

This book is for anyone who wishes to illustrate their mathematical ideas, which in our experience means everyone. It is organized by *material*, rather than by subject area, and purposefully emphasizes the *process* of creating things, including discussions of failures that occurred along the way. As a result, the reader can learn from the experiences of those who came before, and will be inspired to create their own illustrations.

Topics illustrated within include prime numbers, fractals, the Klein bottle, Borromean rings, tilings, space-filling curves, knot theory, billiards, complex dynamics, algebraic surfaces, groups and prime ideals, the Riemann zeta function, quadratic fields, hyperbolic space, and hyperbolic 3-manifolds. Everyone who opens this book should find a type of mathematics with which they identify.

Each contributor explains the mathematics behind their illustration at an accessible level, so that all readers can appreciate the beauty of both the object itself and the mathematics behind it.



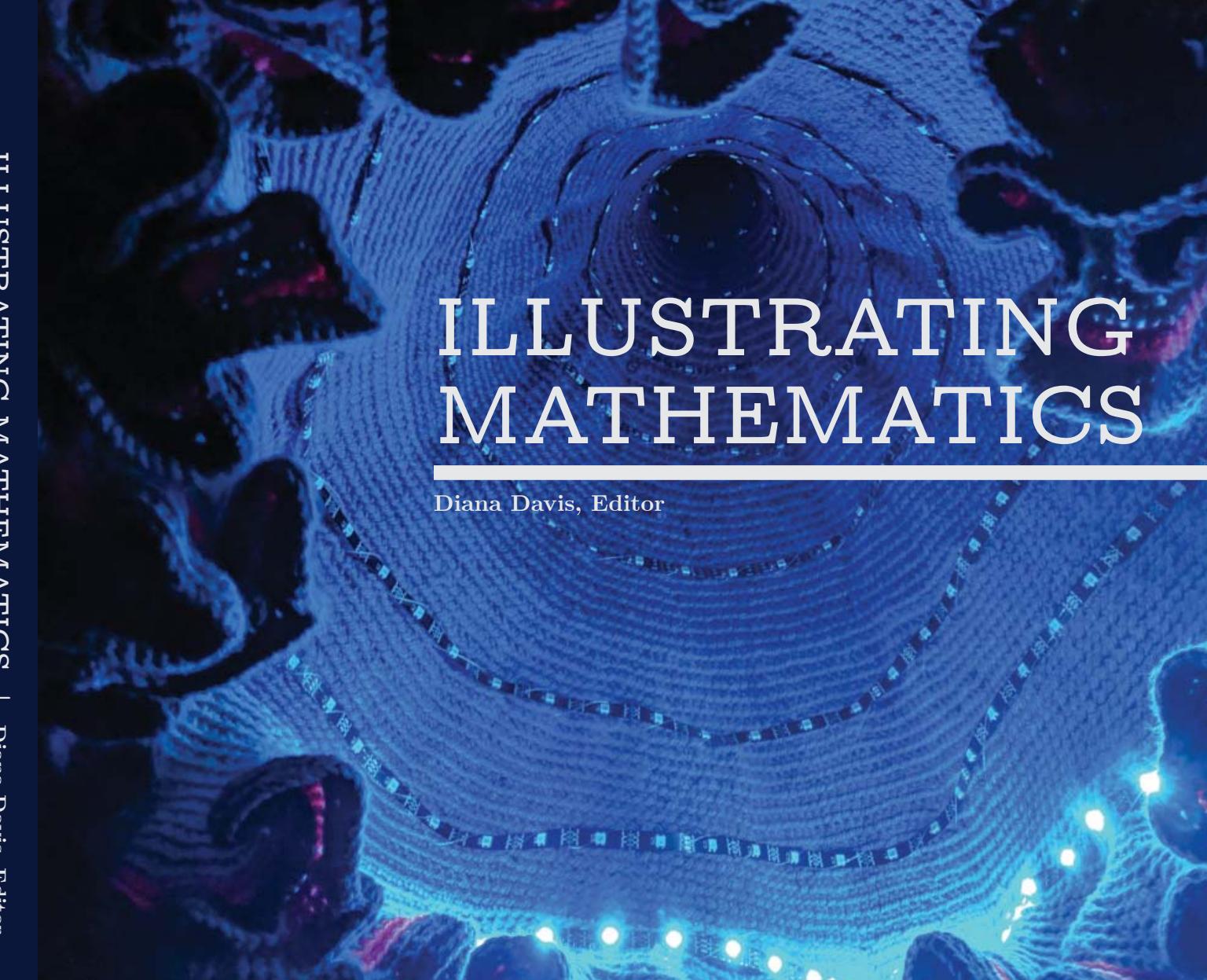
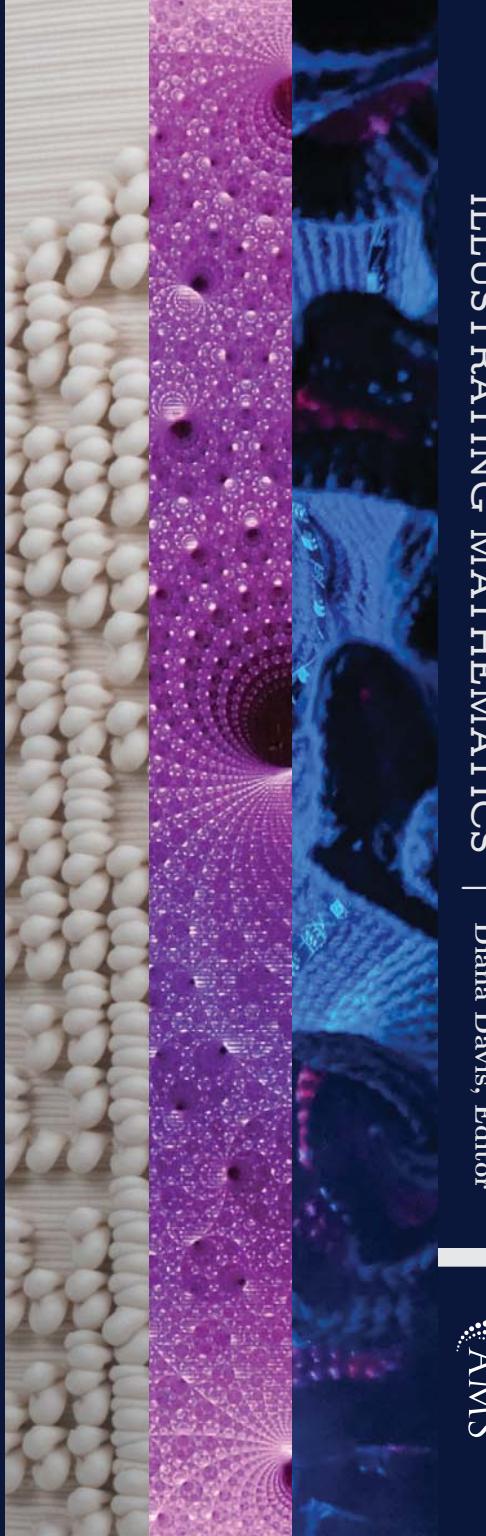
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ILLUSTRATING MATHEMATICS | Diana Davis, Editor

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ILLUSTRATING MATHEMATICS

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CONTENTS

Introduction	1
Drawing	4
Paper & fiber arts	14
Laser cutting	32
Graphics	48
Video and & virtual reality	66
3D printing	82
Mechanical constructions & other materials	128
Multiple ways to illustrate the same thing	150
Acknowledgments	163
Image credits	164
Consent to publish	166
Index	169

INTRODUCTION

It is said that an interpreter – a person who translates from one language to another – does not exchange words for words, but rather exchanges meaning for meaning. As mathematicians, a large part of our job is to explain things to others. We may use words and symbols to do this, but our job is not primarily to convey words and symbols; like an interpreter's, it is to convey *meaning*.

There are many ways to communicate meaning. As the great mathematician William Thurston said, “Mathematics is an art of human understanding. Mathematical concepts are abstract, so it ends up that there are many different ways that they can sit in our brains. A given mathematical concept might be primarily a symbolic equation, a picture, a rhythmic pattern, a short movie – or best of all, an integrated combination of several different representations.”

The purpose of this book, then, is to help you find a good representation of the mathematical concept you wish to illustrate. This book does three things:

- Showcases the great variety of materials for illustrating mathematics,
- Gives voice to people's stories about illustrating their mathematics, so that we can learn from their experience, and
- Shows the variety of ways that different people use the same materials in very different ways.

In addition, it will introduce you to many of the amazing people who spent time at the Institute for Computational and Experimental Research Mathematics (ICERM) in fall 2019 for the Illustrating Mathematics program, in an attempt to capture some of the creative and generous spirit that flowed through our days there.

Diana Davis
May 1, 2020
Bures-sur-Yvette, France



Illustrating Mathematics semester program participants, ICERM, Fall 2019.

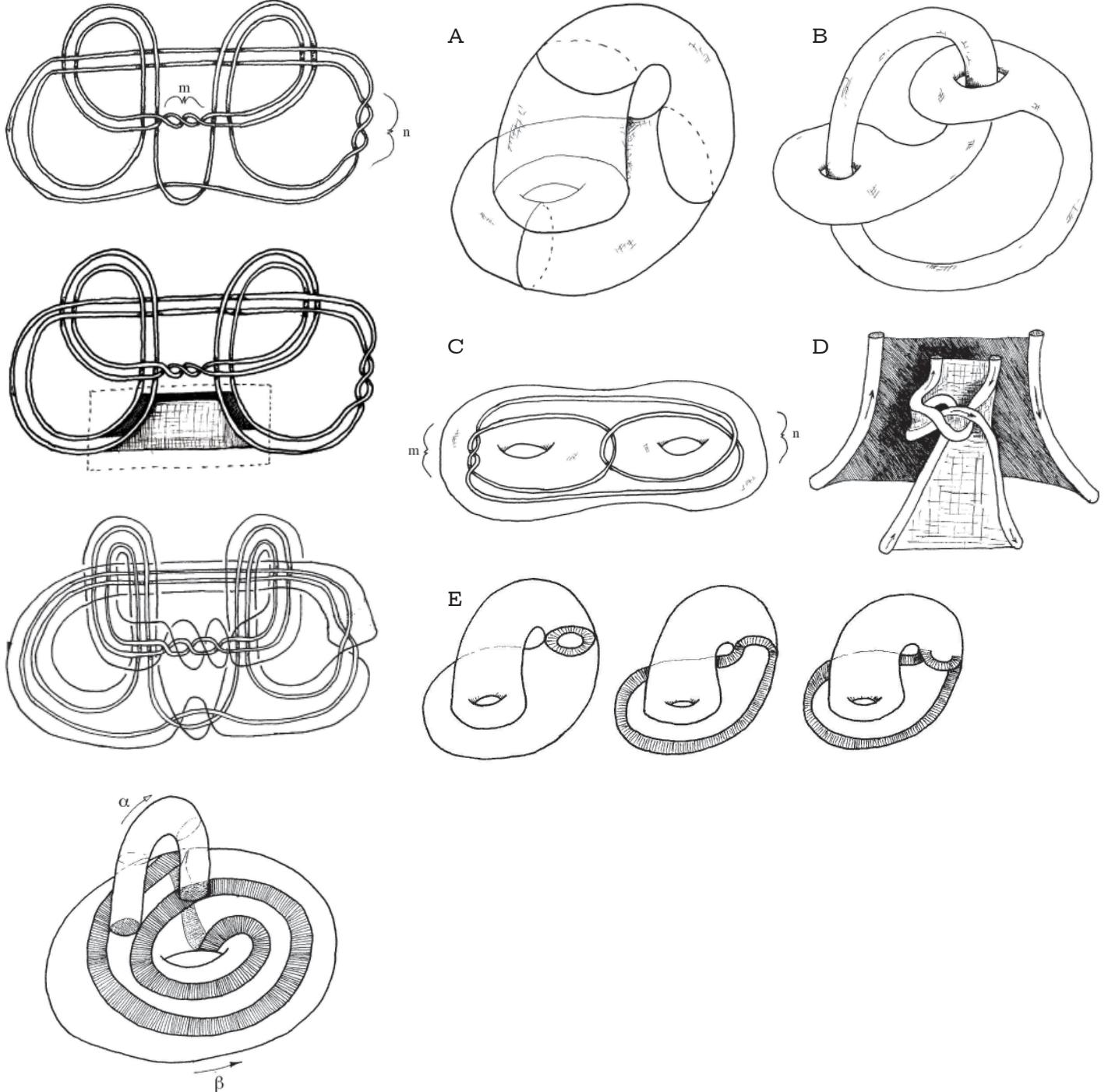


DRAWINGS

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The adage that a picture is worth a thousand words is certainly true in mathematics, in which one carefully chosen figure can eliminate the need for many lines of exposition. For most of us, our most frequent way of illustrating mathematics is by hand – on chalkboards, scrap paper, paper napkins or, increasingly, whiteboards and electronic tablets.



