z)\,dV$ over a 3D solid region R. There are 6 possible orders: $dx\,dy\,dz$, $dx\,dz\,dy$, $dy\,dx\,dz$, ...

Example. Compute $\iiint_{[0,1]\times[0,2]\times[0,3]} 8xyz\,dV$. This integrates $f(x,y,z) = 8xyz$ over the box $\begin{cases} 0\le x\le 1\\ 0\le y\le 2\\ 0\le z\le 3.\end{cases}$ Let's do it in the order $dz\,dy\,dx$.

$$\int_{x=0}^{x=1}\int_{y=0}^{y=2}\int_{z=0}^{z=3} 8xyz\,dz\,dy\,dx = \int_0^1\int_0^2 \left(4z^2xy\Big|_{z=0}^{z=3}\right) dy\,dx = \int_{x=0}^{x=1}\int_{y=0}^{2} 36xy\,dy\,dx = \int_0^1 \left(18xy^2\Big|_{y=0}^{y=2}\right)dx = \int_{x=0}^{x=1} 72x\,dx$$

$$= 36x^2\Big|_{x=0}^{x=1} = 36.$$

What does a triple integral <u>mean</u>? It adds up function values over a solid region.

Special case: $f(x,y,z)=1$ then

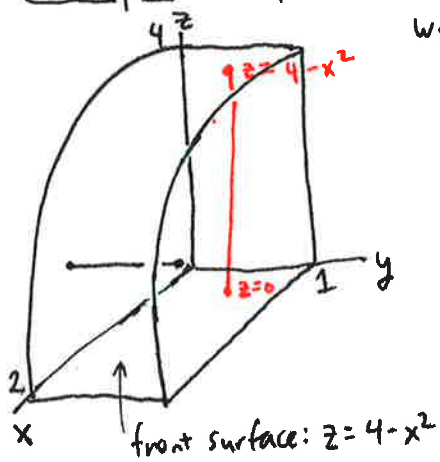
$$\iiint_R dV = \text{volume of region } R.$$

Example: $\iiint_{\text{solid ball of radius } r} dV = \frac{4}{3}\pi r^3$

Example: $\iiint_{[0,a]\times[0,b]\times[0,c]} dV =$ _____

In general: The function $f(x,y,z)$ gives the density of an object, or the electric charge, at each point. Then $\iiint_R f(x,y,z)\,dV$ gives the total mass of the object, or the total electric charge, etc.

Example. Compute the triple integral of $f(x,y,z)=x+y$ of the region R shown below.



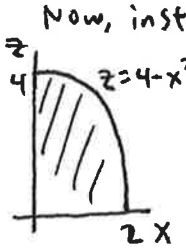
We want to integrate from $z=0$ to $z=4-x^2$ over the rectangle $0\le x\le 2$, $0\le y\le 1$.

$$\iiint_R (x+y)\,dV = \int_{y=0}^{y=1}\int_{x=0}^{x=2}\int_{z=0}^{z=4-x^2}(x+y)\,dz\,dx\,dy = \int_{y=0}^{y=1}\int_{x=0}^{x=1}(x+y)(4-x^2)\,dx\,dy$$

$$= \int_{y=0}^{y=1}\int_{x=0}^{x=2}\left(4x+4y-x^3-x^2y\right)dx\,dy = \int_{y=0}^{y=1}\left(2x^2+4xy-\tfrac{1}{4}x^4-\tfrac{1}{3}x^3y\right)\Big|_{x=0}^{x=2}dy$$

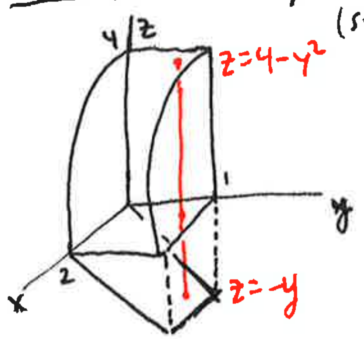
$$= \int_{y=0}^{y=1}\left(\underbrace{8+8y-4}_{8-4=4}-\tfrac{8}{3}y\right)dy = 4y+4y^2-\tfrac{4}{3}y^2\Big|_{y=0}^{y=1} = 4+4-\tfrac{4}{3} = \frac{20}{3}.$$

Now, instead of segments in the z-direction, let's use segments in the y-direction.



$$\iiint_R (x+y)\,dV = \underbrace{\int_{x=0}^{x=2}\int_{z=0}^{z=4-x^2}}_{\text{as in double integrals}}\int_{y=0}^{y=1}(x+y)\,dy\,dz\,dx = \dots = 20/3.$$

Example. Change the region so that the bottom surface is $z=-y$, not $z=0$, and set it up. (so, we just drop down the "floor" so it is slanted.)



So the rectangular region of integration in the xy-plane is the same, but now the z-segment starts lower down:

$$\iiint_R (x+y)\,dV = \int_{y=0}^{y=1}\int_{x=0}^{x=2}\int_{z=-y}^{z=4+y^2}(x+y)\,dz\,dx\,dy.$$

For a triple integral $\iiint_R f(x,y,z)\,dV$: $\int\int\int f(x,y,z)\,d(\text{inner variable})\,d(\text{middle variable})\,d(\text{outer variable})$

inner bounds depend only on middle & outer variables

This is the "shadow" of R in the middle variable-outer var. plane.

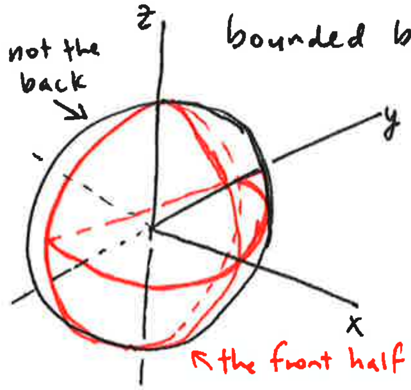
- the outer bounds are always CONSTANT
- the middle bounds depend only on the OUTER variable

Mathematician spotlight: Dusa McDuff, Professor, Barnard College
- symplectic geometry & topology, very eminent mathematician
- applies to even dimensions, e.g. 2, 4, 6, ... how do we visualize these?

Last time: Introduction to triple integrals, for finding "total mass" (or just volume)

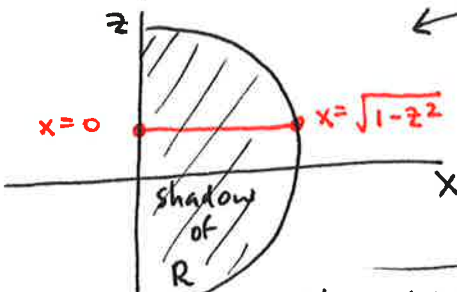
Today: More triple integrals; in particular, how to use the solid region of integration to set up the limits of integration.

Example. Set up the integral $\iiint_R \left(e^{xz^2}\cdot y\cdot \cos(xy)+3\right) dV$, where R is the solid half ball bounded by the yz-plane and the unit sphere, on the positive-x side.



Let's do the order $dy\, \underbrace{dx\, dz}$ so, we integrate over the "shadow" of R in the xz-plane.

$$\underbrace{\int_{z=-1}^{z=1}\int_{x=0}^{x=\sqrt{1-z^2}}}_{\text{determine this just like a double integral}}\underbrace{\int_{y=-\sqrt{1-x^2-z^2}}^{y=\sqrt{1-x^2-z^2}}}\left(e^{xz^2}\cdot y\cdot \cos(xy)+3\right) dy\, dx\, dz$$



for a fixed point (x,z) in the shadow of R, the y-value goes from
- the back surface: $y=-\sqrt{1-x^2-z^2}$

to
- the front surface: $x^2+y^2+z^2=1$

$y=+\sqrt{1-x^2-z^2}$

$y^2=1-x^2-z^2$

$y=\pm\sqrt{1-x^2-z^2}$

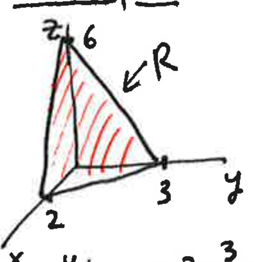
Clever trick: What if you actually wanted to compute this?

Notice that
- R is symmetric with respect to y ($y=0$ is a mirror for it)
- $e^{xz^2}\cdot y\cdot \cos(xy)$ is odd with respect to y (plug in $-y$, get the opposite of when you plug in y)

So $\iiint_R e^{xz^2}\cdot y\cdot \cos(xy)\, dV = 0$

So the value of the integral is $\iiint_R 3\, dV = 3(\text{volume of } R) = 3\cdot\frac{1}{2}\cdot\frac{4}{3}\pi\cdot 1^3 = \underline{\underline{2\pi}}.$

Example. Find the volume of the region bounded by the coordinate planes and $3x+2y+z=6$.

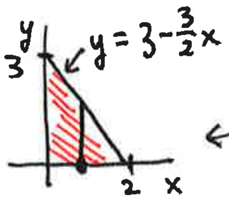


(0) Geometry: volume of pyramid $=\frac{1}{3}(\text{base area})(\text{height}) = \frac{1}{3}\cdot 3\cdot 6 = \underline{\underline{6}}.$ $\Rightarrow z=6-3x-2y$

(1) Let's do $dz\, \underbrace{dy\, dx}$ → use shadow of R in the xy-plane.

$$\underbrace{\int_{x=0}^{x=2}\int_{y=0}^{y=3-\frac{3}{2}x}}_{\text{set up double integral}}\int_{z=0}^{z=6-3x-2y} 1\, dz\, dy\, dx = \int_{x=0}^{x=2}\int_{y=0}^{y=3-\frac{3}{2}x}(6-3x-2y)\, dy\, dx$$

the volume over the triangular region in the xy-plane, under the plane $z=6-3x-2y$.



$= \ldots = \underline{\underline{6}}.$

Now, let's go the other way. Decipher the bounds!

Example. Describe the region of integration for the integral $\displaystyle\int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}}\int_{z=0}^{z=\sqrt{x^2+y^2}} f(x,y,z)\,dz\,\underbrace{dy\,dx}$

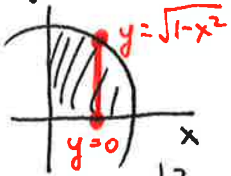
I suggest to always write these in!

shadow plane is xy-plane

① the shadow is in the xy-plane, with

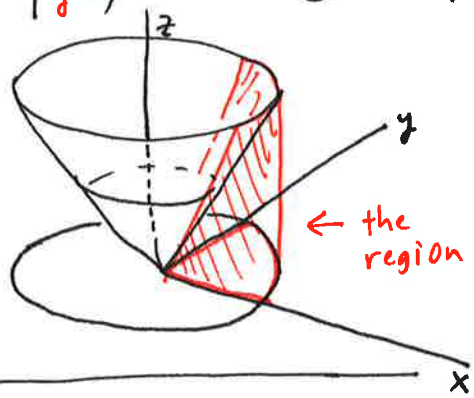
y from 0 to $\sqrt{1-x^2}$ } unit circle

x from 0 to 1 ← helpful to sketch bounding curves $y=0$, $y=\sqrt{1-x^2}$.



② the z-values go from $z=0$ (in xy-plane) up to $z=\sqrt{x^2+y^2}$. ← this surface is a CONE!

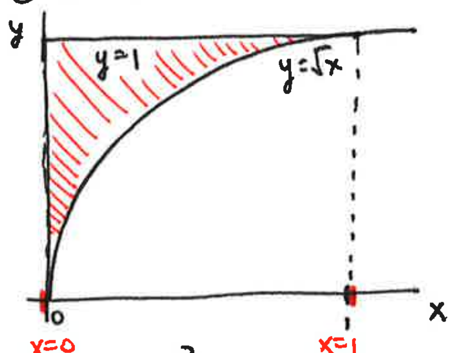
③ Let's put it together: we have the solid region in the first quadrant of the xy-plane, under the cone $z=\sqrt{x^2+y^2}$, inside the cylinder $x^2+y^2=1$.



Example. Sketch the region of integration for the integral $\displaystyle\int_{x=0}^{x=1}\int_{y=\sqrt{x}}^{y=1}\int_{z=0}^{z=1-y} f(x,y,z)\,dz\,\underbrace{dy\,dx}$

shadow plane is xy-plane

① the shadow of R is in the xy-plane, from $y=\sqrt{x}$ to $y=1$, with x from 0 to 1.



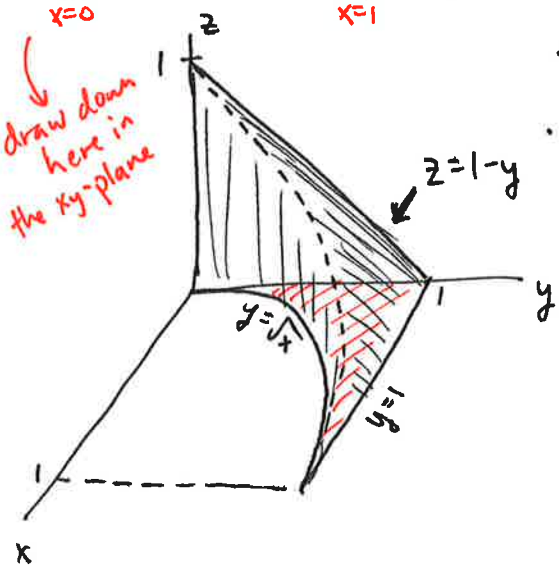
② the z-values go from $z=0$ (so, the xy-plane) up to $z=1-y$, which is a PLANE.

③ To put it together and sketch the solid region, we need to determine where the (vertical) surface $y=\sqrt{x}$ intersects the plane $z=1-y$:

- in the yz-plane: $x=0 \Rightarrow y=\sqrt{x}=0 \Rightarrow z=1-y=1-0=0 \Rightarrow (0,0,1)$
- in the xy-plane: $z=0 \Rightarrow z=1-y=0 \Rightarrow y=1 \Rightarrow 1=\sqrt{x} \Rightarrow x=1 \Rightarrow (1,1,0)$

The intersection of the surfaces $y=\sqrt{x}$ and $z=1-y$ is a curve from $(0,0,1)$ to $(1,1,0)$. (dashed)

Our solid region is the "curved tetrahedron" shape.

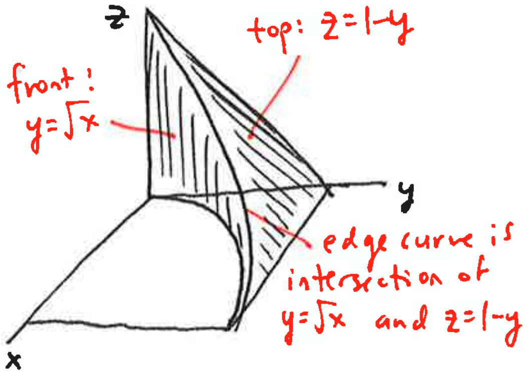


Mathematician spotlight: Jayadev Athreya, Associate Professor, Univ. of Washington
- studies flows on surfaces, billiards
- recent short paper proved existence of vertex-to vertex path on dodecahedron by showing it!

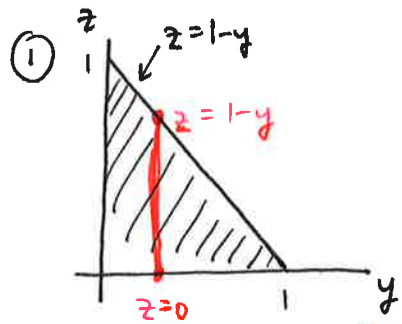
Last time: Setting up and interpreting triple integrals over solid regions

Today: More setting up triple integrals, including breaking a 3D region into two pieces

Recall: Last time, we sketched the region of integration for $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx$.

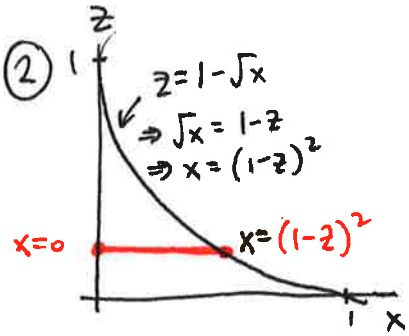


Now, let's rewrite it in the orders ① $dx\, dz\, dy$ and ② $dy\, dx\, dz$.
shadow plane is yz — shadow plane is xz.



$$\int_{y=0}^{y=1} \int_{z=0}^{z=1-y} \int_{x=0}^{x=y^2} f(x,y,z)\, dx\, dz\, dy$$

x goes from back (yz-plane, $x=0$) to front: surface $y=\sqrt{x}$ $\Rightarrow y^2 = x$



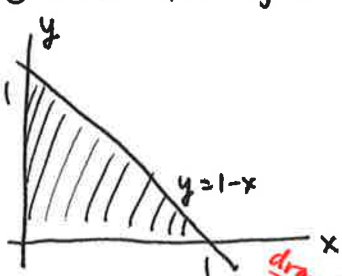
what is the shadow of the curved edge in the xz-plane? Eliminate y from the equations.
$y=\sqrt{x}$ and $z=1-y \Rightarrow z = 1-\sqrt{x}$.

$$\int_{z=0}^{z=1} \int_{x=0}^{x=(1-z)^2} \int_{y=\sqrt{x}}^{y=1-z} f(x,y,z)\, dy\, dx\, dz$$

choose a point in the xz-plane. Draw a line in the y-direction. It pops INTO the solid at the surface $y=\sqrt{x}$, and it pops OUT at the surface $z=1-y \Rightarrow y=1-z$.

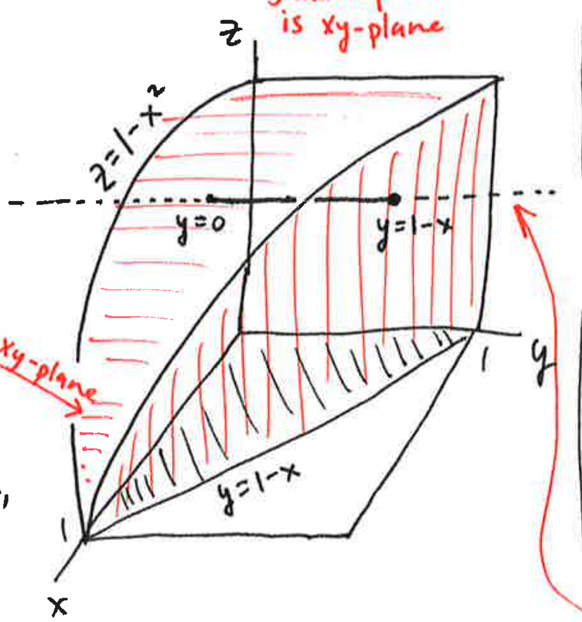
Example. Rewrite $\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\, dz\, dy\, dx$ in orders ① $dy\, dx\, dz$ and ② $dx\, dz\, dy$.
shadow plane is xy-plane — shadow plane is xz

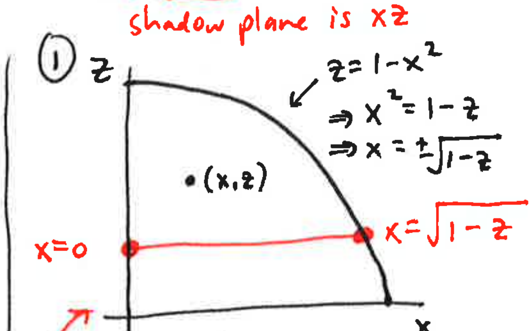
⓪ Sketch the region:



from $y=0$ to $y=1-x$, between $x=0$ and $x=1$

Then over this region, z goes from $z=0$ up to $z=1-x^2$

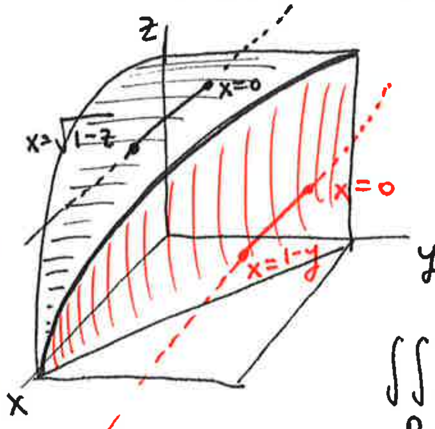


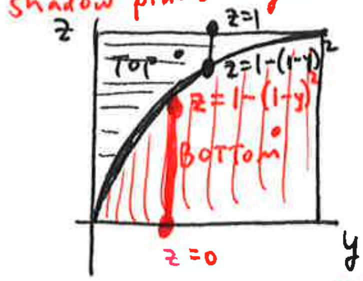


$$\int_{z=0}^{z=1} \int_{x=0}^{x=\sqrt{1-z}} \int_{y=0}^{y=1-x} f(x,y,z)\, dy\, dx\, dz$$

for a point (x,z) in the shadow of R, a line in the y-direction enters R at $y=0$ and exits at $y=1-x$.

(Continued) ② order $dx\,dz\,dy$:
shadow plane is yz





For a point (y,z) in the shadow of R, a line parallel to the x-axis enters R at $x=0$, and exits at

- $x=1-y$ if in the red region (bottom)
- $x=\sqrt{1-z}$ if in the black region (top).

So we need to set up a sum of two integrals.

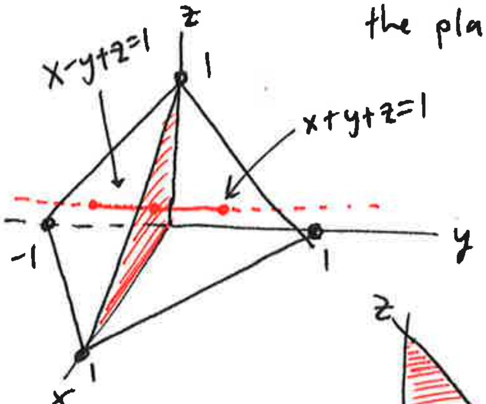
$$\iiint_R f(x,y,z)\,dV = \int_{y=0}^{y=1}\int_{z=0}^{z=1-(1-y)^2}\int_{x=0}^{x=1-y} f(x,y,z)\,dx\,dz\,dy + \int_{y=0}^{y=1}\int_{z=1-(1-y)^2}^{z=1}\int_{x=0}^{x=\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy.$$

The intersection of the surfaces $z=1-x^2$ and $y=1-x$ in the yz-plane: eliminate x.

$y=1-x \Rightarrow x=1-y$

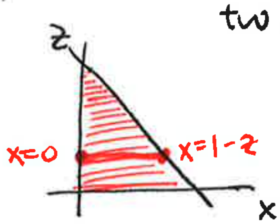
$z=1-x^2=1-(1-y)^2$

Example. Set up a triple integral for the region bounded by the xy-plane, the yz-plane, the plane $\underbrace{x+y+z=1}$, and the plane $\underbrace{x-y+z=1}$.



3 points on plane: $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ $\longrightarrow$ now we can plot these planes $\longleftarrow$ 3 points on plane: $(1,0,0)$, $(0,-1,0)$, $(0,0,1)$

Notice that if your shadow plane is xy or yz, you'll need two ∫ integrals. So let's use xz as the shadow plane.

