THE SHAPE OF THURSTON'S MASTER TEAPOT

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ABSTRACT. We establish basic geometric and topological properties of Thurston's Master Teapot and the Thurston set for superattracting unimodal continuous self-maps of intervals. In particular, the Master Teapot is connected, contains the unit cylinder, and its intersection with a set $\mathbb{D} \times \{c\}$ grows monotonically with c. We show that the Thurston set described above is not equal to the Thurston set for postcritically finite tent maps, and we provide an arithmetic explanation for why certain gaps appear in plots of finite approximations of the Thurston set.

1. INTRODUCTION

In his last paper, unfinished at the time of his death, William Thurston studied piecewise-linear maps of the unit interval [Thu14]. One concept mentioned in this paper is an object that Thurston, in his 2012 course at Cornell University, affectionately called the *Master Teapot*, and which can be defined as follows. A unimodal endomorphism f of a real interval is said to be *critically periodic* if the critical point is a fixed point of some forward iterate of f, and is said to be *postcritically finite* if the forward orbit of the critical point is a finite set. If f is postcritically finite, it is easy to see that the orbit of the critical point determines a Markov partition of the interval. The Perron-Frobenius theorem then implies that the exponential of the topological entropy of f, $e^{h_{top}(f)}$, is a weak Perron number - i.e. a real, positive algebraic integer that is not less than the absolute value of any of its Galois conjugates - which we call the the *growth rate* of f and denote by $\lambda(f)$. Denote by $\mathcal{F}^{cp}$ the family of critically periodic unimodal continuous self-maps of compact real intervals. Then *Thurston's Master Teapot* is the set

$$\Upsilon_2^{cp} := \overline{\{(z, \lambda) \in \mathbb{C} \times \mathbb{R} \mid \lambda = \lambda(f) \text{ for some } f \in \mathcal{F}^{cp}, z \text{ is a Galois conjugate of } \lambda\}}.$$

An application of Υ_2^{cp} is that it can be used as a necessary condition for a weak Perron number to be the growth rate of a critically periodic unimodal map: β being such a number would imply that for each of its Galois conjugates z, $(z, \beta) \in \Upsilon_2^{cp}$. Studying the Master Teapot may also inform the open question of completely classifying the set of dilatations of pseudo-Anosov surface diffeomorphisms, which may be thought of as two-dimensional analogues of uniformly expanding interval self-maps. We call the image of the projection of the Master Teapot to $\mathbb{C}$ the *Thurston set*; this set, which we discuss later, has been the subject of several recent works (e.g. [CKW17, Tio20, Tho17]). Another motivation for studying these sets is that the part of the Thurston set inside the unit disk may be viewed as an analogue of the Mandelbrot set – while the Mandelbrot set may be defined as the set of parameters $c \in \mathbb{C}$ such that 0 belongs to the filled Julia set of the polynomial $z \mapsto z^2 + c$, the part of the Thurston set inside the unit disk coincides with the set of parameters $z \in \mathbb{D}$ for which 0 belongs to the limit set of the iterated function system generated by the maps $x \mapsto zx + 1, x \mapsto zx - 1$. Furthermore, a forthcoming article by the last two authors will show that each horizontal slice of the Master Teapot is an analogue of the Mandelbrot set. This topic is also connected to the theory of "core entropy," which has been the subject of numerous recent works (see, e.g. [Tio15, Tio16, GT17, TBG$^+$19]), as the restriction of a real quadratic polynomial to its Hubbard tree is a unimodal interval self-map.

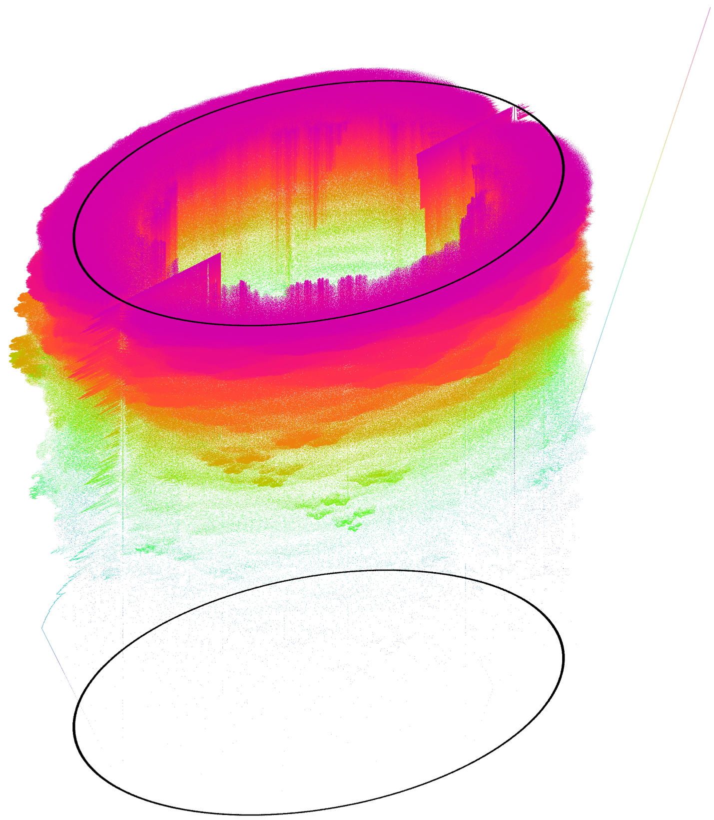

FIGURE 1. A plot of a finite approximation of the Thurston Master Teapot, showing all points coming from maps in $\mathcal{F}^{cp}$ with critical period at most 23. The "spout" of the Teapot in the upper right corner is the line $\{(x, 0, x) : x \in [1, 2] \subset \mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$, although the bottom of the spout is not visible in this finite approximation. The "handle" of the Teapot in the lower left corner lies above the negative real axis. The fact that the plot fades out towards the bottom of the Teapot is due to the finiteness of the approximation; considering maps with longer critical periods would give rise to more points near the bottom of the Teapot. The two black circles are the sets $S^1 \times \{1\}$ and $S^1 \times \{2\}$, and the color gradients is according to the height of the points.

Thurston describes the part of the Master Teapot Υ_2^{cp} *outside* the unit cylinder as "a network of very frizzy hairs, ...sometimes joining and splitting, but always transverse to the horizontal planes," [Thu14, Figure 7.7] and the part *inside* the unit cylinder as "confined to (and dense in) closed sets that include the unit circle and increases [sic] monotonically with λ" [Thu14, Figure 7.8]. The first phenomenon is well-known (see e.g. proof by Tiozzo [Tio20, proof of Theorem 1.3]), but Thurston did not provide any further explanation for the second. A main contribution in our paper is a proof of the second phenomenon, which is that a point in the unit disc $\mathbb{D}$ which is on a horizontal slice of the Master Teapot persists as the height of the slice increases. [1]

Theorem 1 (Persistence)**.** *For any point $z \in \mathbb{C}$ in the open unit disk $\mathbb{D}$, if (z, β) is in the Master Teapot, then every point above it up to height 2 is also in the Master Teapot. In other words,*

$$(z, \lambda) \in \Upsilon_2^{cp} \textit{ implies } \{z\} \times [\lambda, 2] \subset \Upsilon_2^{cp}.$$

Two corollaries of this main theorem are the following:

Theorem 2 (Unit Cylinder)**.** *The Master Teapot Υ_2^{cp} contains the unit cylinder $S^1 \times [1, 2]$.*

Another equivalent way to state the Unit Cylinder Theorem 2 is that $S^1 \times \{1\}$ is contained in the Master Teapot, and the Persistence Theorem 1 holds on the closed unit cylinder.

Theorem 3 (Connectedness)**.** *The Master Teapot Υ_2^{cp} is connected. Furthermore, $\Upsilon_2^{cp} \cap (\overline{\mathbb{D}} \times [1, 2])$ is path-connected.*

We also proved a number of results that are not logically dependent on the main theorem above and concern other sets related to the Master Teapot. Let the *Thurston set*, which we denote Ω_2^{cp}, be the projection of Thurston's Master Teapot onto $\mathbb{C}$:

$$\Omega_2^{cp} := \overline{\{z \in \mathbb{C} \mid \lambda = \lambda(f) \text{ for some } f \in \mathcal{F}^{cp}, z \text{ is a Galois conjugate of } \lambda\}}.$$

In other words, the Thurston set Ω_2^{cp} is the closure of the set containing all Galois conjugates of growth rates of unimodal maps which are critically periodic.

A heretofore mysterious feature of plots of finite approximations of the Thurston set, formed by bounding the length of the postcritical orbits, was the appearance of visible "gaps" or holes at fourth roots of unity, sixth roots of unity, and certain other algebraic numbers.

The gaps on the unit circle get filled in as the length of the postcritical orbits approaches infinity [Tio20, Proposition 6.1]. It is known, however, that $\Omega_2^{cp} \cap \mathbb{D}$ does have a hole other than the large central hole around the origin [CKW17]. The Gap Theorem 4 provides an arithmetic explanation for these visible gaps in finite approximations of Ω_2^{cp}.

Theorem 4 (Gaps)**.** *For $n \in \mathbb{N}$, let ω_n denote the set of Galois conjugates of growth rates of unimodal critically periodic maps with postcritical length at most n. Let R be one of the rings $\mathbb{Z}[\sqrt{-D}]$ or $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ for $D = 1, 2,$ or 5, and set $c = \inf\{|z| : z \in R, z \neq 0\}$. Then for any $x \in R$,*

$$B_{r(x)}(x) \cap \omega_n \subset \{x\},$$

where

$$r(x) = \begin{cases} \min\left\{\frac{c}{(2n^2+3n+1)|x|^n e}, \frac{1}{n+1}\right\} & \textit{if } |x| \geq 1, \\ \min\left\{\frac{c}{(2n^2+3n+1)e}, \frac{1}{n+1}\right\} & \textit{if } |x| \leq 1. \end{cases}$$

[1]To see this phenomenon in action – that roots inside the unit cylinder persist, and also that roots outside the unit cylinder move continuously – see our video `https://vimeo.com/259921275`.

Let $\mathcal{F}^{pcf}$ be the family of unimodal postcritically finite self-maps of real intervals. We define the *postcritically finite Thurston set*, Ω_2^{pcf}, as

$$\Omega_2^{pcf} := \overline{\{z \in \mathbb{C} \mid \lambda = \lambda(f) \text{ for some } f \in \mathcal{F}^{pcf}, z \text{ is a Galois conjugate of } \lambda\}}.$$

In other words, the postcritically finite Thurston set Ω_2^{pcf} is the closure of the set containing all Galois conjugates of growth rates of unimodal maps which are postcritically finite.