These drawings are from my first solo paper, “Bundles, handcuffs, and local freedom.” They illustrate the existence of a knot in the 3-sphere whose complement is hyperbolic, that admits a fibration over the circle, and whose group contains a subgroup that is locally free (finitely generated subgroups are free) and not free. This answered a question from James Anderson: He had observed that if such a thing could not exist, then there would be counterexamples to Thurston’s Virtual Fibration Conjecture (which we now know to be true due to the remarkable work of Ian Agol and Dani Wise).

The fundamental group of the 2-complex X (A) contains a subgroup that is locally free and not free. The way to construct the knots is to find a fibered knot in the complement of X so that X ’s group injects into the knot’s. The complement of X is the handlebody (B). The knot is inside the handlebody (C). The knot is built by taking a square knot (which is fibered), cabling (which is again fibered), and then “plumbing.” Plumbing is the process of gluing a twisted band along a square in a fiber (D). The rest of the figures are part of the proof that the knot is hyperbolic, and that you can perform surgeries to get closed hyperbolic fibered manifolds with groups having locally free non-free subgroups.

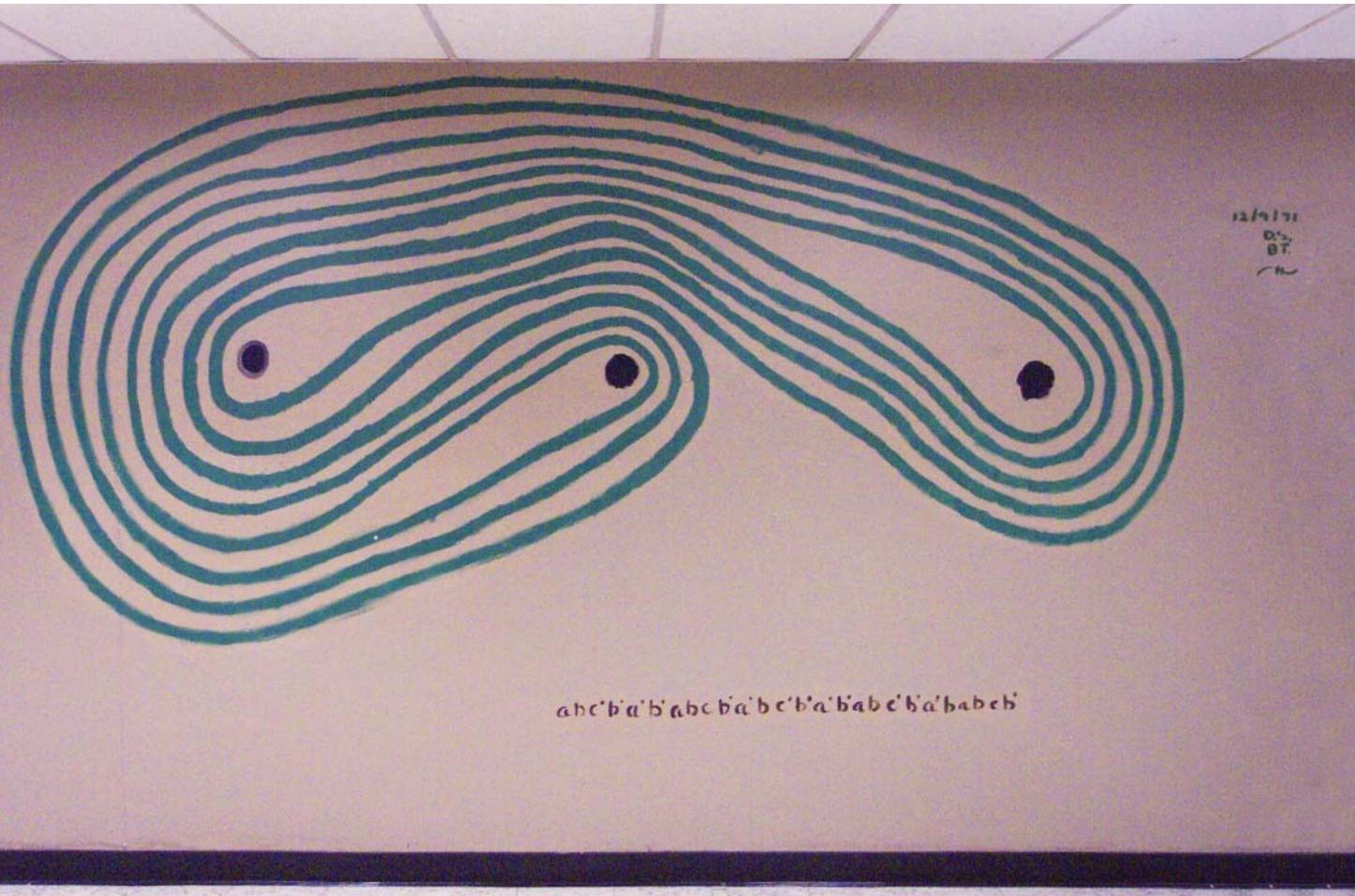
One year while I was in graduate school, Bob Gompf was teaching a course in 4-manifolds that I had lost the thread of. I didn’t want to be rude, so I kept attending. So there I was in class, doodling and thinking about Anderson’s question. I knew that connect sums, cabling, and plumbing preserved fibering. I sketched the complement of X and saw the square knot there. I cabled it for fun. I wanted the result to be hyperbolic, so I needed to do something else. So I plumbed a little band on to get rid of the essential torus that was ruining hyperbolicity, and I left class with a theorem. It took a while to prove that the examples were hyperbolic, but I had found them. Getting lost in a lecture isn’t always bad!

I drew the pictures by hand since I didn’t know how to do it any other way!



AUTUMN KENT
University of Wisconsin
pen drawings

Images reprinted by permission from Springer Nature: Geometriae Dedicata, 106, *Bundles, handcuffs, and local freedom*, Autumn Kent, 2004, <https://www.springer.com/journal/10711/>.



In December of 1971, the grad students at Berkeley invited me (I was also heavily bearded with long hair) to paint math frescoes on the corridor wall separating their offices from the elevator foyer. While I was milling around before painting, one of the grad students, Bill Thurston, came up to ask, “Do you think this is interesting to paint?” It was a complicated maze-like looking smooth one-dimensional object encircling three points in the plane. I asked, “What is it?” and was astonished to hear, “It is a simple closed curve.” I said, “You bet it’s interesting!”

So we proceeded to spend several hours painting this curve on the wall. It was a great learning and bonding experience. For such a curve drawing to look good, it has to be drawn in sections of short, parallel, slightly curved strands that are subsequently smoothly spliced together. It was natural and automatic to do it in terms of bunches of strands at a time – as an approximate foliation – and then connect them up at the end. Thus some years later in '76, when Bill gave an impromptu three-hour lecture about his theory of surface transformations, I absorbed it painlessly at a heuristic level after the experience of several hours of painting in '71.

When I asked how Thurston got such curves, he said by successively applying to a given simple curve a pair of Dehn twists along intersecting curves. The “wall curve painting,” two meters high and four meters wide, dated and signed, lasted on that Berkeley wall with periodic restoration for almost four decades before finally being painted over.



DENNIS SULLIVAN &
WILLIAM THURSTON

University of California, Berkeley
painted mural

Further information: Lee Mosher, *What Is... A Train Track?* Notices of the AMS 50(3), March 2019, pp. 354–356.

A large, abstract geometric sculpture made from numerous folded and crumpled pieces of paper. The sculpture has a complex, fractal-like structure with many sharp edges and facets, resembling a polyhedron or a collection of overlapping Platonic solids. The lighting is dramatic, highlighting the texture and depth of the paper's folds.

PAPER & FIBER ARTS



Paper is by far the most abundant resource in most mathematicians' offices, and it is our most readily available tool for creating mathematical illustrations. Most of us created the Platonic solids out of paper at some point in our early education, and have created numerous other models throughout our mathematical careers. For creating objects that people will handle and manipulate, and especially for those that require some stretch, sewn or knitted fabrics may work better than paper.



The *triacontahedron* is a Catalan solid that has thirty congruent rhombic faces. Its dual polyhedron is the icosidodecahedron, which is the Archimedean solid obtained by deeply truncating the dodecahedron or, alternatively, deeply truncating the icosahedron.

To make a paper model of the triacontahedron, one can easily find a *net* on the internet – a flat connected version of all the faces that, when assembled, gives the polyhedron – to download, print, cut out, and tape together. In this case, I wanted a version where the faces are decorated, and where I could print out or laser-cut each piece. To be able to do that cleanly, I affixed tabs to the edges. Yet tabs introduce an issue, because they require space. Acute angles between adjacent faces of any given net might not allow for the insertion of tabs. I therefore dissected the net into pieces so that on each piece no adjacent faces have exterior edges making acute angles. To make matters more mathematically interesting and less demanding to produce, I made all of the pieces congruent.

Note that this dissection condition came from the medium: paper is an inexpensive way to construct a solid, and it is easy to decorate. I added the congruence condition so that the number of unique files created for the printer or laser cutter is kept to a minimum. Each face could be cut individually with tabs on all four sides, but it is preferable to have more faces connected, because then the paper is automatically hinged between these faces, which constrains their relationship in space and creates structural integrity.

With ten three-face pieces cut and ready, it seems simple to assemble the triacontahedron. Surprisingly, there are many ways to begin assembling with the pieces and get stuck so that the desired polyhedron cannot be completed. The reader might wonder why we don't use five- or six-face pieces, for then only six or five copies would have been needed, respectively. As with many projects in mathematical art, this enterprise has proven to provide mathematically interesting questions as well as illustrations.

For more information on the rhombic triacontahedron:
<http://mathworld.wolfram.com/RhombicTriacontahedron.html>



CAROLYN YACKEL

Mercer University

laser-cut paper



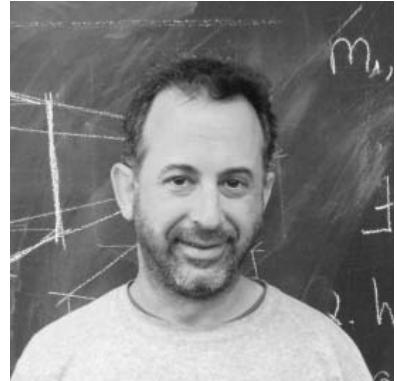
We are making paper with constant negative curvature, in order to demonstrate various features of the hyperbolic plane. A flat piece of paper has zero curvature, a sphere has constant positive curvature, and our paper has constant negative curvature. Our project builds on earlier work of Roger Alperin, Barry Hayes, and Robert Lang.

We hope that this paper can serve as an instructional tool for college-level geometry courses. Our goal is to explore hyperbolic versions of many of the things that we can make from flat paper: straight-edge and compass constructions, tilings, origami, and so on. Paper is a natural medium for this, as, unlike fabric, it has very little elasticity (so the intrinsic geometry is preserved when it is smoothly deformed) and it holds a crease (so we can generalize ideas and methods of flat paper folding to get new hyperbolic designs).

We learned a lot of math in the process of creating this paper! For example, we needed to learn how to isometrically parameterize the pseudosphere (top right) and related constant-curvature surfaces such as Dini's surface (left) from the upper half-plane model of hyperbolic space. This project also enabled us to experience hyperbolic geometry in a physical way, which offered interesting surprises despite our existing theoretical understanding. For instance, we knew from local isometry theorems that the paper should fold over onto itself and fit perfectly, yet when holding the paper in our hands it was hard to believe it would actually work. (It does!)

We are still trying to figure out the best way to make this paper. Our first attempt used papier mâché, but it was too stiff. Following advice of John Edmark, Rotem Tamir, and others, we improved our process by making flat strips from pulp, then introducing curvature by pressing the strips onto a pseudospherical mold (top right) while they were still wet. We hope to make larger, sturdier, more robust paper by starting with higher-quality fibers. On the other hand, we have learned that Alba Málaga Sabogal has had success making hyperbolic paper from toilet paper fibers.

