Godunov's methods for 1D-Euler Equations

Solve Euler equations

$$
\begin{aligned}
&\frac{\partial \boldsymbol{w}}{\partial t}+\frac{\partial \boldsymbol{f}(\boldsymbol{w})}{\partial x}=\mathbf{0} && x \in (0,\ell)\,, t\in(0,T)\,,\\
&\frac{\partial \boldsymbol{w}(0,t)}{\partial x}=\frac{\partial \boldsymbol{w}(\ell,0)}{\partial x}=0 && t\in(0,T)\,,\\
&\boldsymbol{w}(x,0)=\boldsymbol{w}^0(x) && x\in(0,\ell)\,,
\end{aligned}
$$

where
$\boldsymbol{w}=(\rho,\rho u,E)^T, \boldsymbol{f}(\boldsymbol{w})=(\rho u,\rho u^2+p,(E+p)u)^T, p=(\gamma-1)(E-\frac{1}{2}\rho u^2)$.
Jacobi matrix $\boldsymbol{A}(\boldsymbol{w})=D\boldsymbol{f}(\boldsymbol{w})/D\boldsymbol{w}$ is in the form

$$
\boldsymbol{A}(\boldsymbol{w})=\begin{pmatrix} 0 & 1 & 0\\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & \gamma-1\\ u\left[\frac{1}{2}(\gamma-1)u^2-H\right] & H-(\gamma-1)u^2 & \gamma u \end{pmatrix},
$$

where
$H=\dfrac{a^2}{\gamma-1}+\dfrac{1}{2}u^2$ is enthalpy, $a=\sqrt{\dfrac{\gamma p}{\rho}}$ is speed of sound and $\gamma=1,4$ is Poisson adiabatic constant. The eigenvalues and eigenvectors of matrix $\boldsymbol{A}(\boldsymbol{w})$ are

$$
\begin{aligned}
&\lambda_1(\boldsymbol{w})=u-a && \boldsymbol{r}_1(\boldsymbol{w})=(1,u-a,H-au)^T,\\
&\lambda_2(\boldsymbol{w})=u && \boldsymbol{r}_2(\boldsymbol{w})=(1,u,\frac{1}{2}u^2)^T,\\
&\lambda_3(\boldsymbol{w})=u+a && \boldsymbol{r}_3(\boldsymbol{w})=(1,u+a,H+au)^T.
\end{aligned}
$$

We denote by $\boldsymbol{T}=(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)$ the matrix of eigenvectors

$$
\boldsymbol{T}=\begin{pmatrix} 1 & 1 & 1\\ u-a & u & u+a\\ H-au & \frac{1}{2}u^2 & H+au \end{pmatrix},
$$

and the inverse matrix by

$$
\boldsymbol{T}^{-1}=\frac{1}{2a^2}\begin{pmatrix} \frac{1}{2}(\gamma-1)u^2+au & -a-(\gamma-1)u & \gamma-1\\ 2a^2-(\gamma-1)u^2 & 2(\gamma-1)u & -2(\gamma-1)\\ \frac{1}{2}(\gamma-1)u^2-au & a-(\gamma-1)u & \gamma-1 \end{pmatrix}
$$

and the diagonal matrix of eigenvalues

$$
\boldsymbol{D}=\begin{pmatrix} u-a & 0 & 0\\ 0 & u & 0\\ 0 & 0 & u+a \end{pmatrix}.
$$

Then $\boldsymbol{A}(\boldsymbol{w})=\boldsymbol{T}(\boldsymbol{w})\boldsymbol{D}(\boldsymbol{w})\boldsymbol{T}^{-1}(\boldsymbol{w})$.

We take an equidistant mesh on the interval $\langle 0,\ell\rangle$, i.e. for $N>1$ we set $h=\ell/N$ and we denote $x_j=\left(j-\frac{1}{2}\right)h$, $j=1,2,\dots,N$. Interval $\langle 0,\ell\rangle$ is union of N finite volumes

$D_j = \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right] \equiv [(j-1)h, jh]\,, \qquad j = 1, 2, \ldots, N,$
where $x_{j-\frac{1}{2}} = x_j - \frac{1}{2}h, x_{j+\frac{1}{2}} = x_j + \frac{1}{2}h$. On interval $\langle 0, T\rangle$ we take the mesh
$0 < t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots < t_{Q-1} < t_Q = T$
and we denote the time step by $\tau_k = t_k - t_{k-1}$.

