Homework 4 - Timiș Diana (917)

1. Radius of convergence and the convergence set

a) $\sum_{n\geq 0} \frac{n x^n}{2^n}$; $a_n = \frac{n}{2^n}$, $c = 0$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{2n}\right| = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2} \in (0, +\infty) \Rightarrow$

$\Rightarrow R = \frac{1}{L} = \frac{1}{\frac{1}{2}} = 2 \Rightarrow$ The series is convergent on $(c-R, c+R) = (-2, 2)$

Check for: $\begin{cases} x=-2: \sum_{n\geq 0}\frac{n(-2)^n}{2^n} = \sum_{n\geq 0}(-1)^n n \text{ divergent} \\ x = 2: \sum_{n\geq 0}\frac{n\cdot 2^n}{2^n} = \sum_{n\geq 0} n \text{ divergent}\end{cases} \Rightarrow$

$\Rightarrow$ The convergence set is $C = (-2, 2)$

b) $\sum_{n\geq 1}\frac{x^{2n}}{\sqrt{n}} = \sum_{n\geq 1}\frac{(x^2)^n}{\sqrt{n}} := \sum_{n\geq 1}\frac{y^n}{\sqrt{n}}$, where $y = x^2 \geq 0$; $a_n = \frac{1}{\sqrt{n}}$, $c = 0$

$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n}}{\sqrt{n+1}}\right| = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = \sqrt{\lim_{n\to\infty}\frac{n}{n+1}} = \sqrt{1} = 1 \in (0, +\infty) \Rightarrow$

$\Rightarrow R = \frac{1}{L} = \frac{1}{1} = 1 \Rightarrow$ The series is convergent on $(c-R, c+R) = (-1, 1)$

But $y \geq 0$ $\Bigg\} \Rightarrow$

$\Rightarrow$ The series is convergent for $y \in [0, 1)$

Check for $y = 1$ ($\Leftrightarrow x \in \{\pm 1\}$): $\sum_{n\geq 1}\frac{1}{\sqrt{n}} = \sum_{n\geq 1}\frac{1}{n^{\frac{1}{2}}}$ divergent $\left(\frac{1}{2}\leq 1\right)$ $\Bigg\} \Rightarrow$

$\Rightarrow$ The series is abs. conv. for $y \in [0, 1) \Leftrightarrow$ abs. conv. for $x \in (-1, 1)$

c) $\sum_{n\geq 1}(-1)^n\cdot\frac{(x-1)^n}{n}$; $a_n = \frac{(-1)^n}{n}$, $c = 1$

$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{n+1}\cdot\frac{n}{(-1)^n}\right| = \lim_{n\to\infty}\left|-\frac{n}{n+1}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1 \Rightarrow$

$\Rightarrow R = \frac{1}{L} = \frac{1}{1} = 1 \Rightarrow$ The series is convergent on $(c-R, c+R) = (0, 2)$

Check for: $\begin{cases} x = 0: \sum_{n\geq 1}(-1)^n\cdot\frac{(-1)^n}{n} = \sum_{n\geq 1}\frac{1}{n} \text{ divergent} \\ x = 2: \sum_{n\geq 1}(-1)^n\cdot\frac{1^n}{n} = \sum_{n\geq 1}\frac{(-1)^n}{n} = -\sum_{n\geq 1}\frac{(-1)^{n+1}}{n} = -\ln 2 \text{ convergent}\end{cases} \Rightarrow$

$\Rightarrow$ The convergence set is $C = (0, 2]$

2. Study the convergence and compute the sum for the series $\sum_{n\geq 2} \frac{x^n}{n(n-1)}$.

Case I. $|x| > 1$

We apply the ratio test: $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)n} \cdot \frac{n(n-1)}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x(n-1)}{n+1} \right| = |x| > 1 \Rightarrow$

$\Rightarrow$ For $|x| > 1$, the series diverges.

Case II. $x = 1$

$$\sum_{n\geq 2} \frac{1}{n(n-1)}$$

Let (S_n) be a sequence defined by $S_n = \sum_{k=2}^{n} \frac{1}{k(k-1)}$

$$S_n = \sum_{k=2}^{n} \frac{1}{k(k-1)} = \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) = \sum_{k=2}^{n} \frac{1}{k-1} - \sum_{k=2}^{n} \frac{1}{k} = \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=2}^{n} \frac{1}{k} = 1 - \frac{1}{n} = \frac{n-1}{n}$$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n-1}{n} = \frac{1}{1} = 1 \in \mathbb{R} \Rightarrow$

$\Rightarrow \sum_{n\geq 2} \frac{1}{n(n-1)} = 1$ (converges) $\Rightarrow$

$\Rightarrow$ For $x = 1$, the series converges.