Video showing local isometries, rotating and translating paper on the surfaces:
<https://www.youtube.com/watch?v=JRd928WpY9w>



**AARON ABRAMS &
STEPAN PAUL**

Washington & Lee University;
Harvard University

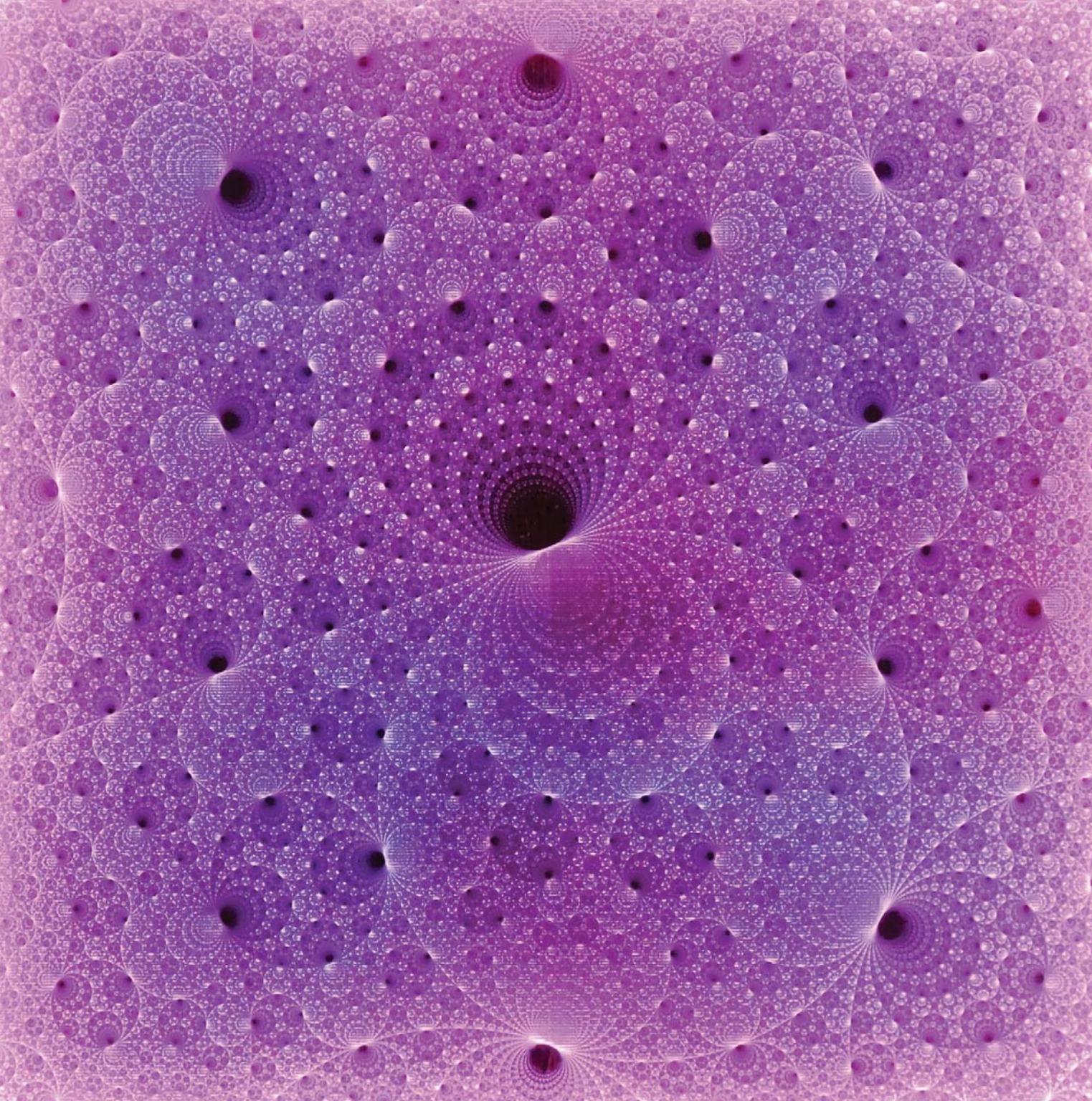
*homemade paper, 3D-printed
plastic, molded silicone*

LASER CUTTING



Since mathematical objects are infinitely precise, it is always a little disappointing to cut out some pieces with scissors or a bandsaw, fit them together, and find that they are not quite right. The laser cutter eliminates much of this issue because it can reproduce a resolution of up to 600 DPI, creating cuts, folds, curves, tabs, holes, and other mathematical elements in, for all practical purposes, *exactly* the right spot.

Of all the techniques presented here, the laser cutter is the most dangerous, as it can melt and warp materials, create toxic fumes, and start fires. It must be watched at all times when in use, and it is not suitable for children. And yet, if you wish to create a physical manifestation of any 2D object, the laser cutter is almost certainly the best tool.



There are many possibilities for the shape of the universe in which we live. To understand these better, topologists often imagine a surface that divides the universe into simpler regions. Imagine a surface that is somewhat translucent and is sitting in the middle of one of these possible universes. As we look out, we see an amazing picture. Because of the way both the universe and the surface fold back on themselves, we see different shades of light and dark wherever we look, creating an amazingly complex and self-similar pattern.

In this piece, we – I, Saul Schleimer, and Henry Segerman – show one such view of one such possible universe, laser-cut in acrylic and side-lit with LED strips. In particular, this picture represents a small part of the boundary of *3D hyperbolic space*. We view this space as coming from the universal cover of some 3-manifold, containing a topologically significant surface. We created a depth map based on how many lifts of the surface you pass through as you move out a fixed distance from some point of hyperbolic space toward the boundary.

This is a laser-etched piece of acrylic, with carefully designed lighting to illuminate the etching. I had never tried edge-lighting an acrylic relief, and thought this would be a perfect way to create a visualization of our results. I was surprised at how beautiful it turned out!

There were a lot of experiments to get the lighting effect correct. First, I built a custom frame that hides an LED strip light. I then learned that the acrylic needed to have a clear boundary where it met the frame so that light propagated through it. Then I needed to cut the acrylic again, make modifications to the frame, and so on.



DAVID BACHMAN

Pitzer College

laser-cut acrylic

GRAPHICS

○ ○ ○

One of the most tantalizing mathematical illustrations ever is what I like to call the “asterisk picture” of the Mandelbrot set. Mathematicians had the idea of applying the function $f(z) = z^2 + c$ over and over, starting with the input $z = 0$, and using each output as the next input. They wanted to know what happens to the image, for different points c in the complex plane. Using a very early computer in 1978, they set it to work on this problem, and for each representative point in the box $[-2, 2] \times [-2, 2]$, they had the computer print an asterisk if the image stayed bounded, and print nothing if the image went off to infinity. The result was a shape that no one had ever seen before. To this day and beyond, exploring mathematics through computer graphics continues to elicit a sense of surprise and awe.



For years, I have struggled to find good techniques for coloring mathematical surfaces with patterns. Coloring a sphere is easy, because it's so round and regular, but the bands in this image have not been easy to color well. My goal is to find *conformal coordinates* on this shape, which means that all the angles in the source pattern are portrayed correctly on the surface. Conformal coordinates make patterns more recognizable. I heard Steve Trettel's talk "What does a torus look like?" and that sent me back to an old project, which involved coloring this set of bands with a wallpaper pattern.

The innovation shown here comes from making the shape in software called Grasshopper (a plug-in for an architectural design software called Rhino). The good news is that I was able to paste a nice wallpaper pattern onto the surface and then make it look nice in Photoshop. The bad news is that the coordinates are not conformal: the pattern is indeed distorted.

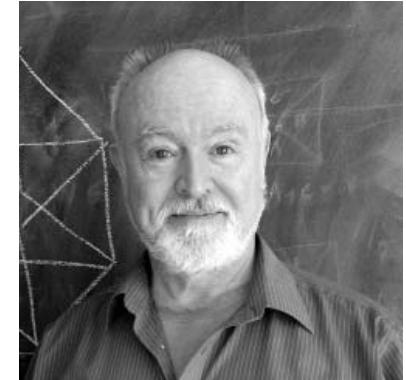
Grasshopper has a growing community of mathematician users who support each other in solving problems, which was very helpful: I had been coloring this shape by coloring a gigantic mesh, with approximately one vertex for each pixel of the coloring pattern. It really bogged down the computer. When I asked the Grasshopper community about it, Daniel Pinker, the creator of Kangaroo (another Rhino plug-in), solved my problem perfectly. I'll be working with this new technique for a long time.

It is still a long road to finding conformal coordinates on this particular structure, but I learned a lot about coloring tori. Surprisingly, the best way to find coordinates on donut shapes is to look for them in four dimensions, where they lie nicely on the 3D sphere.

The alert reader will see that there's an error in the image shown here: the pattern fails to match on the left side of the higher horizontal band. After all my Facebook friends had told me how beautiful it was, two mathematical artists looked at it for two seconds before saying, "Oh, but there's a mistake right there!" It's wonderful to be working in a community of people who take time to look closely at things.

Further information:

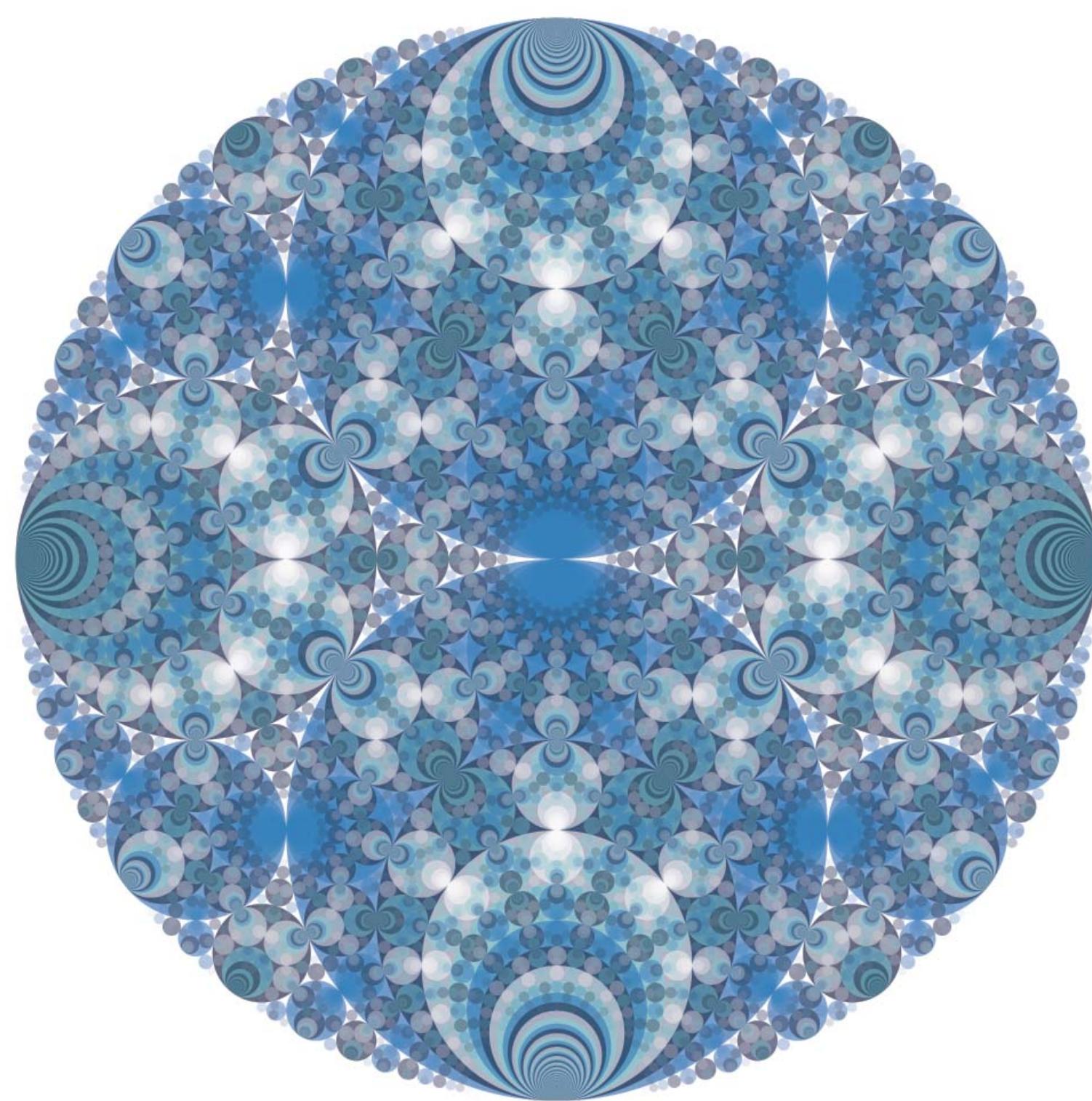
Wilder Boyden and Frank A. Farris, *Polyhedral Symmetry from Bands and Tubes*, Journal of Mathematics and the Arts, to appear.



