

Math 424

Billiards, Surfaces and Geometry

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Williams College
Williamstown, MA
Fall 2016

To the Student

Contents: As you work through this book, you will discover that various topics about geometry, surfaces and billiards have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on ellipses. The curriculum is problem-centered, rather than topic-centered. Techniques, definitions and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records — there are no boxes containing important theorems. You should keep a growing list of important terms in the *Reference* at the end of this book.

Your homework: Each page of this book contains the homework assignment for one night. The first day of class, we will work on the problems on page 1, and your homework is page 2; on the second day of class, we will discuss the problems on page 2, and your homework will be page 3, and so on for each of the 37 class days of the semester. You should plan to spend two to three hours each night solving problems for this class.

Comments on problem-solving: You should approach each problem as an exploration. Reading each question carefully is essential, especially since definitions, highlighted in italics, are routinely inserted into the problem texts. It is important to make large, clear, accurate diagrams, and paper models, whenever appropriate. Useful strategies to keep in mind are: create an easier problem, work backwards, and recall a similar problem. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day.

Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

The problems in this text

This style of problems is based on the curriculum at Phillips Exeter Academy, a private high school in Exeter, NH. Some of the problems are taken from *Geometry and Billiards* by Serge Tabachnikov, and from *Mostly Surfaces* by Richard Schwartz. The rest of the problems were written by Diana Davis, who has been both a student and a faculty member at PEA, for a senior seminar at Williams College. Anyone is welcome to use this text, and these problems, so long as you do not sell the result for profit. If you create your own text using these problems, please give appropriate attribution, as I am doing here.

A note on class discussion

Please be patient with me, and with your classmates, and most of all with yourself, as everyone adapts to working and learning in this method. I have carefully constructed this set of problems, thinking hard about each problem and how they all connect and build the ideas, step by step. Often you will see the connections. Sometimes you won't, but a classmate will, and will explain it to you. Occasionally, everyone will miss the connection and I will have to rewrite future problems to reflect that. This is the first time any class has used these problems, so there are bound to be a few errors, of omission or of overexplanation.

You might wonder, what is my job as your teacher? Part of my job is to give you good problems to think about, which are in this book. During class, my job is to help you learn to talk about math with each other, and help you build a set of problem-solving strategies. At the beginning, I will give you lots of pointers, and as you improve your skills I won't need to help as much.

One way of describing this method is "the student bears the laboring oar." This is a metaphor: You are rowing the boat; you are not merely along for the ride. You do the work, and in this way you do the learning. The next page gives some ideas for ways that you can do this work of moving the "boat," which is our class and your learning, forward.

Just remember that we are all in this together. Our goal is for each student to learn the ideas and skills of geometry, surfaces and billiards, really learn them — and along the way I will learn new things, too. That's the beauty of this teaching and learning method, that it recognizes the humanity in each of us, and allows us to communicate authentically, person to person.

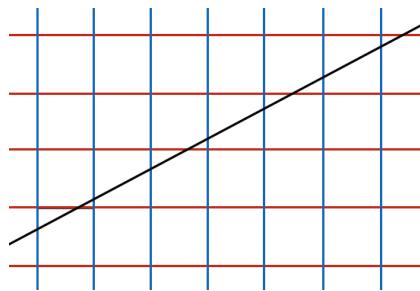
A college course should teach more than the curriculum: students should also learn something about how to be good people and good citizens. Many math courses, in addition to their mathematical content, teach the values of hard work and perseverance. In this course, through talking with your classmates about math, and struggling together to solve hard problems, you will learn that each person has something to contribute, and that the solution may come from the person you thought was least likely. This is a good life lesson.

Discussion Skills

1. Contribute to the class every day
2. Speak to classmates, not to the instructor
3. Put up a difficult problem, even if not correct
4. Use other students' names
5. Ask questions
6. Answer other students' questions
7. Suggest an alternate solution method
8. Draw a picture or build a model
9. Connect to a similar problem
10. Summarize the discussion of a problem

Math 424: Billiards, Surfaces and Geometry

1. Draw a line on an infinite square grid, and record each time the line crosses a horizontal or vertical edge. We will assume that the direction of travel along a line is always left to right. We could record the line to the right with the sequence $\dots \bullet \dots$, or we could assign A to horizontal and B to vertical edges, and record it as $\dots BABBAABBA\dots$.



- (a) What is the slope of the line in the picture?
- (b) Record this *cutting sequence* of colors, or of A s and B s, for several different lines. Describe any patterns you notice. What can you predict about the cutting sequence, from the line?
- (c) What should you do if the line hits a vertex?

2. Consider a ball bouncing around inside a square billiard table. We'll assume that the table has no "pockets" (it's a billiard table, not a pool table!), that the ball is just a point, and that when it hits a wall, it reflects off and the angle of incidence equals the angle of reflection, as in real life. (We'll prove this *billiard reflection law* later.)

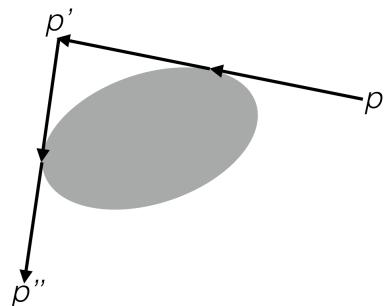
- (a) A billiard path is called *periodic* if it repeats, and the *period* is the number of bounces before repeating. Construct a periodic billiard path of period 2.
- (b) For which other periods can you construct periodic paths?

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1. Draw several examples of billiard trajectories in a circular billiard table. Describe the behavior in general.

2. Consider a billiard “table” in the shape of an infinite sector with a small vertex angle, say 10° . Draw several examples of billiard trajectories in this sector (calculate the angles at each bounce so that your sketch is accurate). Is it possible for the trajectory to go in toward the vertex and get “stuck”? Find an example of a trajectory that does this, or explain why it cannot happen.

3. *Outer billiards.* Though it may seem strange to call it “billiards,” we can also define a billiard map on the *outside* of a billiard table. We choose a starting point p , and a direction, either clockwise or counter-clockwise. Then, draw the tangent line from p to the table in that direction to find the point of tangency. Double the vector from p to the point of tangency, and add this to p to get p' , as in the picture. We repeat the construction to find p'' , and so on.

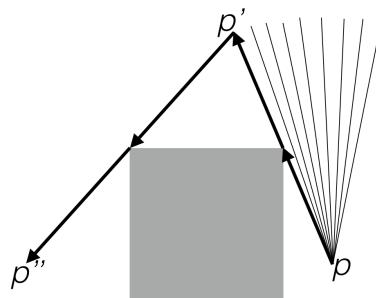


Draw several examples of outer billiards on a circular table, and describe the behavior in general.

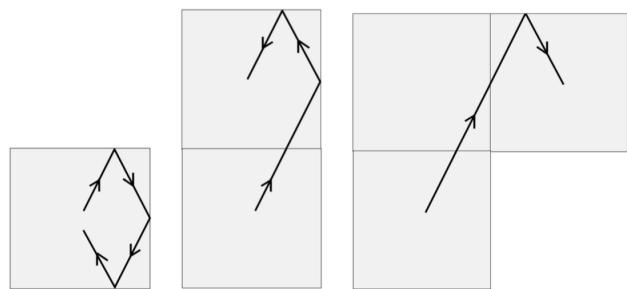
By the way, outer billiards are sometimes called “dual billiards.” When talking about several kinds of billiards, you can use the term “inner billiards” for regular billiards.

4. (Continuation) Outer billiards also works on a polygonal table, of course. Here, the “tangent line” is always through a vertex – you can think of sweeping a line counter-clockwise until it hits a vertex, as shown.

Draw several examples of outer billiards on a square table. Can you find any periodic trajectories? *Hint:* Make a large, clear, accurate diagram. Use graph paper and a ruler. Use a new picture for each trajectory.



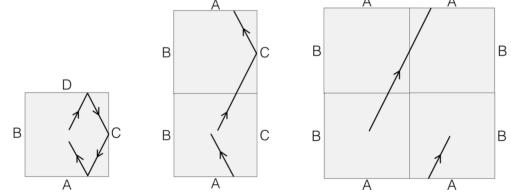
5. A powerful tool for understanding inner billiards is *unfolding* a trajectory into a straight line, by creating a new copy of the billiard table each time the ball hits an edge. Two steps of the unfolding process are shown for a small piece of trajectory. Draw several more steps. Then use this unfolding to prove that any trajectory with slope 2 yields a periodic billiard trajectory on the square. (We always assume that one edge is horizontal.) Which other slopes yield a periodic billiard trajectory?



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1. We have considered a polygonal billiard table. Now consider a billiard ball bouncing inside a *smooth* domain D . Assume that D is also convex. Show that there exist at least two distinct 2-periodic billiard orbits in D .
2. Show that the cutting sequence corresponding to a line of slope $1/2$ on the square grid is periodic. Which other slopes yield periodic cutting sequences? What can you say about the period, from the slope?
3. Prove that every billiard trajectory on the square with irrational slope is non-periodic.

4. Here's another way that we can unfold the square billiard table. First, unfold across the top edge of the table, creating another copy in which the ball keeps going straight. The new top edge is just a copy of the bottom edge, so we now label them both A to remember that they are the same. Similarly, we can unfold across the right edge of the table, creating another copy of the unfolded table. The new right edge is a copy of the left edge, so we now label them both B . When the trajectory hits the top edge A , it reappears in the same place on the bottom edge A and keeps going. Similarly, when the trajectory hits the right edge B , it reappears on the left edge B . This is called *identifying* (or *gluing*) the top and bottom edges, and *identifying* the left and right edges, of the square.

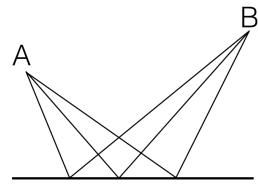


- (a) Label the top and bottom edges of a sheet of paper A , and the left and right edges B , and tape the identified edges together to create a surface. What does this surface look like?
- (b) What does the surface look like to a small bug sitting on edge A ? sitting in the middle of the square? sitting on a corner?
- (c) When we unfolded the trajectory to a line, we created a new copy of the table every time the trajectory crossed an edge. Explain why here we only need 4 copies.
- (d) The partial billiard trajectory shown on the left part of the figure repeats after 6 bounces. Sketch in the rest of the trajectory in each of the three pictures. What is its corresponding *cutting sequence* for the surface on the right part of the figure?

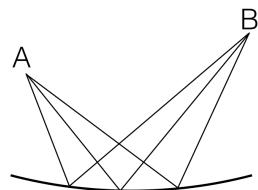
5. Consider again a billiard table in the shape of an infinite sector, with vertex angle α . Show that any billiard on such a table makes finitely many bounces. What is the maximum number of bounces?

Math 424: Billiards, Surfaces and Geometry

- 1.** *A proof of the billiard reflection law.* The Fermat principle says that light propagates from point A to point B along the path that takes the least possible time. Since our paths are in the Euclidean plane, this is just the shortest path. Consider a single reflection in a flat mirror ℓ (the horizontal line in the picture), and find the point X along the line that minimizes the distance $AX + XB$. Explain how to obtain the billiard reflection law (angle of incidence equals angle of reflection) as a consequence.



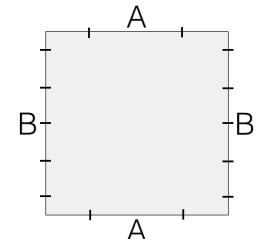
- 2.** (Continuation) Now, let the mirror be a smooth curve ℓ ; again, our goal is to find the point X on ℓ that minimizes the length $AX + XB$. We will use two different methods to deduce the reflection law.



(a) Calculus. Let X be a point in the plane, and define $f(X) = |AX| + |XB|$. The gradient vector of the function $|AX|$ is the unit vector in the direction from A to X , and likewise for the function $|BX|$. By the Lagrange multipliers principle, applied to the function _____ under the constraint _____ (fill these in), X is a critical point if and only if $\nabla f(X)$ is orthogonal to ℓ . Use vectors to deduce the billiard reflection law. *Hint:* You may have to review Lagrange multipliers. Do it!

(b) Mechanics. Let ℓ be a wire, X a small ring that can move along the wire without friction, and AXB an elastic string fixed at A and B and passing through the ring. Use an equilibrium tension argument to deduce the billiard reflection law.

- 3.** In Page 3 # 4, we ended up with a trajectory of slope 2 on the *square torus* surface. The picture to the right shows some scratchwork for drawing a trajectory of slope $2/5$ (or $-2/5$) on the square torus. Starting at the top-left corner, connect the top mark on the left edge to the left-most mark on the top edge with a line segment. Then connect the other six pairs, down to the bottom-right corner.



- (a)** Explain why, on the torus surface, these line segments connect up to form a continuous trajectory. Find the cutting sequence corresponding to this trajectory.
(b) Exactly where on the edges should you place the marks so that all of the segments have the same slope?

- 4.** (Continuation) In the previous problem, we put 2 marks on edge A and 5 marks on edge B and connected up the marks to create a trajectory with slope $2/5$. What if we did the same procedure with 4 marks on edge A and 10 marks on edge B ?

- 5.** Suppose 100 ants are on a log 1 meter long, each moving either to the left or right with unit speed. Assume the ants collide elastically (when they hit each other, each ant immediately turns around and goes the other way), and that when they reach the end of the log, they fall off. What is the longest possible waiting time until all the ants are off the log?

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1. Draw an accurate picture of a trajectory on the square torus with slope $3/4$, and do the same for two other slopes of your choice. For each one, find the corresponding cutting sequence. *Hint:* Use graph paper. You can print out any size you want, from the Internet.

2. We saw that a billiard trajectory on the square table can be *unfolded* to a line on the square torus. Going the other way, a line on the square torus can be *folded* to a billiard trajectory on the square table. Choose one of your trajectories from the previous problem. On a loose piece of paper (or a transparency), draw a large accurate picture on a square, and fold it in half twice, like a paper napkin. Hold it up to the light, and trace through to see the torus trajectory, transformed into a billiard on the square! Find the corresponding cutting sequences on the square torus, and on the square table. What are their periods?

3. Consider again a billiard table in the shape of an infinite sector, with vertex angle α . Use unfolding to show that any billiard on such a table makes at most $\lceil \pi/\alpha \rceil$ bounces. *Hint:* Unfold the sector as many times as you can.