Case III. $x = -1$

$$\sum_{n\geq 2} \frac{(-1)^n}{n(n-1)} = \sum_{n\geq 2} \left(\frac{(-1)^n}{n-1} - \frac{(-1)^n}{n} \right) = \sum_{n\geq 2} \left(\frac{(-1)^n}{n-1} + \frac{(-1)^{n+1}}{n} \right) = \sum_{n\geq 2} \frac{(-1)^n}{n-1} + \sum_{n\geq 2} \frac{(-1)^{n+1}}{n} =$$

$$= \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} + \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} - 1 = 2\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} - 1 =$$

$$= 2\ln 2 - 1 = \ln 2^2 - \ln e = \ln 4 - \ln e = \ln \frac{4}{e} \in \mathbb{R} \text{ (converges)} \Rightarrow$$

$\Rightarrow$ For $x = -1$, the series converges.

Case IV. $|x| < 1$

We apply the ratio test: $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)n} \cdot \frac{n(n-1)}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x(n-1)}{n+1} \right| = |x| < 1 \Rightarrow$

$\Rightarrow$ For $|x| < 1$, the series converges.

$$\left(\frac{x^n}{n(n-1)} \right)' = \frac{1}{n(n-1)} \cdot (x^n)' = \frac{1}{n(n-1)} \cdot n x^{n-1} = \frac{x^{n-1}}{n-1}$$

$$\sum_{n\geq 2} \frac{x^n}{n(n-1)} = \sum_{n\geq 2} \int_0^t \frac{t^{n-1}}{n-1}\, dt = \int_0^x \sum_{n\geq 2} \frac{t^{n-1}}{n-1}\, dt = \int_0^x \sum_{n\geq 1} \frac{t^n}{n}\, dt$$

We know that: $\sum_{n\geq 0} y^n = 1 + y + y^2 + \dots = \frac{1}{1-y}$, $|y| < 1$

We have: $\begin{cases} \int_0^t \sum_{n\geq 0} y^n\, dy = \sum_{n\geq 0} \int_0^t y^n\, dy = \sum_{n\geq 0} \frac{y^{n+1}}{n+1} \Big|_0^t = \sum_{n\geq 0} \frac{t^{n+1}}{n+1} = \sum_{n\geq 1} \frac{t^n}{n} \\ \int_0^t \frac{1}{1-y}\, dy = -\ln(1-y) \Big|_0^t = -\ln(1-t) \end{cases} \Rightarrow$

$$\left.\begin{aligned}
&\Longrightarrow \sum_{n\geq 1} \frac{t^n}{n} = -\ln(1-t)\\
&\sum_{n\geq 2} \frac{x^n}{n(n-1)} = \int_0^x \sum_{n\geq 1} \frac{t^n}{n}\, dt
\end{aligned}\right\} \Longrightarrow$$

$$\Longrightarrow \sum_{n\geq 2} \frac{x^n}{n(n-1)} = \int_0^x -\ln(1-t)\, dt = -\int_0^x \ln(1-t)\, dt = -\left(t\cdot\ln(1-t)\Big|_0^x - \int_0^x \frac{-t}{1-t}\, dt \right) =$$