FRANK A. FARRIS

Santa Clara University

computer-generated graphic



This image shows part of an orbit of a group of matrices over the ring of integers in an imaginary quadratic field. In particular, it is the projective special linear group over the Gaussian integers. These matrices act as Möbius transformations, taking circles to circles. Asmus Schmidt used such arrangements of circles as a way to break up the complex plane into pieces, for the purposes of a continued fraction algorithm.

Schmidt's original work took place before the advent of powerful computer graphics systems, and his papers included some figures showing only half a dozen or a dozen circles, illustrating only one level of the recursive structure. I wanted to more fully understand the structure, and I wanted to find out what happened in other imaginary quadratic fields. Using computer graphics allowed me to work interactively, changing parameters and redrawing frequently. It was also important to me to be able to print out the pictures and work with colored pencil, ruler, and compass on top of them. For example, in some pictures the human eye picks up "ghost circles" that appear to fit in the geometry but are missing. I measured these with a ruler and compass to conjecture their exact form, and then proved that they exist.

Easy access to exploring these pictures led me to conjecture and prove a variety of things about the relationships between the arithmetic of the imaginary quadratic field and the pictures themselves. For example, the circle arrangement is connected if and only if the field is Euclidean.

The experience of programming the pictures was itself enlightening: for example, a naive algorithm exploring the Cayley tree results in varying resolution in different parts of the picture, and inadvertent congruence conditions add or remove extra layers of circles. Often, errors in coding led to useful insights. Later on, I worked harder to make the pictures more beautiful, and learned about color choices and vector graphics, among other things.

Further information: Katherine E. Stange, *Visualizing the Arithmetic of Imaginary Quadratic Fields*, International Mathematics Research Notices, Volume 2018, Issue 12, June 2018, pp. 3908–3938.

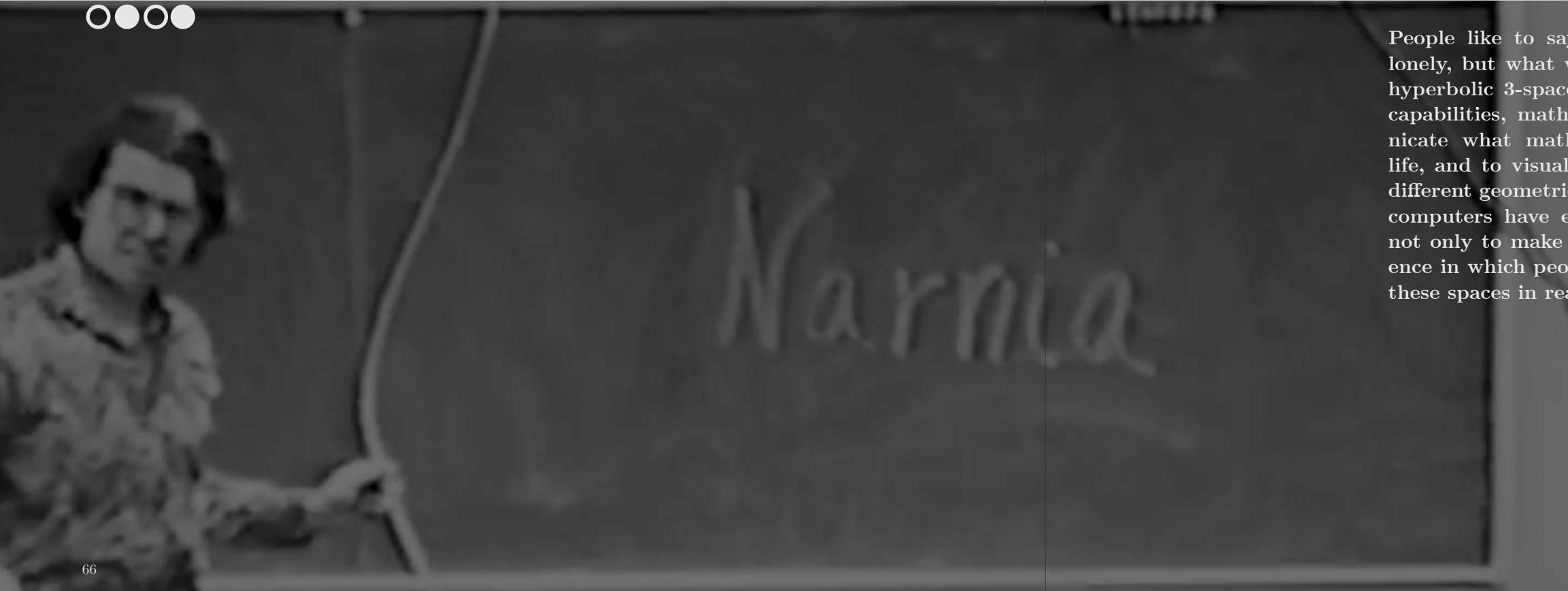


KATHERINE E. STANGE

University of Colorado Boulder

computer-generated graphic

VIDEO AND VIRTUAL REALITY



People like to say that hyperbolic space is cold, dark and lonely, but what would it *really* be like to wander around in hyperbolic 3-space? As soon as computers had the requisite capabilities, mathematicians were making videos to communicate what mathematical objects would look like in real life, and to visualize what it would be like to move through different geometries. Now that processors are much faster and computers have excellent graphic capabilities, it is possible not only to make a video, but to make an interactive experience in which people can direct their own movement through these spaces in real time, in virtual reality.



In 2018, Olga Frolkina of Moscow State University published a proof of the impossibility of packing an uncountably infinite number of Möbius strips into an infinite 3D space. Evelyn Lamb wrote about the proof for *Quanta Magazine*, and I created this animation to accompany the article. This is visual storytelling that hints at Frolkina's result rather than an accurate simulation. Since it's impossible to render an infinite number of anything, we had to make do with "a lot" of Möbius strips in a finite space.

Working in 3D simplifies and expedites certain processes, leaving more time for creativity and ambitious ideas. I would not have tackled this project if I had to draw out dozens of Möbius strips and then animate them frame by frame! Using Cinema 4D, I modeled just one strip, cloned it, and then ran a simulation. Also, in 3D software it's possible to both simulate real-world dynamics accurately and ignore them at will, so you can create situations that are believable but physically impossible – like magic!

Despite being a mathematically-inclined artist, I'm not a mathematician and was not previously aware of this type of packing problem. Although I wouldn't expect anyone to learn much solely from my artwork, I hoped that it would spark intrigue and invite people to read about this fascinating research.

Coincidentally, the difficulties I encountered in creating this illustration echoed the conclusions of the mathematical result – it really is challenging to cram Möbius strips into a 3D space (albeit for different reasons). In theory this should have been straightforward to simulate. In practice, though, all sorts of weird stuff happened. The Möbius strips would glitch and jiggle around wildly, sometimes they'd pop right through the container like ghosts through walls, and getting them to look sufficiently smushed and then pop out dramatically was no picnic either. I had to keep tweaking various parameters to get the whole system to behave. But this is a common enough occurrence – there might as well be some kind of pseudo-scientific "law": "projects that seem simple are usually much more complicated than one might anticipate."

The article with the animation: Evelyn Lamb, Möbius strips defy a link with infinity. Quanta Magazine, February 20, 2019.



OLENA SHMAHALO

Quanta Magazine

computer-generated animation

3D PRINTING



The best way to understand a 3D object is to hold it in your hands. But how can we create an accurate model of a surface or another mathematical object? A hundred years ago, mathematicians made models of surfaces out of plaster, supported by metal rods. Many mathematics departments and mathematical institutes are fortunate to have collections of such surfaces, which are beautiful and instructive. Exactly how people constructed the molds to create these surfaces is somewhat mysterious.

These days, it is possible to 3D-print objects that represent mathematical surfaces, by using math software to plot the surface and then using 3D-printing software to transform the plot into a solid, printable object. Although the process is simple in theory, 3D printing is a relatively new technology, and as such there are usually technical hurdles to resolve along the way, so any project usually requires several failed attempts before the result succeeds in realizing the mathematician's vision.



This set of 3D prints illustrates singular algebraic surfaces. I work on solutions to the problem of physically visualizing nodal singularities, where two or more pieces come together at a single point. My main motivation for 3D-printing them is to illustrate the output of the algorithm for numerical real cellular decomposition implemented in my computer program *Bertini_real*. It computes a union of “cells,” each equipped with a generic point and homotopy that can be used to compute additional points on the real part of a complex variety. This output is naturally 3D-printable.

Using fused filament fabrication (FFF) with thermoplastic polyurethane (TPU) is the right way to produce these objects for a number of reasons. FFF can be done at home or in the office with an inexpensive 3D printer, with no chemically dangerous materials or supplies. The only post-processing step is removing support material, and, since TPU is flexible, printed surfaces can have much thinner connections at nodal singularities without inevitably breaking. Furthermore, the very small earring-sized prints are robust enough to be worn daily, and the skeleton-like object is seemingly delicate but readily able to be carried in a backpack or pocket. I continue to learn about how to engage the public in my research by having cheap, non-breakable objects to show them while wearing earrings at the grocery store. TPU is a challenging material to work with, requiring a high quality hot end and a commitment to printing very slowly, but it's incredibly rewarding.

My early versions of prints of the Barth sextic and other nodal surfaces, such as those coming from the Herwig Hauser gallery of algebraic surfaces like Octldong, broke either during support removal or transport for show. Many of my most experimental polylactic acid (PLA) prints are thus doomed to live forever trapped in their support material, since it's so far inside a cavity in the surface I would destroy it in completing it, or because the material connection between pieces is so small.

Gallery of more images: <https://silviana.org/gallery>

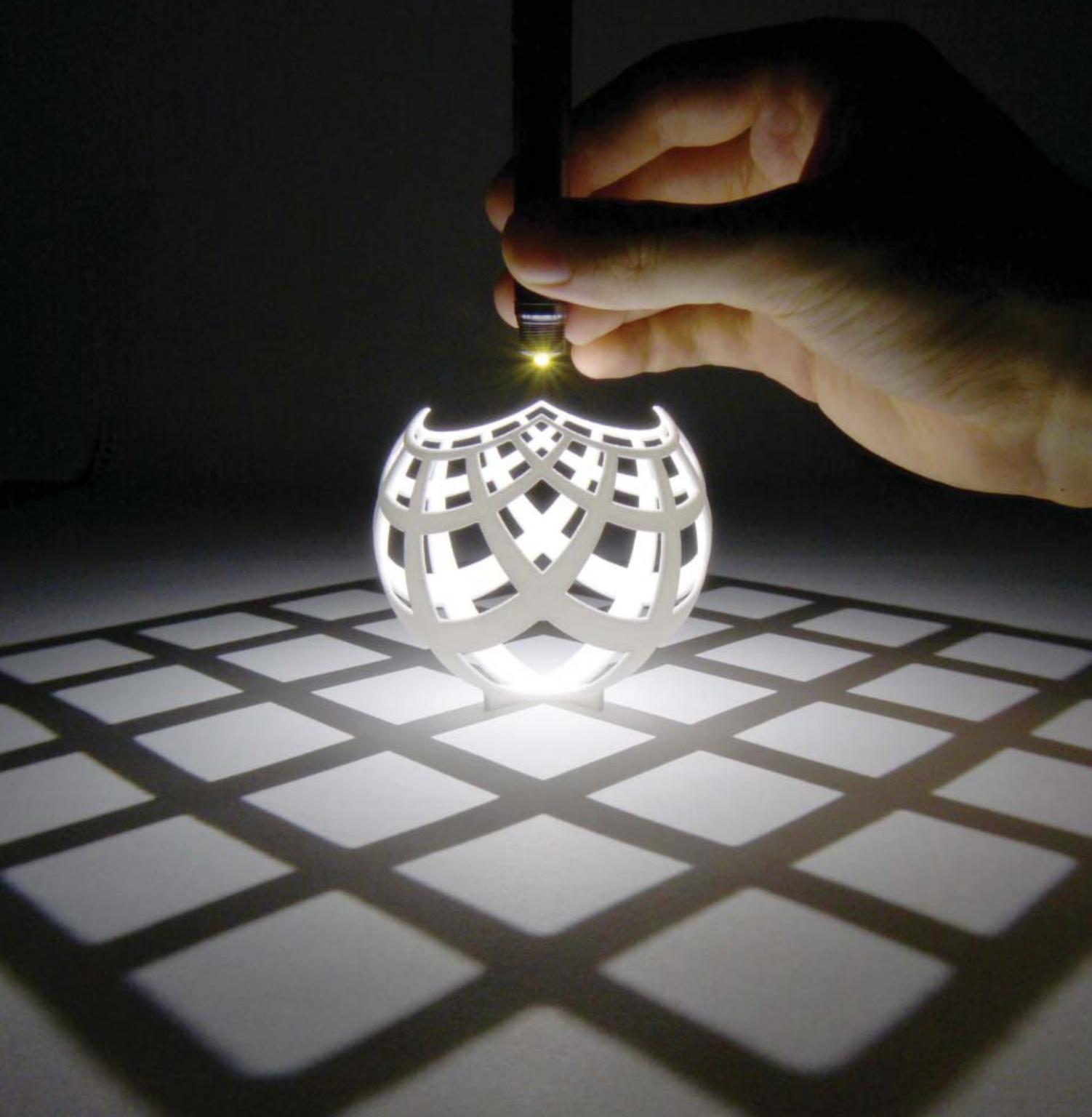
Further information: Silviana Amethyst, Daniel J. Bates, Wenrui Hao, Jonathan D. Hauenstein, Andrew J. Sommese, Charles W. Wampler, *Algorithm 976: Bertini_*
Real: Numerical Decomposition of Real Algebraic Curves and Surfaces, ACM
Transactions on Mathematical Software, 44(1), [10].
<https://doi.org/10.1145/3056528>



