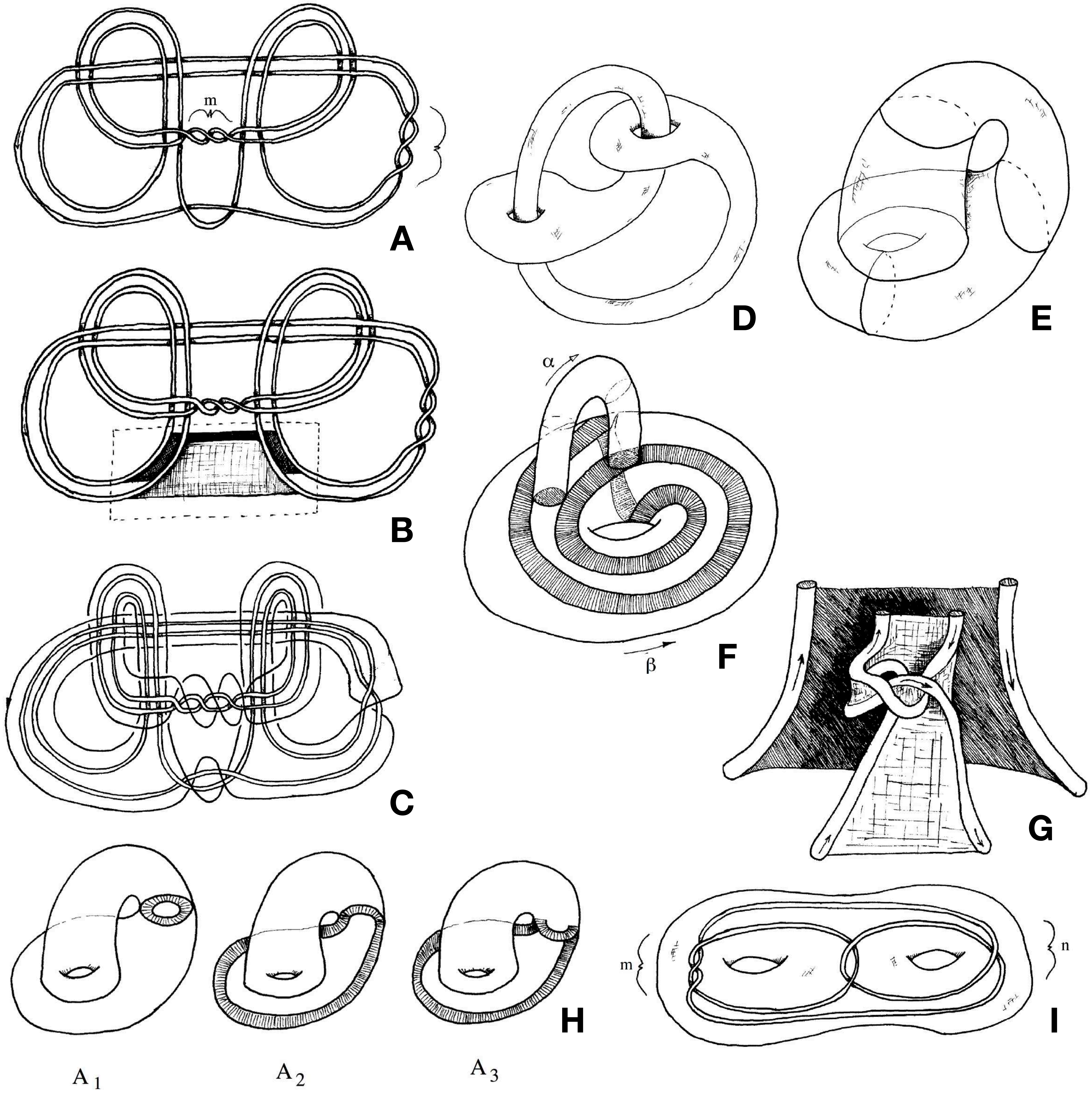
DRAWINGS



The drawings are from my first solo paper , ”Bundles, handcuffs, and local freedom.” They illustrate the existence of a knot in the 3-sphere whose complement is hyperbolic, admits a fibration over the circle, and whose group contains a subgroup that is locally free (finitely generated subgroups are free) and not free. This answered a question of James Anderson. He had observed that if such a thing could not exist, then there would be counterexamples to Thurston's Virtual Fibration Conjecture, which we now know to be true, by remarkable work of Ian Agol and Dani Wise.

The fundamental group of the 2-complex X (E) contains a subgroup that is locally free and not free. The method to construct the knots is to find a fibered knot in the complement of X so that X's group injects into the knot's. The complement of X is the handlebody (D). The knot is inside the handlebody (I). The knot is built by taking a square knot (which is fibered), cabling (which is again fibered), and then “plumbing.” Plumbing is the process of gluing a twisted band along a square in a fiber (G-H). The rest of the figures are part of the proof that the knot is hyperbolic, and that you can perform surgeries to get closed hyperbolic fibered manifolds with groups having locally free non-free subgroups.

One year while I was in graduate school, Bob Gompf was teaching a course in 4-manifolds that I had lost the thread of. I didn't want to be rude, so I kept attending. So there I was in class doodling and thinking about Anderson's question.

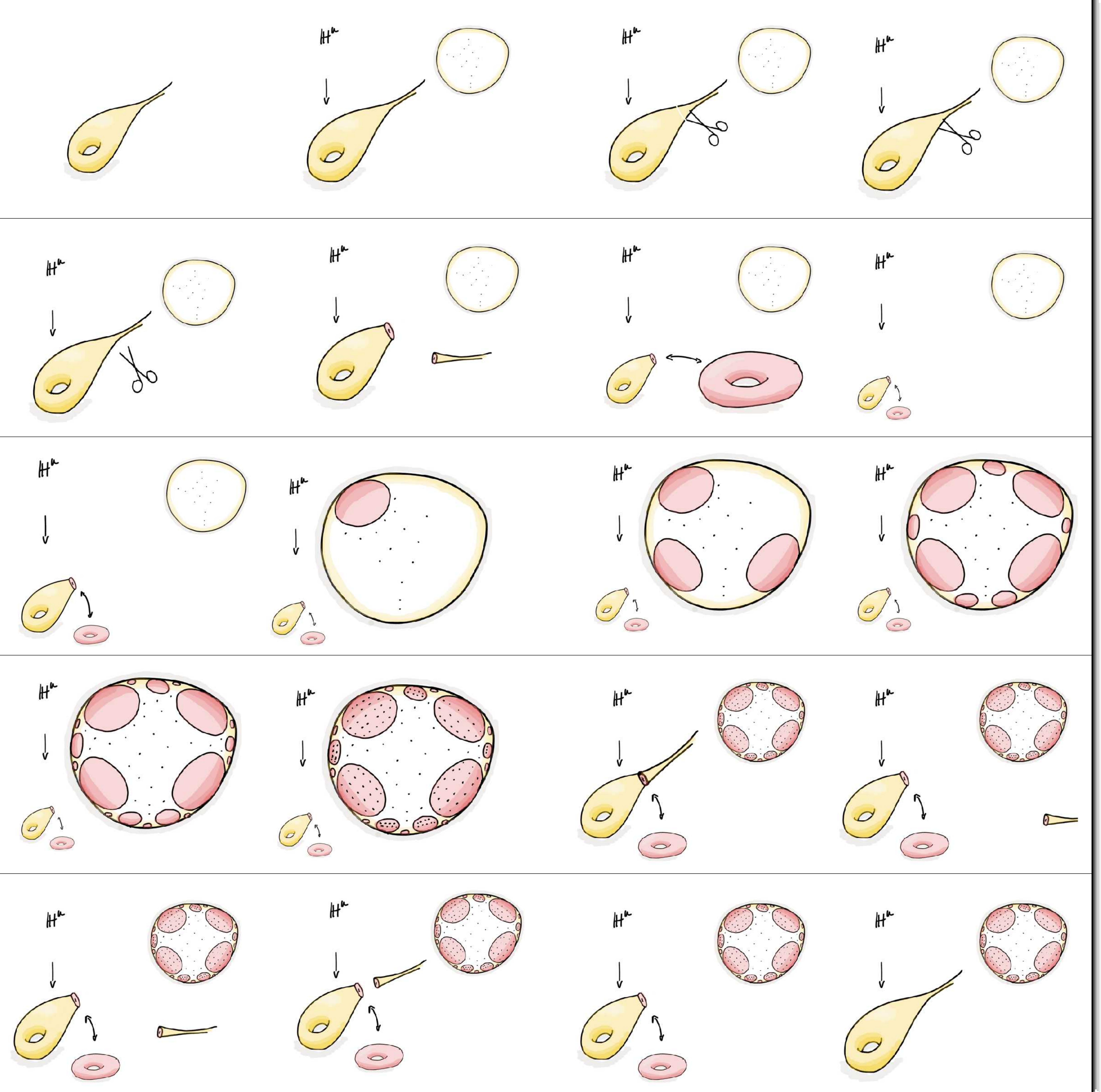
Autumn Kent

University of Wisconsin
drawings

I knew that connect sums, cabling, and plumbing preserved fibering. I sketched the complement of X and saw the square knot there. I cabled it for fun. I wanted the result to be hyperbolic so I needed to do something else. So I plumbed a little band on to get rid of the essential torus that was ruining hyperbolicity and I left class with a theorem. It took a while to prove that the examples were hyperbolic but I had found them. Getting lost in a lecture isn't always bad!

I drew the pictures by hand since I didn't know how to do it any other way!

For further information:
Autumn Kent, *Bundles, handcuffs, and local freedom*, Geometriae Dedicata 106 (2004), 145-159



This is an example of a CAT(0) space with isolated flats. It is the 3-dimensional hyperbolic space with an infinite collection of horoballs removed, called “neutered 3-space.”

The hyperbolic 3-space is the universal cover of a non-compact hyperbolic 3-manifold, here with one cusp. The fundamental group of that manifold doesn’t change when we cut the cusp off, but the manifold is now compact, with a torus as boundary. At the universal cover level, cutting off the cusp corresponds to removing the horoballs.

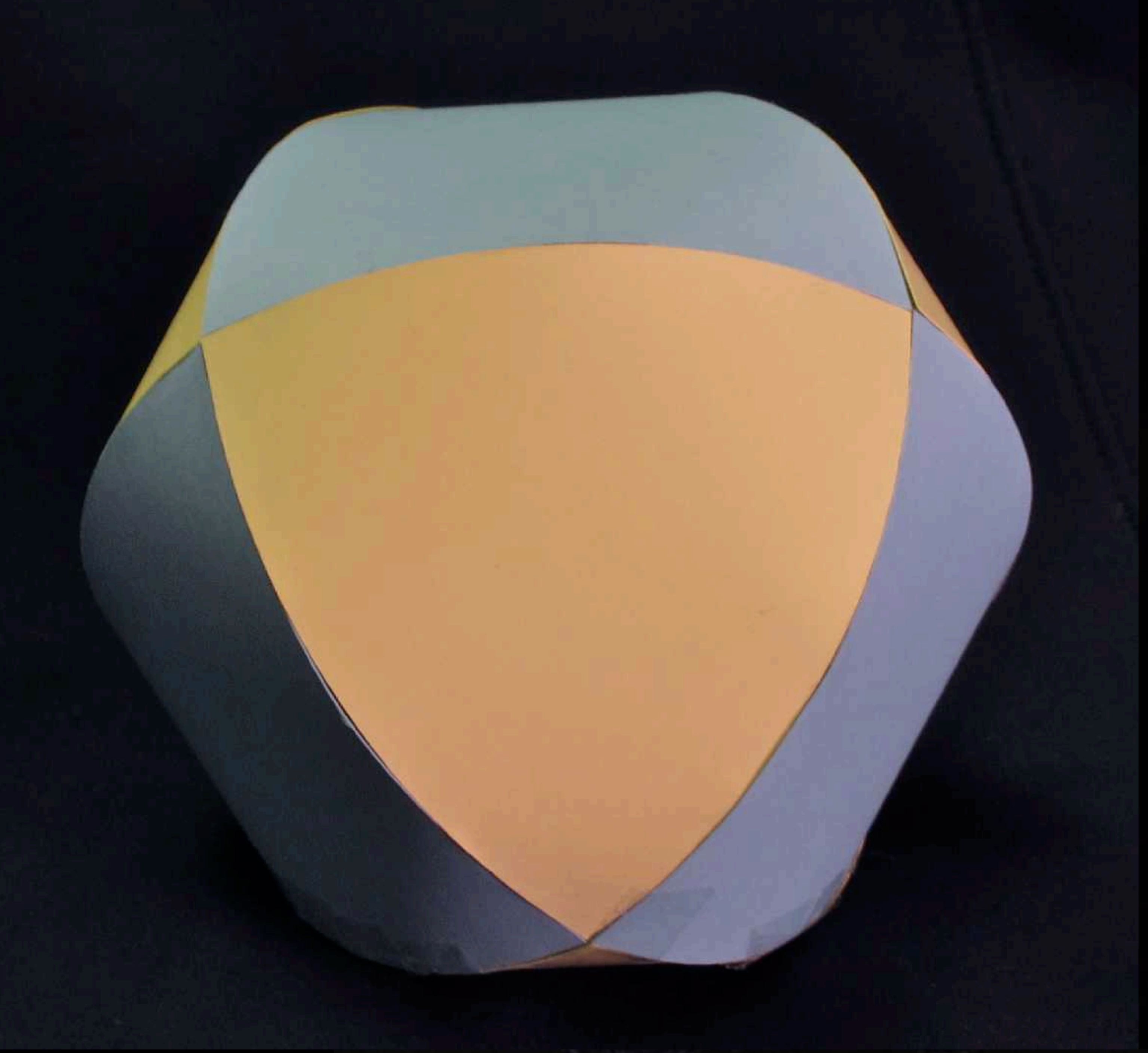
The hand-drawn pictures create an animated gif that illustrates the cutting of the cusp and the correspondence with the removal of the horoballs. The graphics are also an expression of a feeling of castration in my own professional life.

I like drawing, and I can do it easily on my iPad while traveling and on the go.

Indira Chatterji

University of Nice
Material

PAPER



Essentially, my project illustrates a version of the Gauss-Bonnet Theorem. The question I explored (with Bill Thurston), was the following:

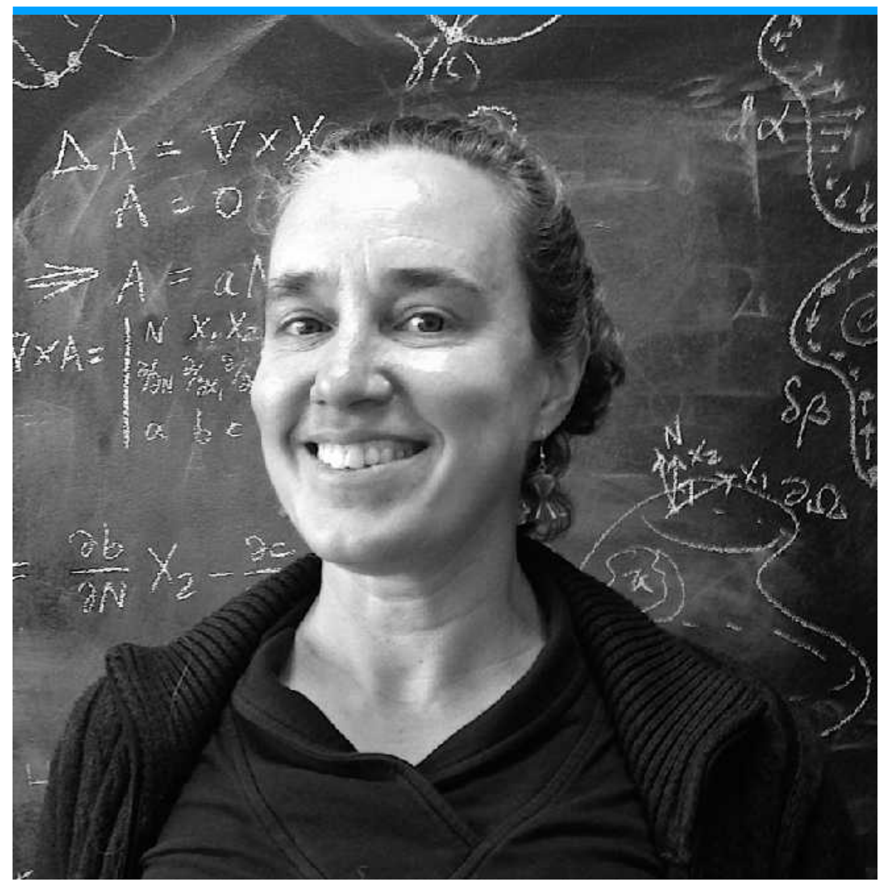
How can we make well-fitting "clothing," out of rigid material, for a sphere and other surfaces?