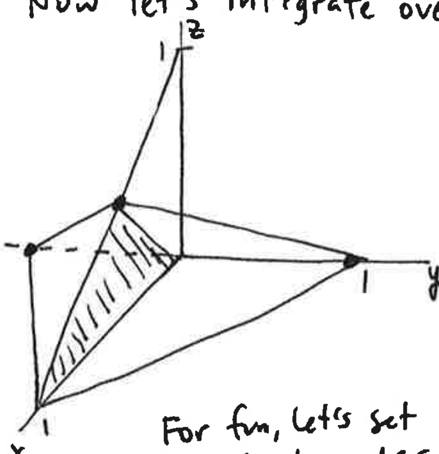


$$\int_{z=0}^{z=1}\int_{x=0}^{x=1-z}\int_{y=x+z-1}^{y=1-x-z} f(x,y,z)\,dy\,dx\,dz$$

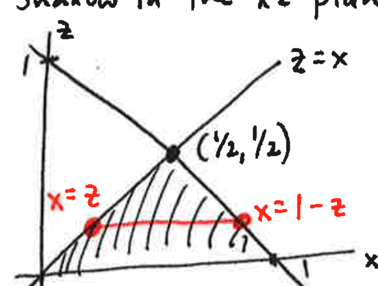
$y=1-x-z$ ← solve right plane $x+y+z=1$ for y

$y=x+z-1$ ← solve left plane $x-y+z=1$ for y

Now let's integrate over the part (half) of that region below the plane $z=x$:



Shadow in the xz plane:

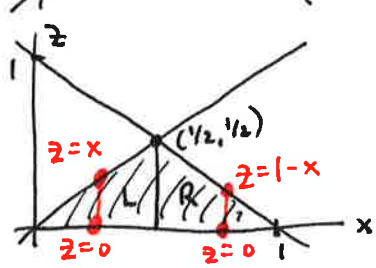


$$\int_{z=0}^{z=1/2}\int_{x=z}^{x=1-z}\int_{y=x+z-1}^{y=1-x-z} f(x,y,z)\,dy\,dx\,dz$$

$y=1-x-z$ ← as above

$y=x+z-1$ ← as above

For fun, let's set it up in the order $dy\,dz\,dx$: So, vertical segments.



$$\underbrace{\int_{x=0}^{x=1/2}\int_{z=0}^{z=x}\int_{y=x+z-1}^{y=1-x-z} f\cdots dy\,dz\,dx}_{\text{LEFT}} + \underbrace{\int_{x=1/2}^{x=1}\int_{z=0}^{z=1-x}\int_{y=x+z-1}^{y=1-x-z} f\cdot dy\,dz\,dx}_{\text{RIGHT}}$$

Mathematician spotlight: Evelyn Lamb, Freelance science writer for Scientific American and others

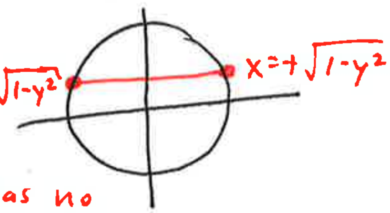
- Studies Teichmüller theory; postdoc at University of Utah
- explains research mathematics clearly & engagingly to a wide audience

The past three classes: Triple integrals, and solid regions of integration.

Today & next time: Tools for changing variables to make the integral easier.

Example: Converting from rectangular to polar coordinates (double integral).

Compute $\iint_D e^{x^2+y^2}\,dA$, where D is the unit disk $x^2+y^2 \le 1$.



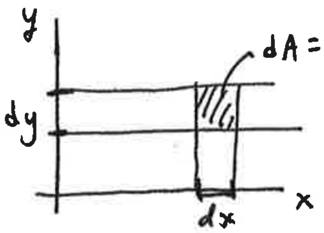
$$= \int_{y=-1}^{y=1} \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy$$

← impossible, because $e^{x^2+y^2}$ has no antiderivative with respect to x.

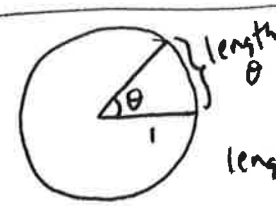
Maybe switch the order? No, that just exchanges x with y everywhere, so it's still impossible.

Idea: Maybe we can convert to polar coordinates. $e^{x^2+y^2} = e^{r^2}$, $D = \begin{cases} 0 \le \theta \le 2\pi, \\ 0 \le r \le 1. \end{cases}$ that seems easier.

OK, but what is dA?

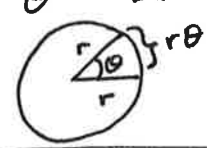


$dA = dx \cdot dy$, the area of an infinitesimal rectangle with side lengths dx and dy.



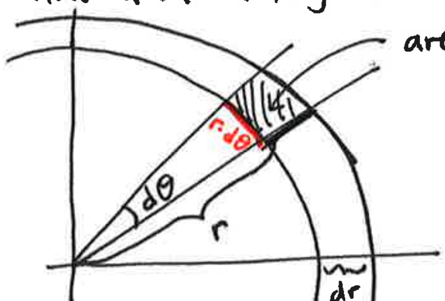
How do we see this? Call the length ℓ.

$$\frac{\ell}{\theta} = \frac{2\pi r}{2\pi} = \frac{2\pi}{2\pi} = 1 \Rightarrow \ell = \theta.$$



when the circle radius is r, the arc length scales up by a factor of r.

How about a tiny area in polar coords?



area of this $\approx$ width $\times$ length
$= dr \times r \cdot d\theta$

$\Rightarrow dA = r \cdot dr \cdot d\theta.$

$$\text{So } \iint_D e^{x^2+y^2}\,dA = \int_{\theta=0}^{\theta=2\pi} \underbrace{\left(\int_{r=0}^{r=1} e^{r^2} \cdot r \cdot dr \right) d\theta}_{\text{usually we use the order } dr\,d\theta} = \int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{2} e^{r^2} \Big|_{r=0}^{r=1} \right) d\theta = \int_{\theta=0}^{2\pi} \left(\frac{1}{2}e - \frac{1}{2}\cdot 1 \right) d\theta$$

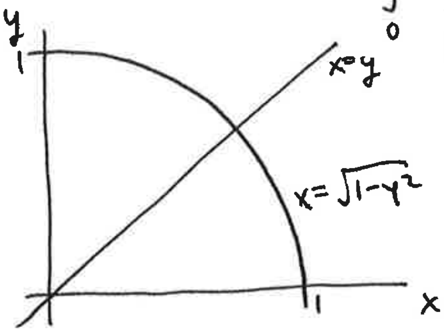
$$= \frac{1}{2}(e-1)\cdot\theta \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{1}{2}(e-1)\cdot 2\pi = \pi(e-1).$$

Converting to polar coordinates made this integral computable because the extra "r" from the dA term made the integrand $e^{r^2}\cdot r$, which has an antiderivative w.r.t. r.

Takeaway message: when converting to polar coordinates, $\boxed{dA = r \cdot dr \cdot d\theta} = r \cdot d\theta \cdot dr.$

Example. Compute $\displaystyle\int_0^{\sqrt{2}/2}\int_y^{\sqrt{1-y^2}} dx\, dy.$



sketch the curves: $x=y$

$x=\sqrt{1-y^2}$

$\Rightarrow x^2 = 1-y^2$

$\Rightarrow x^2+y^2=1$

(0) Geometry: this is 1/8 of a unit disk, and we are finding its area, so the value is $\frac{1}{8}\cdot\pi\cdot 1^2 = \frac{\pi}{8}$.

(1) Compute as written: $\displaystyle\int_0^{\sqrt{2}/2}\left(\sqrt{1-y^2}-y\right)dy = \ldots$ requires a trig substitution :(

(2) Convert to polar coordinates:

$$\int_{\theta=0}^{\theta=\pi/4}\left(\int_{r=0}^{r=1} r\, dr\right)d\theta = \int_{\theta=0}^{\theta=\pi/4}\left(\frac{r^2}{2}\Big|_{r=0}^{r=1}\right)d\theta = \int_{\theta=0}^{\theta=\pi/4}\frac{1}{2}\,d\theta = \frac{1}{2}\theta\Big|_{\theta=0}^{\theta=\pi/4}$$

$= \pi/8.$:)

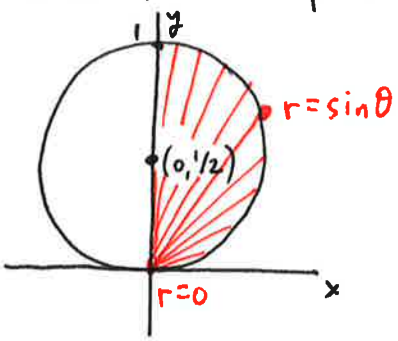
Example. Compute $\displaystyle\int_0^1\int_0^{\sqrt{y-y^2}}\frac{1}{\sqrt{x^2+y^2}}\,dx\,dy.$ ← what does this look like?!

$x=\sqrt{y-y^2}$

$\Rightarrow x^2 = y-y^2 \Rightarrow x^2+\left(y-\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$] circle

$\Rightarrow x^2+y^2=y \Rightarrow r^2 = r\sin\theta$

$\Rightarrow r = \sin\theta$] same circle, in polar

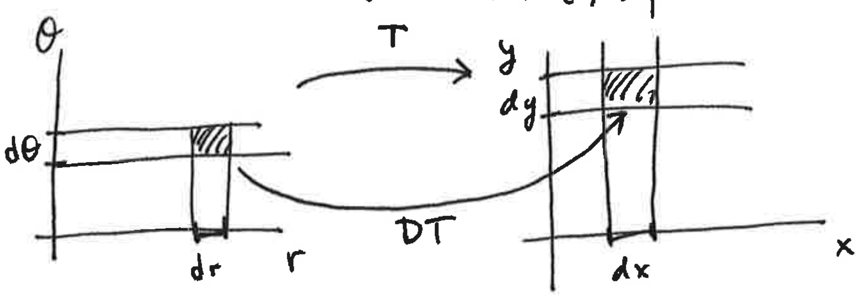


$$= \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=\sin\theta}\frac{1}{r}\cdot r\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi/2}\sin\theta\cdot d\theta = -\cos\theta\Big|_{\theta=0}^{\theta=\pi/2} = -\cos\frac{\pi}{2} - -\cos 0$$

$= -0+1 = 1.$

Here is another way to derive $dA = r\cdot dr\cdot d\theta$, which will generalize to other changes of variables:

View the process of transforming (x,y) coords into (r,θ) coords as a transformation between the (r,θ) plane and the (x,y) plane (easier direction):



here $T(r,\theta) = (r\cdot\cos\theta,\ r\cdot\sin\theta)$

$= (x(r,\theta),\ y(r,\theta))$

The Jacobian DT of T describes the linear transformation sending the $dr \times d\theta$ rectangle to the $dx \times dy$ rectangle.

$\Rightarrow$ determinant of DT tells us the expansion factor!

$dA = |\det(DT)|\,(\text{old area}) = |\det(DT)|\,dr\cdot d\theta$

$$DT = \begin{pmatrix}\partial x/\partial r & \partial x/\partial\theta\\ \partial y/\partial r & \partial y/\partial\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix} \Rightarrow \det(DT) = r\cos^2\theta - -r\sin^2\theta$$

$= r(\cos^2\theta + \sin^2\theta) = r.$

so $\boxed{dA = |\det(DT)|\cdot dr\cdot d\theta = r\cdot dr\cdot d\theta.}$

Diana Davis Math 34 March 26, 2018 Class #24 ①
§5.5

Mathematician spotlight: Colin Adams, Thomas T. Read Professor of Mathematics, Williams
- studies knot theory, hyperbolic geometry
- giving Kitao lecture on Tuesday at 4:30 pm

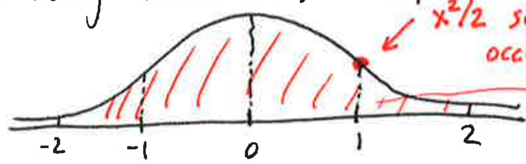
Last time: converting double integrals into polar coordinates

Recall: $dA = dx \cdot dy = r \cdot dr \cdot d\theta$

Today: changing coordinates to other convenient coordinates, other than polar.

Amazing & important application of using double integrals & polar coordinates: the bell curve.

given by $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$:



$x^2/2$ so that inflection points occur at $x = \pm 1$

need to find the area so we can normalize to area 1.

very important in statistics, economics, psychology...

We'll compute $\int_{-\infty}^{\infty} e^{-x^2} dx$ ← essentially the bell curve, but with easier constants. Aww, shucks! It has no antiderivative.

Clever trick:

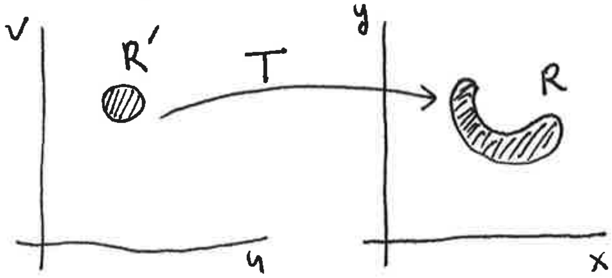
Let $A = \int_{-\infty}^{\infty} e^{x^2} dx$. Then $A^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy \right)$

$$\Rightarrow A^2 = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-x^2-y^2} dy\, dx = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2} \cdot r \cdot dr \cdot d\theta = \int_{\theta=0}^{2\pi} \left(-\frac{1}{2} e^{-r^2} \Big|_{r=0}^{r=\infty} \right) d\theta = \int_{\theta=0}^{2\pi} \frac{1}{2} d\theta = \underline{\underline{\pi}}.$$

This is the easiest way to compute this integral.

Change of variables: When we change from (x,y) coordinates to (u,v) coordinates, we want to know what to do with dA: $dA = dx \cdot dy = \_\_\_ \cdot du \cdot dv$. (for example, $dA = \underline{r} \cdot dr \cdot d\theta$).

Think of a transformation from the uv-plane to the xy-plane:



$T(u,v) = (x(u,v), y(u,v))$

Define the notation:
$$\frac{\partial(x,y)}{\partial(u,v)} = \det(DT) = \det \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

DT is the Jacobian matrix for the transformation T.

Then $\underline{\underline{dA = dx \cdot dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du \cdot dv.}}$

→ For example, we computed that for the polar coordinates transformation $T(r,\theta) = (r \cdot \cos\theta, r \cdot \sin\theta)$, we have $\det(DT) = r$, so $\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = |r| \Rightarrow dA = |r|\, dr\, d\theta$
$= r\, dr\, d\theta$
since $r \geq 0$.

Check this out: "change of variables" is just a two-dimensional u-substitution!

Let's apply the Jacobian method to the one-dimensional case (single-variable calculus):

Compute $\displaystyle\int_{x=0}^{x=4} x\cdot e^{x^2}\cdot dx$ Hmmm. Let's substitute $u=x^2$, so $x=\sqrt{u}=u^{1/2}$, so the Jacobian $\left(\frac{\partial x}{\partial u}\right)=\left(\frac{1}{2\sqrt{u}}\right)$.

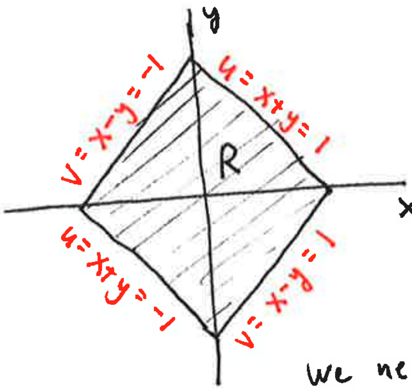
This is a 1×1 matrix with determinant $\frac{1}{2\sqrt{u}}$.

When x goes from 0 to 4, $u=x^2$ goes from 0 to 16, so we have:

$$\int_{x=0}^{x=4} x\cdot e^{x^2}\cdot dx=\int_{u=0}^{u=16}\sqrt{u}\cdot e^{u}\cdot\underbrace{\left(\frac{1}{2\sqrt{u}}\right)}_{\text{Jacobian determinant factor}}du=\int_0^{16}\frac{1}{2}e^{u}\,du,$$

just as you would do for a normal u-substitution! ☺

Example. Compute $\displaystyle\iint_R (x^2-y^2)\,dA$, where R is the diamond with vertices $(\pm1,0),(0,\pm1)$.



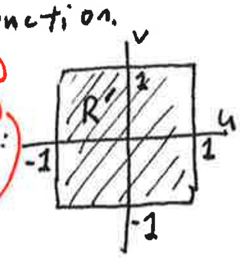
Option 1: Break up the region and compute the integral directly.

Option 2: Choose a change of variables that plays nicely with the region **and** with the integrand function.

Let $u=x+y$ and $v=x-y$. $\Rightarrow \begin{cases} x=\frac{1}{2}(u+v)\\ y=\frac{1}{2}(u-v)\end{cases}$

Then the boundary curves are $u=\pm1$, $v=\pm1$.

and $(x^2-y^2)=(x+y)(x-y)=u\cdot v$.

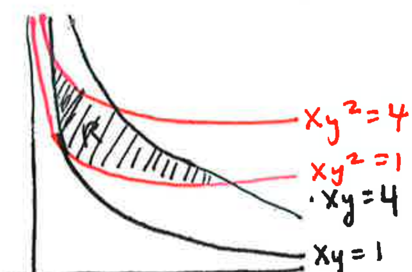


We need one more thing:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|\det\begin{pmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{pmatrix}\right|=\left|\det\begin{pmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{pmatrix}\right|=\left|-\tfrac{1}{2}\right|=\tfrac{1}{2}.$$

So now $$\iint_R (x^2-y^2)\,dA=\iint_{R'} u\cdot v\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv=\int_{v=-1}^{v=1}\int_{u=-1}^{u=1} u\cdot v\cdot\underbrace{\tfrac{1}{2}}_{\text{Jacobian det. factor}}du\,dv=\int_{v=-1}^{v=1}\left(\tfrac{1}{4}u^2v\Big|_{u=-1}^{u=1}\right)dv=\int_{-1}^{1}0\,dv=\underline{\underline{0.}}$$

Example. Compute $\displaystyle\iint_R xy^2\,dA$, where D is in the first quadrant bounded by $\begin{cases} xy=1, & xy^2=1,\\ xy=4, & xy^2=4.\end{cases}$



Let's use the change of variables $u=xy$ and $v=xy^2$.

We'll need $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$, so we have to solve for x and y in terms of u and v.

Ugh! What a job! But wait – $T:(u,v)\to(x,y)$, so $T^{-1}:(x,y)\to(u,v)$.

We want $D(T^{-1})$, which is $(DT)^{-1}$. So let's compute:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}=\left|\det\begin{pmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{pmatrix}\right|^{-1}=\left|\det\begin{pmatrix}y & x\\ y^2 & 2xy\end{pmatrix}\right|^{-1}=\frac{1}{xy^2}=\underline{\underline{\frac{1}{v}.}}$$

So $$\iint_R xy^2\,dA=\int_{v=1}^{v=4}\int_{u=1}^{u=4} v\cdot\underbrace{\frac{1}{v}}_{\text{Jacobian determinant factor}}du\,dv=\int_{v=1}^{v=4}\int_{u=1}^{u=4}1\,du\,dv=\text{area of }3\times3\text{ rectangle}=\underline{\underline{9.}}$$ ☺

> So, a nice trick: if you've defined u and v in terms of x and y, use $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$.

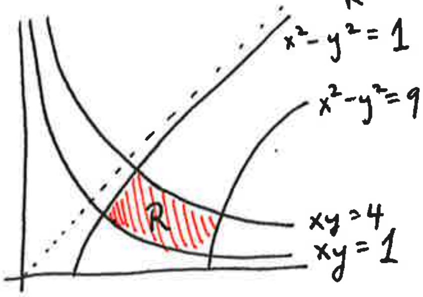
Diana Davis Math 34 March 27, 2018 Class # 25 ① §5.5

Mathematician spotlight: Amie Wilkinson, Professor, University of Chicago
- dynamical systems, ergodic theory
- studies spaces of surfaces

Last time: change of variables, to polar coordinates (r,θ) and to general coordinates (u,v)

Today: integration in cylindrical & spherical coordinates.

Example: Compute $\iint_R (x^2+y^2)e^{x^2-y^2}\,dA$, where R is bounded by $x^2-y^2=1$, $x^2-y^2=9$, $xy=1$, $xy=4$:



Based on the region, let's try the change of variables
$\begin{cases} u=xy \\ v=x^2-y^2. \end{cases}$ Then $\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\det\begin{pmatrix} y & x \\ 2x & -2y\end{pmatrix}\right| = |-2y^2-2x^2| = 2(x^2+y^2)$, since $x,y>0$ in R.

our desired Jacobian expansion factor for area $\rightarrow$ So $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{2(x^2+y^2)}$ ← we can't easily convert this to u & v, but luckily it cancels out in the integral.

So now $\iint_R (x^2+y^2)e^{x^2-y^2}\,dA = \iint_{R'} \cancel{(x^2+y^2)}\, e^{x^2-y^2}\cdot \underbrace{\frac{1}{2\cancel{(x^2+y^2)}}}_{\text{Jacobian}}\,du\,dv = \int_{v=1}^{v=9}\int_{u=1}^{u=4} \frac{1}{2}e^v\,du\,dv = \int_{v=1}^{v=9}\frac{3}{2}e^v\,dv = \underline{\underline{\frac{3}{2}(e^9-e)}}.$

For triple integrals, we use the 3×3 Jacobian expansion factor for volume:

Cylindrical coordinates: $\begin{cases} x=r\cdot\cos\theta \\ y=r\cdot\sin\theta \\ z=z \end{cases} \Rightarrow \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \left|\det\begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial z\end{pmatrix}\right| = \left|\det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1\end{pmatrix}\right| = |r| = \underline{\underline{r}}.$

So $\boxed{dV = r\cdot dz\cdot dr\cdot d\theta.}$ This makes sense because only x and y are affected when converting to cylindrical coordinates, so the expansion factor is the same as for polar coordinates.

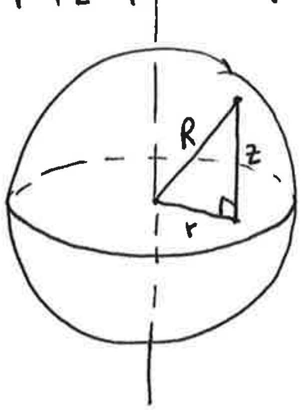
Spherical coordinates: $\begin{cases} x=\rho\cdot\sin\phi\cdot\cos\theta \\ y=\rho\cdot\sin\phi\cdot\sin\theta \\ z=\rho\cdot\cos\phi \end{cases} \Rightarrow \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \left|\det\begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0\end{pmatrix}\right|$

$= |\rho^2\sin\phi| = \rho^2\sin\phi$ ← $0\le\phi\le\pi$, so $\sin\phi\ge 0$.

So $\boxed{dV = \rho^2\sin\phi\cdot d\rho\cdot d\phi\cdot d\theta.}$

Example. Let's test this out by finding the volume of the solid ball B of radius R (centered at $\vec{0}$).