SILVIANA AMETHYST

University of Wisconsin–Eau Claire

3D-printed plastic



Stereographic projection is a map from the sphere to the plane. Place a sphere on the plane, and draw a straight line from the north pole of the sphere down to the plane. The line goes inside the sphere, hits the sphere, then continues on to hit the plane. Stereographic projection sends the point on the sphere to the point on the plane. This is precisely what the light rays do, so the grid pattern on the plane is the stereographic projection of the pattern on the sphere.

I chose to illustrate this to fill a gap in my book *Visualizing Mathematics with 3D Printing*. I already had some prints showing the “3D shadow” you get as the result of a higher-dimensional version of stereographic projection: from the sphere in 4D space to 3D space. So I needed to explain what stereographic projection is, and the lower-dimensional version seemed like it would be easier to understand.

3D printing has a lot of nice features – it can be very precise, which I needed in order to get a good shadow. A print is also physical – you can touch and play with it, unlike a computer graphics render. Also unlike a computer render, there’s no way to “cheat.” People are rightly suspicious that some effect could be faked with computer graphics, but they (think they) understand everything about a piece of plastic and a flashlight – so it must be a real phenomenon.

I started with the design on the plane, then made cones from the design up to the north pole. Then I used the cones to cut out the windows. There are some tricky issues to do with the thickness of the sphere. Ideally, the sphere would have zero thickness, but of course that would be impossible to print. Cutting out the cones works, but leaves very sharp angles near the north pole of the sphere. So I needed to cut a second time to remove these sharp angles.

Video of the object in action: <https://youtu.be/VX-0Laeczgk>

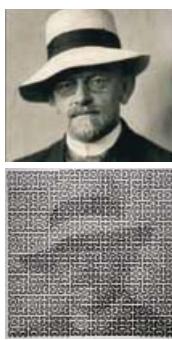
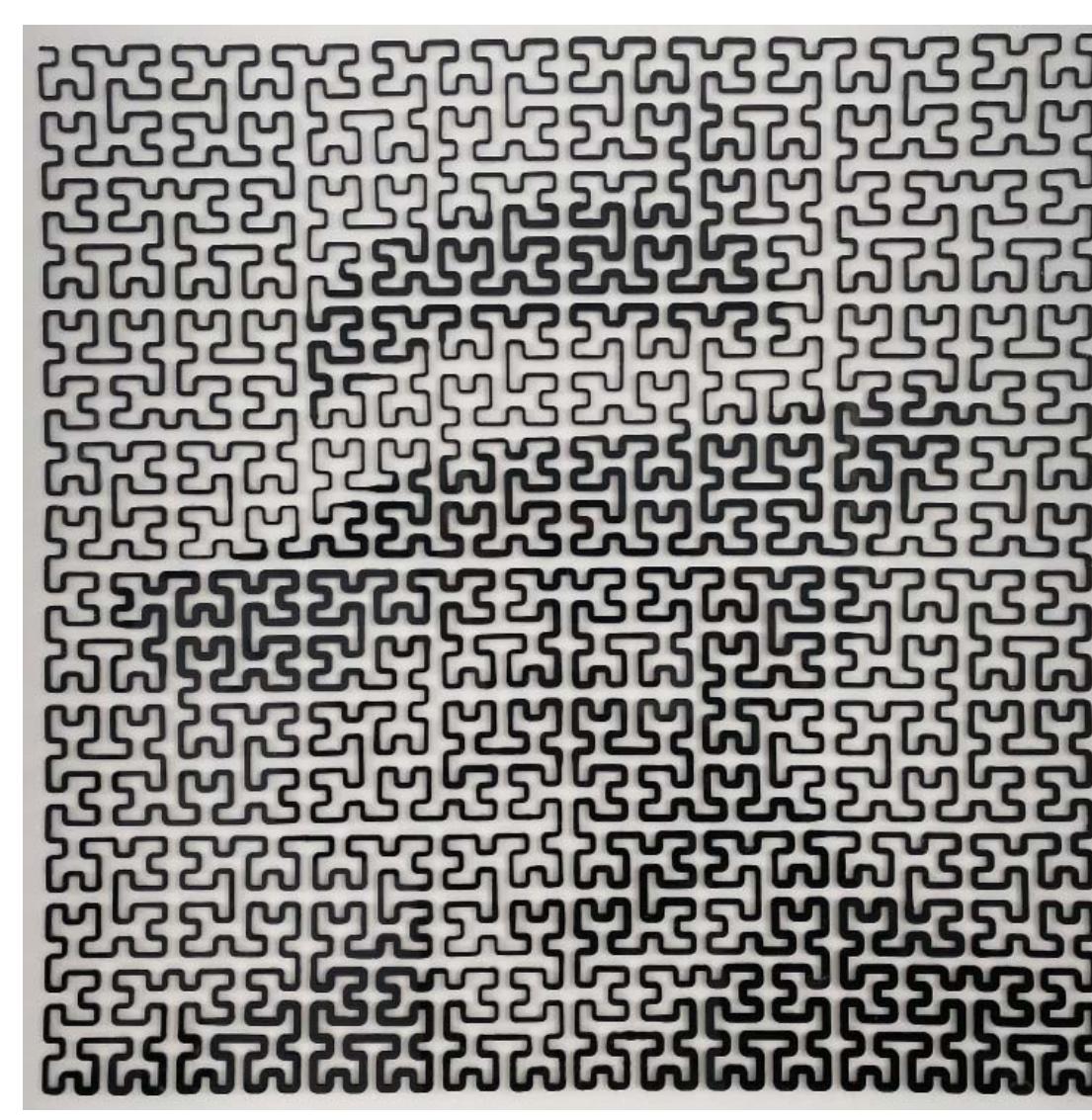
Further information: Henry Segerman, *Visualizing Mathematics with 3D printing*, Johns Hopkins University Press (2016).



HENRY SEGERMAN

Oklahoma State University

*3D-printed plastic
...plus a flashlight*



More than anything, my work illustrates the human brain's remarkable ability to fill in missing information as it works to make sense of the data it receives from the surrounding world. By varying the thickness of piping generated along the sixth iterate of Hilbert's famous space-filling curve, I have rendered Hilbert's 2D portrait as a 3D object. Using a Python script within the modeling program Rhino, my code reads in an image and uses pixel data to generate rectangular piping along the curve. The width of the rectangular profile of the piping at a point depends on the intensity of the pixel at that point; it is wider when the intensity of the pixel is lower (where the image is darker) and thinner when the intensity of the pixel is higher (where the image is brighter).

I am new to 3D printing, so I chose this medium in large part to gain expertise in the modeling and printing processes of a 3D object. I worked in stages, first learning how to print a Hilbert curve with square piping, then incorporating a rectangular profile with variable width. I used a Python script within Rhino to carry out the process, reading in image data from the famous (cropped) picture of Hilbert in his white hat.

I created this portrait using the sixth iterate of the Hilbert curve, which I constructed in pieces – printing off 16 subsquares of the curve on an Ultimaker 3. Each subsquare is approximately five inches in width, and they were generated using the fourth iterate of the Hilbert curve. Hoping to render a higher resolution of the image, I was initially planning to generate the portrait using the seventh iterate of the curve. However, such an approach would have involved well over 140 hours of printing.

I decided I was happy with a lower resolution, which may require some squinting on the part of the viewer. However, the precision improves dramatically when the image is scaled down (see thumbnail top right). An unexpected discovery was the difference in finish and texture between the top of each printed curve segment and the bottom. Preferring the shiny finish of the bottom, I ultimately decided to print each subsquare of the curve upside down.

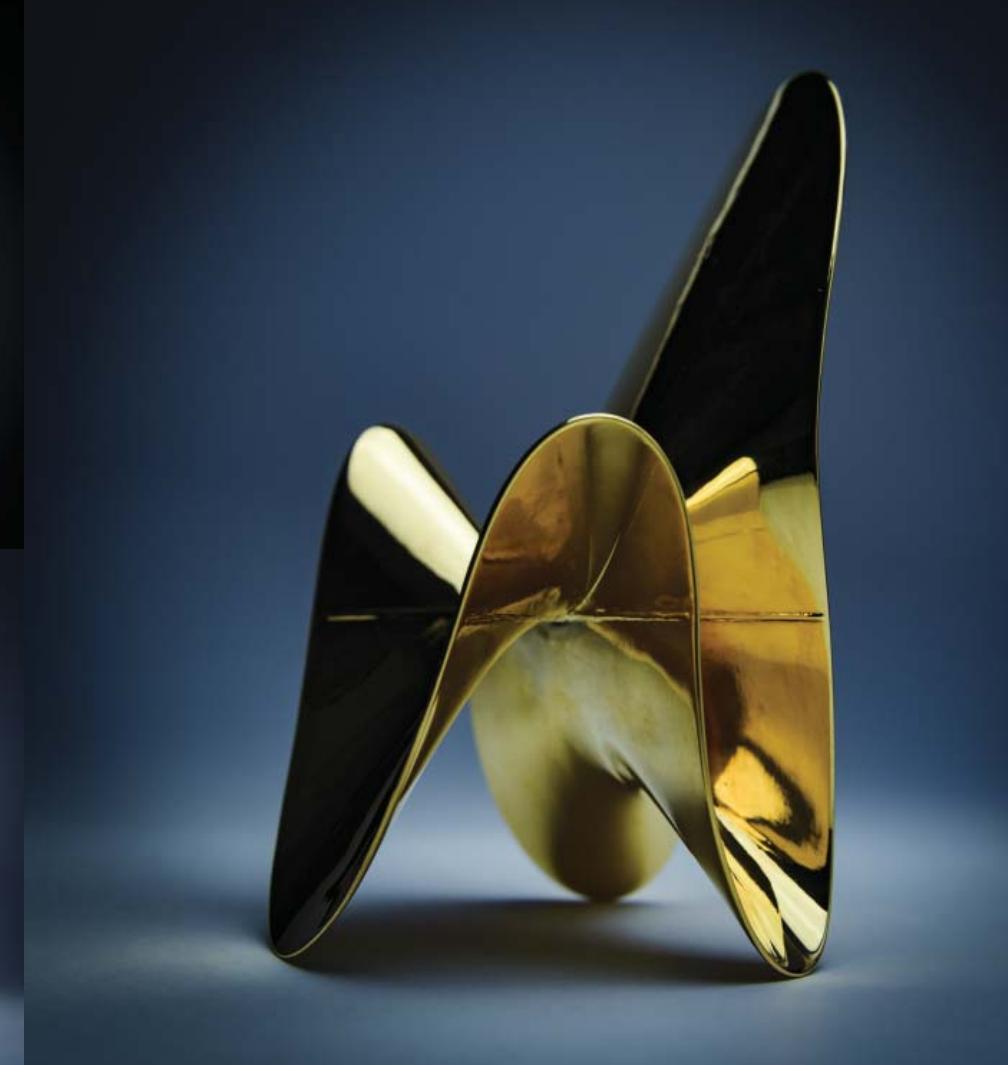
Further information: Judy Holdener, *Hilbert's Portrait via his Space-Filling Curve*. Proceedings of Bridges 2020: Mathematics, Music, Art, Architecture, Culture (2020), to appear.



