Problem 1.

$$F_{\alpha\beta} = \frac{1}{2} tr\left(C^{-1}\frac{\partial C}{\partial\theta^\alpha} C^{-1}\frac{\partial C}{\partial\theta^\beta} + 2C^{-1}\frac{\partial\mu}{\partial\theta^\alpha}\frac{\partial\mu^T}{\partial\theta^\beta}\right)$$

$$\frac{\partial C^{-1}}{\partial\theta^\alpha} = -C^{-1}\frac{\partial C}{\partial\theta^\alpha}C^{-1}$$

The Fisher Matrix is given by :

$$F_{\alpha\beta} = -\left\langle \frac{\partial \ln L(D|\theta)}{\partial\theta^\alpha}\frac{\partial \ln L(D|\theta)}{\partial\theta^\beta}\right\rangle$$

The negative of the log-likelihood is given by:

$$-\ln L = \frac{1}{2} tr(\ln C + C^{-1}D) + \text{const}$$

Takin 1st derivative: $\leftarrow \frac{\partial}{\partial\alpha} = \frac{\partial}{\partial\theta^\alpha}$

$$-\frac{\partial \ln L}{\partial\theta^\alpha} = \frac{1}{2} tr\left(C^{-1}\frac{\partial C}{\partial\alpha} + \frac{\partial C^{-1}}{\partial\alpha}D + C^{-1}\frac{\partial D}{\partial\alpha}\right)$$

$$-\frac{\partial^2 \ln L}{\partial\theta^\beta\theta^\alpha} = \frac{1}{2} tr\left(\frac{\partial C^{-1}}{\partial\beta}\frac{\partial C}{\partial\alpha} + C^{-1}\frac{\partial^2 C}{\partial\beta\partial\alpha} + \frac{\partial^2 C^{-1}}{\partial\beta\partial\alpha}D + \frac{\partial C^{-1}}{\partial\alpha}\frac{\partial D}{\partial\beta} + \frac{\partial C^{-1}}{\partial\beta}\frac{\partial D}{\partial\alpha} + C^{-1}\frac{\partial^2 D}{\partial\beta\partial\alpha}\right)$$

$$= \frac{1}{2} tr\left(\frac{\partial C^{-1}}{\partial\beta}\left(\frac{\partial C}{\partial\alpha} + \frac{\partial D}{\partial\alpha}\right) + C^{-1}\left(\frac{\partial^2 C}{\partial\beta\partial\alpha} + \frac{\partial^2 D}{\partial\beta\partial\alpha}\right) + \frac{\partial^2 C^{-1}}{\partial\beta\partial\alpha}D + \frac{\partial C^{-1}}{\partial\alpha}\frac{\partial D}{\partial\beta}\right)$$

Lets compute the expectation values for:

$$\left\langle\frac{\partial D}{\partial\alpha}\right\rangle = \frac{\partial}{\partial\alpha}\langle D\rangle = \frac{\partial C}{\partial\alpha} = \left\langle \frac{\partial}{\partial\alpha}\left((x-\mu)(x-\mu)^T\right)\right\rangle = \left\langle -\frac{\partial\mu}{\partial\alpha}(x-\mu)^T - (x-\mu)\left(\frac{\partial\mu}{\partial\alpha}\right)^T\right\rangle$$

$$\left\langle\frac{\partial^2 D}{\partial\alpha\partial\beta}\right\rangle = \frac{\partial^2 C}{\partial\beta\partial\alpha} = \left\langle \cancelto{0}{-\frac{\partial^2\mu}{\partial\beta\partial\alpha}}(x-\mu)^T + \cancelto{0}{\frac{\partial\mu^T}{\partial\beta}}\frac{\partial\mu}{\partial\alpha} + \frac{\partial\mu}{\cancelto{0}{\partial\beta}}\frac{\partial\mu^T}{\partial\alpha} - \frac{\partial^2\mu^T}{\partial\beta\partial\alpha}\cancelto{0}{(x-\mu)}\right\rangle = 0 \leftarrow \text{since } \langle(x-\mu)\rangle = 0 \text{ and } \frac{\partial\mu}{\partial\theta} = 0$$

$$\frac{\partial^2 C^{-1}}{\partial\beta\partial\alpha} = \frac{\partial}{\partial\beta}\left(-C^{-1}\frac{\partial C}{\partial\alpha}C^{-1}\right) = C^{-1}\frac{\partial C}{\partial\beta}C^{-1}\frac{\partial C}{\partial\alpha}C^{-1} - C^{-1}\cancelto{0}{\frac{\partial^2 C}{\partial\beta\partial\alpha}}C^{-1} + C^{-1}\frac{\partial C}{\partial\alpha}\left(C^{-1}\frac{\partial C}{\partial\beta}C^{-1}\right) = 2C^{-1}\frac{\partial C}{\partial\beta}C^{-1}\frac{\partial C}{\partial\alpha}C^{-1}$$