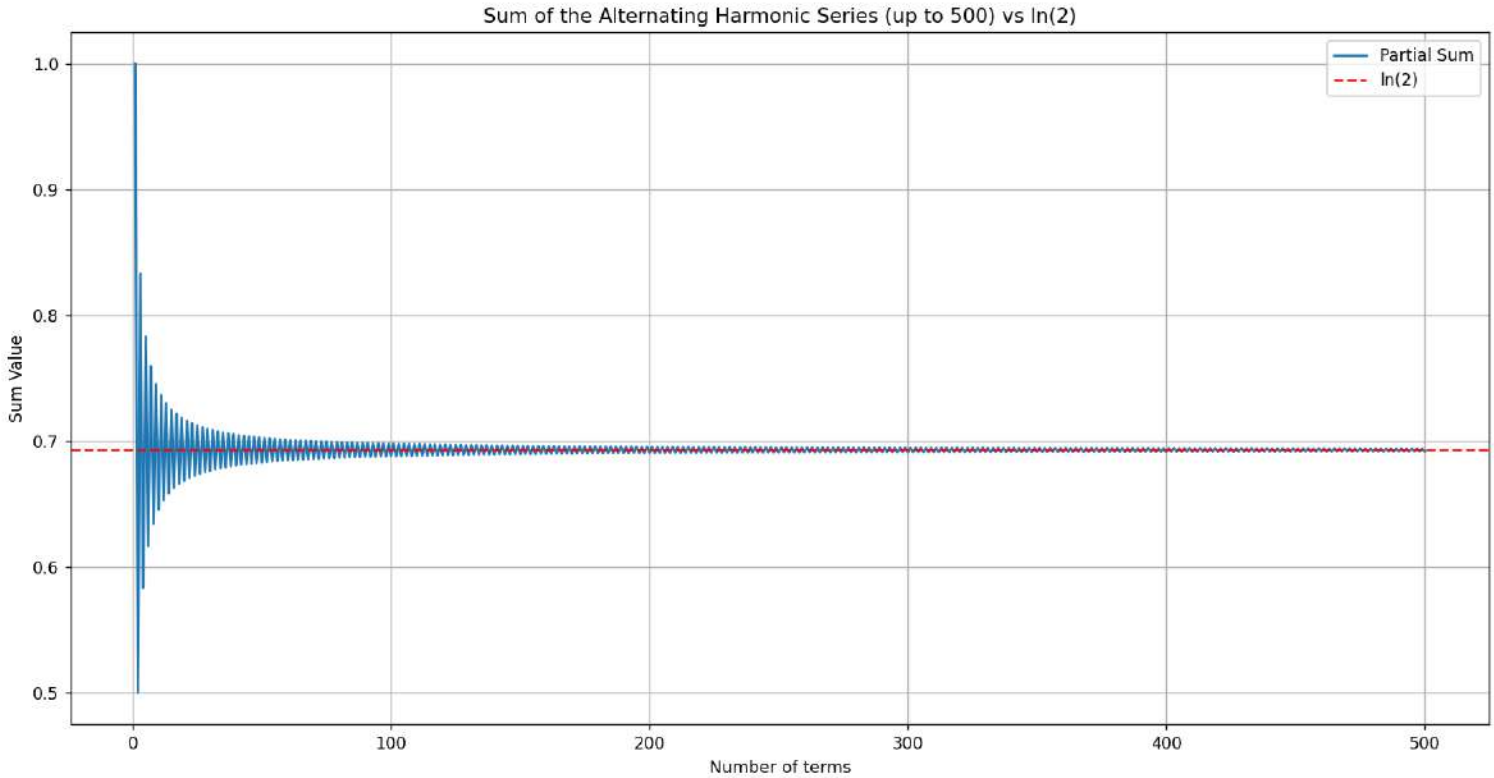
$$= -\left(x\ln(1-x) + \int_0^x \frac{t}{1-t}\, dt \right) = -\left(x\ln(1-x) - \int_0^x dt + \int_0^x \frac{1}{1-t}\, dt \right) =$$

$$= -\left(x\ln(1-x) - t\Big|_0^x - \ln(1-t)\Big|_0^x \right) = -\left(x\ln(1-x) - x - \ln(1-x) \right) =$$