Paper was a fun, inexpensive medium to work with. Also, the craft paper cutters are relatively inexpensive, and can easily cut complicated curves.

I learned about planar curvature of curves, and Gaussian curvature of surfaces.

Our first try at making clothing for a sphere was the version using "fat triangles" to make an octahedral pattern (top left). This is the pattern made from 8 "triangles" that have edges made from circle arcs, that make right angled triangles. This sort of pattern works great when made from cloth, which has some stretch, but when made from paper, created a "sphere" with large flat faces. By making longer, meandering seams, the curvature was distributed over a larger portion of the sphere, which made a very satisfying sphere (bottom). An intermediate sphere was made to further illustrate how seams can distribute the curvature (top right).

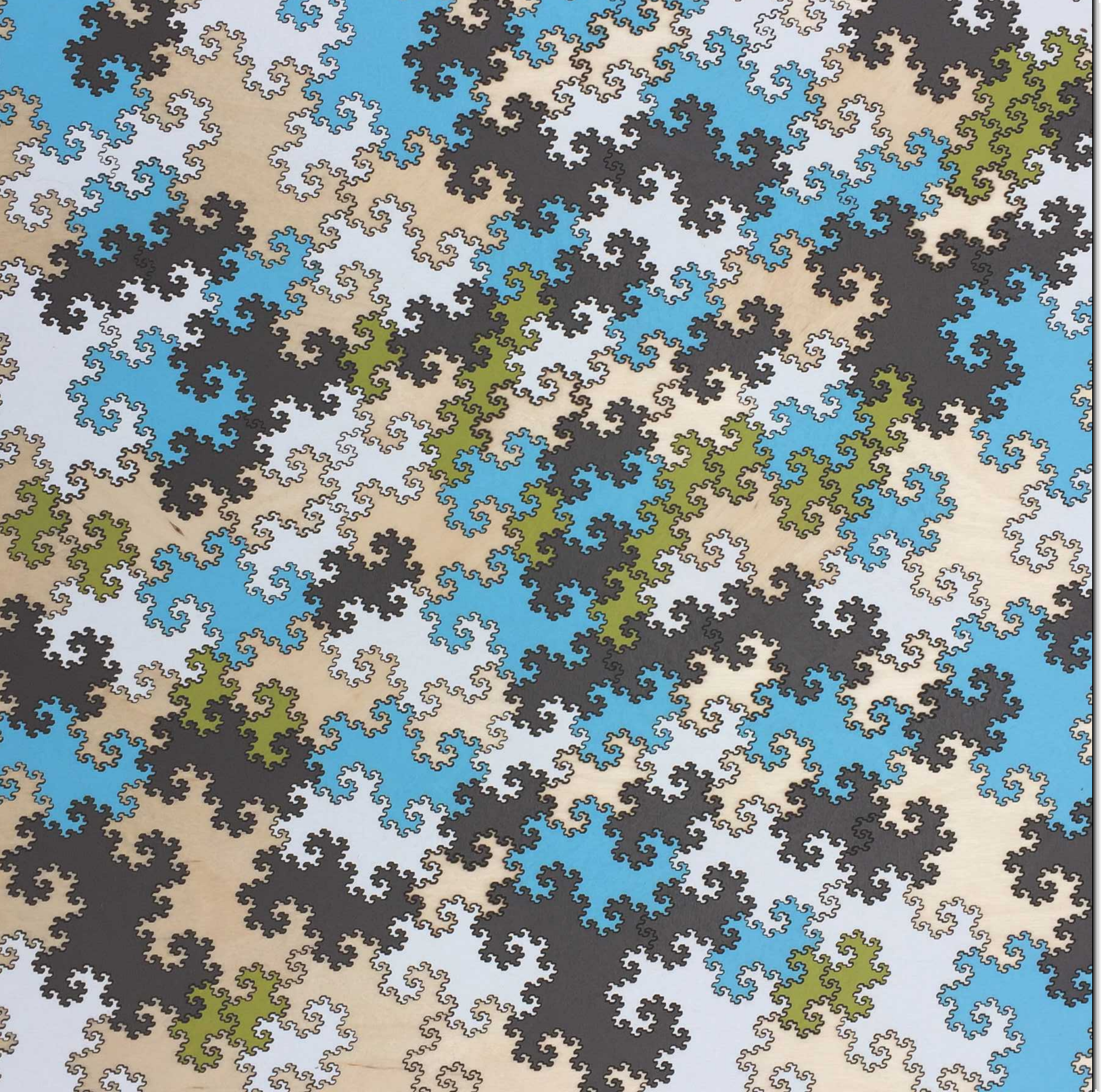
For further information:
https://archive.bridgesmathart.org/2011/bridges2011-1.html



Kelly Deep

Cornell University
paper

LASER CUTTING



This is a tesselation with twin dragons. The *twin dragon* is a 2-reptile fractal, meaning that it can be replicated from two smaller copies of itself. It illustrates that all pieces of the same size are translations of each other. We look for things that are mathematically well-known, beautiful, and accessible and engaging with the general public, and the twin dragon fits our criteria well.

We chose, as we often do, painted wood. People like the feel of wood, and we love the flexibility of an unlimited color palette.

We first made *terdragons*, which are closely related 3-reptiles. Tesselated terdragon pieces of a given size appear in up to six orientations. We were surprised to discover that twin dragons of a given size all have the same orientation.

We could only find descriptions of how to construct the interior of the twin dragon, and we used Adobe Illustrator to trace the boundary, but it wasn't good enough to create interlocking tiles. We were fortunate to be introduced to Dylan Thurston via Sarah Koch while Dylan was visiting the University of Michigan. We discussed the problem, and Dylan devised a clever substitution scheme for constructing the twin dragon starting with a single skewed hexagon. From there, it was obvious how to use an L-system to directly construct a boundary with whatever level of detail we wanted, and the pieces interlocked perfectly.

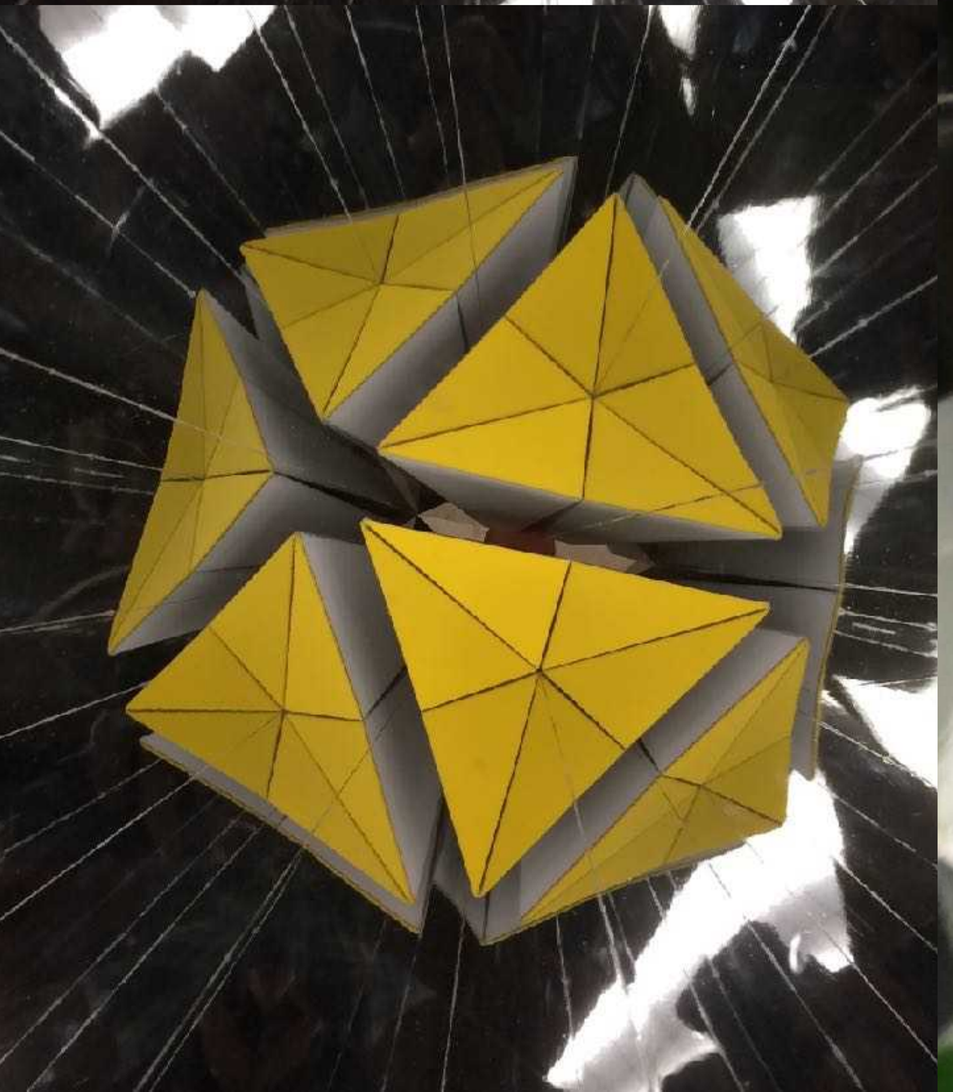
Our first efforts at cutting actually had too much detail. The pieces had fantastical lacy edges that interlocked, but as you tiled with them, the lacy bits would break off. We then dialed in just enough detail to make the pieces sturdy.

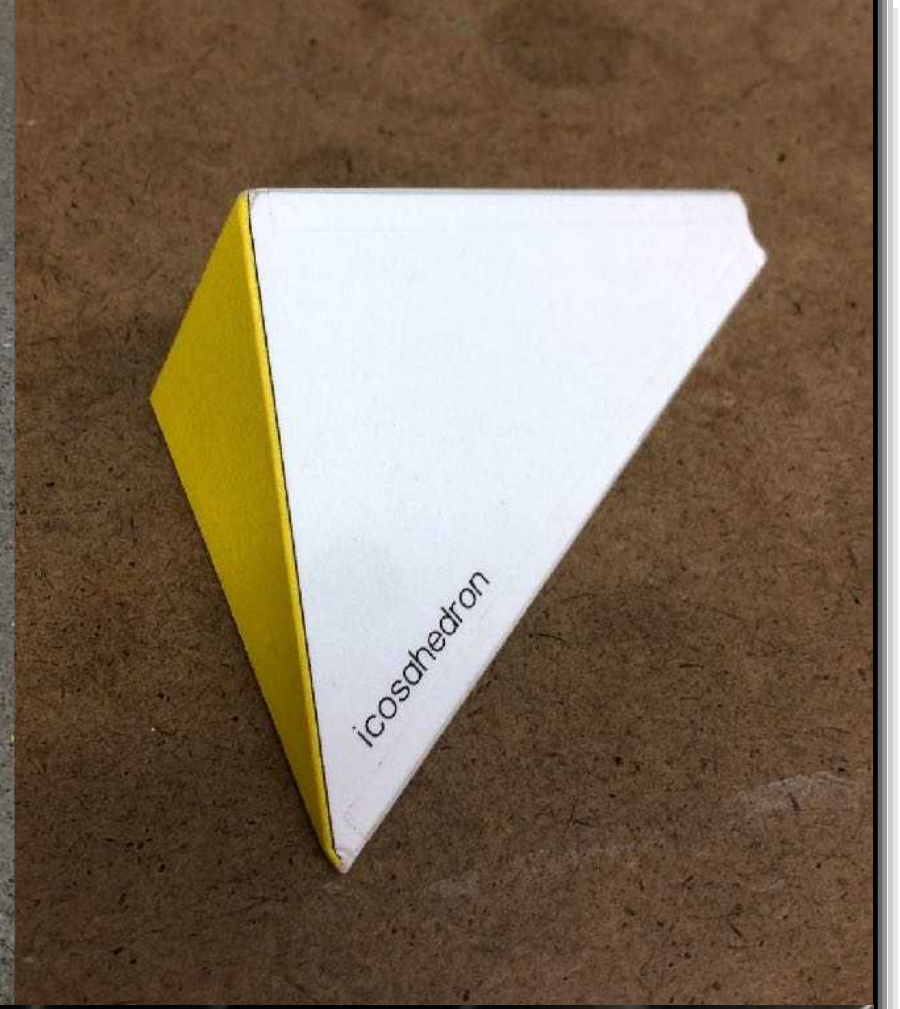
For further information:
https://www.cherryarbordesign.com/tessellating-with-twin-dragon-fractals/

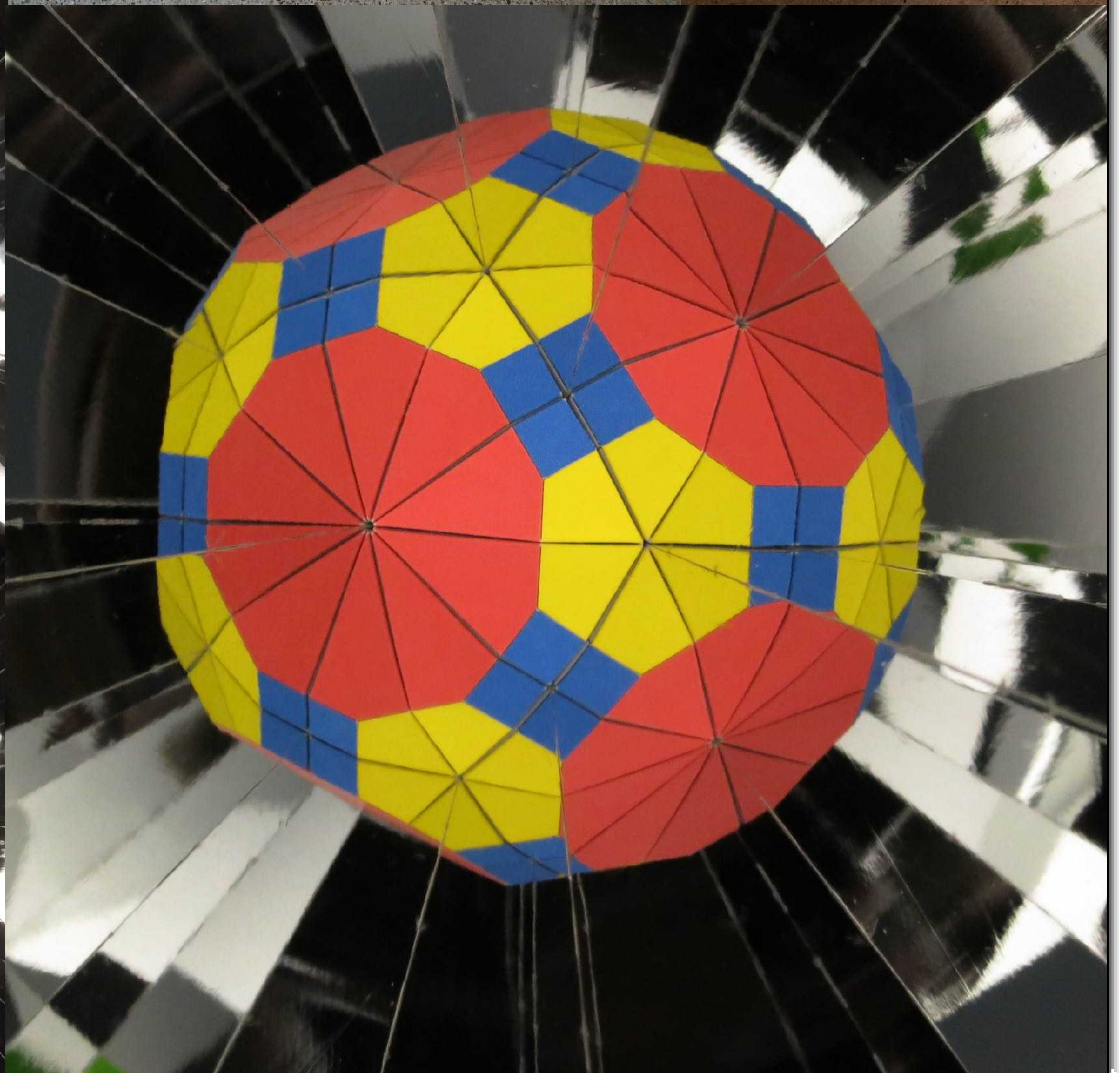
Heidi Robb and Peter Benson

Cherry Arbor Design, LLC

laser-cut wood, painted







These polyhedral kaleidoscopes allow one to create and view objects with icosahedral, octahedral, and tetrahedral point symmetries. The accompanying fundamental domains allow one to create the Platonic solids, and most of the Archimedean and Catalan solids. One can also invent new fundamental domains, and place them in the kaleidoscopes to see the resulting form.