In cylindrical coordinates:
$r^2+z^2=R^2 \Rightarrow z=\pm\sqrt{R^2-r^2}$



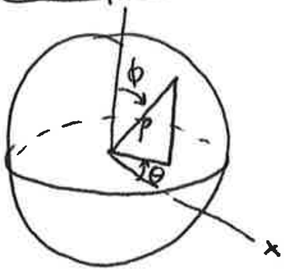
Shadow in xy-plane (or $r\theta$-plane): the disk of radius R, $r=0$ to $r=R$ and $\theta=0$ to $\theta=2\pi$.

So volume $= \iiint_B 1\,dV = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R}\int_{z=-\sqrt{R^2-r^2}}^{z=+\sqrt{R^2-r^2}} 1\cdot \underbrace{r}_{\text{Jacobian}}\cdot dz\cdot dr\cdot d\theta$

$= \int_{\theta=0}^{2\pi}\int_{r=0}^{R} 2r\sqrt{R^2-r^2}\,dr\,d\theta = \underbrace{\int_{\theta=0}^{2\pi} d\theta}_{2\pi}\cdot\underbrace{\int_{r=0}^{R} 2r\sqrt{R^2-r^2}\,dr}_{\text{compute this}} = 2\pi\left(-\frac{2}{3}(R^2-r^2)^{3/2}\Big|_{r=0}^{r=R}\right)$

$= 2\pi\left(\frac{2}{3}R^3\right) = \underline{\underline{\frac{4}{3}\pi R^3}}.$ ☺

Example. Volume of the solid ball B of radius R again, now in spherical coordinates.



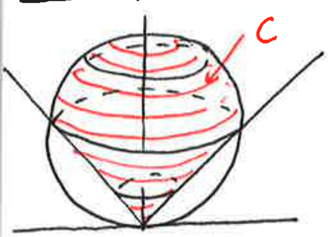
$$\iiint_B 1\,dV = \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi}\int_{\rho=0}^{\rho=R} 1\cdot \underbrace{\rho^2\sin\phi}_{\text{Jacobian}}\cdot d\rho\cdot d\phi\cdot d\theta$$

$$= \underbrace{\int_{\theta=0}^{2\pi} d\theta}_{2\pi}\cdot\int_{\phi=0}^{\phi=\pi}\sin\phi\,d\phi\cdot\int_{\rho=0}^{\rho=R}\rho^2\,d\rho = (2\pi)\left(-\cos\phi\Big|_{\phi=0}^{\phi=\pi}\right)\left(\frac{\rho^3}{3}\Big|_{\rho=0}^{\rho=R}\right)$$

$$= 2\pi\cdot 2\cdot\frac{R^3}{3} = \frac{4}{3}\pi R^3.$$ ☺

Okay, so we have some faith that it works. Now let's integrate something more exotic.

Example. Integrate $f(x,y,z)=xz$ over the "ice cream cone" C bounded by the surfaces $z=\sqrt{x^2+y^2}$ and $x^2+y^2+(z-1)^2=1$, in all three coordinate systems.



$\Rightarrow z=\sqrt{r^2}=r$

$\Rightarrow r^2+(z-1)^2=1$

$(z-1)^2=1-r^2 \Rightarrow z=1+\sqrt{1-r^2}$

$\Rightarrow r^2=1-(z-1)^2 \Rightarrow r=\sqrt{1-(z-1)^2}$

1. Cylindrical, in order $dz\,dr\,d\theta$:

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=r}^{z=1+\sqrt{1-r^2}}\underbrace{(r\cdot\cos\theta\cdot z)}_{xz}\cdot\underbrace{r}_{\text{Jacobian}}\cdot dz\cdot dr\cdot d\theta = \underbrace{\int_{\theta=0}^{2\pi}\cos\theta\cdot d\theta}_{=0}\cdot\int_{r=0}^{r=1}\int_{z=r}^{z=1+\sqrt{1-r^2}} r^2 z\,dz\,dr = 0.$$

2. Cylindrical, in order $dr\,dz\,d\theta$:

$$\underbrace{\int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=1}\int_{r=0}^{r=z}(r\cdot\cos\theta\cdot z)\cdot r\cdot dz\cdot dr\cdot d\theta}_{\text{cone part}} + \underbrace{\int_{\theta=0}^{\theta=2\pi}\int_{z=1}^{z=2}\int_{r=0}^{r=\sqrt{1-(z-1)^2}}(r\cdot\cos\theta\cdot z)\cdot r\cdot dz\cdot dr\cdot d\theta}_{\text{ice cream (sphere) part}}$$

3. Spherical $d\rho\,d\phi\,d\theta$:

$$\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/4}\int_{\rho=0}^{\rho=2\cos\phi}\underbrace{\rho\sin\phi\cos\theta}_{x}\cdot\underbrace{\rho\cos\phi}_{z}\cdot\underbrace{\rho^2\sin\phi}_{\text{Jacobian}}\cdot d\rho\,d\phi\,d\theta.$$

$$= \underbrace{\int_{\theta=0}^{2\pi}\cos\theta\,d\theta}_{0}\cdot\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{\rho=2\cos\phi}\rho^4\cdot\sin^2\phi\cdot\cos\phi\cdot d\rho\cdot d\phi$$

$x^2+y^2+(z-1)^2=1$
$x^2+y^2+z^2-2z+1=1$
$x^2+y^2+z^2=2z$
$\rho^2=2\rho\cos\phi$
$\rho=2\cos\phi$
Solve for sphere equation in terms of ρ

Explain how you could figure out that $\iiint_C xz\,dV=0$, without doing any calculations.

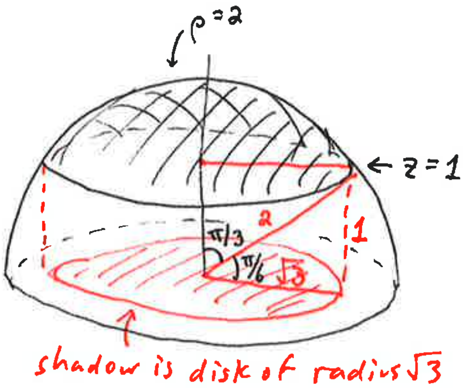
Diana Davis Math 34 March 30, 2018 Class #26 ① §3.1

Mathematician spotlight: Federico Ardila, Associate Professor, San Francisco State Univ.

- combinatorics of objects in algebra, geometry, topology, etc.
- uses polyhedra to understand power series e.g. $a_0 + a_1 x + a_2 x^2 + \cdots$ (!)

Example. Set up an integral for the solid "spherical cap" inside $\rho = 2$ and above $z = 1$.

$x^2+y^2+z^2=2^2$
$z^2 = 4-x^2-y^2$
$z = \sqrt{4-x^2-y^2}$



Rectangular: $$\int_{x=-\sqrt{3}}^{x=\sqrt{3}} \int_{y=-\sqrt{3-x^2}}^{+\sqrt{3-x^2}} \int_{z=1}^{z=\sqrt{4-x^2-y^2}} f(x,y,z)\, dz\, dy\, dx$$

Cylindrical: $$\int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{3}} \int_{z=1}^{\sqrt{4-r^2}} f(x,y,z)\cdot r \cdot dz \cdot dr \cdot d\theta$$

$= \sqrt{4-r^2}$

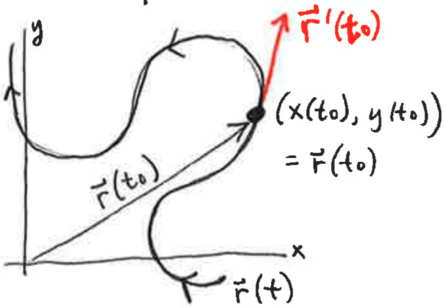
Spherical: $$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/3} \int_{\rho=1/\cos\phi}^{2} f(x,y,z)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

plane $z=1$:
$\Rightarrow \rho \cdot \cos\phi = 1$
$\Rightarrow \rho = \frac{1}{\cos\phi}$

Curves! Any curve can be described parametrically by $\vec{r}(t) = (x(t), y(t))$ (ant on paper)

$\vec{r}(t) = (x(t), y(t), z(t))$ (fly in air)

For example, in 2D:



At any time t, the location of the fly is $\vec{r}(t) = (x(t), y(t))$; the direction of travel is given by $\vec{r}\,'(t) = (x'(t), y'(t))$, the tangent vector to the curve at the point $\vec{r}(t)$.

The magnitude of the tangent vector, $|\vec{r}\,'(t)|$, is the speed of the fly.

So $\vec{r}\,'(t)$ is the velocity (direction and speed) vector, and $\vec{r}\,''(t)$ is the acceleration vector.

Example. Suppose you fire a pebble from a slingshot at the origin, at an angle of 45° with a speed of $\sqrt{2}$ meters/sec. Assume that the only force acting on the pebble is gravity, at g meters/sec². Find parametric equations for its position.

when and where does the pebble hit the ground?
when $y(t)=0$.

- start with acceleration: $\vec{r}\,''(t) = [0, -g]$
- integrate to get velocity: $\vec{r}\,'(t) = [0 + c_1, -gt + c_2]$ ← c_1, c_2 are the "+c" integration constants
  - we know $\vec{r}\,'(0) = [1,1]$: $\vec{r}\,'(0) = [c_1, c_2] = [1,1] \Rightarrow c_1 = 1, c_2 = 1$
  - $\Rightarrow \underline{\vec{r}\,'(t) = [1, -gt+1]}$
- integrate to get position: $\vec{r}(t) = [t + c_3, -\frac{1}{2}gt^2 + t + c_4]$ ← c_3, c_4 are the integration constants
  - we know $\vec{r}(0) = [0,0]$: $\vec{r}(0) = [c_3, c_4] = [0,0] \Rightarrow c_3 = 0, c_4 = 0$
  - $\Rightarrow \vec{r}(t) = [t, -\frac{1}{2}gt^2 + t]$.

$$\Rightarrow \begin{cases} x(t) = t \\ y(t) = t - \frac{1}{2}gt^2 \end{cases} = 0$$ to see when it hits the ground $\Rightarrow t(1 - \frac{1}{2}gt) = 0$

$\Rightarrow t = 0$ or $t = 2/g$

start — hits at ≈ 0.2 sec.

How far does the pebble / bug / fly travel? We need to find the <u>arclength.</u>

<u>Recall:</u> distance = rate × time, so distance traveled in time "dt" is speed × dt = $\|\vec{r}\,'(t)\| \cdot dt$.

magnitude of velocity vector

So the total distance (arclength) traveled from time $t=a$ to $t=b$ is

$$\int_{t=a}^{t=b} \underbrace{\|\vec{r}\,'(t)\|}_{\text{rate}} \underbrace{dt}_{\text{time}}.$$

Let's test it out on a circle to see if it works.

<u>Example:</u> Find the length of a circle of radius R. (we expect the answer to be $2\pi R$.)

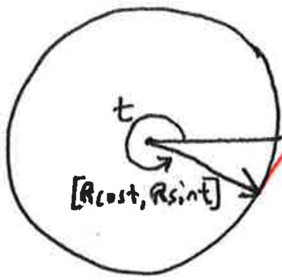
$\vec{r}(t) = [R\cos(t), R\sin(t)]$

$\Rightarrow \vec{r}\,'(t) = [-R\cdot\sin(t), R\cdot\cos(t)]$

$\Rightarrow \|\vec{r}\,'(t)\| = \sqrt{(-R\cdot\sin(t))^2 + (R\cdot\cos(t))^2} = R.$

So its length is $\displaystyle\int_{t=0}^{t=2\pi} R\cdot dt = Rt\Big|_{t=0}^{t=2\pi} = \underline{\underline{2\pi R}}.$ ☺ (it works!)

Picture:

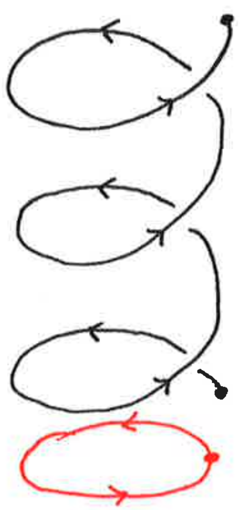


$\vec{r}\,'(t) = [-R\sin t, R\cos t]$ is the direction vector of the particle. It has a constant speed of $\|\vec{r}\,'(t)\| = R$.

Check this out: $\vec{r}\,''(t) = [-r\cos t, -r\sin t] = -\vec{r}(t)$. So the acceleration vector points toward the center of the circle.

acceleration

<u>Example.</u> A lazy housefly is languidly circling toward the ceiling light in a helix (corkscrew), with her path described by $\vec{r}(t) = [\cos 3t, \sin 3t, t]$, with distance in feet and time in minutes. Find how far she travels from time $t=0$ to time $t=2\pi$.



Ok, we have $\vec{r}(t) = [\cos 3t, \sin 3t, t]$

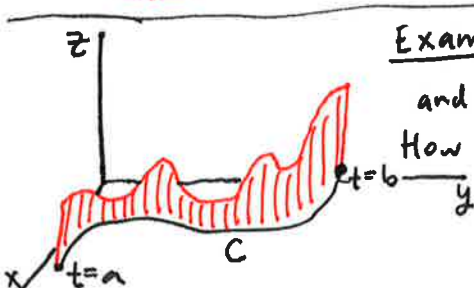
so $\vec{r}\,'(t) = [-3\sin 3t, 3\cos 3t, 1]$

$\Rightarrow \|\vec{r}\,'(t)\| = \sqrt{(-3\sin 3t)^2 + (3\cos 3t)^2 + 1^2} = \sqrt{9+1} = \sqrt{10}.$ ← constant speed of $\sqrt{10}$ ft/min

So the length is $\displaystyle\int_{t=0}^{t=2\pi} \underset{\text{ft/min}}{\|\vec{r}\,'(t)\|}\ \underset{\text{min}}{dt} = \int_{t=0}^{t=2\pi} \sqrt{10}\, dt = \sqrt{10}\,t\Big|_{t=0}^{t=2\pi} = 2\pi\sqrt{10}$. feet.

Check this out: $\vec{r}\,''(t) = [-9\cos 3t, -9\sin 3t, 0]$ ← not accelerating in the z-direction.

← shadow of path in the xy-plane is $(\cos 3t, \sin 3t)$ – a triple-speed unit circle.



<u>Example:</u> Suppose $f(x,y)$ = depth of gold dust at point (x,y) and you vacuum along the curve $C = \{\vec{r}(t)$ from $t=a$ to $t=b\}$. How much gold dust will you suck up?

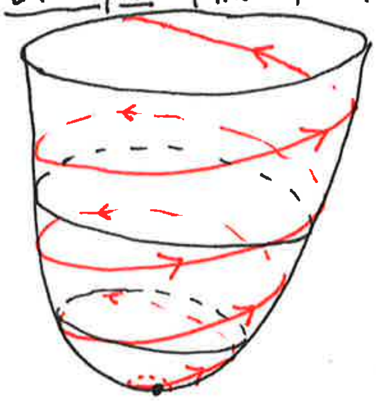
$$\int_C \underset{\text{depth}}{f(x,y)}\, dS$$ ← distance parameter along the curve C

Diana Davis   Math 34   April 2, 2018   Class #27
§3.3 ①

Mathematician spotlight: Rachel Epstein, Assistant Professor, Georgia College
(B.A. Reed College, PhD U. of Chicago, postdoc Harvard, Vis. A.P. Swarthmore)
- Studies computably enumerable sets: those whose elements can be listed by some algorithm.
- amazingly, there are (way) more numbers that cannot be listed by an algorithm, than numbers that can be.

Today: A bit more on curves, and then vector fields! (like wind)

Example: Find the tangent line to the curve $\vec{r}(t) = [t\cdot\cos t,\ t\cdot\sin t,\ t^2]$ at $t = \pi$.



First, what does it look like? • x and y are spiraling outward
• z is increasing
• $x^2+y^2 = z$, so it lies on a paraboloid!

To find the tangent line, we need a point and a direction vector.

point: $\vec{r}(\pi) = [\pi\cdot\cos\pi,\ \pi\cdot\sin\pi,\ \pi] = [-\pi, 0, \pi^2]$.

direction vector: $\vec{r}\,'(t) = [\cos t + t\cdot\sin t,\ \sin t - t\cos t,\ 2t]$

so $\vec{r}\,'(\pi) = [\cos\pi + \pi\cdot\sin\pi,\ \sin\pi - \pi\cdot\cos\pi,\ 2\pi] = [-1, \pi, 2\pi]$

So the tangent line equation is $[x(t), y(t), z(t)] = [-\pi, 0, \pi^2] + t[-1, \pi, 2\pi] = \begin{bmatrix} -\pi - t \\ \pi t \\ \pi^2 + 2\pi t \end{bmatrix}$.

Check this out: For a circle, or for any curve on a sphere, $\vec{r}(t)$ is perpendicular to the tangent vector $\vec{r}\,'(t)$. Are circles and spheres the only case where this happens? Yes!

Proposition: If $\vec{r}(t)$ is a curve with the property that $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$, then $\|\vec{r}(t)\|$ is a constant, i.e. $\vec{r}(t)$ lies on a circle, sphere, etc.

Proof: $\|\vec{r}(t)\|^2 = \vec{r}(t)\cdot\vec{r}(t)$ ← a vector dotted w/ itself gives its length squared

$\Rightarrow \frac{d}{dt}\|\vec{r}(t)\|^2 = \frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t)$ ← product rule

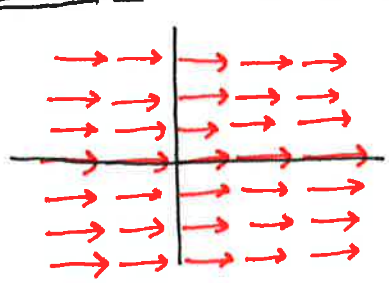
$= 2\left(\vec{r}(t)\cdot\vec{r}\,'(t)\right)$ ← combine like terms

$= 0$ ← we assumed $\vec{r}(t)$ is perp. to $\vec{r}\,'(t)$

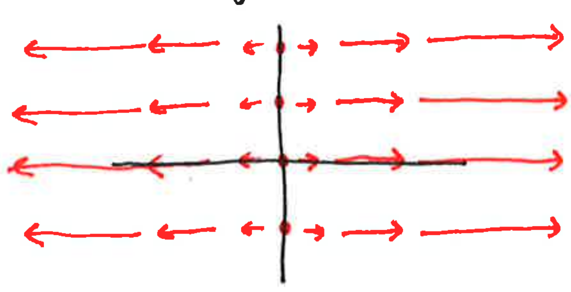
$\Rightarrow \frac{d}{dt}\|\vec{r}(t)\| = 0 \Rightarrow \|\vec{r}(t)\|$ is a constant, $\Rightarrow \vec{r}(t)$ lies on a circle/sphere!

Vector fields: Associates a vector to each point in a space.

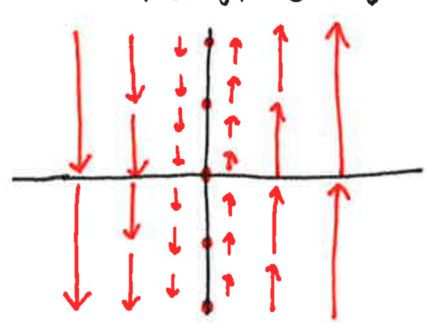
Example: $\vec{F}(x,y) = [1,0]$



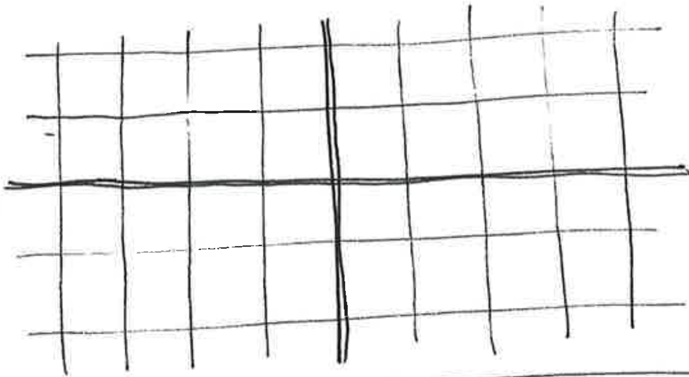
$\vec{F}(x,y) = [x,0]$



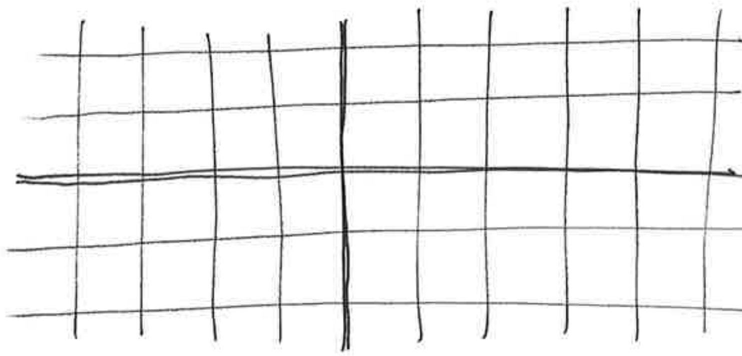
$\vec{F}(x,y) = [0,x]$



<u>Now you try:</u> Sketch $\vec{F}(x,y) = [x, y]$



Sketch $\vec{F}(x,y) = [-y, x]$

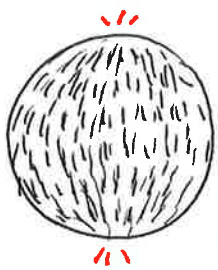


The above examples will be our favorites, which we will use repeatedly to understand new ideas.

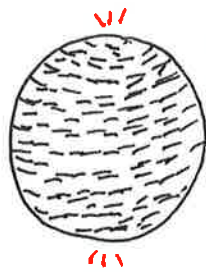
Application of vector fields: <u>The Hairy Ball Theorem:</u> Given any continuous tangent vector field on a sphere, there must be at least one point on the sphere where the tangent vector is $\vec{0}$.

<u>Corollary:</u> If you have a hairy ball (koosh ball, coconut, etc.) and you wish to comb down all the hair so that it lies flat, there will be at least one point where it goes wrong — this goal is impossible to achieve.

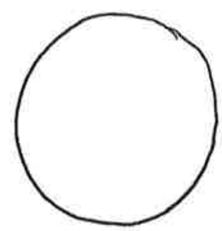
<u>Examples:</u>



comb all the hair "down": problem at north & south poles ☹



comb all the hair "east": problem at the poles again ☹



You try!

<u>Corollary:</u> There is always at least one point on Earth where the wind isn't blowing.

<u>Application to economics:</u> Suppose you have three products x_1, x_2, x_3.

Put their prices into a vector: $[p_1, p_2, p_3] = \vec{p}$.

Corresponding to each product there is a demand, and in particular an excess demand, with vector $[d_1, d_2, d_3] = \vec{d}$.

↑ we want the excess demand to be $\vec{0}$.