$$= -\left((x-1)\ln(1-x) - x \right) = x - (x-1)\ln(1-x)$$

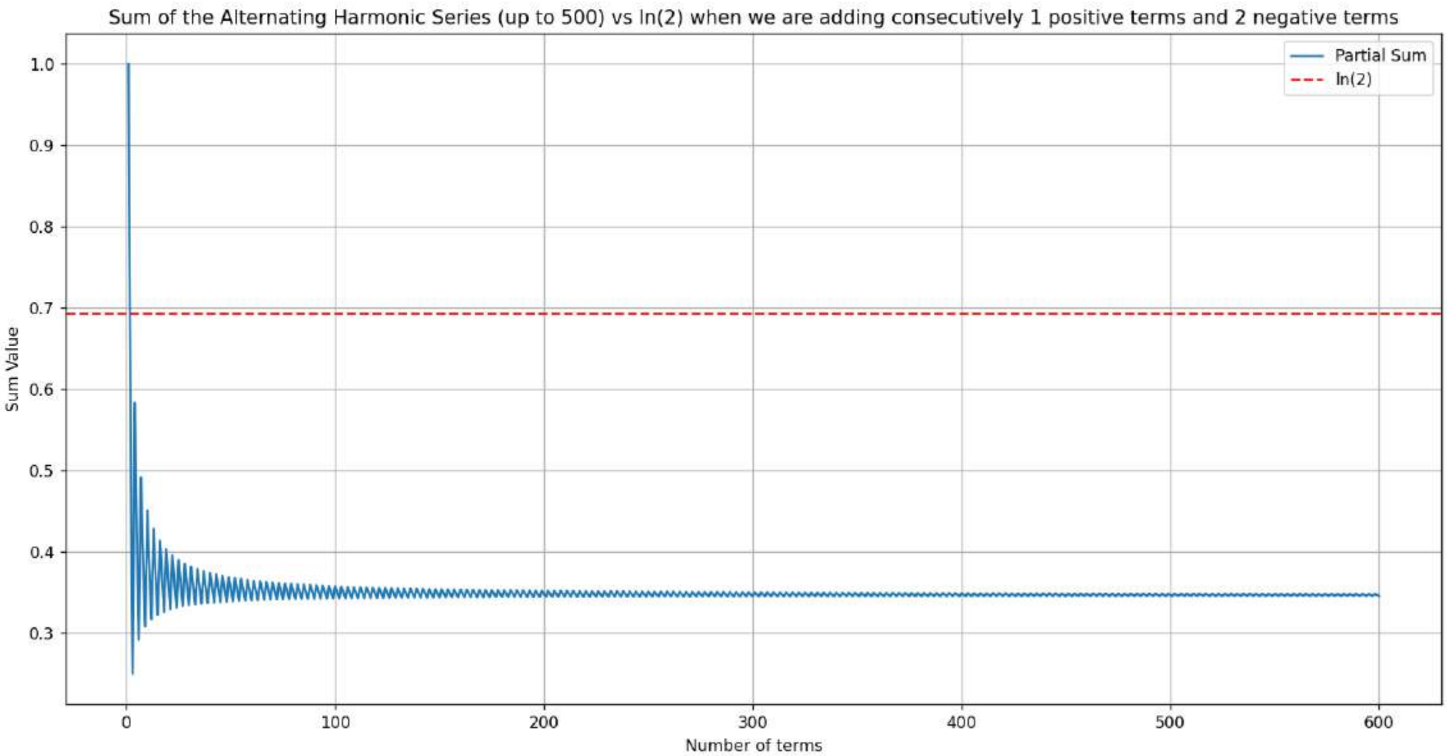
main.py

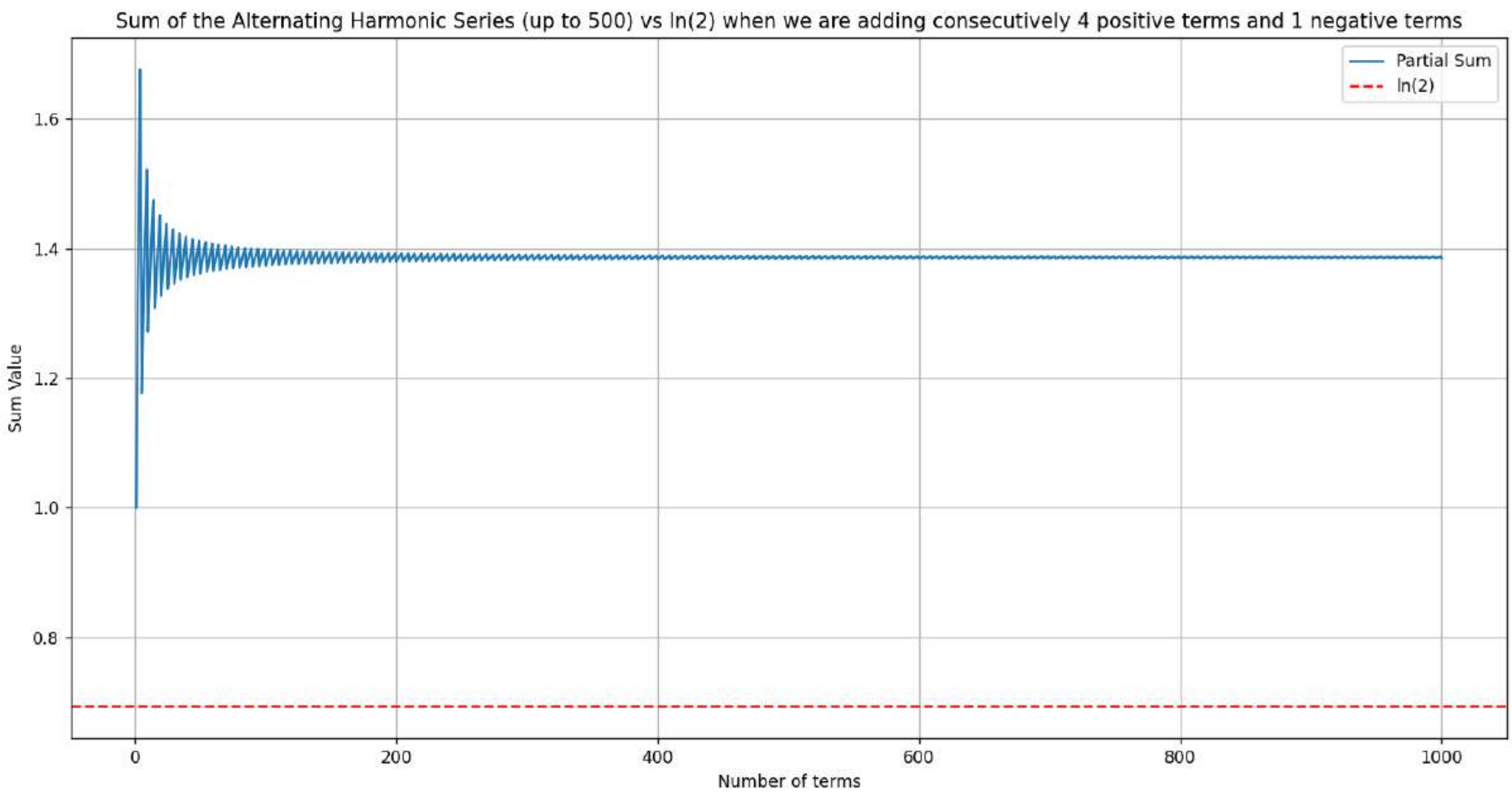
```python
import math
import matplotlib.pyplot as plt

maximumNumberOfTerms = 500
sumValues = []
partialSum = 0

for i in range(1, maximumNumberOfTerms + 1):
    currentTerm = ((-1) ** (i + 1)) * (1 / i)
    partialSum += currentTerm
    sumValues.append(partialSum)

ln2 = math.log(2)

plt.plot(range(1, maximumNumberOfTerms + 1), sumValues, label="Partial Sum")
plt.axhline(ln2, color='red', linestyle='--', label="ln(2)")

plt.xlabel("Number of terms")
plt.ylabel("Sum Value")
plt.title("Sum of the Alternating Harmonic Series (up to 500) vs ln(2)")
plt.legend()
plt.grid(True)
plt.show()
```