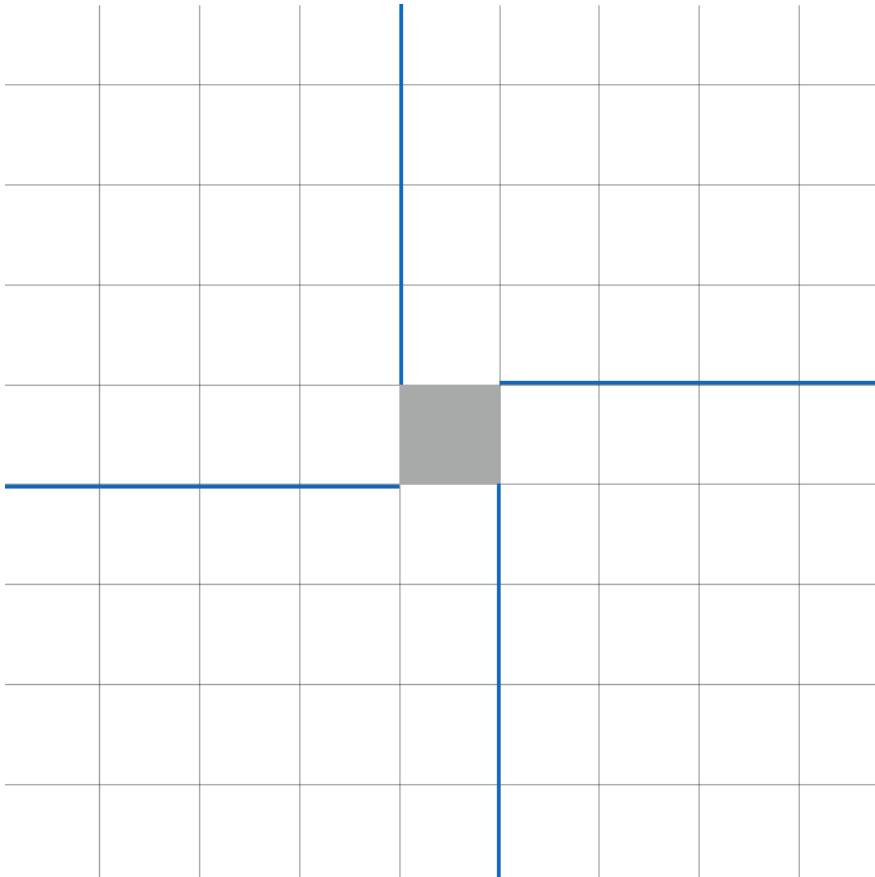
4. Consider again outer billiards on the square table, in the counter-clockwise direction.

(a) Points p on the blue lines are not allowed, because their images p' are ambiguously defined. Explain.

(b) Points p whose image p' is on a blue line are also not allowed. Explain. These are the *inverse images* of the blue points. Color these points red.

(c) The inverse images of the red lines are also not allowed. Explain. Color these green.

(d) Color the inverse images of the green points black. Keep going, with different colors at each step. Describe the full set of disallowed points.



Note: The resemblance of the (incomplete stages of the) diagram to a swastika is unintentional. By the definition of the outer billiard map, it is unfortunately impossible to avoid. This symbol was first used 12,000 years ago; the fact that it arises in outer billiards shows that it is a natural construction that has sadly become synonymous with an odious regime.

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1. Prove that a trajectory on the square torus is periodic if and only if its slope is rational.

2. *Billiards in other polygonal tables.* You have constructed several periodic billiard paths in the square billiard table. Other polygons also have periodic paths. We will show that the triangle connecting the base points of the three altitudes of a triangle is a 3-periodic billiard trajectory, by showing that angles ARP and CRQ are equal; the argument is the same for the other bounces.

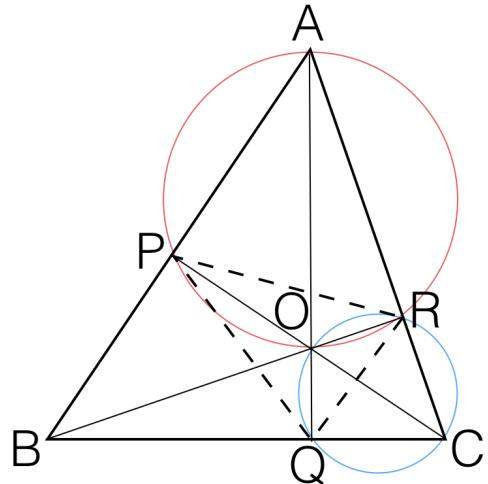
(a) A classic theorem of geometry says the following: Opposite angles of a quadrilateral add up to π if and only if the quadrilateral is *cyclic*, that is, its four vertices all lie on the same circle. Make a sketch showing that that a rectangle is cyclic, while a non-rectangular parallelogram is not.

(b) Use that theorem to show that quadrilaterals $APOR$ and $CROQ$ are cyclic, as suggested by the diagram.

(c) Another classic theorem of geometry says that two angles supporting the same circular arc are equal. Use this to show that $\angle PAO = \angle PRO$, and $\angle ORQ = \angle OCQ$.

(d) Use triangles BAQ and BCP to show that $\angle PAO = \angle OCQ$.

(e) Show that $\angle ARP = \angle CRQ$, as desired.



3. In Page 5 # 5, you showed that for outer billiards on the square, all of the points on the square grid lines are not allowed. Choose a point p that is *not* on one of the grid lines. Under the outer billiard map, this point reflects through a sequence of vertices v_1, v_2, \dots where each v_i is one of the four vertices of the square table. Explain why *every* point that is in the same (open) square as p reflects through that *same* sequence of vertices.

4. The *continued fraction expansion* gives an expanded expression of a given number, say $15/11$. To obtain the continued fraction expansion for a number, we do the following:

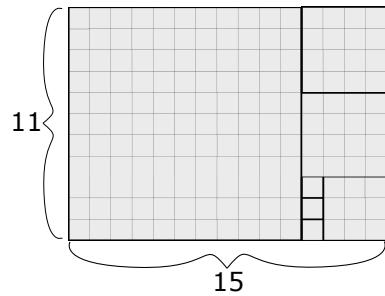
$$\frac{15}{11} = 1 + \frac{4}{11} = 1 + \frac{1}{11/4} = 1 + \frac{1}{1 + 7/4} = 1 + \frac{1}{2 + 3/4} = 1 + \frac{1}{2 + \frac{1}{4/3}} = \mathbf{1} + \frac{1}{\mathbf{2} + \frac{1}{\mathbf{1} + \frac{1}{3}}}.$$

The idea is to pull off 1s until the number is less than 1, and then take the reciprocal of what is left, and repeat. Find the continued fraction expansion of $5/7$, showing all your steps as above. By the way, since all the numerators are 1, we can denote the continued fraction expansion more compactly by recording only the bolded numbers, so $15/11 = [1, 2, 1, 3]$. (Soon, we will relate continued fraction expansions to trajectories on the square torus, in a surprising and beautiful way.)

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- 1.** Geometrically, the continued fraction algorithm for a number x is:

1. Begin with a $1 \times x$ rectangle (or $p \times q$ if $x = p/q$).
2. Cut off the largest possible square, as many times as possible. Count how many squares you cut off; this is a_1 .
3. With the remaining rectangle, cut off the largest possible squares; the number of these is a_2 .
4. Continue until there is no remaining rectangle. The continued fraction expansion of x is then $[a_1, a_2, \dots]$.



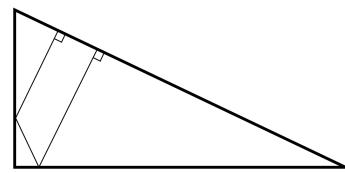
Draw the rectangle picture for $5/7$ and geometrically compute its continued fraction expansion, and check that it agrees with your previous results. Explain why this geometric method is equivalent to the fraction method previously explained, for determining the continued fraction expansion.

- 2.** You have seen that studying a line on the square grid is essentially the same as studying a straight trajectory on the square torus. Explain how to go from one to the other, in both directions.
- 3.** An *automorphism* of a surface is a bijective action that takes the surface to itself. Two types of automorphisms of the square *torus* come from symmetries of the *square* itself: reflections and rotations. (Remember these from the beginning of your abstract algebra course, when you studied the dihedral groups: they are the rigid motions that take the square to itself.) How many symmetries of the square are there? Which of them can you visualize on the “surface of a donut” 3D torus surface?
- 4.** An active area of research is to describe all possible cutting sequences on a given surface. On the square torus, that question is: “Which infinite sequences of *As* and *Bs* are cutting sequences corresponding to a trajectory?” Let’s answer an easier question: How can you tell that a given infinite sequence of *As* and *Bs* is *not* a cutting sequence? You have computed many examples of cutting sequences that *do* correspond to a line on the square grid or square torus. Write down four of them. Then make up an example of an infinite sequence of *As* and *Bs* that *cannot* be a cutting sequence on the square grid or square torus, and justify your answer.
- 5.** Consider again the altitude construction of a 3-periodic path in a triangle. Sketch a trajectory that is parallel to the one in the construction, but that starts a small distance from the base point of the altitude. Show that this trajectory is also periodic, and find its period.

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1. *Periodic trajectories in triangles.*

(a) Explain why the altitude construction of a 3-periodic path only works for acute triangles.



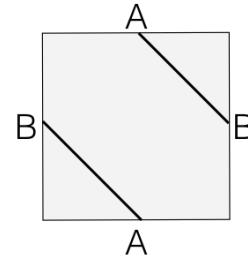
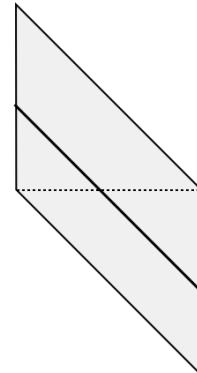
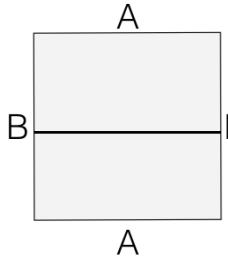
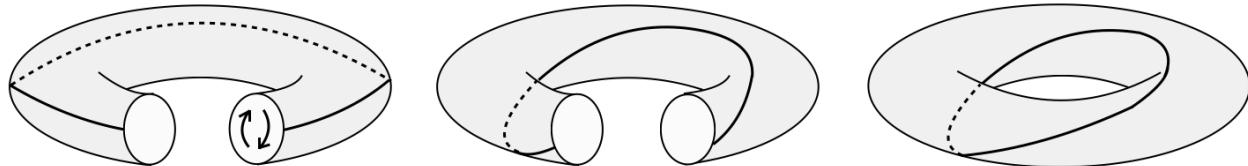
(b) Prove that the figure does indeed depict a 6-periodic billiard trajectory in a right triangle. This construction was found by Rich Schwartz (DD's Ph.D. advisor). He calls it "shooting into the corner."

(c) Does every obtuse triangle have a periodic billiard path? This question is open, and is the subject of current research. The answer is known to be "yes" for many cases. Find an example of a periodic path in an obtuse triangle.

2. Construct a periodic billiard path on the square table with an odd period, or show that it is not possible to do so.

3. Explain why a cutting sequence on the square torus can have blocks of multiple As separated by single Bs, or blocks of multiple Bs separated by single As, but not both.

4. It turns out that there are three types of automorphisms of the square torus: reflections and rotations, which you explored in Page 7 # 3, plus one more type that is not a symmetry of the square: a *shear*. The shear is shown below on the square on and the 3D surface, where its effect is to twist the torus. Explain. What 2×2 matrix, applied to the "unit square" $[0, 1] \times [0, 1]$ shown in the left picture, gives the parallelogram shown in the middle picture? (Recall from linear algebra that the first column of a 2×2 matrix is the image of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the second column is the image of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Assume that $(0, 0)$ is at the square's lower-left corner.)



5. We have identified the top and bottom edges, and the left and right edges, of a square to obtain a surface: the square torus. If we identify opposite parallel edges of a parallelogram, what surface do we get?

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1. Given a trajectory on the square torus, we want to know what happens to that trajectory under an automorphism of the surface. One way to answer this question is to sketch the trajectory before and after applying the automorphism. Another way is by comparing their cutting sequences: the cutting sequence $c(\tau)$ corresponding to the original trajectory τ , and the cutting sequence $c(\tau')$ corresponding to the transformed trajectory τ' .

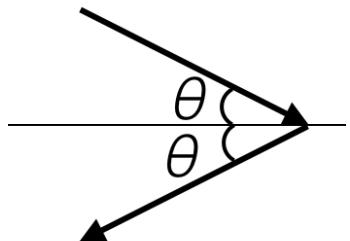
- (a) Let τ_2 be the trajectory of slope 2. Sketch τ_2 , and find $c(\tau_2)$.
 - (b) For each automorphism (1)-(5) below, sketch τ'_2 , and compute $c(\tau'_2)$.
 - (c) Explain how to obtain $c(\tau')$ from $c(\tau)$ for a general trajectory τ , for each automorphism.
- (1) reflection across a horizontal line;
 - (2) reflection across a vertical line;
 - (3) reflection across the positive diagonal;
 - (4) reflection across the negative diagonal;
 - (5) rotation by 90° .

2. (Continuation) Find the 2×2 matrix that performs each of these five automorphisms. For the purpose of this question, assume that the square torus is centered at the origin, so it is $[-1, 1] \times [-1, 1]$ or a dilate thereof. Find the determinant of each matrix and explain why they are all ± 1 .

3. Show that a periodic trajectory on a polygonal billiard table is never isolated: an even-periodic trajectory belongs to a 1-parameter family of parallel periodic trajectories of the same period and length, and an odd-periodic trajectory is contained in a strip consisting of trajectories whose period and length is twice as great. *Hint:* Think about a wide ribbon whose center line is the trajectory, wrapping around the triangle.

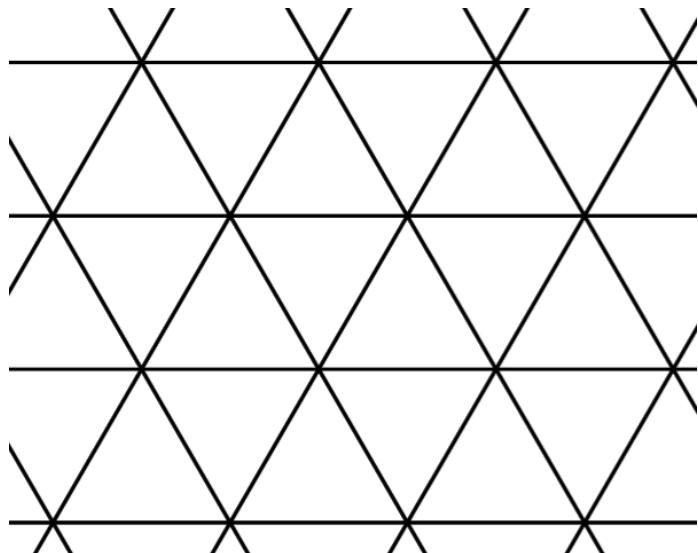
4. Find the continued fraction expansion of $\sqrt{2} - 1$. (Use a calculator to deal with the decimals.) Then solve the equation $x = \frac{1}{2+x}$ and explain how these are related.