Walras's Law says $\vec{p} \cdot \vec{d} = 0$.

So, on the sphere of possible price vectors, the excess demand vector forms a (continuous) tangent vector field!

By the Hairy Ball Theorem, there is some point on the sphere where $\vec{d} = \vec{0}$! At this point, there is no excess demand!

So that point gives the optimal prices for the products. ☺

Diana Davis Math 34 April 4, 2018 Class #28 ① §3.4

Mathematician spotlight: Dylan Thurston, Professor, University of Indiana (oops)

- studies topology, homology, geometry, ...
- is the (biological) child of my (mathematical) grandfather, William (Bill) Thurston, one of the greatest mathematicians of the 20th century.

Today: "divergence" & "curl" of a vector field will help us describe the behavior of flows.

Vector fields! Example: Sketch the vector field $\vec{F}(x,y) = [y, x]$.

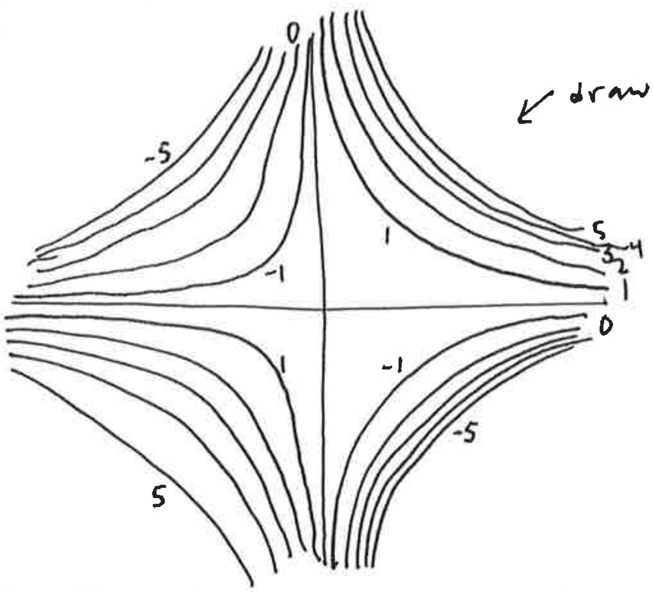
- Option 1: plot the vectors at some chosen points to get an idea of the picture.
- Option 2: view $\vec{F}$ as the gradient of some other function $f(x,y)$, ← not every $\vec{F}$ is the gradient of an f. plot level curves of $f(x,y)$, and draw in gradient vectors. ← this one is direction of steepest ascent

→ New quest: find $f(x,y)$ so that $\nabla f = \vec{F}$, i.e. $[f_x, f_y] = [y, x]$.

ok, so: $f_x = y \Rightarrow f(x,y) = x \cdot y + g_1(y)$ ← some function of y

$f_y = x \Rightarrow f(x,y) = y \cdot x + g_2(x)$ ← some f of x

$\Rightarrow \vec{F} = \nabla f(x,y)$, where $f(x,y) = x \cdot y$.



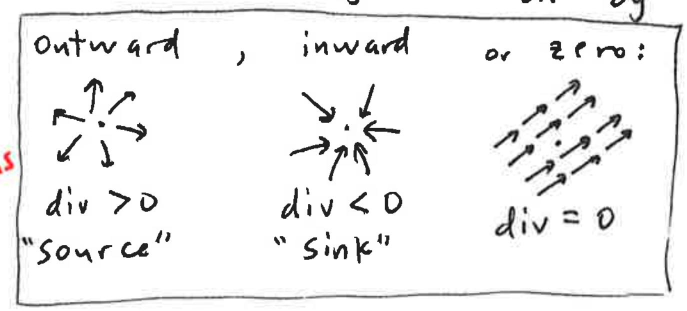
↙ draw in gradient vectors showing the direction of greatest ascent, at many example points of your choice.

Remember: the magnitude of the vector represents how steeply you are ascending.

Think! what would change about this picture if we chose $f(x,y) = xy + 10$, another function for which $\nabla f = \vec{F}$?

Divergence: The divergence of a vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

It measures whether the "net flow" is



new notation: $\nabla = [\partial/\partial x, \partial/\partial y, \partial/\partial z]$, ↖ treat this like a vector

so $\nabla \cdot \vec{F} = [\partial/\partial x, \partial/\partial y, \partial/\partial z] \cdot [P, Q, R]$

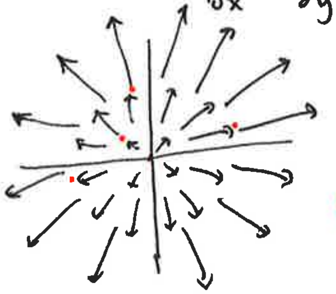
$= \frac{\partial}{\partial x} P + \frac{\partial}{\partial y} Q + \frac{\partial}{\partial z} R = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Using this notation, $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$ ← a dot product of a "vector" with a vector.

Example: Compute the divergence of our two favorite vector fields from last time.

$\vec{F}(x,y) = [x,y]$ so $P(x,y) = x$, $Q(x,y) = y$

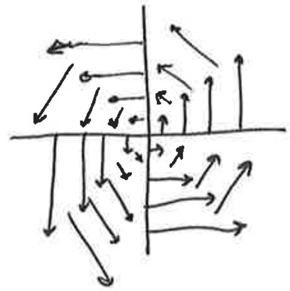
$\Rightarrow \operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 1 + 1 = 2.$



at every point, the vectors going "out" are longer than the vectors coming "in."

$\vec{G}(x,y) = [-y,x]$ so $P(x,y) = -y$, $Q(x,y) = x$

$\Rightarrow \operatorname{div} \vec{G} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 + 0 = 0.$

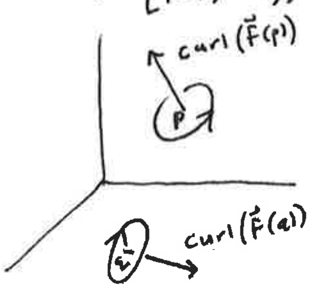


at every point, the vectors going "out" are the same size as the vectors going "in" — nothing is created or destroyed; the "water" just goes in a circle.
↳ we should quantify that:...

Curl. The curl of a vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is $\operatorname{curl}(\vec{F}) = (R_y - Q_z)\vec{i} - (R_x - P_z)\vec{j} + (Q_y - P_x)\vec{k}$.

how will I ever remember this?!

using our new notation $\nabla = [\partial/\partial x, \partial/\partial y, \partial/\partial z]$,
$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$
as the determinant of this matrix!



Geometrically: curl is a vector, measuring "circulation."
- its direction gives the axis of rotation (right hand rule)
- its length gives the speed/strength of rotation.

If $\vec{F}(x,y) = P\vec{i} + Q\vec{j}$ is a 2D vector field, $\operatorname{curl}\vec{F}$ is in z-direction: $[0,0,c]$.
$c > 0$ if the curl is CCW
$c < 0$ " " " CW

Let's find the curl of our favorite vector fields. For $\vec{G}$, we expect $c > 0$.

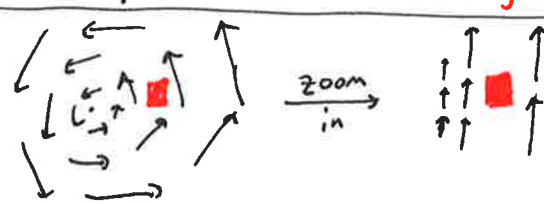
① $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}.$

no rotation anywhere!

② $\operatorname{curl}(\vec{G}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 2\vec{k} = [0,0,2].$

CCW rotation everywhere!

How is it possible that it has CCW rotation everywhere? Imagine putting a small block of wood in a whirlpool.



the current is stronger on the right than on the left, so it will spin CCW!

Do some algebra for a more complicated vector field:
$$\vec{F} = [x \cdot y \cdot \sin(z) + y,\ y - x \cdot e^z,\ x \cdot y \cdot z]$$

notice that div and curl are both functions of x, y and z.
$\operatorname{div}(\vec{F})$ is a scalar function.
$\operatorname{curl}(\vec{F})$ is a vector function.

$\operatorname{div}(\vec{F}) =$

$\operatorname{curl}(\vec{F}) =$

Diana Davis | Math 34 | April 9, 2018 | Class # 29 ①
§6.1

Mathematician spotlight: Steve Robinson, Professor, Wake Forest University

- Studies partial differential equations (PDEs) - similar to Profs. Maringa & Morris
- applies the fact that the game Hex always has a winner, to prove mathematical theorems! → Colloquium Tuesday 4:30pm SCI 199

For the last few weeks of the course, we will study how to integrate a $\begin{cases}\text{function} \\ \text{vector field}\end{cases}$ over a $\begin{cases}\text{curve} \\ \text{surface}\end{cases}$ ← $2\times 2 = 4$ things.

today

From last time: divergence $= \operatorname{div}(\vec F) =$ "amount of stuff created" a scalar function.

curl $= \operatorname{curl}(\vec F) =$ "direction of the axis of rotation" a vector function.

Suppose that $\vec F$ and $\vec G$ are vector fields, and f is a function. Which of the following make sense?

1. $\operatorname{div}(\operatorname{curl}(\vec F))$
2. $\operatorname{curl}(\operatorname{div}(\vec F))$
3. $\operatorname{curl}(\operatorname{div}(f))$
4. $\operatorname{div}(\vec F \bullet \vec G)$
5. $\operatorname{curl}(\nabla f)$
6. $\operatorname{div}(\nabla f)$

Let's compute: For $f(x,y,z)$, and for $F = [P,Q,R]$,

(assuming f is reasonably nice)

$$\operatorname{curl}(\nabla f) = \operatorname{curl}([f_x, f_y, f_z]) = \begin{vmatrix} \vec i & \vec j & \vec k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix} = [f_{zy} - f_{yz}, -(f_{zx} - f_{xz}), f_{yx} - f_{xy}] = [0,0,0]$$

by Clairaut's Theorem!

(assuming P, Q, R are reasonably nice)

$$\operatorname{div}(\operatorname{curl}(\vec F)) = \operatorname{div}(R_y - Q_z, P_z - R_x, Q_x - P_y) = \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}} = 0$$

by Clairaut's Theorem!

<u>Theorem.</u> For any function f, $\operatorname{curl}(\nabla f) = \vec 0$. ← vector

For any vector field $\vec F$, $\operatorname{div}(\operatorname{curl}(\vec F)) = 0$. ← number

---

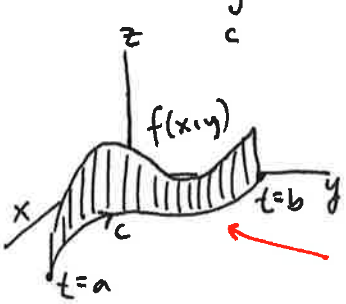
<u>Scalar line integrals!</u> Integrating a function over a curve.

"charge" "snow" "wire" "path"

Given a function $f(x,y)$ and a curve C in the xy-plane with parametric equations $\vec r(t)$ for $a \le t \le b$, the scalar line integral of f over C is

$$\int_C f(x,y)\, ds = \int_{t=a}^{t=b} \underbrace{f(\vec r(t))}_{\text{the height of the function at the point where you are at time } t} \underbrace{\|\vec r'(t)\|\, dt}_{ds}$$

↳ ds, an infinitesimal piece of arclength = the distance you travel in "dt" seconds

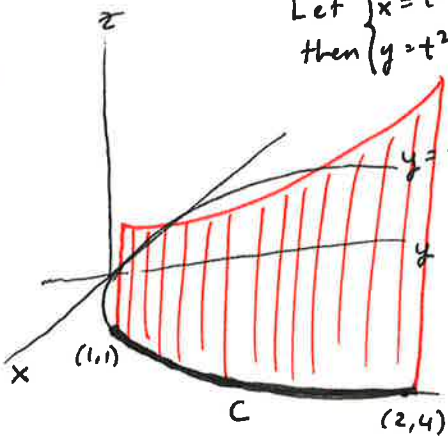


the integral gives the "total area of the fence."
charge wire
snow path

Example. Compute the line integral of $f(x)=8x$, along the curve C consisting of the part of the parabola $y=x^2$ from $x=1$ to $x=2$.

First step: find parametric equations for C. tip: make one variable "t", then solve for the other variables in terms of t.

Let $\begin{cases} x=t \\ y=t^2 \end{cases}$ from $t=1$ to $t=2$.
then



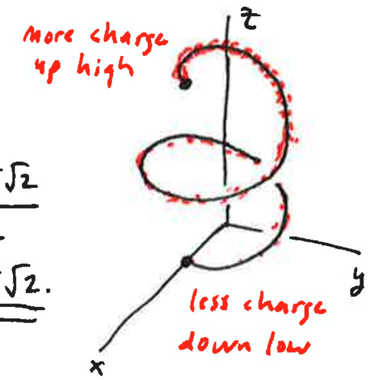
So $\displaystyle\int_C f(x,y)\,ds = \int_{t=1}^{t=2} f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$

$$= \int_{t=1}^{t=2} f(t,t^2)\,\|[1,2t]\|\,dt = \int_{t=1}^{t=2} 8t\sqrt{1+4t^2}\,dt$$

$$= \frac{2}{3}\left(1+4t^2\right)^{3/2}\Big|_{t=1}^{t=2} = \underline{\underline{\frac{2}{3}\left(17^{3/2}-5^{3/2}\right)}}. \approx 39$$

estimate answer:
length ≈ 3
height ≈ 12
$\Rightarrow$ expect answer ≈ 36

Example. A coil of wire (spring) is shaped like two turns of the helix $\vec{x}(t)=(\cos t, \sin t, t)$, so from $t=0$ to $t=4\pi$. The amount of charge at any point is given by $f(x,y,z)=z$, due to a nearby charged object. Compute the total charge on the wire.



Let's compute $\vec{x}\,'(t) = [-\sin t, \cos t, 1]$
$\Rightarrow \|\vec{x}\,'(t)\| = \sqrt{(-\sin t)^2+(\cos t)^2+1^2} = \sqrt{1+1} = \sqrt{2}.$

So $\displaystyle\int_C f(x,y,z)\,ds = \int_{t=0}^{t=4\pi} f(\cos t, \sin t, t)\,\|\vec{x}\,'(t)\|\,dt = \int_{t=0}^{t=4\pi} t\sqrt{2}\,dt = \frac{t^2\sqrt{2}}{2}\Big|_{t=0}^{t=4\pi} = \frac{(4\pi)^2\sqrt{2}}{2}$

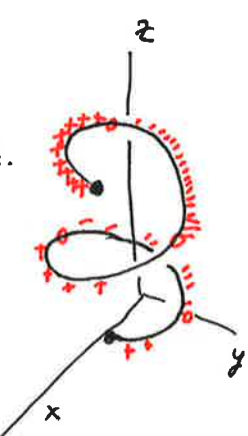
$$= \underline{\underline{8\pi^2\sqrt{2}}}.$$

Example. If we integrate $\displaystyle\int_C 1\,ds$, what does it mean? The <u>length</u> of C.

For the curved wire C from above, its length is

$$\underbrace{\int_{t=0}^{t=4\pi} 1\,\|\vec{x}\,'(t)\|\,dt}_{\text{same as our arclength formula}} = \int_{t=0}^{4\pi} \sqrt{2}\,dt = \underbrace{4\pi}_{\text{circumference of 2 unit circles}}\,\underbrace{\sqrt{2}}_{\text{from being stretched "up" in the z-direction}}.$$

Example. Find the total charge on the wire if charge is given by $f(x,y,z)=xz$.

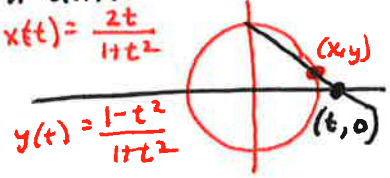


$$\int_C f(x,y,z)\,ds = \int_{t=0}^{t=4\pi} t\cdot\cos t\cdot\sqrt{2}\,dt = \sqrt{2}\int_{t=0}^{t=4\pi} t\cdot\cos t\cdot dt = \sqrt{2}\left(t\sin t+\cos t\right)\Big|_{t=0}^{t=4\pi}$$

integration by parts!

$$= \sqrt{2}\left(4\pi\sin 4\pi + \cos 4\pi - (0\sin 0 + \cos 0)\right) = \sqrt{2}(0) = \underline{\underline{0.}}$$

Mathematician spotlight: Yajnaseni Dutta, graduate student, Northwestern Univ.
- studies birational geometry & Hodge theory
- birational geometry: studying which curves can be mapped to each other via "rational functions": $x(t) = \frac{2t}{1+t^2}$, $y(t) = \frac{1-t^2}{1+t^2}$



Recall: plan for rest of semester: integrate a scalar / vector function along a curve / surface
scalar function along a curve: last time
vector function along a curve: this time, and also the next two (Green's Thm).

First, review: scalar line integrals

Example. Compute $\int_C (z^2 - 12y + z)\, ds$, where C is defined by $\vec{x}(t) = [t^3, 3t^2, 6t]$, $-1 \le t \le 2$.

① have parametric equations for C! ② we'll need $\|\vec{x}\,'(t)\|$: $\vec{x}\,'(t) = [3t^2, 6t, 6]$

$$\Rightarrow \|\vec{x}\,'(t)\| = \sqrt{9t^4 + 36t^2 + 36} = \sqrt{(3t^2+6)^2} = \underline{3t^2 + 6}.$$

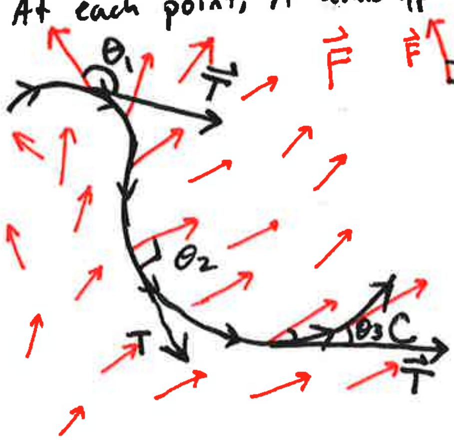
③ we'll need $f(\vec{x}(t))$: $f(x,y,z) = z^2 - 12y + z$ and $[x(t), y(t), z(t)] = [t^3, 3t^2, 6t]$, so

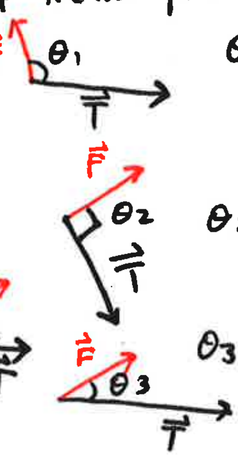
$$f(\vec{x}(t)) = (6t)^2 - 12(3t^2) + (6t) = 36t^2 - 36t^2 + 6t = \underline{6t}.$$

④ Now set up the integral:

$$\int_C f\, ds = \int_{t=-1}^{t=2} f(\vec{x}(t))\, \|\vec{x}\,'(t)\|\, dt = \int_{t=-1}^{t=2} \underbrace{6t}_{f(\vec{x}(t))} \overbrace{(3t^2+6)}^{\|\vec{x}\,'(t)\|} dt = \frac{1}{2}(3t^2+6)^2 \Big|_{t=-1}^{t=2} = \frac{1}{2}(18^2 - 9^2) = \frac{243}{2}.$$

Vector line integrals! Given a vector field $\vec{F}$ and a curve C with its direction specified, the vector line integral of $\vec{F}$ measures how much $\vec{F}$ "points in the same direction" as C. At each point, it adds up the dot product of $\vec{F}$ with the unit tangent vector to C.





$\theta_1 > \frac{\pi}{2}$ so $\vec{F} \cdot \vec{T} < 0$: $\vec{F}$ and C go different directions. "$\vec{F}$ goes against the flow"

$\theta_2 = \frac{\pi}{2}$ so $\vec{F} \cdot \vec{T} = 0$: $\vec{F}$ and C are perpendicular.

$\theta_3 < \frac{\pi}{2}$, so $\vec{F} \cdot \vec{T} > 0$: $\vec{F}$ and C go in the same direction. "F goes with the flow"

Symbolically, we compute $\int_C \vec{F} \cdot d\vec{s} = \int_C \vec{F} \cdot \vec{T}\, ds$, where $\vec{T}$ is the unit tangent vector at each point. We use the unit tangent vector because the "speed" along C should not affect how much C moves with/against $\vec{F}$.

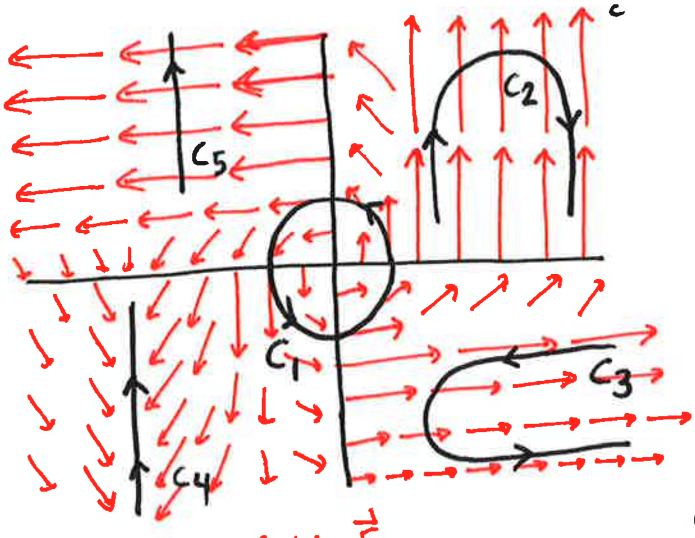
What if you reverse the direction of the curve?

Let $-C$ be the curve C, traversed in the opposite direction as C.

$$\int_{-C} \vec{F} \cdot d\vec{s} = -\int_C \vec{F} \cdot d\vec{s}$$

because it changes the sign of all the computations — "with the flow" becomes "against the flow," and vice-versa.

Example. Determine whether $\int_C \vec{F}\cdot d\vec{s}$ is positive, negative or zero for each curve.



C_1: The curve goes with the flow, so $\int_{C_1} \vec{F}\cdot d\vec{s}$ ___ 0.

C_2: The curve goes with the flow at the beginning, and against the flow at the end, and they are equal and opposite, so $\int_{C_2} \vec{F}\cdot d\vec{s}$ ___ 0.

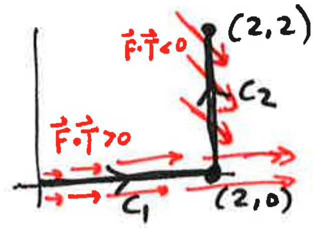
C_3: Curve goes against the flow at the beginning and with it at the end, but the flow against is stronger, so $\int_{C_3} \vec{F}\cdot d\vec{s}$ ___ 0.

C_4: Curve is always against the flow, so $\int_{C_4} \vec{F}\cdot d\vec{s}$ ___ 0.

C_5: Curve is always perpendicular to the flow, so $\int_{C_5} \vec{F}\cdot d\vec{s}$ ___ 0.