From the initial condition we set $\boldsymbol{w}_j^0 = \boldsymbol{w}^0(x_j), j = 1, \ldots, N$. The boundary conditions will be used the extrapolate B.C., so in fictional finite volumes $D_0 = [-h, 0]$ and $D_{N+1} = [\ell, \ell + h]$ we prescibe values $\boldsymbol{w}_0^k = \boldsymbol{w}_1^k$ and $\boldsymbol{w}_{N+1}^k = \boldsymbol{w}_N^k$ for $k = 0, 1, 2 \ldots$.

The general numerical scheme is
$\boldsymbol{w}_j^{k+1} = \boldsymbol{w}_j^k - \frac{\tau_k}{h}\left(\boldsymbol{H}(\boldsymbol{w}_j^k, \boldsymbol{w}_{j+1}^k) - \boldsymbol{H}(\boldsymbol{w}_{j-1}^k, \boldsymbol{w}_j^k)\right), \quad j = 1, 2 \ldots, N, \quad k = 0, 1, \ldots,$
where $\boldsymbol{H}(\boldsymbol{w}_L, \boldsymbol{w}_R)$ is numerical flux.

Roe numerical flux $\boldsymbol{H}_{Roe}(\boldsymbol{w}_L, \boldsymbol{w}_R)$ with Harten-Hyman entropy fix:
Firstly, for $\boldsymbol{w}_L = (\rho_L, \rho_L u_L, E_L)^T$ and $\boldsymbol{w}_R = (\rho_R, \rho_R u_R, E_R)^T$ we define $\hat{\boldsymbol{w}} \equiv \hat{\boldsymbol{w}}(\boldsymbol{w}_L, \boldsymbol{w}_R) = (\hat{\rho}, \hat{\rho}\hat{u}, \hat{E})^T$, where

$$\hat{\rho} = \left[\frac{1}{2}\left(\sqrt{\rho_L} + \sqrt{\rho_R}\right)\right]^2, \qquad \hat{u} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}},$$
$$\hat{H} = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \qquad \hat{E} = \frac{1}{\gamma}\hat{\rho}\hat{H} + \frac{\gamma - 1}{2\gamma}\hat{\rho}\hat{u}^2.$$

The speed of sound in new variable $\hat{\boldsymbol{w}}$ is

$$\hat{a} = \sqrt{(\gamma - 1)\left(H - \tfrac{1}{2}\hat{u}^2\right)}.$$

Let's $\hat{\boldsymbol{\gamma}} = \boldsymbol{T}^{-1}(\hat{\boldsymbol{w}})(\boldsymbol{w}_R - \boldsymbol{w}_L) \equiv (\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3)^T$, where the coefficients $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3$ can be calculated:

$$\hat{\gamma}_2 = \frac{\gamma - 1}{\hat{a}^2}\left[(\hat{H} - \hat{u}^2)\delta_1 + \hat{u}\delta_2 - \delta_3\right],$$
$$\hat{\gamma}_1 = \frac{1}{2\hat{a}}\left[(\hat{u} + \hat{a})\delta_1 - \delta_2 - \hat{a}\hat{\gamma}_2\right],$$
$$\hat{\gamma}_3 = \delta_1 - \hat{\gamma}_2 - \hat{\gamma}_3,$$

where $\delta_1, \delta_2, \delta_3$ are components of vector $\boldsymbol{w}_R - \boldsymbol{w}_L = (\delta_1, \delta_2, \delta_3)^T$. The Roe numerical flux is calculated by:

a) If $\lambda_2(\hat{\boldsymbol{w}}) > 0$, we calculate $\boldsymbol{w}_L^\star = \boldsymbol{w}_L + \hat{\gamma}_1 \boldsymbol{r}_1(\hat{\boldsymbol{w}})$ and we set

