Extra homework 2 - Timiș Diana (917)

1. $f:[a,b]\to[a,b]$ a contraction: $\exists\, \mathcal{L}\in(0,1)$ such that
$|f(x)-f(y)|\le \mathcal{L}|x-y|$, $\forall x,y\in[a,b]$
$x_1\in[a,b]$ arbitrary, (x_n) a sequence given by $x_{n+1}=f(x_n)$, $\forall n\in\mathbb{N}$
a) (x_n) Cauchy
b) $\exists x^*\in[a,b]$ s.t. $f(x^*)=x^*$ and $x^*=\lim\limits_{n\to\infty}x_n$

a) Let $p\in\mathbb{N}$. We have:

$$0\le|x_{n+p}-x_n|=|x_{n+p}-x_{n+p-1}+x_{n+p-1}-x_{n+p-2}+\dots+x_{n+2}-x_{n+1}+x_{n+1}-x_n|\le$$
$$\le|x_{n+p}-x_{n+p-1}|+|x_{n+p-1}-x_{n+p-2}|+\dots+|x_{n+2}-x_{n+1}|+|x_{n+1}-x_n|=$$
$$=\mathcal{L}|f(x_{n+p-1})-f(x_{n+p-2})|+\mathcal{L}|f(x_{n+p-2})-f(x_{n+p-3})|+\dots+\mathcal{L}|f(x_n)-f(x_{n-1})|\le$$
$$\le\mathcal{L}|x_{n+p-1}-x_{n+p-2}|+\mathcal{L}|x_{n+p-2}-x_{n+p-3}|+\dots+\mathcal{L}|x_n-x_{n-1}|\le\dots\le$$
$$\le\mathcal{L}^{n-1}(|x_{p+1}-x_p|+|x_p-x_{p-1}|+\dots+|x_2-x_1|)$$

$$\lim_{n\to\infty}(\mathcal{L}^{n-1}(|x_{p+1}-x_p|+|x_p-x_{p-1}|+\dots+|x_2-x_1|))=$$
$$=(|x_{p+1}-x_p|+\dots+|x_2-x_1|)\cdot\lim_{n\to\infty}\mathcal{L}^{n-1}=(|x_{p+1}-x_p|+\dots+|x_2-x_1|)\cdot 0=0$$

Using the Sandwich Theorem, we obtain:

$$\lim_{n\to\infty}|x_{n+p}-x_n|=0\implies\forall\varepsilon>0,\ \exists N_\varepsilon\in\mathbb{N}\text{ s.t. }|x_{n+p}-x_n|<\varepsilon,\ \forall n\ge N_\varepsilon\implies$$
$\implies(x_n)$ is Cauchy

b) We know that (x_n) is Cauchy $\implies(x)_n$ is convergent
We denote $x^*=\lim\limits_{n\to\infty}x_n$
We have:

$$0\le|f(x_n)-x_n|=|x_{n+1}-x_n|=|f(x_n)-f(x_{n-1})|\le\mathcal{L}|x_n-x_{n-1}|\le\dots\le$$
$$\le\mathcal{L}^{n-1}|x_2-x_1|$$
$$\lim_{n\to\infty}(\mathcal{L}^{n-1}|x_2-x_1|)=|x_2-x_1|\lim_{n\to\infty}\mathcal{L}^{n-1}=|x_2-x_1|\cdot0=0$$
$\xrightarrow{\text{Sandwich Theorem}}$

$$\implies\lim_{n\to\infty}|f(x_n)-x_n|=0\implies\lim_{n\to\infty}(f(x_n)-x_n)=0\iff$$
$$\iff\lim_{n\to\infty}f(x_n)-\lim_{n\to\infty}x_n=0\iff\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}x_n\iff\lim_{n\to\infty}f(x_n)=x^*$$

Knowing that f is a contraction ($|f(x)-f(y)|\le\mathcal{L}|x-y|$, $\forall x,y\in[a,b]$), we can consider y as a fixed point and $\delta>0$ such that $|x-y|<\delta$, $\forall x\in[a,b]$ (We can make this assumption, because $[a,b]$ bounded).

