PARAMETRIC ANALYSIS

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CONTENTS

I. PARAMETRIC ANALYSIS

PARAMETRIC ANALYSIS

- Used quite often in large-scale programming and in nonlinear optimization
- One way of analyzing a model by examining its behavior as one or more of its input parameters is varied
- Find a direction along which the objective function gradient (cost vector) or the right-hand-side vector of the constraints are perturbed
- Ascertain the resulting trajectory of optimal solutions
- Interested in determining optimal solutions to a class of problems by perturbing either the objective vector or the right-hand-side vector along a fixed direction
- In our case, we will be dealing with two types of perturbation:
    1. Perturbation of the cost vector (on the objective function)
    2. Perturbation of the right-hand-side vector (on the constraints)

II. PERTURBATION OF THE COST VECTOR

PERTURBATION OF THE COST VECTOR

- Consider the following problem:

$$\begin{aligned} &\text{Minimize} && cx \;\rightarrow\; (c + \lambda c')x \\ &\text{subject to} && Ax = b \\ & && x \geq 0 \end{aligned}$$

- Suppose that the cost vector c is perturbed along the cost direction c'
- That is, c is replaced by $c + \lambda c'$ where $\lambda \geq 0$
- We are interested in finding optimal solutions and corresponding objective values as a function of $\lambda \geq 0$.

EFFECT OF PERTURBATION OF THE COST VECTOR

- We want to find the optimal solutions and their trajectories as λ is varied:
  - Same feasible region (there are no changes in the constraints)
  - Objective function changes as λ is varied (since the cost vector changes)

$$c = c + \lambda c'$$

$$Ax = b$$

$$c_1 x_1 + c_2 x_2 = z \rightarrow x_2 = \frac{-c_1}{c_2} x_1 + \frac{z}{c_2}$$

$$\Downarrow$$

$$x_2 = -\frac{(c_1 + \lambda c_1{}')}{(c_2 + \lambda c_2{}')} x_1 + \frac{z}{(c_2 + \lambda c_2{}')}$$

SAMPLE MODEL

- Consider the following model:

Minimize

$$-x_1 - 3x_2 \qquad \text{(objective function z)}$$

subject to

$$x_1 + x_2 \leq 6$$

$$-x_1 + 2x_2 \leq 6 \qquad \text{(constraints)}$$

$$x_1, x_2 \geq 0$$

PERTURBATION OF THE COST VECTOR FOR THE SAMPLE MODEL

- From the sample model, we have:

$$\begin{array}{ll} \text{Minimize} & -x_1 - 3x_2 \\ \text{subject to} & x_1 + x_2 \leq 6 \\ & -x_1 + 2x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

- Perturb the cost vector $c = (-1, -3)$ along the vector $c' = (2,1)$
- Find optimal solutions and optimal objective values of the class of problems whose objective function is given by $(-1 + 2\lambda, -3 + \lambda)$ for $\lambda \geq 0$.

MATLAB IMPLEMENTATION - CODES

```
    % constraints setup
    A = [1,1;-1,2];
    b = [6,6];
    ub = [inf inf];
    lb = [0,0];
    Aeq=[];
    beq=[];

    % different values of lambda
    lambda = [0 0.25 0.5 0.75 1 1.25 1.5 1.75 2 2.25 2.5 2.75 3 4 5 6 7];

    c = [-1 -3]; % cost vector
    cp = [2 1]; % cost pertubation direction
    solx=[];
    soly=[];

    % minimizing objective function based on lambda
for i=1:length(lambda)
        l = lambda(i);
        cf = [c(1)+cp(1)*l c(2)+cp(2)*l]; %coefficients of the objective function
        xmin = linprog(cf,A,b,Aeq,beq,lb,ub);
        z = cf*xmin;
        solx(i) = xmin(1);
        soly(i) = xmin(2);

        % plotting
        xp = linspace(0,10,100);
        yp = (-cf(1)/cf(2))*xp + z/cf(2);
        axis([0 7 0 7]);
        hold on
        txt=strcat('{\lambda}=  ', num2str(l));
        plot(xp,yp,'LineWidth',2,'DisplayName',txt);
end
```

Minimize

$$-x_1 - 3x_2$$

subject to

$$x_1 + x_2 \leq 6$$

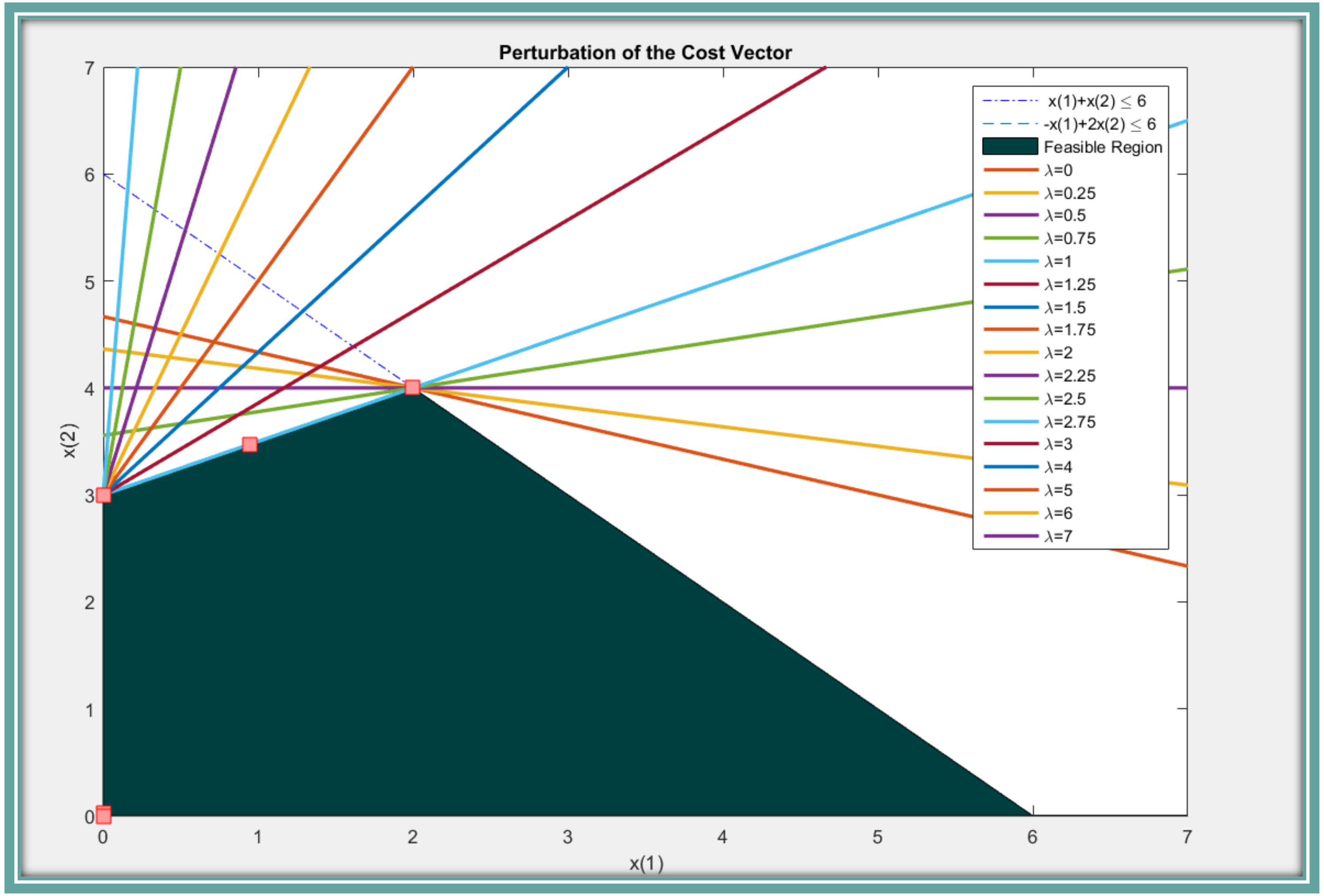
$$-x_1 + 2x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

