Problem 1. $F_{\alpha\beta} = \frac{1}{2} tr\left(C^{-1} \frac{\partial C}{\partial \theta^\alpha} C^{-1} \frac{\partial C}{\partial \theta^\beta} + 2C^{-1} \frac{\partial \mu}{\partial \theta^\alpha} \frac{\partial \mu^T}{\partial \theta^\beta} \right)$

$\frac{\partial C^{-1}}{\partial \theta^\alpha} = -C^{-1} \frac{\partial C}{\partial \theta^\alpha} C^{-1}$

The Fisher Matrix is given by : $F_{\alpha\beta} = -\left\langle \frac{\partial \ln L(D|\theta)}{\partial \theta^\alpha} \frac{\partial \ln L(D|\theta)}{\partial \theta^\beta} \right\rangle$

The negative of the log-likelihood is given by:

$$-\ln L = \frac{1}{2} tr(\ln C + C^{-1} D) + \text{const}$$

Takin 1st derivative: $\frac{\partial}{\partial \alpha} = \frac{\partial}{\partial \theta^\alpha}$

$$-\frac{\partial \ln L}{\partial \theta^\alpha} = \frac{1}{2} tr\left(C^{-1} \frac{\partial C}{\partial \alpha} + \frac{\partial C^{-1}}{\partial \alpha} D + C^{-1} \frac{\partial D}{\partial \alpha} \right)$$

$$-\frac{\partial^2 \ln L}{\partial \theta^\beta \theta^\alpha} = \frac{1}{2} tr\left(\frac{\partial C^{-1}}{\partial \beta} \frac{\partial C}{\partial \alpha} + C^{-1} \frac{\partial^2 C}{\partial \beta \partial \alpha} + \frac{\partial^2 C^{-1}}{\partial \beta \partial \alpha} D + \frac{\partial C^{-1}}{\partial \alpha} \frac{\partial D}{\partial \beta} + \frac{\partial C^{-1}}{\partial \beta} \frac{\partial D}{\partial \alpha} + C^{-1} \frac{\partial^2 D}{\partial \beta \partial \alpha} \right)$$

$$= \frac{1}{2} tr\left(\frac{\partial C^{-1}}{\partial \beta} \left(\frac{\partial C}{\partial \alpha} + \frac{\partial D}{\partial \alpha} \right) + C^{-1} \left(\frac{\partial^2 C}{\partial \beta \partial \alpha} + \frac{\partial^2 D}{\partial \beta \partial \alpha} \right) + \frac{\partial^2 C^{-1}}{\partial \beta \partial \alpha} D + \frac{\partial C^{-1}}{\partial \alpha} \frac{\partial D}{\partial \beta} \right)$$

Lets compute the expectation values for:

$$\left\langle \frac{\partial D}{\partial \alpha} \right\rangle = \frac{\partial}{\partial \alpha} \langle D \rangle = \frac{\partial C}{\partial \alpha} = \left\langle \frac{\partial}{\partial \alpha} \left((x-\mu)(x-\mu)^T \right) \right\rangle = \left\langle -\frac{\partial \mu}{\partial \alpha} (x-\mu)^T - (x-\mu) \left(\frac{\partial \mu}{\partial \alpha} \right)^T \right\rangle$$

$$\left\langle \frac{\partial^2 D}{\partial \alpha \partial \beta} \right\rangle = \frac{\partial^2 C}{\partial \beta \partial \alpha} = \left\langle -\cancel{\frac{\partial^2 \mu}{\partial \beta \partial \alpha}} (x-\mu)^T + \cancel{\frac{\partial \mu^T}{\partial \beta}} \frac{\partial \mu}{\partial \alpha} + \frac{\partial \mu}{\cancel{\partial \beta}} \frac{\partial \mu^T}{\partial \alpha} - \frac{\partial^2 \mu^T}{\partial \beta \partial \alpha} \cancel{(x-\mu)} \right\rangle = 0$$

since $\langle x-\mu \rangle = 0$ and $\frac{\partial \mu}{\partial \theta} = 0$

$$\frac{\partial^2 C^{-1}}{\partial \beta \partial \alpha} = \frac{\partial}{\partial \beta} \left(C^{-1} \frac{\partial C}{\partial \alpha} C^{-1} \right) = C^{-1} \frac{\partial C}{\partial \beta} C^{-1} \frac{\partial C}{\partial \alpha} C^{-1} - C^{-1} \cancel{\frac{\partial^2 C}{\partial \beta \partial \alpha}} C^{-1} + C^{-1} \frac{\partial C}{\partial \alpha} \left(C^{-1} \frac{\partial C}{\partial \beta} C^{-1} \right) = 2C^{-1} \frac{\partial C}{\partial \beta} C^{-1} \frac{\partial C}{\partial \alpha} C^{-1}$$

All crossed terms are zero, it follows

$$-\ln L = \frac{1}{2} tr\left(-C^{-1} \frac{\partial C}{\partial \beta} C^{-1} \frac{\partial C}{\partial \alpha} - C^{-1} \frac{\partial C}{\partial \beta} C^{-1} \overset{\text{sym}}{\frac{\partial D}{\partial \alpha}} + 2C^{-1} \frac{\partial C}{\partial \alpha} C^{-1} \frac{\partial C^{-1}}{\partial \beta} D - C^{-1} \frac{\partial C}{\partial \alpha} C^{-1} \overset{\text{sym}}{\frac{\partial D}{\partial \beta}} \right)$$

$$= \frac{1}{2} tr\left(-C^{-1} \frac{\partial C}{\partial \beta} C^{-1} \frac{\partial C}{\partial \alpha} - 2C^{-1} \frac{\partial C}{\partial \beta} C^{-1} \frac{\partial D}{\partial \alpha} + 2C^{-1} \frac{\partial C}{\partial \alpha} C^{-1} \frac{\partial C^{-1}}{\partial \beta} D \right)$$

$$= \frac{1}{2} tr\left(-C^{-1} \frac{\partial C}{\partial \beta} C^{-1} \frac{\partial C}{\partial \alpha} - 2C^{-1} \frac{\partial C}{\partial \beta} C^{-1} \frac{\partial D}{\partial \alpha} - 2C^{-1} \frac{\partial C}{\partial \alpha} C^{-1} \left(C^{-1} \frac{\partial C}{\partial \beta} C^{-1} \right) D \right)$$

$$= \frac{1}{2} tr\left(-C^{-1} \frac{\partial C}{\partial \beta} C^{-1} \frac{\partial C}{\partial \alpha} - 2C^{-1} \frac{\partial C}{\partial \beta} C^{-1} \left(\frac{\partial D}{\partial \alpha} + \frac{\partial C}{\partial \alpha} C^{-1} D \right) \right)$$

Where the 2nd term

$$\left\langle \frac{\partial D}{\partial \alpha} \right\rangle = \frac{\partial C}{\partial \alpha} \Rightarrow \underbrace{\left\langle \left(-\frac{\partial \mu}{\partial \beta} (x-\mu)^T - (x-\mu) \frac{\partial \mu^T}{\partial \beta} \right) C^{-1} \left(-\frac{\partial \mu}{\partial \alpha} (x-\mu)^T - (x-\mu) \frac{\partial \mu^T}{\partial \alpha} - \frac{\partial \mu}{\partial \alpha} (x-\mu)^T C^{-1} D - (x-\mu) \frac{\partial \mu^T}{\partial \alpha} C^{-1} D \right) \right.}_{\substack{\text{should} \\ \text{be equal}}}$$

$$\Rightarrow \frac{\partial \mu}{\partial \alpha} \frac{\partial \mu^T}{\partial \beta}$$