Training augmented interpolation

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Introduction

In this projet we explored the resolution of transport equations, for this we implemented the Semi Lagrangian scheme, using the classical Lagrange interpolation and deep Lagrange interpolation.

Transport equation:

$$\begin{cases} \partial_t u + a\partial_x u = 0 \\ u(t=0,x) = u_0(x,\mu) \end{cases}$$

General context

This internship project was supervised by the INRIA and the research team on the development of numerical methods to solve partial differential equations for the simulation of physical phenomena of the University of Strasbourg.

Objectives

The main objective of this project is to:

The other objectives are:

- To study the convergence of the semi-Lagrangian method with the deep Lagrange interpolator.
- To study the convergence of the semi-Lagrangian method with the deep Lagrange interpolator with PINNs.
- To study the convergence of the semi-Lagrangian method with the deep Lagrange interpolator with PINNs and the use of the adjoint method.

Semi-Lagrangian Scheme

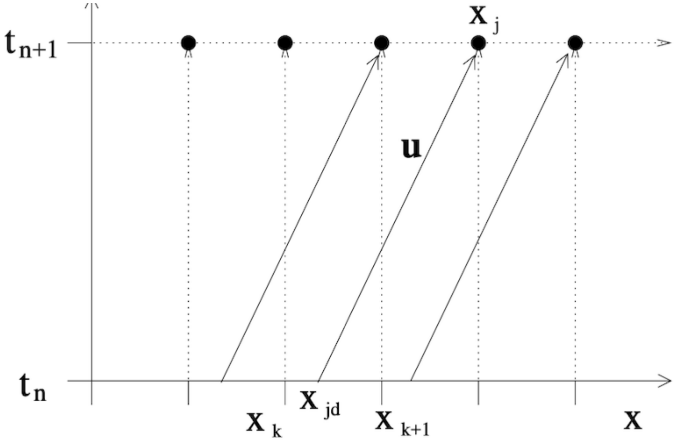


Figure: Semi-Lagrangian in 1D

Lagrange Interpolation Operators

The Lagrange interpolation operator:

$$\mathcal{I}_h^m(f)(x) = \sum_{i=1}^{m} f(x_i) P_i(x)$$

with $P_i(x_j) = \delta_{ij}$

The Deep Lagrange Interpolation:

$$\mathcal{I}_d^m(f) = \sum_{i=1}^{n} \frac{f(x_i)}{u_\theta(x_i)} P_i(x) u_\theta(x) = \mathcal{I}^m\left(\frac{f}{u_\theta}\right) u_\theta(x)$$

With $P_i(x_j) = \delta_{ij}$ Using this choice, we obtain that $\mathcal{I}_d(f)(x_i) = f(x_i)$ as the classical interpolator.

u_θ Function:

How do we choose u_θ ?
We use a neural network which will approximate the $u_\theta(x)$ function.

$$u_\theta(x, t, \mu, \sigma, a)$$

We train a neural network with a **Physics Informed Neural Network** strategy, and we use the previous interpolation to approximate the solution of the PDE.

Universal Approximation Theorem

Any continuous function $f : [0, 1]^n \to [0, 1]$ can be approximated arbitrarily well by a neural network with at least 1 hidden layer with a finite number of weights.

Even if neural networks can express very complex functions compactly, determining the precise parameters (weights and biases) required to solve a specific PDE can be difficult.

Physics informed neural network

- PINNs are a type of universal function approximators that can embed the knowledge of any physical laws that govern a given data-set in the learning process, and can be described by PDEs.
- They approximate PDE solutions by training a neural network to minimize a loss function; it includes terms reflecting the initial and boundary conditions along the space-time domain's boundary and PDE residual and data points.
- PINNs training can be thought of as a supervised and/or supervised learning approach.