$$c' = (2,1)$$

$$c = (-1 + 2\lambda, -3 + \lambda)$$

MATLAB IMPLEMENTATION - RESULTS



Hence, the optimal solution for any λ in the interval $[0,1]$ is given by the following tableau:

	z	x_1	x_2	x_3	x_4	RHS
z	1	0	0	$-5/3 + (5/3)\lambda$	$-2/3 - (1/3)\lambda$	$-14 + 8\lambda$
x_1	0	1	0	$2/3$	$-1/3$	2
x_2	0	0	1	$1/3$	$1/3$	4

Hence, the optimal tableau for $\lambda \in [1, 3]$ is given as follows:

	z	x_1	x_2	x_3	x_4	RHS
z	1	$5/2 - (5/2)\lambda$	0	0	$-3/2 + (1/2)\lambda$	$-9 + 3\lambda$
x_3	0	$3/2$	0	1	$-1/2$	3
x_2	0	$-1/2$	1	0	$1/2$	3

At $\lambda = 3$ the coefficient of x_4 in row 0 is equal to zero, and x_4 is introduced in the basis leading to the following tableau:

	z	x_1	x_2	x_3	x_4	RHS
z	1	-5	0	0	0	0
x_3	0	1	1	1	0	6
x_4	0	-1	2	0	1	6

GAMS IMPLEMENTATION - CODES

```
$OFFLISTING
$EOLCOM //
$INLINECOM { }
$TITLE Parametric Analysis for determining optimal solutions when pertubations are done on the cost vector or on the right-hand-side vector.

* Options:
OPTION LIMROW = 0; OPTION LIMCOL = 0;
OPTION SOLPRINT = OFF; OPTION SYSOUT = OFF;

SETS
J indices of variables / 1 * 2 /,
L indices of pertubation rate / L01 * L29 /;

// Pertubation on the cost vector
PARAMETER
C(J) cost vector / 1 -1, 2 -3 /, // c = (-1, -3)
Cp(J) cost direction of pertubation / 1 2, 2 1 /, // c' = (2, 1)
Cof2(J) second constraint coefficients /1 -1, 2 2 /,
lambdaL(L) pertubation rate scenarios
/ L01 0, L02 0.1, L03 0.5, L04 0.9,
  L05 1, L06 1.1, L07 1.5, L08 1.9,
  L09 2, L10 2.1, L11 2.5, L12 2.9,
  L13 3, L14 3.1, L15 3.5, L16 3.9,
  L17 4, L18 4.1, L19 4.5, L20 4.9,
  L21 5, L22 5.1, L23 5.5, L24 5.9,
  L25 6, L26 6.1, L27 6.5, L28 6.9,
  L29 7 /
Cf(J) final pertubed cost vector / 1 -1, 2 -3 /;

SCALAR
lambda pertubation rate;

VARIABLE
Z objective function;

POSITIVE VARIABLE X(J) optimal solution variable;
```

GAMS IMPLEMENTATION - CODES

```
EQUATIONS objfuncCost, constr1Cost, constr2Cost;
//objfuncCost.. Z =e= SUM(J, (C(J) + (lambda * Cp(J))) * X(J));
objfuncCost.. Z =e= SUM(J, Cf(J) * X(J));
constr1Cost.. SUM(J, X(J)) =l= 6;
constr2Cost.. SUM(J, Cof2(J) * X(J)) =l= 6;

MODEL ParmAnalysisCost / objfuncCost, constr1Cost, constr2Cost /;

// Output File
FILE OUT / "OUT-PMTR-ANLYS.TXT" /;
PUT OUT;
// File Heading
PUT @1 "Parametric Analysis Results" /;
PUT @1 "===============================" / /;
PUT @1 "Pertubation of the Cost Vector:" /;
PUT @1 "------------------------------------" /;
// Column Headers
PUT @1 "Scen.:",@10"Lambda:",@20"Obj.f:",@30"Type",@35"Sol?",@43;
LOOP(J, PUT" X(",J.TL:1:0,'):    '; );
LOOP(J, PUT" C(",J.TL:1:0,'):   '; );
PUT /;
// Solves model and enter values
LOOP(L, // for different values of lambda
lambda = lambdaL(L);
Cf(J) = C(J) + (lambda * Cp(J));
SOLVE ParmAnalysisCost MINIMIZING Z using LP;
PUT @1 L.TL:8,@10 lambda:8:6, @20 Z.L:9:4, @30 ParmAnalysisCost.MODELSTAT:3:0,
@35 ParmAnalysisCost.SOLVESTAT:3:0, @40;
LOOP(J, PUT X.L(J):9:4, ' ';);
LOOP(J, PUT Cf(J):9:4, ' ';); PUT /;
);
```

GAMS IMPLEMENTATION- RESULTS

```
Parametric Analysis Results
==============================

Pertubation of the Cost Vector:
----------------------------------
```

Scen.:	Lambda:	Obj.f:	Type	Sol?	X(1):	X(2):	C(1):	C(2):
L01	0.000000	-14.0000	1	1	2.0000	4.0000	-1.0000	-3.0000
L02	0.100000	-13.2000	1	1	2.0000	4.0000	-0.8000	-2.9000
L03	0.500000	-10.0000	1	1	2.0000	4.0000	0.0000	-2.5000
L04	0.900000	-6.8000	1	1	2.0000	4.0000	0.8000	-2.1000
L05	1.000000	-6.0000	1	1	2.0000	4.0000	1.0000	-2.0000
L06	1.100000	-5.7000	1	1	0.0000	3.0000	1.2000	-1.9000
L07	1.500000	-4.5000	1	1	0.0000	3.0000	2.0000	-1.5000
L08	1.900000	-3.3000	1	1	0.0000	3.0000	2.8000	-1.1000
L09	2.000000	-3.0000	1	1	0.0000	3.0000	3.0000	-1.0000
L10	2.100000	-2.7000	1	1	0.0000	3.0000	3.2000	-0.9000
L11	2.500000	-1.5000	1	1	0.0000	3.0000	4.0000	-0.5000
L12	2.900000	-0.3000	1	1	0.0000	3.0000	4.8000	-0.1000
L13	3.000000	0.0000	1	1	0.0000	0.0000	5.0000	0.0000
L14	3.100000	0.0000	1	1	0.0000	0.0000	5.2000	0.1000
L15	3.500000	0.0000	1	1	0.0000	0.0000	6.0000	0.5000
L16	3.900000	0.0000	1	1	0.0000	0.0000	6.8000	0.9000
L17	4.000000	0.0000	1	1	0.0000	0.0000	7.0000	1.0000
L18	4.100000	0.0000	1	1	0.0000	0.0000	7.2000	1.1000
L19	4.500000	0.0000	1	1	0.0000	0.0000	8.0000	1.5000
L20	4.900000	0.0000	1	1	0.0000	0.0000	8.8000	1.9000
L21	5.000000	0.0000	1	1	0.0000	0.0000	9.0000	2.0000
L22	5.100000	0.0000	1	1	0.0000	0.0000	9.2000	2.1000
L23	5.500000	0.0000	1	1	0.0000	0.0000	10.0000	2.5000
L24	5.900000	0.0000	1	1	0.0000	0.0000	10.8000	2.9000
L25	6.000000	0.0000	1	1	0.0000	0.0000	11.0000	3.0000
L26	6.100000	0.0000	1	1	0.0000	0.0000	11.2000	3.1000
L27	6.500000	0.0000	1	1	0.0000	0.0000	12.0000	3.5000
L28	6.900000	0.0000	1	1	0.0000	0.0000	12.8000	3.9000
L29	7.000000	0.0000	1	1	0.0000	0.0000	13.0000	4.0000