We proved that:

Theorem 5 (Two Thurston Sets). *The Thurston set Ω_2^{cp} and the postcritically finite Thurston set Ω_2^{pcf} are not equal.*

The caption of Thurston's image [Thu14, Figure 1.1] states that the image shows the roots of the defining polynomials for "a sample of about 10^7 postcritically finite quadratic maps

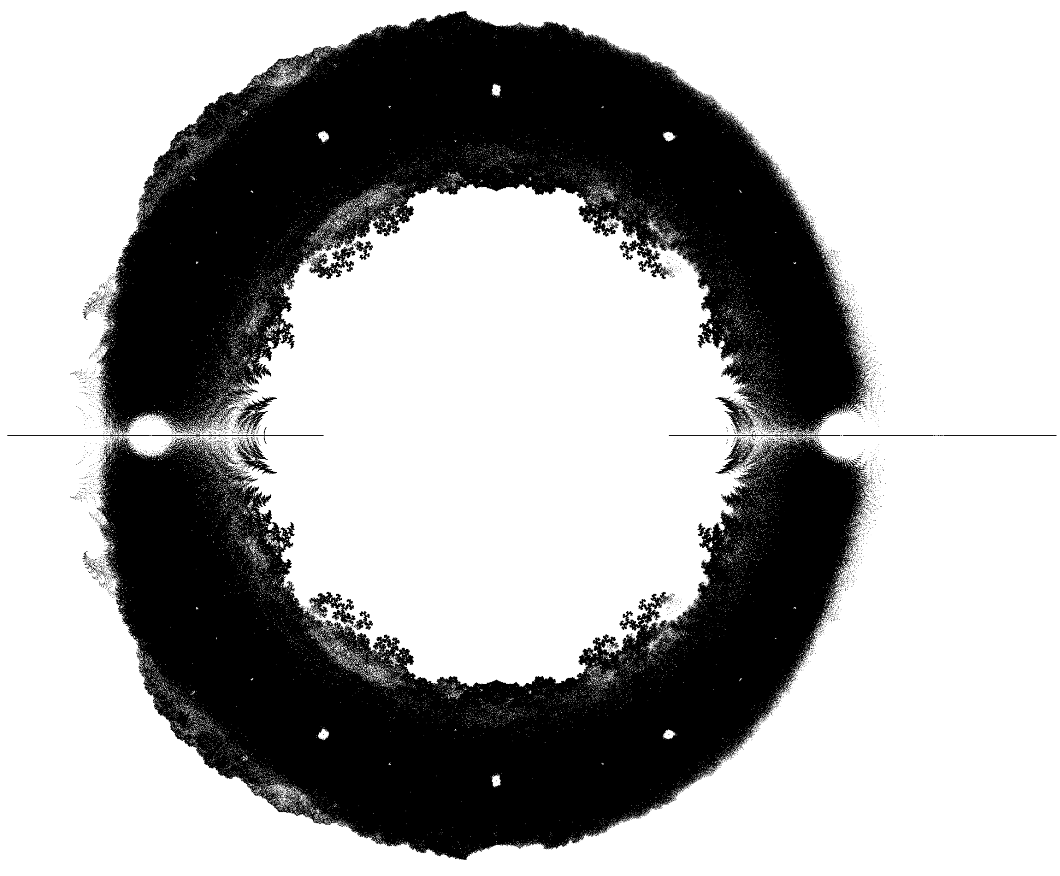

FIGURE 2. An approximation of the Thurston set, Ω_2^{cp}, consisting of the roots of all minimal polynomials associated to postcritically finite tent maps for which the post-critical period is at most 25

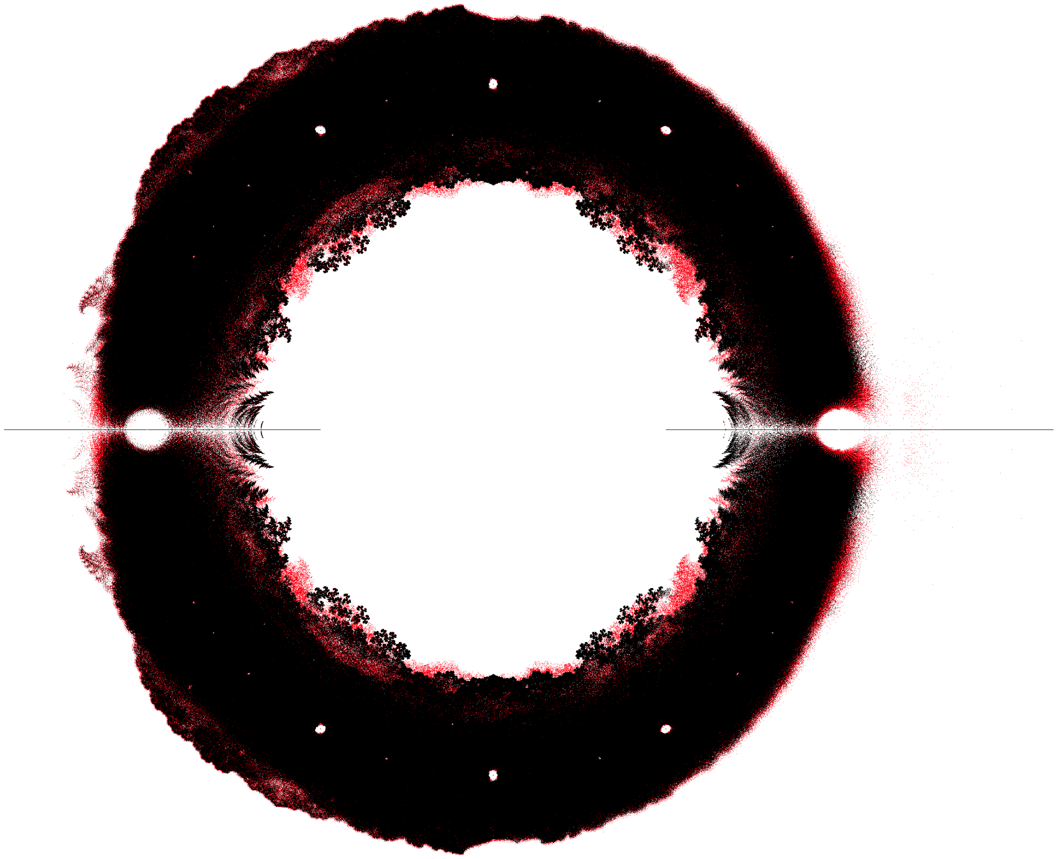

FIGURE 3. Here is an overlay of finite approximations of the two Thurston sets: the black image contains points in Ω_2^{cp} corresponding to period length up to 25, and the red image contains points in Ω_2^{pcf} corresponding to preperiod plus period up to 22. The set Ω_2^{pcf} is shown on its own in Figure 5.

of the interval with postcritical orbit of length ≤ 80." We suspect, based on visual comparison of plots, that Thurston's image shows only roots of critically periodic tent maps, i.e. shows Ω_2^{cp} and not Ω_2^{pcf} (c.f. Figure 3).

At the moment, we do not have a good understanding of the shape of the postcritically finite Thurston set Ω_2^{pcf} and the analogously defined "teapot" Υ_2^{pcf}; for example, we do not know if they exhibit persistence (as in the Persistence Theorem 1) or connectivity (as in the Connectedness Theorem 3).

1.1. **Perspectives on the Thurston set.** Our main tool for the study of Ω_2^{cp}, and unimodal maps on intervals in general, is the Milnor-Thurston kneading theory. The Milnor-Thurston kneading theory [MT88] (also cf. [Guc79]) provides the connection between general unimodal maps, real quadratic maps and subshifts in certain symbolic dynamical systems via entropy-preserving semi-conjugacies, and connect them to the study of infinite power series with prescribed coefficients called kneading determinants. As a result, there are numerous characterizations of Ω_2^{cp} from different points of view, and our results build (directly or indirectly) on a long history of research in each of these areas.

1. *Polynomials and power series with prescribed coefficients* An alternative way to describe the kneading determinant and kneading polynomials, which predates Milnor-Thurston kneading theory, is β-expansions and Parry polynomials, which were first introduced in [Par60] for maps of the form $x \mapsto \beta x \mod 1$ and later extended to a larger class of piecewise linear interval self-maps (e.g. [G0́7, IS09, DMP11, Ste13, LSS16]). Solomyak [Sol94] used Parry polynomial to study the closure of the Galois conjugates of β such that $x \mapsto \beta x \mod 1$ has finite critical orbit, Thompson [Tho17] used it to study a set that contains the Thurston set, and the distribution of roots of Parry polynomials was studied in [VG08a, VG08b]. More generally, there is a large body of literature that investigating the roots of polynomials and power series with all coefficients in a prescribed set (see, for example, [OP93, BBBP98, BEK99, Kon99, SS06, BEL08]). The polynomials most closely related to the Thurston set are perhaps Littlewood, Newman and Borwein polynomials, polynomials whose coefficients belong to the sets $\{\pm 1\}$, $\{0, 1\}$ and $\{-1, 0, +1\}$ respectively.

2. *Complex dynamics.* Since the study of unimodal maps can be reduced to the study of real quadratic maps, the study of entropies of critically periodic unimodal maps is reduced to the study of core entropy on superattracting parameters on the real slice of the Mandelbrot set. The study of the core entropy on the Mandelbrot set is a rich subject, cf. [DH85, Poi09, Li07, MS13, Thu16, Tio16, Tio15].

3. *Symbolic dynamics and Iterated function systems (IFS).* The kind of symbolic dynamical systems semiconjugate to a real quadratic map was described in [MT88] via a combinatorial "admissibility criteria". Using this, Tiozzo [Tio20] proved that the Thurston set Ω_2^{cp} is connected, locally connected, and contains a uniform neighborhood of the unit circle. In particular, [Tio20] shows that a point z with absolute value less than 1 is in the Thuston set Ω_2^{cp} if and only if 0 is in the limit set of the iterated function system generated by the two maps $x \mapsto zx + 1$ and $x \mapsto zx - 1$. This and some other related IFS are the focus of numerous works, including [BH85, Bou88, Bou92, Ban02, SX03, Sol04, Sol05]. In [CKW17], Calegari, Koch and Walker used this and a related IFS to prove that the Thurston set has a hole, in addition to the obvious, large hole of radius $1/2$ centered at 0.

1.2. **Structure of the paper.** A major consequence of the Milnor-Thurston theory is that unimodal maps on intervals are semiconjugate to *tent maps* with the same entropy. This tool is essential in our method of proof.

§2: Preliminaries We define the *β-itinerary* of a point under a tent map, *Parry polynomials*, and give the *admissibility criterion* for itineraries, which are key tools in our arguments.

§3: Quadractic maps, iterated function systems, and renormalization provides background on Milnor-Thurston kneading theory and reviews the concept of *renormalization*.

§4: Dominant words reviews the definition and properties of dominant words from Tiozzo's work [Tio15].

§5: Persistence proves the main theorem, Persistence Theorem 1.

§6: The unit cylinder and connectivity shows that the Master Teapot is connected inside the unit cylinder, and uses this structure to prove the Unit Cylinder Theorem 2 and the Connectedness Theorem 3.

§7: Gaps in the Thurston set explains why there appear to be "holes" near primitive roots of unity in the finite approximations of the Thurston set. We show that these holes are associated to discrete subgroups, proving the Gap Theorem 4.

§8: Ω_2^{cp} and Ω_2^{pcf} are not equal shows that the periodic and preperiodic Thurston sets are not equal, proving the Two Thurston Sets Theorem 5.

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2. Preliminaries

Theorem 2.1. [MT88, Theorem 7.4] *Every continuous self-map g of an interval with finitely many turning points and with $h_{top}(g) > 0$ is semi-conjugate to a uniform λ-expander $PL(g)$ with the same topological entropy $h_{top}(g) = \log \lambda$. If g is postcritically finite, so is $PL(g)$.*

Thus, to understand Thurston's Master Teapot, it will suffice to study these more rigid dynamical system.