5. *Tiling billiards.* The final type of billiards (besides inner and outer) that we will study is *tiling billiards*, where a trajectory refracts through a tiling of the plane. The *refraction rule* is that when the trajectory (thick) hits an edge of the tiling (thin), it passes through in such a way that the angle of incidence is equal to the angle of reflection, and the trajectory has been reflected across the edge, as shown to the right. Sketch some (clear, accurate) trajectories on the square grid tiling. What kinds of behaviors can you find? Can you prove that these are the only ones?



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1. We saw that for tiling billiards on the square grid, there are only two types of trajectories: those that go to the opposite edge and zig-zag, and those that go to the adjacent edge and make a 4-periodic path. How many types of trajectories are there for tiling billiards on the triangular grid?



2. How many billiard paths of period 10 are there on the square billiard table? Of period 14? Construct (make a mathematically accurate sketch of) each of these.

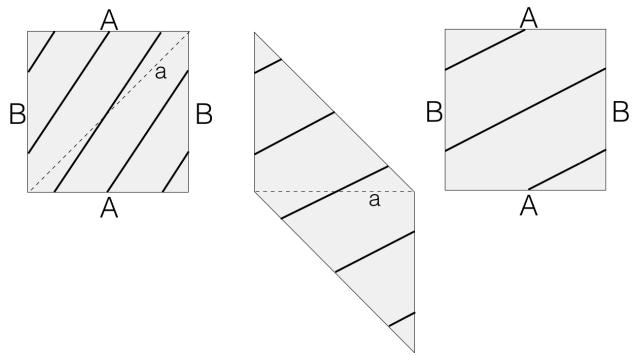
3. Find the continued fraction expansions of $3/2, 5/3, 8/5$, and $13/8$. Describe any patterns you notice, and explain why they occur.

In the following problems, we will determine the effect of the shearing automorphism from Page 8 # 4 on a trajectory τ and its cutting sequence $c(\tau)$.

4. First, we will apply symmetry to reduce our work to just one set of trajectories. Show that, given a linear trajectory in any direction on the square torus, we can apply rotations and reflections so that it is going left to right with slope ≥ 1 .

5. Since we have reduced to the case of slopes that are ≥ 1 , we will analyze the effect of the shear $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, because they work nicely together. Later we will show that everything else can be reduced to this case.

As an example, we'll use the trajectory τ with slope $3/2$, with corresponding cutting sequence $c(\tau) = \overline{BAABA}$ (left picture). We shear it via $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, which transforms the square into a parallelogram (middle picture), and then we reassemble the two triangles back into a square torus, while respecting the edge identifications (right picture). The new cutting sequence is $c(\tau') = \overline{BAB}$.

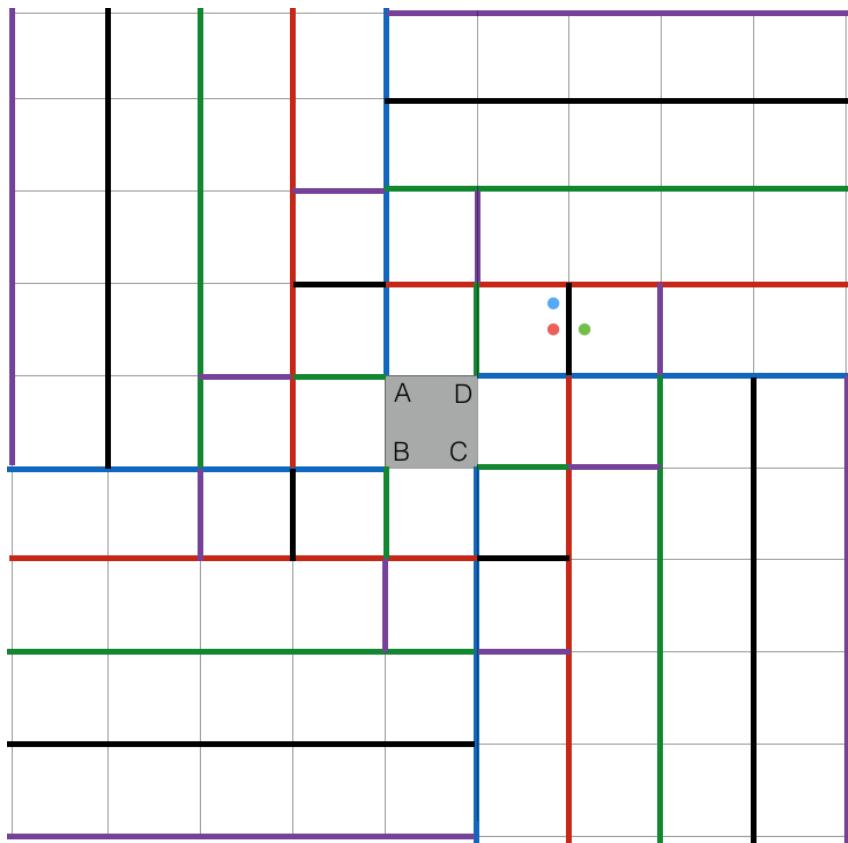


Do this geometric process for three different trajectories τ of your choice: Sketch a trajectory τ , sketch its image as a parallelogram after shearing by $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, and then sketch the reassembled square with the new trajectory τ' . For each, record $c(\tau)$ and $c(\tau')$. Try to find the pattern: a rule to get $c(\tau')$ from $c(\tau)$. Hint: You can use the “edge marks” technique from Page 4 # 3 on the parallelogram edges to make an accurate picture.

Math 424: Billiards, Surfaces and Geometry

- 1.** The purpose of this problem is to prove the “decomposition into necklaces” shown on the next page, for outer billiards on the square.

In Page 5 # 4, we showed that all of the lines in the square grid are not allowed: the images of points on the blue lines are ambiguously defined, because they could reflect through two different vertices of the square. The inverse images of the blue points (red) are thus not allowed, nor the inverse images (green) of those, nor the inverse images (black) of those, nor the inverse images (purple) of those...



- (a) Plot the trajectories of the blue, pink and green points under the outer billiards map. They are all periodic, so keep going until you get back to the original point.
- (b) Write down the sequence of corners reflected through, for each point: A, B, D, \dots
- (c) Prove the following statement: Two points in the same square reflect through the same sequence of corners. Points in different squares reflect through a different sequence of corners.
- (d) How do you know the first place where the sequences will differ?

- 2.** Prove that a trajectory with slope p/q (in lowest terms) on the square billiard table has period $2(p+q)$. *Note:* We have stated this in class several times, with several different arguments; the goal here is to write down a clear, rigorous proof.

- 3.** (Continuation of Page 10 # 5) Show that if we apply the shear $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ to the square torus:

- (a) The effect on the slope of a trajectory is to decrease it by 1.
 - (b) The effect on the cutting sequence corresponding to a trajectory whose slope is greater than 1 is to remove one A between each pair of Bs .
- 4.** Find the first few steps of the continued fraction expansion of π . Explain why the common approximation $22/7$ is a good choice. Then find the best fraction to use, if you want a fractional approximation for π using integers of three digits or fewer.

Math 424: Billiards, Surfaces and Geometry

1. In Page 11 # 1, we showed that for the outer billiard map on the square, points in a given square in the grid move together. Now we will explore *how* they move.

(a) Plot the complete orbit (meaning, until you get back to where you started) of the R and of the winky face under the outer billiard map. One step is shown for the R.

(b) Prove that the square of the outer billiard map (this means that you apply it twice) is a translation.

2. We need one more piece in order to relate trajectories on the square torus, continued fractions, and cutting sequences.

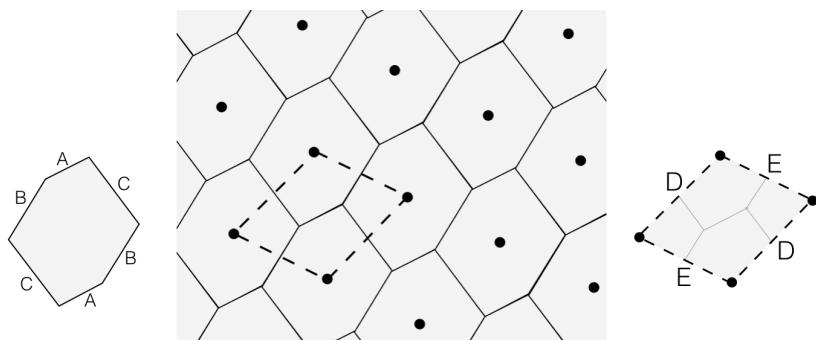
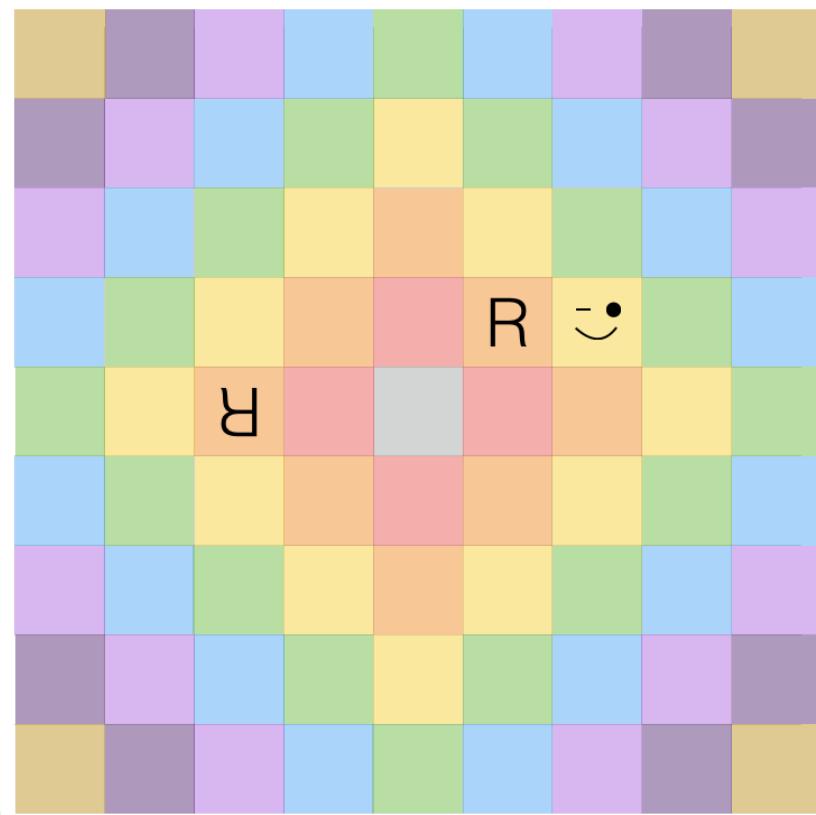
Show that if we apply the flip $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to the square torus:

(a) The effect on the slope of a trajectory is to take its reciprocal.

(b) The effect on the cutting sequence corresponding to a trajectory is to switch *A*s and *B*s.

3. If we identify opposite parallel edges of a hexagon, what surface do we get? The figure to the right shows one way to figure it out, via a *cut-and-paste* approach. Explain.

An alternative approach is to sketch what it looks like to glue identified edges together, assuming that the hexagon is made out of stretchy material. Try this, too.

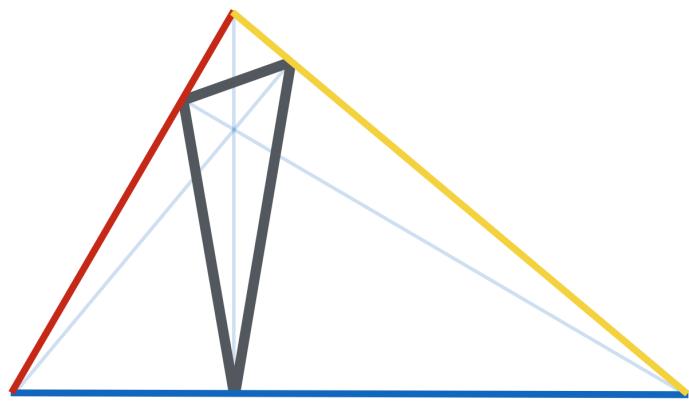


4. In the picture above, we tiled the plane with a hexagon that has three pairs of opposite parallel edges. Our “random” hexagon happened to be convex. Does a non-convex hexagon with three pairs of opposite parallel edges still tile the plane?

5. Prove that the continued fraction expansion of a number terminates (stops) if and only if it is rational.

Math 424: Billiards, Surfaces and Geometry

1. The figure to the right shows the altitude construction of a 3-periodic billiard path in the 40-60-80 triangle. In Page 9 # 3 we claimed that this path is part of a “strip,” or “family,” of nearby parallel billiard trajectories of period 6. Using a cutout of this triangle and a large blank piece of paper, unfold this trajectory, tracing a new copy of the triangle at each edge crossing, until the new triangle is a translation of the original triangle.

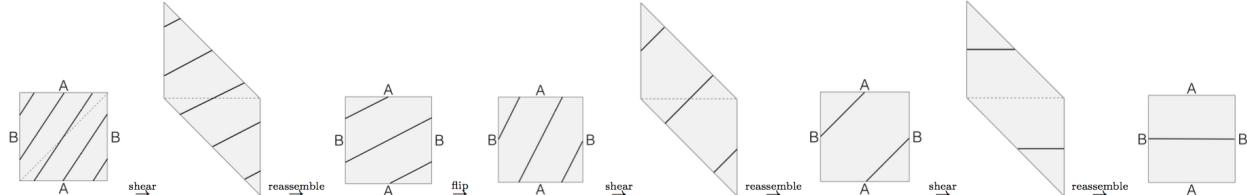


- (a) Sketch the “strip” of parallel billiard trajectories on your unfolding. How wide can you make the strip – what constrains its width?
- (b) Choose one trajectory in this strip, other than the original billiard path, and “fold” it back up, i.e. sketch it on the triangle above. Comment on any patterns you notice.

2. Starting with a trajectory on the square torus with positive slope, apply the following algorithm:

1. If the slope is greater than or equal to 1, apply the shear $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.
2. If the slope is between 0 and 1, apply the flip $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
3. If the slope is 0, stop.

An example is shown below.



We can note down the steps we took: shear, flip, shear, shear. We ended with a slope of 0. Work backwards, using this information and your work in Page 11 # 3a and Page 12 # 2a, to determine the slope of the initial trajectory. Keep track of each step.

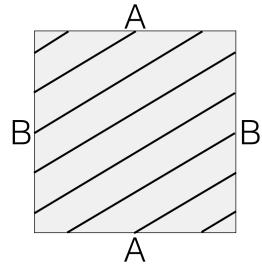
3. (Continuation) Translate the algorithm in the previous problem, which uses the *slope* of a trajectory, into an algorithm that uses the *cutting sequence* corresponding to a trajectory. You should translate each of the four sentences (“Starting with...,” 1, 2 and 3.) Then apply your algorithm to the cutting sequence \overline{ABAAB} and check that your results are consistent with the pictures in the figure above. *Hint:* use the part b’s.

4. Suppose that a given periodic cutting sequence on the square torus has period n . Are there any values of n for which you can determine the cutting sequence (perhaps up to some symmetry) from this information?