Hence, the optimal solution for any λ in the interval [0,1] is given by the following tableau:

	z	x_1	x_2	x_3	x_4	RHS
z	1	0	0	$-5/3+(5/3)\lambda$	$-2/3-(1/3)\lambda$	$-14+8\lambda$
x_1	0	1	0	$2/3$	$-1/3$	2
x_2	0	0	1	$1/3$	$1/3$	4

Hence, the optimal tableau for $\lambda \in [1, 3]$ is given as follows:

	z	x_1	x_2	x_3	x_4	RHS
z	1	$5/2-(5/2)\lambda$	0	0	$-3/2+(1/2)\lambda$	$-9+3\lambda$
x_3	0	$3/2$	0	1	$-1/2$	3
x_2	0	$-1/2$	1	0	$1/2$	3

At $\lambda = 3$ the coefficient of x_4 in row 0 is equal to zero, and x_4 is introduced in the basis leading to the following tableau:

	z	x_1	x_2	x_3	x_4	RHS
z	1	-5	0	0	0	0
x_3	0	1	1	1	0	6
x_4	0	-1	2	0	1	6

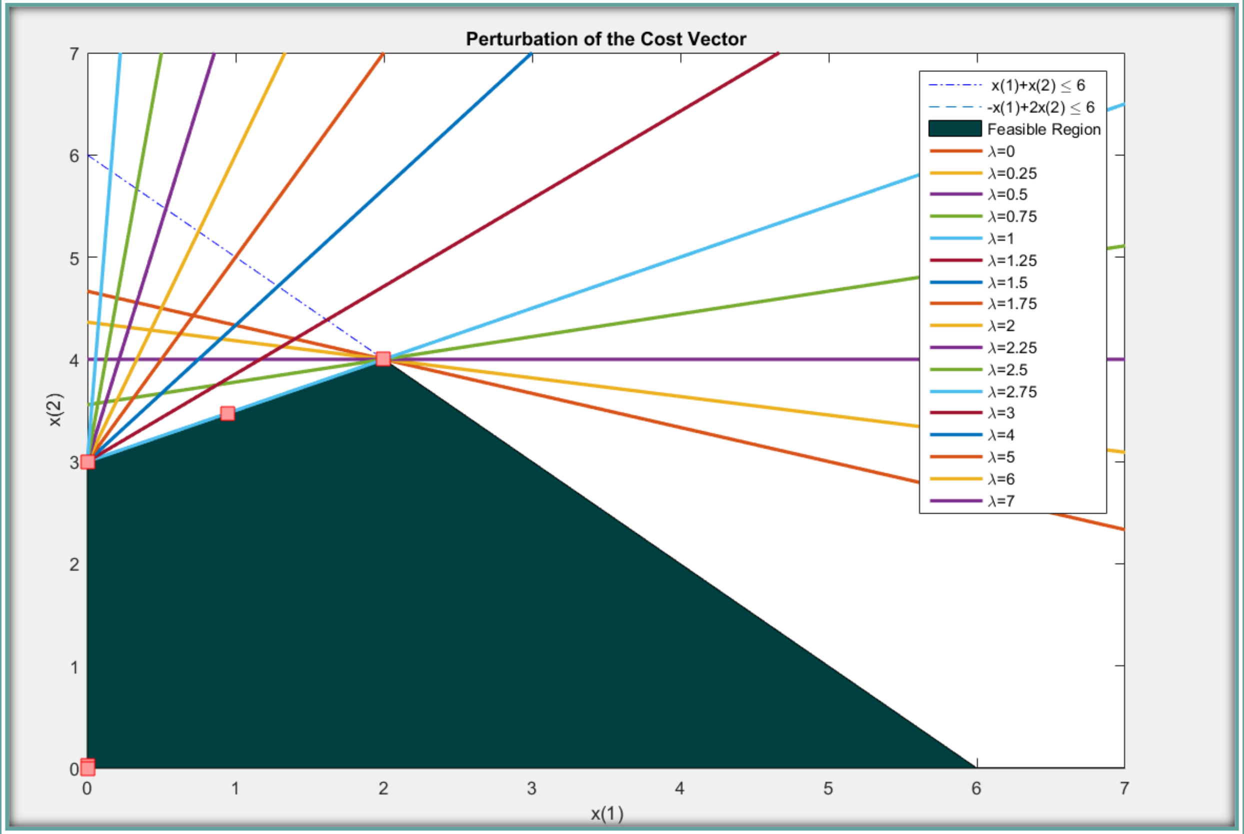
Therefore, $S = \varnothing$ and hence, the basis $[\mathbf{a}_3, \mathbf{a}_4]$ is optimal for all $\lambda \in [3, \infty]$.

MATLAB AND GAMS – COMPARISON OF RESULTS

Parametric Analysis Results
==============================

Pertubation of the Cost Vector:

Scen.:	Lambda:	Obj.f:	Type	Sol?	X(1):	X(2):	C(1):	C(2):
L01	0.000000	-14.0000	1	1	2.0000	4.0000	-1.0000	-3.0000
L02	0.100000	-13.2000	1	1	2.0000	4.0000	-0.8000	-2.9000
L03	0.500000	-10.0000	1	1	2.0000	4.0000	0.0000	-2.5000
L04	0.900000	-6.8000	1	1	2.0000	4.0000	0.8000	-2.1000
L05	1.000000	-6.0000	1	1	2.0000	4.0000	1.0000	-2.0000
L06	1.100000	-5.7000	1	1	0.0000	3.0000	1.2000	-1.9000
L07	1.500000	-4.5000	1	1	0.0000	3.0000	2.0000	-1.5000
L08	1.900000	-3.3000	1	1	0.0000	3.0000	2.8000	-1.1000
L09	2.000000	-3.0000	1	1	0.0000	3.0000	3.0000	-1.0000
L10	2.100000	-2.7000	1	1	0.0000	3.0000	3.2000	-0.9000
L11	2.500000	-1.5000	1	1	0.0000	3.0000	4.0000	-0.5000
L12	2.900000	-0.3000	1	1	0.0000	3.0000	4.8000	-0.1000
L13	3.000000	0.0000	1	1	0.0000	0.0000	5.0000	0.0000
L14	3.100000	0.0000	1	1	0.0000	0.0000	5.2000	0.1000
L15	3.500000	0.0000	1	1	0.0000	0.0000	6.0000	0.5000
L16	3.900000	0.0000	1	1	0.0000	0.0000	6.8000	0.9000
L17	4.000000	0.0000	1	1	0.0000	0.0000	7.0000	1.0000
L18	4.100000	0.0000	1	1	0.0000	0.0000	7.2000	1.1000
L19	4.500000	0.0000	1	1	0.0000	0.0000	8.0000	1.5000
L20	4.900000	0.0000	1	1	0.0000	0.0000	8.8000	1.9000
L21	5.000000	0.0000	1	1	0.0000	0.0000	9.0000	2.0000
L22	5.100000	0.0000	1	1	0.0000	0.0000	9.2000	2.1000
L23	5.500000	0.0000	1	1	0.0000	0.0000	10.0000	2.5000
L24	5.900000	0.0000	1	1	0.0000	0.0000	10.8000	2.9000
L25	6.000000	0.0000	1	1	0.0000	0.0000	11.0000	3.0000
L26	6.100000	0.0000	1	1	0.0000	0.0000	11.2000	3.1000
L27	6.500000	0.0000	1	1	0.0000	0.0000	12.0000	3.5000
L28	6.900000	0.0000	1	1	0.0000	0.0000	12.8000	3.9000
L29	7.000000	0.0000	1	1	0.0000	0.0000	13.0000	4.0000



III. PERTURBATION OF THE RIGHT-HAND-SIDE VECTOR

PERTURBATION OF THE RIGHT-HAND-SIDE VECTOR

- Consider the following problem:

$$\begin{array}{ll} \text{Minimize} & cx \\ \text{subject to} & Ax = b \rightarrow Ax = b + \lambda b' \\ & x \geq 0 \end{array}$$

- Suppose that the right-hand-side vector b is perturbed along the direction b'
- That is, b is replaced by $b + \lambda b'$ where $\lambda \geq 0$
- We are interested in finding optimal solutions and corresponding objective values as a function of $\lambda \geq 0$.

EFFECT OF PERTURBATION ON THE RIGHT-HAND-SIDE VECTOR

$$\boxed{b = b + \lambda b'}$$

- We want to find the optimal solutions and their trajectories as λ is varied:
  - Form of the objective function remains the same (no changes in cost vector)
  - Feasible region changes as λ is varied

$$c_1 x_1 + c_2 x_2 = z$$

$$\Downarrow$$

$$x_2 = \frac{-c_1}{c_2} x_1 + \frac{z}{c_2}$$

$$a_{11}x_1 + a_{12}x_2 \leq b_1 \rightarrow x_2 \leq \frac{-a_{11}}{a_{12}} x_1 + \frac{b_1}{a_{12}}$$

$$a_{21}x_1 + a_{22}x_2 \geq b_2 \rightarrow x_2 \geq \frac{-a_{21}}{a_{22}} x_1 + \frac{b_2}{a_{22}}$$

$$\Rightarrow$$

$$x_2 \leq \frac{-a_{11}}{a_{12}} x_1 + \frac{b_1 + \lambda {b_1}'}{a_{12}}$$

$$x_2 \geq \frac{-a_{21}}{a_{22}} x_1 + \frac{b_2 + \lambda {b_2}'}{a_{22}}$$

SAMPLE MODEL

- Consider the following model:

Minimize

$$-x_1 - 3x_2 \qquad \text{(objective function z)}$$

subject to

$$x_1 + x_2 \leq 6$$

$$-x_1 + 2x_2 \leq 6 \qquad \text{(constraints)}$$

$$x_1, x_2 \geq 0$$

PERTURBATION OF THE RIGHT-HAND-SIDE VECTOR FOR THE MODEL

- From the sample model, we have

$$
\begin{array}{ll}
\text{Minimize} & -x_1 - 3x_2 \\
\text{subject to} & x_1 + x_2 \leq 6 \\
 & -x_1 + 2x_2 \leq 6 \\
 & x_1, x_2 \geq 0
\end{array}
$$

- Perturb the cost vector $b = (6,6)$ along the vector $b' = (-1,1)$
- Find optimal solutions and optimal objective values of the class of problems whose constraints have the right-hand-side values given by $(6 - \lambda, 6 + \lambda)$ for $\lambda \geq 0$.

MATLAB IMPLEMENTATION - CODES

```
bp = [-1 1]; %rhs perturbation direction

% minimizing objective function based on lambda
for i=1:length(lambda)
    l = lambda(i)

    %plotting feasible region
    x=linspace(0,10,100);
    y=linspace(0,10,100);
    f1=(bi(1)+(l*bp(1)))-x;
    f2=((bi(2)+(l*bp(2)))/2)+.5.*x;

    for i=1:length(x)
        r(i)=min([f1(i),f2(i)]);
    end

    figure(1)
    plot(x,f1,'-.b','HandleVisibility','off');
    axis([0 10 0 10])
    hold on
    plot(x,f2,'--','HandleVisibility','off');
    rtxt=strcat('{\lambda}=  ', num2str(l));
    h3=area(x,r,'DisplayName',rtxt);
    p=l/10;
    h3.FaceColor = [0 0.25+p 0.5];

     % constraints setup
    A = [1,1;-1,2];
    b = [bi(1)+(l*bp(1)),bi(2)+(l*bp(2))];
    ub = [inf inf];
    lb = [0,0];
    Aeq=[];
    beq=[];

    xmin = linprog(c,A,b,Aeq,beq,lb,ub);
```