I chose to use front-surface acrylic mirror and cardstock because they are laser-cuttable.

I made several iterations of the kaleidoscopes, working to refine the design so that they could easily be fabricated by someone with little or no crafting abilities. I hope they will be useful tools for others to learn about polyhedral symmetries.

John Edmark

Stanford University
laser-cut glass, cardboard

For further information:
https://www.instructables.com/id/Easy-to-Make-Polyhedral-Kaleidoscopes/

This collection of wooden tiles is a way to visualize the semi-direct product of the symmetric group S_3 acting on three copies of Z/2Z. Each tile represents one group element --- the three lines give an element of S_3 and the three dots give an element of (Z/2Z)^3 --- and we can illustrate the group action by concatenating a series of tiles and then reading off the resulting element from the lines formed between the tiles and the dots (mod 2) along each line.

These tiles originated in a project with Jonah Ostroff and Lucas Van Meter where we created variations of the classic card game SET for non-abelian groups. Such a game requires illustrations of group actions that can be easily understood, and so we made these tiles, along with versions for several other non-abelian groups, in order to experiment with the design of these non-abelian SET games.

This medium allows for tactile interaction with what is usually a purely abstract concept. By physical rearranging the wooden pieces, the players can test and visualize the multiplication in this group; small wooden pieces are ideal for this type of manipulation. We have also tried playing with paper cards, and while the mathematical content is the same, the laser-cut wooden tiles are much more aesthetically pleasing to touch and move.

I learned lots of interesting details about permutation groups and their group actions by creating this illustration. For example, I discovered that if you order the three transpositions in S_3, then the first two transpositions always compose to the same 3-cycle as the second two transpositions. More generally, I found that playing with these tiles gave an intuitive understanding of the algebra used to construct their underlying groups, especially in the case of semi-direct products.

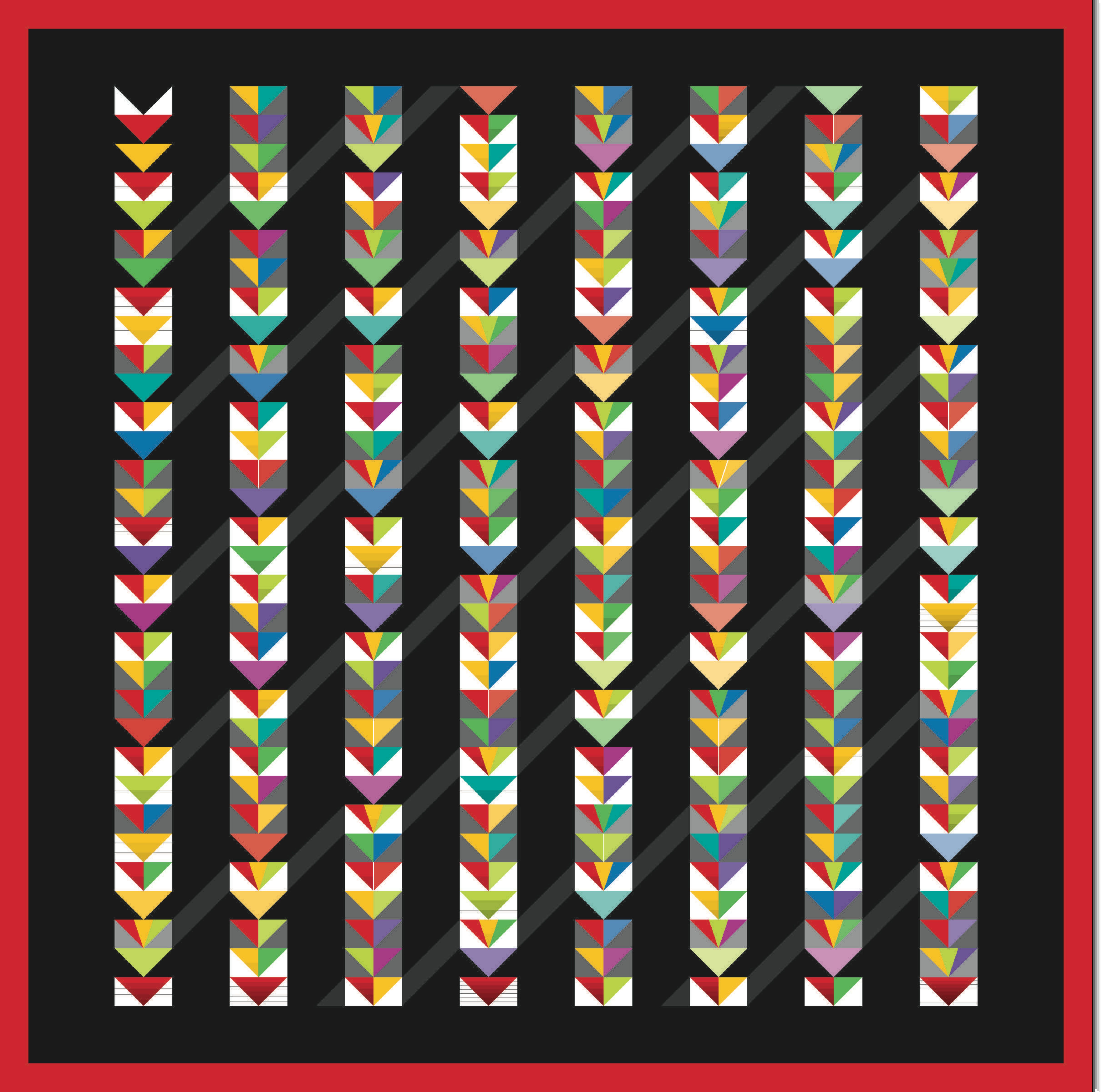
For further information:

Catherine Hsu

University of Bristol

laser-cut wood

GRAPHICS



This print, titled “Prime Goose Chase,” has a structure based on a traditional quilt pattern called *Wild Goose Chase*, while its content relates to the Fundamental Theorem of Arithmetic. The integers from 1 to 256 are the “geese,” and the prime decomposition of each integer is shown using colored triangles. There are 8 columns of numbers, starting with a black triangle representing 1 at the upper left. Solid triangles are used for primes, and each prime is assigned a unique color: 2 = red, 3 = gold, 5 = yellow-green, ... 19 = magenta. As larger primes are needed, more colors are created by adding white to these basic 8 hues. Composite numbers are represented by subdivided triangles. Since 6 = 2 x 3, it is half red and half gold. Powers of primes are shown using horizontal shades of the base color.

I believe integers have personalities, largely based on their prime factors. In this design, my goal was to produce a visual table that would show the factoring of individual integers, and also allow number patterns to be observed.

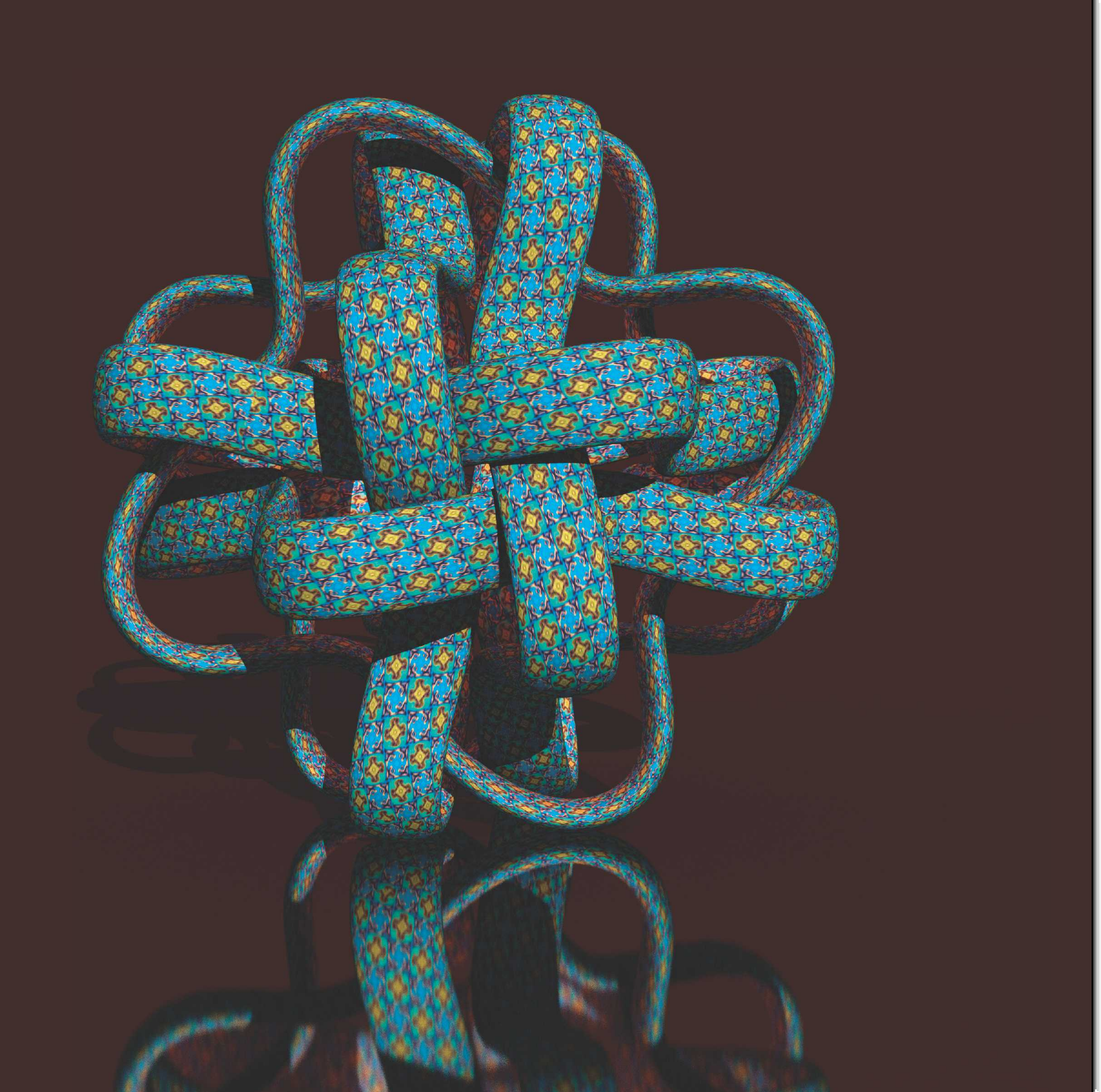
This work was produced as a digital print using Adobe Illustrator. This medium provided the precise shapes and range of colors my design required. By developing this piece, my appreciation of the integers and their inherent rhythms was improved.

In my first draft, the black spaces between the columns seemed too static. I solved this by adding sloping gray bands connecting multiples of 6. This also illustrated the fact that all primes, after 2 and 3, are plus/minus 1 from a multiple of 6. I enjoy finding visual elements that reinforce mathematical concepts.

For further information:
http://mekvisysuals.yolasite.com

Margaret Kepner

Independent artist
graphics



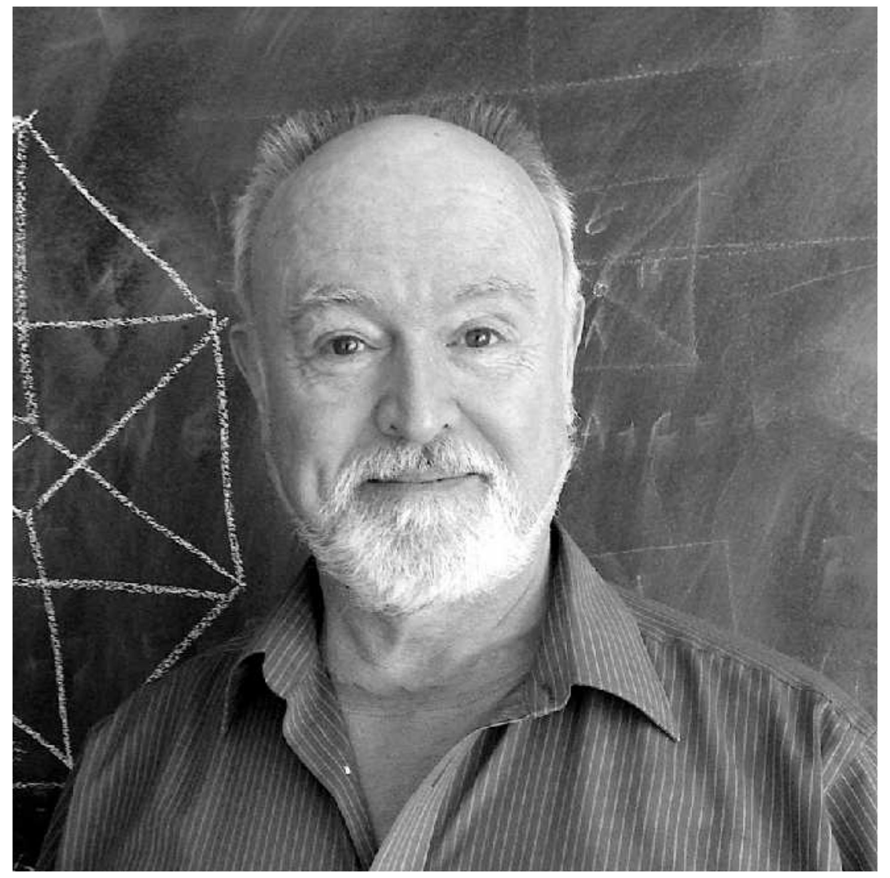
For years, I have struggled to find good techniques for coloring mathematical surfaces with patterns. Coloring a sphere is easy, because it's so round and regular, but the bands in this image have not been easy to color well. My goal is to find what are called "conformal" coordinates on this shape; this means that all the angles in the source pattern are portrayed correctly on the surface. Conformal coordinates make patterns more recognizable. I heard Steve Trettel's talk, "What does a torus look like?" and that sent me back to an old project, which involved coloring this set of bands with a wallpaper pattern.

The innovation shown here comes from making the shape in software called *Grasshopper* (a plug-in for an architectural design software called *Rhino*). The good news is that I was able to paste a nice wallpaper pattern onto the surface and then make it look nice in Photoshop. The bad news is that the coordinates are not conformal: the pattern is indeed distorted.

Grasshopper has a growing community of mathematicians who support each other in solving problems. Also, I learned how helpful the Grasshopper community is. I had been coloring this shape by coloring a gigantic mesh, with approximately one vertex for each pixel of the coloring pattern. It really bogged down the computer. When I asked the Grasshopper community, my questions was answered perfectly by Daniel Pinker, the creator of *Kangaroo*, another Rhino plug-in. I'll be working with this new technique for a long time.

It is still a long road to finding conformal coordinates on this particular structure, but I learned a lot about coloring tori. Surprisingly, the best way to find coordinates on donut shapes is to look for them in *four* dimensions, where they lie nicely on the three-dimensional sphere.

For further information:
Polyhedral Symmetry from Bands and Tubes, Wilder Boyden and Frank A. Farris, submitted to the Journal of Mathematics and the Arts, Fall 2019.



Frank A. Farris

Santa Clara University
graphics

The alert reader will see that there's an error in the image shown here: the pattern fails to match on the left side of the higher horizontal band. After all my Facebook friends had told me how beautiful is was, two mathematical artists looked at it for two seconds before saying, "Oh, but there's a mistake right there!" It's wonderful to be working in a community of people who take time to look closely at things.

In my multivariable calculus video course, one of the capstone projects is the computation of volumes of balls and spheres in arbitrary dimensions. The familiar πr^2 in 2D and $4/3\pi r^3$ in 3D for balls of radius r becomes an elegant interlaced sequence of recursive formulae in arbitrary dimensions.

How do you illustrate high-dimensional spheres? I wanted an image that would convey a sense of interleaving, as equatorial discs are revolved to spin out higher-dimensional balls. I also wanted to capture the beauty and mystery of these objects: the "music of the spheres" that animated the imagination of cosmologists of old.