5. Let P be a convex quadrilateral that has a 4-periodic inner billiard trajectory that reflects consecutively in all four sides. Prove that P is cyclic. *Hint:* reread Page 6 # 2a.

Math 424: Billiards, Surfaces and Geometry

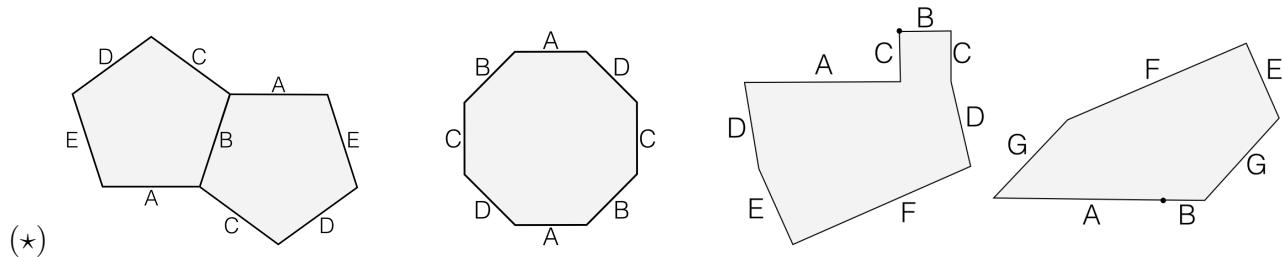
1. Apply the geometric algorithm from Page 13 # 2 to the trajectory shown to the right. Note down the steps you take (shears and flips) and use this information to work backwards from an ending slope of 0 to determine the slope of the initial trajectory. Show all of your steps.



2. (Continuation) Explain how shears and flips on the square torus are related to continued fraction expansions.

3. (Continuation) Find the cutting sequence corresponding to the trajectory above. Apply your algorithm from Page 13 # 3 to it, and check that your results at each step are consistent with each step of your work in problem 1.

4. We can create a surface by identifying opposite parallel edges of a single polygon, as we have done with the square and hexagon. We can do the same with two polygons, or with any number of polygons. We'll call such a surface a *Polygon Surface*. Some examples are below. Edges with the same letter are identified, as with *A* and *B* on the square torus. The two polygons on the right side together form a single surface.



Make up your own example of a polygon surface that is different from everyone else's, made from *three* polygons. We will call your new surface *S*. How many edges does *S* have?

(For a sense of what it would be like to live on the double pentagon surface, watch the short video <http://www.tinyurl.com/doublepentagon>.)

5. Consider a billiard trajectory in the unit circle, where at each impact the trajectory makes angle α with the circle.

- (a) Find the central angle θ from the circle's center, between each impact point and the next one, as a function of α .
- (b) Prove that if $\theta = 2\pi p/q$ for some $p, q \in \mathbb{N}$, then every billiard orbit is q -periodic and makes p turns around the circle before repeating.
- (c) Prove that if θ is *not* a rational multiple of π , then the orbit of every point is *dense*: every interval on the circle contains points of its orbit. *Hint:* Review your Real Analysis book for examples of one set that is dense in another, to make sure that you understand the question.

Math 424: Billiards, Surfaces and Geometry

1. For a natural number p , how many periodic billiard paths (up to symmetry) of period $2p$ are there on the square billiard table? Check your answer with your previous results.

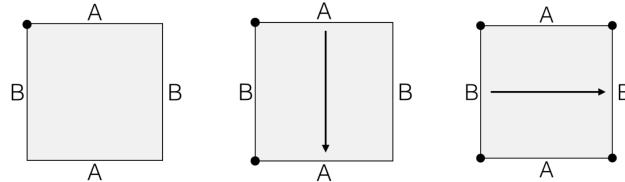
2. Another way to count billiard trajectories in the square is to ask how many periodic trajectories of length less than L it has. This question should be understood properly: Periodic trajectories appear in parallel families; we will count the number of such families.

(a) How long is the trajectory of slope 2? The trajectory of slope $1/3$?

(b) Show that the number of periodic families of length less than L is approximately $\pi L^2/8$.

3. Given that the continued fraction expansion of a particular number is $[0; 1, 2, 2]$, find the cutting sequence corresponding to a trajectory on the square torus with this slope.

4. *Vertex chasing.* To explain how to count the vertices of a surface, we will use the square torus. First, mark any vertex (say, the top left). We want to see which other vertices are the same as this one. The marked vertex is at the left end of edge A , so we also mark the left end of the bottom edge A . We can see that the top and bottom ends of edge B on the left are now both marked, so we mark the top and bottom ends of edge B on the right, as well. Now all of the vertices are marked, so the square torus has just one vertex. (We already knew that – how?)



Determine the number of vertices for a hexagon with opposite parallel edges identified. Do the same for each surface in Page 14 # 4, and for your surface S created in the same problem.

5. In Page 12 # 3, we showed how a hexagon can be cut into 4 pieces and reassembled into a parallelogram, while respecting the edge identifications. For the moment, let's forget about surfaces and just cut and reassemble polygons. Show how to dissect a 1×3 rectangle into pieces that can be reassembled into a square. Use as few pieces as possible.

6. As mentioned in Page 7#4, an active area of research is to describe, or “characterize,” all possible cutting sequences on a given surface. Now we can do this for the square torus.

Theorem. Valid cutting sequences on the square torus are those that do not fail under the following algorithm:

Starting with an infinite sequence of As and Bs , apply the following algorithm:

1. If there are multiple Bs separated by single As , switch As and Bs .
2. If there are multiple As separated by single Bs , remove an A between each pair of Bs .
3. If the sequence has AA somewhere and BB somewhere else, stop; it fails to be a valid cutting sequence.

Earlier in the course, students conjectured that a cutting sequence could only have two consecutive numbers of As , such as 2 and 3, between each pair of Bs , e.g. \overline{BABAAA} is not allowed. Use the theorem to prove this conjecture true. Start by writing down a clear statement of the result. Make sure that your argument works for aperiodic sequences corresponding to irrational slopes.

Math 424: Billiards, Surfaces and Geometry

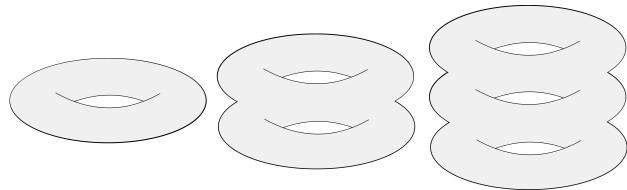
1. Consider again the theorem in Page 15 # 6.

(a) The vexing part of this characterization is that it doesn't have a step saying, "Stop! Congratulations; you have a valid cutting sequence." It only says, "Keep going; your cutting sequence hasn't proven to be invalid yet." But it turns out that it's the best we can do. Explain why this algorithm *does* stop for a *periodic* cutting sequence.

(b) I left out one technical point of the theorem: It actually characterizes the *closure* of the space of all cutting sequences. Valid cutting sequences are in the interior of the space, and cutting sequences such as $\dots AAABAAA \dots$ are on the boundary of the space. Explain why this cutting sequence does not fail in the algorithm, and why it is not a valid cutting sequence. Find another cutting sequence on the boundary, other than $\dots BBBABBB \dots$

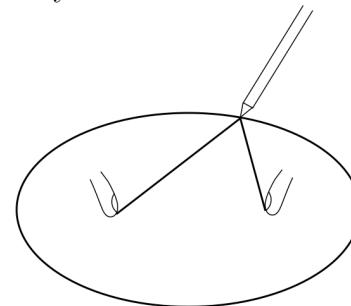
2. Once we've made a surface, the *Euler characteristic* gives us a way of easily determining what kind of surface we obtain, without needing to come up with a clever trick like cutting up and reassembling hexagons into parallelograms. Given a surface S made by identifying edges of polygons, with V vertices, E edges, and F faces, its Euler characteristic is $\chi(S) = V - E + F$. Find the Euler characteristic of the square torus, the cube, the tetrahedron, and the hexagon surface.

3. (Continuation) One of the main goals of the field of *topology* is to classify surfaces by their *genus*, which, informally speaking, is the number of "holes" it has. The surfaces to the right have genus 1, 2 and 3.



We can use the Euler characteristic to determine the genus of a surface: A surface S with genus g has Euler characteristic $\chi(S) = 2 - 2g$. Use this to find the genus of the square torus, the cube, the tetrahedron, the surfaces in Page 14 # 4, and your surface S .

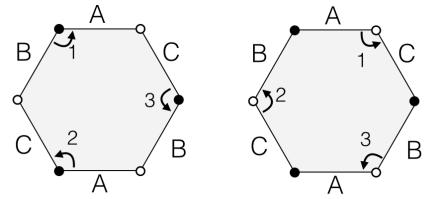
4. In previous courses, you have described an *ellipse* algebraically. Geometrically, you can construct it as follows (actually do this): Take a length of string and have your friend hold down the two endpoints, so that the string is somewhat loose. With your pencil, pull out the string until it is taut and trace out all the points the pencil can reach, as shown. Each of the two endpoints of the string is called a *focus* of the ellipse. Show that a billiard trajectory through one focus reflects through the other focus.



5. Imagine that you are a tiny creature living on a smooth surface, and you are trying to determine whether you are living on a sphere or a torus. Each day, you plant a stake into the ground, tie one end of a ball of yarn to the stake, and go for a very long walk, letting out yarn as you go. When you return to the stake, you wind the yarn back onto the ball. Explain how this type of experiment would help you determine the genus of your planet. Can you devise an experiment to differentiate between planets of genus 1 and 2?

Math 424: Billiards, Surfaces and Geometry

- 1.** *Walking around a vertex.* We can determine the angle around a vertex by “walking around” it, as shown in the figure for a hexagon with opposite parallel edges identified. The left picture shows that the angle around the black vertex is $3 \cdot \frac{2\pi}{3}$, and the right picture shows the same for the white vertex. Explain what is going on.



Since the black and white vertices each have 2π of angle around them, all the corners of the hexagon surface come together in a flat plane. (We already knew that – how?)

Find the angle around each vertex of the surfaces in Page 14 # 4, and for your surface S .

- 2.** (Continuation) A surface is called *flat* if it looks like the flat plane everywhere, except possibly at finitely many *cone points* (vertices), where the angle around each vertex is a multiple of 2π . Prove that if a surface is created by identifying opposite parallel edges of a collection of polygons, then it is flat. (If you were not able to do problem 1 for your surface S because you didn’t know its angles precisely, go back and do it with this result in hand!)

- 3.** Reconsider the question about billiard trajectories of length L .

- (a) Explain why the number of lattice points inside a disc of radius L is approximately πL^2 , especially when L is large.
 (b) Use this to prove Page 15 # 2(b).

- 4.** A *parabola* is the set of points that are equidistant from a given point (the focus) and a given line (the directrix).

- (a) What are the focus and the directrix for our favorite parabola, $y = x^2$?
 (b) Prove that a billiard trajectory through the focus of a parabola reflects to a ray parallel to the axis of the parabola.

- 5.** Here is our dream: To understand the effect of *every* automorphism of the square torus, on the cutting sequence corresponding to a trajectory. Here is our progress so far:

1. There are three types of automorphisms: rotations, reflections and shears. We understood the effects of rotations and reflections in Page ____ # _____. (fill these in)
2. Using rotations and reflections, we reduced our work, now only for shears, to the case of trajectories whose slope is in $[1, \infty)$ in Page ____ # _____.
 3. We understood the effect of the matrix $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ on a trajectory on the square torus in Page ____ # _____. By the way, we used $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ because it works nicely with trajectories whose slope is in $[1, \infty)$: it makes them simpler, like taking a derivative in calculus, while $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ makes them more complicated, like taking an integral.

Find the analogous effects on slopes of trajectories, of the matrices $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

- 6.** (Continuation: a challenging problem) There is just one more step, to show that every shear can be reduced to the ones we understand. Prove the following:

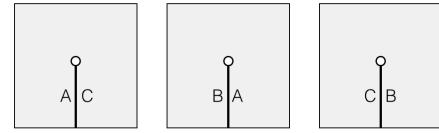
4. Every 2×2 matrix with nonnegative integer entries and determinant 1 is a product of powers of the shears $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. For example, $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2$.

Math 424: Billiards, Surfaces and Geometry

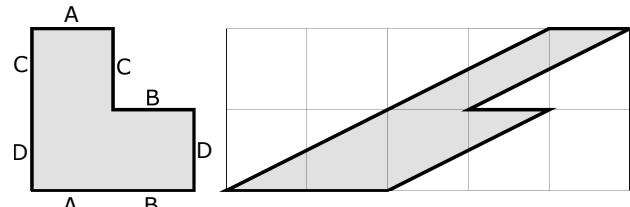
1. For each of the following, construct a surface, made from polygons with parallel edges identified, with the given property, or explain why it is not possible to do so:

- (a) One of the vertices has angle 6π around it.
- (b) One of the vertices has angle π around it.
- (c) Two of the vertices have different angles around them.

2. What does it look like to have 6π of angle at a vertex? Cut slits in three sheets of paper, and tape the edges together as shown. The vertex angle at the white point is now that of three planes, which is 6π . Bring your taped paper in to class.



3. In Page 8 # 4, we sheared the square torus by the matrix $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, which transformed it into a parallelogram, and then we reassembled the pieces back into a square, which was a twist of the torus surface. We can apply the same kind of transformation to many other surfaces. Consider the L-shaped surface made of three squares, with edge identifications as shown in the left picture. We shear it by the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, cut it into pieces, and then reassemble, *by translation only*, the pieces back into the L shape. Show how to do this. Make sure your reassembly respects the edge identifications.



4. One reason that people study cutting sequences on the square torus is that they have very low *complexity*: The *complexity function* $f(n)$ on a sequence is the number of different “words” of length n in the sequence. One way to think about complexity is that there is a “window” n letters wide that you slide along the sequence, and you count how many different things appear in the window.

- (a) What is the highest possible complexity for a sequence of *As* and *Bs*? For this question, consider all possible sequences of *As* and *Bs*, not just cutting sequences.
- (b) Confirm that the cutting sequence \overline{ABABB} has complexity $f(n) = n+1$ for $n = 1, 2, 3, 4$ and complexity $f(n) = n$ for $n \geq 5$.
- (c) Prove that a periodic cutting sequence on the square torus with period p has complexity $f(n) = n + 1$ for $n < p$ and complexity $f(n) = n$ for $n \geq p$.
- (d) Aperiodic sequences on the square torus are called *Sturmian sequences*. Show that Sturmian sequences have complexity $f(n) = n + 1$.

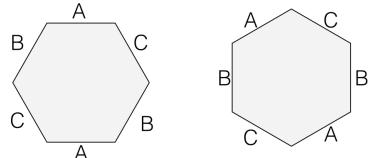
5. For a polygon surface (Page 14 # 4), a *cylinder direction* is a direction of any trajectory that goes from a vertex to another vertex, possibly crossing many polygons.