2.1. **Tent maps.** Denote the unit interval by $I = [0, 1]$. For fixed $\beta \in (1, 2]$, the *tent map* of slope β is the continuous, piecewise linear map f_β of the unit interval I defined by:

$$f_\beta = \begin{cases} \beta x & x \leq \frac{1}{\beta} \\ 2 - \beta x & x > \frac{1}{\beta} \end{cases} .$$

For a continuous self-map f of an interval with finitely many turning points, the topological entropy $h(f)$ is equal to the following limit:

$$h(f) = \lim_{n \to \infty} \frac{1}{n} \log(\mathrm{Var}(f^n)), \tag{1}$$

where $\mathrm{Var}(f)$ denotes the total variation of f [MS80]. Then a straightfoward calculation confirms that for a tent map f_β, the *growth rate*, which is the exponential of the topological entropy, is equal to β. In other words, $e^{h(f_\beta)} = \beta$.

Via Milnor-Thurston's entropy-preserving semi-conjugacy [MT88], the critical point of a unimodal map is sent to the unique preimage of 1 under the associated tent map. Then a tent map f_β is said to be *postcritically finite* if the f_β-orbit of 1 is finite, and f_β is said to be *critically periodic* if 1 is a periodic point for f_β. Now we have alternative interpretations of Thurston's Master Teapot, the Thurston set, and the postcritically finite Thurston set:

- The Master Teapot Υ_2^{cp} is the closure of the set of all pairs (z, β) in $\mathbb{C} \times \mathbb{R}$ for which z is a Galois conjugate of β, and β is the growth rate of a critically periodic tent map;
- the Thurston set Ω_2^{cp} is the closure of the set of all Galois conjugates of growth rates of critically periodic tent maps;
- the postcritically finite Thurston set Ω_2^{pcf} is the closure of the set of all Galois conjugates of growth rates of postcritically finite tent maps.

2.2. **Combinatorial itineraries.** The dynamics of the tent map f_β can be represented by a relatively simple Markov coding: there is a Markov partition of the unit interval into two subintervals labeled 0 and 1, and we represent the f_β-orbit of any point x with an itinerary sequence whose n-th term is 0 or 1 depending on which subinterval contains the nth iterate of x. We will make this representation more precise later in this section, since we will extensively use the Markov coding of a tent map in this work. First, we will define essential abstract data of sequences and words in the alphabet $\{0,1\}$.

Definition 2.2. We will use the term *string* to refer to an ordered list of letters in some alphabet, and this list may be either finite or infinite. We adopt the convention that a *word* is always a finite string, and a *sequence* is always an infinite string. An itinerary is also assumed to be an infinite string. We often concatenate a word w with the notation w^n, which is the word created from repeating w exactly n times. Similarly, w^∞ is the sequence created by repeating w infinitely many times.

Definition 2.3. The sequence of *signs* associated to a sequence $w = (w_1w_2\dots) \in \{0,1\}^{\mathbb{N}}$ is the sequence $e_w : \mathbb{N} \to \{-1,+1\}$ defined by

$$e_w(j) = \begin{cases} +1 & \text{if } w_j = 0, \\ -1 & \text{if } w_j = 1. \end{cases}$$

The sequence of *cumulative signs* associated to a sequence $w = (w_1w_2\dots) \in \{0,1\}^{\mathbb{N}}$ is the sequence $s_w \colon \mathbb{N} \to \{+1,-1\}$ defined by $s_w(1) = 1$ and

$$s_w(j+1) = \prod_{k=1}^{j} e_w(k) \tag{2}$$

for $j \geq 1$. In other words, the $(k+1)^{\text{st}}$ sign $s_w(k+1)$ is equal to 1 if and only if the sum of the first k entries of the sequence w is even. If w is a finite string, the *cumulative sign* of w is defined as $\prod_k e_w(k)$. The sequence of *digits* associated to a sequence $w = (w_1w_2\dots) \in \{0,1\}^{\mathbb{N}}$ is the sequence $d_w : \mathbb{N} \to \{0,2\}$ defined by $d_w(i) = 2w_i$.

2.3. **Ordering on the set of strings.**

Definition 2.4 (Twisted lexicographic ordering)**.**

(1) Define the ordering $\leq_E$ on the set of sequences in $\{0,1\}^{\mathbb{N}}$ as follows. Given two distinct sequences $w = (w_1w_2\dots)$ and $v = (v_1v_2\dots)$ in $\{0,1\}^{\mathbb{N}}$, define $w <_E v$ if and only if at the first integer n such that $w_n \neq v_n$,

$$\begin{cases} w_n < v_n & \text{if } s_w(n) = +1, \\ w_n > v_n & \text{if } s_w(n) = -1. \end{cases}$$

Note that $s_w(n) = s_v(n)$ by definition since n is the first index at which the sequences w and v differ.

(2) Define the ordering $\leq_E$ on the set of words in the alphabet $\{0,1\}$ as follows. Given two words w and v, write $w <_E v$ if and only if $w^\infty <_E v^\infty$.

Remark 2.5. It is straightforward to check from the definition of twisted lexicographic ordering that if a word a has positive cumulative sign, then for any strings v, w, we have $w <_E v$ if and only if $aw <_E av$. Similarly, if a has negative cumulative sign, then $w <_E v$ if and only if $aw >_E av$.

Now we can define the concept of β-itinerary as below:

Definition 2.6 (β-itinerary)**.** Let $I_0^\beta = [0, 1/\beta]$, $I_1^\beta = [1/\beta, 1]$. The β-itinerary of the tent map f_β is the sequence $w = (w_1 w_2 \dots)$ satisfying the following two conditions:

(1) $f^n(1) \in I^\beta_{w_{n+1}}$.
(2) Among all the sequences satisfying the preceding condition (1), w is the minimal such sequence under the twisted lexicographical ordering.

It is obvious that if f_β is not critically periodic, there is a unique sequence satisfying condition (1) which has to be the β-itinerary. If f_β is critically periodic, one can easily check that the β-itinerary can be equivalently defined explicitly as follows: if $f^k_\beta(1) = 1$ and k is minimal, then this is the itinerary w^∞ where w has length k and the last digit of w is chosen such that w has positive cumulative sign. From this observation one can see that this definition is consistent with the standard kneading theory definition of the itinerary of 1 under f_β, which is the limit of the itineraries of x_i under f_β, where $x_i \in [0, 1]$, $\lim_i x_i = 1$, and the forward orbit of x_i never hits any critical point.

2.4. **Parry polynomials.** The definition below for a Parry polynomial is motivated by the concept of β-expansion.

Definition 2.7. Let w be a word in the alphabet $\{0, 1\}$. Set $f^z_0(x) = zx$ and $f^z_1(x) = 2 - zx$. Then the *Parry polynomial* for w is

$$\begin{aligned} P_w(z) &:= s_w(p+1)(f^z_{w_p} \circ f^z_{w_{p-1}} \circ \cdots \circ f^z_{w_1}(1) - 1) \\ &= z^p - s_w(1)d_w(1)z^{p-1} - \cdots - s_w(p)d_w(p) - s_w(p+1) \\ &= (z-1)(z^{p-1} + s_w(2)z^{p-2} + \cdots + s_w(p)). \end{aligned} \tag{3}$$

When f_β is critically periodic, the first line of equation (3) confirms that for any word w for which w^∞ is a β-itinerary, we have that β is a root of the Parry polynomial P_w. Thus, the minimal polynomial for β is a factor of P_w for any word w such that w^∞ is a β-itinerary. As a final observation, P_w is also never irreducible over the integers, as it always has a factor of $(z - 1)$. At times it will be important for our arguments to ensure that the Parry polynomial has only this one extra factor of $(z - 1)$, i.e. has exactly two irreducible factors.

2.5. **Admissible itineraries.** Let $\sigma\colon \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ be the standard *shift map*, defined by $\sigma(w_1 w_2 w_3 \dots) = (w_2 w_3 \dots)$.

Milnor-Thurston developed a combinatorial criterion for a sequence in $\{0, 1\}^{\mathbb{N}}$ to be realized as an itinerary of the critical value under a quadratic map from the family $g_c\colon x \mapsto x^2 + c$, where c is real. A quadratic map is given an itinerary in the same procedure as for a tent map; partition the domain of the map into two intervals whose intersection is the critical point, and the left interval receives a coding value of 1 while the right interval receives a coding value of 0.

Theorem 2.8. [MT88, Theorem 12.1] *A sequence $a = (a_n)$ in $\{0, 1\}^{\mathbb{N}}$ is an itinerary of the critical value of a quadratic map if and only if $\sigma^j(a) \le_E a$ for all $j \in \mathbb{N}$.*

As a corollary of Milnor-Thurston's semi-conjugacy from Theorem 2.1, a sequence in $\{0, 1\}^{\mathbb{N}}$ which is realizable as an itinerary of of 1 under a tent map is also realizable as an itinerary of the critical value of a quadratic map, as 1 is the image of the critical value under semi-conjugacy. Thus, Theorem 2.8 introduces a necessary combinatorial condition on β-itineraries which we call admissibility:

Definition 2.9 (Admissibility)**.** A sequence $a = (a_1 a_2 \dots)$ in the alphabet $\{0, 1\}$ is *admissible* in the Milnor-Thurston sense if for all positive integers j, the shifted sequence satisfies the inequality

$$\sigma^j(a) \leq_E a.$$

Then a word w is admissible if and only if the sequence w^∞ is admissible.

On the other hand, the converse is more subtle because the Milnor-Thurston semi-conjugacy is not a true conjugacy, the reason being that a critically periodic tent map is semi-conjugate to infinitely many quadratic maps with different post-critical itineraries. However we do have the partial converse which will be sufficient for our purposes:

Proposition 2.10. *Let w be a word in the alphabet $\{0, 1\}$ with positive cumulative sign. If w is admissible and the Parry polynomial associated with w can be factored into $z - 1$ and another irreducible factor, then w^∞ is the β-itinerary for some $\beta \in (1, 2]$.*

Proof. Theorem 12.1 of [MT88] tells us that if w^∞ is admissible, it must be the itinerary of c under some quadratic map $g_c : x \mapsto x^2 + c$ (here we let $I_1 = (-\infty, 0]$, $I_0 = [0, \infty)$). Because a quadratic map is unimodal, we can find some tent map f_β semi-conjugate to the quadratic map g_c. Suppose $(w')^\infty$ is the β-itinerary, and w' has minimal length. The proof of Lemma 12.2 in [MT88] implies (which can also be checked by bookkeeping) that the itinerary of the critical value c under any quadratic map g_c which is semi-conjugate to f_β must lie between $(w')^\infty$ and $(w'')^\infty$, where w'' has the same length as w' and agrees with w' except for the last letter. Hence, $(w')^\infty \leq_E w^\infty \leq_E (w'')^\infty$, which implies that the length of w must be a multiple of the length of w'. Because β is a root of the Parry polynomial associated with w, w' and w'', the fact that the Parry polynomial of w has only two irreducible factors implies that w and w' have the same length. Hence $w = w'$ or $w = w''$. The condition that w has positive cumulative sign precludes $w = w''$, so $w = w'$. ☐

Note that the converse of Proposition 2.10 is false; if we write the β-itinerary as w^∞, this string is admissible in the Milnor-Thurston sense, but the Parry polynomial P_w may have more than two irreducible factors, even if w is minimal length and has positive cumulative sign.

For a critically periodic tent map f_β, we call the Parry polynomial associated with the β-itinerary *the Parry polynomial of* f_β and denote it by P_β. In the case that f_β is postcritically finite, a similar procedure using the sum of a power series produces a polynomial associated to a preperiodic word, and hence to the preperiodic β-itinerary.