$$\tilde{\lambda}_1 = \begin{cases} \lambda_1(\boldsymbol{w}_L)\dfrac{\lambda_1(\boldsymbol{w}_L^\star) - \lambda_1(\hat{\boldsymbol{w}})}{\lambda_1(\boldsymbol{w}_L^\star) - \lambda_1(\boldsymbol{w}_L)} & \text{for} \quad \lambda_1(\boldsymbol{w}_L) < 0 < \lambda_1(\boldsymbol{w}_L^\star) \\ \lambda_1(\hat{\boldsymbol{w}}) \quad \text{otherwise} \end{cases}$$

and then the numerical flux is

$$\boldsymbol{H}_{Roe}(\boldsymbol{w}_L, \boldsymbol{w}_R) = \boldsymbol{f}(\boldsymbol{w}_L) + \hat{\gamma}_1 \tilde{\lambda}_1^- \boldsymbol{r}_1(\hat{\boldsymbol{w}}).$$

b) If $\lambda_2(\hat{\boldsymbol{w}}) \leq 0$, we calculate $\boldsymbol{w}_R^\star = \boldsymbol{w}_R - \hat{\gamma}_3 \boldsymbol{r}_3(\hat{\boldsymbol{w}})$ and we set

$$\tilde{\lambda}_3 = \begin{cases} \lambda_3(\boldsymbol{w}_R)\dfrac{\lambda_3(\hat{\boldsymbol{w}}) - \lambda_3(\boldsymbol{w}_R^\star)}{\lambda_3(\boldsymbol{w}_R) - \lambda_3(\boldsymbol{w}_R^\star)} & \text{for} \quad \lambda_3(\boldsymbol{w}_R^\star) < 0 < \lambda_3(\boldsymbol{w}_R) \\ \lambda_3(\hat{\boldsymbol{w}}) \qquad \text{otherwise} \end{cases}$$

and then the numerical flux is

$$\boldsymbol{H}_{Roe}(\boldsymbol{w}_L, \boldsymbol{w}_R) = \boldsymbol{f}(\boldsymbol{w}_R) - \hat{\gamma}_3 \tilde{\lambda}_3^+ \boldsymbol{r}_3(\hat{\boldsymbol{w}}).$$

The right upper index $\pm$ is used in usual sense, i.e. $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

The time step τ_k is calculated by

$$\tau_k \leq Ch \left(\max_{j=0,\ldots,N} \left(|\hat{u}(\boldsymbol{w}_j^k, \boldsymbol{w}_{j+1}^k)| + \hat{a}(\boldsymbol{w}_j^k, \boldsymbol{w}_{j+1}^k) \right) \right)^{-1},$$

where $C \leq 1$ is so called CFL constant, for example $C = 0,9$.

Vijayasundaram numerical flux:

$$\boldsymbol{H}_V(\boldsymbol{w}_L, \boldsymbol{w}_R) = \boldsymbol{A}^+(\hat{\boldsymbol{w}})\boldsymbol{w}_L + \boldsymbol{A}^-(\hat{\boldsymbol{w}})\boldsymbol{w}_R,$$

where $\hat{\boldsymbol{w}} = \dfrac{\boldsymbol{w}_L + \boldsymbol{w}_R}{2}$, $\boldsymbol{A}^\pm = \boldsymbol{T}\boldsymbol{D}^\pm\boldsymbol{T}^{-1}$, $\boldsymbol{D}^\pm = \mathrm{diag}(\lambda_1^\pm, \lambda_2^\pm, \lambda_3^\pm)$.

The time step is calculated in the same manner as in Roe flux, the velocity $\hat{u}(\boldsymbol{w}_j^k, \boldsymbol{w}_{j+1}^k)$ and the speed of sound $\hat{a}(\boldsymbol{w}_j^k, \boldsymbol{w}_{j+1}^k)$ we obtain from $\hat{\boldsymbol{w}}(\boldsymbol{w}_j^k, \boldsymbol{w}_{j+1}^k) := \frac{1}{2}\left(\boldsymbol{w}_j^k + \boldsymbol{w}_{j+1}^k\right)$, $j = 0, 1, \ldots, N$.

