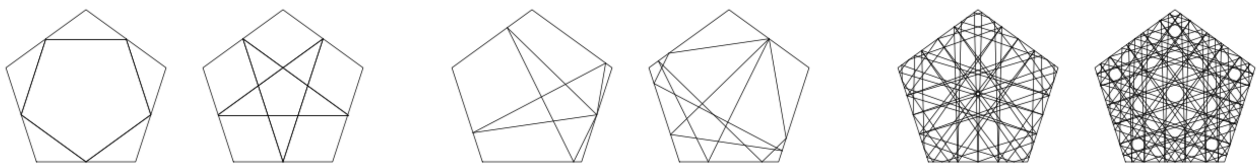
You have in your hands a periodic billiard path on a pentagonal billiard table. The idea is that you put your billiard ball down on one of the lines on your pentagon, and shoot it in the direction of that line. The ball bounces around for a while, following the lines, and then comes back to where it started and repeats the same path.

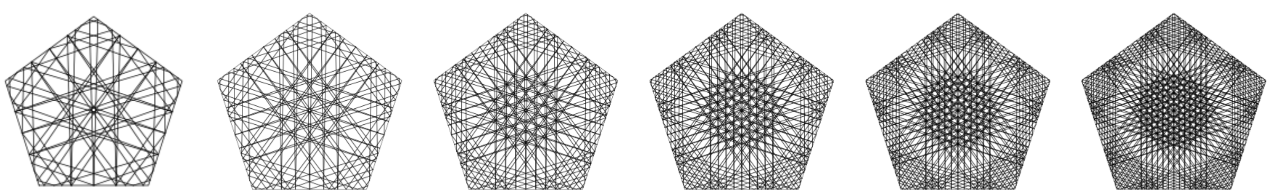
My friend Samuel Lelièvre and I created these pictures by first putting a *tree structure* on the set of periodic directions on the regular pentagon. The idea is to organize the periodic directions by complexity. The first three levels of the tree appear on the other side of this page.

For each periodic direction, there are actually *two* different periodic billiard paths, a short one and a long one. Which one you get depends on where you start. The long one is the golden ratio ($\varphi \approx 1.618$) times as long as the short one. The tree on the other side shows the short trajectory for each direction. Some examples of pairs of short and long "buddies" are below.



You may notice that most of the patterns have rotational symmetry, but some do not. We know that 5/6 of the paths have rotational symmetry and 1/6 don't. We can prove mathematically which paths have the symmetry, but we don't have an intuitive explanation of it yet.

We noticed that there are "families" of similar trajectories with increasing complexity. We recently figured out where to find such families. Aren't they beautiful? As the trajectories in the family get longer and longer, from far away they look like shading patterns.



The first billiard table you would think to study is the square, which has been well understood since the 1800s. The pentagon is the next one you'd want to study, partly just because it has one more side than the square, but mostly because things related to pentagons tend to work out in an elegant way. The ratio of the length of the diagonal to the length of the edge is φ, and when the golden ratio is around, things tend to be beautiful. I hope you agree.

Your trajectory is as follows:

Best wishes for lifelong pentagon enjoyment,
Diana

Tree of periodic directions on the regular pentagon: the beginning

Diana Davis and Samuel Lelièvre