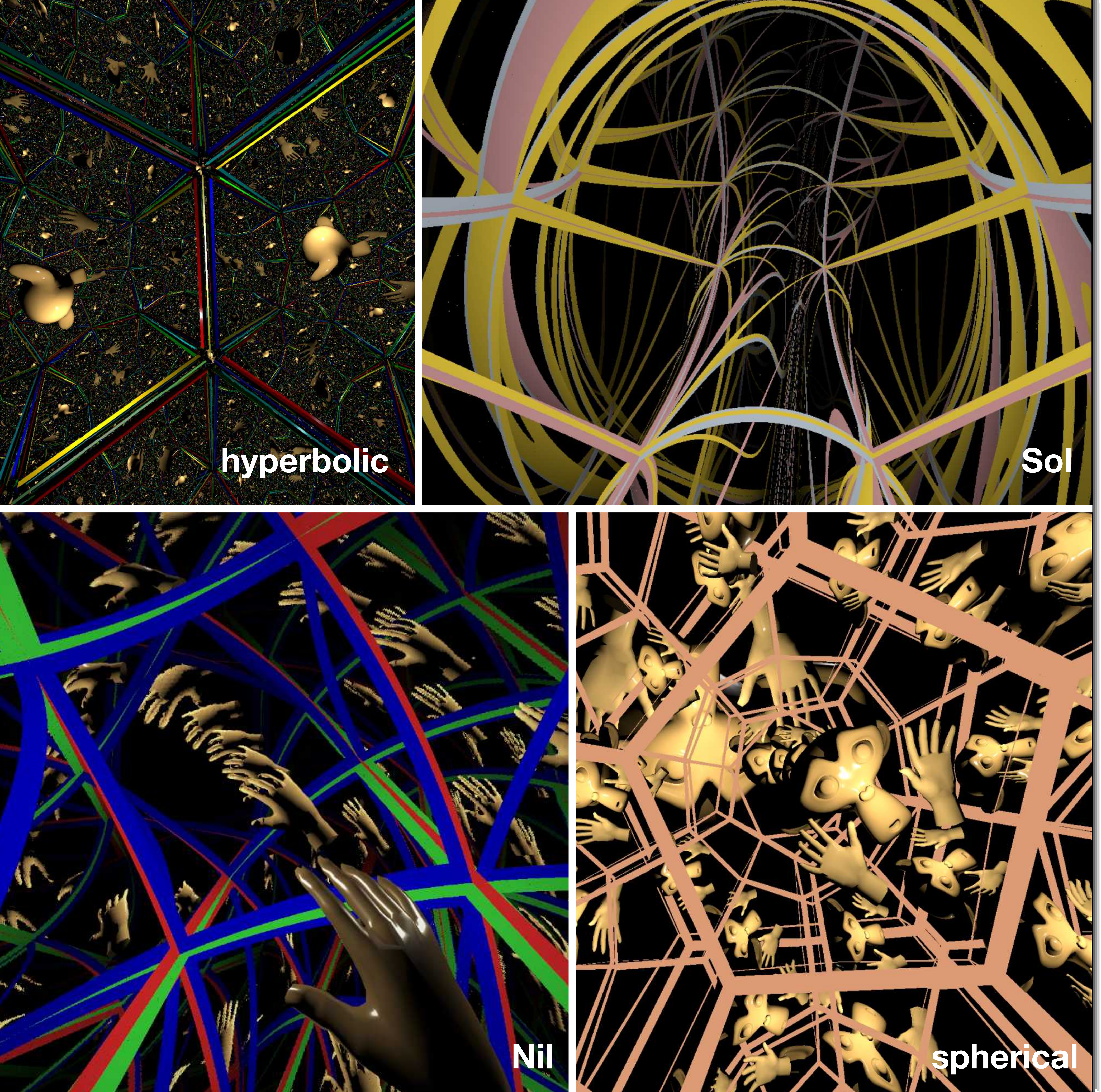
Computer graphics has no limits. All the geometric and topological entities I imagine can be cast into colors and forms on the screen. Color is the ultimate hard choice. Given that the calculus YouTube series is called "Calculus BLUE", I changed the initial greyscale images to an oversaturated blue-themed scheme.

As a result of rendering and, especially, animating this sequence, I fell in love with high-dimensional spheres in a more sensory as opposed to intellectual manner. It's one thing to "do the math" and another altogether to see the thing unfurl.

For further information:
www.youtube.com/c/ProfGhristMath/

Robert Ghrist

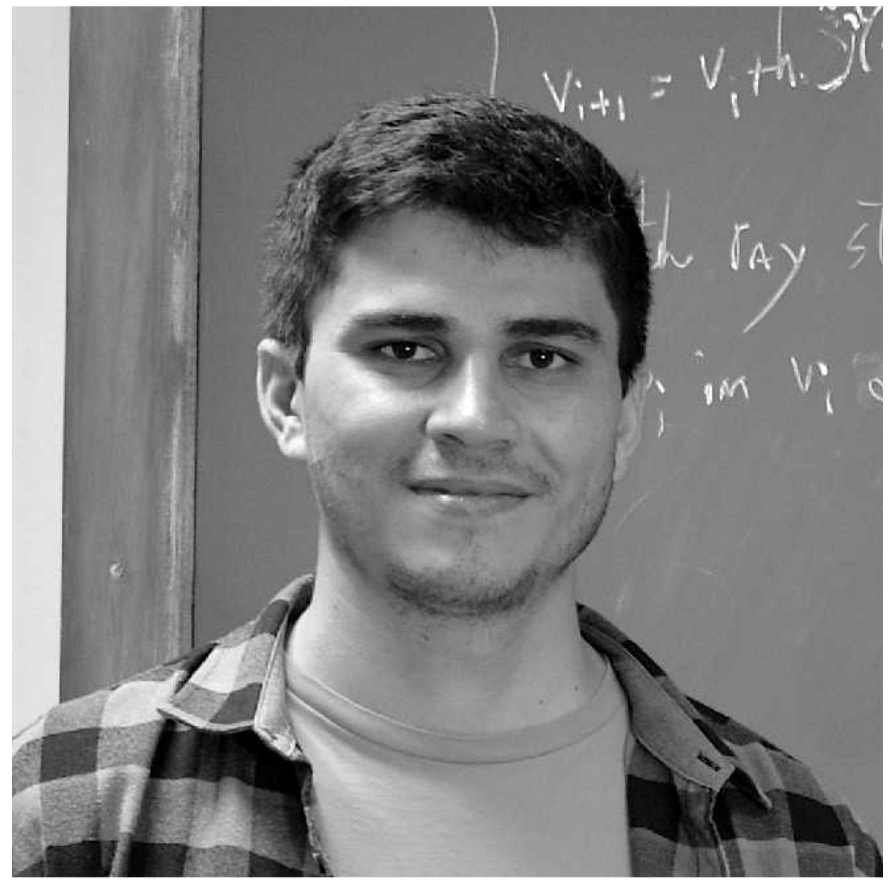
University of Pennsylvania
graphics



These images are inside views of famous 3-manifolds, with their geometries modeled by Thurston geometries. Such spaces date back to the famous *Thurston geometrization conjecture*, proved in 2003 by Grigori Perelman. The theorem states that every compact three-dimensional manifold decomposes into pieces whose geometry is modeled by Thurston geometries. The *Poincaré conjecture* is a corollary of this theorem.

it is difficult to visualize such beautiful spaces due to complications imposed by topology and geometry. The main effort for visualization such spaces was at the Geometry Center from 1994 to 1998 under the leadership of William Thurston. This program studied and disseminated modern geometry using interactive visualization based on the traditional rasterization pipeline (from computer graphics). Two problems arise: the scene must be replicated to "unroll" spaces with complicated topology, and to rasterize a scene it is necessary to compute rays between the scene points and camera position (a hard problem). To produce our images we use a ray tracing pipeline that overcomes such difficulties operating intrinsically in the geometry and topology. This consists of searching for the scene objects by tracing rays from the viewer towards the pixels.

Our images illustrate four three manifolds with their geometries modeled by hyperbolic, spherical, Nil, and Sol geometries — we think these are the most interesting Thurston geometries. Such manifolds are extremely important in the study of 3-manifolds, and we can visualize being immersed in a scene where powerful modern mathematics was developed.

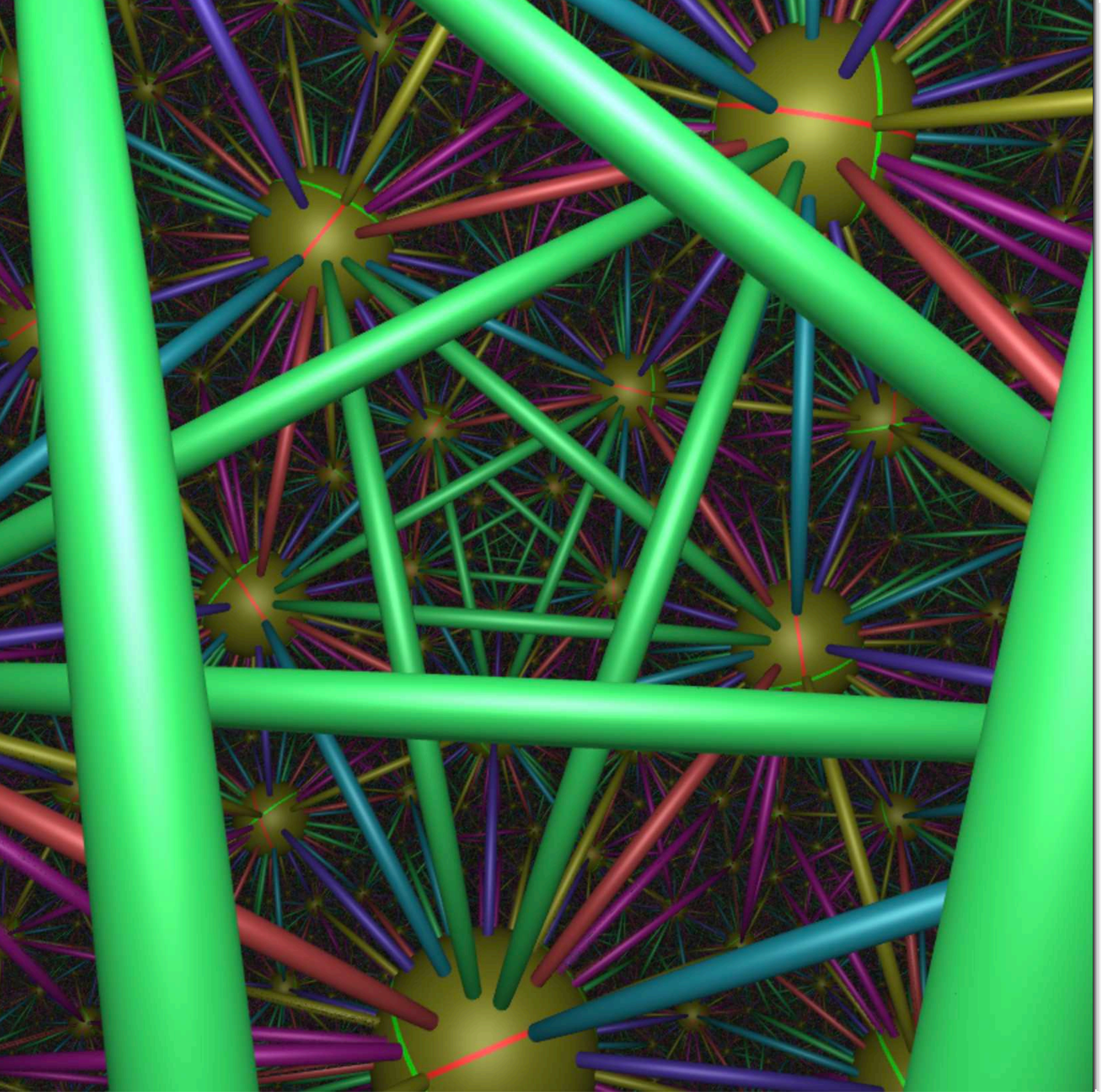


Tiago Novello de Brito

IMPA - Instituto de Matemática Pura e Aplicada
graphics — virtual reality

The hyperbolic and spherical manifolds are both obtained by gluing, with appropriate rotations, opposite faces of special regular dodecahedrons in hyperbolic and spherical spaces. A monkey's hand, or face and hands, are added to the scene to give more context. The Nil and Sol manifolds are obtained by identifying faces of the unit cube. In both cases, two faces are identified by translation, and we identify the remaining faces in such a way to give rise to the Nil and Sol manifolds. In the Nil manifold, there is one hand plus the cube's edges, and in the Sol manifold only edges. The face pairing makes the rays iterate giving a tessellation of Nil and Sol spaces by cubes.

See videos, papers, technical reports and texts at https://www.visgraf.impa.br/ray-vr/

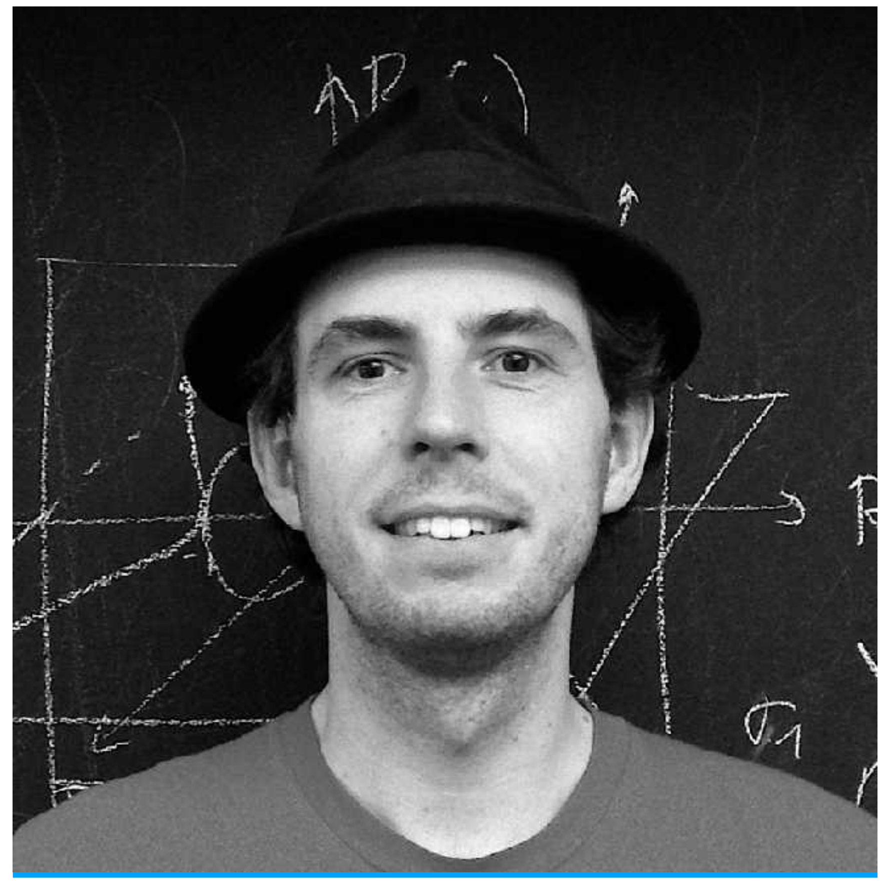


I added a feature to the 3-manifold software SnapPy that shows you the view you have when being inside a hyperbolic 3-manifold. I just tried out some random census manifolds to see whether I get an image that would make a nice demo. I was pleasantly surprised to find this one (s431). The image shows the edges of the triangulation and a cusp neighborhood as you look down a closed geodesic in the hyperbolic 3-manifold. Walking down the closed geodesic once rotates the hyperbolic 3-manifold by 2/5 of a turn, and you see a beautiful helix formed by one of the edges.

A 3D print of a tiling by fundamental domains is always distorted in some way because we live in a Euclidean world (at least on human scales). However, the image you see here is the true (undistorted) inside view of the hyperbolic 3-manifold.

This is still work in progress. For example, I am still trying to come up with a good way to show which tetrahedra are the same in the manifold. I first tried to do this by coloring a small ball about their incenters. It turns out that our Euclidean intuition about where we the expect the incenter of a tetrahedron is completely off in the hyperbolic world.

For further information:
https://snappy.math.uic.edu/



Matthias Goerner

graphics

3D PRINTING

This set of 3d prints illustrates singular algebraic surfaces. I work on solutions to the particular problem of physically visualizing nodal singularities, where two or more pieces come together at a single point. My main motivation for 3d printing them is to illustrate the output of the algorithm for numerical real cellular decomposition implemented in my computer program Bertini_real. It computes a union of "cells", each equipped with a generic point and homotopy which can be used to compute additional points on the real part of a complex variety. This output is naturally 3d printable.