2.6. **Irreducibility.** To check that a Parry polynomial has only two irreducible factors, we will use two lemmas from [Tio20], which are derived from Eisenstein's criterion.

Lemma 2.11. [Tio20, Lemma 4.1] *Let $d = 2^n - 1$ with $n \geq 1$, and choose a sequence $\epsilon_0, \epsilon_1, \dots, \epsilon_n$ with each $\epsilon_k \in \{\pm 1\}$ such that $\sum_{k=0}^{d} \epsilon_k \equiv 2 \bmod 4$. Then the polynomial*

$$f(x) := \epsilon_0 + \epsilon_1 x + \cdots + \epsilon_d x^d$$

is irreducible in $\mathbb{Z}[x]$.

Lemma 2.12. [Tio20, Lemma 4.2] *Let $f(x) = 1 + \sum_{k=1}^{d} \epsilon_k x^k$ be a polynomial with $\epsilon_k \in \{\pm 1\}$ for all $1 \leq k \leq d$ and $\epsilon_k = -1$ for some k. If $f(x)$ is irreducible in $\mathbb{Z}[x]$, then for all $n \geq 1$, the polynomial $f(x^{2^n})$ is irreducible in $\mathbb{Z}[x]$.*

3. Quadractic maps, iterated function systems, and renormalization

In this section we elaborate on the connections to quadratic maps and iterated function systems, including consequences of kneading theory for the combinatorial itineraries we will study in this work. We close the section with Proposition 3.10, which is a sufficient criterion for the Persistence Theorem 1.

3.1. **Iterated function system description of the Thurston set.** Here we associate a limit set Λ_z to a nonzero complex parameter z in the open unit disk $\mathbb{D}$. A point $z \in \mathbb{D} \setminus \{0\}$ defines a contracting iterated function system (IFS) generated by the two maps

$$f_0^z : x \mapsto zx + 1, \quad f_1^z : x \mapsto zx - 1.$$

The *attractor* or *limit set* Λ_z of this IFS is defined to be the unique fixed, nonempty, compact set $S \subset \mathbb{C}$ such that $S = f_0^z(S) \cup g_1^z(S)$. The existence and uniqueness of this attractor and statements of some of its fundamental properties are due to Hutchinson [Hut81]. It is straightforward to see that:

Lemma 3.1. [CKW17, Lemma 3.1.1] *The limit set Λ_z associated to $z \in \mathbb{D} \setminus \{0\}$ is contained in the open ball of radius $\frac{1}{1-|z|}$ around the origin in the compex plane.*

Our work is motivated by Tiozzo's description of $\Omega_2^{cp} \cap \mathbb{D}$ in [Tio20], which is as follows:

Theorem 3.2. [Tio20] *The Thurston set Ω_2^{cp} intersected with $\mathbb{D}$ is equal to the set of all compex numbers z whose associated limit set Λ_z contains the origin.*

Milnor and Thurston showed that any tent map f_β is semiconjugate to a quadratic map $g_c : x \mapsto x^2 + c$ for $c \in [-2, 1/4]$, and this semiconjugacy preserves the data of the Markov coding and hence the entropy [MT88]. The *kneading series* of a quadratic map g_c and a number x is a power series

$$K(x,t) = 1 + \sum_{n=1}^{\infty} \eta_n t^n,$$

where $\eta_n(1)$ is the cumulative sign $\eta_n(x) = \prod_{i=0}^{n-1} sign(g_c^i(x))$.
The *kneading determinant* is

$$K_c(t) = \begin{cases} K(c,t) \text{ if the critical point is not periodic under } f_c, \\ \lim_{C \to c^+} K(C,t) \text{ if the critical point is periodic under } f_c \end{cases} .$$

When g_c has periodic critical orbit, $K_c(t) = \frac{P_{c,\text{knead}}(t)}{1-t^n}$, where $P_{c,\text{knead}}$ is called the *kneading polynomial*.

Remark 3.3. The semiconjugacy between f_β and g_c sends the critical value 1 to c, and intervals I_o^β and I_1^β to $[0,\infty)$ and $(-\infty, 0]$ respectively. Hence, $\eta_n(c) = s_w(n)$, which implies that:

$$(t-1)t^{n-1} P_{c,\text{knead}}(t^{-1}) = P_\beta(t).$$

The following are classical from kneading theory; see [MT88]:

Proposition 3.4. *Fix a parameter c in $[-2, 1/4]$ with associated quadratic map $g_c : x \mapsto x^2 + c$ with entropy h, and let $s = e^h$ be the growth rate of this map.*

(1) [MT88, Theorem 13.1, Corollary 13.2] *The growth rate s is a continuous function of c.*
(2) [MT88, Theorem 13.1, Corollary 13.2] $s > \sqrt{2}$ *if and only if* $c < -1$.

(3) [MT88, Theorem 6.3] *If $s > 1$, $1/s$ is the smallest positive root of the kneading polynomial $P_{c,knead}$.*

□

Furthermore, Milnor and Thurston introduce an ordering on the additive group $\mathbb{Z}[[t]]$ of formal power series with integer coefficients is defined by setting $\alpha = a_0 + a_1 t + \cdots > 0$ whenever $a_0 = \cdots = a_{n-1} = 0$ but $a_n > 0$ for some $n \geq 0$ [MT88]. This induces an ordering on formal power series as follows: if a, b are distinct formal power series with integer coefficients, then $a > b$ if and only if $a - b > 0$.

The following is an immediate consequence of [MT88, Section 13]. We include proofs for completeness.

Lemma 3.5. *For tent maps, the kneading determinant is a monotonically decreasing function of the growth rate.*

Proof. For the real one-parameter family of maps $f_a(x) = (x^2 - a)/2$, [MT88, Theorem 13.1] asserts that the kneading determinant $D(f_a) \in \mathbb{Z}[[t]]$ is monotonically decreasing as a function of the parameter a; and Corollary 13.2 asserts the growth rate is monotonically increasing as a function of a. The family of maps $\{f_a\}$ takes on all possible growth rates; this can be seen from the fact that f_a is conjugate to the map $g_{(-a/4)}(z) = z^2 + (-a/4)$ via the conjugation map $h(z) = z/2$, growth rate is a continuous function of c (Proposition 3.4(1)), and the Intermediate Value Theorem. □

Lemma 3.6. *Let f_β be a tent map with kneading determinant a and let w be the β-itinerary; let $f_{\beta'}$ be a tent map with kneading determinant b and let w' be the β'-itinerary. If $a > b$, then $w <_E w'$.*

Proof. From Remark 3.3, $a = 1 + \sum_{i=1}^{\infty} s_w(i) t^i$, $b = 1 + \sum_{i=1}^{\infty} s_{w'}(i) t^i$. Let n be the smallest natural number such that $s_w(n) \neq s_{w'}(n)$. We must have $s_w(1) = s_{w'}(1)$, so we may assume $n \geq 2$. By definition, $a > b$ implies $s_w(k) = s_{w'}(k)$ for all $k = 1, \ldots, n-1$, and $s_w(k) = (-1)^{w_{k-1}} s_w(k-1)$, so we must have $w_j = w'_j$ for $1 \leq j \leq n-2$ and $w_{n-1} \neq w'_{n-1}$. Since $s_w(n) > s_{w'}(n)$, the two possibilities are:

$$s_w(n-1) = s_{w'}(n-1) = +1, \quad w_{n-1} = 0, \quad w'_{n-1} = 1, \text{ or}$$
$$s_w(1, n-1) = s_{w'}(n-1) = -1, \quad w_{n-1} = 1, \quad w'_{n-1} = 0.$$

In both cases, we have $w <_E w'$. □

Combining Lemma 3.5 and Lemma 3.6, we have:

Corollary 3.7. *If $1 < \beta < \beta' \leq 2$, the β-itinerary w and β'-itinerary w' satisfy the inequality $w <_E w'$.*

3.2. **Renormalization.** We will develop the proof of the Persistence Theorem 1 using a combinatorial approach of Tiozzo [Tio20]. Certain tent maps f_β admit itineraries with strong combinatorial properties. Due to the renormalization phenomenon, if the slope β is at most $\sqrt{2}$ then it is impossible for any associated itinerary to satisfy this strong combinatorial property. *Renormalization* is how we and Tiozzo compensate for this obstruction.

One of the renormalization or "tuning" procedures on the Mandelbrot set (see e.g. [Tio20, § 7.2]) implies the following:

Lemma 3.8 (Renormalization Lemma)**.** *If $\beta \in (1, \sqrt{2}]$, then f_β has periodic critical orbit of length $2k$ if and only if f_{β^2} has periodic critical orbit of length k.*

This is a well-known fact and we give a short proof below for completeness.

Proof. Consider the intervals $J_1^\beta = \left[2-\beta, \frac{2}{1+\beta}\right]$, $J_2^\beta = \left[\frac{2}{1+\beta}, 1\right]$. Then it is straightforward to check that $f(J_1) \subset J_2$, $f(J_2) \subset J_1$, and $f^2\colon J_2 \to J_2$ is a unimodal piecewise linear map with constant slope β^2 and critical value equal to 1, and is clearly conjugate to the tent map of slope f_{β^2}. Hence, f_β^n, the nth iterate of f_β, fixes 1 if and only if $(f_\beta^2\big|_{J_2})^{n/2}$, the $n/2$-iterate of the restriction, also fixes 1. □

The Renormalization Lemma 3.8 motivates the following definition:

Definition 3.9. A tent map f_β with growth rate β in the interval $(1,2]$ is defined to be *renormalizable* if $\beta \le \sqrt{2}$, and is otherwise *nonrenormalizable*.

With the Renormalization Lemma 3.8, we reduce the Persistence Theorem 1 to the following proposition, which is essentially persistence restricted to growth rates of nonrenormalizable tent maps:

Proposition 3.10. *Let $\beta \in (1,2]$ be the growth rate of a critically periodic tent map and let $z \in \mathbb{D}$ be a root of the Parry polynomial of β. Then for any real number y satisfying $2 > y > \max\{\beta, \sqrt{2}\}$ and any real number $\epsilon > 0$, there exist a real number β' within ϵ of y such that*

(1) one of the Galois conjugates of β' is within distance ϵ of z, and
(2) a Parry polynomial of β' is of the form $(x-1)f(x)$, where $f(x^{2^n})$ is irreducible in $\mathbb{Z}[x]$ for all natural numbers n.

The proof of Proposition 3.10 will appear at the end of Section 5. We confirm here that this proposition is indeed sufficient to prove the Persistence Theorem 1.

Proof of the Persistence Theorem 1 from Proposition 3.10. Suppose (z,β) is a pair in the Master Teapot Υ_2^{cp}. We want to show that if $y \in [\beta, 2]$, then $(z,y) \in \Upsilon_2^{cp}$. Note that it suffices to consider $y > \beta$.

Let $\epsilon > 0$. By definition of the Master Teapot, there exists a β_0 that is the growth rate of some tent map f_{β_0} with periodic critical orbit, z_0 a Galois conjugate of β, such that $|z_0 - z| < \epsilon$ and $|\beta_0 - \beta| < \epsilon$. Then we may choose ϵ small enough that $y > \beta_0$ as well.

If $y > \sqrt{2}$ then Proposition 3.10 along with the triangle inequality directly implies existence of β' within ϵ of y, such that one of the Galois conjugates of β' is within 2ϵ distance from z, as desired.