Now that we know what vector line integrals mean, let's compute them!

Method 1: compute by considering function and curve together.



Example: Compute the line integral of $\vec{F} = [x^2+y, -(x+1)y]$ over $C = C_1 C_2$:

Along C_1, $y=0$, so $\vec{F} = [x^2, 0]$, and $\vec{T} = [1,0]$, so $\vec{F}\cdot\vec{T} = [x^2,0]\cdot[1,0] = \underline{\underline{x^2}}$.

Along C_2, $x=2$, so $\vec{F} = [4+y, -3y]$, and $\vec{T} = [0,1]$, so $\vec{F}\cdot\vec{T} = [4+y,-3y]\cdot[0,1] = \underline{-3y}$.

$$\text{So } \int_C \vec{F}\cdot d\vec{s} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_{C_1}\vec{F}\cdot\vec{T}\,ds + \int_{C_2}\vec{F}\cdot\vec{T}\,ds = \int_{x=0}^{x=2} x^2\,dx + \int_{y=0}^{y=2} -3y\,dy = \underbrace{+\frac{8}{3}}_{\vec{F}\text{ is with }C_1} + \overbrace{-6}^{\vec{F}\text{ is against }C_2} = \underbrace{-\frac{10}{3}}_{\text{net flow is against } C=C_1C_2.}$$

Method 2: Equation that always works. If C is defined by $\vec{x}(t)$, the tangent vector is $\vec{x}'(t)$, so the unit tangent vector is $\vec{T} = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}$.

$$\text{So } \int_C \vec{F}\cdot\vec{T}\,ds = \int_{t=a}^{t=b} \underbrace{\vec{F}(\vec{x}(t))}_{\vec{F}} \underbrace{\cdot}_{\cdot} \underbrace{\frac{\vec{x}'(t)}{\cancel{\|\vec{x}'(t)\|}}}_{\vec{T}} \underbrace{\cancel{\|\vec{x}'(t)\|}\,dt}_{ds} = \int_{t=a}^{t=b} \vec{F}(\vec{x}(t))\cdot\vec{x}'(t)\,dt.$$

← if you do it this way, you don't have to worry about making a unit tangent vector!

Example: Compute the line integral of $\vec{F} = \left[\frac{-y\cdot\sin x}{x^2}, \frac{\cos x}{2x}\right]$ over the piece of $y=x^2$ from $\left(\frac{\pi}{2}, \frac{\pi^2}{4}\right)$ to $\left(\frac{5\pi}{4}, \frac{25\pi^2}{16}\right)$.

Let $\vec{x}(t) = [t, t^2]$, $\frac{\pi}{2} \le t \le \frac{5\pi}{4}$

$\Rightarrow \vec{x}'(t) = [1, 2t]$.

Also, $\vec{F}(\vec{x}(t)) = \vec{F}(t,t^2) = \left[\frac{-\cancel{t^2}\sin t}{\cancel{t^2}}, \frac{\cos t}{2t}\right] = \left[-\sin t, \frac{\cos t}{2t}\right]$.

$$\text{So } \int_C \vec{F}\cdot\vec{T}\,ds = \int_{t=\pi/2}^{t=5\pi/4} \vec{F}(\vec{x}(t))\cdot\vec{x}'(t)\,dt = \int_{t=\pi/2}^{t=5\pi/4} \left[-\sin t, \frac{\cos t}{2t}\right]\cdot[1,2t]\,dt = \int_{t=\pi/2}^{t=5\pi/4} (-\sin t + \cos t)\,dt = \cos t + \sin t \Big|_{t=\pi/2}^{t=5\pi/4}$$

$$= \underline{\underline{-\sqrt{2}-1.}}$$

Diana Davis Math 34 April 13, 2018 Class #31 ①
§6.2

Mathematician spotlight: Pamela Harris, Assistant Professor, Williams College
- studies algebraic combinatorics, representation theory
- example: visibility problems

☺ you
• trees you can see
• trees you can't see (blocked)

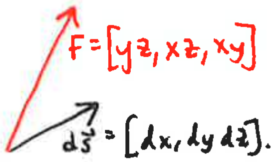
In the whole orchard, what proportion of trees can you see? $\frac{6}{\pi^2}$ 😮

Today: integrating vector fields over (closed) curves $\xleftrightarrow[\text{Green's Theorem, as}]{\text{is the same, by}}$ integrating scalar functions over the area region inside

First, two warm-up vector line integrals.

Example. Compute $\underbrace{\int_C yz\,dx + xz\,dy + xy\,dz}$ where C is $\vec{x}(t) = [t, t^2, t^3]$ for $0 \le t \le 2$.

What is this?! Rewrite: $yz\,dx + xz\,dy + xy\,dz = [yz, xz, xy] \cdot [dx, dy, dz]$
$= \vec{F} \cdot d\vec{s}$. Oh. Familiar!

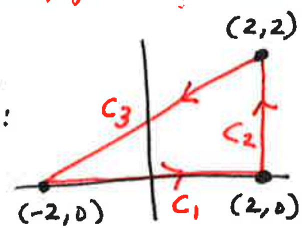


Here we have $x = t$, $y = t^2$, $z = t^3$
$\Rightarrow dx = dt$, $dy = 2t\,dt$, $dz = 3t^2\,dt$.

So compute: $$\int_C yz\,dx + xz\,dy + xy\,dz = \int_{t=0}^{t=2} \underbrace{\overset{y}{(t^2)}\overset{z}{(t^3)}}_{t^5}\overset{dx}{dt} + \underbrace{\overset{x}{(t)}\overset{z}{(t^3)}\overset{dy}{2t}}_{2t^5}dt + \underbrace{\overset{x}{(t)}\overset{y}{(t^2)}\overset{dz}{3t^2}}_{3t^5}dt = \int_{t=0}^{t=2} 6t^5\,dt = t^6\Big|_0^2 = \underline{\underline{64.}}$$

This notation is actually very nice, because it tells us exactly what we need to multiply and add up, to get the line integral over our vector field for our curve.

Example. Compute the line integral of $\vec{F} = [2x^2 - 3y^2, 2x + 3y^2]$ over the curve C:



Let's do each of these separately, C_1 and C_2 cleverly and C_3 with the equation:

• On C_1, $y = 0$, so $\vec{F} = [2x^2, 2x]$. Also, C is in the positive x-direction, so $\vec{T} = [1, 0]$ and $ds = dx$. So
$$\int_{C_1} \vec{F} \cdot \vec{T}\,ds = \int_{x=-2}^{x=2} [2x^2, 2x] \cdot [1, 0]\,dx = \int_{-2}^{2} 2x^2\,dx = \frac{2}{3}x^3\Big|_{x=-2}^{x=2} = \underline{\underline{\frac{32}{3}.}}$$

• On C_2, $x = 2$, so $\vec{F} = [8 - 3y^2, 4 + 3y^2]$. Also, C is in the positive y-direction, so $\vec{T} = [0, 1]$ and $ds = dy$.
$$\int_{C_2} \vec{F} \cdot \vec{T}\,ds = \int_{y=0}^{y=2} [8 - 3y^2, 4 + 3y^2] \cdot [0, 1]\,dy = \int_{y=0}^{y=2} (4 + 3y^2)\,dy = 4y + y^3\Big|_{y=0}^{y=2} = \underline{\underline{16.}}$$

• We need to parameterize C_3. We start at $(2,2)$ and go in direction $[-4, -2]$, so we have
$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + t\begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{pmatrix} 2 - 4t \\ 2 - 2t \end{pmatrix} \text{ for } 0 \le t \le 1.$$
Check: $t = 0 \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $t = 1 \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ ☺

$\Rightarrow \vec{x}'(t) = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$ and $\vec{F}(x,y) = [2x^2 - 3y^2, 2x + 3y^2] \Rightarrow \vec{F}(2-4t, 2-2t) = [2(2-4t)^2 - 3(2-2t)^2, 2(2-4t) + 3(2-2t)^2]$.

$$\int_{C_3} \vec{F} \cdot \vec{T}\,ds = \int_{t=a}^{t=b} \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\,dt = \int_{t=0}^{t=1} [32t^2 - 32t + 8 - 12t^2 + 24t - 12, 4 - 8t + 12t^2 - 24t + 12] \cdot [-4, -2]\,dt$$
$$= \int_{t=0}^{t=1} (-104t^2 + 96t - 16)\,dt = \underline{\underline{\frac{-8}{3}.}}$$

$$\Rightarrow \int_C \vec{F} \cdot \vec{T}\,ds = \frac{32}{3} + 16 - \frac{8}{3} = \underline{\underline{24.}}$$

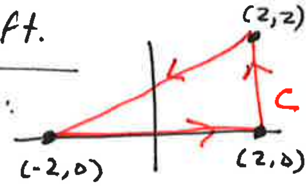
Wow, that was so tedious. There must be a better way. Yes!

Green's Theorem. For a vector field $\vec{F} = [P, Q]$, where P and Q have continuous partial derivatives throughout a region D in the xy-plane whose boundary ∂D consists of simple, closed curves, we have

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA,$$

no self intersections.   has to enclose a region!

when ∂D is oriented so that, moving along it, you always have D on the left.



First, let's try it! For our previous example: $\vec{F} = [\underbrace{2x^2 - 3y^2}_{P}, \underbrace{2x + 3y^2}_{Q}]$ on C:

$Q_x = 2$ and $P_y = -6y$, so

$$\iint_D (Q_x - P_y)\,dA = \iint_D (2+6y)\,dA = \int_{y=0}^{y=2}\int_{x=2y-2}^{x=2} (2+6y)\,dx\,dy = \int_{y=0}^{y=2} (2+6y)(2-(2y-2))\,dy = \int_0^2 (-12y^2 + 20y + 8)\,dy = 24.$$

$= -3$

So much easier!

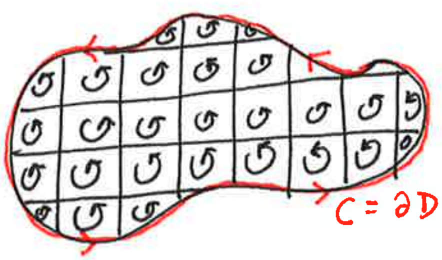
OK, now why does Green's Theorem work?

- If $\vec{F} = [P, Q]$, we can write $\vec{F} = [P, Q, 0]$ and compute $\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = [\underbrace{0, 0, Q_x - P_y}]$.

  a vector in the z-direction

- So Green's Theorem says $\displaystyle\int_C \vec{F}\cdot\vec{T}\,ds \xrightarrow{\text{rewrite}} \int_{\partial D} P\,dx + Q\,dy \xrightarrow{\text{Green's Theorem}} \iint_D (Q_x - P_y)\,dA \xrightarrow{\text{rewrite}} \iint_D \text{``}\operatorname{curl}(\vec{F})\text{''}\,dA$

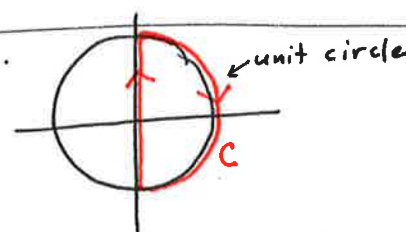
  just the z-component



- $\iint_D \operatorname{curl}(\vec{F})\,dA$ adds up the circulation at each point in D. It cancels out along each interior boundary, so all you get is the circulation around the outside curve $C = \partial D$!

Let's do one more example, and use the theorem.

Example. Compute $\displaystyle\int_C -y\,dx + x\,dy$ where C is:



Estimate: $[-y, x]$ is one of our favorite vector fields, CCW circulation, so we expect our answer to be ___ 0.

Check the conditions of Green's Theorem:

- $P = -y$ and $Q = x$ have continuous partial derivatives throughout the enclosed region: ✓
- The boundary consists of simple, closed curves: ✓
- The boundary curve is oriented so that, moving along it, the enclosed region is on the left: X
- The boundary curve is oriented so that, moving along it, the enclosed region is on the left: X

  → so we need to change the sign of the double integral. No problem!

$$\int_C -y\,dx + x\,dy = -\int_{-C} -y\,dx + x\,dy = -\iint_D (Q_x - P_y)\,dA = -\iint_D (1-(-1))\,dA = -\iint_D 2\,dA = -2\,(\text{area of } D)$$
$$= -2\left(\tfrac{\pi}{2}\right) = -\pi.$$

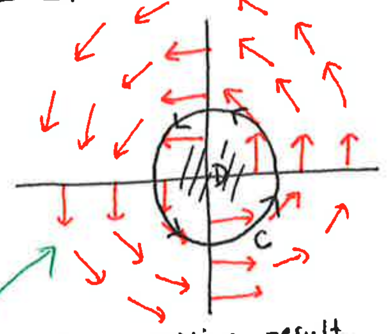
Note: this one would not have been too hard to do directly as a vector line integral, setting it up over the curved part and over the line part, but Green's Theorem allows us to do it all in one step.

Diana Davis   Math 34   April 16, 2018   Marathon Monday   Class #32 1 §6.3

Mathematician spotlight: Harrison Bray, postdoc, University of Michigan
Hamilton College undergrad, Tufts Univ. PhD (Boston!)
- Studies geometric structures, geodesic flow
- studying the Thurston set in the complex plane - beautiful pictures

Today: More on Green's Theorem, and another instance where vector line integrals become easier - conservative vector fields.

Example: (danger zone) Compute $\int_C \frac{-y\,dx + x\,dy}{x^2+y^2}$, where C is the CCW unit circle.



Green's Theorem says: $\int_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$.   Let D = unit disk, so $\partial D = C$.

Here, $P = \frac{-y}{x^2+y^2} \Rightarrow \underset{\text{quotient rule}}{\ldots} \Rightarrow P_y = \frac{y^2-x^2}{(x^2+y^2)^2}$, $Q = \frac{x}{x^2+y^2} \Rightarrow \underset{\text{quotient rule.}}{\ldots} \Rightarrow Q_x = \frac{y^2-x^2}{(x^2+y^2)^2}$.

Expect a positive result.

error: these vectors should get smaller as you go further out (not unit vectors as shown)

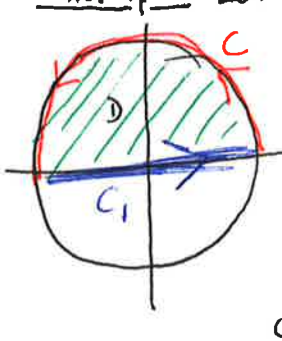
So $\int_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \iint_D \left(\left(\frac{y^2-x^2}{x^2+y^2}\right) - \left(\frac{y^2-x^2}{x^2+y^2}\right)\right) dA = \iint_D 0\,dA = 0.$?! what went wrong??

The problem: Green's Theorem only holds if the vector field $F = [P, Q]$ is continuous everywhere in the region enclosed by D. This is not the case — here we are dividing by 0 at the origin, and $[P, Q]$ is not continuous there. (which direction would it point??)

Note: It would be totally ok to integrate this vector field on a curve not enclosing $\vec{0}$, using Green's Theorem.

Closing off a curve to apply Green's Theorem:

Example. Let $\vec{F} = [xy^2 + x^2,\ x^2y + x - y^{\sin(e^y)}]$ and let C be the top part of the CCW unit circle:



Find $\int_C \vec{F}\cdot\vec{T}\,ds$. Hmm, we want to use Green's Theorem, but C is not closed?!

Idea: 1 close off C with another curve C_1, to enclose a region (D).
2 use Green's Thm over D, and subtract off $\int_{C_1} \vec{F}\cdot\vec{T}\,ds$ to get $\int_C \vec{F}\cdot\vec{T}\,ds$.

$$\int_{C+C_1} \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{T}\,ds + \int_{C_1} \vec{F}\cdot\vec{T}\,ds = \iint_D (Q_x - P_y)\,dA \rightarrow \underbrace{\int_C \vec{F}\cdot\vec{T}\,ds}_{\text{want}} = \underbrace{\iint_D (Q_x-P_y)\,dA}_{\text{Green's Thm}} - \underbrace{\int_{C_1} \vec{F}\cdot\vec{T}\,ds}_{\text{compute directly}}.$$

Let's compute each part:

$Q(x,y) = x^2y + x - y^{\sin(e^y)} \Rightarrow Q_x = 2xy + 1$
$P(x,y) = xy^2 + x^2 \Rightarrow P_y = 2xy$

$\Rightarrow \iint_D (Q_x - P_y)\,dA = \iint_D (2xy + 1 - 2xy)\,dA = \iint_D 1\,dA = \text{area of } D = \frac{\pi}{2}.$

On C_1, $y = 0 \Rightarrow \vec{F} = [x^2, x]$
and $\vec{T} = [1, 0]$
and $ds = dx$, so

$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{x=-1}^{x=1} [x^2, x]\cdot[1,0]\,dx = \int_{x=-1}^{x=1} x^2\,dx = \frac{2}{3}.$$

So $\int_C = \iint_D - \int_{C_1}$
$= \frac{\pi}{2} - \frac{2}{3}.$

Some strategies for computing vector line integrals:

(0) Just do it — parameterize the curve, and compute the integral: $\int_{t=a}^{t=b} \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\, dt$.

(1) If the curve is horizontal or vertical, simplify $\vec{F}$ and $\vec{T}$ and compute it directly.

(2) If the curve is closed, or if you can close it off, check that $\vec{F}$, C satisfy Green's Thm and use it.

NEW (3) If $\vec{F}$ is a conservative vector field, apply the Fundamental Theorem of Line Integrals.

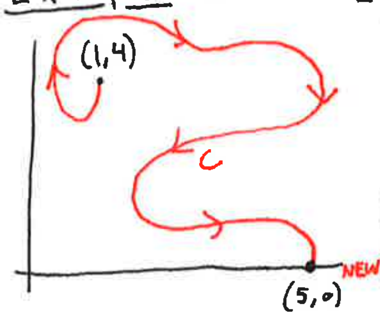
Fundamental Theorem of Line Integrals: If $\vec{F} = \nabla f$ for some function $f(x,y)$ or $f(x,y,z)$ that is defined everywhere on C and the region it encloses, then

$$\int_C \vec{F}\cdot\vec{T}\, ds \underset{\text{rewrite}}{=} \int_C \nabla f \cdot \vec{T}\, ds = f(\text{end point of } C) - f(\text{start point of } C).$$

$f(x,y)$ is called the potential function.

just like Fundamental Theorem of Calculus, $\int_a^b f(x)dx = F(b) - F(a)$, if $F'(x) = f(x)$.

Example. Let $\vec{F} = [e^y + y^2 + 1, xe^y + 2xy + \cos y]$ be the wind vector field in Swarthmore.



The classroom is located at (1,4) and the dining hall at (5,0).

Let C be the path shown, taken to avoid puddles. Compute $\int_C \vec{F}\cdot\vec{T}\, ds$, which is the amount the wind helped/hurt us get to lunch.

(0) Can't use any previous method, since we don't know enough about C.

(3) See if we can find f so that $\vec{F} = \nabla f$, i.e. hope that $\vec{F}$ is conservative.

Ok, we want $f_x = e^y + y^2 + 1$ and $f_y = xe^y + 2xy + \cos y$

$\Rightarrow f(x,y) = xe^y + xy^2 + x + g(y)$ ← any function of y would have disappeared

$\Rightarrow f(x,y) = xe^y + xy^2 + \sin(y) + h(x)$ ← any function of x would have disappeared

Combining these, the function that works for both is

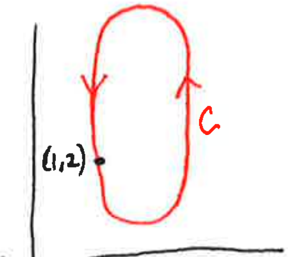
$f(x,y) = xe^y + xy^2 + x + \sin(y)$. → check: $f_x = e^y + y^2 + 1$ ✓ $f_y = xe^y + 2xy + \cos y$ ✓

Ok, so $\vec{F}$ is conservative, and its potential function is this $f(x,y)$ we just found.

So by the Fund. Thm. of Line Integrals, $\int_C \vec{F}\cdot\vec{T}\, ds = f(5,0) - f(1,4) = (5\cdot e^0 + 5\cdot 0^2 + 5 + \sin(0)) - (1\cdot e^4 + 1\cdot 4^2 + 1 + \sin(4))$

$= -7 - e^4 - \sin(4)$. A tough walk to lunch!

Example. Suppose that, under the same weather conditions as above, you decide to run/swim 10,000m around the track. Thus C consists of 25 ccw laps of the track, as shown.



Compute $\int_C \vec{F}\cdot\vec{T}\, ds$.

We already showed that $\vec{F}$ is a conservative vector field, with $\vec{F} = \nabla f$ for the potential function $f(x,y) = xe^y + xy^2 + x + \sin(y)$ above.

So $\int_C \vec{F}\cdot\vec{T}\, ds = f(\text{end point}) - f(\text{start point})$.

Since the race starts and ends in the same place, $f(\text{end point}) = f(\text{start point})$,

So $\int_C \vec{F}\cdot\vec{T}\, ds = 0$.

Mathematician spotlight: Siddhi Krishna, graduate student, Boston College

- studies topology: the shape of stretchy surfaces
- also knot theory, and 3- and 4-manifolds (surfaces one or two dimensions up)
- teaches at BEAM in the summers, math for smart inner-city kids.

We are learning to integrate $\begin{cases}\text{functions}\\ \text{vector fields}\end{cases}$ over $\begin{cases}\text{curves} & \leftarrow \text{we can do this now! yay!}\\ \text{surfaces.} & \leftarrow \text{we'll study this from now to the end.}\end{cases}$

From last time: If $\vec{F}$ is a gradient vector field, i.e. $\vec{F} = \nabla f$ for some function f, then f is a conservative vector field: $\int_C \vec{F}\cdot\vec{T}\,ds = 0$ for a closed curve C, and $\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2}\vec{F}\cdot\vec{T}\,ds$ if .

Check this out: Even if $\vec{F}$ is not conservative, part of it might be; break it into cons. & non-cons.:

Also: To see if $\vec{F}$ is conservative, check if $P_y - Q_x = 0$.
Because if $\vec{F} = \nabla f = [f_x, f_y]$, then (by Clairaut's Theorem) $f_{xy} = f_{yx} \Rightarrow f_{xy} - f_{yx} = 0$.