Minimize

$$-x_1 - 3x_2$$

subject to

$$x_1 + x_2 \leq 6$$

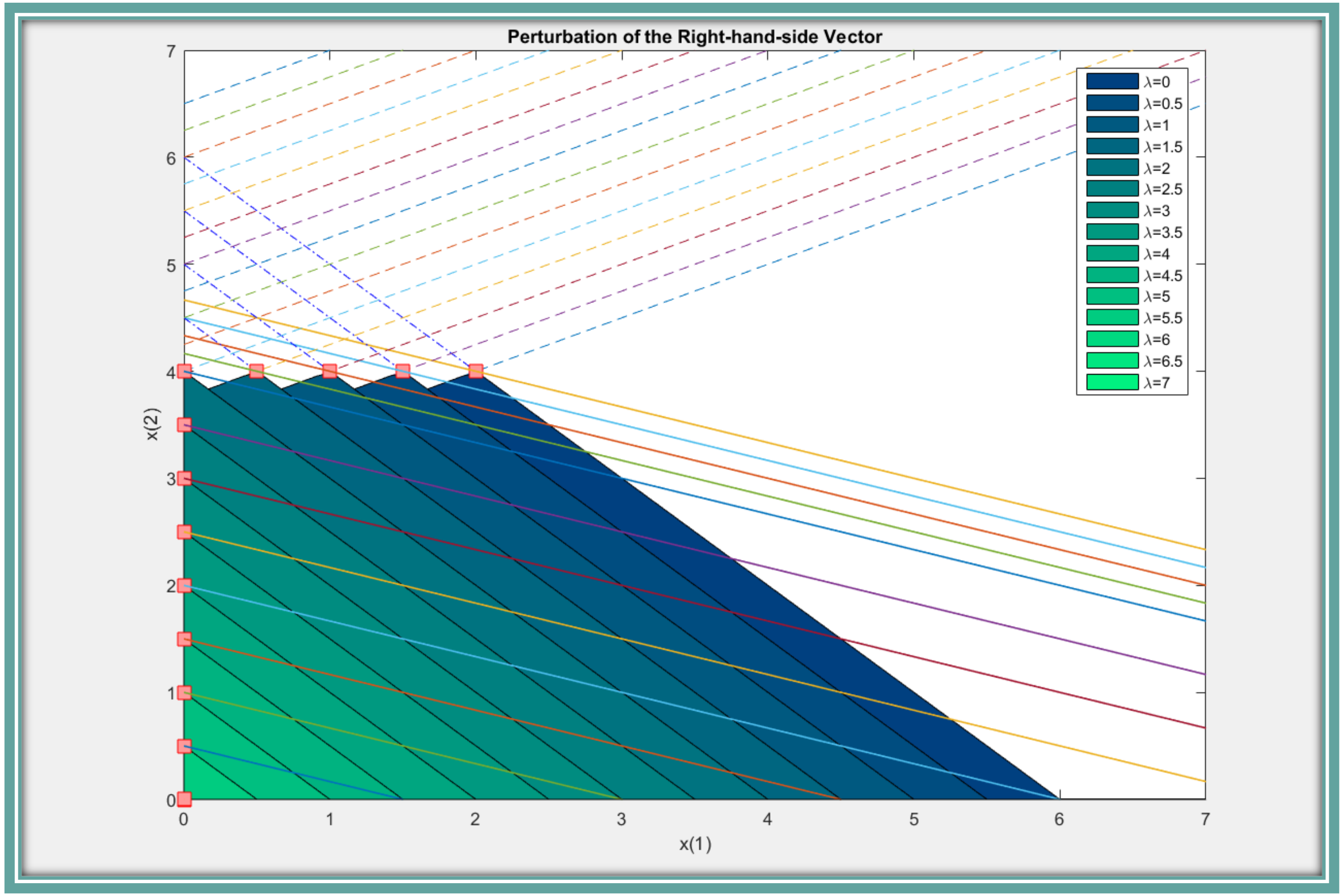
$$-x_1 + 2x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

$$b' = (-1,1)$$

$$b = (6-\lambda, 6+\lambda)$$

MATLAB IMPLEMENTATION - RESULTS



Therefore, the basis $[\mathbf{a}_1, \mathbf{a}_2]$ remains optimal over the interval [0, 2]. In particular, for any $\lambda \in [0,2]$ the objective value and the right–hand–side are given by

$$\begin{aligned} z(\lambda) &= \mathbf{c}_B\bar{\mathbf{b}} + \lambda\mathbf{c}_B\bar{\mathbf{b}}' \\ &= (-1,-3)\begin{pmatrix}2\\4\end{pmatrix} + \lambda(-1,-3)\begin{pmatrix}-1\\0\end{pmatrix} = -14+\lambda \\ \bar{\mathbf{b}} + \lambda\bar{\mathbf{b}}' &= \begin{pmatrix}2\\4\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}2-\lambda\\4\end{pmatrix}, \end{aligned}$$

and the simplex tableau appears as follows:

	z	x_1	x_2	x_3	x_4	RHS
z	1	0	0	$-5/3$	$-2/3$	$-14 + \lambda$
x_1	0	1	0	$2/3$	$-1/3$	$2 - \lambda$
x_2	0	0	1	$1/3$	$1/3$	4

The optimal tableau over the interval [2, 6] is depicted below:

	z	x_1	x_2	x_3	x_4	RHS
z	1	-2	0	-3	0	$-18 + 3\lambda$
x_4	0	-3	0	-2	1	$-6 + 3\lambda$
x_2	0	1	1	1	0	$6 - \lambda$

GAMS IMPLEMENTATION - CODES

```
// Additional Codes for Pertubation on the Right-hand-side vector b
PARAMETER
b(J) initial right-hand-side vector / 1 6, 2 6/,
bp(J) right-hand-side direction of pertubation / 1 -1, 2 1 /,
bf(J) final right-hand-side vector / 1 6, 2 6/;

EQUATIONS objfuncRHS, constr1RHS, constr2RHS;
objfuncRHS.. Z =e= SUM(J, C(J) * X(J));
constr1RHS.. SUM(J, X(J)) =l= bf("1");
constr2RHS.. SUM(J, Cof2(J) * X(J)) =l= bf("2");

MODEL ParmAnalysisRHS / objfuncRHS, constr1RHS, constr2RHS /;

// Output File
PUT / @1 "Pertubation of the Right-hand-side Vector:" /;
PUT @1 "-----------------------------------" /;
// Column Headers
PUT @1 "Scen.:",@10"Lambda:",@20"Obj.f:",@30"Type",@35"Sol?",@43;
LOOP(J, PUT" X(",J.TL:1:0,'):    '; );
LOOP(J, PUT" b(",J.TL:1:0,'):   '; );
PUT /;
// Solves model and enter values
LOOP(L, // for different values of lambda
lambda = lambdaL(L);
bf(J) = b(J) + (lambda * bp(J));
SOLVE ParmAnalysisRHS MINIMIZING Z using LP;
PUT @1 L.TL:8,@10 lambda:8:6, @20 Z.L:9:4, @30 ParmAnalysisRHS.MODELSTAT:3:0,
@35 ParmAnalysisRHS.SOLVESTAT:3:0, @40;
LOOP(J, PUT X.L(J):9:4, ' ';);
LOOP(J, PUT bf(J):9:4, ' ';); PUT /;
);
```

GAMS IMPLEMENTATION - RESULTS

Pertubation of the Right-hand-side Vector:

Scen.:	Lambda:	Obj.f:	Type	Sol?	X(1):	X(2):	b(1):	b(2):
L01	0.000000	-14.0000	1	1	2.0000	4.0000	6.0000	6.0000
L02	0.100000	-13.9000	1	1	1.9000	4.0000	5.9000	6.1000
L03	0.500000	-13.5000	1	1	1.5000	4.0000	5.5000	6.5000
L04	0.900000	-13.1000	1	1	1.1000	4.0000	5.1000	6.9000
L05	1.000000	-13.0000	1	1	1.0000	4.0000	5.0000	7.0000
L06	1.100000	-12.9000	1	1	0.9000	4.0000	4.9000	7.1000
L07	1.500000	-12.5000	1	1	0.5000	4.0000	4.5000	7.5000
L08	1.900000	-12.1000	1	1	0.1000	4.0000	4.1000	7.9000
L09	2.000000	-12.0000	1	1	0.0000	4.0000	4.0000	8.0000
L10	2.100000	-11.7000	1	1	0.0000	3.9000	3.9000	8.1000
L11	2.500000	-10.5000	1	1	0.0000	3.5000	3.5000	8.5000
L12	2.900000	-9.3000	1	1	0.0000	3.1000	3.1000	8.9000
L13	3.000000	-9.0000	1	1	0.0000	3.0000	3.0000	9.0000
L14	3.100000	-8.7000	1	1	0.0000	2.9000	2.9000	9.1000
L15	3.500000	-7.5000	1	1	0.0000	2.5000	2.5000	9.5000
L16	3.900000	-6.3000	1	1	0.0000	2.1000	2.1000	9.9000
L17	4.000000	-6.0000	1	1	0.0000	2.0000	2.0000	10.0000
L18	4.100000	-5.7000	1	1	0.0000	1.9000	1.9000	10.1000
L19	4.500000	-4.5000	1	1	0.0000	1.5000	1.5000	10.5000
L20	4.900000	-3.3000	1	1	0.0000	1.1000	1.1000	10.9000
L21	5.000000	-3.0000	1	1	0.0000	1.0000	1.0000	11.0000
L22	5.100000	-2.7000	1	1	0.0000	0.9000	0.9000	11.1000
L23	5.500000	-1.5000	1	1	0.0000	0.5000	0.5000	11.5000
L24	5.900000	-0.3000	1	1	0.0000	0.1000	0.1000	11.9000
L25	6.000000	0.0000	1	1	0.0000	0.0000	0.0000	12.0000
L26	6.100000	0.1000	4	1	0.0000	-0.1000	-0.1000	12.1000
L27	6.500000	0.5000	4	1	0.0000	-0.5000	-0.5000	12.5000
L28	6.900000	0.9000	4	1	0.0000	-0.9000	-0.9000	12.9000
L29	7.000000	1.0000	4	1	0.0000	-1.0000	-1.0000	13.0000

Therefore, the basis $[\mathbf{a}_1, \mathbf{a}_2]$ remains optimal over the interval [0, 2]. In particular, for any $\lambda \in [0,2]$ the objective value and the right–hand–side are given by

$$\begin{aligned} z(\lambda) &= \mathbf{c}_B\bar{\mathbf{b}} + \lambda \mathbf{c}_B\bar{\mathbf{b}}' \\ &= (-1,-3)\begin{pmatrix}2\\4\end{pmatrix} + \lambda(-1,-3)\begin{pmatrix}-1\\0\end{pmatrix} = -14+\lambda \\ \bar{\mathbf{b}} + \lambda\bar{\mathbf{b}}' &= \begin{pmatrix}2\\4\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}2-\lambda\\4\end{pmatrix}, \end{aligned}$$

and the simplex tableau appears as follows:

	z	x_1	x_2	x_3	x_4	RHS
z	1	0	0	$-5/3$	$-2/3$	$-14+\lambda$
x_1	0	1	0	$2/3$	$-1/3$	$2-\lambda$
x_2	0	0	1	$1/3$	$1/3$	4

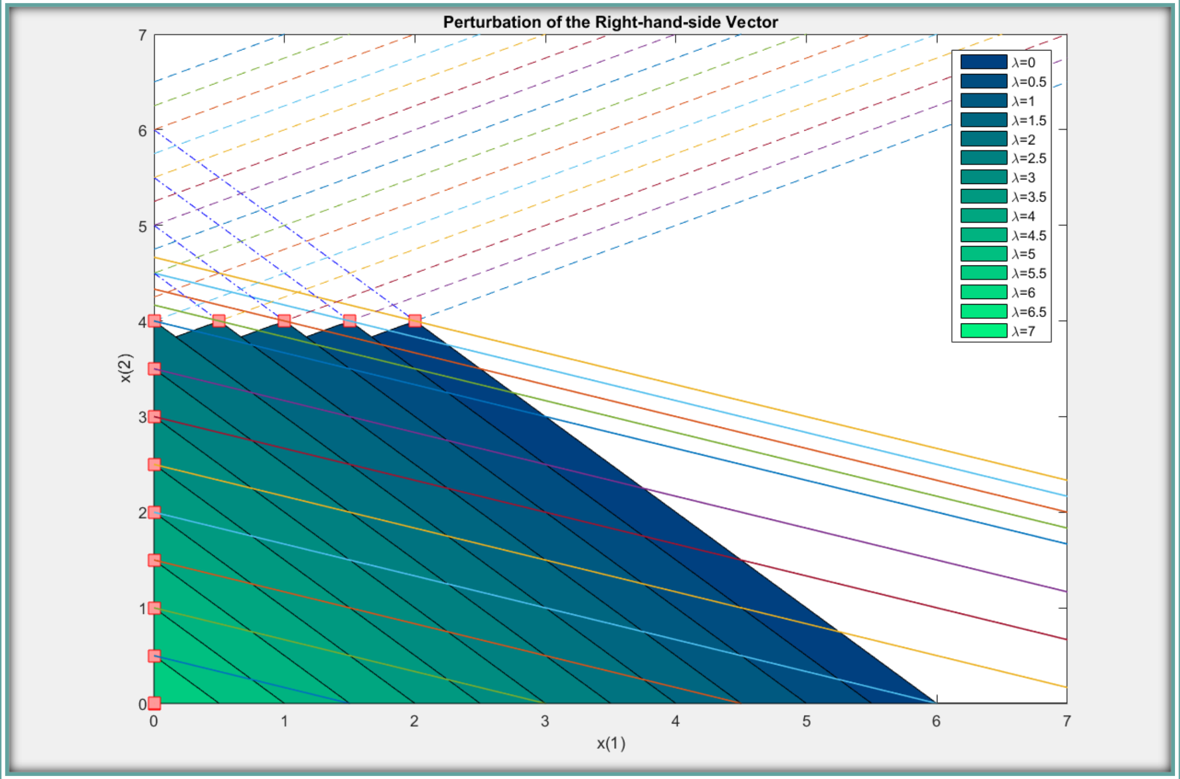
The optimal tableau over the interval [2, 6] is depicted below:

	z	x_1	x_2	x_3	x_4	RHS
z	1	-2	0	-3	0	$-18+3\lambda$
x_4	0	-3	0	-2	1	$-6+3\lambda$
x_2	0	1	1	1	0	$6-\lambda$