It remains to consider $y < \sqrt{2}$. Fix an integer n such that $y \in \left(2^{\frac{1}{2^{n+1}}}, 2^{\frac{1}{2^n}}\right]$. By the Renormalization Lemma 3.8, $\beta_0^{2^n}$ is the growth rate of some tent map with periodic critical orbit, is clearly less than y^{2^n}, and $z_0^{2^n}$ is a Galois conjugate of $\beta_0^{2^n}$ because the Galois group consists of field automorphisms. Since $\sqrt{2} < y^{2^n} \le 2$, again by Proposition 3.10, we can find some β' within ϵ of y^{2^n}, such that one of the Galois conjugates z' of β' is within 2ϵ of z^{2^n}. The second condition in Proposition 3.10 implies that all the 2^n-th roots of z' are Galois conjugates of $(\beta')^{\frac{1}{2^n}}$. The conclusion follows. □

4. DOMINANT WORDS

In this section we will review Tiozzo's definition of dominant words in [Tio15] and the properties of dominant words proved in [Tio15].

In [Tio15], Tiozzo considers the tent map with slope 2 f_2, then any points x in unit interval has a correspondence with a infinite sequence w_x in $\{0,1\}^{\mathbb{N}}$, which is its Markov coding, where the two subintervals are $I_0 = [0, 1/2]$ and $I_1 = [1/2, 0]$. It is obvious that

f_2 is semiconjugate to the shift map σ under this correspondence, and $x < y$ implies that $w_x <_E w_y$ where $<_E$ is the twisted lexicographical order defined earlier.

For any finite word w of length n, let x be the point on the unit interval labeled by w^∞, then the *Cylinder set* C_x is defined as the subinterval which contains x and f_2^{n+1} sends it homeomorphically to the unit interval I; in other words, C_x is the closure of points with a coding starting with $w \cdot w_1$. Here w_1 is the first letter of w.

Definition 4.1. [Tio15, Definition 10.4] A finite word w in the alphabet $\{0, 1\}$ is called *dominant* if and only if it satisfies the following two conditions:

(1) w has positive cumulative sign.
(2) For any $1 \leq k \leq n-1$, $f_2^k(C_x)$ lies to the left of C_x and their interiors are disjoint.

Now, because $x < y$ implies $w_x <_E w_y$, and $f_2^k(C_x)$ is the closure of points whose itineraries start with a suffix of w followed by the first letter of w, we can make the definition of dominance more explicit as below:

Lemma 4.2. *Let w be a word in the alphabet $\{0, 1\}$ that starts with 10 and has positive cumulative sign. Then w is dominant if and only if for any proper suffix b of w, the word $(b1)$ is (strictly) smaller than the prefix of w of length $|b| + 1$ in the twisted lexicographical ordering $<_E$.* □

Definition 4.3. A word w in the alphabet $\{0, 1\}$ is *irreducible* if there exists no shorter word w_0 in the alphabet $\{0, 1\}$ and integer $n \geq 2$ such that $w = (w_0)^n$.

The main result we will cite from Tiozzo's work is the following, which can be read from the proof of [Tio15, Theorem 10.5] in [Tio15, Section 10.1]. One can read it from the first paragraph of the proof of Lemma 10.6 on page 689, and the third paragraph of the proof of Proposition 10.5 on page 692.

Proposition 4.4 ([Tio15, Section 11.2])**.** *If $\beta \in (\sqrt{2}, 2]$ and w is a word in the alphabet $\{0, 1\}$ such that w^∞ is a β-itinerary, then for any positive integer n there exists a word w' in the alphabet $\{0, 1\}$ that is a power of some dominant word such that $w^n w'$ is also a dominant word.*

Tiozzo uses this observation to prove that the growth rates associated to dominant words are dense in the interval $[\sqrt{2}, 2]$.

The following lemma is straightforward to show from calculation:

Lemma 4.5. *Any β-itineraries for $\beta \in (1, 2]$ start with 10.* □

Because of Lemma 4.5, the dominant itineraries obtained in Proposition 4.4 must satisfies the assumptions of Lemma 4.2.

5. Persistence

The goal of this section is to prove Proposition 3.10, which is a version of persistence restricted to growth rates of nonrenormalizable tent maps; that is, growth rates that are larger than $\sqrt{2}$. As discussed in Section 3 following the statement of the proposition, the Persistence Theorem 1 follows by renormalization.

5.1. **Constructing dominant extensions.** The development of persistence for growth rates of nonrenormalizable tent maps hinges on a series of technical combinatorial lemmas. The goal of this subsection is the proof of the Extension Lemma 5.3.

Proposition 5.1. *Assume w_1 is dominant, w_2 is admissible and irreducible, n is a positive integer such that*

$$2n|w_2| > |w_1| > n|w_2|,$$

$w_1^\infty >_E w_2^\infty$, and w_2^n has positive cumulative sign. Then $(w_1 w_2^n)^\infty$ is admissible.

Proof. It suffices to show that

$$\sigma^k (w_1 w_2^n)^\infty \leq_E (w_1 w_2^n)^\infty$$

for all $k < |w_1| + n|w_2|$. If $1 < k < |w_1|$, denote by b the proper suffix of w_1 of length $|w_1| - k$. Then $(b1)$ is a prefix of $\sigma^k(w_1 w_2^n)$ because the first letter of w_2 is 1 by admissibility and Lemma 4.5. By dominance of w_1 and Lemma 4.2, $(b1)$ is smaller than the prefix of w_1 of length $|b| + 1$ in the twisted lexicographical ordering, which proves

$$\sigma^k(w_1 w_2^n) = b w_2^n <_E w_1$$

and provides the desired inequality.

If $k = |w_1|$, for contradiction, see that existence of n such that $w_2^n \geq_E w_1$ implies

$$w_2^\infty <_E w_1^\infty \leq_E (w_2^n)^\infty = w_2^\infty,$$

which is impossible given the assumption that w_2^∞ is smaller than w_1^∞ in the twisted lexicographical ordering. Thus,

$$\sigma^{|w_1|}(w_1 w_2^n)^\infty = w_2^n (w_1 w_2^n)^\infty <_E (w_1 w_2^n)^\infty.$$

Lastly, we consider the shift by k where $|w_1| < k < |w_1| + n|w_2|$. Let $r = k - |w_1|$, so that $1 < r < n|w_2|$. See that $\sigma^r w_2^n >_E w_1$ is impossible, because $\sigma^r w_2^n >_E w_1$ and admissibility of w_2 implies

$$w_1^\infty <_E \sigma^r (w_2)^\infty \leq_E w_2^\infty,$$

a contradiction. We conclude that $\sigma^r w_2^n \leq_E w_1$. If this inequality is strict, we are done: we would have

$$\sigma^k (w_1 w_2^n) = \sigma^{|w_1| + r}(w_1 w_2^n) = \sigma^r w_2^n <_E w_1$$

as desired.

We must now consider when this inequality is not strict; in other words, $\sigma^r w_2^n$ is a prefix of w_1. We will need to prove that such a string must always have cumulative negative sign. If it does, then $|w_1| - r < |w_1|$ implies

$$\sigma^{|w_1| - r}(w_1 w_2^n)^\infty \leq_E (w_1 w_2^n)^\infty$$

by dominance of w_1 discussed above. Then by Remark 2.5 and the fact that $\sigma^r w_2^n$ has negative cumulative sign,

$$\begin{aligned}(w_1 w_2)^\infty = \sigma^r w_2^n \sigma^{|w_1| - r} w_1 w_2^n (w_1 w_2^n)^\infty &\geq_E \sigma^r w_2^n (w_1 w_2^n)^\infty \\ &= \sigma^{k - |w_1|} w_2^n (w_1 w_2^n)^\infty \\ &= \sigma^k (w_1 w_2^n)^\infty.\end{aligned}$$

It remains to prove that if $\sigma^r w_2^n$ is a prefix of w_1, then it cannot have cumulative positive sign. Consider the suffix $b = \sigma^r w_2^n$ of w_2^n. Since w_2^n is admissible, $b \leq_E a$, where a is the prefix of w_2^n of the same length. Since $w_2^\infty <_E w_1^\infty$, moreover a is smaller than or equal to the prefix of w_1 of the same length, which is assumed to be equal to b. Then $b \leq_E a \leq_E b$ implies equality, and we conclude $w_2^n = ac = db = da$.

Now

$$w_2^\infty = (ac)^\infty = (da)^\infty \geq_E a \cdot (da)^\infty,$$

implying
$$(ca)^\infty \geq_E (da)^\infty = w_2^\infty \geq_E (ca)^\infty$$
because we assumed that a has positive cumulative sign (see Remark 2.5) and w_2 is admissible, hence $(ca)^\infty = (ac)^\infty$. Then
$$w_2^\infty = (ac)^\infty = a \cdot (ca)^\infty = a \cdot (ac)^\infty = a^2(ca)^\infty = \cdots = a^\infty$$
implies $a = w_2^m$ for some m because w_2 is irreducible. Then $w_1 = af = w_2^m f$ for some suffix f, and again by dominance of w_1 and Lemma 4.2,
$$w_1^\infty = (w_2^m f)^\infty = w_2^m(fw_2^m)^\infty \leq_E (w_2^m w_1)^\infty = w_2^{2m}(fw_2^m)^\infty \leq_E \cdots \leq_E w_2^\infty,$$
which contradicts the assumption that $w_1^\infty >_E w_2^\infty$. □

Now we want to further make sure that $(w_1 w_2^n)^\infty$ is a β-itinerary and has a Parry polynomial with a large irreducible factor. However, this would require some slight modification of the construction as below:

Definition 5.2. We say that a string v is an *extension* of a word w if w is a proper prefix of v. If v is finite then such a v is a *finite extension* of w.

Lemma 5.3 (Extension). *Let w_1 be dominant such that $w_1 >_E 10 \cdot 1^{|w_1|-2}$ and w_2 be admissible and irreducible, $w_1^\infty >_E w_2^\infty$, and assume there exists an m such that*
$$2m|w_2| > |w_1| > m|w_2|.$$
Then there exists a finite extension w_1' of w_1 and an integer $m' \geq m$ such that $(w_1' w_2^{m'})^\infty$ is admissible, $|w_1'| > m'|w_2|$, and $P(z^{2^k})/(z^{2^k} - 1)$ is an irreducible polynomial for any $k \geq 0$, where P is the Parry polynomial of $(w_1' w_2^{m'})$.

The following Lemma will give us a recipe for extending w_1, as needed for the Extension Lemma.

Lemma 5.4. *Let w be a dominant word, such that $w_1 >_E 10 \cdot 1^{|w_1|-2}$. Then the words*
$$w \cdot 10 \cdot 1^\kappa \cdot 10 \cdot 1^{|w|} \cdot \mathbf{01} \cdot 1^{|w|} \text{ and } w \cdot 10 \cdot 1^\kappa \cdot 10 \cdot 1^{|w|} \cdot \mathbf{10} \cdot 1^{|w|}$$
for any odd natural number $\kappa > |w|$, and
$$w \cdot 1^\kappa \cdot 10 \cdot 1^{|w|} \cdot \mathbf{01} \cdot 1^{|w|} \text{ and } w \cdot 1^\kappa \cdot 10 \cdot 1^{|w|} \cdot \mathbf{10} \cdot 1^{|w|}$$
for any even natural number $\kappa > |w|$, are all dominant extensions of w.