Fused filament fabrication (FFF) using thermoplastic polyurethane (TPU) is the right material for these objects for a number of reasons. FFF can be done at home or in the office with an inexpensive 3d printer, with no chemically dangerous materials or supplies. Post-processing consists only of removing support material, and since TPU is flexible, printed surfaces can have much thinner connections at nodal singularities without inevitably breaking. Furthermore, the very small earring-sized prints are robust enough to be worn daily, and the skeleton-like object is seemingly delicate, but readily able to be carried in a backpack or pocket. TPU is a challenging material to work with, requiring a high quality hotend and a commitment to printing very slowly, but it's incredibly rewarding.

I have really come to understand the differences in the inherent singularities in a surface, as opposed to those coming from projections -- such as fictitious self-crossings when printing a 4-variable complex curve into three spatial variables. Also, the Barth Sextic exhibits really fascinating movement when printed with very thin singularities, something impossible to see when printing from a rigid material. I also continue to learn about how to engage the public in my research by having cheap non-breakable objects to show them while wearing earrings at the grocery store.

Silviana Amethyst

University of Wisconsin-Eau Claire
3D printing

My early versions of prints of the Barth Sextic and other nodal surfaces, such as those coming from the Herwig Hauser gallery of algebraic surfaces like Octdong, broke either during support removal or transport for show. Many of my most experimental PLA prints are thus doomed to live forever trapped in their support material, since it's so far inside a cavity in the surface I would destroy it in completing it, or the material connection between pieces is so small.

For further information:
https://silviana.org/gallery

Danielle A Brake, Daniel J. Bates, Wenrui Has, Jonathan D. Hauenstein, Andrew J. Sommese, Charles W. Wampler, *Algorithm 976: Bertini_Real: Numerical Decomposition of Real Algebraic Curves and Surfaces*, ACM Trans. Math. Softw., July 2017.

These objects solve the following minimization problem: among all polytopes with given number of vertices and given volume, find the one which has the least surface area. We call them *Akiyama polyhedra*, after Shigeki Akiyama, who discovered them. From left to right, bottom to top, these are the Akiyama polyhedra with 3, 4, 5, ..., 12 vertices. All of the polyhedra in the picture have the same volume.

I do a lot of mathematical outreach for children, and I made these to bring to math festivals. When you can hold an object in your hand, you get aa better understanding of it. For example, in some cases, the symmetry of the shape is obvious (the tetrahedron and icosahedron) but several of them are not at all obvious, so it helps to be able to look at it from all sides.

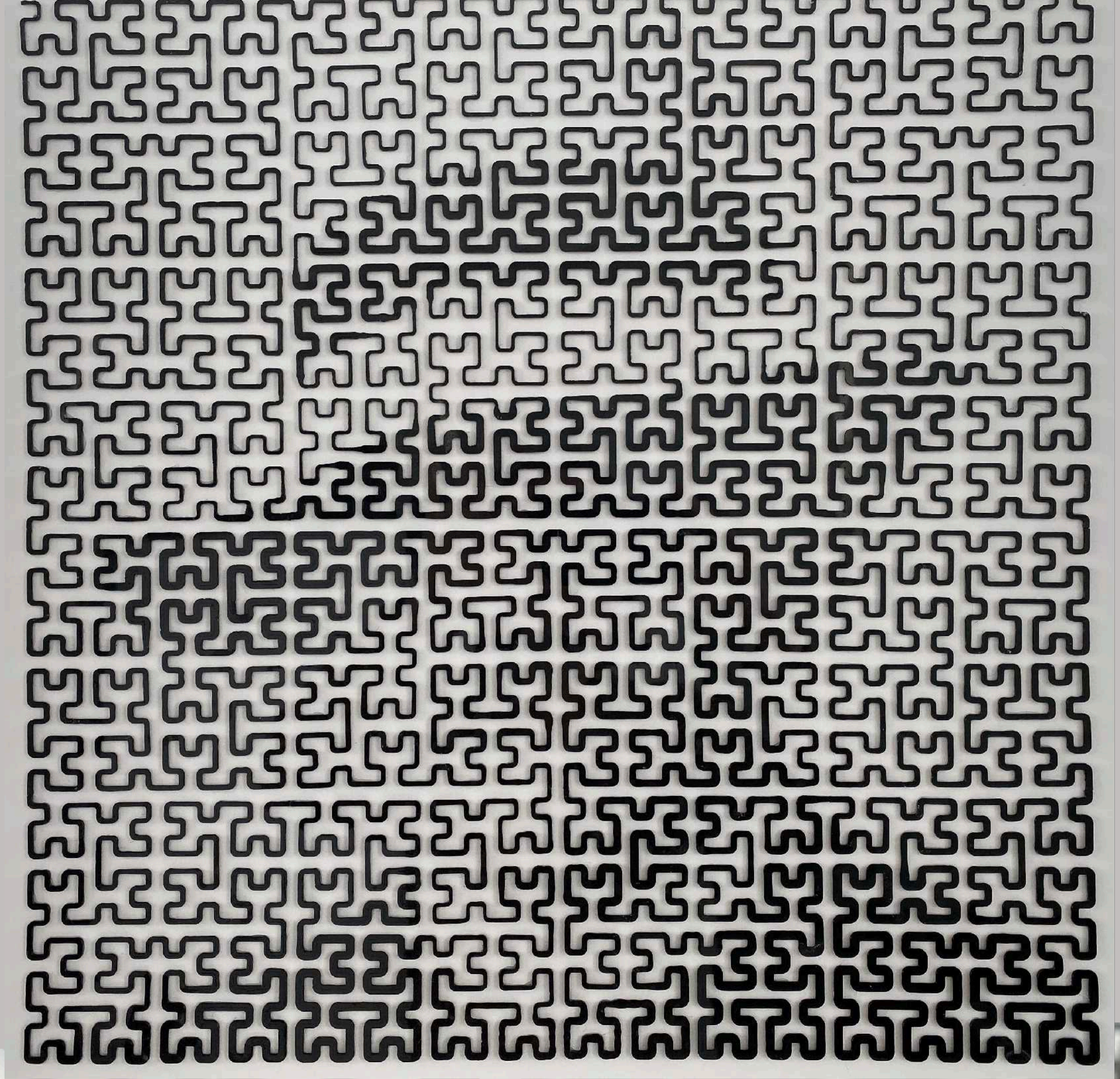
When I 3D printed these, I printed them with 100% infill, so that they have not only the same volume, but the same weight. Visually, some of the polyhedra look a lot bigger than others, so making them the same weight emphasizes that they all have the same volume. This was easier to do using 3D printing technology than with any other method.

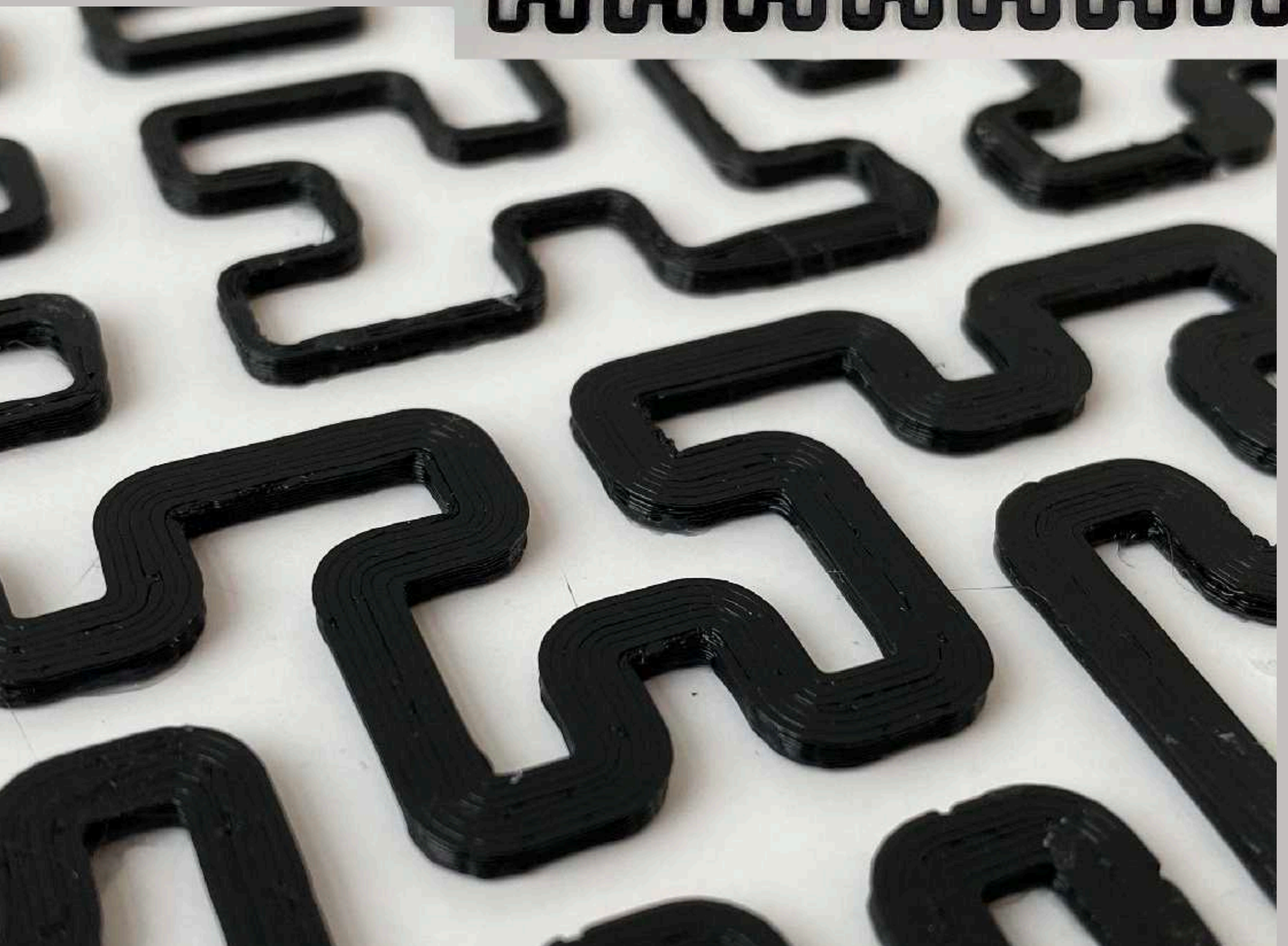
Alba Marina Málaga Sabogal

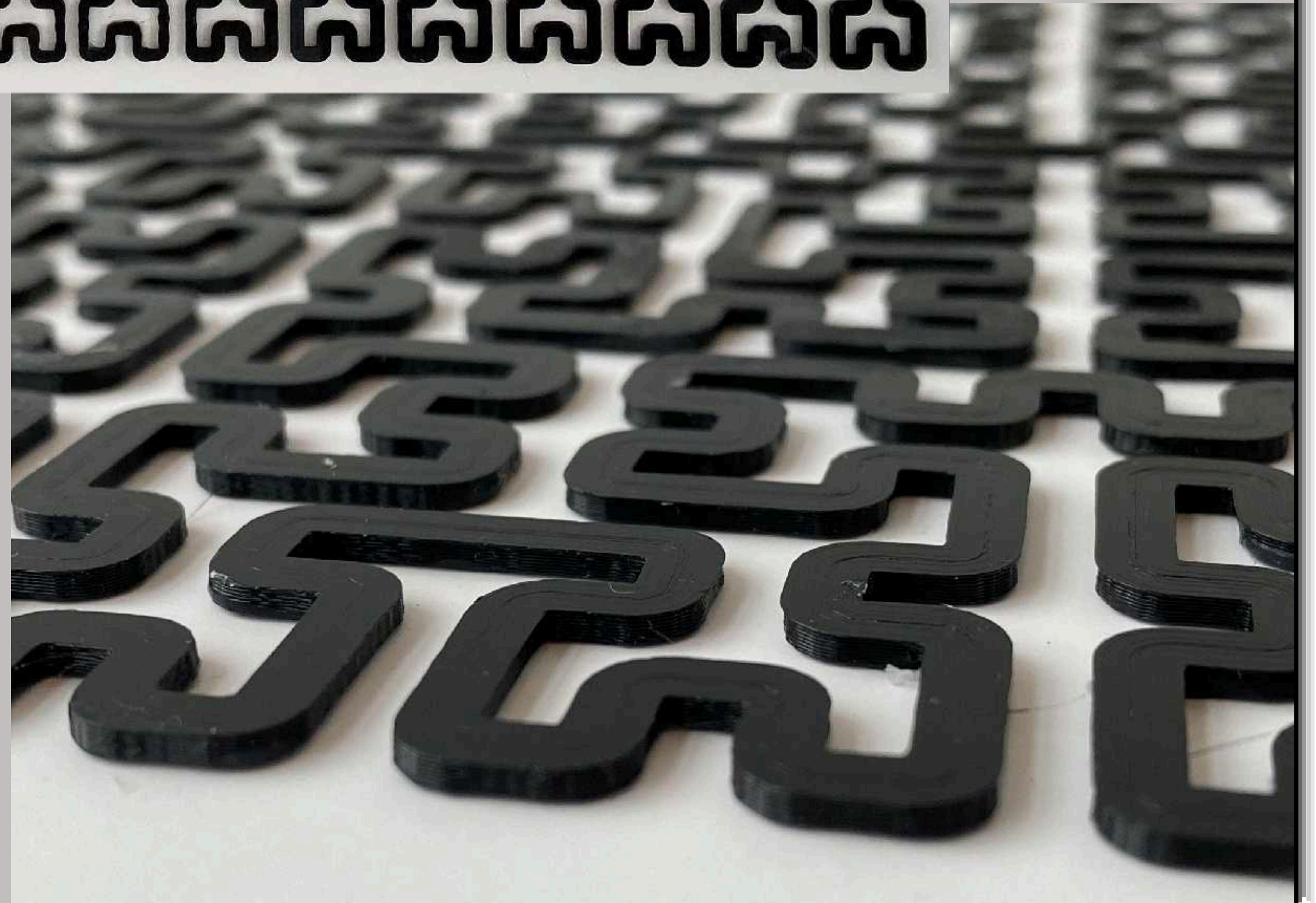
ICERM

3D-printed plastic

For further information:
https://imaginary.org/hands-on/minimizing-polyhedra
Minimum polyhedron with n vertices, Shigeki Akiyama, preprint, 2017.







More than anything, my work illustrates the human brain's remarkable ability to fill in missing information as it works to make sense of the data it receives from the surrounding world. By varying the thickness of piping generated along the sixth iterate of Hilbert's famous space-filling curve, I have rendered Hilbert's 2D portrait as a 3D object. Using Python script within the modeling program Rhino, my code reads in an image and uses pixel data to generate a rectangular piping along the curve. The width of the rectangular profile of the piping at a point depends on the intensity of the pixel at that point; it is wider when the intensity of the pixel is lower (where the image is darker) and thinner when the intensity of the pixel is higher (where the image is brighter).

I am new to 3D printing, so I chose this medium in large part to gain expertise in the modeling and printing processes of a 3D object. I worked in stages, first learning how to print a Hilbert curve with square piping, before incorporating a rectangular profile with variable width. I used a Python script within Rhino to carry out the process, reading in image data from the famous (cropped) picture of Hilbert in his white hat.

I created this portrait using the sixth iterate of the Hilbert curve, which I constructed in pieces â€“ printing off 16 â€œsubsquaresâ€ of the curve on an Ultimaker 3. Each subsquare is approximately 5 inches in width and was generated using the fourth iterate of the Hilbert curve. Hoping to render a higher resolution of the image, I was initially planning to generate the portrait using the seventh iterate of the curve. However, such an approach would have involved well over 140 hours of printing.

Judy Holdener

Kenyon College
3D printed plastic, paper

I decided I was happy with a lower resolution, which may require some squinting on the part of the viewer. An unexpected discovery was the difference in finish and texture between the top of each printed curve segment and the bottom. Preferring the shiny finish of the bottom, I ultimately decided to print each subsquare of the curve upside down.

For further information:
Wikimedia Commons image "File:Hilbert.jpg" https://commons.wikimedia.org/wiki/File:Hilbert.jpg

The 2-adic solenoid is quite the elusive pop star, featured on many covers of dynamical systems textbooks but rarely seen as a tangible object in our world. Fortunately, this piece can be held in the palm of your hand! It consists of three solid tori 3D printed in nickel, representing the geometric construction of the 2-adic solenoid. The second torus is stretched and wrapped inside the first, and the third is stretched, twice, and wrapped inside the second. The process can be repeated ad infinitum. The intersection of all transformed solid tori gives the solenoid. The cross-section of the piece shows a system of intersecting disks forming a Cantor set, illustrating the fact that locally, the solenoid is a Cantor set cross an interval.

