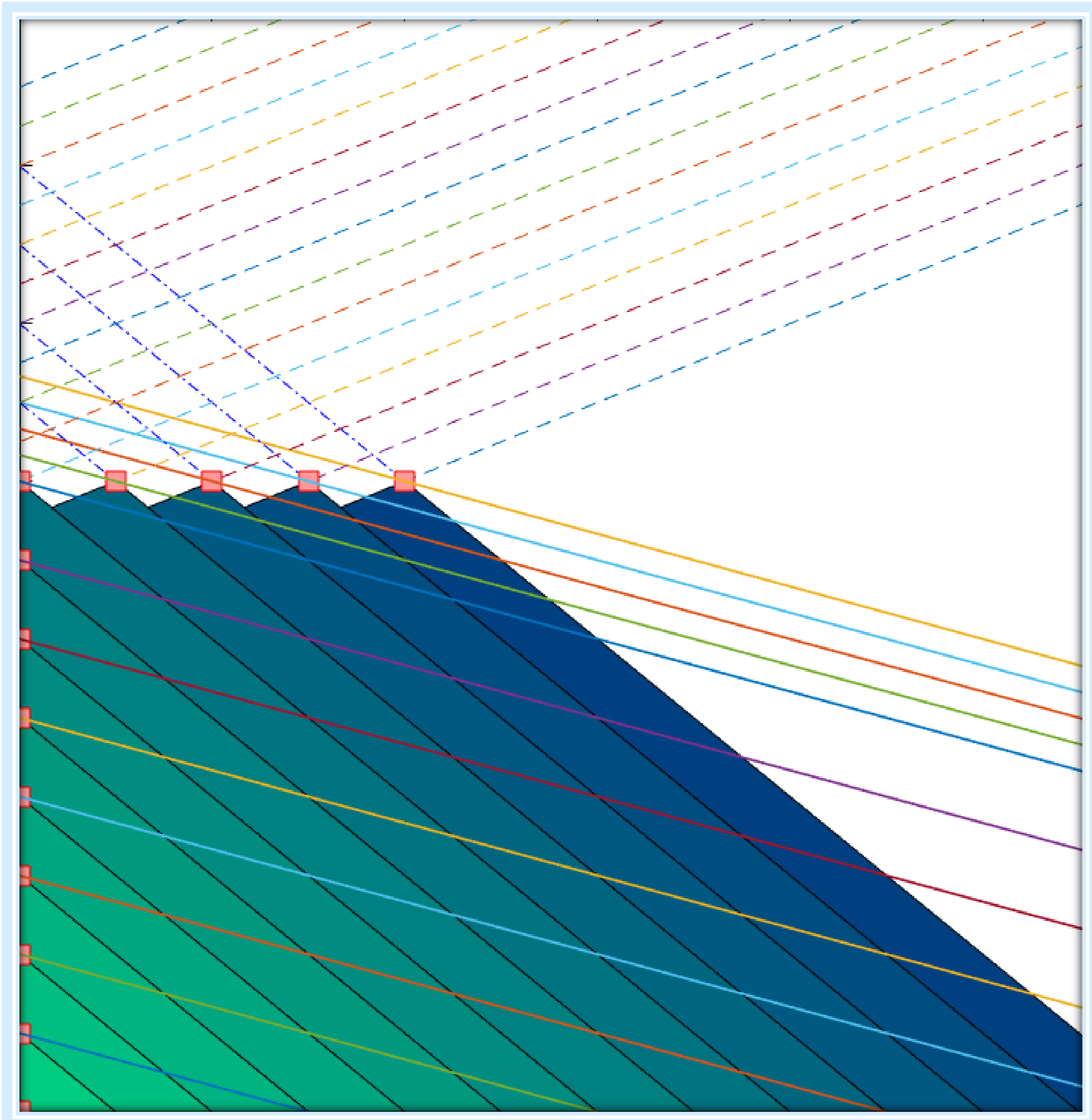

Parametric Analysis

RESEARCH PAPER

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Parametric Analysis

Parametric Analysis is often used to analyze behavior of a model within different parameter sets. It is where certain model parameters are modified and the effect of these variations is studied across the entire model in an iterative process. By varying these parameters, it will be possible to investigate different model design possibilities and find the most optimal solution to different scenarios.