What are the cylinder directions for the square torus?

Math 424: Billiards, Surfaces and Geometry

1. A *caustic* of a planar billiard table is a curve such that, if a billiard trajectory is tangent to it, then the billiard trajectory is tangent to it after every reflection. Show that the caustic of a circle is another circle, and find its radius as a function of the angle α between the billiard trajectory and the (tangent line to the) circle.

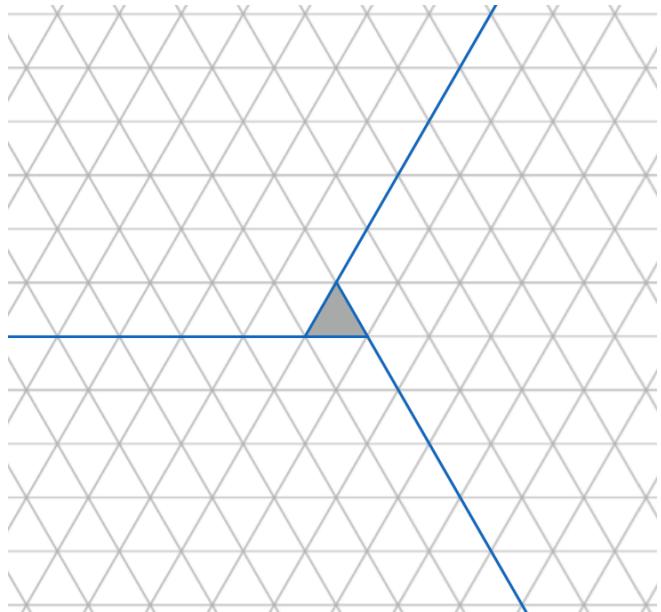
2. We can partition a polygon surface into *cylinders*. The boundary of a cylinder is a cylinder direction (Page 18 # 5), and there are no vertices inside a given cylinder. To construct the cylinders, draw a line in the cylinder direction through each vertex of the surface, which might pass through many polygons before it reaches its ending vertex. These lines cut the surface up into strips, and then you can follow the edge identifications to see which strips are glued together. Several examples are shown on the next page for the double pentagon.



Sketch the horizontal *cylinder decomposition*, by shading each cylinder a different color, of each of the following surfaces: the square torus, the regular octagon, the L-shaped table from Page 18 # 3, and the regular hexagon for each of the orientations shown.

3. (Continuation) The *modulus* of a cylinder is the ratio of its width to its height. The width and height are measured parallel to, and perpendicular to, the cylinder direction. Find the modulus of each cylinder for each cylinder decomposition in the previous problem.

4. Consider the counter-clockwise outer billiard map on the *triangular* billiard table, as shown. Explain why points on the blue lines are not allowed. Then sketch the inverse images (red) of the blue lines, the inverse images (green) of the red lines, the inverse images (black) of the green lines, the inverse images (purple) of the black lines, and so on.



5. Consider a tiling billiards trajectory that crosses an edge e of the tiling.

(a) Show that, if you fold the tiling along edge e , the two pieces of trajectory that intersect edge e lie on top of each other.

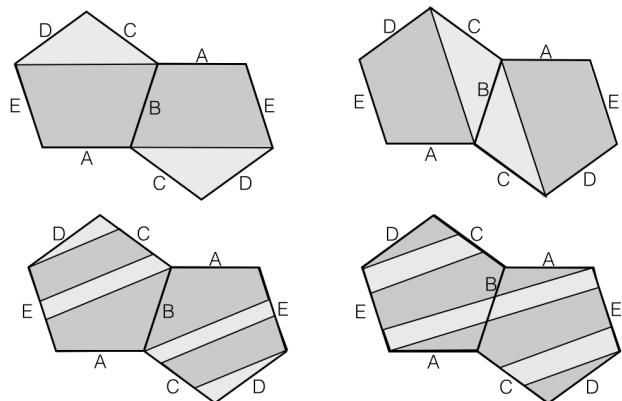
(b) Check this result by folding up your trajectories on the equilateral triangle tiling from Page 10 # 1. Can you fold *all* the edges of a trajectory onto each other at once?

6. (Challenge) Alice and Bob are in a square room with mirrored walls. They hate each other, and they don't want to see each other at all, through the room or in any reflection in the walls, looking in any direction. Show that it is possible for Alice and Bob to position a finite number of their friends in the room so that they cannot see each other (their friends block their view of the other person). *From the 1989 Leningrad Olympiad.*

Math 424: Billiards, Surfaces and Geometry

1. Here are cylinder decompositions of the double pentagon surface in four different directions.

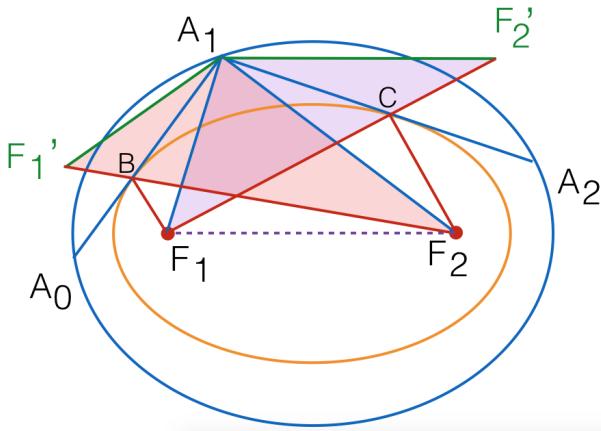
(a) For each decomposition shown, consider a trajectory on the surface, in the cylinder direction. Write down the cutting sequence for the trajectory in the light cylinder and for the trajectory in the dark cylinder. Think about similarities and differences with our work on the square torus.



(b) The two cylinder decompositions in the top line of the picture are essentially the same, just in a different direction. Construct a vertical cylinder decomposition of the surface. Is it the same as either of those in the bottom line of the picture?

2. **Theorem (billiards in an ellipse).**

Consider an ellipse E with foci F_1, F_2 . If some segment of a billiard trajectory does not intersect the focal segment F_1F_2 of E , then no segment of this trajectory intersects F_1F_2 , and all segments are tangent to the same ellipse E' with foci F_1 and F_2 .



(a) Consider the billiard trajectory $A_0A_1A_2$ in the larger ellipse E shown in the figure. Explain why $\angle A_0A_1F_1 = \angle A_2A_1F_2$.

(b) Reflect F_1 across $\overline{A_0A_1}$ to create F'_1 , and reflect F_2 across $\overline{A_1A_2}$ to create F'_2 . Explain why $\angle A_0A_1F'_1 = \angle A_0A_1F_1$ and $\angle A_2A_1F'_2 = \angle A_2A_1F_2$.

(c) Show that $\Delta F'_1A_1F_2$ and $\Delta F_1A_1F'_2$ are congruent.

(d) Mark the intersection of $\overline{F'_1F_2}$ with $\overline{A_0A_1}$ as B , and the intersection of $\overline{F_1F'_2}$ with $\overline{A_1A_2}$ as C . Show that the string length $|F_1B| + |BF_2|$ is the same as the string length $|F_1C| + |CF_2|$.

(e) Prove the theorem as stated above.

3. A *dissection* of a polygon P is a description of P as the union $P = P_1 \cup \dots \cup P_n$ of smaller polygons, no two of which overlap. In other words, we cut up P into n polygonal pieces. Two polygons P and P' are *dissection equivalent* if there are dissections $P = \bigcup_{i=1}^n P_i$ and $P' = \bigcup_{i=1}^n P'_i$ such that P_i and P'_i are congruent for each i . In other words, you can cut up P into polygonal pieces and reassemble them into P' . In this case we say $P \sim P'$.

(a) Explain why the 1×3 rectangle is dissection equivalent to a $\sqrt{3} \times \sqrt{3}$ square.

(b) Prove that \sim is an *equivalence relation*. (This requires proving that the relation is reflexive, symmetric and transitive. If you don't remember the definitions, look it up.)

(c) Show that every triangle is dissection equivalent to a parallelogram. What is the fewest number of pieces you can use?

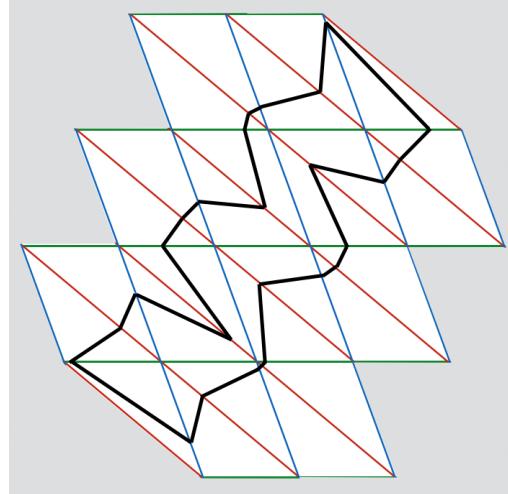
Math 424: Billiards, Surfaces and Geometry

1. The shears $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ *twist* the square torus once, as we have seen (Page 8 # 4). Draw a picture showing the effect of $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ on the square torus, including how to cut up and reassemble the pieces back into the square torus while respecting edge identifications. Then explain why the shears $\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$ twist the square torus m times.

2. (Continuation) Consider again the effect of the shear $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ on the L-shaped table – how many times does this shear twist each of its cylinders? Why?

3. The figure shows a periodic tiling billiards trajectory on a triangle tiling. Cut off (on the separate page you received) the shaded part and then fold along all the edges of the tiling, in such a way that every part of the trajectory lies on a single line. *Notes:* This may take you a while. Don't give up. Bring your folded triangles in to class. Save your folded triangles, as we will use them in subsequent problems.

4. Show that every parallelogram is dissection equivalent to a rectangle. What is the fewest number of pieces you can use?



5. To *triangulate* a polygon P means to partition it into triangles, whose vertices are the vertices of P , that don't overlap and whose union is P .

(a) How many different triangulations are there of a hexagon?

(b) Prove that every polygon, including non-convex polygons, can be triangulated.

6. For a planar domain with reflective boundary, the *illumination problem* asks: is it possible to illuminate the domain with a point source of light that emits rays in all directions? You can think of this as a candle in a mirrored room.

(a) Explain why this problem is only interesting for non-convex domains.

(b) Prove that the C-shaped domain made from five squares can be illuminated. Is there a particular place where you have to put your candle, to illuminate the whole thing, or does any point work?

(c) *Big challenge.* Try to make up an example of a domain that cannot be illuminated. This means that, no matter where you put your candle, some part of the room will still be dark. *Note:* We will see one later; the point here is not to search the internet, but to construct some examples to see that the problem is hard.

7. What is the caustic of a billiard trajectory in an ellipse?

Math 424: Billiards, Surfaces and Geometry

1. A set of numbers is *rationally related* if all of the numbers are rational multiples of each other. Give an example of a set that is rationally related, and of one that is not.

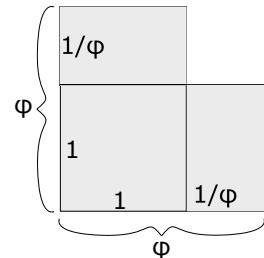
2. A particularly nice flat surface is the “Golden L,” whose opposite parallel edges are identified horizontally and vertically in the same way as in Page 18 # 3, and whose edge lengths are as shown in the picture.

(a) Find the number φ satisfying the property that when you cut off the largest possible square from a $1 \times \varphi$ rectangle, the leftover rectangle has the same proportions as the original.

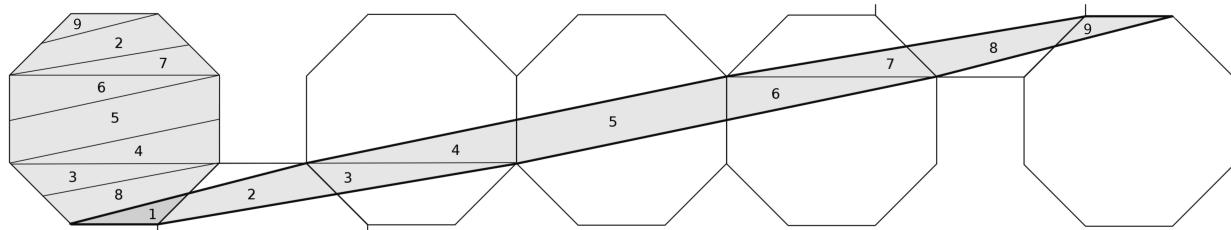
(b) Show that this number satisfies the relation $\varphi = 1 + 1/\varphi$.

(c) Are the moduli of the cylinders of the Golden L rationally related?

(d) Find the continued fraction expansion of φ .

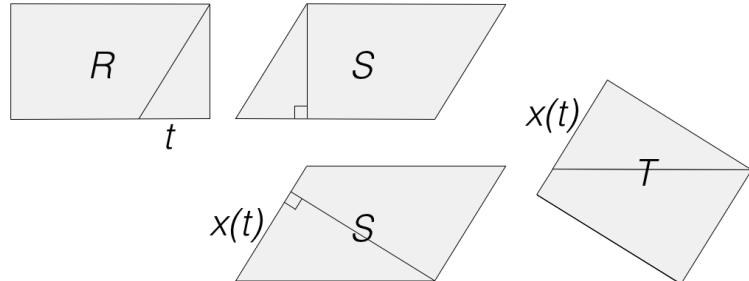


3. Start with a parallelogram, and add a diagonal to break it into two triangles T_1 and T_2 . Fold the parallelogram along this diagonal. Prove that, in this folded state, the circumcenter of T_1 and the circumcenter of T_2 coincide. (*Circumcenter*: if you don’t know, look it up.)



4. Amazingly, many surfaces made from regular polygons can be sheared, cut up and reassembled back into the original surface in the same way that we have done with square-tiled surfaces. One example is the regular octagon surface, shown above. What shearing matrix was applied to the surface? How many times was each cylinder twisted? Why?

5. The picture shows a two-step construction of dissection equivalence, based on a parameter t . The first step shows that $R \sim S$, and the second step shows that $S \sim T$. The middle two pictures are the same, but emphasize a different decomposition in each copy.



(a) Explain why the shape of the rectangle T varies continuously with t .

(b) Start with a 1×2 rectangle. For which values of t can you perform this construction? How about for a $1 \times r$ rectangle? *Hint:* Actually do this, with paper and scissors and tape.

(c) What does the rectangle T look like, at the minimum and maximum values of t ?

(d) Prove that every rectangle is dissection equivalent to a square.

Math 424: Billiards, Surfaces and Geometry

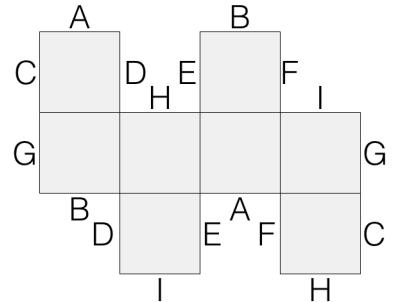
1. The figure shows a surface called the *eierlegende Wollmilchsau* (for a fun time, Google image search this term).