Example: Is $\vec{F} = [1-2y+2xe^{x^2}y^2,\ 2ye^{x^2}+2x]$ conservative?

$$\frac{d}{dy}(1-2y+2xe^{x^2}y^2) - \frac{d}{dx}(2ye^{x^2}+2x) \stackrel{?}{=} 0$$

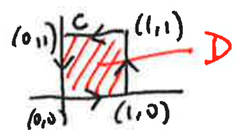
$$-2 + \cancel{4xy}e^{x^2} - \cancel{4xy}e^{x^2} - 2 = -4 \neq 0 \therefore \text{ not conserative.}$$

But part of it is!

$$\vec{F} = \underbrace{[1+2xe^{x^2}y^2,\ 2ye^{x^2}]}_{\text{conservative! } = \nabla f \text{ for}} + \underbrace{[-2y, 2x]}_{\text{not conservative.}}$$

$f(x,y) = x + e^{x^2}\cdot y^2$

Let's do this: Integrate $\vec{F}$ over the ccw unit square, C:



$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C (\nabla f + [-2y, 2x])\cdot\vec{T}\,ds = \int_C \nabla f\cdot\vec{T}\,ds + \int_C [\overbrace{-2y}^{P}, \overbrace{2x}^{Q}]\cdot\vec{T}\,ds$$

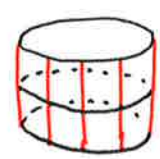
$$= 0 + \iint_D (2 - -2)\,dA = 0 + \iint_D 4\,dA = 4\,(\text{area of } D) = \underline{\underline{4.}}$$

Parameterized surfaces

We can describe any curve in space by $\vec{x}(t) = [x(t), y(t), z(t)]$ ← one variable: a 1-dim'l thing

We can describe any surface in space by $\vec{X}(s,t) = [x(s,t), y(s,t), z(s,t)]$ ← two variables: a 2-dim'l thing

Example: Sketch the surface described by $\vec{X}(s,t) = [\cos t, \sin t, s]$



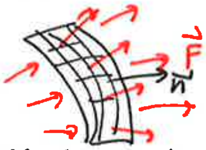
- hold s constant, vary t: get a unit circle at height s
- hold t constant, vary s: get a vertical line

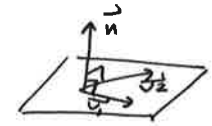
→ (infinite, vertical) cylinder!

Example: Find parametric equations for the sphere of radius R centered at the origin.

- First idea: $\vec{X}(x,y) = (x, y, \pm\sqrt{R^2-x^2-y^2})$ for x, y in the unit disk. ← Yuck!
- Better idea: $\vec{X}(r,\theta) = (r\cdot\cos\theta, r\cdot\sin\theta, \pm\sqrt{R^2-r^2})$, $0 \le r \le R$, $0 \le \theta < 2\pi$. ← Better, but...
- Best idea: $\vec{X}(\phi,\theta) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. ☺

We will frequently want the normal vector to a surface, so that we can use a dot product to measure how much a vector field points through it:





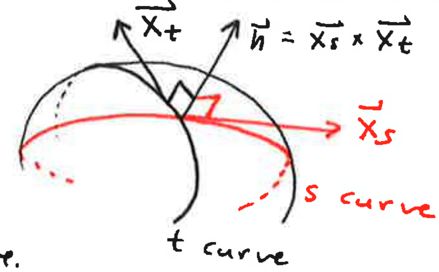
Recall: To find a vector that is perpendicular to a plane, find two vectors $\vec{v}_1, \vec{v}_2$ in the plane and take their cross product: $\vec{n} = \vec{v}_1 \times \vec{v}_2$.

We need to find two vectors that are tangent to the surface and take their cross product.

- We'll get one from s ($\vec{X}_s$) and one from t ($\vec{X}_t$) and get a normal vector $\vec{n} = \vec{X}_s \times \vec{X}_t$.

→ Hold t fixed and vary s: get an "s curve." Differentiate $\vec{X}$ with respect to s and get a vector tangent to the s curve, $\vec{X}_s$.

→ Hold s fixed and vary t: get a "t curve." Differentiate $\vec{X}$ with respect to t and get a vector tangent to the t curve, $\vec{X}_t$.



Definition: A surface is smooth at a given point if $\vec{X}_s \times \vec{X}_t \neq \vec{0}$ there.

Example: Find a normal vector to the sphere $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$. Expect: vector in radial direction, a multiple of $\vec{X}(\phi,\theta)$.

$\vec{X}_\phi = [R\cos\phi\cos\theta, R\cos\phi\sin\theta, -R\sin\phi]$

$\vec{X}_\theta = [-R\cos\phi\sin\theta, R\cos\phi\cos\theta, 0]$

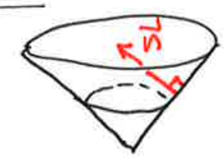
$$\Rightarrow \vec{X}_\phi \times \vec{X}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\theta \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= [R^2\sin^2\phi\cos\theta, R^2\sin^2\phi\sin\theta, R^2\sin\phi\cos\theta]$$

$$= (R\sin\phi)[R\sin\phi\cos\phi, R\sin\phi\sin\theta, R\cos\theta]$$

$$= (R\sin\phi)\,\vec{X}(\phi,\theta)$$ ← a multiple of $\vec{X}(\phi,\theta)$ as expected.

Example: Parameterize the cone $z=r$ and find the normal vector at $(0,1,1)$.



Let's use polar: $\vec{X}(r,\theta) = [r\cos\theta, r\sin\theta, r]$.

Which r,θ get us to $[0, 1, 1]$? $r =$ _____, $\theta =$ _____.

- The r-curve through $(0,1,1)$ is $\vec{X}(r,\frac{\pi}{2}) = [r\cos\frac{\pi}{2}, r\sin\frac{\pi}{2}, r] = [0,r,r]$.

  $\Rightarrow \vec{X}_r = [0,1,1]$ (everywhere)

- The θ-curve through $(0,1,1)$ is $\vec{X}(1,\theta) = [\cos\theta, \sin\theta, 1]$

  $\Rightarrow \vec{X}_\theta = [-\sin\theta, \cos\theta, 1]$, so X_θ at $\theta = \frac{\pi}{2}$ is $[-1,0,0]$.

- Now take the cross product: $\vec{X}_r \times \vec{X}_\theta = [0,1,1]\times[-1,0,0] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = [0,-1,1]$ ← this points "inward"

For a general point (r,θ), $\vec{X}_r = [\cos\theta, \sin\theta, 1]$, $\vec{X}_\theta = [-r\sin\theta, r\cos\theta, 0]$

$\Rightarrow \vec{n} = \vec{X}_r \times \vec{X}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = [-r\cos\theta, -r\sin\theta, r]$ inward, towards the z-axis.

$\vec{n} = \vec{0}$ at $r=0$, so not smooth at origin.

Orientation: For the cone and sphere, there is a notion of "inward" and "outward."

For a plane, there is no "inside" and "outside," but there is a well-defined notion of sides - you could paint one side red and the other side blue.

For some surfaces, such as a Möbius strip or Klein bottle, they have only one side! They are "nonorientable." We won't study them. :-)

Diana Davis | Math 34 | April 20, 2018 | Class #34 ① §7.1

Mathematician spotlight: Henry Segerman, Assistant Professor, Oklahoma State University
- Studies 3-manifolds, triangulations, hyperbolic geometry
- does mathematical visualization, 3D printing, virtual reality for visualization
- creates 3D printed objects that blend math and art (IMHO)

Last time: We discovered how to describe a surface parametrically, with two parameters: $\vec{x}(s,t)$, $\vec{x}(r,\theta)$ etc.
We determined that the normal vector to the surface is $\vec{n} = \vec{X_s} \times \vec{X_t}$ (etc.)

Today: We are integrating $\begin{Bmatrix}\text{scalar}\\ \text{vector}\end{Bmatrix}$ functions over $\begin{Bmatrix}\text{curves}\\ \text{surfaces}\end{Bmatrix}$: Scalar surface integrals!

First, let's find a tangent plane.

Example: Find an equation for the tangent plane to DD's favorite surface,
$\vec{x}(r,\theta) = [r\cdot\cos\theta,\ r\cdot\sin\theta,\ \cos r]$, for $0 \le r \le \infty$, $0 \le \theta \le 2\pi$, at $(1, 0, \cos 1)$, i.e. when $r=1, \theta=0$.

r-curves:
- hold θ constant
- vary r

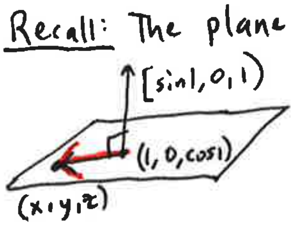
$\theta = 0 \Rightarrow \vec{x}(r,0) = [r, 0, \cos r]$
$\theta = \frac{\pi}{2} \Rightarrow \vec{x}(r,\frac{\pi}{2}) = [0, r, \cos r]$ } cosine curves radiating out from 0

θ-curves:
- hold r constant
- vary θ

$r=1 \Rightarrow \vec{x}(1,\theta) = [\cos\theta, \sin\theta, \cos 1]$
$r=10 \Rightarrow \vec{x}(10,\theta) = [10\cos\theta, 10\sin\theta, \cos 10]$ } circles at undulating heights

$$\vec{X_r} = [\cos\theta, \sin\theta, -\sin r] \Rightarrow \vec{X_r}(1,0) = [1, 0, -\sin 1]$$
$$\vec{X_\theta} = [-r\sin\theta, r\cos\theta, 0] \Rightarrow \vec{X_\theta}(1,0) = [0, 1, 0]$$
$$\Rightarrow \vec{X_r} \times \vec{X_\theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\sin 1 \\ 0 & 1 & 0 \end{vmatrix} = \underbrace{[\sin 1, 0, 1]}_{\text{normal vector to surface at } (1,0,\cos 1)} = \vec{n}$$

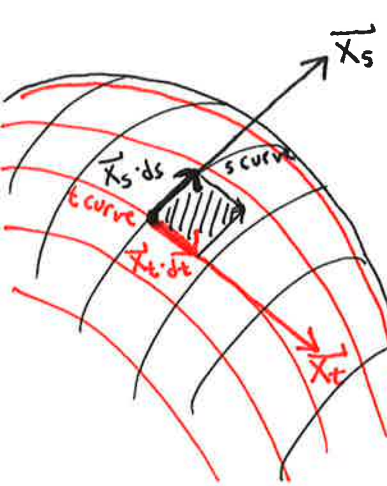
Recall: The plane passing through $(1,0,\cos 1)$ with normal vector $[\sin 1, 0, 1]$ is the collection of points (x,y,z) such that



$$\begin{bmatrix}\sin 1\\ 0\\ 1\end{bmatrix} \cdot \begin{bmatrix}x-1\\ y-0\\ z-\cos 1\end{bmatrix} = 0 \Rightarrow \sin 1(x-1) + 0(y-0) + 1(z-\cos 1) = 0$$
$$\Rightarrow x\cdot\sin 1 - \sin 1 + z - \cos 1 = 0 \Rightarrow \underline{\underline{x\cdot\sin 1 + z = \cos 1 - \sin 1.}}$$
tangent plane equation.

What a cool surface! How would we ever calculate its surface area, for instance?

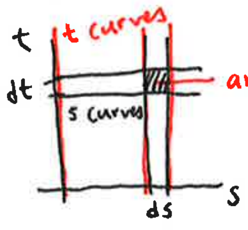
Surface area. Consider a surface S parameterized by $\vec{X}(s,t)$, defined over a region D of the st-plane.
Last time, we showed that $\vec{X_s}$ and $\vec{X_t}$ are tangent to the surface in different directions.
Recall that $\|\vec{u} \times \vec{v}\|$ is the area of the parallelogram spanned by vectors $\vec{u}$ and $\vec{v}$,
so $\|\vec{X_s} \times \vec{X_t}\|$ " " " " " " " $\vec{X_s}$ and $\vec{X_t}$.



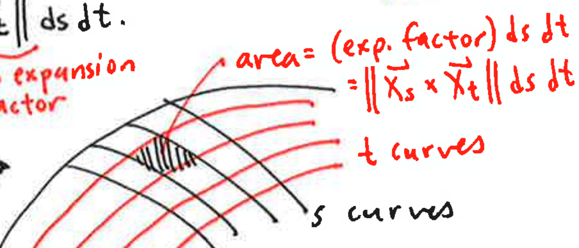
Idea: Make a tiny vector in the s-direction: $\vec{X_s}\cdot ds$ ← tiny distance
and " " " " t-direction: $\vec{X_t}\cdot dt$ ←

Then $\|\vec{X_s}\,ds \times \vec{X_t}\,dt\| = \text{area of tiny parallelogram} = \|\vec{X_s} \times \vec{X_t}\|\underbrace{ds\,dt}_{\text{pull out constants}}$

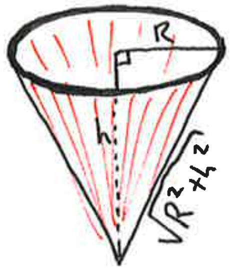
So Surface area of $S = \iint_D \underbrace{\|\vec{X_s} \times \vec{X_t}\|}_{\text{Jacobian expansion factor}} ds\,dt.$



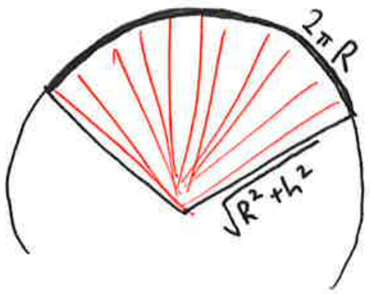
$\xrightarrow{\vec{X}(s,t)}$



Example. Find the surface area of a cone of radius R and height h.



Method 1: geometry. Cut open the cone and lay it flat.



$$\text{area} = \begin{pmatrix}\text{area of disk}\\ \text{of radius}\\ \sqrt{R^2+h^2}\end{pmatrix} \times \begin{pmatrix}\text{proportion}\\ \text{of the disk}\\ \text{that we have}\end{pmatrix}$$

$$= \pi\left(\sqrt{R^2+h^2}\right)^{\cancel{2}} \times \left(\frac{\cancel{2\pi} R}{\cancel{2\pi}\cancel{\sqrt{R^2+h^2}}}\right)$$

$$= \pi R\sqrt{R^2+h^2}.$$

Method 2: Parameterize the surface and integrate $\iint_D \|\vec{X}_r \times \vec{X}_\theta\|\, dr\, d\theta$.

$$\vec{X}(r,\theta) = \left[r\cdot\cos\theta,\ r\cdot\sin\theta,\ \frac{h}{R}\cdot r\right]$$

$$\vec{X}_r = \left[\cos\theta,\ \sin\theta,\ \frac{h}{R}\right] \Rightarrow \vec{X}_r \times \vec{X}_\theta = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \cos\theta & \sin\theta & h/R\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix} = \left[-\frac{h}{R}\cdot r\cos\theta,\ -\frac{h}{R} r\sin\theta,\ r\right]$$

$$\vec{X}_\theta = \left[-r\sin\theta,\ r\cos\theta,\ 0\right] \qquad \Rightarrow \|\vec{X}_r \times \vec{X}_\theta\| = \underbrace{r}_{\text{variable}}\ \underbrace{\frac{\sqrt{R^2+h^2}}{R}}_{\text{constant}}$$

$$\text{So surface area} = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} \overbrace{r\frac{\sqrt{R^2+h^2}}{R}}^{\|\vec{X}_r \times \vec{X}_\theta\|}\, dr\, d\theta = \frac{\sqrt{R^2+h^2}}{R}\cdot\int_{\theta=0}^{2\pi} d\theta\cdot\int_{r=0}^{R} r\,dr = \frac{\sqrt{R^2+h^2}}{\cancel{R}}\cdot\cancel{2}\pi\cdot\frac{R^{\cancel{2}}}{\cancel{2}} = \pi R\sqrt{R^2+h^2}.$$

☺

Scalar surface integrals! Integrating a scalar function over a surface.

$$\text{Surface area} = \iint_D 1\, dS = \iint_D 1\underbrace{\|\vec{X}_s \times \vec{X}_t\|\, ds\, dt}_{dS}$$

dS = tiny piece of area on our surface S

we can replace 1 with a function that assigns a value (temperature, density, charge) to each point, and integrate it.

Scalar surface integral of $f(x,y,z)$ over $S = \vec{X}(s,t)$ on D:

$$\iint_D f(x,y,z)\, dS = \iint_D \underbrace{f(\vec{X}(s,t))}_{\text{function at each point}}\underbrace{\|\vec{X}_s \times \vec{X}_t\|\, ds\, dt}_{dS}.$$

Example. Compute $\iint_S xy\, dS$, where S is the unit disk in the xy-plane.

Here, the "surface" we are integrating over is a part of our familiar xy-plane.

Method 1: set it up in polar coords:

$$\iint_S xy\cdot dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \underbrace{(r\cdot\cos\theta)}_{x}\underbrace{(r\cdot\sin\theta)}_{y}\cdot\underbrace{r\cdot dr\cdot d\theta}_{dA} = \ldots$$

Method 2: set it up as a surface integral:

$$\vec{X}(r,\theta) = [r\cdot\cos\theta,\ r\cdot\sin\theta,\ 0] \text{ for } \left.\begin{matrix}0\le\theta\le 2\pi\\ 0\le r\le 1\end{matrix}\right\} D$$

$$\vec{X}_r = [\cos\theta,\ \sin\theta,\ 0] \Rightarrow \|\vec{X}_r \times \vec{X}_\theta\| = \|[0,0,r]\| = r.$$

$$\vec{X}_\theta = [-r\sin\theta,\ r\cos\theta,\ 0]$$

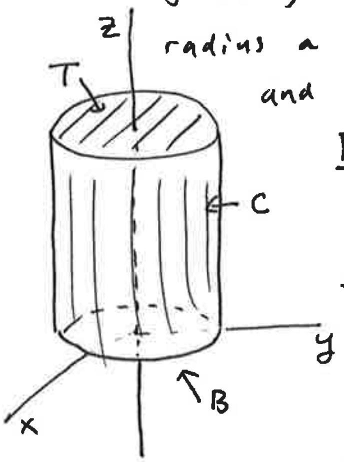
$$\text{so } \iint_S xy\cdot dS = \iint_D f(r\cdot\cos\theta,\ r\cdot\sin\theta,0)\|\vec{X}_r \times \vec{X}_\theta\|\, dr\, d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \underbrace{(r\cos\theta)(r\sin\theta)}_{f(x,y,z)=xy}\cdot\overbrace{r}^{\|\vec{X}_r \times \vec{X}_\theta\|}\cdot dr\, d\theta$$

as before!

Mathematician spotlight: Katherine Johnson, mathematician at NASA (currently age 99)
- calculated flight trajectories for the first Americans in space
- created the backup system that helped the Apollo 13 crew return safely
- helped NASA transition from human computers to digital computing machines

Plan for last two weeks: Scalar surface integrals ← last time, and again today
vector surface integrals ← start today, and study until the end.

Example. Suppose that the amount of mold at the point (x,y,z) on an old can of beans is given by $f(x,y,z) = x^2+y^2+z$. Further suppose that the can is a cylinder of radius a and height h, centered on the z-axis from $z=0$ to $z=h$, with the top and bottom disks attached. How much total mold is on the can?



Plan: Compute the scalar surface integral of f over the three surfaces — the top T, bottom B and cylinder C — and add them.

Bottom:
$$\iint_B \overbrace{(x^2+y^2+z)}^{f(x,y,z)}\,dS = \iint_D (x^2+y^2+\underset{\uparrow\ z=0\text{ on }B}{0})\,\underset{\uparrow\ \text{xy-plane}}{dA} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\cdot r\,dr\,d\theta = 2\pi\cdot\frac{a^4}{4} = \frac{\pi a^4}{2}.$$
shadow is unit disk → D

Top:
$$\iint_T (x^2+y^2+z)\,dS = \iint_D (x^2+y^2+\underset{\uparrow\ z=h\text{ on }B}{h})\,\underset{\uparrow\ \text{xy-plane}}{dA} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}(r^2+h)\,dr\,d\theta = 2\pi\left(\frac{a^4}{4}+\frac{ha^2}{2}\right).$$
shadow is unit disk → D

Cylinder: We need to set up a scalar surface integral. First, we need to parameterize C.

$$\vec{X}(\theta,z) = [a\cdot\cos\theta,\ a\cdot\sin\theta,\ z] \quad \text{for} \quad \begin{matrix} 0 \le \theta \le 2\pi, \\ 0 \le z \le h. \end{matrix}$$

then
$$\left.\begin{matrix} \vec{X}_\theta = [-a\cdot\sin\theta,\ a\cdot\cos\theta,\ 0] \\ \vec{X}_z = [0,\ 0,\ 1] \end{matrix}\right\} \Rightarrow \vec{X}_\theta \times \vec{X}_z = [a\cdot\cos\theta,\ a\cdot\sin\theta,\ 0] \Rightarrow \|\vec{X}_\theta\times\vec{X}_z\| = a.$$

So we can compute the scalar surface integral:
$$\iint_C f(x,y,z)\,dS = \iint_R f(\vec{X}(\theta,z))\,\|\vec{X}_\theta\times\vec{X}_z\|\,dz\,d\theta = \int_{\theta=0}^{2\pi}\int_{z=0}^{h} \overbrace{\Big(\underbrace{(a\cdot\cos\theta)^2}_{x^2} + \underbrace{(a\cdot\sin\theta)^2}_{y^2} + z\Big)}^{f(\vec{X}(\theta,z))}\cdot \overbrace{a}^{\|\vec{X}_\theta\times\vec{X}_z\|}\cdot dz\,d\theta$$
region in the θz-plane → R

$$= \int_{\theta=0}^{2\pi}\int_{z=0}^{h}(a^2+z)\cdot a\cdot dz\cdot d\theta = \int_{\theta=0}^{2\pi}\int_{z=0}^{h} a^3+za = 2\pi\left(a^3h + a\frac{h^2}{2}\right).$$

Total:
$$\iint_{\text{can}} f\,dS = \iint_B f\,dS + \iint_T f\,dS + \iint_C f\,dS = \underbrace{\frac{\pi a^4}{2}}_{B} + \underbrace{\frac{\pi a^4}{2} + \pi h a^2}_{T} + \underbrace{2\pi a^3 h + \pi a h^2}_{C}.$$

Diana Davis   Math 34   April 23, 2018   Class #35 ②

All integrals are variations on the same idea: adding up function values ($1, f(x,y), f(x,y,z), \dots$) over some geometric object (interval, region, solid, curve, surface...).

Vector surface integrals!

Suppose there is a rapidly flowing stream full of tiny fishes, ← a vector field $\vec{F}$

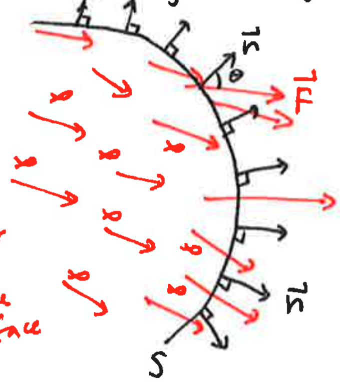
and you are holding a net that has an inside and an outside, ← a surface S with a chosen orientation

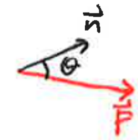
and you wish to measure how many fish you catch. ← how much $\vec{F}$ flows through S.

Use a vector surface integral:

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS$$

vector field; unit normal at each point of surface; add it up over the whole surface





$\vec{F}\cdot\vec{n}$ measures how much $\vec{F}$ points in the same direction as $\vec{n}$.