As an application, this type of analyzation helps with the improvement of original model designs by continuously improving it based on the result of each iterations. Varying model parameters for a what-if scenarios may lead to making small modifications on some model decision variables that may produce better quality and minimized cost for the model where it can still retain its purpose.

DESCRIPTION

Parametric Analysis begins with finding a direction along which the objective function gradient (the cost vector) or the right-hand-side vector of the constraints are perturbed. Then, it is necessary to ascertain the resulting trajectory of the corresponding optimal solutions based on these perturbations. Therefore, in parametric analysis, we are interested in determining optimal solutions to a class of problems by perturbing any of its given parameters along a fixed direction.

This research will mostly deal with the following two types of perturbations:

- Perturbation of the Cost Vector (on the objective function)
- Perturbation of the Right-hand-side Vector (on the constraints)

Perturbation of the Cost Vector

We consider the following problem:

$$\begin{array}{ll} \text{Minimize} & \boldsymbol{cx} \\ \text{subject to} & \boldsymbol{Ax} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

For this perturbation, we suppose that the cost vector **c** is perturbed along some fixed direction **c'**. Therefore, we are replacing **c** by $\boldsymbol{c} + \boldsymbol{\lambda c'}$ where $\boldsymbol{\lambda} \geq \boldsymbol{0}$. In parametric analysis, we are interested in finding optimal solutions and their corresponding objective values as a function of $\boldsymbol{\lambda} \geq \boldsymbol{0}$.

EFFECT OF PERTURBATION OF THE COST VECTOR

We want to find the optimal solutions and their trajectories as $\boldsymbol{\lambda}$ is varied. We begin with analyzing the effect of this perturbation to our model. Since we are only replacing $\boldsymbol{c} = \boldsymbol{c} + \boldsymbol{\lambda c'}$, then our feasible region given by $\mathbf{Ax} = \mathbf{b}$ will not change since the constraint coefficients nor the right-hand-side vector have any relationship with $\boldsymbol{c}$. On the other hand, our objective function **z** which is given as the product of the cost vector **c** and the decision variable **x** will certainly change.

Given $\mathbf{c} = [c_1\ c_2]$ and $\mathbf{x} = [x_1\ x_2]$ we have,

$$\mathbf{z} = \mathbf{cx} = c_1 x_1 + c_2 x_2$$

Manipulating this equation to be able to represent the behavior of the objective function in two dimensions, we have

$$x_2 = \frac{-c_1}{c_2} x_1 + \frac{z}{c_2}$$

Applying the perturbation of the cost vector on c_1 and c_2, we will now have a new function for x_2,

$$x_2 = -\frac{(c_1 + \lambda c_1{}')}{(c_2 + \lambda c_2{}')} x_1 + \frac{z}{(c_2 + \lambda c_2{}')}$$

Note that we substituted c_1 with $c_1 + \lambda c_1{}'$ and c_2 with $c_2 + \lambda c_2{}'$.