All crossed terms are zero, it follows

$$-\ln L = \frac{1}{2} tr\left(-C^{-1}\frac{\partial C}{\partial\beta}C^{-1}\frac{\partial C}{\partial\alpha} - C^{-1}\frac{\partial C}{\partial\beta}C^{-1}\overset{\text{sym}}{\frac{\partial D}{\partial\alpha}} + 2C^{-1}\frac{\partial C}{\partial\alpha}C^{-1}\frac{\partial C^{-1}}{\partial\beta}D - \overset{\text{sym}}{C^{-1}}\frac{\partial C}{\partial\alpha}C^{-1}\frac{\partial D}{\partial\beta}\right)$$

$$= \frac{1}{2} tr\left(-C^{-1}\frac{\partial C}{\partial\beta}C^{-1}\frac{\partial C}{\partial\alpha} - 2C^{-1}\frac{\partial C}{\partial\beta}C^{-1}\frac{\partial D}{\partial\alpha} + 2C^{-1}\frac{\partial C}{\partial\alpha}C^{-1}\frac{\partial C^{-1}}{\partial\beta}D\right)$$

$$= \frac{1}{2} tr\left(-C^{-1}\frac{\partial C}{\partial\beta}C^{-1}\frac{\partial C}{\partial\alpha} - 2C^{-1}\frac{\partial C}{\partial\beta}C^{-1}\frac{\partial D}{\partial\alpha} - 2C^{-1}\frac{\partial C}{\partial\alpha}C^{-1}\left(C^{-1}\frac{\partial C}{\partial\beta}C^{-1}\right)D\right)$$

$$= \frac{1}{2} tr\left(-C^{-1}\frac{\partial C}{\partial\beta}C^{-1}\frac{\partial C}{\partial\alpha} - 2C^{-1}\frac{\partial C}{\partial\beta}C^{-1}\left(\frac{\partial D}{\partial\alpha} + \frac{\partial C}{\partial\alpha}C^{-1}D\right)\right)$$

Where the 2nd term

$$\left\langle\frac{\partial D}{\partial\alpha}\right\rangle = \frac{\partial C}{\partial\alpha} \Rightarrow \underbrace{\left\langle\left(-\frac{\partial\mu}{\partial\beta}(x-\mu)^T - (x-\mu)\frac{\partial\mu^T}{\partial\beta}\right)C^{-1}\left(-\frac{\partial\mu}{\partial\alpha}(x-\mu)^T - (x-\mu)\frac{\partial\mu^T}{\partial\alpha} - \frac{\partial\mu}{\partial\alpha}(x-\mu)^T C^{-1}D - (x-\mu)\frac{\partial\mu^T}{\partial\alpha}C^{-1}D\right)\right.}_{\text{should be equal}} \Rightarrow \frac{\partial\mu}{\partial\alpha}\frac{\partial\mu^T}{\partial\beta}$$

2. $\vec{y} = A\vec{x} \Rightarrow$ $C' = \mathrm{cov}(\vec{y}) = \langle (y - E(y))(y - E(y))^T \rangle = \langle A(\vec{x} - \vec{\mu}) \cdot [A(\vec{x} - \vec{\mu})]^T \rangle = A \cdot C \cdot A^T$

$$C'^{-1} = (ACA^T)^{-1} = A^{-T} C^{-1} A^{-1}$$

$$\mu' = A\mu \, , \quad \mu'^T = (A\mu)^T = \mu^T A^T$$

Then $C'^{-1} \frac{\partial C'}{\partial \theta^\alpha} C'^{-1} \frac{\partial C}{\partial \theta^\beta}$ $= A^{-T} C^{-1} A^{-1} \cdot \frac{\partial}{\partial \theta^\alpha} (ACA^T) \cdot (A^{-T} C^{-1} A^{-1}) \frac{\partial}{\partial \theta^\beta} (ACA^T)$

$$= A^{-T} C^{-1} (A^{-1} A) \frac{\partial C}{\partial \theta^\alpha} (A^T A^{-T}) C^{-1} (A^{-1} A) \frac{\partial C}{\partial \theta^\beta} \cdot A^T$$

$$= A^{-T} C^{-1} \frac{\partial C}{\partial \theta^\alpha} C^{-1} \frac{\partial C}{\partial \theta^\beta} A^T = C^{-1} \frac{\partial C}{\partial \theta^\alpha} C^{-1} \frac{\partial C}{\partial \theta^\beta}$$

$$2C'^{-1} \frac{\partial \mu'}{\partial \theta^\alpha} \frac{\partial \mu'^T}{\partial \theta^\beta} = 2 \cdot A^{-T} C^{-1} A^{-1} \cdot \frac{\partial (A\mu)}{\partial \theta^\alpha} \cdot \frac{\partial (\mu^T A^T)}{\partial \theta^\beta}$$

$$= 2 \cdot A^{-T} C^{-1} (A^{-1} A) \frac{\partial \mu}{\partial \theta^\alpha} \cdot \frac{\partial \mu^T}{\partial \theta^\beta} A^T$$

$$= 2 \cdot A^{-T} C^{-1} \frac{\partial \mu}{\partial \theta^\alpha} \frac{\partial \mu^T}{\partial \theta^\beta} A^T$$

$$= 2 C^{-1} \cdot \frac{\partial \mu}{\partial \theta^\alpha} \frac{\partial \mu^T}{\partial \theta^\beta}$$

Therefore, any linear transformation of data leaves Fisher matrix unchanged.