JUDY HOLDENER

Kenyon College

3D-printed plastic



These are two so-called *cubic surfaces*: the points satisfy a certain equation in x , y , z of degree three. These are some of the most fascinating objects in the intersection of algebra and geometry, and they are from the 19th century.

I have created many different 3D-printed versions of such objects over the past 20 years. Classical sculptures show them in plaster, and my modern versions, which are exhibited in museums, are usually 3D-printed in white plastics. But these objects in gold-plated brass visualize their fascinating geometry in a much more obvious way, mainly because of the reflections of the material. In particular, in some interesting light situations with lighter and darker colors around, these reflections yield interesting visual effects on the surfaces of the sculpture. The curvature of these surfaces is much more interesting than you might think when just computing them, and printing them in brass allows us to observe these features.

Over the years, I have created many different versions of cubic surface sculptures using 3D printing, until finally, now, I like them. Earlier versions I created consisted of the part of each surface lying inside a certain sphere. This is a natural way to cut a finite part of an infinite object – at least when one does not know much about the object and wants to focus a region around the origin. But I worked with these objects for a long time, and thought a lot about which representation in space I prefer. My final choice is close to the classical one: cutting by a cylinder. But the decision about the position and size of the cylinder is not always an obvious one, and making this decision has been quite time-consuming for some of the cubic surfaces, as has the choice of the exact equations.

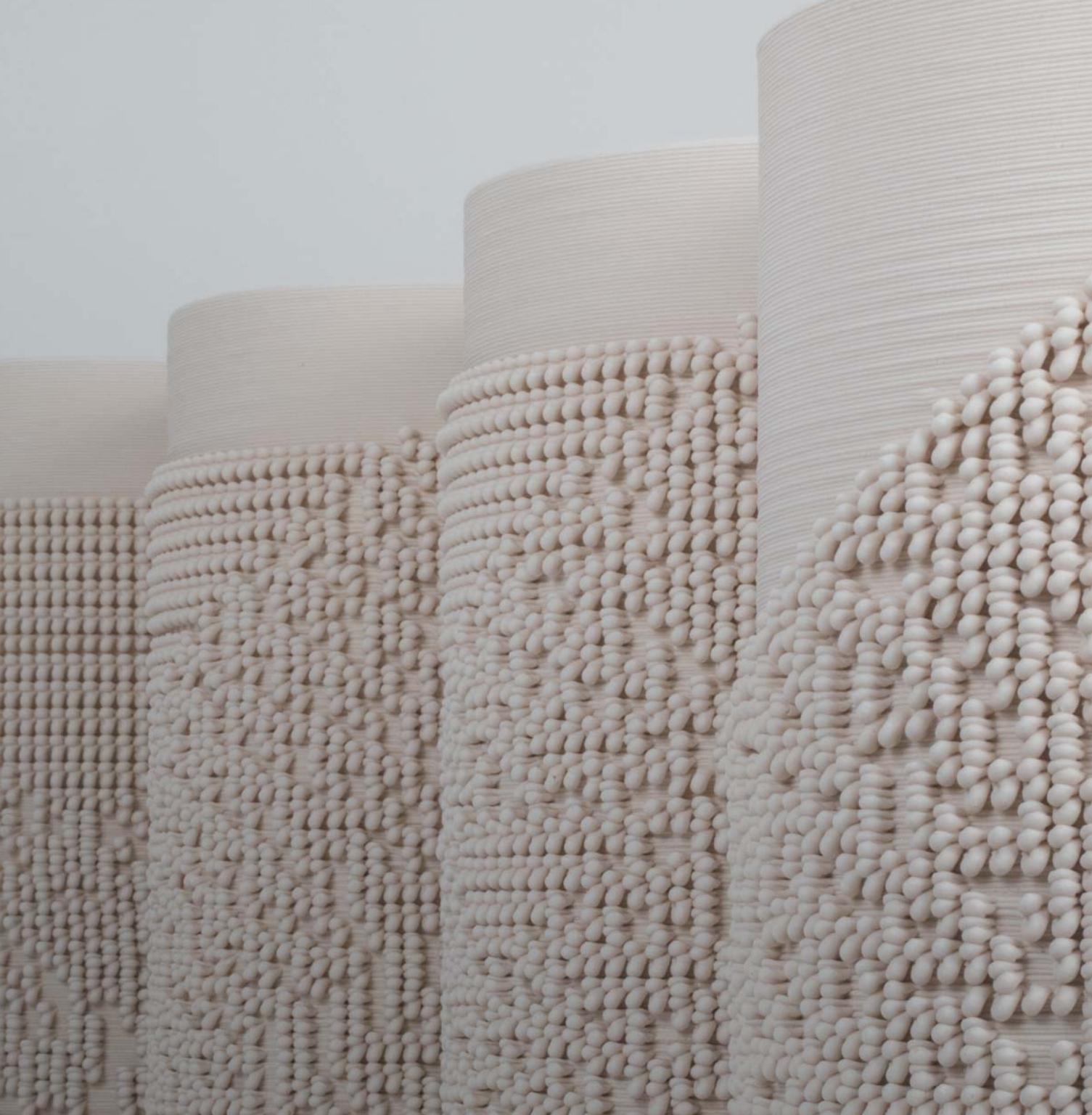
For more sculptures, and to purchase them: <https://math-sculpture.com/>
Further information: Oliver Labs, *Straight Lines on Models of Curved Surfaces*,
The Mathematical Intelligencer, Volume 39, (2017), pp. 15–26.



OLIVER LABS

MO-Labs

3D-printed gold-plated brass



This piece is an illustration of elementary cellular automata, made from 3D-printed ceramic. A *cellular automaton* consists of a grid of cells, in which the state of each cell depends on the states of those around it. Simple examples of cellular automata are *Pascal's triangle*, which has rows of numbers each of which is the sum of the two above it, and John Conway's *Game of Life*, in which each square in a grid is colored white or black at each time step, depending on the colors of the cells around it in the previous time step.

I'm not a mathematician, so I learned everything about cellular automata through this project. Once I was doing the work and the research, I started to be even more interested in exploring complex systems that are based on iterating simple rules. I think about these rule-based mathematical systems as games that can be played for an infinitely long time.

My workflow from the math to the object can be summarized by the following steps: After picking the game rules, we coded them in Sage, using CoCalc (see link below). I worked on coding this with mathematician Sara Billey from the University of Washington, then altered Billey's code-kernel with the help of Daria Mićović. The code generated a 2D plot of a distribution matrix in black and white. We then used a 3D-modeling program, Rhino, to turn this matrix into printable 3D geometries of cubes and cylinders.

This process required careful calculations and many tests, which were reverse engineered from the relationship between the printing layer height, nozzle size, and cell size, and the desired scale of the finished object. A precise reflection of the original matrix was my main goal, reflecting on the algorithmic nature of the first half of the process. On the other hand, I enjoy seeing how physical reality – the qualities of the material and the physics of gravity, time, etc. – makes subtle unexpected changes on the otherwise perfect model.

I work by testing and constantly refining until I have a very solid prototype that can be 3D-printed reliably. Still, I might lose 50% of the pieces because what I'm asking from the fragile porcelain is kind of difficult already.

More of my work: <https://www.timeatihanyi.com/>

Information about 3D-printed sculpture: <https://www.sliprabit.org/studio>

CoCalc is a web-based platform for computational math, founded by mathematician William Stein: <http://cocalc.com/>



TIMEA TIHANYI

Slip Rabbit Studio

3D-printed ceramic

MECHANICAL CONSTRUCTIONS & OTHER MATERIALS



Linkages, tensegrities, and other mathematical constructions are plenty interesting to study in theory, but it's far better to construct them, hold them in your hands, and see how they move. As you will see, you can make an illustrative mathematical object out of just about anything.

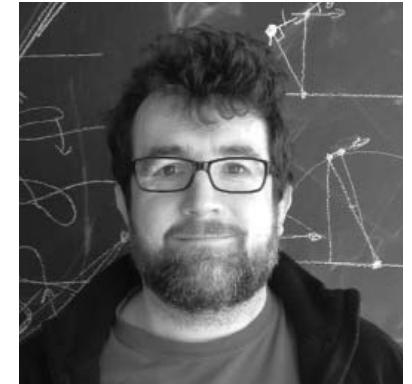


This is a hyperboloid, cut from a brick.

This item was the result of a project with Nick Bruscia and Dan Vrana from University at Buffalo and OMAX corporation, who make the water jet. The challenge was simply to explore what was possible with a five-axis water jet. As the cutting “tool” is a straight (well, spiraling) jet of water, *ruled surfaces* are the natural thing to start experimenting with.

A surface is *ruled* if, for every point on the surface, there is a line through that point that is completely contained in the surface. A hyperboloid is a ruled surface, so to cut it out we need the jet of water to follow the ruling lines. This object therefore starts as an illustration of an application of geometry: how to control and think about the forms that can be made by controlling the form of a general tool.

It also presents a compelling version of a classic mathematical object. The surprising material (brick) draws you in, with the question of how it could be made. Even knowing the tool used might not help, as the curved surface seems impossible to make with such a straight tool. This leads directly to an understanding of the behavior of ruled surfaces and the hyperboloid.



EDMUND HARRISS

University of Arkansas

Waterjet-cut brick



This is a large window, about 30 feet wide (bottom image), installed at the Mathematics Tutoring and Teaching Center at the University of Arkansas. It shows a self-similar hierarchical pattern in the plane, made by a recursive *tiling substitution rule*: all of the arcs are copies of one another, and each arc sits beneath four smaller arcs, always in the same manner. Conversely, each arc is one of four sitting above a larger arc, themselves a group of four upon an even larger one. In the mathematician's mind, this continues forever, smaller and smaller, and larger and larger, ad infinitum, always in a regular manner.

I made this image in Adobe Illustrator. When drawing a mathematical illustration in such a program, the main task is figuring out *how* to do the drawing, coming up with a process that will give the desired result. The mathematical illustrator must be willing to start from scratch several times, refining this process. I drew this window four or five times over, ultimately leading to a simple method that was accurate to 1/100th inch across a nearly 30 foot span.

The complexity of the details of the drawing was severely restricted by its exponential growth each scale of refinement – as drawn, the illustration nearly overwhelmed my computer!

I work with tiling substitution rules in my research, and the opportunity to draw this window led me to explore this one. There's a long literature on how *undecidability* and other aspects of the Theory of Computation arise as matching rules on these hierarchical tilings, and certain aspects of this particular system seem to point in some new directions for "programming" in these tilings.