EXAMPLE

Now, consider that we have the following model

Minimize

$$-x_1 - 3x_2 \qquad \textit{(objective function z)}$$

subject to

$$x_1 + x_2 \le 6$$

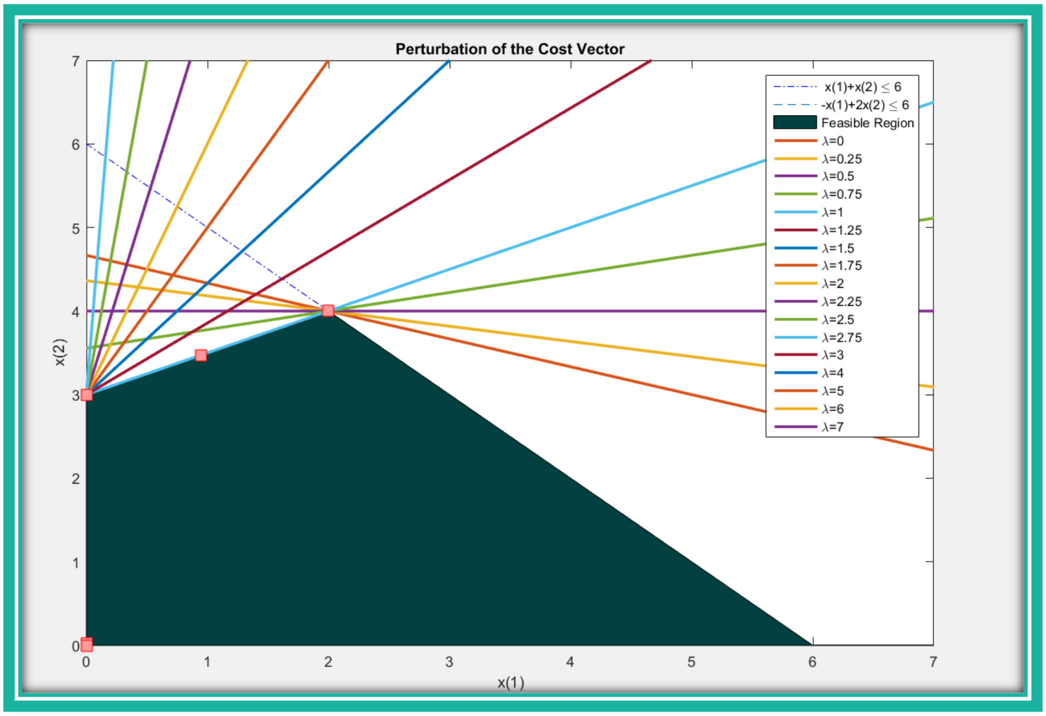
$$-x_1 + 2x_2 \le 6 \qquad \textit{(constraints)}$$

$$x_1, x_2 \ge 0$$

We want to perturb the cost vector $c = (-1, -3)$ along the vector $c' = (2,1)$ **.**We want to find the optimal solutions and optimal objective values of the class of problems whose objective function is then given by $(-1 + 2\lambda,\ -3 + \lambda)$ **for** $\lambda \ge 0$**.**

IMPLEMENTATION IN MATLAB

Using the **linprog** command, we are able to solve this linear programming problem:



It can be seen from the plot that we have the feasible region shaded and the lines correspond to the objective function values as λ is varied. The optimal points are marked and it is shown that for certain range of λ, we may have the same optimal points as shown by $\lambda = [0,1]$ and $\lambda = [1,3]$. By $\lambda = 3$, our objective function and optimal solutions go to zero.

IMPLEMENTATION IN GAMS

Parametric Analysis Results
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Pertubation of the Cost Vector:

Scen.:	Lambda:	Obj.f:	Type	Sol?	X(1):	X(2):	C(1):	C(2):
L01	0.000000	-14.0000	1	1	2.0000	4.0000	-1.0000	-3.0000
L02	0.100000	-13.2000	1	1	2.0000	4.0000	-0.8000	-2.9000
L03	0.500000	-10.0000	1	1	2.0000	4.0000	0.0000	-2.5000
L04	0.900000	-6.8000	1	1	2.0000	4.0000	0.8000	-2.1000
L05	1.000000	-6.0000	1	1	2.0000	4.0000	1.0000	-2.0000
L06	1.100000	-5.7000	1	1	0.0000	3.0000	1.2000	-1.9000
L07	1.500000	-4.5000	1	1	0.0000	3.0000	2.0000	-1.5000
L08	1.900000	-3.3000	1	1	0.0000	3.0000	2.8000	-1.1000
L09	2.000000	-3.0000	1	1	0.0000	3.0000	3.0000	-1.0000
L10	2.100000	-2.7000	1	1	0.0000	3.0000	3.2000	-0.9000
L11	2.500000	-1.5000	1	1	0.0000	3.0000	4.0000	-0.5000
L12	2.900000	-0.3000	1	1	0.0000	3.0000	4.8000	-0.1000
L13	3.000000	0.0000	1	1	0.0000	0.0000	5.0000	0.0000
L14	3.100000	0.0000	1	1	0.0000	0.0000	5.2000	0.1000
L15	3.500000	0.0000	1	1	0.0000	0.0000	6.0000	0.5000
L16	3.900000	0.0000	1	1	0.0000	0.0000	6.8000	0.9000
L17	4.000000	0.0000	1	1	0.0000	0.0000	7.0000	1.0000
L18	4.100000	0.0000	1	1	0.0000	0.0000	7.2000	1.1000
L19	4.500000	0.0000	1	1	0.0000	0.0000	8.0000	1.5000
L20	4.900000	0.0000	1	1	0.0000	0.0000	8.8000	1.9000
L21	5.000000	0.0000	1	1	0.0000	0.0000	9.0000	2.0000
L22	5.100000	0.0000	1	1	0.0000	0.0000	9.2000	2.1000
L23	5.500000	0.0000	1	1	0.0000	0.0000	10.0000	2.5000
L24	5.900000	0.0000	1	1	0.0000	0.0000	10.8000	2.9000
L25	6.000000	0.0000	1	1	0.0000	0.0000	11.0000	3.0000
L26	6.100000	0.0000	1	1	0.0000	0.0000	11.2000	3.1000
L27	6.500000	0.0000	1	1	0.0000	0.0000	12.0000	3.5000
L28	6.900000	0.0000	1	1	0.0000	0.0000	12.8000	3.9000
L29	7.000000	0.0000	1	1	0.0000	0.0000	13.0000	4.0000