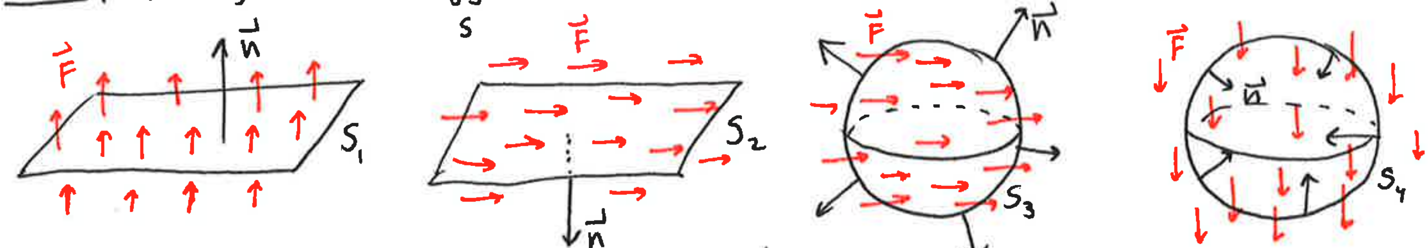
$\vec{F}\cdot\vec{n} > 0$: same direction! catch fish. :)

$\vec{F}\cdot\vec{n} < 0$: opposite direction! lose fish. :(

$\vec{F}\cdot\vec{n} = 0$: perpendicular! fish swim along net.

This quantity is called the "flux" of $\vec{F}$ through S.

Example. Say whether $\iint_S \vec{F}\cdot\vec{n}\, dS$ is positive, negative or 0 for each:



$\iint_{S_1} \vec{F}\cdot\vec{n}\, dS$ ___ 0 because: | $\iint_{S_2} \vec{F}\cdot\vec{n}\, dS$ ___ 0 because: | $\iint_{S_3} \vec{F}\cdot\vec{n}\, dS$ ___ 0 because: | $\iint_{S_4} \vec{F}\cdot\vec{n}\, dS$ ___ 0 because:

Example. Let S be the unit square in the xy-plane with upward unit normal, and $\vec{F} = [\cos(xz), e^y, 5]$.
Compute $\iint_S \vec{F}\cdot\vec{n}\, dS$. Here we can see $\vec{n} = [0,0,1]$, so $\vec{F}\cdot\vec{n} = [\sim, \sim, 5]\cdot[0,0,1] = 5$.

So $\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S 5\, dS = 5(\text{area of } S) = 5\cdot 1 = \underline{\underline{5}}$. ← makes sense because, as for S_1 above, all the $\vec{F}$ vectors point (generally) "up", so in the same direction as $\vec{n}$.

How to compute vector surface integrals in general:

- For a surface $S: \vec{X}(s,t)$, normal vector is $\vec{X_s}\times\vec{X_t}$, so unit normal vector is $\dfrac{\vec{X_s}\times\vec{X_t}}{\|\vec{X_s}\times\vec{X_t}\|}$.
- dS = area of parallelogram spanned by $\vec{X_s}\, ds$ and $\vec{X_t}\, dt$ $= \|X_s \times X_t\|\, ds\, dt$.

So
$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}\cdot\underbrace{\left(\frac{\vec{X_s}\times\vec{X_t}}{\cancel{\|\vec{X_s}\times\vec{X_t}\|}}\right)}_{\vec{n}}\underbrace{\cancel{\|X_s\times X_t\|}\, ds\, dt}_{dS} = \boxed{\iint_R \vec{F}(\vec{X}(s,t))\cdot(\vec{X_s}\times\vec{X_t})\, ds\, dt}$$

region in st-plane → R

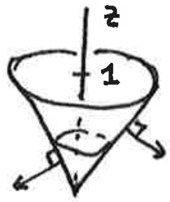
We'll use this next time!

Diana Davis   Math 34   April 25, 2018   Class #36 (1)

Mathematician spotlight: Edgar Duéñez, senior software engineer, Google
- math contests in high school; math major; PhD in C.S.
- Studied search algorithms & evolutionary bases of social behavior
- at Google, developed machine learning algorithms to identify images.

Today: more on vector surface integrals
- Stokes' Theorem: relates a vector surface integral to the vector line integral on the boundary of the surface. Is just Green's Theorem in 3D space instead of the plane.

Example. The electric force field $\vec{E}$ from the backup power generator is $\vec{E} = [xz, yz, y]$. You are standing upside down wearing a tin hat shaped like the cone $z = \sqrt{x^2+y^2}$ from $z=0$ to $z=1$.



Compute the electric flux of $\vec{E}$ flowing out of the hat.

- First, parameterize the hat: $\vec{X}(r,\theta) = [r\cos\theta, r\sin\theta, r]$ for $0 \le \theta \le 2\pi$, $0 \le r \le 1$.
- Now compute tangent vectors: $\vec{X_r} = [\cos\theta, \sin\theta, 1]$
  $\vec{X_\theta} = [-r\sin\theta, r\cos\theta, 0]$, now use them to find normal.

$$\Rightarrow \vec{X_r} \times \vec{X_\theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = [-r\cos\theta, -r\sin\theta, r]$$

← z is positive, so this is the upward (inward) normal vector. So take the opposite. $\vec{X_\theta} \times \vec{X_r}$

we'll use $-\vec{X_r} \times \vec{X_\theta} = \vec{X_\theta} \times \vec{X_r} = [r\cos\theta, r\sin\theta, -r]$.

Now
$$\iint_{\text{hat}} \vec{E} \cdot \vec{n}\, dS = \iint_R \vec{E}(\vec{X}(r,\theta)) \cdot (\vec{X_\theta} \times \vec{X_r})\, dr\, d\theta = \int_0^{2\pi} \int_0^1 [\underbrace{r^2\cos\theta}_{xz}, \underbrace{r^2\sin\theta}_{yz}, \underbrace{r\sin\theta}_{y}] \cdot \underbrace{[r\cos\theta, r\sin\theta, -r]}_{\vec{X_\theta} \times \vec{X_r}}\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(r^3(\cos^2\theta + \sin^2\theta) - r^2\sin\theta \right) dr\, d\theta = \int_0^{2\pi} \int_0^1 (r^3 - r^2\sin\theta)\, dr\, d\theta = \ldots = \frac{\pi}{2}.$$

positive, so the flux is outward!

Example. Let S be the flying pancake formed by slicing the plane $x+z=5$ ($z = 5-x$) with the cylinder $x^2+y^2=9$, like a cookie cutter. Give the pancake upward orientation, and compute $\iint_S (-4\vec{i} - x\vec{k}) \cdot d\vec{S}$.

- First, parameterize the surface: $\vec{X}(x,y) = [x, y, 5-x]$ for x, y in the disk of radius 5 centered at the origin
- Now, compute tangent vectors: $\vec{X_x} = [1, 0, -1]$
  $\vec{X_y} = [0, 1, 0]$; now find a normal vector:

$$\vec{X_x} \times \vec{X_y} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = [1, 0, 1]$$

← check: is upward! yay
is the normal vector that you can pull out of the plane coefficients: $1x + 0y + 1z = 5$.

$$\text{Now } \iint_S (-4\vec{i} - x\vec{k}) \cdot d\vec{S} = \iint_{\substack{\text{disk of} \\ \text{radius 3}}} [-4, 0, -x] \cdot [1, 0, 1]\, dA = \iint_{\substack{\text{disk of} \\ \text{radius 3}}} (-4-x)\, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} (-4 - r\cos\theta) \cdot r \cdot dr \cdot d\theta$$

$$= \ldots = -36\pi.$$

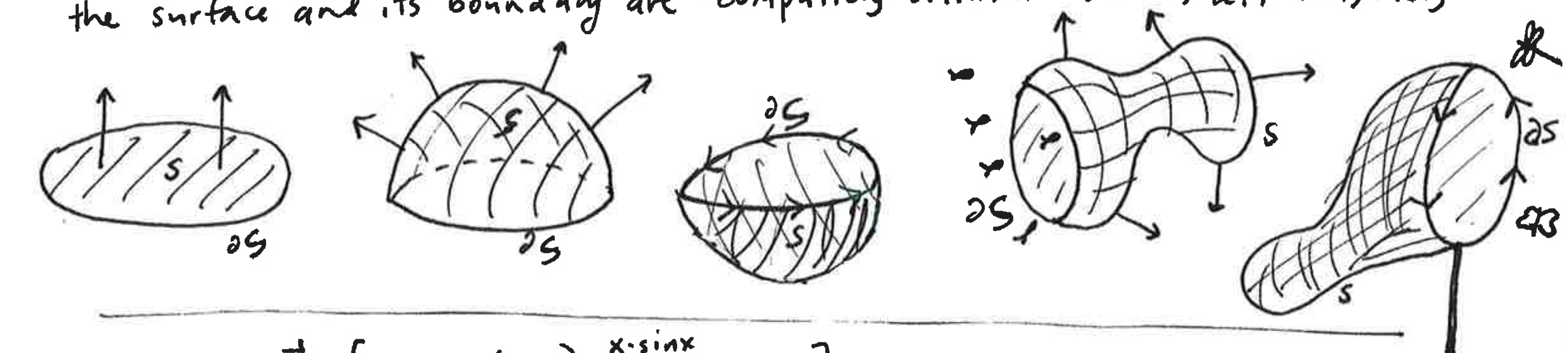
negative, so the vector field tends to flow down through the pancake.

Stokes' Theorem. Let $\vec{F}$ be a vector field, and let S be an oriented surface, whose boundary (if any) is ∂S, oriented so that if you walk along ∂S with your head in the direction of the normal vectors for the chosen orientation of S, your left arm is over S.

Then $\displaystyle\int_{\partial S} \vec{F}\cdot d\vec{s} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}.$

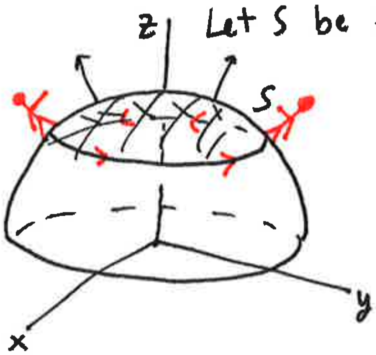
"The flux of the curl of $\vec{F}$ across a surface S is equal to the line integral of $\vec{F}$ along its boundary."

Draw in normal vectors to the surface, or arrows along the boundary curve, so that the surface and its boundary are "compatibly oriented" (head, left arm, etc.)



Example. Let $\vec{F} = [-y,\ x+(z-1)x^{x\cdot\sin x},\ x^2+y^2]$.

Let S be the piece of the sphere $x^2+y^2=z^2=2$ above $z=1$, with outward orientation.



Compute $\displaystyle\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$.

Method 1: Just do it! Compute $\operatorname{curl}\vec{F}$, and integrate.

ok, $\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x+(z-1)x^{x\cdot\sin x} & x^2+y^2 \end{vmatrix} = \left[\_\_\_,\ \_\_\_,\ \frac{\partial}{\partial x}\left(x+(z-1)x^{x\cdot\sin x} - \ldots\right)\right]$

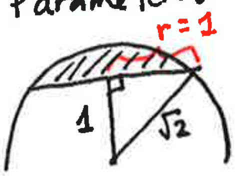
what a mess! let's not.

Method 2: Apply Stokes' Theorem.

So $\displaystyle\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S}\vec{F}\cdot d\vec{s}$, as long as ∂S is oriented in the correct direction.

Drawing in a left-armed stick figure above, it seems that we should orient the boundary circle as shown: its shadow is CCW in the xy-plane.

Parameterize the boundary curve ∂S. What is its radius? 1. ← a ccw circle of radius 1 at height $z=1$.



side view

So $\vec{x}(\theta) = [\cos\theta, \sin\theta, 1]$ for $0\le\theta\le 2\pi$

$\Rightarrow \vec{x}'(\theta) = [-\sin\theta, \cos\theta, 0]$

Note that on the boundary, $z=1$, so $x+(z-1)x^{x\sin x} = x+(1-1)\ldots = x$.

So now $\displaystyle\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S}\vec{F}\cdot d\vec{s} = \int_{\theta=0}^{2\pi}[\underbrace{-\sin\theta}_{-y},\ \underbrace{\cos\theta}_{x},\ \underbrace{1}_{x^2+y^2}]\cdot\underbrace{[-\sin\theta, \cos\theta, 0]}_{\vec{x}'(\theta)}\,d\theta = \int_0^{2\pi}(\sin^2\theta+\cos^2\theta)\,d\theta = \int_0^{2\pi}d\theta = \underline{\underline{2\pi}}.$

Positive, so the vector field $\vec{F}$ goes (net) CCW, and $\operatorname{curl}\vec{F}$ is (net) outward.

Diana Davis   Math 34   April 27, 2018   Class # 37 ①
§ 7.3

Some intentional themes of the mathematicians I decided to spotlight:

- they are all alive
- they alternated male and female
- at most two per week were caucasian

Happy national day of silence! Lots of adults, lots of professors, lots of mathematicians are LGBTQ+. Including:

- me (Diana Davis)
- Moon Duchin
- Emily Riehl
- Autumn Kent
- Dylan Thurston
- Harrison Bray

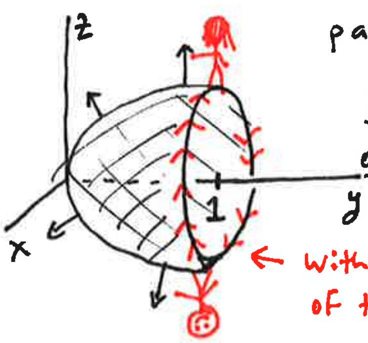
... and many more!

<u>Last time:</u> Stokes' Theorem: $\int_{\partial S} \vec{F}\cdot d\vec{s} = \iint_S (\text{curl}\,\vec{F})\cdot d\vec{S}$, when S and ∂S have compatible orientation.

<u>This time:</u>

- Lots of examples
- "surface independence" property of curl vector fields, analogous to path independence of conservative vector fields.

<u>Example.</u> Let $\vec{F} = [(y-1)\sin(e^{xz^y}),\ xyz\,e^{xyz},\ xz+y]$, and let S be the piece of the paraboloid $y = x^2+z^2$ with $y \le 1$, oriented outwards. Compute $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$.

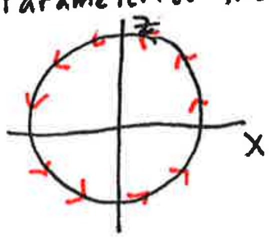


<u>Option 1:</u> Just do it! But this is tough, because the $\vec{i}$ & $\vec{j}$ functions are tough.

<u>Option 2:</u> Stokes' Theorem! $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{s}$, as long as we orient ∂S compatibly.

← with this orientation, if you walk along ∂S with your head in the direction of the chosen normal vectors (orientation) of S, your left arm is over S.

Parameterize the boundary curve:



$x(t) = \cos t$
$z(t) = \sin t$
and
$y(t) = 1$

$\Rightarrow \vec{x}(t) = [\cos t,\ 1,\ \sin t],\quad 0 \le t \le 2\pi$

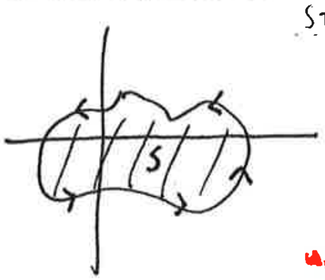
$\Rightarrow \vec{x}'(t) = [-\sin t,\ 0,\ \cos t]$.

so $\vec{F}(\vec{x}(t)) = [0,\ \underbrace{\cancel{\ \ }}_{\text{something messy}},\ \cos t\cdot\sin t + 1]$.

Now $\int_{\partial S} \vec{F}\cdot d\vec{s} = \int_{t=0}^{2\pi} F(\vec{x}(t))\cdot\vec{x}'(t)\,dt = \int_0^{2\pi} [0, \cancel{\ \ }, \cos t\cdot\sin t + 1]\cdot[-\sin t, 0, \cos t]\,dt = \int_0^{2\pi} (\cos^2 t\,\sin t + \cos t)\,dt = 0$

$\Rightarrow \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \underline{\underline{0}}$.

<u>Check this out!</u> If S and ∂S are in the xy-plane, then Stokes' Theorem is just Green's Theorem:



Stokes' Theorem says: $\int_{\partial S} \vec{F}\cdot d\vec{s} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

same

$= \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$

because with CCW orientation your head is always pointing up out of the page

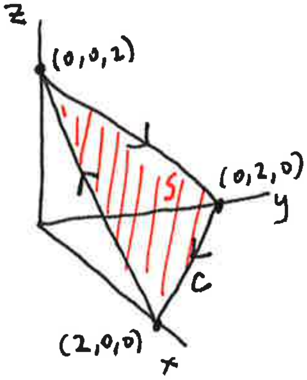
xy plane

$= \iint_S \text{curl}\,\vec{F}\cdot\vec{k}\,dA$

$= \iint_S [-Q_z,\ P_z,\ Q_x - P_y]\cdot[0,0,1]\,dA$

$= \iint_S (Q_x - P_y)\,dA$ ← Green's Theorem! ☺

Example of going the other way: Let $\vec{F} = [x\cdot\sin(e^x) - xz, -2xy, z^2+y]$ and let C be the triangular path from $(2,0,0) \to (0,0,2) \to (0,2,0) \to (2,0,0)$.



Compute $\int_C \vec{F}\cdot d\vec{s}$.

Option 1: Parameterize each of the three parts, and deal with $x\cdot\sin(e^x)$. ☹

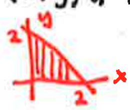
Option 2: Stokes' Theorem! $\int_C \vec{F}\cdot d\vec{s} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$ ← where S is any surface whose boundary is C, with appropriate orientation.

Let's use the simplest surface bounded by C: the flat triangle, which is part of the plane $x+y+z=2$.

Orientation of S: upward normal or downward normal? ______

Parameterize S: $\vec{X}(x,y) = (x, y, 2-x-y) \Rightarrow \vec{X}_x = [1,0,-1]$, $\vec{X}_y = [0,1,-1]$ $\Rightarrow \vec{X}_x \times \vec{X}_y = [1,1,1]$ ← we want so use the opposite: $\vec{X}_y \times \vec{X}_x = [-1,-1,-1]$.

for x, y in



Compute $\operatorname{curl}\vec{F}$: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x\cdot\sin(e^x) - xz & -2xy & z^2+y \end{vmatrix} = [1, x, -2y]$.

So $\int_C \vec{F}\cdot d\vec{s} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_D \operatorname{curl}\vec{F}(\vec{X}(x,y))\cdot(\vec{X}_y\times\vec{X}_x)\,dx\,dy = \int_{y=0}^{2}\int_{x=0}^{2-y} [1, x, -2y]\cdot[-1,-1,-1]\,dx\,dy = \int_0^2\int_0^{2-y}(-1-x+2y)\,dx\,dy$

D ← region in the xy-plane parameterizing S

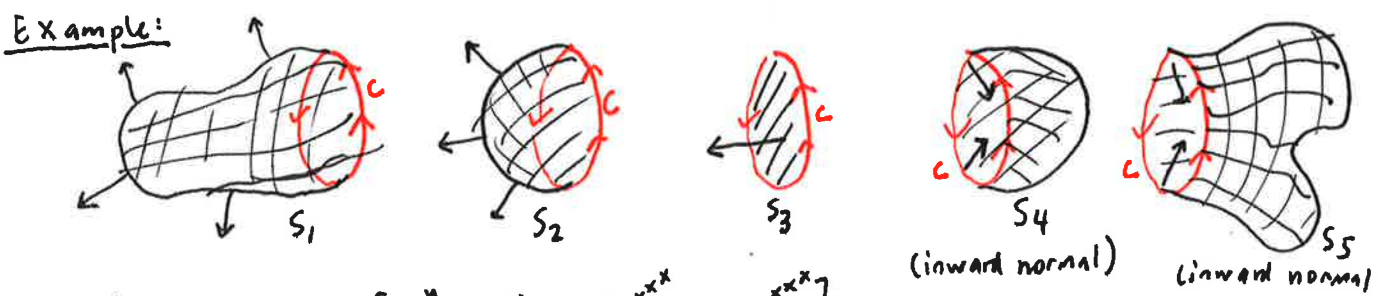
$= \underline{\underline{2.}}$

But actually, let's think about this: Stokes' Theorem says $\int_C \vec{F}\cdot d\vec{s} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$, where S is any surface whose boundary is C (with correct orientation).

So if two surfaces S_1, S_2 have the same boundary, $\iint_{S_1} \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_2} \operatorname{curl}\vec{F}\cdot d\vec{S}$. Wow!

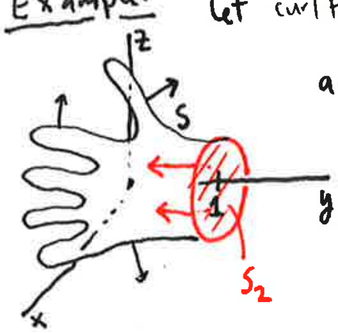
(and compatible/same orientation)

Example:



$\iint_{S_1} \operatorname{curl}\vec{F}\cdot d\vec{S} = \vdots = \iint_{S_5} \operatorname{curl}\vec{F}\cdot d\vec{S}$

Example. Let $\operatorname{curl}\vec{F} = \left[y^{y^y}\sin(z^2), (y-1)e^{x^{x^x}}+2, z e^{x^{x^x}}\right]$ and compute $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$ over the "hand surface" shown. No thank you!



Method 1: do it! No.

Method 2: Compute $\int_{\partial S} \vec{F}\cdot d\vec{s}$ using Stokes' Thm. But we don't have $\vec{F}$. ☹

Method 3: Replace S with a simpler surface with the same boundary!

$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_2} \operatorname{curl}\vec{F}\cdot d\vec{S}$ ← unit disk at $y=1$ with "left" normal

$= \iint_{S_2} \operatorname{curl}\vec{F}\cdot\vec{n}\,dS = \iint_{S_2} \operatorname{curl}\vec{F}\cdot[0,-1,0]\,dS$

$= \iint_{S_2} \left((y-1)e^{x^{x^x}}+2\right)dS = \iint_{S_2} -2\,dS = \underline{\underline{-2\pi.}}$ ☺

↖ $y=1$ on S_2

Mathematician spotlight: Gwyn Coogan, math teacher, Phillips Exeter Academy
- math major at Smith College
- ran in 1992 Olympics in Barcelona (10,000m)
- PhD at UC Boulder in number theory and postdoc at Wisconsin
- now teaches, coaches, and teaches other math teachers new ways to teach.

Recently, we've been exploring Stokes' Theorem:

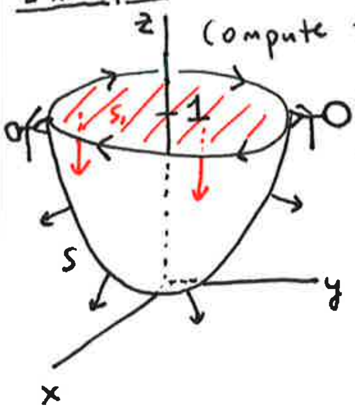
surface integral of curl $\longleftrightarrow$ boundary curve integral of vector field.
2D integral of "derivative" $\longleftrightarrow$ 1D integral of function

Today, we'll start exploring our last theorem of the semester, Gauss's Theorem:

solid integral of divergence $\longleftrightarrow$ boundary surface integral of vector field.
3D integral of "derivative" $\longleftrightarrow$ 2D integral of function.