Moreover, for each κ, the sums of the coefficients of the kneading polynomials for the two extensions differ by 2.

Proof. The parity condition on κ is to guarantee that the new word has an even number of 1s, which is part of the definition of dominance.

We apply the alternate definition of dominance from Lemma 4.2. Let w' be one of the possible extensions in the statement of the Lemma. Let b be any suffix of w'. If a prefix of b is a suffix of w, then $(b1)$ is smaller than the prefix of w' of the same length in the twisted lexicographical ordering by dominance of w and the construction of w'. If not, then if b starts with 0 or 11, and the desired inequality is immediate, so the interesting case is if b starts with 10 and no prefix of b is a suffix of w. By construction, including our choice of $\kappa > |w|$ in the κ odd case, we are comparing $10 \cdot 1^{|w|-1}$ with $w \cdot 1$, and the former must be smaller by assumption.

For any natural number κ, odd or even, there are now two choices to extend w to a dominant word. The two choices only differ by an exchange of 01 with 10 in one position.

This exchange will change the sum of the coefficients of the kneading polynomials by a factor of 2. □

5.1.1. *Proof of extension lemma.*

Proof the Extension Lemma 5.3. We need to choose for w_1' one of the extensions of w_1 from Lemma 5.4, and select n, κ, and m' so that $|w_1'|$ has length $2^n - 1 - m'|w_2|$ and

$$2m'|w_2| > |w_1'| > m'|w_2|.$$

To do so, first define constants $C_1 = 1 + |w_1| + m|w_2|$ and $C_2 = |w_2|$. Then choose n for which

$$2^n > \max\{C_2(10m+3) + C_1, 18C_2 + C_1\}$$

and define

$$k_n = \left\lceil \frac{2^n - C_1}{2C_2} \right\rceil - 2, \qquad k_n' = \left\lceil \frac{2^n - C_1}{2C_2} \right\rceil - 3. \tag{4}$$

The two options k_n and k_n' are needed for parity reasons. Choosing $2^n > C_2(10m+3) + C_1$ ensures that

$$k_n > k_n' > 10m, \tag{5}$$

which becomes useful later in the proof when we define the length of the extension. The choice of $2^n > 18C_2 + C_1$ and the definition of k_n, k_n' ensures (respectively) that

$$3k_n > 3k_n' > \frac{2^n - C_1}{C_2} > 2k_n > 2k_n'. \tag{6}$$

Let $m' = k_n + m$ if this is even, and else, replace k_n with k_n'. We will proceed with the notational choice $m' = k_n + m$ and assume m' is even, but note that the needed inequalities hold for both k_n and k_n'.

Now, replacing C_1, C_2 with their definitions, applying Equation (6), and invoking the assumed relationship between $|w_1|$ and $|w_2|$, we see that

$$3m'|w_2| > 3k_n|w_2| + m|w_2| + |w_1| > 2^n - 1 > 2k_n|w_2| + m|w_2| + |w_1| > 2m'|w_2|,$$

which implies

$$2m'|w_2| > 2^n - 1 - m'|w_2| > m'|w_2|. \tag{7}$$

We now adjust the extension w_1' of w_1 to have length $|w_1'| = 2^n - 1 - m'|w_2|$, so that $(w_1' w_2^{m'})$ has total length $2^n - 1$.

If $|w_1|$ is odd, then $\kappa = (2^n - 1 - m'|w_2|) - 6 - 3|w_1|$ is even, as needed for

$$w \cdot 1^\kappa \cdot 10 \cdot 1^{|w|} \cdot \mathbf{01} \cdot 1^{|w|} \quad \text{and} \quad w \cdot 1^\kappa \cdot 10 \cdot 1^{|w|} \cdot \mathbf{10} \cdot 1^{|w|}$$

to both be dominant extensions of w_1 by Lemma 5.4, each of length $2^n - 1 - m'|w_2|$.

If $|w_1|$ is even, then $\kappa = (2^n - 1 - m'|w_2|) - 4 - 3|w_1|$ is odd, as needed for

$$w_1 \cdot 1^\kappa \cdot 10 \cdot 1^{|w_1|} \cdot \mathbf{01} \cdot 1^{|w_1|} \quad \text{and} \quad w_1 \cdot 1^\kappa \cdot 10 \cdot 1^{|w_1|} \cdot \mathbf{10} \cdot 1^{|w_1|}$$

to both be dominant extensions of w_1 by Lemma 5.4, each of length $2^n - 1 - m'|w_2|$. In all the above cases, $\kappa > |w_1|$ follows from Equation (5).

For each choice, $w_1^\infty >_E w_2^\infty$ implies $w_1'^\infty >_E w_2^\infty$, and $w_2^{m'}$ has positive cumulative sign because we ensured that m' is even. Combined with Equation (7), we have all the necessary hypotheses to apply Proposition 5.1 and conclude that $(w_1' w_2^{m'})^\infty$ is admissible. We also designed w_1' so that $|w_1'| > m'|w_2|$.

The sum of the coefficients of the kneading polynomial of $w_1' w_2^{m'}$ is even, because it has 2^n coefficients, each of which is either -1 or $+1$. By the final observation in Lemma 5.4, we can choose the extension so that the sum of the coefficients of the kneading polynomial for $w_1' w_2^{m'}$ is equivalent to $2 \mod 4$. Since the kneading polynomial has degree $2^n - 1$, we apply Lemma 2.12 to conclude irreducibility. □

5.2. **Controlling Galois conjugates and entropies of concatenations.**

Lemma 5.5 (Nearby roots). *Let w_2 be a word whose Parry polynomial has a root at $z_0 \in \mathbb{D}$. Then for any $\epsilon > 0$, there exists an integer $N = N(\epsilon, w_2) \in \mathbb{N}$ such that $n > N$ implies that for every word w_1 for which $w_1 w_2^n$ is admissible, the Parry polynomial associated to $(w_1 w_2^n)$ has a root within distance ϵ of z_0.*

Proof. First, for any word w, denote the Parry polynomial associated to w by P_w. Let D be the closed disk of radius ϵ centered at z_0, and let C be the boundary of D. Without loss of generality, assume that ϵ is small enough that $D \subset \mathbb{D}$, and that D contains no root of P_{w_2} except z_0.

For any $n \in \mathbb{N}$, it is straightforward to see that

$$P_{w_1 w_2^n}(z) = z^{n|w_2|} P_{w_1}(z) + \left(z^{(n-1)|w_2|} + z^{(n-2)|w_2|} + \cdots + 1 \right) P_{w_2}(z).$$

Set $\alpha = \min_{z \in C} |P_{w_2}(z)|$, which exists and is positive by compactness and the assumption that D contains no root of P_{w_2} except z_0. Set

$$0 < \beta := \min_{z \in C} \left(1 - |z|^{|w_2|} \right) / \left(1 + |z|^{|w_2|} \right).$$

Then for all $z \in C$, we have

$$\left| \left(z^{(n-1)|w_2|} + z^{(n-2)|w_2|} + \cdots + 1 \right) P_{w_2}(z) \right| \geq \left| \frac{1 - (z^{|w_2|})^n}{1 - z^{|w_2|}} \right| \alpha \geq \frac{1 - |z^{|w_2|}|}{1 + |z^{|w_2|}|} \alpha \geq \beta\alpha > 0,$$

where the middle nonstrict inequality follows the triangle inequality and that $\left| z^{|w_2|} \right| < 1$.

Set $1 > m := \max_{z \in D} |z|$. Also for all $z \in C$, since all coefficients of P_{w_1} have absolute value at most 3,

$$\left| z^{n|w_2|} P_{w_1}(z) \right| \leq |z|^{n|w_2|} \left(1 + 3 \sum_{i=0}^{\infty} |z|^i \right) \leq m^{n|w_2|} \left(1 + 3 \sum_{i=0}^{\infty} m^i \right).$$

Therefore, for sufficiently large $n \in \mathbb{N}$ depending only on w_2, we have

$$\left| z^{(n-1)|w_2|} P_{w_1}(z) \right| < \frac{\beta\alpha}{2}.$$

Consequently, the winding number around 0 of the image of C under $P_{w_1 w_2^n}$ equals the winding number around 0 of the image of C under the map

$$z \mapsto \left(z^{(n-1)|w_2|} + z^{(n-2)|w_2|} + \cdots + 1 \right) P_{w_2}(z).$$

The winding number of the image around 0 is related to number of zeros via the Argument Principle; for a holomorphic function f and a simple closed contour Γ, the number N of zeros of f inside Γ is given by

$$N = \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{f(\Gamma)} \frac{dw}{w}, \tag{8}$$

where $w = f(z)$. Since P_{w_2} has a root in D, this implies $P_{w_1 w_2^n}$ also has a root in D for sufficiently large n. □

5.3. **Dominant approximations of growth rates.** Finally, we prove that we can approximate $y \in (\sqrt{2}, 2)$ with growth rates corresponding to extensions of dominant strings.

Lemma 5.6. *Let w_1 be an admissible word such that w_1^∞ is a β-itinerary for some $\beta > 1$. For any $\epsilon > 0$, there exists an integer $N = N(\epsilon, w_1)$ such $n > N$ implies that for every word w_2 for which $(w_1^n w_2)^\infty$ is a β'-itinerary, β' is within distance ϵ of β.*

Proof. Let w' be the $(\beta - \epsilon)$-itinerary, and let w'' be the $(\beta + \epsilon)$-itinerary. Let N be large enough that the $N|w_1|$-prefix of w', w_1^∞ and w'' are all distinct. Then we must have $w' <_E (w_1^n w_2)^\infty <_E w''$ so $\beta' \in (\beta - \epsilon, \beta + \epsilon)$. □

Combining Lemma 5.6, Proposition 4.4, and the fact that slopes of tent maps with periodic critical orbits are dense, the following result is evident:

Lemma 5.7 (Dominant approximations)**.** *For all $y \in (\sqrt{2}, 2)$ and all $\epsilon > 0$, there exists a sequence of dominant words $(w_n)_{i=1}^\infty$ such that for any admissible extension w_n' of w_n, including the empty extension, if $(w_n')^\infty$ is a β-itinerary, then β is within ϵ of y.*

Remark 5.8. Because w_n is constructed using Proposition 4.4, and $y > \sqrt{2}$, together with Lemma 3.6 and the fact that $it_{\sqrt{2}} = 10 \cdot 1^\infty$, we can further assume that $w_n >_E 10 \cdot 1^{|w_n|-2}$.

5.4. **Proof of Proposition 3.10.** Now we prove Proposition 3.10, which will finish the proof of the Persistence Theorem 1.

Proof of Proposition 3.10. Let w be an irreducible word in the alphabet $\{0, 1\}$ such that w^∞ is the β-itinerary. Then z is a root of the Parry polynomial of w, since this is equal to the Parry polynomial of β. If $y = \beta$ the statement is trivial, so assume $y > \beta$. Fix

$$0 < \epsilon < \frac{y - \beta}{2}.$$

Construct the sequence of dominant words (w_n) as in the Dominant Approximations Lemma 5.7 and Remark 5.8; the words w_n satisfy that for any admissible extension w_n' of w_n, if $(w_n')^\infty$ is some β'-itinerary, then β' is within ϵ of y. We will show there is a subsequence of (w_n) with corresponding extensions (w_n') which meet this criteria and whose corresponding growth rates have controlled Galois conjugates.

Passing to subsequences as needed, we may assume that $|w_n| \to \infty$ as $n \to \infty$, since there are only finitely many words of bounded length.