The results in GAMS are clearly the same as in MATLAB. The only difference are the intervals of λ. We can see better on this graph the exact optimal points and exact objective function values as λ is varied. It is also shown here the effect of the perturbation on the values of the cost vectors c(1) and c(2). For $\lambda = [0,1]$, it can be seen that the optimal solutions are the same and are given by (2,4). For $\lambda = [1,3]$, we have optimal solutions (0,3) and for $\lambda = [3, \infty]$ we have (0,0). In our sample model, we can see that as λ becomes larger, our objective function increases and stays at 0. Our optimal solution coordinate values also decreases and seems to stay at 0.

Perturbation of the Right-hand-side Vector

We consider the same problem as above, the difference is that in this case we suppose that the right-hand-side vector $\boldsymbol{b}$ is perturbed along some fixed direction $\boldsymbol{b'}$. Therefore, we are now replacing $\boldsymbol{b}$ by $\boldsymbol{b} + \boldsymbol{\lambda b'}$ where $\boldsymbol{\lambda} \geq \mathbf{0}$. Our goal is still to find the optimal solutions and their corresponding objective values as a function of $\boldsymbol{\lambda} \geq \mathbf{0}$.

EFFECT OF PERTURBATION OF THE RIGHT-HAND-SIDE VECTOR

We want to find the optimal solutions and their trajectories as $\boldsymbol{\lambda}$ is varied. We begin with analyzing the effect of this perturbation to our model. Since we are only changing $\boldsymbol{b} = \boldsymbol{b} + \boldsymbol{\lambda b'}$ then our objective function $\boldsymbol{z}$ will not change since it doesn't depend on $\boldsymbol{b}$. On the other hand, our feasible region given by $\mathbf{Ax} = \mathbf{b}$ will certainly change.

Given $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} \end{bmatrix}$, $\mathbf{x} = [\boldsymbol{x_1}\ \boldsymbol{x_2}]$ and $\mathbf{b} = [\boldsymbol{b_1}\ \boldsymbol{b_2}]$, for $\mathbf{Ax} = \mathbf{b}$, we have,

$$\boldsymbol{a_{11}x_1 + a_{12}x_2 \leq b_1}$$

$$\boldsymbol{a_{21}x_1 + a_{22}x_2 \geq b_2}$$

Manipulating this equation to be able to represent the changes in the equations of the constraints, we have

$$\boldsymbol{x_2 \leq \frac{-a_{11}}{a_{12}}x_1 + \frac{b_1}{a_{12}}}$$

$$\boldsymbol{x_2 \geq \frac{-a_{21}}{a_{22}}x_1 + \frac{b_2}{a_{22}}}$$

Applying the perturbation of the right-hand-side vector on $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$, we will now have a new function for the constraints,

$$\boldsymbol{x_2 \leq \frac{-a_{11}}{a_{12}}x_1 + \frac{b_1 + \lambda {b_1}'}{a_{12}}}$$

$$\boldsymbol{x_2 \geq \frac{-a_{21}}{a_{22}}x_1 + \frac{b_2 + \lambda {b_2}'}{a_{22}}}$$