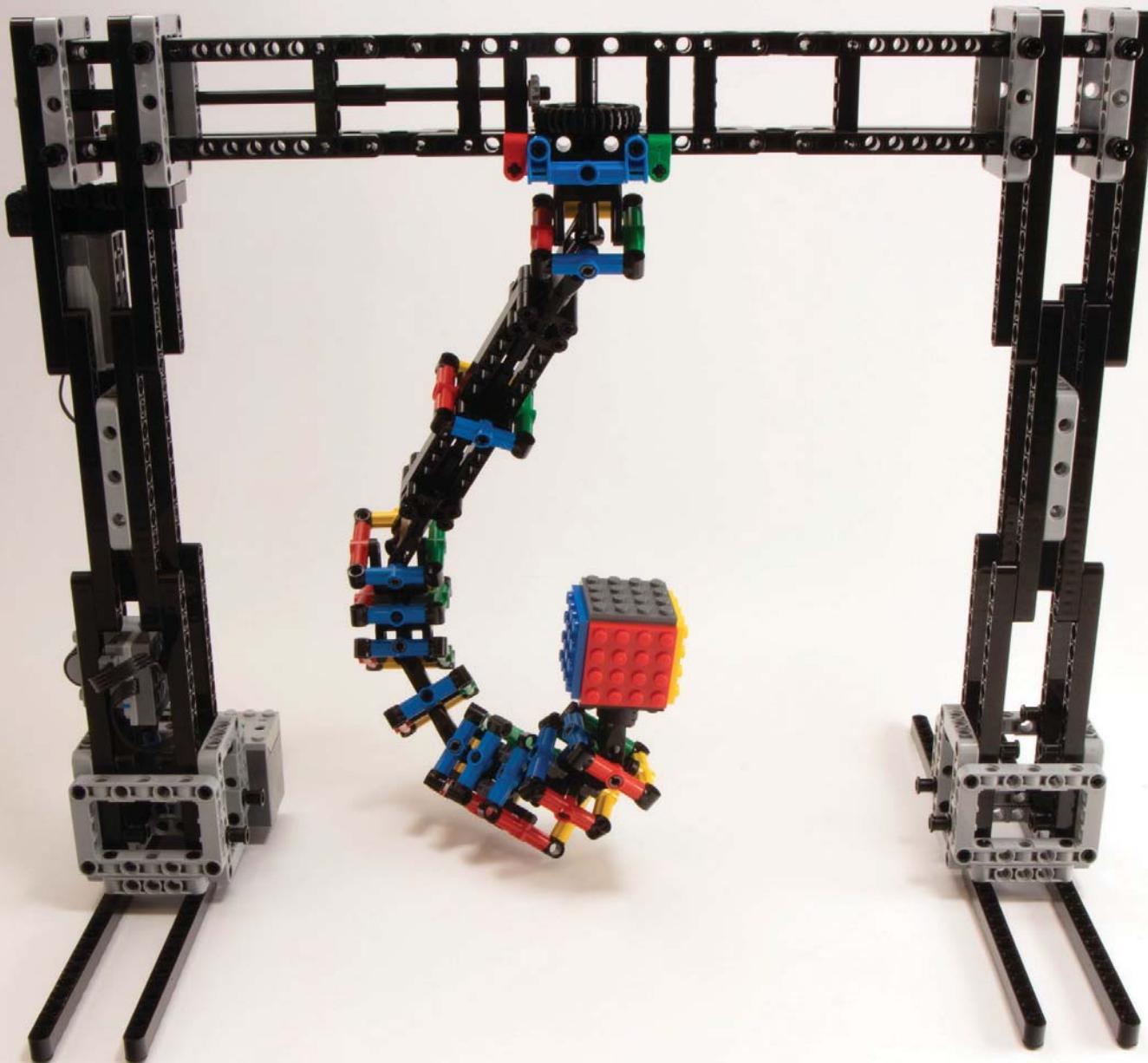
The design was printed in three layers: a sandwich of two layers of colored ink around a layer of white ink. The central white layer was printed in various opacities, reflecting light and color in some areas, and translucent in others, with bright zings of color moving through the window.



CHAIM GOODMAN-STRAUSS

University of Arkansas

laminated glass with printed interlayer



This is a mechanical illustration of the “Dirac belt trick” or “plate trick.” The central cube rotates continuously around a vertical axis, yet it is connected to the fixed outer frame via a sequence of hinge joints, each pivoting by at most 45° each way. The arm returns to its original state after every *two* complete turns of the cube. It can serve as an anti-twist mechanism: one can run a piece of thread or an electrical wire along the arm, fixed to the cube and the frame at its ends, which turns continuously with the arm, yet never gets twisted up. This reflects the two-to-one map from the (simply connected) space of unit quaternions to the (not simply connected) space of 3D rotations. To my knowledge, this is the first mechanical linkage that demonstrates the phenomenon. The rotation can be powered by an electric motor or by turning a handle.

Lego is the ideal medium for experimentation with mechanisms. One can go from an idea to a working model in minutes, and then to a more polished art piece with a bit more investment and effort.

I had wondered for some time whether such a linkage was possible. When I had the idea for the particular mechanism, it was not all clear to me whether it would work, even theoretically. The Lego model quickly provided the answer. Moreover, the model has provided new insight into the mathematical and physical phenomenon: the arm can be separated naturally into upper and lower parts, each of which on its own moves in an intuitive way; all that remains is to combine them.



ALEXANDER HOLROYD

University of Bristol

Lego construction pieces

Video of the object in motion: <http://youtu.be/byi5Gzjc04Q>

Double version with lights: https://youtu.be/1x_oQv_qj_U

Further information: https://en.wikipedia.org/wiki/Plate_trick

MULTIPLE WAYS TO ILLUSTRATE THE SAME THING

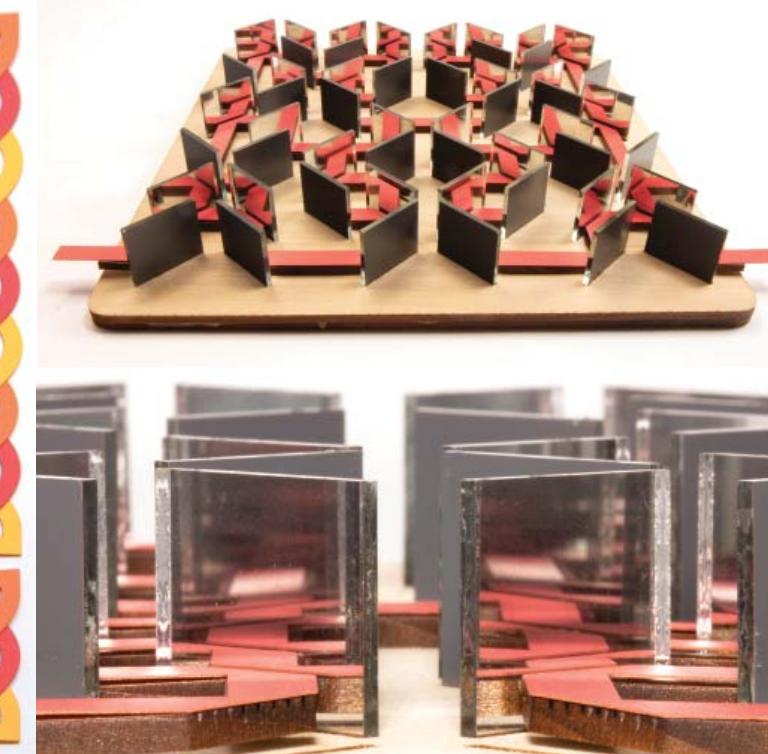
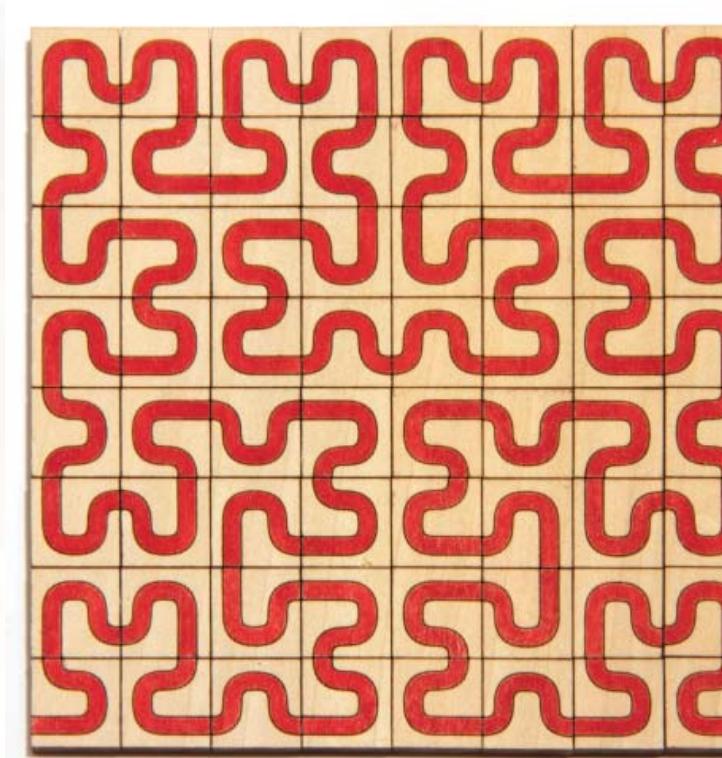
●○○○

150

gear into the North Atlantic. One of the biggest disputes on a sword boat is whether to set out or not. Crews have hauled in a fresh gale because their captain misjudged the weather.

One of the great joys of a career in mathematics is the opportunity to sit with a problem for a long time. Over the months, years and decades, our perspective on a mathematical idea may change several times, so that the way we come to understand it later is very different from the way we initially learned it. When we personalize a problem, an object, or an idea in this way, we own it and, in so doing, we give it new life. After all, no mathematical idea can be said to “exist” in its own right; an idea only exists in the minds of those who understand it.

151



In order to understand something, we should look at it from different perspectives. The Hilbert curve is a continuous fractal space-filling curve first described by the German mathematician David Hilbert in 1891. There are several different but equivalent ways of defining such curves: One way is “external” or “plotter-based,” given by absolute directions: up, down, left, and right. Another is “internal” or “turtle-based,” given by commands like “turn left” or “move forward” without any knowledge of absolute direction. Here are some illustrations of both.

Top left: This shows three iterations of the curve, illustrating the recursive step from one level to the next, created with a laser cutter. Laser cutting opens up an enormous space of possibilities by its extreme precision.

Top right: This is a tiling system, with only three types of tiles that trace out the curve. The tiles correspond precisely to the instructions “turn left,” “turn right,” and “move forward.” These wooden tiles were created with a laser cutter and colored by hand.

Bottom left: This is a rendition in paper of a Celtic knot: a 192-crossing, 3-component Brunnian link. Each strand has its own color and is equivalent to the unknot. If any one of the strands is removed, the other two are left unconnected. Because strands are overlapping, I cut this from three different pieces of paper and braided them together. Paper is a fun medium, but also very fragile. I made early versions with a mechanical paper cutter, but had better luck with a laser cutter.

Bottom right: This is a 52-mirror labyrinth, with a “laser beam” that traces out the curve. The top layer of the base has slits precisely cut to hold the 52 mirrors so that their reflective back faces are on the appropriate diagonals. The symbolic “laser beam” consists of a red piece of paper resting on a wooden platform of the same shape. This illustrates the internal point of view, as the “laser beam” has no knowledge of absolute direction, and the mirrors (or their absence) serve as the local commands. Originally I wanted to use an actual laser beam to trace out the curve, but the mirrors absorbed too much light, and the beam was invisible after a dozen bounces.

Further information:

Michael Bader, *An Introduction with Applications in Scientific Computing*, Springer-Verlag Berlin Heidelberg (2013).