I chose 3D printing because I wanted the model to be as precise as possible. Nickel was chosen because I wanted the object to act as a paperweight, sitting heavy atop some papers or in a person's hand. When making the piece I knew very little about the actual mathematical object, but the making of the tangible piece spurred a lot of research and discovery. I created several talks explaining the different constructions which I labeled as *geometric, topological* and *algebraic*. Through this research I learned that the solenoid arises in several mathematical fields, and takes on many forms that seem different but are ultimately equivalent dynamical systems.

Creating the design in Mathematica did not take too long once Ian Putnam helped me with the surface formula. Printing a rough sketch model at UVic's Design Scholarship Commons also happened fairly quickly. But, making the model printable in metal through Shapeways was a process of trial and error over the course of many frustrating months. Rendering the model in Mathematica and uploading to Shapeways often took several minutes, with failure being the final result. Ultimately, tinkering and persistence won Shapeways over and my design was printed!

Dina Buric

University of Victoria

3D printed nickel

For further information:

http://www.math.uvic.ca/~buricd/projects.html

Lecture Notes on Smale Spaces
http://www.math.uvic.ca/faculty/putnam/ln/Smale_spaces.pdf

An Introduction to Dynamical Systems by R. Clark Robinson

The Klein bottle is one of the simplest examples of a *fiber bundle*: a space built by putting copies of one shape (the fiber) at each point of another space (the base space) in such a way that adjacent copies glue together nicely to form the total space.

The commonly used embedding of the Klein bottle into 3-space is quite good at illustrating how to construct the Klein bottle: through gluing one pair of sides forming a tube, and then gluing the other pair with a flip. This shape of the Klein bottle, however, highlights precisely the fiber bundle structure itself. We can see and follow the base space around the figure, and a selection of fibers are printed so we can see their circular structure as well.

3D printing was a natural choice, since the construction I envisioned was relatively straightforward to implement algorithmically. Where a hand-constructed or sculpted approach would have suffered from variations across the piece, a 3D-printed approach gave it the uniformity I sought.

Choosing to print a wireframe highlighted the fibers and base space structure I wanted to illustrate, and makes it easy to trace these features as they wind around the shape.

My first attempt (lower right) was printed on an extrusion printer. A combination of extreme overhangs and issues with temperature control failed that print, with drooping strands of printing material not connecting to each other. Based on that failure, I switched to sintered type printing methods —and sent my models to Shapeways rather than trying to print them myself. By using sintering, the overhangs are supported by the unprinted materials, and cease to be a problem for the printing. The first instance of that, seen here in grey, was printed in alumide and used the features of that printing method to print a disconnected ball in place inside the Klein Bottle.

Mikael Vejdemo-Johansson

CUNY College of Staten Island /
CUNY Graduate Center

3D printed plastic, alumide, steel/bronze composite

The modeling for that had aesthetic issues. Along a curve winding twice around the Klein bottle, several tubes in the model bundle up together. Furthermore, the fibers themselves were directly self-intersecting - somewhat obscuring the fact that they are all topologically circles. Revisiting the modeling script I created a new version which includes the base space as one of the longitudinal wireframe wires, and that dodges the fibers out of the way of each other. This way, the circularity of the fibers is more clearly visible and the self-intersection of the Klein bottle along the base space can be seen in the alternating intersections of fibers with the base space circle.

These are two so-called *cubic surfaces:* the points satisfy a certain equation in *x,y,z* of degree three. These are some of the most fascinating objects between algebra and geometry from the 19th century.

I have been creating many different 3D-printed versions of such objects in the past 20 years. Classical sculptures show them in plaster, and my modern versions, which are exhibited in museums, are usually 3D-printed in white plastics. But these objects in gold-plated brass visualize their fascinating geometry in a much more obvious way, mainly because of the reflections of the material. In particular, in some interesting light situations with lighter and darker colors around, these reflections yield interesting visual effects on the surfaces of the sculpture.

The curvature of these surfaces is much more interesting than you might think when just computing it.

I only made earlier versions of these objects using 3D printing. Over the years, I made so many different versions, until, finally, now, I like them. Earlier versions that I created were the part of the surface inside a sphere, which is a natural way to cut a finite part of an infinite object — at least when one does not know anything about the object. But I worked with this object for a long time, and thought a lot about what the best representation of this object in space is, and I found a better way to choose a piece of this infinite surface, than just what is inside a sphere.

For further information:
https://math-sculpture.com/
https://link.springer.com/article/10.1007/s00283-017-9709-y

Oliver Labs

MO-Labs Dr. Oliver Labs
3D-printed gold-plated brass

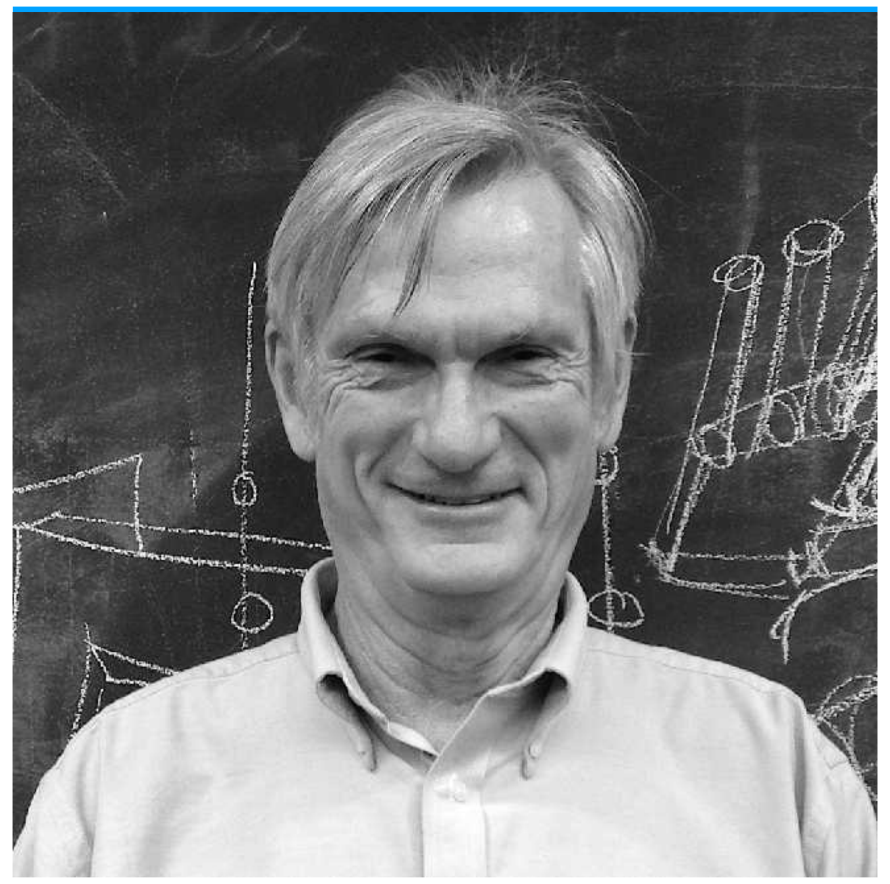
These are fundamental building blocks for knotted and braided surfaces in 4-dimensional space.

My goal was to create standard sized models that could be snapped together in a lego-like fashion so that a variety of surface braids could be made as children's toys. Prototypes needed to be printed via 3-dimensional printers.

Singularities require care when being depicted in the real world.

In the 3-dimensional printing world, I learned to cope with unifying normal vectors, in particular, but in more generality I learned, without much surprise, that curves and surfaces are fictions. My goal is to depict surfaces as they appear when projected into 3-dimensional space from 4-dimensional space, but to do so, the surfaces that I draw must be thickened to create 3-dimensional models which can be sent to the printer. (3) Some effort had to be made to accommodate the transition from 2-dimensional graphics to 3-dimensional graphics. Splines and handles of curves require more care on a 3-dimensional palette that is projected to a computer screen. On the other hand, some needed operations became easier. Finally, as I move forward, I will have to change my paradigm from drawing surfaces to sculpting them as I develop facility with Blender.

For further information:
arXiv:1907.01899, arXiv:1905.04838



J. Scott Carter

3D printed plastic

FABRIC

The Fabric of Spacetime is an installation serving as an interactive model of a young universe, combining crochet and electronics to create a dynamic and luminescent experience.

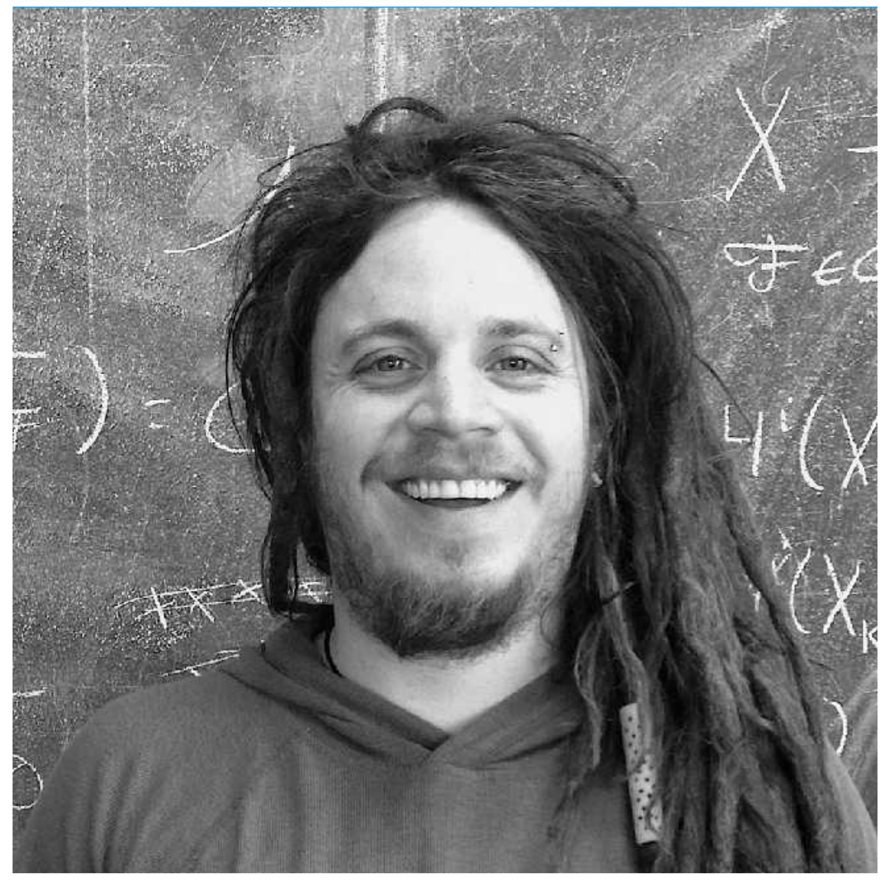
The main physical component is a large, hand-crocheted hyperbolic manifold, where stitches were added at an exponential rate. In this way, the circumference grows exponentially faster than the length, introducing negative curvature and resulting in the many folds. This technique was pioneered in 1997 by Dr. Daina Taimina, a mathematician at Cornell. The curvature is analogous to the geometry of a very young universe (much less than one second old), where the spacial dimensions grew exponentially with respect to the time dimension, introducing curvature in the geometry of spacetime itself.

Sewed into the fabric model are 264 individually programmable neopixel LEDs, forming a spiral pattern around the inside, and mounted throughout room are 6 servo motors, each connected by fishing wire to a fold in the crocheted model. There is also a PIR motion sensor directed toward the underbelly of the piece. All of these are wired to an Arduino MEGA microcontroller.

While undisturbed, the servos pull the model open and closed in a regular breathing motion, shifting to red while opening, and to blue while closing. This is an homage to the cosmic red shift and blue shift of the universe. Indeed, since red has the longest wavelength of the visible spectrum, things moving away from us at relativistic velocities gain a slight red tint in color, and things moving away gain a slight blue tint. In this way the color shift of the piece represents the actual shift that would be visible in an expanding and contracting universe.

For further information:

https://github.com/gdorfsmanhopkins/fabric-of-spacetime
http://www.gabrieldorfsmanhopkins.com/art.html



Gabriel Dorfsman-Hopkins

ICERM / Berkeley

fabric, LEDs

I find crochet to be a very flexible medium, both literally and figuratively, and am fascinated at the way it can create various geometries and symmetry patterns, and also how easy it is to combine with electronics,

.

I really tangibly felt the intensity of exponential growth. To create hyperbolic crochet, you add stitches in such a way that the circumference of the crocheted piece grows exponentially. In practice, this means the workflow really starts to slow down. This piece took hundreds of hours of crochet, and weighs 10 pounds. The circumference of the piece at the base is over 80 feet (all hidden in the folds).

MECHANICAL CONSTRUCTIONS

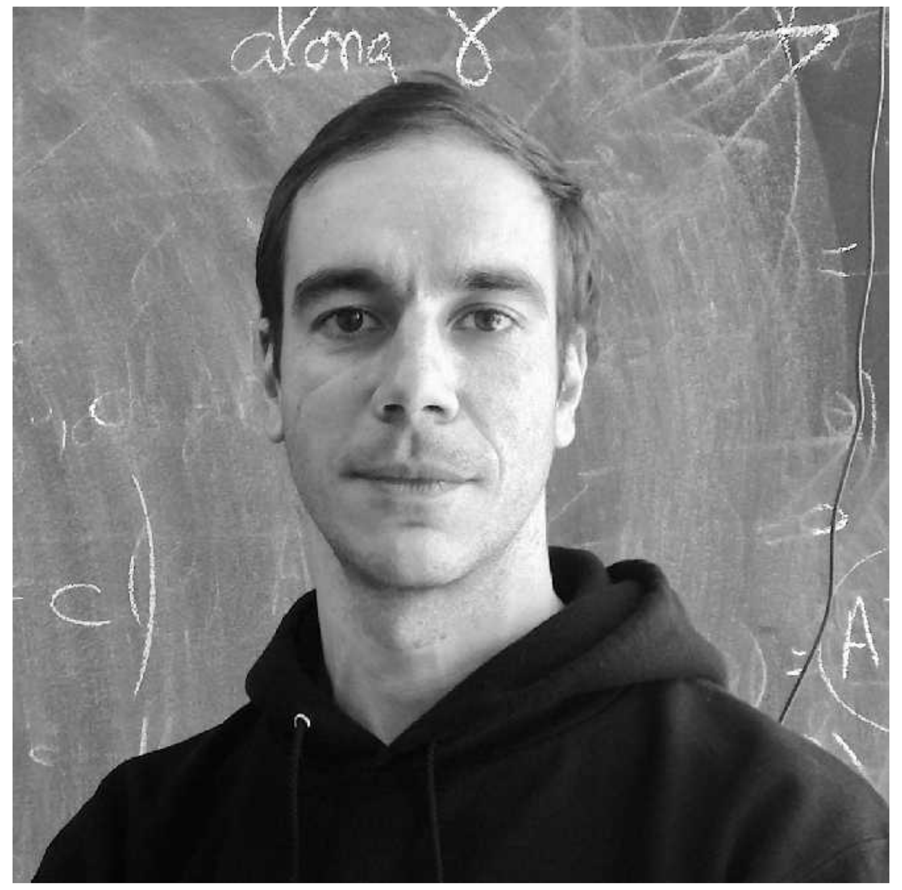
The *Cantor set* is one of the simplest fractal objects. Its ternary representation is obtained by an iterative procedure: start with a segment of length 1 and remove the middle third; for each of the two remaining segments, remove the middle third; and so on. After an infinite number of steps, the remaining object is the Cantor set, and looks like dust. More abstractly, it is the unique non-empty, metrizable, compact, totally disconnected set, without any isolated points.

For a long time, topologists were intrigued by the embeddings of a Cantor set in 3-dimensional Euclidean space. In particular, they asked if it is possible to find such an embedding in $\mathbf{R}^3$ so that there exists a loop that cannot be shrunk to a point without crossing the Cantor set. Said differently, does there exist an embedding of the Cantor set in $\mathbf{R}^3$ whose complement is simply connected?

The ternary Cantor set described above does not satisfy this property. Nevertheless, the question was answered positively by Louis Antoine at the beginning of the 20th century. This embedding is now known as *Antoine's necklace*. It is a necklace, whose links are themselves a necklaces, whose links are themselves necklaces, whose links...