(a) Sketch each vertical cylinder a different color, and find the modulus of each.

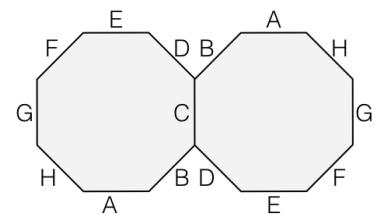
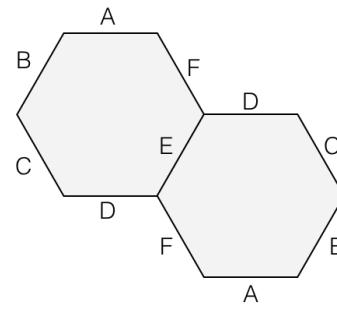
(b) Redraw the surface so that at least one vertical cylinder is arranged all in a vertical column, just as the horizontal cylinder with a G on each end is arranged all in a horizontal row in the figure.

(c) Find the number of vertices, edges and faces of the surface, and its genus.

(d) For which values of m is the shear $\begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$ an automorphism of the surface? and for $\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$?



2. Definition. The *double regular n -gon surface* is the surface made by identifying opposite parallel edges of two regular n -gons, one of which is the reflection of the other. We assume that each n -gon has a horizontal edge, and that all edges have unit length.



(a) Find the modulus of each horizontal cylinder, for the double hexagon surface and for the double octagon surface (shown above).

(b) Repeat part (a), for cylinder directions $\theta = \pi/6$ and $\theta = \pi/8$, respectively.

(c) Choose another double n -gon surface, with $n \geq 5$, and sketch it. Find the modulus of each of its horizontal cylinders.

3. Hilbert's 3rd problem asks: *Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces that can be reassembled into the second?*

(a) Read the history of Hilbert's 23 problems online or in a book, find out why they are important, and find something that is unique about the 3rd problem.

(b) Answer Hilbert's question for the two-dimensional case, of *polygons* with equal *area*: either prove that it is always possible, or give a counterexample with justification.

4. An *ellipse* with foci F_1, F_2 and string length ℓ is the locus of all points X satisfying $|F_1X| + |XF_2| = \ell$. A *hyperbola* with foci F_1, F_2 and “imaginary string length” ℓ is the locus of all points X satisfying $|F_1X| - |XF_2| = \pm\ell$.

In Page 20 # 2, we showed that every segment of a billiard trajectory in an ellipse that does *not* pass through the focal segment F_1F_2 is tangent to a confocal ellipse. Show that every segment of a billiard trajectory in an ellipse that *does* pass through the focal segment F_1F_2 is tangent to a confocal hyperbola.

Math 424: Billiards, Surfaces and Geometry

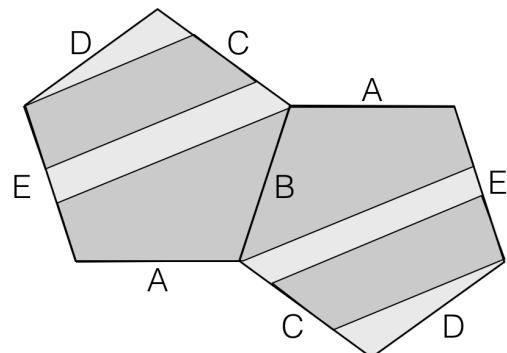
1. We showed that every triangle is dissection equivalent to a parallelogram in Page ____ #____, that every parallelogram is dissection equivalent to a rectangle in Page ____ #____, and that every rectangle is dissection equivalent to a square in Page ____ #____. Using this strategy, cut up an equilateral triangle into pieces and reassemble them into a square of the same area. Bring your pieces in to class. Can you find a dissection using fewer pieces?

2. An automorphism of a polygon surface must take vertices to vertices. In order for the shear $\begin{bmatrix} 1 & M \\ 0 & 1 \end{bmatrix}$ to be an automorphism of a particular surface, what must the relationship be between M and the moduli $m_1, m_2 \dots m_k$ of its horizontal cylinders c_1, c_2, \dots, c_k ? How many times does this shear twist each cylinder?

3. The figure shows a copy of the lower-left picture of Page 20 # 1.

(a) Some trajectories in the cylinder direction have cutting sequence \overline{DCEC} . Sketch one of them. Then describe the set of *all* such trajectories.

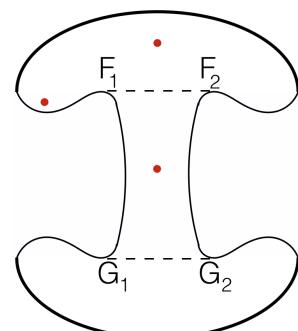
(b) Do the same for the other cutting sequence corresponding to a trajectory in the cylinder direction.



4. Given a tiling billiards trajectory on an obtuse triangle tiling (such as your folded triangles from Page 21 # 3), show that, if the tiling is folded along each edge that the trajectory crosses, all of the triangles that the trajectory crosses are inscribed in the same circumcircle in their folded state. Is the same true on non-obtuse triangle tiling?

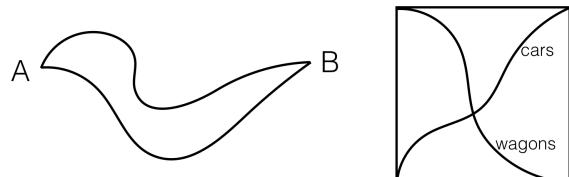
5. One way to pose the illumination problem is: “Is every region illuminable from *some* point in the region?” R and L Penrose constructed a counterexample, a region that cannot be illuminated from *any* point inside, shown to the right. The top and bottom are half-ellipses (the thick curves), with foci F_1, F_2 and G_1, G_2 respectively. Explain why this example works, by explaining which part of the region is illuminated when the light source is placed:

- (a) in the interior of a half-ellipse,
- (b) in the middle section of the region, and
- (c) in one of the lobes.



Roger Penrose is best known for creating the *Penrose tiling*, a set of two tiles, a “kite” and a “dart,” that *only* tile the plane aperiodically.

6. *Configuration spaces.* Towns A and B are connected by two roads. Suppose that two cars, connected by a rope of length $2r$, can go from A to B without breaking the rope. Prove that two circular wagons of radius r moving along these roads in opposite directions will necessarily collide. Do this by parameterizing the two roads and constructing the configuration space shown above.



Math 424: Billiards, Surfaces and Geometry

1. **Theorem (Modulus Miracle).** Every horizontal cylinder of a double regular n -gon surface has modulus $2 \cot \pi/n$.

(a) Compute this number for $n = 3, 4, 6, 8$ and whichever n you used in Page 23 # 2c, and check that it agrees with your previous calculations of the moduli.

(b) Explain why $\begin{bmatrix} 1 & 2 \cot \pi/n \\ 0 & 1 \end{bmatrix}$ is an automorphism of the double regular n -gon surface.

(c) In Page 8 # 4, we said that there are three types of automorphisms of the square torus: reflections, rotations, and shears. The same is true for all double regular n -gon surfaces. For a general n , what are the reflection and rotation automorphisms of the surface?

2. In the triangle tiling that you folded up in Page 21 # 3, there are “right-side-up triangles” R_i (shade these in), and “upside-down triangles” U_i (leave these white). You have shown that, in their folded state, all of the triangles end up inscribed in the same circle.

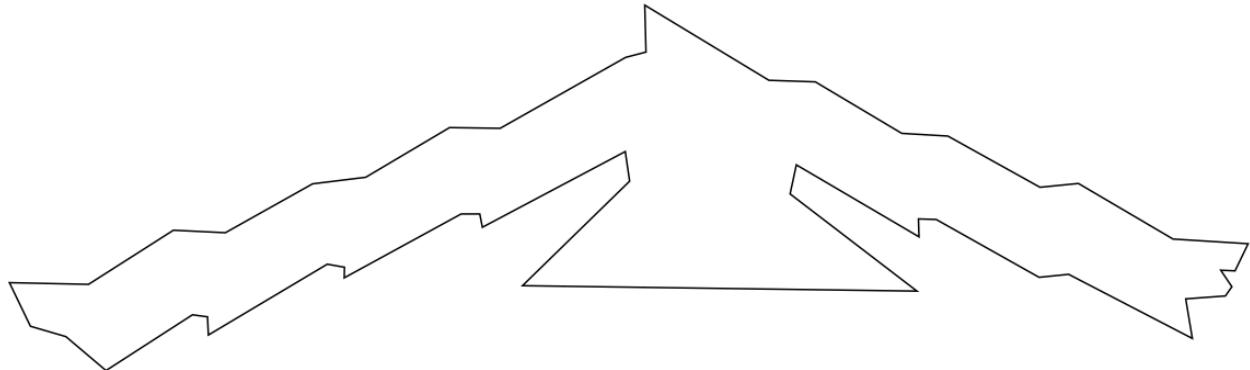
(a) Show that, in their folded state, all of the R_i are rotations of each other.

(b) Suppose that a trajectory passes from some right-side-up triangle R_1 , to an upside-down triangle U_1 , to another right-side-up triangle R_2 . By what angle do you rotate R_1 to get R_2 ?

3. Show that a given surface has a shearing automorphism in a given direction only if the moduli of its cylinders in that direction are rationally related.

4. *The art gallery problem.* Suppose you have an art gallery with priceless masterpieces on all of the walls, so you must ensure that each wall is in the view of a security guard. You wish to employ the smallest possible number of security guards to accomplish this goal.

(a) For the Mission Park shape below, show that 8 guards are sufficient. Can you do better?



(b) Explain how this *art gallery problem* is different from the *illumination problem*.

5. Do each of the following, by actually cutting up paper, and bring your pieces in to class:

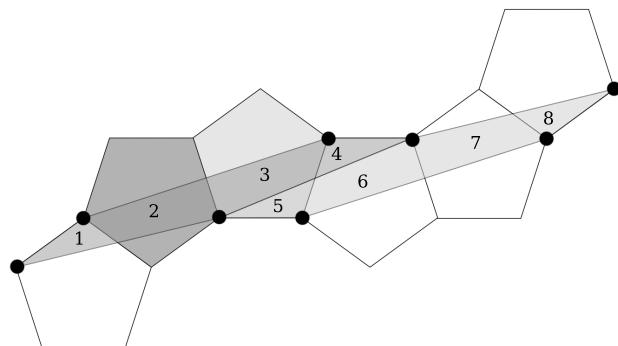
(a) Cut up and reassemble a 6×1 rectangle into a square.

(b) Cut up and reassemble a 30° - 60° - 90° triangle into a 3-4-5 triangle.

What is the smallest number of pieces you can use in each case?

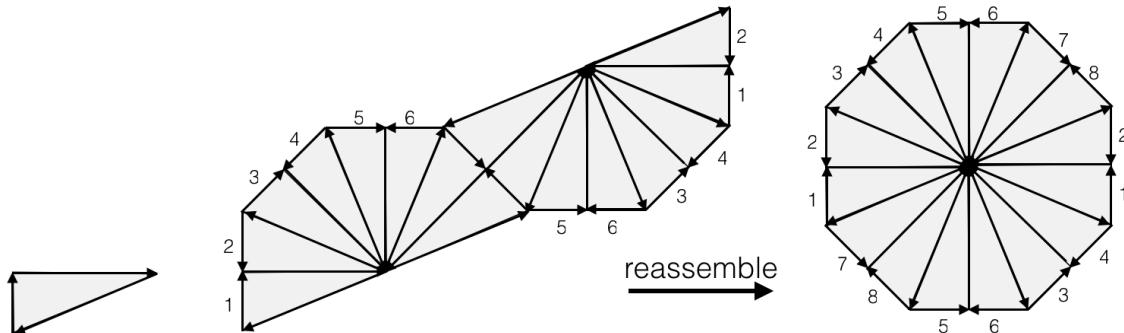
Math 424: Billiards, Surfaces and Geometry

- 1.** The picture shows the image of the double pentagon surface under the shear $\begin{bmatrix} 1 & 2 \cot \pi/5 \\ 0 & 1 \end{bmatrix}$.



- (a) Show how to reassemble (by translation, and respecting the edge identifications) the pieces back into the double pentagon.
- (b) Explain why the even-numbered pieces end up in one pentagon and the odd-numbered pieces in the other.
- (c) By comparing the widths (or heights) of its two cylinders, show that the double pentagon surface is a cut, reassembled, and horizontally stretched version of the Golden L.

- 2.** Our original motivation for studying the square torus was that it was the unfolding of the square billiard table. In fact, we can view *all* regular polygon surfaces as unfoldings of *triangular* billiard tables. We unfold the $(\pi/2, \pi/8, 3\pi/8)$ triangular billiard table until every edge is paired with a parallel, oppositely-oriented partner edge:

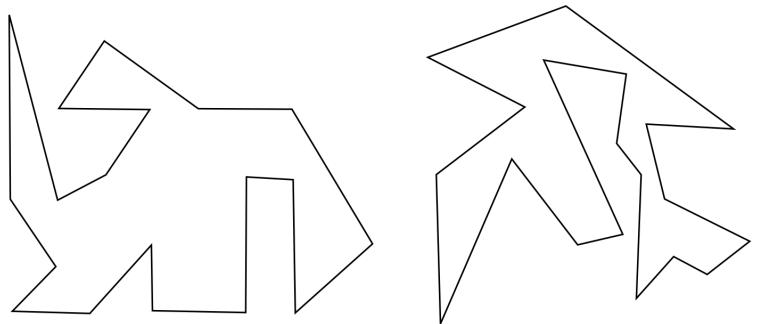


This gives us the regular octagon surface! So the regular octagon surface is the unfolding of the $(\pi/2, \pi/8, 3\pi/8)$ triangle.

- (a) What triangle unfolds to the double regular pentagon surface? Make a guess, then draw the unfolding as above.
- (b) Draw the “shooting into the corner” period-6 trajectory in the triangular billiard table on the left above. Then unfold it to a periodic trajectory on the regular octagon surface.

- 3.** In this problem, we will give an *upper bound* on the number of guards required to guard the priceless artwork on the walls of an n -sided polygon.

- (a) Explain why a guard at the corner of a triangle can see the entire triangle.
- (b) By triangulating your polygon and coloring each of the three vertices of each triangle a different color, prove that an upper bound for the number of guards required is $\lfloor n/3 \rfloor$.
- (c) Find the minimum number of guards for each polygon in the figure above.