MATLAB AND GAMS – COMPARISON OF RESULTS

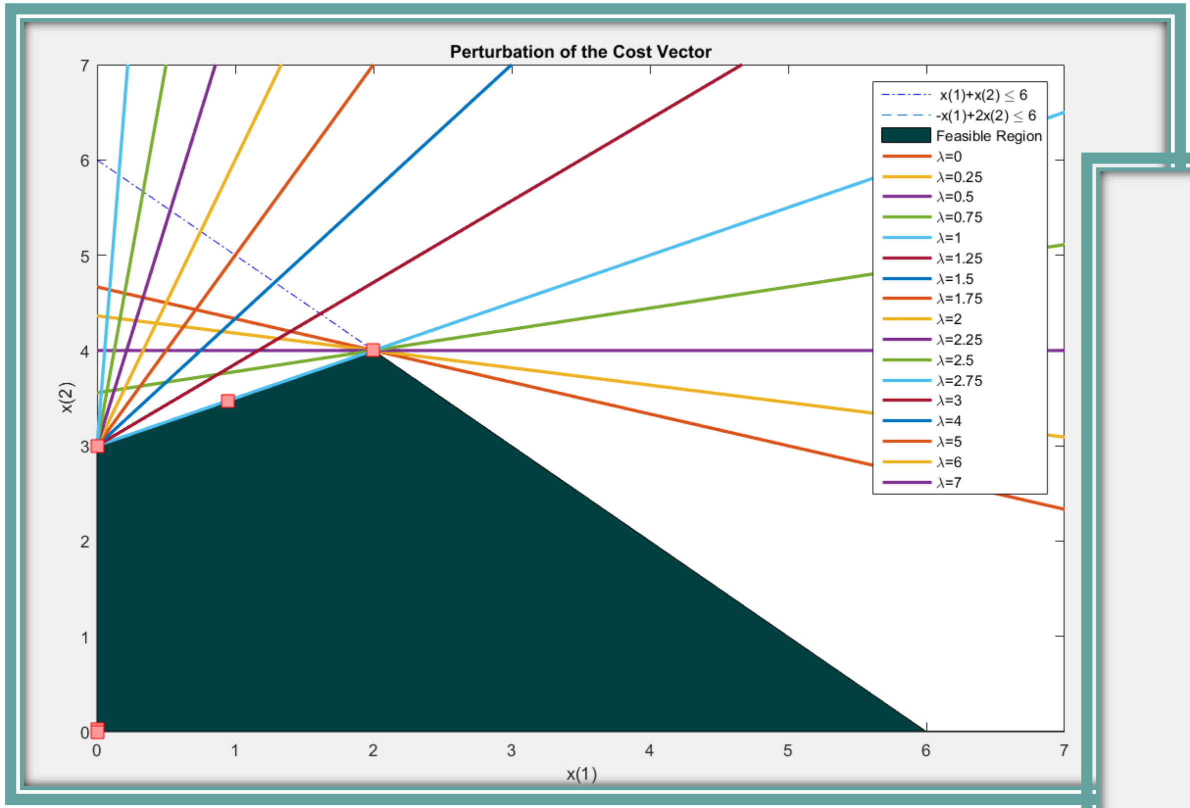
Pertubation of the Right-hand-side Vector:

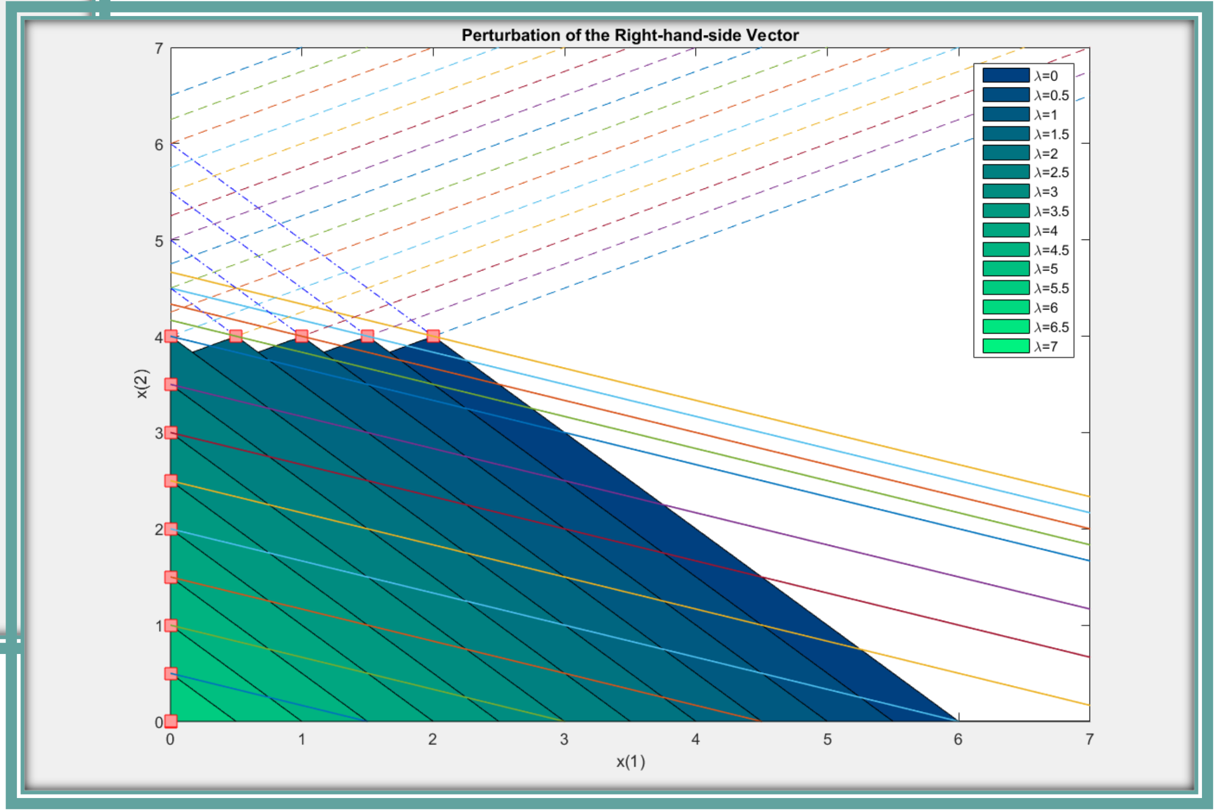
Scen.:	Lambda:	Obj.f:	Type	Sol?	X(1):	X(2):	b(1):	b(2):
L01	0.000000	-14.0000	1	1	2.0000	4.0000	6.0000	6.0000
L02	0.100000	-13.9000	1	1	1.9000	4.0000	5.9000	6.1000
L03	0.500000	-13.5000	1	1	1.5000	4.0000	5.5000	6.5000
L04	0.900000	-13.1000	1	1	1.1000	4.0000	5.1000	6.9000
L05	1.000000	-13.0000	1	1	1.0000	4.0000	5.0000	7.0000
L06	1.100000	-12.9000	1	1	0.9000	4.0000	4.9000	7.1000
L07	1.500000	-12.5000	1	1	0.5000	4.0000	4.5000	7.5000
L08	1.900000	-12.1000	1	1	0.1000	4.0000	4.1000	7.9000
L09	2.000000	-12.0000	1	1	0.0000	4.0000	4.0000	8.0000
L10	2.100000	-11.7000	1	1	0.0000	3.9000	3.9000	8.1000
L11	2.500000	-10.5000	1	1	0.0000	3.5000	3.5000	8.5000
L12	2.900000	-9.3000	1	1	0.0000	3.1000	3.1000	8.9000
L13	3.000000	-9.0000	1	1	0.0000	3.0000	3.0000	9.0000
L14	3.100000	-8.7000	1	1	0.0000	2.9000	2.9000	9.1000
L15	3.500000	-7.5000	1	1	0.0000	2.5000	2.5000	9.5000
L16	3.900000	-6.3000	1	1	0.0000	2.1000	2.1000	9.9000
L17	4.000000	-6.0000	1	1	0.0000	2.0000	2.0000	10.0000
L18	4.100000	-5.7000	1	1	0.0000	1.9000	1.9000	10.1000
L19	4.500000	-4.5000	1	1	0.0000	1.5000	1.5000	10.5000
L20	4.900000	-3.3000	1	1	0.0000	1.1000	1.1000	10.9000
L21	5.000000	-3.0000	1	1	0.0000	1.0000	1.0000	11.0000
L22	5.100000	-2.7000	1	1	0.0000	0.9000	0.9000	11.1000
L23	5.500000	-1.5000	1	1	0.0000	0.5000	0.5000	11.5000
L24	5.900000	-0.3000	1	1	0.0000	0.1000	0.1000	11.9000
L25	6.000000	0.0000	1	1	0.0000	0.0000	0.0000	12.0000
L26	6.100000	0.1000	4	1	0.0000	-0.1000	-0.1000	12.1000
L27	6.500000	0.5000	4	1	0.0000	-0.5000	-0.5000	12.5000
L28	6.900000	0.9000	4	1	0.0000	-0.9000	-0.9000	12.9000
L29	7.000000	1.0000	4	1	0.0000	-1.0000	-1.0000	13.0000