Steger-Warming numerical flux:

$$\boldsymbol{H}_{SW}(\boldsymbol{w}_L, \boldsymbol{w}_R) = \boldsymbol{A}^+(\boldsymbol{w}_L)\boldsymbol{w}_L + \boldsymbol{A}^-(\boldsymbol{w}_R)\boldsymbol{w}_R$$

The time step is calculated in the same manner as in Roe flux, the velocity $\hat{u}(\boldsymbol{w}_j^k, \boldsymbol{w}_{j+1}^k)$ and the speed of sound $\hat{a}(\boldsymbol{w}_j^k, \boldsymbol{w}_{j+1}^k)$ we obtain from $\hat{\boldsymbol{w}}(\boldsymbol{w}_j^k, \boldsymbol{w}_{j+1}^k) := \boldsymbol{w}_j^k$, $j = 0, 1, \ldots, N$.

Van Leer numerical flux with Harten entropy fix:

$$\boldsymbol{H}_{VL}(\boldsymbol{w}_L, \boldsymbol{w}_R) = \tfrac{1}{2}\left[\boldsymbol{f}(\boldsymbol{w}_L) + \boldsymbol{f}(\boldsymbol{w}_R) - |\boldsymbol{A}(\hat{\boldsymbol{w}})|(\boldsymbol{w}_R - \boldsymbol{w}_L)\right],$$

where $\hat{\boldsymbol{w}} = \dfrac{\boldsymbol{w}_L + \boldsymbol{w}_R}{2}$, $|\boldsymbol{A}| = \boldsymbol{T}|\boldsymbol{D}|\boldsymbol{T}^{-1}$, $|\boldsymbol{D}| = \mathrm{diag}(\phi_\delta(\lambda_1), \phi_\delta(\lambda_2), \phi_\delta(\lambda_3))$, where

$$\phi_\delta(\lambda) = \begin{cases} \dfrac{\lambda^2 + \delta^2}{2\delta} & \text{for} \quad |\lambda| < \delta \\ |\lambda| & \text{for} \quad |\lambda| \geq \delta \end{cases}$$

and $\delta \in \langle 0.1, 1 \rangle$ is parameter, which depend on solved problem and must be well-corrected.

The time step is calculated in the same manner as in Roe flux, the velocity $\hat{u}(\boldsymbol{w}_j^k, \boldsymbol{w}_{j+1}^k)$ and the speed of sound $\hat{a}(\boldsymbol{w}_j^k, \boldsymbol{w}_{j+1}^k)$ we obtain from $\hat{\boldsymbol{w}}(\boldsymbol{w}_j^k, \boldsymbol{w}_{j+1}^k) = \frac{1}{2}(\boldsymbol{w}_j^k + \boldsymbol{w}_{j+1}^k)$, $j = 0, 1, \ldots, N$.

Test problems:

$$\frac{\partial \boldsymbol{w}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{w})}{\partial x} = \mathbf{0}, \qquad \boldsymbol{w}(x,0) = \begin{cases} \boldsymbol{w}_L & \text{for} \quad x < x_0 \\ \boldsymbol{w}_R & \text{for} \quad x > x_0, \end{cases}$$

where $x \in (0, \ell)$, $t \in (0, T)$. The initial data for $\boldsymbol{w}_L = (\rho_L, \rho_L u_L, E_L)^T$ and $\boldsymbol{w}_R = (\rho_R, \rho_R u_R, E_R)^T$ are in the following table.

test	ℓ	T	x_0	ρ_L	u_L	p_L	ρ_R	u_R	p_l
0	1	0.2	0.5	1	0	1	0.125	0	0.1
1	1	0.2	0.3	1	0.75	1	0.125	0	0.1
2	1	0.15	0.5	1	-2	0.4	1	2	0.4
3	1	0.012	0.5	1	0	1000	1	0	0.01
4	1	0.035	0.4	5.99924	19.5975	460.894	5.99942	-6.19633	46.0950
5	1	0.012	0.8	1	-19.59754	1000	1	-19.59745	0.01