This piece represents the first four levels of the Antoine's necklace. Louis Antoine was a professor at the Université de Rennes, where I currently work. When my colleagues discovered that I was interested in illustrating mathematics, they challenged me to realize an Antoine's necklace.

Because the links of the necklace are so intertwined, it was not possible create this object with a 3D printer or a laser cutter. Inspired by jewelry, I decided to build the object from brass chain. This piece is made of 10,000 links, which I opened and closed individually.



Rémi Coulon

CNRS / Université de Rennes 1
Brass links

The whole process let me experiment the self-similar structure of fractals.

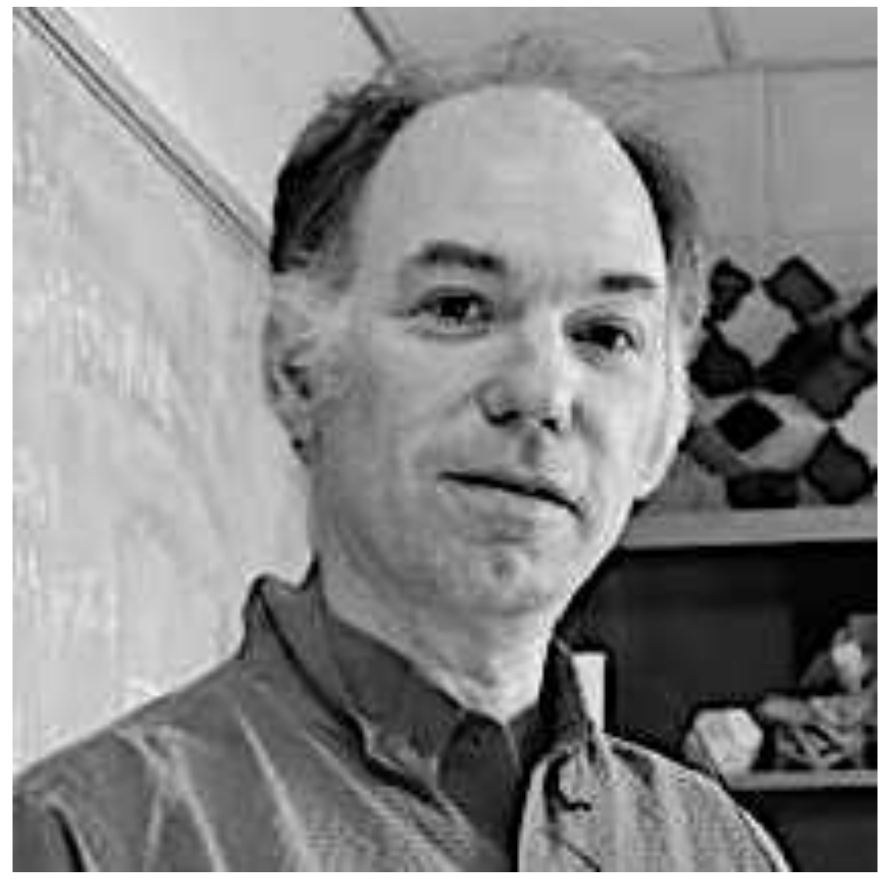
In particular, at the end, a single link is used to close simultaneously one copy of each level of the necklace. The first versions were unicolor. However, it was not easy to distinguish the smallest levels. Using two different plating solves this problem. The next challenge is to build a stand to expose the necklace.

For further information:

Louis Antoine, Sur l'homémorphisme de deux figures et leurs voisinages. Journal de Mathématiques Pures et Appliquées 4 (1921) pp. 221-325.

Surfaces of negative curvature are familiar in everyday life: a surface has “negative curvature” at a point if it is saddle-like there, and the more negative it is, the more extreme this saddle is. A surface with negative curvature is ruffly, like lettuce, and has a tremendous amount of surface area for the volume it occupies. Though a lot of the mathematics of these surfaces has been well understood for more than 150 years, there remain many open, unexplored questions of just how these surfaces actually sit in space and the dramatic changes they undergo when they are manipulated. Partly, this may be because there aren't many ways to actually build such a surface — crochet [[as on page … , presumably]] is one technology —and mathematicians haven't generally played with many physical examples. This method aims to help.

This surface is made of parallel, constant width, constantly turning strips following “horocyclic" paths. You can make a surface of constant negative curvature out of annular strips pieced together in this way — in fact, Beltrami himself made a paper model like this one, more than 150 years ago! But this is an inefficient use of material. It's impossible to cut very many annular strips from a flat sheet without a great deal of waste. Straight strips are far more efficient to cut out, but strips of flat materials such as steel or paper cannot readily change their intrinsic turning radius: though a straight strip may be coiled or twisted, they can only be bent in the direction perpendicular to its surface, and the strip must remain a geodesic on any surface it sits upon. But foam strips are squishy and can be bent in other ways. EVA foam is just right for this piece. Moreover, it is fun to use and look at. It looks like playroom playmats, and that's exactly what this sculpture is made from!



Chaim Goodman-Strauss

University of Arkansas

EVA (ethylene-vinyl acetate) foam

[[might add a lot here if there is room and the piece is accepted]]
The pieces for this were cut using a router on a CNC machine —but EVA foam makes a fine crumbly, nasty dust of plastic microparticles, which must be rinsed off. It is much better to invest in a CNC knife that cuts cleanly.

For further information:
Crocheting Adventures with Hyperbolic Planes: Tactile Mathematics, Art and Craft for All to Explore, Daina Taimina, CRC Press, 2nd Ed 2018

Yana Mohanty and Bjoern Muetzel

Imathgination LLC and Dartmouth College

photograph of mirrors attached to plastic tiles

What happens when you attach mirror surfaces to the inside faces of a rhombicosidodecahedron? We set out to find this out by placing a spherical camera near the center of the solid and then stereographically projecting the output. In this image, the “pole” of the projection is the face opposite to the one through which the camera was placed. Careful placement of the camera allows us to see reflections of pentagons in other pentagons and squares.

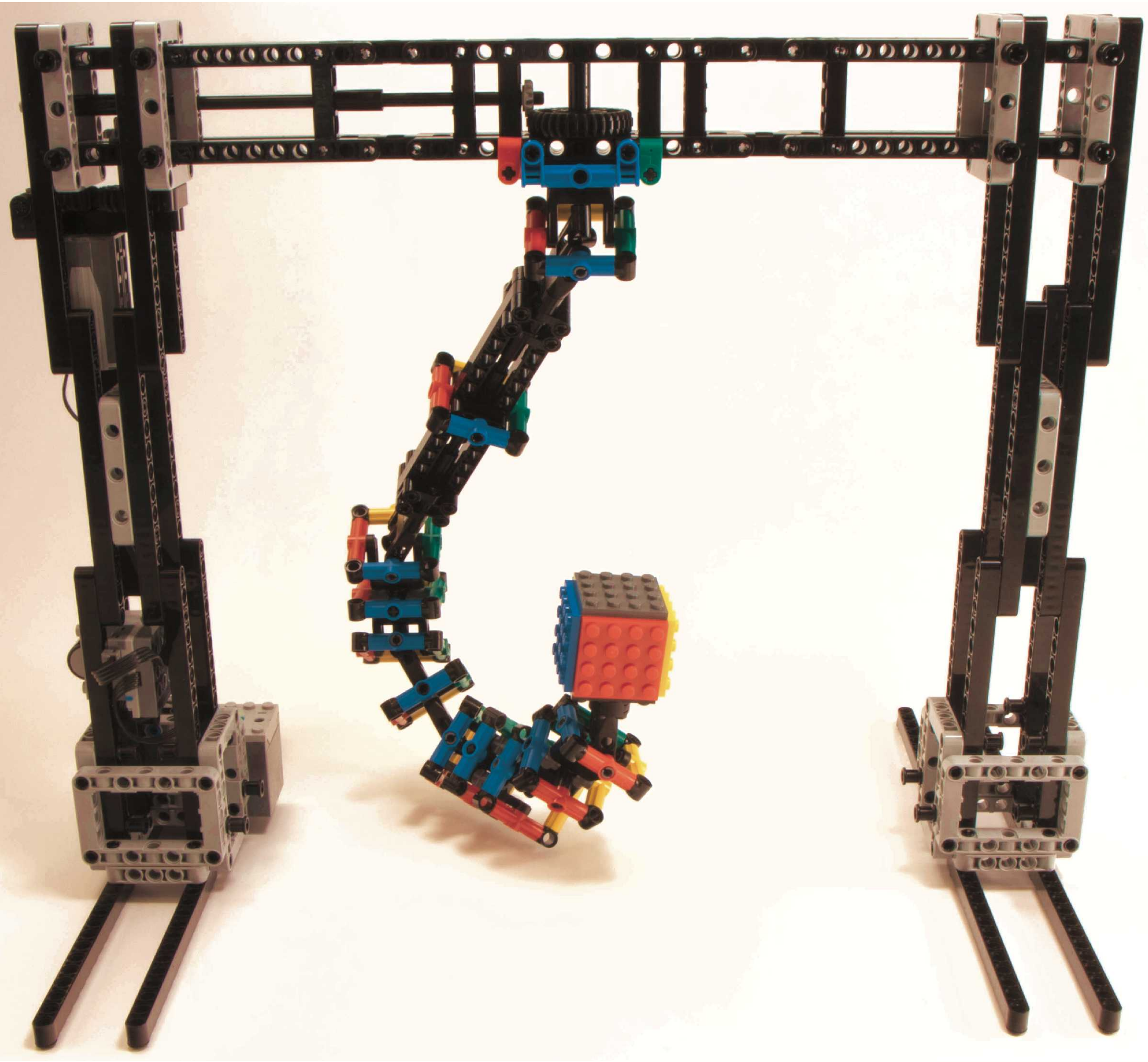
With all the freedom that computer-generated images allow, photography has almost a “retro” appeal. The spherical camera, of course, allows for very unconventional photography by taking in the entire scene around it. Stereographically projecting this view onto a two-dimensional plane gives a unique interpretation of what it’s like to live inside the intriguing mirrored polyhedron.

An unintentional byproduct of this project was seeing what happens when there are slight deviations from the stereographic projection. When using a spherical camera inside an almost spherical object, these deviations are expressed as the lens of the camera being slightly below or above the center of the object. As a result, the projections look different depending on which pole from which you project.

In order to get the best possible image, we had to experiment with the position of the camera in the polyhedron, as well with as the ambient light. The latter is tricky, since light can come into the Geometiles rhombicosidodecahedron only through the thin crevices between the tiles, and from the bottom face where the camera is placed.

For further information:
https://geometiles.com/archimedean-billiards/
https://math.dartmouth.edu/~mutzel/gallery.php

We say “we” even though it was Bjoern taking the pictures at Dartmouth College, and Yana was processing the raw images in California. At some point, things got downright comical: Yana was making recommendations on where to shoot the pictures based on pictures of rooms on the Dartmouth campus that she had only seen online.



This is a mechanical illustration of the "Dirac belt trick" or "plate trick". The central cube rotates continuously around a vertical axis, yet it is connected to the fixed outer frame via a sequence of hinge joints, each pivoting by at most 45 degrees each way. The arm returns to its original state for every TWO complete turns of the cube. It can serve as an anti-twist mechanism - one can run a piece of thread or an electrical wire along the arm, fixed to the cube and the frame at its ends, and yet it never gets twisted up. This reflects the two-to-one map from the (simply connected) space of unit quaternions to the (not simply connected) space of three-dimensional rotations. To my knowledge this is the first mechanical linkage that demonstrates the phenomenon. The rotation is powered by an electric motor, or by turning a handle.

LEGO is the ideal medium for experimentation with mechanisms. One can go from an idea to a working model in minutes, and then to a more polished art piece with a bit more investment and effort.
I had wondered for some time whether such a linkage was possible. When I had the idea for the particular mechanism, it was not all clear to me whether it would work, even theoretically. The LEGO model quickly provided the answer. Moreover, the model has provided new insight into the mathematical and physical phenomenon: the arm can be separated naturally into upper and lower parts, each of which on its own moves in an intuitive way; all that remains is to combine them.

For further information:
http://en.wikipedia.org/wiki/Plate_trick

Alexander Holroyd

University of Washington
Material

OTHER MATERIALS

This piece shows rational points on a cubic surface. This is related to Fermat's famous last theorem.

Away from the 27 straight lines contained in the surface, there is only a small finite number of rational points with numerator and denominator below 100 (on the part of the surface shown). To visualize these points in real life, we needed some support material to hold them at their position, because the points themselves are not connected — there is an empty space between them. Laser-in-glass allows to do this with a support material (glass!) that is almost invisible. This is perfect for getting a good idea of the fascinating structure and geometry of these points.

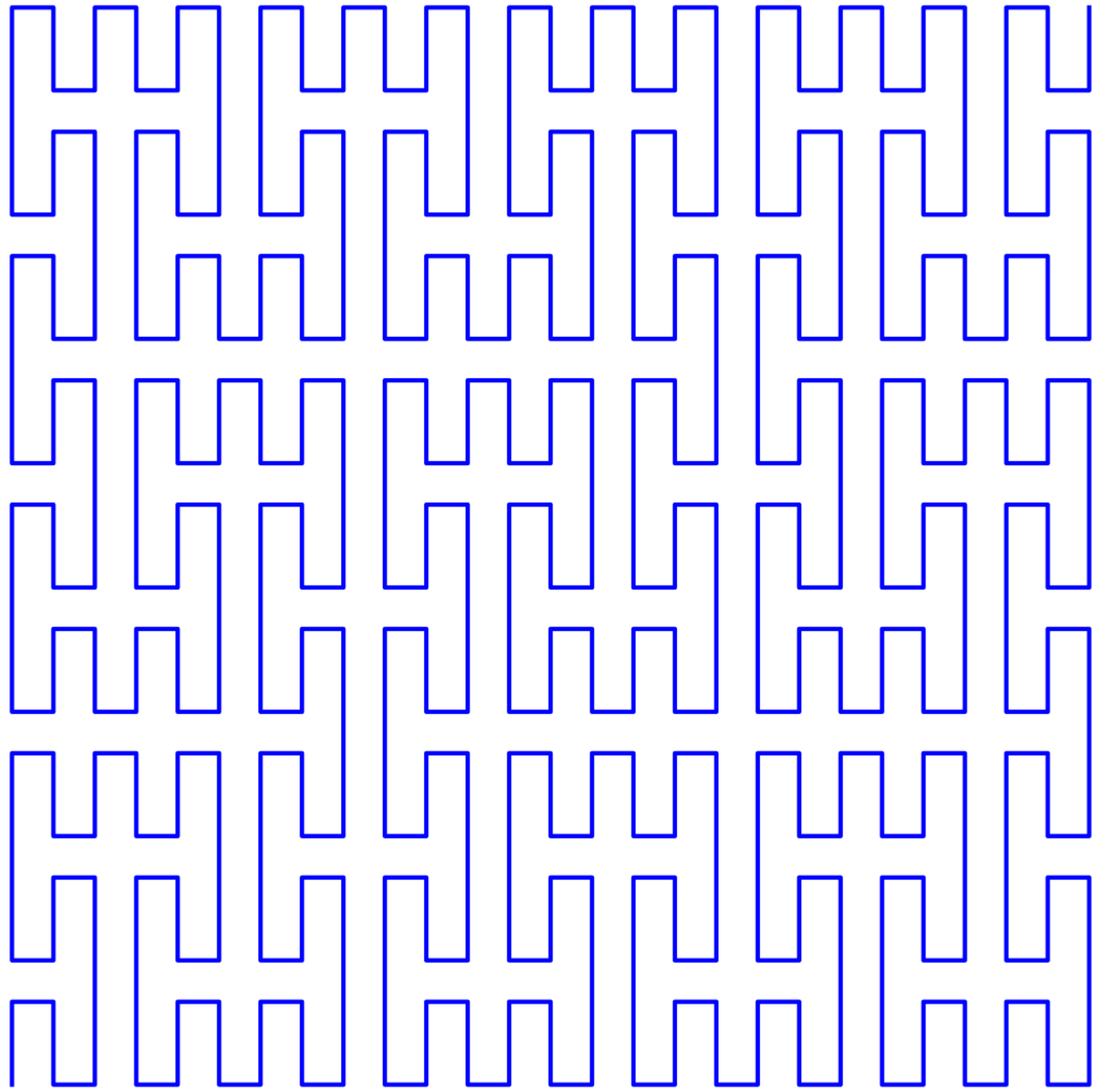
There is some complicated hidden structure in the positions of the points. Understanding it mathematically is not easy, but visually, one quickly gets a rough feeling for it.

