One-dimensional convection-diffusion equation

Solve the one-dimensional convection-diffusion equation

$$
\begin{aligned}
&-\varepsilon u''(x) + u'(x) = 1 \qquad x \in (0,1) \\
&u(0) = 0, \quad u(1) = 0.
\end{aligned}
$$

The exact solution to this problem is

$$u(x) = x + \frac{1 - e^{x/\varepsilon}}{e^{1/\varepsilon} - 1}.$$

It is clear that $\lim_{\varepsilon \to 0} u(x) = u$ for $x \in < 0, 1)$.

Discretize equations by means of the finite difference method. Take an equidistant mesh on the interval $(0,1)$ with the mesh size h and with nodes $x_i = ih$, $i = 0, 1, \dots, N$. The approximate solution in node x_i we denote by U_i.

The difference scheme for approximation of the first derivation:

(1) $u'(x_i) = \dfrac{u(x_i) - u(x_{i-1})}{h} + O(h), \qquad i = 1, \dots, N-1 \qquad$ backward

(2) $u'(x_i) = \dfrac{u(x_{i+1}) - u(x_i)}{h} + O(h), \qquad i = 1, \dots, N-1 \qquad$ forward

(3) $u'(x_i) = \dfrac{u(x_{i+1}) - u(x_{i-1})}{2h} + O(h^2), \qquad i = 1, \dots, N-1 \qquad$ central

(4) $u'(x_i) = \begin{cases} \dfrac{u(x_{i+1}) - u(x_{i-1})}{2h} + O(h^2), & i = 1 \\ \dfrac{3u(x_i) - 4u(x_{i-1}) + u(x_{i-2})}{2h} + O(h^2) & i = 2, \dots, N-1 \end{cases} \qquad$ upwind

(5) $u'(x_i) = \begin{cases} \dfrac{-2u(x_{i-1}) - 3u(x_i) + 6u(x_{i+1}) - u(x_{i+2})}{6h} + O(h^3), & i = 1 \\ \dfrac{u(x_{i-2}) - 6u(x_{i-1}) + 3u(x_i) + 2u(x_{i+1})}{6h} + O(h^3), & i = 2 \\ \dfrac{-2u(x_{i-3}) + 9u(x_{i-2}) - 18u(x_{i-1}) + 11u(x_i)}{6h} + O(h^3), & i = 3, \dots, N-1 \end{cases} \qquad$ upwind

The difference scheme for approximation of the second derivation:

(6) $u''(x_i) = \dfrac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} + O(h^2), \qquad i = 1, \dots, N-1$

$$(7)\ u''(x_i) = \begin{cases} \dfrac{11u(x_{i-1}) - 20u(x_i) + 6u(x_{i+1}) + 4u(x_{i+2}) - u(x_{i+3})}{12h^2} + O(h^3), & i = 1 \\ \dfrac{-u(x_{i-2}) + 16u(x_{i-1}) - 30u(x_i) + 16u(x_{i+1}) - u(x_{i+2})}{12h^2} + O(h^4), & i = 2, \dots, N-2 \\ \dfrac{-u(x_{i-3}) + 4u(x_{i-2}) + 6u(x_{i-1}) - 20u(x_i) + 11u(x_{i+1})}{12h^2} + O(h^3), & i = N-1 \end{cases}$$

Try these combinations of schemes:

1. (1)+(6), order 1, h without restrictions
2. (2)+(6), order 1, $h < \varepsilon$
3. (3)+(6), order 2, $h < 2\varepsilon$
4. (4)+(6), order 2, h without restrictions
5. (5)+(7), order 3, h without restrictions

Test that the order of given method is p. You have two posibilities:

a) $e_k \equiv ||\boldsymbol{U}_k - \boldsymbol{u}_k||_\infty \doteq C h_k^p$, where
$\boldsymbol{U}_k = (U_0^{(k)}, U_1^{(k)}, \dots, U_{N_k}^{(k)})^T$ is approximate solution on equidistant mesh with the mesh size $h_k = 1/N_k$ and
$\boldsymbol{u}_k = (u(0), u(h_k), \dots, u(N_k h_k))^T$ is exact solution on the same mesh. The number $\{N_k\}_{k=1}^n$ we choose
$N_k = a + bk, \qquad k = 1, \dots, n$
for some constants a, b and n. For example $a = 50, b = 25, n = 10$.

b) testing of numerical order of method:
$\ln(e_k) = \ln(C) + p\ln(h_k), \qquad k = 1, \dots, n$ and then $\dfrac{e_{k+1}}{e_k} \doteq \left(\dfrac{h_{k+1}}{h_k}\right)^p$
we use logarithm on this equation and we get sequence
$q_k = \dfrac{\ln(e_{k+1}/e_k)}{\ln(h_{k+1}/h_k)}, \qquad k = 1, \dots, n-1,$
which has to converge to order p.