Math 424: Billiards, Surfaces and Geometry

1. Show that, given a linear trajectory on the double regular n -gon surface, we can apply rotations and reflections so that it is traveling left to right in a direction θ with $0 \leq \theta \leq \pi/n$.
 2. When we sheared, cut and reassembled the square torus, we gave a rule for the effect of this action on a cutting sequence corresponding to a trajectory on the surface: *Given a trajectory τ on the square torus with slope greater than 1, and its corresponding cutting sequence $c(\tau)$, let τ' be result of applying $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ to τ . To obtain $c(\tau')$ from $c(\tau)$, shorten each string of As by 1.* We can do the same for the double pentagon, octagon, and other regular polygon surfaces:

Theorem. Given a trajectory τ on the double regular n -gon surface for n odd, or on the regular n -gon surface for n even, where the direction of τ is between 0 and π/n , and its corresponding cutting sequence is $c(\tau)$, let τ' be result of applying $\begin{bmatrix} -1 & 2\cot\pi/n \\ 0 & 1 \end{bmatrix}$ to τ . To obtain $c(\tau')$ from $c(\tau)$, keep only the sandwiched letters. (A sandwiched letter is one where the same letter both precedes and follows it.)

John Smillie and Corinna Ulcigrai proved the even case in 2011, and DD proved the odd case in 2013. It also holds for double regular n -gons for n even.

In the figure for Page 20 # 1, for the one in the lower right corner, you found cutting sequences $c(\tau_1) = \overline{BEC\bar{E}}$ and $c(\tau_2) = \overline{ABECD\bar{C}\bar{E}B}$ corresponding to the two trajectories in the cylinder direction.

- (a)** Apply the Theorem above to each of these cutting sequences to find $c(\tau'_1)$ and $c(\tau'_2)$.

(b) τ_1 and τ_2 are parallel, so τ'_1 and τ'_2 should be parallel as well. Explain. Then find the cylinder decomposition of the double pentagon in the direction parallel to τ'_1 and τ'_2 .

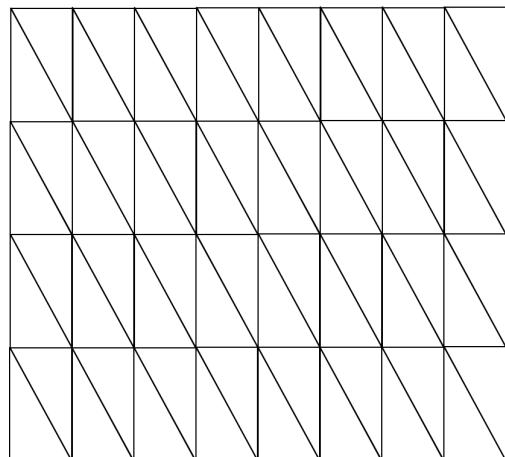
3. (Continuation) One rule is “shorten each string of A s by 1,” and the other is “keep only the sandwiched letters.” For a cutting sequence on the square torus, are these equivalent? If not, can you reconcile them?

4. The *defect* of a vertex is 2π minus the sum of all the angles at the vertex. The *total defect* of a polyhedron is the sum of the defects of all of its vertices. Find the total defect of the cube, the dodecahedron, the square torus, and the double pentagon surface.

5. Prove that, for a tiling billiards trajectory on an edge-to-edge right triangle tiling with parallel diagonals:

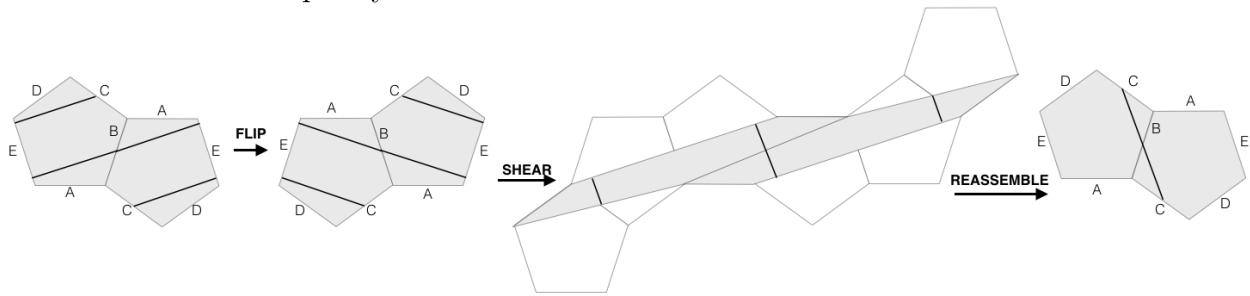
- (a)** If the trajectory never intersects the two legs of a triangle in a row, then it escapes. (*Escapes* means that it eventually leaves any ball of finite radius, or in other words that it is unbounded.)

(b) If a trajectory hits a hypotenuse at its midpoint, then it hits the midpoint of *every* hypotenuse it intersects.



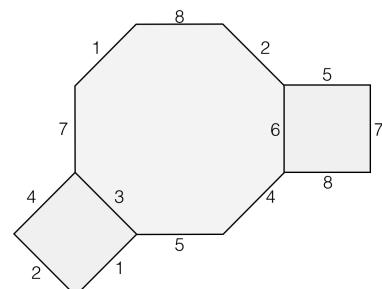
Math 424: Billiards, Surfaces and Geometry

1. The *Gauss-Bonnet Theorem* says that the total (Gaussian) curvature K of a closed surface S is $\int_{\partial S} K \, dA = 2\pi \chi(S)$, where $\chi(S)$ is the Euler characteristic.
- (a) Compute each side of this equation for the unit sphere S .
- (b) Descartes' special case of the Gauss-Bonnet Theorem says that the total defect of a polyhedron is $2\pi \chi(S)$. Check this formula for each of the four “polyhedra” in Page 27 # 4.
2. Geometrically, the matrix $\begin{bmatrix} -1 & 2 \cot \pi/5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \cot \pi/5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is a horizontal flip followed by a horizontal shear. The picture shows how the trajectory \overline{BECE} is flipped, sheared and reassembled into the trajectory \overline{BC} . Draw the same picture for the trajectory $\overline{ABECDCEB}$, to check the rule “keep only the sandwiched letters.”



3. You might wonder why we use $\begin{bmatrix} -1 & 2 \cot \pi/n \\ 0 & 1 \end{bmatrix}$ instead of $\begin{bmatrix} 1 & 2 \cot \pi/n \\ 0 & 1 \end{bmatrix}$. It is because its effect is simpler to describe: The matrix $\begin{bmatrix} -1 & 2 \cot \pi/n \\ 0 & 1 \end{bmatrix}$ induces the effect “keep only sandwiched letters” on the associated cutting sequence, while the matrix $\begin{bmatrix} 1 & 2 \cot \pi/n \\ 0 & 1 \end{bmatrix}$ induces the effect “keep only sandwiched letters, and also permute the edge labels.” What is this permutation?

4. For some time regular polygon surfaces (discovered by William Veech) and square-tiled surfaces (such as the eierlegende Wollmilchsau) were the only known examples of surfaces that have all of the symmetries we listed for the square torus: rotation, reflection, and the shear. Then Veech's student, Clayton Ward, discovered a larger family of such surfaces, now known as *Ward surfaces*. One way to describe a Ward surface is as a regular $2n$ -gon with two regular n -gons, where alternating edges of the $2n$ -gon are glued to one of the n -gons, and the remaining edges of the $2n$ -gon are glued to the other n -gon.

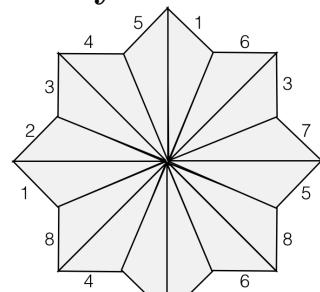


For $n = 4$, the Ward surface is an octagon and two squares, with edges identified as shown. Decompose this surface into horizontal cylinders, and find the modulus of each. Simplify your answers to a form where you can compare them. Are they rationally related?

5. Prove that every edge-to-edge right triangle tiling with parallel diagonals has an escaping tiling billiards trajectory.

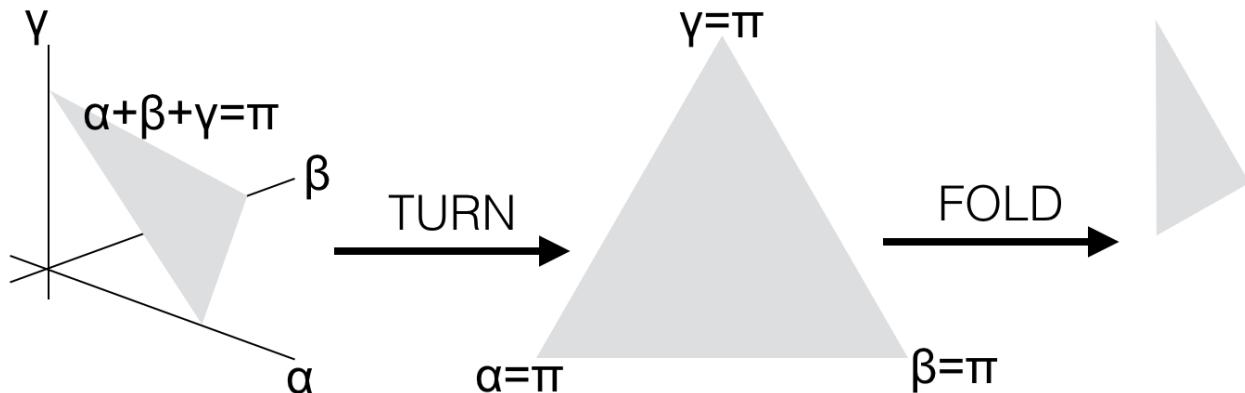
Math 424: Billiards, Surfaces and Geometry

1. The other way to describe a Ward surface is as the unfolding of the $(\pi/n, \pi/2n, \pi - \pi/n - \pi/2n)$ triangle. The picture shows the unfolding of this triangle for $n = 4$ into a surface. Show how to cut it up and reassemble this surface into the “octagon with two squares” presentation. In what sense are these “the same surface”?



2. Up to symmetry, a triangle can be uniquely specified by its three angles α, β, γ . There are two restrictions on the angles: $\alpha + \beta + \gamma = \pi$ and $\alpha, \beta, \gamma > 0$. So we can represent the space of all possible triangles (up to similarity) by the triangular part of the plane $x + y + z = \pi$ that lies in the first octant, as shown. In this picture, each *point* of the space represents a *triangle*. So the space of triangles is itself a triangle! It's easier to see the picture if we lay the triangle flat, as in the middle picture.

Make a **large** version of the middle picture in your notebook. Then sketch the following sets: the set of all right triangles (dashed), the set of all isosceles triangles (solid), all triangles with angles $0.12\pi, 0.35\pi, 0.53\pi$ (black dots), the set of all acute triangles (shaded).



3. (Continuation) In this representation of the space of all triangles, the angles are *marked* – we keep track of which angle is α and which is β , so the $(0.12\pi, 0.35\pi, 0.53\pi)$ triangle is different from the $(0.35\pi, 0.53\pi, 0.12\pi)$ triangle. This is somewhat redundant, so we can instead represent the space of triangles with *unmarked* angles. This takes advantage of the *symmetries* of the space of triangles to “fold up” the space so that each triangle is only represented once (the right part of the figure).

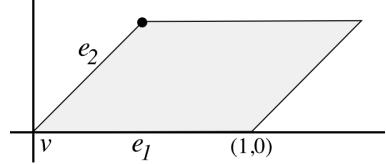
- (a) Sketch all of the sets from problem 2 on a folded picture in your notebook.
- (b) The triangles with the most symmetry form the edges of the space, and form the lines of symmetry that we fold on. Explain.
- (c) As mentioned in problem 1, triangles with angles of π/n and $\pi/2n$ unfold to Ward surfaces. Sketch the set of these Ward triangles in red on your picture.
- (d) Similarly, right triangles with a vertex angle of π/n unfold to regular polygon surfaces. Sketch the set of these triangles in blue on your picture as well.
- (e) Show that these sets are *discrete* in the space of triangles: for each point of the set, it is possible to find an open set containing that point, that does not contain any other point of the set.

Math 424: Billiards, Surfaces and Geometry

We glued up opposite parallel edges of a square to create a torus, and then we showed that gluing opposite parallel edges of *any* parallelogram creates a torus. If we glue up the edges of two different parallelograms, we get the “same” surface (in each case, a torus with genus 1), but in some important way they are “different” surfaces: one may be thin, like a necklace, while one is fat, like a donut. We can make this difference precise:

Two parallelograms are *equivalent* if there is an orientation-preserving similarity (rotation, dilation, translation, or compositions of these) that transforms one into the other.

We only want one representative of each parallelogram, so we will choose one in each equivalent group as follows: Given a parallelogram with two edges e_1, e_2 meeting counter-clockwise at v , we can translate it so that v is at the origin, we can rotate it so that e_1 is along the positive x -axis and the parallelogram lies above the x -axis, and we can dilate it so that e_1 has unit length, as shown. The marked point is the point in the upper halfplane that represents the parallelogram.



The other endpoint of e_2 lies somewhere in the upper halfplane, and this point uniquely determines the parallelogram. We can represent the space of parallelograms as the upper halfplane. With this representation, each point in the upper halfplane corresponds to a parallelogram, just as every point in the space of triangles corresponds to a triangle.

1. Identify the point(s) in the upper halfplane that correspond(s) to: a square torus; a parallelogram with side lengths 2 and 3 and an angle of $\pi/4$; rhombuses; the images of the square torus under the shear $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ for all real numbers t . (When the analogous shearing action is considered on more complicated kinds of translation surfaces, the result, which is called the *horocycle flow*, is an active area of research.)

If we call a parallelogram with opposite parallel edges identified a *parallelogram torus*, then *Teichmüller space* \mathcal{T}_1 is the space of all parallelogram tori (up to equivalence). We can represent \mathcal{T}_1 as the upper halfplane.

2. It is also possible to define Teichmüller space \mathcal{T}_g for surfaces of any genus g . It is trickier in higher genus, because there is no easy way to select a representative for each equivalent surface as we did for the parallelograms.

(a) Our favorite surfaces of genus 2 are the octagon and the double pentagon. Draw a “movie” of continuously deforming one into the other, while keeping identified edges parallel.

(b) We draw the double pentagon with a shared middle edge. Contract this edge, like tightening a girdle around its waist, in such a way that identified edges stay parallel throughout the whole continuous process. What genus surface do you get when the edge is length 0?

The set of Veech surfaces is *discrete* in Teichmüller space, which makes them hard to find.

3. (Continuation of Page 28 # 2) Another way to draw the space of all possible triangles is to take a line segment and break it into three pieces, and form a triangle out of the pieces.

- (a) Draw the space of triangles under this method. *Hint:* This takes some thought.
 (b) Shade in the region corresponding to acute triangles.