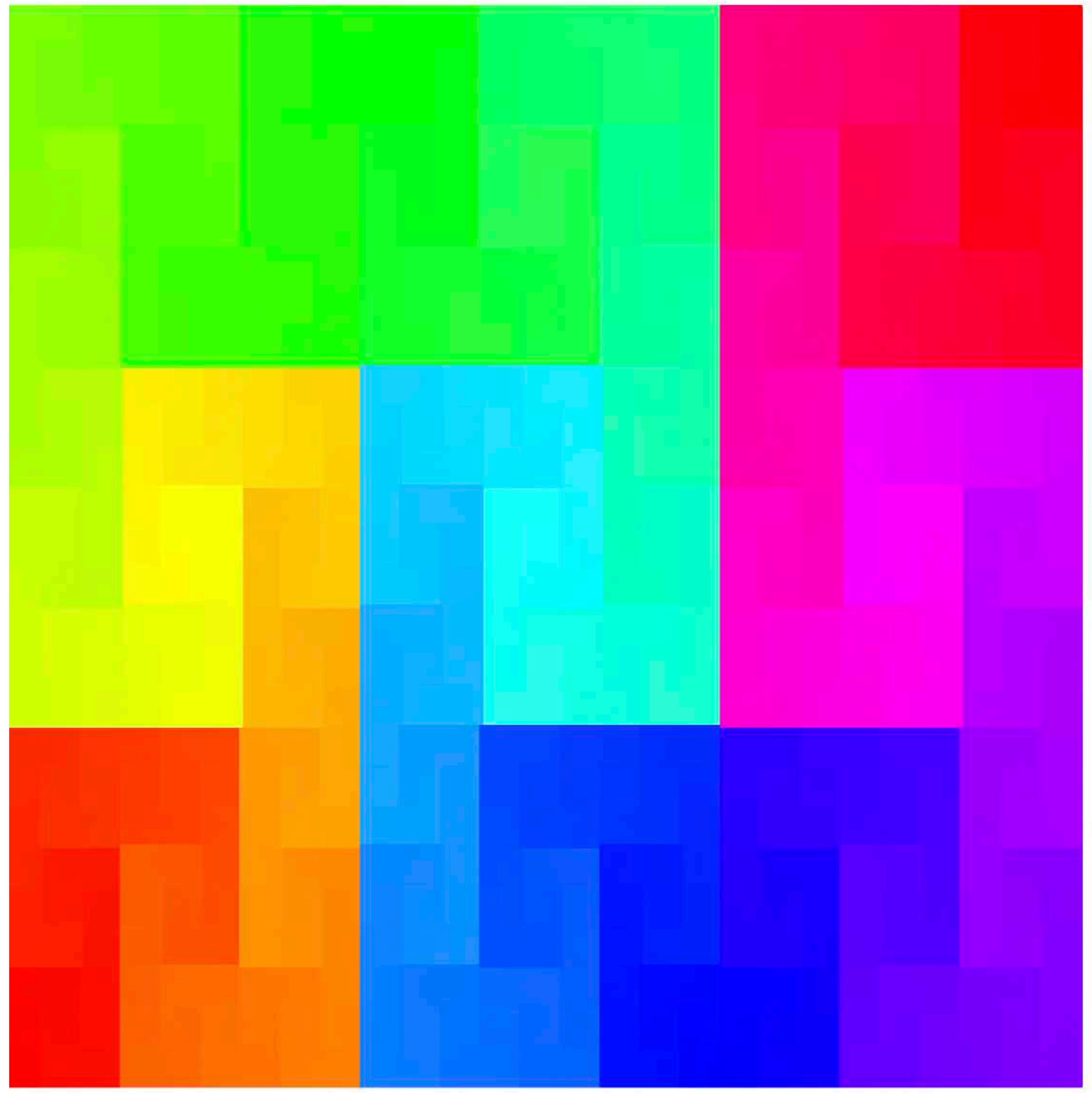
Choosing the correct size for the small balls representing the points was a challenge. Also, choosing a good bound for the numerator and denominator was a lengthy experimental process. Working with Ulrich Derenthal, we essentially had to choose a compromise between enough points to see certain geometrical features, without the points being too numerous or too large. Too many points almost cause the whole space to be filled.

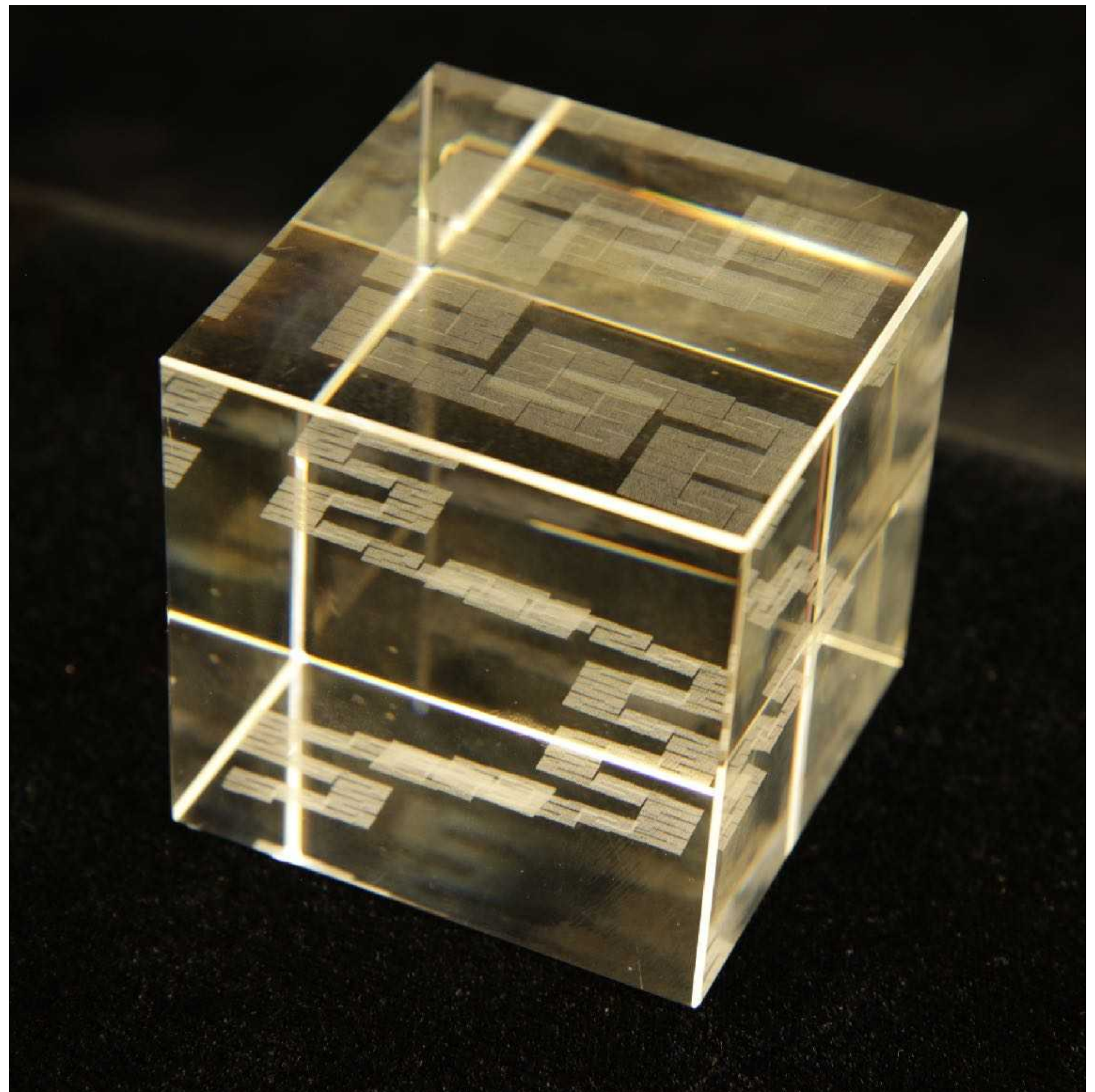
Oliver Labs

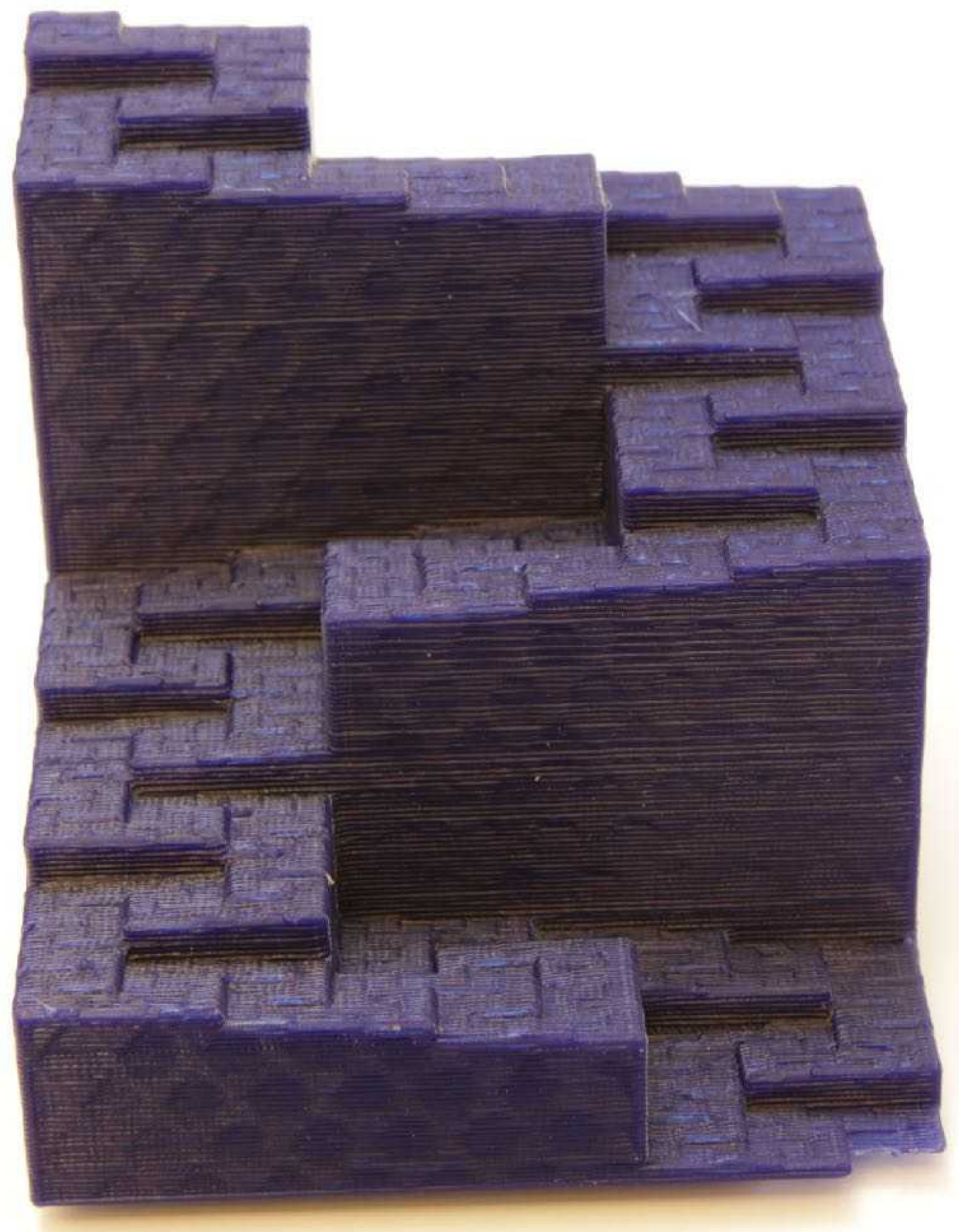
MO-Labs Dr. Oliver Labs
laser-engraved glass

FOUR WAYS TO ILLUSTRATE THE SAME THING







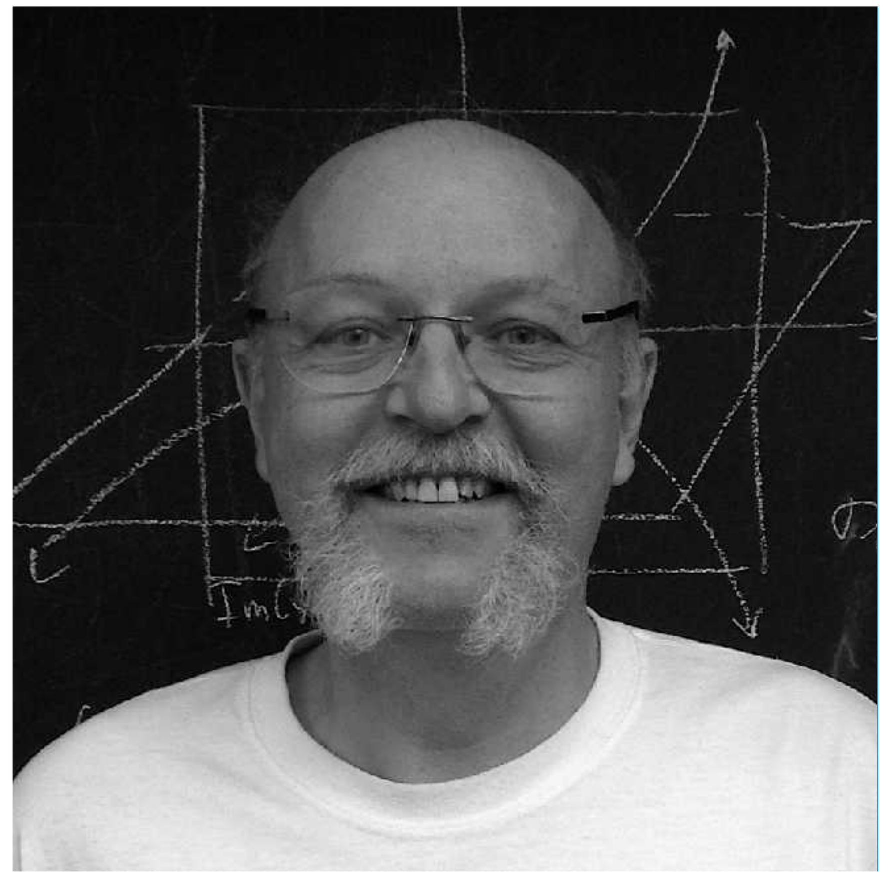


It has always been evident to geometers that curves and surfaces are different objects. But, in 1890, Giuseppe Peano, in the famous paper *Sur une courbe qui remplit toute une aire plane,* showed a very counterintuitive example of a continuous "space-filling" curve that completely fills the unit square.

The paper of Peano has no figures; when I was a student, the conventional wisdom was that such a curve was impossible to represent, as the figure would have been a black square! To get intuition about this curve, the best way was to draw a finite approximation, as shown in the top left. However, this does not give a good intuition of the curve itself, as it is very homogeneous, and makes it difficult to understand the dynamics of the curve.

In 1986, I found a dynamically motivated space-filling curve — once monsters are discovered, they appear everywhere! I had the idea later of making a representation such that the color of each point depends on the parameter of the curve; in this way, one can see the movement along the curve, by the the direction of continuous change of colour (top right).

A few years later, Xavier Bressaud suggested that it might be clearer to show the 3-dimensional graph of the function, and we made some interesting figures showing perspective views of this graph for a paper in *Experimental Mathematics*. But this representation was not complete: why not built the real object in dimension 3? There is an obvious problem : a fractal curve in $\R^3$ is too fragile to 3D print. But laser engraving in glass is a technology perfectly adapted to an object of this type, and in 2017, with Eltarr Loukman, an energetic undergraduate student of Aix-Marseille University, we succeeded to create this object, which shows how a topological line can have Hausdorff dimension 2, and project onto a square in one direction, but not in all directions (bottom left).



Pierre Arnoux

Université d'Aix-Marseille / CNRS
graphics, 3D printed plastic

Upon seeing this curve in glass, I realized that the set of points below the graph formed a kind of strange hill, with an ascending path passing through all points of the hill, and that this object could easily be 3D printed, which I did at ICERM this fall (bottom right). This gives another viewpoint on this remarkable object. Similar models can be made for any plane-filling curve, and it will be interesting to make them to get a better intuition about the properties of these curves, and the differences between them.