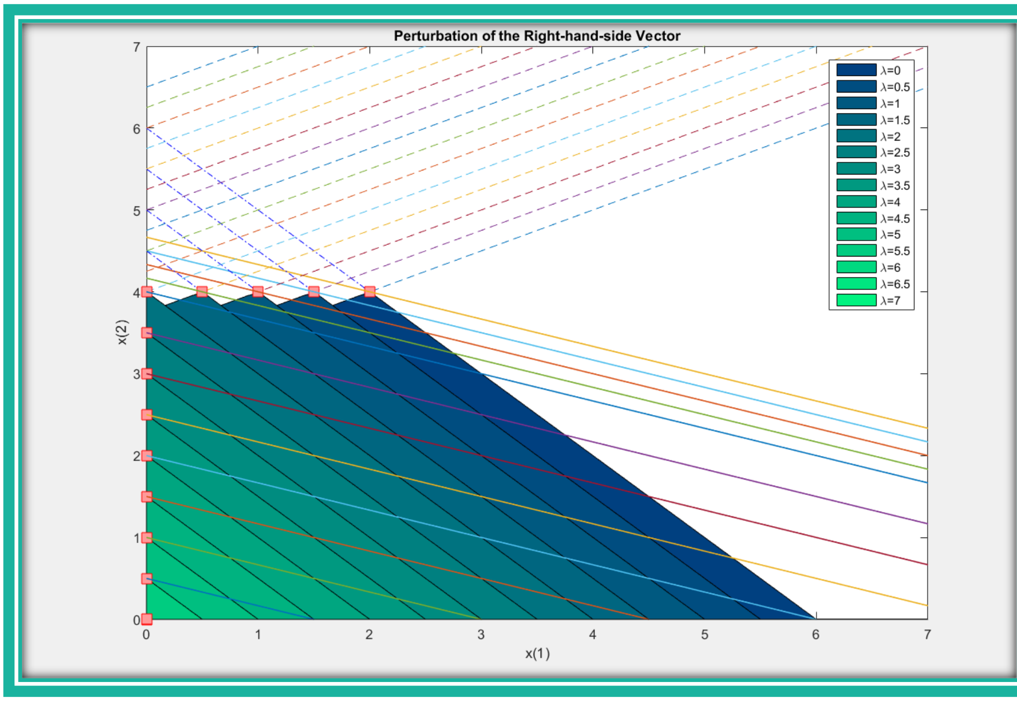
Note that we substituted $\boldsymbol{b_1}$ with $\boldsymbol{b_1 + \lambda {b_1}'}$ and $\boldsymbol{b_2}$ with $\boldsymbol{b_2 + \lambda {b_2}'}$.

EXAMPLE

Using the same example as above, we now want to perturb the right-hand-side vector $\boldsymbol{b} = (\mathbf{6}, \mathbf{6})$ along the vector $\boldsymbol{b'} = (-\mathbf{1}, \mathbf{1})$. We want to find the optimal solutions and optimal objective values of the class of problems whose constraints have the right-hand-side values given by $(\mathbf{6} - \boldsymbol{\lambda},\ \mathbf{6} + \boldsymbol{\lambda})$ **for** $\boldsymbol{\lambda} \geq \mathbf{0}$.

IMPLEMENTATION IN MATLAB

Using the **linprog** command, we are able to solve this linear programming problem:



Based from the plots of the regions, it is easy to see that as λ changes, then we will also have different feasible regions. It can also be seen that we also have different optimal values for each variations of λ unlike the previous example when it is still possible to have the same optimal solution for a certain range of λ.