Physics Informed Neural Network

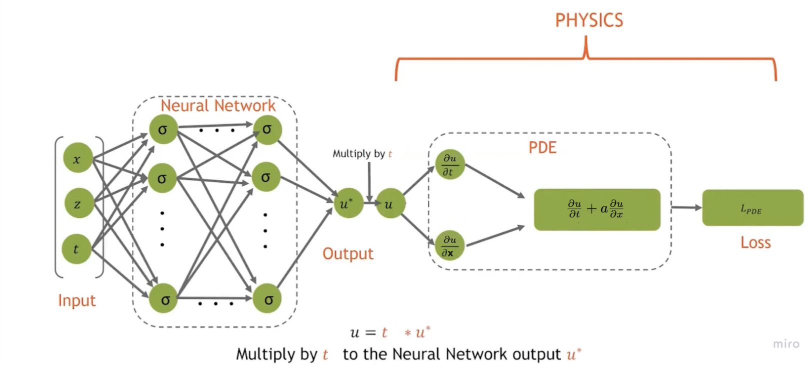


Figure: PINN strategy

Results

Results

Using Deep Lagrange Interpolation

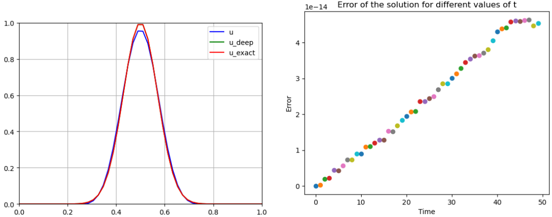


Figure: Solution and error with nx=50, nt=100, a=1, \epsilon=0.0

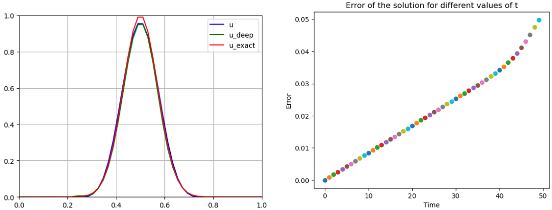
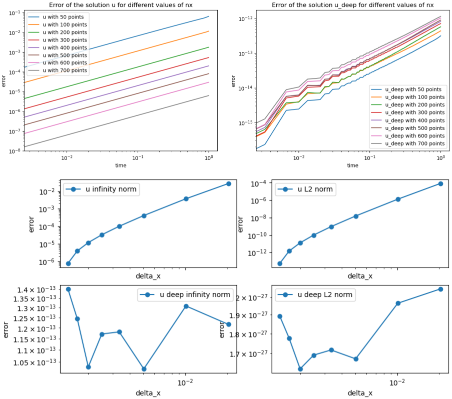


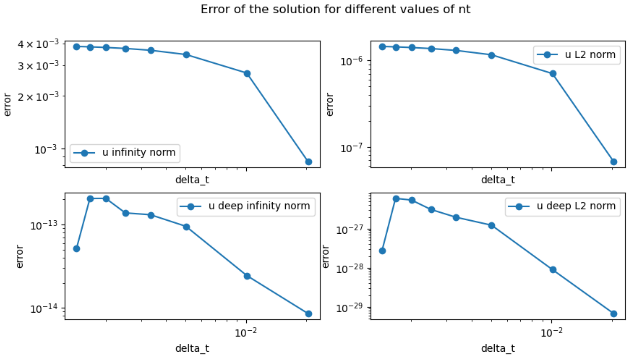
Figure: Solution and error with nx=50, nt=100, a=1, \epsilon=0.085

Results

Convergence Results
In space



Convergence Results
In time



Conclusions

Our findings demonstrate that the Semi-Lagrangian scheme combined with deep learning approaches shows promise in improving the accuracy of transport equation solutions. However, there is still further research to be done implementing the deep Lagrange interpolation with PINNs to optimize the deep learning models and explore their full potential in solving complex transport equations.

Bibliography

1 Semi Lagrangian Scheme - A visual explanation.
`https://youtu.be/kvBRFxRIJuY`

2 Tutorial on Semi-Lagrangian schemes, BIRS workshop 11w5086.

3 Chapitre 4. Méthodes numériques pour Vlasov.