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(This page is from Tabachnikov, *Geometry and Billiards*, Example 1.2.) One reason to study billiards is because they model certain systems from classical mechanics. Consider the mechanical system consisting of two point masses m_1 and m_2 on the positive half-line $x \geq 0$. The collision between the points is *elastic*: that is, the energy and momentum are conserved. The reflection of m_1 off the left endpoint of the half-line is also elastic: if the point hits the “wall” $x = 0$, its velocity changes sign. Let the coordinates of the points be x_1 and x_2 .

1. Explain why and how the *configuration space* of the system can be described by the infinite sector $0 \leq x_1 \leq x_2$ with angle $\pi/4$.
2. Let v_1 and v_2 be the speeds of the point masses. As long as the points do not collide, the *phase point* (x_1, x_2) in configuration space moves with constant velocity $[v_1, v_2]$. Consider the instant of **collision of the point masses**, and let u_1, u_2 be the speeds after the collision. Use your knowledge of physics to justify each of the following equations: (If you need to review this, look up momentum and kinetic energy.)

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2, \quad \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1 u_1^2}{2} + \frac{m_2 u_2^2}{2}. \quad (1)$$

3. Now we introduce new variables to simplify the calculations: Let $\bar{x}_i = \sqrt{m_i} x_i, i = 1, 2$.
 - (a) In these variables, explain why the configuration space is the sector whose lower boundary is $\bar{x}_1/\sqrt{m_1} = \bar{x}_2/\sqrt{m_2}$.
 - (b) Show that the angle measure of this sector is $\arctan \sqrt{m_1/m_2}$.
4. In the new coordinate system, the speeds rescale the same way as the coordinates: $\bar{v}_i = \sqrt{m_i} v_i, i = 1, 2$. Rewrite the two equations (1) in these variables, to show that:
 - (a) the dot product of the velocity vector with the vector $[\sqrt{m_1}, \sqrt{m_2}]$ is preserved, and
 - (b) the magnitude of the velocity vector does not change in the collision.

5. Justify and explain each of the following statements:
 - (a) The vector $[\sqrt{m_1}, \sqrt{m_2}]$ is tangent to the boundary line $\bar{x}_1/\sqrt{m_1} = \bar{x}_2/\sqrt{m_2}$ of the configuration space, so the tangential component of the velocity vector does not change.
 - (b) When the two point masses collide, it corresponds to the configuration trajectory bouncing off of the boundary of the configuration space, according to the billiard reflection law.
6. Show that a **collision of the left point mass with the wall** $x = 0$ corresponds to the configuration trajectory bouncing off of the vertical boundary of the configuration space.

7. Summarize what we have shown here: “The configurations of _____ correspond to billiards in a _____.”
8. Show that, for a system of two point masses on the half-line, the maximum possible number of collisions is $\left\lceil \frac{\pi}{\arctan \sqrt{m_1/m_2}} \right\rceil$.

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Rainbows (billiards in a raindrop). This page is adapted from Tabachnikov, *Geometry and Billiards*, Digression 5.2, and from Phillips Exeter Academy Math 4-5, Page 57 # 1-2.

The diagram shows a ray of sunlight entering a raindrop. The change of direction at the air-water interface is governed by *Snell's Law of Refraction* (1) $\sin \alpha = k \sin \beta$, where k is the refraction coefficient, with $k \approx 4/3$ for air/water. Angles α and β are measured with respect to a line perpendicular to the surface of the drop, which passes through its center. The dashed line is parallel to the incoming ray.

1. Show that when $\alpha = 0.8$, we have $\beta = 0.5681$ (radians). Then show that when $\alpha = 0.8$, we have $\psi = 0.6723$, and that (2) $\psi = 4\beta - 2\alpha$ in general.

2. First, use Snell's Law (1) to express ψ as a function of α . As α increases from 0 to $\pi/2$, the angle ψ increases from 0 to a maximum value, and then decreases. Find this maximum value of ψ . This angle, which should be about 42° for $k = 4/3$, determines where a rainbow appears in the sky after a late-afternoon thunderstorm.

The next problems use a different calculus method to find the angle of the rainbow, by finding ψ in terms of the general refraction coefficient k , without reference to α or β .

3. A rainbow occurs at a maximum value of ψ , which occurs when $\frac{d\psi}{d\alpha} = 0$. (See *G&B*, pp. 80-81 for why this is.) Use this, and differentiate equation (2), to show that (3) $\frac{d\beta}{d\alpha} = \frac{1}{2}$.

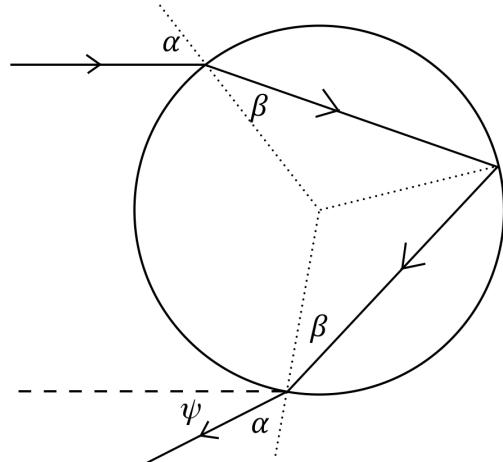
4. Differentiate Snell's Law (1), and combine it with equation (3) to obtain (4) $2 \cos \alpha = k \cos \beta$. Combine this with (1) to eliminate β , and show that (5) $\cos \alpha = \sqrt{\frac{k^2 - 1}{3}}$. Hint: This requires several lines of algebra. Don't give up.

5. Use (4) and (5) to find β in terms of k , and then plug in to (2) to get ψ in terms of k .

6. The refractive index k depends on the *color* of the incident light. The index $k = 4/3$ actually belongs to yellow light. The indices that belong to the extreme colors of the visual spectrum are $k = 1.331$ for red and 1.345 for violet. For each, find the corresponding value of ψ . Then show that the “apparent width” of a rainbow is about 2 degrees.

7. It so happens that if you close one eye and hold an aspirin tablet, whose diameter is 0.7cm, at arm's length, which is approximately 80cm from your eye, the tablet just barely covers the sun. Find the *apparent size* of the sun, which is the size of the angle *subtended* by the sun. Which is wider, the sun or a rainbow?

8. When you are looking directly at a rainbow, where is the sun in relation to you?



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5. Let L be the line with slope the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. Show that φ is an eigenvalue of the linear transformation $A = [\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}]$, and explain why L is invariant under A .

3. Does there exist a smooth domain D in the plane *without* a 2-periodic billiard trajectory?

n. For each integer $n \geq 3$, there is a Ward surface, consisting of a regular $2n$ -gon and two regular n -gons, or equivalently the unfolding of the $(\pi/n, \pi/2n, \pi - \pi/n - \pi/2n)$ triangle into a kind of “sunburst” figure. Draw the $n = 3$ Ward surface, whichever presentation you choose. Label the edge identifications.

n. Let’s define the *regular $2n$ -gon surface* to be the surface made by identifying opposite parallel edges of a regular $2n$ -gon. We assume that the $2n$ -gon has a pair of horizontal edges, and that all edges have unit length.

Modulus Miracle, single even polygon case. Let n be even. Every horizontal cylinder of a regular n -gon surface has modulus $2 \cot \pi/n$, except a central rectangular cylinder if there is one, which has modulus $\cot \pi/n$.

(a) This tells us that we can shear a regular n -gon surface, for n even, using the matrix $\begin{bmatrix} 1 & 2 \cot \pi/n \\ 0 & 1 \end{bmatrix}$ and cut and reassemble the pieces. Calculate $2 \cot \pi/n$ for the regular octagon surface, and check with your earlier calculation.

(b) The proof of this result (and the next) requires tedious arithmetic, so we will not undertake it. Instead, draw the regular decagon surface and shade its horizontal cylinder decomposition. Translate copies of each piece, so that each cylinder is shown as a full parallelogram. Does the statement above appear to be true for this example?

n. A *rational polygon* is a polygon whose angles are all rational multiples of π .

Explain why, for a billiard table that is a rational polygon, the unfolding requires a finite number of copies of the table in order to end up with pairs of oppositely-oriented parallel copies of the same edge.

n. (Continuation) Show that every rational polygon billiard table has a periodic billiard path. You may assume the previous result, even if you were not able to prove it.

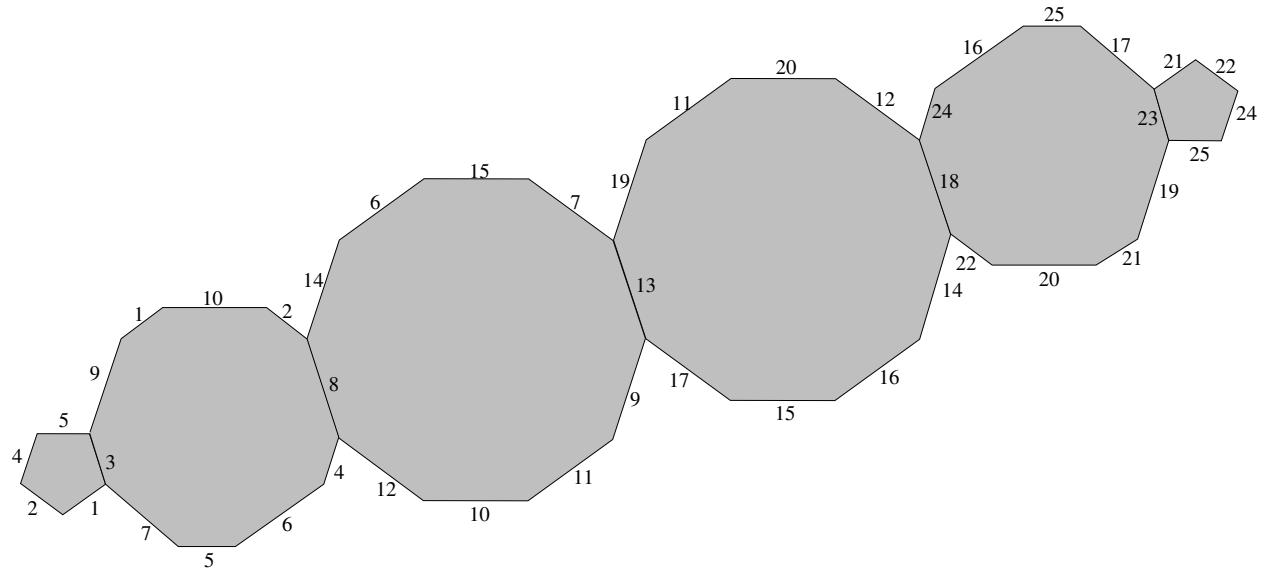
n. In 2006, Irene Bouw and Martin Möller discovered a larger family of lattice surfaces, now called Bouw-Möller surfaces in their honor. The regular polygon surfaces and the Ward surfaces are special cases of Bouw-Möller surfaces. Bouw and Möller gave an algebraic description of the surfaces, and later, Pat Hooper found a polygon decomposition for the surfaces, which we present here. I (DD) studied Bouw-Möller surfaces for my Ph.D. thesis, after studying the double pentagon.

For any $m \geq 2$, and any $n \geq 3$, the (m, n) Bouw-Möller surface is created by identifying opposite parallel edges of m semi-regular $2n$ -gons. A *semi-regular polygon* is an equiangular polygon with an even number of sides. Edge lengths alternate between two different values, which may be equal and may be 0. Their edge lengths are carefully chosen so that the cylinders all have the same modulus. In lieu of giving the precise Bouw-Möller surface definition

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here, we give some examples.

- (a) The $m = 2, n = 5$ Bouw-Möller surface is the double pentagon.
- (b) The $m = 3, n = 4$ Bouw-Möller surface is the octagon with two squares, studied [earlier problem](#).
- (c) The $m = 6, n = 5$ Bouw-Möller surface is shown below.



For the surface shown above, shade each horizontal cylinder a different color. Does it seem plausible that all of the cylinders have the same modulus?

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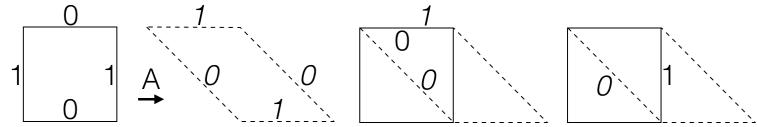
- n. Find the maximum possible number of collisions of two point-masses m_1 and m_2 on the half-line.
- n. Interpret the system of two point masses on a line segment, subject to elastic collisions with each other and with the endpoints of the segment, as a billiard on a right triangle.
- n. Consider a problem on Benford's law.
- n. (Continuation) We will show that the cutting sequence $w = \dots 0100101001001\dots$ corresponding to L is invariant under the substitution

$$\sigma : 0 \rightarrow 01, \quad 1 \rightarrow 0.$$

(a) Apply σ to the part of the cutting sequence w listed above. Does it seem reasonable that the sequence is invariant under σ ?

(b) Let $w_n = \sigma^n(0)$. (This means that you repeatedly apply the substitution rule above, starting with the string 0.) Prove that the lengths of w_n are the Fibonacci numbers.

(c) The map A transforms the square grid into a grid of parallelograms. Let w' be the cutting sequence of L with respect to the new grid. Since A takes one grid



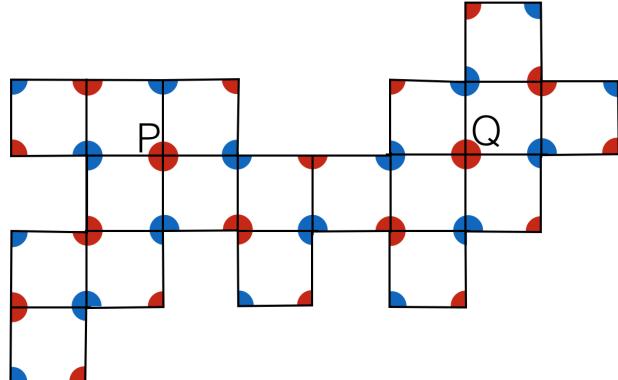
to the other, we have $w' = w$. On the other hand, it follows from the figure that each 0 in w corresponds to 01 in w' , and each 1 in w corresponds to 0 in w' . Explain. **add arrows**

n. Showing that Sturmian sequences are the non-periodic sequences with lowest possible complexity.

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- n. Prove that a billiard trajectory through the foci of an ellipse converges to its major axis.
- n. A *hyperbola* is defined . . .

n. *Tokarsky's example.* In the early 1950s, Ernst Straus asked, "Is every region illuminable from *every* point in the region?" G.W. Tokarsky gave several counterexamples, polygons where there is *some* point in the room, from which *some other* point is not illuminated. One example is to the right. Use unfolding to argue that point Q cannot be illuminated from point P . The idea is that blue vertices block paths from red to red.



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n.

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Reference: fill this in as you encounter new definitions.

<i>term</i>	<i>definition</i>	<i>where it first appears</i>
cutting sequence	the bi-infinite sequence of edges intersected by a line	page #

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Reference (continued)

<i>term</i>	<i>definition</i>	<i>where it first appears</i>
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