IMPLEMENTATION IN GAMS

Pertubation of the Right-hand-side Vector:

Scen.:	Lambda:	Obj.f:	Type	Sol?	X(1):	X(2):	b(1):	b(2):
L01	0.000000	-14.0000	1	1	2.0000	4.0000	6.0000	6.0000
L02	0.100000	-13.9000	1	1	1.9000	4.0000	5.9000	6.1000
L03	0.500000	-13.5000	1	1	1.5000	4.0000	5.5000	6.5000
L04	0.900000	-13.1000	1	1	1.1000	4.0000	5.1000	6.9000
L05	1.000000	-13.0000	1	1	1.0000	4.0000	5.0000	7.0000
L06	1.100000	-12.9000	1	1	0.9000	4.0000	4.9000	7.1000
L07	1.500000	-12.5000	1	1	0.5000	4.0000	4.5000	7.5000
L08	1.900000	-12.1000	1	1	0.1000	4.0000	4.1000	7.9000
L09	2.000000	-12.0000	1	1	0.0000	4.0000	4.0000	8.0000
L10	2.100000	-11.7000	1	1	0.0000	3.9000	3.9000	8.1000
L11	2.500000	-10.5000	1	1	0.0000	3.5000	3.5000	8.5000
L12	2.900000	-9.3000	1	1	0.0000	3.1000	3.1000	8.9000
L13	3.000000	-9.0000	1	1	0.0000	3.0000	3.0000	9.0000
L14	3.100000	-8.7000	1	1	0.0000	2.9000	2.9000	9.1000
L15	3.500000	-7.5000	1	1	0.0000	2.5000	2.5000	9.5000
L16	3.900000	-6.3000	1	1	0.0000	2.1000	2.1000	9.9000
L17	4.000000	-6.0000	1	1	0.0000	2.0000	2.0000	10.0000
L18	4.100000	-5.7000	1	1	0.0000	1.9000	1.9000	10.1000
L19	4.500000	-4.5000	1	1	0.0000	1.5000	1.5000	10.5000
L20	4.900000	-3.3000	1	1	0.0000	1.1000	1.1000	10.9000
L21	5.000000	-3.0000	1	1	0.0000	1.0000	1.0000	11.0000
L22	5.100000	-2.7000	1	1	0.0000	0.9000	0.9000	11.1000
L23	5.500000	-1.5000	1	1	0.0000	0.5000	0.5000	11.5000
L24	5.900000	-0.3000	1	1	0.0000	0.1000	0.1000	11.9000
L25	6.000000	0.0000	1	1	0.0000	0.0000	0.0000	12.0000
L26	6.100000	0.1000	4	1	0.0000	-0.1000	-0.1000	12.1000
L27	6.500000	0.5000	4	1	0.0000	-0.5000	-0.5000	12.5000
L28	6.900000	0.9000	4	1	0.0000	-0.9000	-0.9000	12.9000
L29	7.000000	1.0000	4	1	0.0000	-1.0000	-1.0000	13.0000

We have the same results as MATLAB and the only difference is the interval of λ. We can see from this table that there are no cases when an objective function will have the same optimal values even if λ is very small. More importantly, from $\lambda = (6, \infty]$, it can be seen that we now have negative values for $\boldsymbol{x_2}$. Since we require our $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ to be nonnegative then it means that from this range of λ there are no optimal solutions and the objective function cannot be minimized.

Conclusions

- Parametric Analysis is useful for studying the effect of variations in the data such as:
  - ✓ the cost coefficients,
  - ✓ the right-hand-side coefficients
  - ✓ the constraint coefficients

on the optimal solution to a linear program.

- Perturbations on the right-hand-side vector are more prone to changes in the optimal solutions than perturbations on the cost vector even for small values of λ.
- Useful in determining optimal solutions in cases where there are possible linear adjustments to known variables.

References

- Bazaara M.S, Jarvis J.J, Sherali H.D (2011) *Linear Programming and Network Flows – 4th edition*. Wiley
- Parametric Analysis: The Key to Rapid, Robust Design (https://www.ansys.com/-/media/ansys/corporate/resourcelibrary/whitepaper/wp-parametric-analysis.pdf)
- Solve Linear Programming Problems – MATLAB (https://www.mathworks.com/help/optim/ug/linprog.html)