Sum of the Alternating Harmonic Series (up to 500) vs ln(2)
Partial Sum
ln(2)
1.0
0.9
0.8
0.7
0.6
0.5
Sum Value
0
100
200
300
400
500
Number of terms

main2.py

```python
import math
import matplotlib.pyplot as plt

numberOfPositiveTerms = int(input("Enter the number of positive terms added consecutively to the alternating sum: "))
numberOfNegativeTerms = int(input("Enter the number of negative terms added consecutively to the alternating sum: "))

maximumNumberOfTerms = 200 * (numberOfPositiveTerms + numberOfNegativeTerms)
sumValues = []
partialSum = 0

n = 1
oddNumber = 1
evenNumber = 2
while n <= maximumNumberOfTerms:
    for i in range(numberOfPositiveTerms):
        currentTerm = 1 / oddNumber
        partialSum += currentTerm
        n += 1
        oddNumber += 2
        sumValues.append(partialSum)
    for i in range(numberOfNegativeTerms):
        currentTerm = -1 / evenNumber
        partialSum += currentTerm
        n += 1
        evenNumber += 2
        sumValues.append(partialSum)

ln2 = math.log(2)

plt.plot(range(1, maximumNumberOfTerms + 1), sumValues, label="Partial Sum")
plt.axhline(ln2, color='red', linestyle='--', label="ln(2)")

plt.xlabel("Number of terms")
plt.ylabel("Sum Value")
plt.title("Sum of the Alternating Harmonic Series (up to 500) vs ln(2) when we are adding consecutively " +
          str(numberOfPositiveTerms) + " positive terms and " +
          str(numberOfNegativeTerms) + " negative terms")
plt.legend()
plt.grid(True)
plt.show()
```


Sum of the Alternating Harmonic Series (up to 500) vs ln(2) when we are adding consecutively 1 positive terms and 2 negative terms
Partial Sum
ln(2)
Sum Value
Number of terms
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0
100
200
300
400
500
600


Sum of the Alternating Harmonic Series (up to 500) vs ln(2) when we are adding consecutively 4 positive terms and 1 negative terms
Partial Sum
ln(2)
1.6
1.4
1.2
1.0
0.8
Sum Value
0
200
400
600
800
1000
Number of terms