For each n, let $M_n = \left\lceil \frac{|w_n|}{|w|} \right\rceil - 2$. Then

$$2M_n|w| \geq 2\left(\frac{|w_n|}{|w|} - 2\right)|w| = 2|w_n| - 4|w|. \tag{9}$$

Since $2|w_n| - 4|w| > |w_n|$ if and only if $|w_n| > 4|w|$, we have from equation (9) that

$$|w_n| > 4|w| \implies 2M_n|w| > |w_n|. \tag{10}$$

Observe that

$$|w_n| = \frac{|w_n|}{|w|}|w| > \left(\left\lceil \frac{|w_n|}{|w|} \right\rceil - 2\right)|w| = M_n|w| \quad \text{for all } n$$

and $|w_n| \to \infty$. Therefore, for all n large enough that $|w_n| > 4|w|$, there exists a positive integer M_n such that

$$2M_n|w| > |w_n| > M_n|w|.$$

Note also that $M_n \to \infty$ as $n \to \infty$. Thus, for sufficiently large n, the hypotheses of the Extension Lemma 5.3 hold, using w_n in place of w_1 and w in place of w_2. Then we conclude there exists an integer $m'_n > M_n$ and a dominant extension w'_n of w_n so that $(w'_n w^{m'_n})^\infty$ is admissible and the polynomial

$$\frac{P_{w'_n w^{m'_n}}(x^{2^k})}{1 - x^{2^k}}$$

is irreducible for all $k \geq 0$, where $P_{w'_n w^{m'_n}}$ is the Parry polynomial of the admissible word $w'_n w^{m'_n}$. Hence, by Proposition 2.10, $(w'_n w^{m'_n})^\infty$ is a β'-itinerary, and we have the criteria needed to apply the final conclusion of the Dominant Approximations Lemma 5.7); that is, $\beta' \in [y - \epsilon, y + \epsilon]$. Since $M_n \to \infty$ as $n \to \infty$ and $m'_n > M_n$, we have $m'_n \to \infty$ as $n \to \infty$.

We know z is a root of P_w so by the Nearby Roots Lemma 5.5), for sufficiently large $n \in \mathbb{N}$, we conclude $P_{w'_n w^{M'_n}}$ has a root within ϵ of z. □

6. THE UNIT CYLINDER AND CONNECTIVITY

Proposition 6.1. *The bottom level of the Master Teapot is the unit circle, i.e.*

$$\Upsilon_2^{cp} \cap (\mathbb{C} \times \{1\}) = S^1 \times \{1\}.$$

Proof. We will first show $S^1 \times \{1\} \subset \Upsilon_2^{cp}$. By Proposition 3.10, there exists some growth rate $\beta \in (1, 2)$ of a critically periodic tent map such that β has a Galois conjugate $z \in \mathbb{D}$ satisfying the condition that, for all $k \in \mathbb{N}$, every 2^k-th root of z is a Galois conjugate of $\beta^{\frac{1}{2^k}}$. Repeatedly applying renormalization to $(z, \beta) \in \Upsilon_2^{cp}$ - by which we mean considering the set of 2^k points of the form $(z^{1/2^k}, \beta^{1/2^k})$, all of which are in Υ_2^{cp}, as $k \to \infty$ - and then taking the set of limit points, we get that $S^1 \times \{1\} \subset \Upsilon_2^{cp}$.

To show $\Upsilon_2^{cp} \cap (\mathbb{C} \times \{1\}) \subset S^1 \times \{1\}$, suppose there exists a point $(y, 1) \in \Upsilon_2^{cp}$ such that $|y| \neq 1$. Since 1 has no nontrivial Galois conjugates, $(y, 1) \in \mathbb{C} \times \mathbb{R}$ must be the the limit of a sequence of points $(\alpha_n, \beta_n) \in \mathbb{C} \times \mathbb{R}$ such that β_n is the growth rate of a superattracting tent map and α_n is a Galois conjugate of β. Thus, reindexing the sequence as necessary, we have that for any $k > 0$, there exists β_k with $1 < \beta_k < 1 + \frac{1}{k}$ with Galois conjugate α_k, so that $|\alpha_k - y| < \epsilon$. Now by renormalization, $\beta_k^{2^{n_k}} \leq 2$ is the slope of a critically periodic tent map, where n_k is the maximal value of N for which $\beta_k^{2^N} \leq 2$. The fact that $\alpha_k^{2^{n_k}}$ is a Galois conjugate of $\beta_k^{2^{n_k}}$ follows immediately from the definition of a Galois automorphism. Thus $(\alpha_k^{2^{n_k}}, \beta_k^{2^{n_k}}) \subset \Upsilon_2^{cp}$.

Now, $|\alpha_k|$ is bounded away from 1 for k sufficiently large (because $\alpha_k \to y$), and $n_k \to \infty$ as $k \to \infty$, since $\beta_k \to 1$ as $k \to \infty$. Consequently, either $\alpha_k^{2^{n_k}} \to 0$ or $\alpha_k^{2^{n_k}} \to \infty$ as $k \to \infty$. This is a contradiction because

$$\Omega_2^{cp} \subset \{z \in \mathbb{C} : 1/2 \leq z \leq 2\}$$

by [Tio20, Lemma 2.4] and that the projection of Υ_2^{cp} onto the first coordinate is Ω_2^{cp}. □

Proof of the Unit Cylinder Theorem 2. As in the proof of Proposition 6.1, let β be the slope of a critically periodic tent map such that β has a Galois conjugate $z \in \mathbb{D}$ and for all $k \in \mathbb{N}$, and every 2^k-th root of z is a Galois conjugate of $\beta^{\frac{1}{2^k}}$. Then we have

$$\left\{(z', \beta') \in \mathbb{C} \times \mathbb{R}^+ : (z')^{2^k} = z \text{ and } \beta' = \beta^{\frac{1}{2^k}} \text{ for some } k \in \mathbb{N}\right\} \subset \Upsilon_2^{cp}.$$

The Persistence Theorem 1 then implies that

$$\left\{(z', y) \in \mathbb{C} \times \mathbb{R}^+ : (z')^{2^k} = z, y \geq \beta^{\frac{1}{2^k}} \text{ for some } k \in \mathbb{N}\right\} \subset \Upsilon_2^{cp},$$

Taking the closure of the above set, we obtain that the unit cylinder $\mathbb{S}^1 \times [1, 2]$ is a subset of Υ_2^{cp}. □

Proof of the Connectedness Theorem 3. Connectivity of the part of the Master Teapot outside of the unit cylinder is due to Tiozzo [Tio20]. More specifically, by [Tio20, Lemma 7.3], for any point $(z, \beta) \in \mathbb{C} \times \mathbb{R}$ such that β is the growth rate of a critically periodic tent map, z is a Galois of β, and $|z| > 1$, there exists a continuous path $(\gamma(x), x)$ in Υ_2^{cp} connecting (z, β) to a point $(w, 1)$ with $w \in S^1$. Consequently, since the unit cylinder is a subset of Υ_2^{cp} by the Unit Cylinder Theorem 2, and since Υ_2^{cp} is closed, this implies $\Upsilon_2^{cp} \cap (\{z : |z| \geq 1\} \times \mathbb{R})$ is connected. By [Tio20], the Thurston Set is connected and contains an open annulus containing S^1. By the Persistence Theorem 1, the projection to $\mathbb{C}$ of part of the top level of the Master Teapot that is inside the unit cylinder agrees with the Thurston Set, i.e. $\Upsilon_2^{cp} \cap (\mathbb{D} \times \{2\}) = (\Omega_2 \cap \mathbb{D}) \times \{2\}$. Also by the Persistence Theorem 1, the part of the Master Teapot inside the unit cylinder is connected. Thus, the entire Master Teapot, Υ_2^{cp}, is connected. □

7. Gaps in the Thurston set

Plots of finite approximations of the Thurston set consisting of the roots of all defining polynomials associated to superattracting tent maps of critical orbit length at most n, for fixed $n \in \mathbb{N}$, have "gaps" at certain algebraic integers, some of which are on the unit circle. The Thurston set contains a neighborhood of the unit circle [Tio20], but these gaps get filled in more slowly with n than some other regions. See Figure 2 for a picture of a finite approximation of the Thurston set, and Figure 1 for a closeup of one such gap. In this section, we prove an arithmetic justification for gaps:

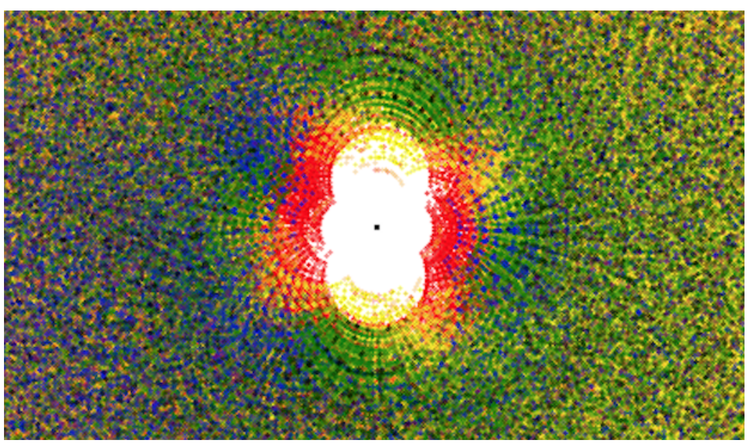

FIGURE 4. A closeup of the how the "gap" around the point i fills in as postcritical length increases, for an approximation of the Thurston set. The points are color-coded by the length of the associated post-critical orbit. Blue is the shortest, followed by green, yellow, orange, and finally red with the longest orbit, of length 23.

Theorem 7.1. *Let α be an algebraic integer such that $\mathbb{Z}[\alpha]$ is a discrete subgroup of $\mathbb{C}$ and let $x \in \mathbb{Z}[\alpha]$. Set $c = \min\{|z| : z \in \mathbb{Z}[\alpha], z \neq 0\}$. Suppose there exists a superattracting tent map* *with postcritical length n whose growth rate has a Galois conjugate of the form $x+\epsilon$ for some $\epsilon \in \mathbb{C}$ with $|\epsilon| \leq \frac{1}{n+1}$. Then*

(1) *if $|x| \geq 1$, then* $\dfrac{c}{(2n^2+3n+1)|x|^n e} \leq \epsilon$.

(2) *if $|x| \leq 1$, then* $\dfrac{c}{(2n^2+3n+1)e} \leq \epsilon$.

Proof. Fix $x \in \mathbb{Z}[\alpha]$ and suppose there exists a real number β associated to a generalized PCF β-map with m intervals and postcritical length n that has a Galois conjugate of the form $x+\epsilon$ for some $\epsilon \in \mathbb{C}$ with $|\epsilon| \leq 1$.