IV. CONCLUSIONS AND REFERENCES

COMPILATION OF RESULTS (IN MATLAB)





COMPILATION OF RESULTS (IN GAMS)

Parametric Analysis Results
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Pertubation of the Cost Vector:

Scen.:	Lambda:	Obj.f:	Type	Sol?	X(1):	X(2):	C(1):	C(2):
L01	0.000000	-14.0000	1	1	2.0000	4.0000	-1.0000	-3.0000
L02	0.100000	-13.2000	1	1	2.0000	4.0000	-0.8000	-2.9000
L03	0.500000	-10.0000	1	1	2.0000	4.0000	0.0000	-2.5000
L04	0.900000	-6.8000	1	1	2.0000	4.0000	0.8000	-2.1000
L05	1.000000	-6.0000	1	1	2.0000	4.0000	1.0000	-2.0000
L06	1.100000	-5.7000	1	1	0.0000	3.0000	1.2000	-1.9000
L07	1.500000	-4.5000	1	1	0.0000	3.0000	2.0000	-1.5000
L08	1.900000	-3.3000	1	1	0.0000	3.0000	2.8000	-1.1000
L09	2.000000	-3.0000	1	1	0.0000	3.0000	3.0000	-1.0000
L10	2.100000	-2.7000	1	1	0.0000	3.0000	3.2000	-0.9000
L11	2.500000	-1.5000	1	1	0.0000	3.0000	4.0000	-0.5000
L12	2.900000	-0.3000	1	1	0.0000	3.0000	4.8000	-0.1000
L13	3.000000	0.0000	1	1	0.0000	0.0000	5.0000	0.0000
L14	3.100000	0.0000	1	1	0.0000	0.0000	5.2000	0.1000
L15	3.500000	0.0000	1	1	0.0000	0.0000	6.0000	0.5000
L16	3.900000	0.0000	1	1	0.0000	0.0000	6.8000	0.9000
L17	4.000000	0.0000	1	1	0.0000	0.0000	7.0000	1.0000
L18	4.100000	0.0000	1	1	0.0000	0.0000	7.2000	1.1000
L19	4.500000	0.0000	1	1	0.0000	0.0000	8.0000	1.5000
L20	4.900000	0.0000	1	1	0.0000	0.0000	8.8000	1.9000
L21	5.000000	0.0000	1	1	0.0000	0.0000	9.0000	2.0000
L22	5.100000	0.0000	1	1	0.0000	0.0000	9.2000	2.1000
L23	5.500000	0.0000	1	1	0.0000	0.0000	10.0000	2.5000
L24	5.900000	0.0000	1	1	0.0000	0.0000	10.8000	2.9000
L25	6.000000	0.0000	1	1	0.0000	0.0000	11.0000	3.0000
L26	6.100000	0.0000	1	1	0.0000	0.0000	11.2000	3.1000
L27	6.500000	0.0000	1	1	0.0000	0.0000	12.0000	3.5000
L28	6.900000	0.0000	1	1	0.0000	0.0000	12.8000	3.9000
L29	7.000000	0.0000	1	1	0.0000	0.0000	13.0000	4.0000

Pertubation of the Right-hand-side Vector:

Scen.:	Lambda:	Obj.f:	Type	Sol?	X(1):	X(2):	b(1):	b(2):
L01	0.000000	-14.0000	1	1	2.0000	4.0000	6.0000	6.0000
L02	0.100000	-13.9000	1	1	1.9000	4.0000	5.9000	6.1000
L03	0.500000	-13.5000	1	1	1.5000	4.0000	5.5000	6.5000
L04	0.900000	-13.1000	1	1	1.1000	4.0000	5.1000	6.9000
L05	1.000000	-13.0000	1	1	1.0000	4.0000	5.0000	7.0000
L06	1.100000	-12.9000	1	1	0.9000	4.0000	4.9000	7.1000
L07	1.500000	-12.5000	1	1	0.5000	4.0000	4.5000	7.5000
L08	1.900000	-12.1000	1	1	0.1000	4.0000	4.1000	7.9000
L09	2.000000	-12.0000	1	1	0.0000	4.0000	4.0000	8.0000
L10	2.100000	-11.7000	1	1	0.0000	3.9000	3.9000	8.1000
L11	2.500000	-10.5000	1	1	0.0000	3.5000	3.5000	8.5000
L12	2.900000	-9.3000	1	1	0.0000	3.1000	3.1000	8.9000
L13	3.000000	-9.0000	1	1	0.0000	3.0000	3.0000	9.0000
L14	3.100000	-8.7000	1	1	0.0000	2.9000	2.9000	9.1000
L15	3.500000	-7.5000	1	1	0.0000	2.5000	2.5000	9.5000
L16	3.900000	-6.3000	1	1	0.0000	2.1000	2.1000	9.9000
L17	4.000000	-6.0000	1	1	0.0000	2.0000	2.0000	10.0000
L18	4.100000	-5.7000	1	1	0.0000	1.9000	1.9000	10.1000
L19	4.500000	-4.5000	1	1	0.0000	1.5000	1.5000	10.5000
L20	4.900000	-3.3000	1	1	0.0000	1.1000	1.1000	10.9000
L21	5.000000	-3.0000	1	1	0.0000	1.0000	1.0000	11.0000
L22	5.100000	-2.7000	1	1	0.0000	0.9000	0.9000	11.1000
L23	5.500000	-1.5000	1	1	0.0000	0.5000	0.5000	11.5000
L24	5.900000	-0.3000	1	1	0.0000	0.1000	0.1000	11.9000
L25	6.000000	0.0000	1	1	0.0000	0.0000	0.0000	12.0000
L26	6.100000	0.1000	4	1	0.0000	-0.1000	-0.1000	12.1000
L27	6.500000	0.5000	4	1	0.0000	-0.5000	-0.5000	12.5000
L28	6.900000	0.9000	4	1	0.0000	-0.9000	-0.9000	12.9000
L29	7.000000	1.0000	4	1	0.0000	-1.0000	-1.0000	13.0000

CONCLUSIONS

- Parametric Analysis is useful for studying the effect of variations in the data such as:
    - ✓ the cost coefficients,
    - ✓ the right-hand-side coefficients
    - ✓ the constraint coefficients

on the optimal solution to a linear program

- From the results, we can see that perturbations on the right-hand-side vector are more prone to changes in the optimal solutions than perturbations on the cost vector even for small values of λ.

- Varying these model parameters for a what-if scenarios may lead to making small modifications on some model decision variables that may produce better quality and minimized cost for the model where it can still retain its purpose.

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THANK YOU FOR LISTENING!