But first, another example of Stokes' Theorem, using the "surface independence" from last time.

Example. Let $\vec{F}$ be a field such that $\operatorname{curl}\vec{F} = [y e^{y^{z^2}}, xy\sin(\cos z) - y^2, 2yz]$. ← we'll see later why such an $\vec{F}$ must exist.



Compute the flux of $\operatorname{curl}\vec{F}$ over S: $z = x^2 + y^2$ with $z \le 1$, outward normal.

Option 1: Just do it! Parametrize S, integrate $\operatorname{curl}\vec{F}$. ☹ looks maybe impossible.

Option 2: Stokes' Thm says $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_S \vec{F}\cdot d\vec{s}$, so find $\vec{F}$ and integrate over the boundary circle. ☹ Did you say find $\vec{F}$??

Option 3: Replace S by a simpler surface with same boundary & compatible orientation.
Let's use the unit disk at $z=1$, oriented down.
So on this surface S_1, $\vec{n} = [0,0,-1]$.

$z=1$ on S_1

Now $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_1} \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_1} \operatorname{curl}\vec{F}\cdot\vec{n}\, dS = \iint_{S_1} [\text{m}, \text{m}, 2yz]\cdot[0,0,-1]\, dS = \iint_{S_1} -2yz\, dS = \iint_{S_1} -2y\, dS.$

To compute this (wonderfully simple!) integral, either

- convert to polar coordinates: $\int_{\theta=0}^{2\pi}\int_{r=0}^{1} -2\cdot r\sin\theta\cdot r\,dr\,d\theta = \int_0^{2\pi}\sin\theta\,d\theta \cdot \int_0^1 -2r^2\,dr = 0\cdot \text{m} = \underline{\underline{0}}.$ (something finite)
- Notice that S_1 is symmetric across $y=0$ and $-2y$ is odd with respect to y, so integral is 0.

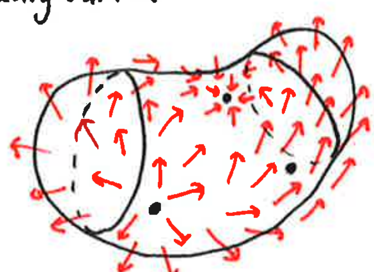
Gauss's Theorem! Suppose that $\vec{F}$ is a vector field with continuous partial derivatives throughout some solid region E in $\mathbb{R}^3$, where the boundary surface ∂E of E is oriented outward.

Then $\iiint_E (\operatorname{div}\vec{F})\, dV = \iint_{\partial E} \vec{F}\cdot d\vec{S}.$

divergence is a scalar function, so this is a scalar triple integral.

$\vec{F}$ is a vector field, so this is a vector surface integral.

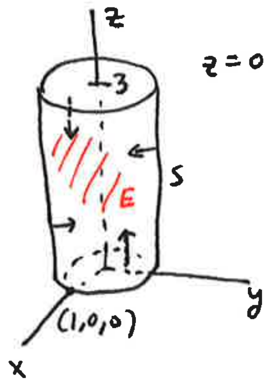
"the amount of stuff that flows into or out of each point enclosed by surface" = "the (net) amount that the vector field $\vec{F}$ flows into or out of the surface"



the double integral adds up $\vec{F}\cdot d\vec{S}$ at each point on the surface of the solid. (Surface is oriented out.)

the triple integral adds up $\operatorname{div}\vec{F}$ at each point of the solid interior

Example. Compute $\iint_S \underbrace{\left[y^{123} e^{\sin(yz)}, y - x^{z^x}, z^2 - z\right]}_{\vec{F}} \cdot d\vec{S}$, where S is the cylinder $r=1$ from $z=0$ to $z=3$, with top and bottom disks attached, and inward normal.



Can we apply Gauss's Theorem? - $\vec{F}$ has continuous partial derivatives everywhere, so ok ✓
- boundary surface is oriented outward ✗
don't worry! just change the sign. ok.

$$\iint_S \vec{F}\cdot d\vec{S} = -\iiint_E \operatorname{div}\vec{F}\, dV = -\iiint_E (0+1+2z-1)\, dV = -\iiint_E 2z\, dV.$$

E ← solid region inside cylinder

Estimate: should $-\iiint_E 2v\, dV$ be positive, negative or zero? ______

Convert to cylindrical coordinates:

$$\iiint_E 2z \cdot dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{3} 2z \cdot r \cdot dz \cdot dr \cdot d\theta = \int_0^{2\pi} d\theta \cdot \int_0^1 r\, dr \cdot \int_0^3 2z\, dz = 2\pi \cdot \frac{1}{2} \cdot 9 = \underline{\underline{9\pi}},$$

(r ← expansion factor!)

so $\iint_S \vec{F}\cdot d\vec{S} = -\iiint_E 2z\, dV = \underline{\underline{-9\pi}}$. negative, as expected! :)

Here are some things we can deduce now.

Analogy: curls : surface integrals :: cons. vector fields : line integrals

<u>The flux of a curl vector field through any closed surface is 0.</u>

Let $\vec{F}$ be continuous everywhere and let S be any closed surface.

Then $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \pm \iiint_E \operatorname{div}(\operatorname{curl}\vec{F})\, dV = \pm\iiint_E 0\, dV = \underline{\underline{0}}.$

($\pm$ ← depending on orientation of S) (E ← the solid enclosed by the surface S) (← we proved that $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$ using Clairaut's Theorem)

This is similar to how the line integral of a conservative vector field over any closed path is 0.

You can prove the same result using Stokes' Theorem, if you agree to think about the empty set: S is closed, so S has no boundary, so $\partial S = \emptyset$, (← notation for the empty set)

$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{s}$, and the integral of <u>any</u> function over the empty set is 0.

<u>A given field is the curl of another if its divergence is 0.</u>

- Is there a field $\vec{F}$ so that $\operatorname{curl}\vec{F} = \left[y e^{y^{z^2}}, xz\sin(\cos(z)) - y^2, 2yz\right]$?
  Well, the divergence of this field is $0 + 0 - 2y + 2y = 0$, so yes!
- Is there a field $\vec{G}$ so that $\operatorname{curl}\vec{G} = \left[y^{199}\sin(\cos(z)), x e^{z^4}, z^3 x\right]$?
  Compute the divergence of this field: ______________ Is there such a $\vec{G}$? ____

Amazing! You can know that such a field does or does not exist, without explicitly finding it!

(this resolves the "we'll see later why..." from the first example on the first page.)

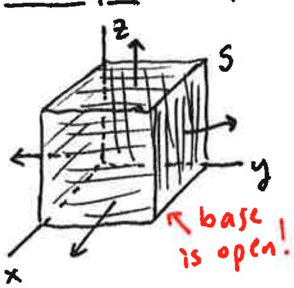
Mathematician spotlight: Illya V. Hicks, Professor of Computational & Applied Math., Rice Univ.
- math major at Texas State University - aced calculus - also played football
- studies combinatorial optimization, graph theory, integer programming
- applications to cancer treatment, social networks, network design

<u>Today</u>: More examples of Gauss's Theorem, including "closing off" a surface.

<u>Friday</u>: An example that combines Stokes' Theorem and Gauss's Theorem, plus a wrap-up overview of the last part of the course.

<u>Example</u>. Compute $\iint_S \overbrace{[P(y,z), Q(x,z), 3z]}^{\vec{F}} \cdot d\vec{S}$, where S is the surface of the unit cube $[0,1]\times[0,1]\times[0,1]$ <u>without</u> the base, oriented <u>out</u>.

↑horrible function of y and z   ↖horrible function of x and z



<u>Option 1</u>: Just do it! Parameterize each of the 5 faces (not too hard) & compute.
→ but we don't know P and Q, and we're told they're horrible. :(

<u>Option 2</u>: Apply Gauss's Theorem and integrate the divergence over the enclosed solid.
→ but S is not a closed surface. :(

<u>Idea</u>: Close off S, apply Gauss's Theorem, then subtract off the vector surface integral of the surface we added.

Let S_1 denote the bottom square face, oriented ______, so $\vec{n} = [\ ,\ ,\ ]$. By Gauss's Theorem,

$$\iint_{\underbrace{S+S_1}_{\text{the whole closed cube surface}}} \vec{F}\cdot d\vec{S} = \iiint_{E \leftarrow \text{solid unit cube}} \operatorname{div}\vec{F}\, dV \Rightarrow \iint_{S \leftarrow \text{5 faces}} \vec{F}\cdot d\vec{S} + \iint_{S_1 \leftarrow \text{bottom face}} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV - \iint_{S_1} \vec{F}\cdot d\vec{S}$$

what we want = integral of div $\vec{F}$ over solid − vector surface integral over bottom face.

OK, let's compute these two parts.

$\vec{F} = [P(y,z), Q(x,z), 3z] \Rightarrow \operatorname{div}\vec{F} = 0+0+3 = 3.$

So $\iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E 3\, dV = 3\iiint_E dV = 3(\text{volume of unit cube}) = 3\times 1 = \underline{\underline{3}}.$

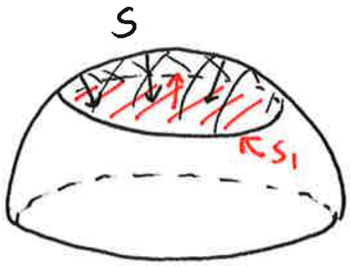
z=0 on the base of the cube, which is the surface S_1

And $\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot\vec{n}\, dS = \iint_{S_1} [P,Q,3z]\cdot[0,0,-1]\, dS = \iint_{S_1} -3z\, dS = \iint_{S_1} 0\, dS = \underline{\underline{0}}.$

So $\iint_S [P,Q,3z]\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV - \iint_{S_1} [P,Q,3z]\cdot d\vec{S} = 3 - 0 = \underline{\underline{3}}.$

Example. Compute $\iint_S \underbrace{[2xy-z^2,\ y-y^2,\ -z]}_{\vec F}\cdot d\vec S$, where S is the "spherical cap" $x^2+y^2+z^2=2$, $z\geq 1$, oriented down.

↑ this function looks tough to integrate; we'd rather use the derivative, which is easier.



we want to use Gauss's Theorem, so we need to do two things:

① close off S by attaching any surface with the same boundary as S:

→ let's use the unit disk at height $z=1$, oriented ______.

This orientation is chosen so the closed surface has consistent orientation.

② Change the sign, since S has inward orientation.

So now by Gauss's Theorem, $\underbrace{\iint_{S+S_1}}_{\text{closed surface}} \vec F\cdot d\vec S = -\iiint_{E \leftarrow \text{enclosed region}} \operatorname{div}\vec F\,dV \Rightarrow \underbrace{\iint_S \vec F\cdot d\vec S}_{\text{(what we want)}} = -\iiint_E \operatorname{div}\vec F\,dV - \iint_{S_1}\vec F\cdot d\vec S.$

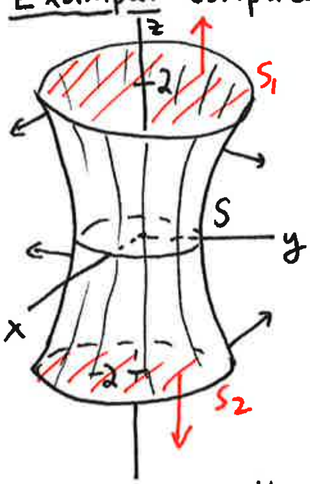
Let's compute these:

$$\iiint_E \operatorname{div}\vec F\,dV = \iiint_E (2y+1-2y-1)\,dV = \iiint_E 0\,dV = 0.$$

$$\iint_{S_1}\vec F\cdot d\vec S = \iint_{S_1}\vec F\cdot\vec n\,dS = \iint_{S_1}[m, m, -z]\cdot[0,0,1]\,dS = \iint_{S_1} -z\,dS = \iint_{S_1} -1\,dS = -\iint_{S_1} dS = -(\text{area of } S_1) = -\pi.$$

$$= -0-(-\pi) = \pi.$$

Example. Compute $\iint_S \underbrace{[ye^{\cos(\sin z)},\ x^{100}e^{z^{xz}},\ x-z^2]}_{F}\cdot d\vec S$, where S is part of the hyperboloid $x^2+y^2-z^2=1$ between $z=-2$ and $z=2$, oriented out.



To use Gauss's Theorem, we have to attach two surfaces!

S_1: the disk of radius $\sqrt5$* in the plane $z=2$, with $\vec n=[0,0,1]$

S_2: " " " " , $z=-2$, " $\vec n=[0,0,-1]$.

* $x^2+y^2-z^2=1$
$x^2+y^2-2^2=1$
$x^2+y^2=5$
$r=\sqrt{x^2+y^2}=\sqrt5$

Then $\underbrace{\iint_S \vec F\cdot d\vec S}_{\text{what we want}} = \iiint_{E\leftarrow\text{solid enclosed}} \operatorname{div}\vec F\,dV - \underbrace{\iint_{S_1}\vec F\cdot d\vec S}_{\text{flux over top cap}} - \underbrace{\iint_{S_2}\vec F\cdot d\vec S}_{\text{flux over bottom cap.}}$, by Gauss's Theorem.

Let's compute them:

$\iiint_E \operatorname{div}\vec F\,dV = \iiint_E (0+0-2z)\,dV = \iiint_E -2z\,dV = 0$ because the solid region E is symmetric across $z=0$ and $-2z$ is odd with respect to z.

S_1 is symmetric with respect to x

$$\iint_{S_1}\vec F\cdot d\vec S = \iint_{S_1}\vec F\cdot\vec n\,dS = \iint_{S_1}[m, m, x-z^2]\cdot[0,0,1]\,dS = \iint_{S_1}(x-z^2)\,dS = \iint_{S_1} x\,dS - \iint_{S_1} z^2\,dS = 0 - \iint_{S_1} 4\,dS$$
$$= -4(\text{area of } S_1)$$

$z=2$ on S_1

similarly,

because $\vec n=[0,0,-1]$ for S_2.

$$\iint_{S_2}\vec F\cdot d\vec S = 0 - \iint_{S_2}(-4)\,dS = 4(\text{area of } S_2).$$

So $\iint_S \vec F\cdot d\vec S = \iiint_E \operatorname{div}\vec F\,dV - \iint_{S_1}\vec F\cdot d\vec S - \iint_{S_2}\vec F\cdot d\vec S = 0 - 4(\text{area of } S_1) + 4(\text{area of } S_2)$
$= 0.$ (have same area)

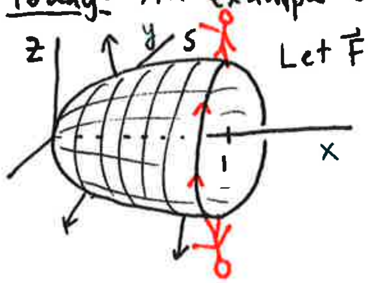
Diana Davis Math 34 May 4, 2018 Class #40 ①

§7.3

Mathematician spotlight: Lila Fontes, Assistant Professor of Computer Science, Swarthmore

- math major at Harvard, PhD at Toronto, postdoc in Paris
- studies privacy and its relationship to communication cost, accuracy & optimality.
- amazingly, it is possible to communicate much while revealing little.

Today: An example bringing together Gauss's and Stokes' Theorems, and then an integral summary!



Let $\vec{F} = [(1-x)e^{\sin(\cos e^y)} - z^2 - y^2,\ y,\ z e^{\sin(\cos e^y)}]$, and let S be $x = y^2 + z^2$, $x \le 1$, oriented outward.

Compute $\iint_S (\vec{F} + \text{curl}(\text{curl}\vec{F})) \cdot d\vec{S}$. break into two parts

$= \underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{①} + \underbrace{\iint_S \text{curl}(\text{curl}\vec{F}) \cdot d\vec{S}}_{②}$ ← let's compute these separately.

① Let's apply Gauss's Theorem. We need to close the surface, so we add the unit disk at $x = 1$, S_1, oriented in positive x-direction: $\vec{n} = [1,0,0]$ on S_1.

Then $\iint_{S+S_1} \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\vec{F}\, dV \Rightarrow \iint_S \vec{F} \cdot d\vec{S} + \iint_{S_1} \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\vec{F}\, dV$

these cancel out since both are $e^{\sin(\cos e^y)}$

$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \underbrace{\iiint_E \text{div}\vec{F}\, dV}_{Ⓐ} - \underbrace{\iint_{S_1} \vec{F} \cdot d\vec{S}}_{Ⓑ}. \quad = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$

Ⓐ $\iiint_E \text{div}\vec{F}\, dV = \iiint_E (\cancel{-e}+1+\cancel{e})\, dV = \iiint_E 1\, dV = \text{volume of } E$

volume of E: $x = x$, $y = r\cos\theta$, $z = r\sin\theta$; $r^2 \le x \le 1$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$.

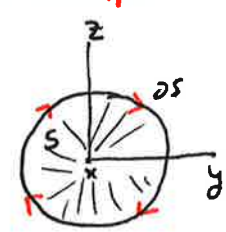
$\Rightarrow dV = r \cdot dx \cdot dr \cdot d\theta$,

$\Rightarrow \text{volume} = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{x=r^2}^{1} r\, dx\, dr\, d\theta = \int_0^{2\pi} d\theta \cdot \int_{r=0}^{1} (r - r^3)\, dr = 2\pi \times \frac{1}{4} = \pi/2.$

Ⓑ $\iint_{S_1} (y^2 - z^2)\, dS = \iint_{S_1} y^2\, dS - \iint_{S_1} z^2\, dS = 0.$

← we could set these up and compute them, but they would come out the same due to the symmetry of the shape. So they cancel out!

② By Stokes' Theorem, $\iint_S \text{curl}(\text{curl}\vec{F}) \cdot d\vec{S} = \int_{\partial S} \text{curl}\vec{F} \cdot d\vec{S}$, if ∂S is oriented correctly: ∂S needs to be clockwise.



So parameterize S by $\vec{x}(t) = [1, \cos(-t), \sin(-t)] = [1, \cos t, -\sin t]$

$\Rightarrow \vec{x}'(t) = [0, -\sin t, -\cos t]$.

Now $\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (1-x)e^{\sin(\cos e^y)} - z^2 - y^2 & y & z e^{\sin(\cos e^y)} \end{vmatrix} = [\sim\sim, -(0 - 2z), 0 - (-2y)]$

$= [\sim\sim, -2z, 2y] = [\sim\sim, 2\sin t, 2\cos t]$.

we don't care because we're going to take the dot product with $\vec{x}'(t)$, and its first component is 0.

So $\displaystyle\iint_S \operatorname{curl}(\operatorname{curl}\vec F)\cdot d\vec S = \int_{\partial S} \operatorname{curl}\vec F\cdot d\vec s = \int_{t=0}^{2\pi} \operatorname{curl}\vec F(\vec x(t))\bullet \vec x\,'(t)\,dt = \int_{t=0}^{2\pi}[\text{—}, +2\sin t, 2\cos t]\bullet[0,-\sin t,-\cos t]\,dt$

$$= \int_{t=0}^{2\pi}(-2\sin^2 t - 2\cos^2 t)\,dt = \int_{t=0}^{2\pi} -2(\cos^2 t+\sin^2 t)\,dt = \int_{t=0}^{2\pi} -2\,dt = \underline{\underline{-4\pi}}.$$

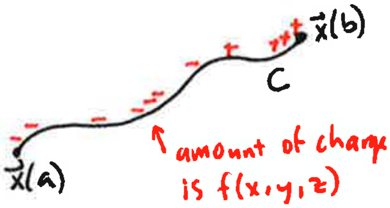
So now, $\displaystyle\iint_S (\vec F + \operatorname{curl}(\operatorname{curl}\vec F))\cdot d\vec S = \iint_S \vec F\cdot d\vec S + \iint_S \operatorname{curl}(\operatorname{curl}\vec F)\cdot d\vec S = \frac{\pi}{2} - 4\pi = -\frac{7}{2}\pi.$

To do this, we needed:
- two big theorems (Gauss's Theorem and Stokes' Theorem)
- a triple integral
- a vector line integral

} all things that we can do! ← strong math.

We can now integrate $\begin{matrix}\text{scalar}\\ \text{vector}\end{matrix}$ functions over $\begin{matrix}\text{curves}\\ \text{surfaces.}\end{matrix}$ All four!

1. Scalar line integrals. <u>Example:</u> adding up total amount of charge on a wire.



Parameterize the curve C: $\vec x(t) = [x(t), y(t), z(t)]$ for $a \le t \le b$.

Then $\displaystyle\int_C f\,ds = \int_{t=a}^{t=b} \underbrace{f(\vec x(t))}_{\text{value of function}}\ \underbrace{\|\vec x\,'(t)\|\,dt}_{\text{distance along the curve}}.$

value of function

distance along the curve for which your function takes that value.

2. Vector line integrals. <u>Example:</u> adding up how much the wind helps/hurts you on your walk.

Parameterize the curve as above. Now direction matters!

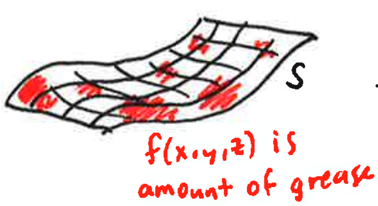
Then $\displaystyle\int_C \vec F\bullet d\vec s = \int_{t=a}^{t=b} \vec F(\vec x(t))\bullet \vec x\,'(t)\,dt.$

← direction of vector field

← direction curve is going

The dot product measures how much $\vec F$ and $\vec x\,'$ point in the same direction.

3. Scalar surface integrals. <u>Example:</u> How much total grease is on your greasy pizza napkin.



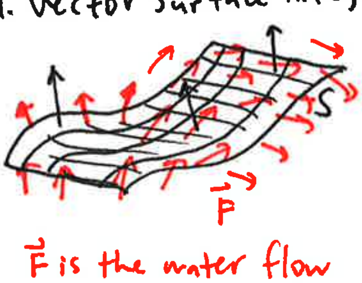
Parameterize the surface S: $\vec X(s,t) = [x(s,t), y(s,t), z(s,t)]$ for s,t in the region R in the st-plane.

Then $\displaystyle\iint_S f\,dS = \iint_R f(\vec X(s,t))\,\|\vec X_s\times\vec X_t\|\,ds\,dt.$

value of function

area of the surface for which your function takes that value.

4. Vector surface integrals. <u>Example:</u> How much (net) water flows into your fishing net.



Parameterize the surface as above.

Then $\displaystyle\iint_S \vec F\cdot d\vec S = \iint_R \vec F(\vec X(s,t))\bullet(\vec X_s\times\vec X_t)\,ds\,dt$ → normal vector to the surface.

direction of vector field

or maybe $\vec X_t\times\vec X_s$, depending on orientation of S.

Thank you for a great semester! Please let me know about your future endeavors!