Then β is the root of the associated Parry polynomial $P_{\beta,E}$;

$$0 = z^{n+1} - (a_0 z^n + a_1 z^{n-1} + \cdots + a_n) - 1,$$

where $a_i \in \{-2, 0, 2\}$. Hence $(x+\epsilon)$ is also a root of $P_{\beta,E}$:

$$0 = (x+\epsilon)^{n+1} - (a_0(x+\epsilon)^n + a_1(x+\epsilon)^{n-1} + \cdots + a_n) - 1.$$

Therefore

$$\begin{aligned} 1 - x^{n+1} + a_0 x^n + \cdots + a_n = (x+\epsilon)^{n+1} - x^{n+1} - \big(a_0((x+\epsilon)^n - x^n) \\ + a_1((x+\epsilon)^{n-1} - x^{n-1}) + \cdots + a_{n-1}((x+\epsilon) - x)\big). \end{aligned}$$

We have $1 - x^{n+1} + a_0 x^n + \cdots + a_n \in \mathbb{Z}[\alpha]$, so $c \leq |1 - x^{n+1} + a_0 x^n + \cdots + a_n|$. Then by the triangle inequality,

(11)

$$\begin{aligned} &c \leq |1 - x^{n+1} + a_0 x^n + \cdots + a_n| \\ &\leq |(x+\epsilon)^{n+1} - x^{n+1}| + |a_0||(x+\epsilon)^n - x^n| + |a_1||(x+\epsilon)^{n-1} - x^{n-1}| \ldots |a_{n-1}||(x+\epsilon) - x|. \end{aligned}$$

We now restrict to the case $|x| \geq 1$. For any $k \leq n+1$, by the binomial theorem, the triangle inequality, and $|\epsilon| \leq \frac{1}{n+1}$,

(12)

$$\begin{aligned} |(x+\epsilon)^k - x^k| = & \left| \sum_{i=1}^{k} \binom{k}{i} x^{k-i} \epsilon^i \right| \leq \sum_{i=1}^{k} \left| \binom{k}{i} x^{k-i} \epsilon^i \right| \\ \leq & \sum_{i=1}^{k} \left| \frac{k^i}{(k-i)!} x^{k-i} \frac{1}{(n+1)^{i-1}} \epsilon \right| = \sum_{i=1}^{k} \left| \left(\frac{k}{n+1} \right)^{i-1} \frac{k}{(k-i)!}\, \epsilon\, x^{k-i} \right| \\ \leq & \epsilon k |x|^{k-1} \sum_{i=1}^{k} \frac{1}{(k-i)!} = \epsilon k |x|^{k-1} \sum_{i=0}^{k-1} \frac{1}{i!} \\ \leq & \epsilon k |x|^{k-1} \sum_{i=0}^{\infty} \frac{1}{i!} = \epsilon k |x|^{k-1} e. \end{aligned}$$

Combining equations (11) and (12) yields

$$\begin{aligned} c &\leq \epsilon(n+1)e|x|^n + |a_0|\epsilon n e|x|^{n-1} + \cdots + |a_{n-1}|\epsilon 1|x|^0 e| \\ &\leq \epsilon(n+1)e|x|^n \left(1 + |a_0| + \cdots + |a_{n-1}|\right) \\ &\leq \epsilon(n+1)e|x|^n \left(1 + 2n\right). \end{aligned}$$

Thus for $|x| \geq 1$,

$$\frac{c}{e(1+2n)(n+1)|x|^n} \leq \epsilon.$$

We now restrict to the case $|x| \leq 1$. In this case, the estimate (12) becomes

$$\left|(x+\epsilon)^k - x^k\right| \leq \epsilon k e. \tag{13}$$

Combining equations (11) and (13) yields

$$c \leq \epsilon(n+1)e(1+|a_0|+|a_1|+\cdots+|a_{n-1}|) \leq \epsilon(n+1)e(1+2n).$$

Hence, for $|x| \leq 1$,

$$\frac{c}{(n+1)(1+2n)e} \leq \epsilon.$$

□

Proof of the Gap Theorem 4. In view of Theorem 7.1, it suffices to classify the discrete subgroups of $\mathbb{C}$. The classification of discrete subrings of $\mathbb{C}$ is well-known, and we include it for completeness: firstly, because it is a discrete additive subgroup, it is either $\mathbb{Z}$ or a lattice of rank 2. If it is the latter case, let $\{1, a\}$ be a basis of the lattice, then a must be an algebraic integer of degree 2, in other words, the discrete subring must be of the form $\mathbb{Z}[a]$ where a is an algebraic integer of degree 2, hence it must be contained in the ring of integers of an algebraic field of degree 2. There are only 4 such rings of integers that contains some element not on the real line and has absolute value less than 2, which are $\mathbb{Z}[\sqrt{-1}]$, $\mathbb{Z}[\sqrt{-2}]$, $\mathbb{Z}[\sqrt{-5}]$, or $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. □

8. Ω_2^{cp} AND Ω_2^{pcf} ARE NOT EQUAL

In this section we prove the Two Thurston Sets Theorem 5, that Ω_2^{cp} and Ω_2^{pcf} are not equal. A finite approximation of Ω_2^{cp} is shown in Figure 2, and a finite approximation of Ω_2^{pcf} is shown in Figure 5.

As outlined in section §3.1, a point $z \in \mathbb{D}$ is in Ω_2^{cp} if and only if 0 is in the limit set of the iterated function system generated by f_z, g_z, where

$$f_z : x \mapsto zx + 1, \quad g_z : x \mapsto zx - 1.$$

Denote the alphabet $\{f_z, g_z\}$ by $\mathcal{F}_z$ and denote the alphabet of inverses $\{f_z^{-1}, g_z^{-1}\}$ by $\mathcal{F}_z^{-1}$. For a word $w = w_1, \ldots, w_n$ in the alphabet $\mathcal{F}_z$ or in the alphabet $\mathcal{F}_z^{-1}$, define the action of w on $\mathbb{C}$ by

$$w(x) = w_n \circ \cdots \circ w_1(x).$$

Lemma 8.1. *Fix $z \in \mathbb{D} \setminus \{0\}$. If there exists $n \in \mathbb{N}$ such that*

$$\min\left\{|v(0)| : v \in (\mathcal{F}_z^{-1})^n\right\} > \frac{1}{1-|z|},$$

then $z \notin \Omega_2^{cp}$.

Proof. Suppose $z \in \mathbb{D} \cap \Omega_2^{cp}$. Then 0 is in the limit set Λ_z. Since $\Lambda_z = f_z(\Lambda_z) \cup g_z(\Lambda_z)$, it follows that Λ_z is fixed by taking the union of the images of Λ_z under all words of length n, for any $n \in \mathbb{N}$:

$$\Lambda_z = \bigcup_{w \in (\mathcal{F}_z)^n} w(\Lambda_z).$$

Hence, for any $n \in \mathbb{N}$, each point in Λ_z is the image of a point Λ_z under some word in $\mathcal{F}_z$ of length n. In particular, 0 is the the image of a point in Λ_z under some word in $\mathcal{F}_z$ of length n. Since $\Lambda_z \subset B_{\frac{1}{1-|z|}}(0)$ by Lemma 3.1, this implies that for any $n \in \mathbb{N}$,

$$\left(\bigcup_{v \in (\mathcal{F}_z^{-1})^n} v(0) \right) \cap B_{\frac{1}{1-|z|}}(0) \neq \emptyset.$$

$\square$

Proof of the Two Thurston Sets Theorem 5. Let β be the leading root of the polynomial

$$P(x) = x^{12} - 2x^{11} + x^{10} - 2x^9 + x^8 - 2x^7 + 2x^6 - 2x^5 + 4x^4 - 2x^3 + 4x^2 - 4x + 2.$$

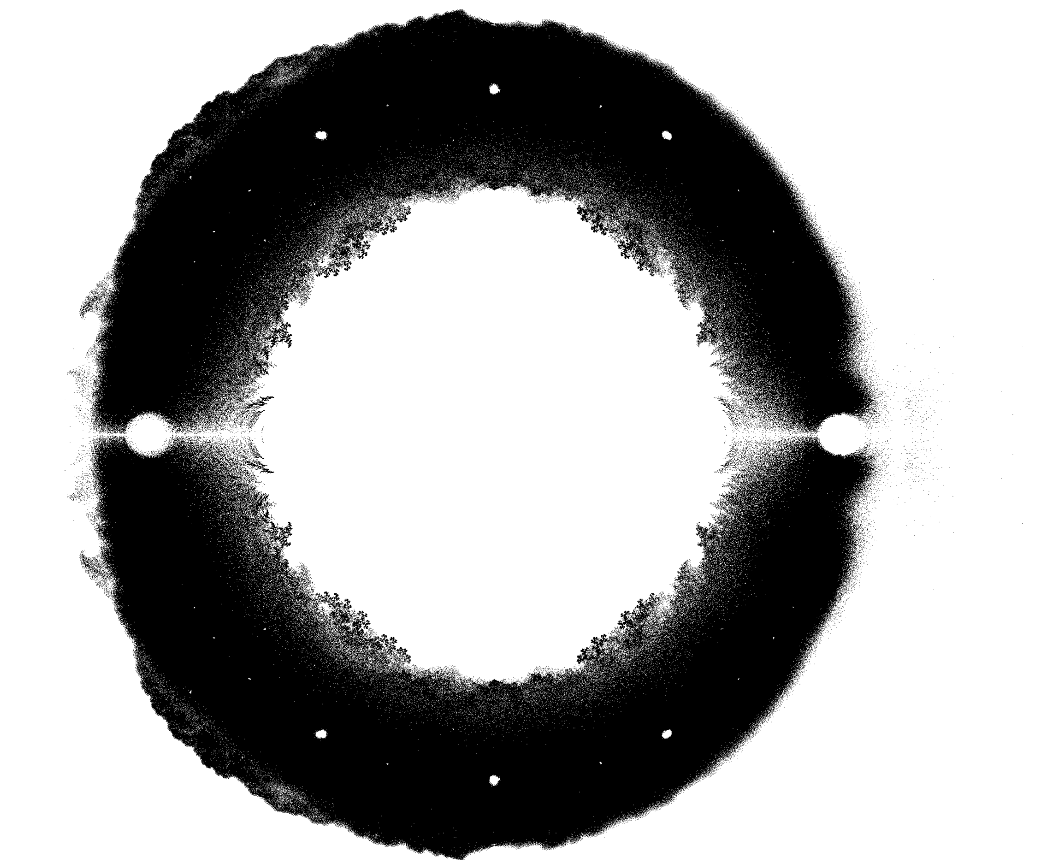

FIGURE 5. An approximation of the preperiodic Thurston set, Ω_2^{pcf}, consisting of the roots of all minimal polynomials associated to postcritically finite tent maps for which the sum of the pre-critical length and the period is at most 22. This set is shown in red in Figure 3. Compare this with the Thurston set Ω_2^{cp} in Figure 2, and note in particular the difference in a large neighborhood of the point 1.

(The value of β is approximately 1.94848.) By computation, the minimal β-itinerary is

$$w = 1000011100(101000)^{\infty}.$$

Because P is irreducible, any roots of P must be in Ω^{pcf}. Let p be the root of P with approximate value

$$p \approx 0.5393738531461442 + 0.4050155839374199i.$$

Since $|p|$ is approximately 0.674509, $p \in \mathbb{D} \cap \Omega_2^{pcf}$.

Let $\mathcal{F}_p^{-1}$ be the alphabet consisting of the two maps f_p^{-1} and g_p^{-1}, where

$$f_p^{-1} : x \mapsto \frac{x-1}{p}, \quad g_p^{-1} : x \mapsto \frac{x+1}{p}.$$

Computation shows that

$$\min \left\{ |v(0)| : v \in (\mathcal{F}_p^{-1})^5 \right\} \approx 4.3792,$$

which is much bigger than $\frac{1}{1-|p|} \approx 3.07228$. Consequently, Lemma 8.1 implies that $p \notin \Omega_2^{cp}$. $\square$

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