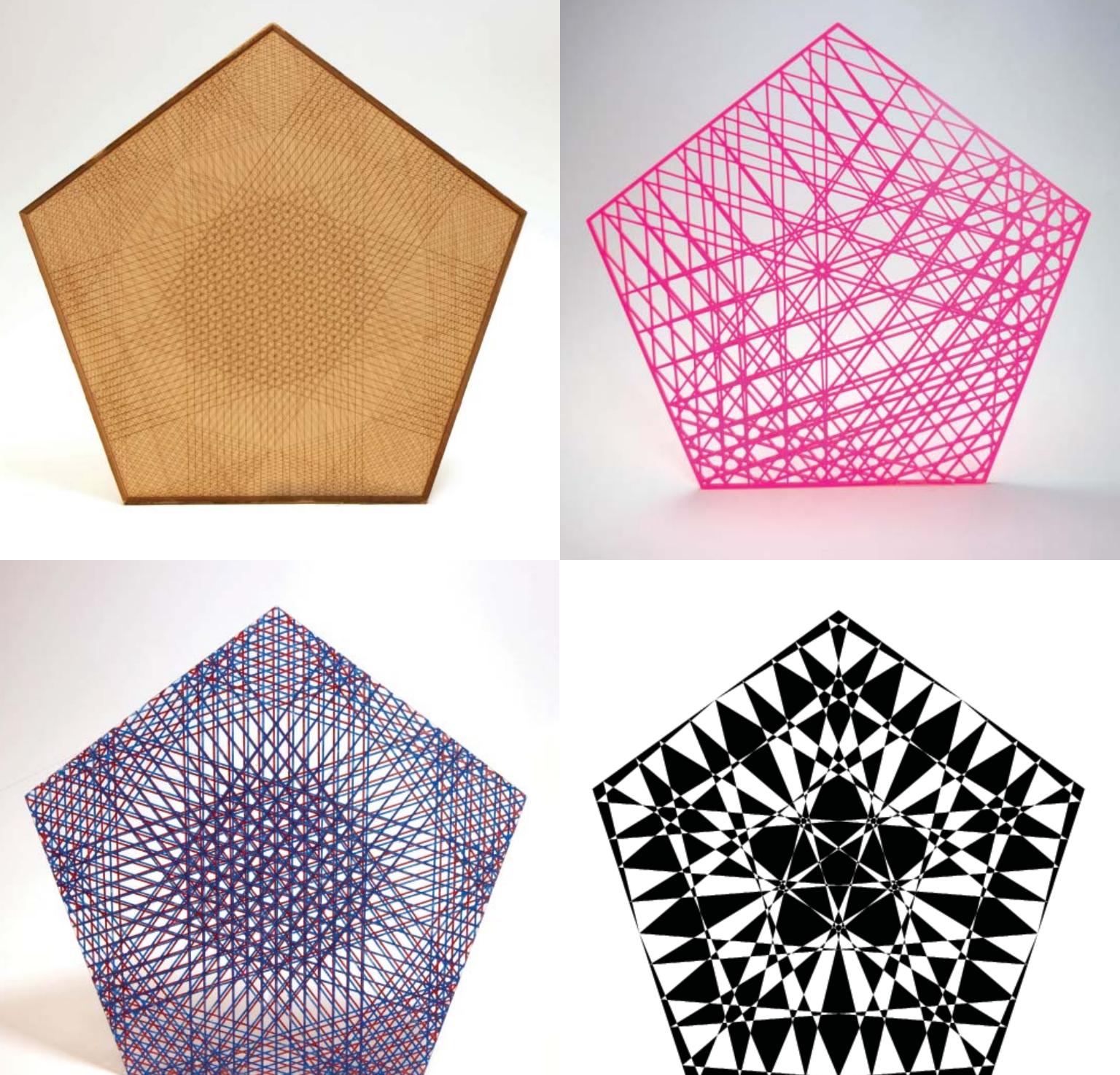
Doug McKenna, *Hilbert Curves: Outside-In and Inside Gone*, Mathemæsthetics, Inc. (2019).



ROGER ANTONSEN

University of Oslo

laser-cut wood, paper, and mirrors



These show periodic billiard paths on the regular pentagon: You imagine that you have a pentagon-shaped billiard table, and you shoot the ball so that it bounces around, eventually returning to where it started and repeating the same path forever. We studied and classified all of the periodic billiard paths on the pentagon, of which there are (countably) infinitely many, and we also wrote a program in Sage to draw them.

Over the past couple of years, we have tried multiple ways of illustrating these paths. The simplest way is just to draw the path itself, as in the upper left; this is laser-engraved in wood, which allows you to draw a very intricate path. Incidentally, the laser follows the path of the billiard as it engraves the path, and it's interesting to see the pattern emerge in real time.

Next, we thickened up the path and removed the negative space, as in the upper right; this is 1/8-inch laser-cut acrylic plastic. While this requires a much simpler trajectory, this method emphasizes the path more. Diana has made small versions of this into dangly earrings, which are excellent conversation starters about our research.

For the picture in the lower left, you imagine that when the path hits the edge of the table, it changes color, from red, to blue, to red, etc. To do this, Diana used clear acrylic with protective paper adhered to both sides, and laser-engraved the even-numbered pieces of the path on one side and the odd-numbered ones on the other side. Since the laser beam is very narrow, she engraved ten nearby parallel paths, removing the protective paper in strips. At this point, the protective paper only remained in the negative space. Then she painted one side red and the other side blue, and removed the painted paper, to leave only the painted path. Removing the paper was *much* more painful and tedious than expected, but the result is beautiful.

Lastly, we noticed that the path partitions the pentagon in a way that is two-colorable, so we colored the regions black and white (bottom right). We coded this on a deadline: when we started, Diana had a train to catch in 30 minutes, and we managed to produce a working version before she had to leave. This picture de-emphasizes the path but raises interesting questions, like whether the ratio of black to white might approach the golden ratio.

Further information: Diana Davis and Samuel Lelièvre, *Periodic paths on the pentagon, double pentagon and golden L*, preprint (2019).



**DIANA DAVIS &
SAMUEL LELIÈVRE**

Swarthmore College;
Université Paris-Saclay
*laser-cut acrylic and wood,
computer-generated graphics*



Surfaces with negative curvature are familiar in everyday life: a surface has *negative curvature* at a point if it is saddle-like there, and the more negative it is, the more extreme this saddle is. A surface with negative curvature is ruffly, like lettuce or curly kale, and has a tremendous amount of surface area for the volume it occupies. Though a lot of the mathematics of these surfaces has been well understood for more than 150 years, there remain many open, unexplored questions about just how these surfaces actually sit in space and the dramatic changes they undergo when they are manipulated. Partly, this may be because there aren't many ways to actually build such a surface – crochet (see pages 17 and 25) is one technology – and mathematicians haven't generally played with many physical examples.

Top: This surface of constant negative curvature is made out of annular strips connected edge to edge. Eugenio Beltrami made a model like this one out of paper, more than 150 years ago! But that is an inefficient use of material. It's impossible to cut very many annular strips from a flat sheet without a great deal of waste. Straight strips are far more efficient, but strips of flat materials such as steel or paper can only be bent in the direction perpendicular to the surface, and the strip must remain a geodesic on any surface it sits upon. But foam strips are squishy and can be bent in other ways, so they are just right for this piece. The pieces for this were cut using a router on a CNC machine, which creates a lot of dust, so it would be much better to invest in a CNC knife that cuts cleanly.

Bottom left: This is a spherical piece of the *gyroid*, an infinite periodic minimal surface. The gyroid has one of the more fascinating of the 3D Euclidean discrete symmetries – it is difficult to understand even when you are looking at it! Steel is solid, durable, and upgrades a piece. Mathematical illustrators should know: Welding is easy, and basic metal work is a fast and forgiving medium! (Welding *well* is another matter.) Because straight strips of steel do not bend into annular ones, these straight strips of steel lie along geodesics on the gyroid.

Bottom middle and right: These are surfaces with constant negative curvature made from flat strips of paper and steel, respectively. The steel object is about ten times the height of the paper one!

Further information: Daina Taimina, *Crocheting Adventures with Hyperbolic Planes: Tactile Mathematics, Art and Craft for All to Explore*, CRC Press (2018).



CHAIM GOODMAN-STRAUSS

University of Arkansas

Ethylene-vinyl acetate (EVA) foam, steel, paper

INDEX

Abrams, Aaron 29
 affine chart 75
 Akiyama polyhedra 105
Alatorre, Darío 81
 algebraic surface 75
 singular 85
 alternating group 43
Amethyst, Silvana 85
 Antoine's necklace 135
Antonsen, Roger 93, 153
 Archimedean solid 27, 139
Arnoux, Pierre 155
Athreya, Jayadev 107

Bachman, David 37
Barnes, Julie 57
Benson, Peter 35
 billiards 45, 157
 blue shift 17
 Borromean rings 161
Bozlee, Sebastian 75
 braided surface 111
 branched cover 69
 bubble 137
Buric, Dina 119

calculus, multivariable 113
 Cantor set 119, 135
Carter, J. Scott 111
 Catalan solid 27, 139
 cellular automaton 125
Chatterji, Indira 13
Chéritat, Arnaud 91
 chip firing 63

Clifford torus 127
 coloring 39, 157
 complex dynamics 51, 99, 101
 complex tree 97
 complex variety 85
 conformal coordinates 53
Coulon, Rémi 11, 135
 crossings, avoiding 93
 cubic differential 107
 curvature 19, 23, 117
 negative 17, 29, 159
 curve painting 9

Davis, Diana 39, 157
 deformation 47, 101
Delp, Kelly 19
 dihedral group 69
 Dini's surface 29
 Dirac belt trick 149
 dodecahedron 43
Dorfsman-Hopkins, Gabriel 17, 101
 duality 109
 dynamical system 81, 119
 chaotic 99

Edmark, John 139
Espigulé, Bernat 97
 eversion
 cylinder 79
 handlebody 161
 sphere 91
 exponential growth 17, 143

Farris, Frank A. 53
Felzenszwalb, Pedro F. 63
Fenyes, Aaron 145
 fiber bundle 123
 fractal 51, 135, 153
 Pythagorean tree 101
 Rauzy 97
 two-reptile 35
Francis, George K. 79
 function
 complex 49, 57, 99, 101
 cubic 113
 Z^3 -periodic 145
 fundamental domain 61, 139
 Fundamental Theorem of Arithmetic 59

Gangl, Herbert 121
 Gauss-Bonnet Theorem 19, 23
Ghrist, Robert 77
Goerner, Matthias 61
Goodman-Strauss, Chaim 143, 159
 group actions 41, 121
 gyroid 159

Harriss, Edmund 23, 131
 Hausdorff dimension 155
 Hilbert space-filling curve 89, 153
Holdener, Judy 89
Holroyd, Alexander 149
 homotopy 79
 Hopf fibration 127
 horoball 13

Hsu, Catherine 41
hyperbolic
 3-manifold 13, 61, 73
 3-space 37, 67, 121
 surface 17, 25
hyperbolic plane 47, 65
 Poincaré disk model 47, 65
 upper hyperboloid model 65
hyperboloid 131

iterated function system 97

Julia set 99, 101

kaleidoscope 139
Kent, Autumn 7
Kepner, Margaret 59
Klein bottle 79, 123
Klivans, Caroline J. 63
knot 7, 69, 103
 Celtic 153
 trefoil 69
knotted surface 111

Labs, Oliver 117, 133
Lelièvre, Samuel 21, 45, 157
level curve 57
linear map 97

Málaga Sabogal, Alba Marina
 21, 45, 105

Mandelbrot set 49
Matsumoto, Sabetta 31, 127
Maynard, Meghan 17
Mcshane, Greg 43, 99
minimal
 polyhedra 105
 surface 127, 137, 159
Miura map fold 95
Möbius strip 71, 79, 127
Möbius transformation 55
Mohanty, Yana 141
Muetzel, Bjoern 141

Nash-Kuiper C¹ embedding theorem
 25

net 27
Nil geometry 73
non-abelian group 41
Novello de Brito, Tiago 73

octahedron 107
origami 95

parameter perturbation 97, 99, 101,

115

Paul, Stepan 29, 113

Peano space-filling curve 155

pentagon 115, 157

 15 families of tiling pentagons 115

periodic

 billiards 45, 157

 surface, infinite polyhedral 107

Perko Pair 103

Platonic solid 43, 109
polyhedron 141
polynomial, cubic 113
prime decomposition 59
prime ideal 147
projection 65, 111, 127
 stereographic 87, 141
projective
 geometry 47
 space 75
special linear group 55
pseudosphere 29

quadratic field, imaginary 55
quaternions 149

Rainford, David 51
rational map 57
rational points 133
recursion 153
red shift 17
rhombicosidodecahedron 141
Riemann zeta function 51
ring 147
Robb, Heidi 35
ruled surface 131

Sander, Evelyn 109
Schaubroeck, Beth 57
Segerman, Henry 87
semi-direct product 41
Séquin, Carlo 161
Sharma, Tashrika 147

Shmahalo, Olena 71
shuffling, card 93
Skrodska, Martin 63
solenoid, 2-adic 119
Sol geometry 11, 73
sphere
 3D 127
 high-dimensional 77
 Sol geometry 11
spherical geometry 73
Stange, Katherine E. 55
substitution rule 143
Sullivan, Dennis 9
Sümmermann, Moritz L. 69
surface, cubic 117, 133
surface, self-intersecting 91
symmetric group 41
symmetry 105, 121, 139, 161

Taalman, Laura 95, 103, 115
tessellation. *See* tiling
Thurston geometries 11, 73

Thurston, William 9, 67;

 1, 7, 19, 69

Tihanyi, Timea 125

tiling 65, 115, 153

 2D 35, 39

 3D 61, 73

 3D hyperbolic 121

 aperiodic 81, 145

 hierachial 143

torus
 3-hole 161
 flat 21
 solid 119
train tracks 9
translation surface 31
Trettel, Steve 47
triicontahedron 27
triangulation 25, 61
Trnkova, Maria 25

uncountability 71
universe, shape of 17, 37, 43

Vejdemo-Johansson, Mikael 123

wallpaper pattern 53
Weaire-Phelan foam 137
Whitney, Glen 137

Yackel, Carolyn 27
Yoshizawa butterfly 95

zippered rectangle construction 31

Zucker, Matt 65