MATH10232: EXAMPLE SHEET 3

Questions for supervision classes

Please hand in solutions to questions 1, 2a and 3a. Attempt **all** other questions and raise any problems with your supervisor.

1. **Existence, uniqueness and graphical solutions**
   Consider the initial value problem

$$yy' = x - 1,$$

   subject to the initial condition

$$y(x = X) = Y,$$

   where X and Y are constants.

   (a) For which values of X and Y does the existence and uniqueness theorem ensure the existence of a unique solution? If a unique solution can be shown to exist, will it exist for *all* values of x?

   (b) Without solving the ODE explicitly, sketch its isoclines and integral curves.

   (c) Find the *general* solution of the ODE, using any suitable method.

   (d) How do the integral curves sketched in part 1b relate to the solution obtained in part 1c.

   (e) Are the sketched solutions consistent with the existence and uniqueness results obtained in part 1a?

2. **More separable ODEs**
   For each of the following differential equations, find the general solution and sketch the integral curves.

   [1] (a)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1+x^2}},$$

   (b)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x}{(1+x^2)^{1/3}},$$

   [2] (c)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2y}{x-2},$$

[1] Recall that integrals involving $\sqrt{1+x^2}$ are best done using the substitution $x = \sinh z$.
[2] Be careful not to overlook any solutions.

[3] (d)

$$\sqrt{1+x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = y,$$

3. **Initial value problems**
   Using the results from question 2, solve the following initial value problems:

   (a)
   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1+x^2}} \quad \text{subject to } y(0) = 5.$$

   (b)
   $$\sqrt{1+x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = y \quad \text{subject to } y(0) = -3.$$

4. **Computer Exercises**

   (a) Generate a 2D grid of points in MATLAB using the command

   ```
   >> [x,y] = meshgrid(-1:0.1:1,-1:0.1:1);
   ```

   This generates the x and y coordinates corresponding to a grid of points in the square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ with spacing 0.1 in the x and y directions.

   (b) Generate a `quiver` (vector) plot by using the command

   ```
   >> quiver(x,y,x,y);
   ```

   What do you notice about your plot?

   (c) Try plotting `>> quiver(x,y,x,x);`, `>> quiver(x,y,y,y);` and `>> quiver(x,y,-y,x);`. Can you work out what the `quiver` command is doing?

   (d) Use your new found knowledge of the `quiver` command to plot the direction fields associated with the ODEs in questions 2(a-c).[4]

[3] Again, be careful not to overlook any solutions. Sketching the solution is easier if you write $\operatorname{arcsinh}(x) + C$ as $\ln(x + \sqrt{1+x^2}) + \ln|K|$.

[4] Hint: The command `ones(21)` generates a 21×21 matrix in which every entry has the value 1.