If we take $\varepsilon > 0$ arbitrary, $\varepsilon < \delta$, and we consider $\mathcal{L} = \frac{\varepsilon}{\delta}$, we will have:

$$|f(x) - f(y)| \le \mathcal{L}|x - y| < \mathcal{L} \cdot \delta = \frac{\varepsilon}{\delta} \cdot \delta = \varepsilon, \quad \forall x \in [a, b] \iff$$

$$\iff |f(x) - f(y)| < \varepsilon, \quad \forall x \in [a, b] \xRightarrow{\text{definition } \varepsilon-\delta}$$

$$\Rightarrow f \text{ is a continuous function} \Rightarrow \lim_{x \to x_0} = f(x_0), \ \forall x_0 \in [a, b] \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x^*)$$

But we proved that $\lim_{n\to\infty} f(x_n) = x^*$ $\Rightarrow f(x^*) = x^*$ (x^* is a fixed point)

Suppose that x^* is not an unique fixed point of f. $\Rightarrow$

$\Rightarrow \exists x' \in [a, b]$, $x' \neq x^*$, such that $f(x') = x'$.

We have: $|f(x^*) - f(x')| \le \mathcal{L}|x^* - x'| \iff$

$\iff |x^* - x'| \le \mathcal{L}|x^* - x'| \quad \Big| : |x^* - x'| \quad (x^* \neq x' \Rightarrow x^* - x' \neq 0 \Rightarrow |x^* - x'| \neq 0)$

$1 \le \mathcal{L}$

But we know that $\mathcal{L} \in (0, 1)$ $\Big\}$ contradiction $\Rightarrow$

$\Rightarrow f$ has an unique fixed point x^* $(f(x^*) = x^*)$ and $x^* = \lim_{n\to\infty} x_n$.

2. $\mathcal{L} \in (0, 1)$, $x_1, x_2 \in \mathbb{R}$, (x_n) given by $x_{n+2} = \mathcal{L} x_{n+1} + (1 - \mathcal{L}) x_n$, $\forall n \in \mathbb{N}$

? (x_n) convergent. Find its limit in terms of $\mathcal{L}$, x_1, x_2.

We have:

$$x_{n+2} = \mathcal{L} x_{n+1} + (1 - \mathcal{L}) x_n, \ \forall n \in \mathbb{N} \iff x_n = \mathcal{L} x_{n-1} + (1 - \mathcal{L}) x_{n-2}, \ \forall n \ge 2 \Rightarrow$$

$$\Rightarrow x_n - x_{n-1} = (\mathcal{L} - 1)(x_{n-1} - x_{n-2}) = (\mathcal{L} - 1)^2 (x_{n-2} - x_{n-3}) = \dots = (\mathcal{L} - 1)^{n-2}(x_2 - x_1), \ \forall n \ge 2$$

$$x_{n-1} - x_{n-2} = (\mathcal{L} - 1)^{n-3}(x_2 - x_1)$$

$$\vdots$$

$$x_2 - x_1 = (\mathcal{L} - 1)^0 (x_2 - x_1)$$

Summing these equalities we obtain:

$$x_n - x_1 = (x_2 - x_1) \cdot \left((\mathcal{L} - 1)^{n-1} + (\mathcal{L} - 1)^{n-2} + \dots + (\mathcal{L} - 1) + 1\right)$$

$$= (x_2 - x_1) \cdot \frac{1 - (\mathcal{L} - 1)^n}{1 - (\mathcal{L} - 1)} = (x_2 - x_1) \cdot \frac{1 - (\mathcal{L} - 1)^n}{2 - \mathcal{L}} \iff$$

$$\iff x_n = x_1 + (x_2 - x_1) \cdot \frac{1 - (\mathcal{L} - 1)^n}{2 - \mathcal{L}}, \quad \forall n \ge 2$$

We have:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(x_1 + (x_2 - x_1)\cdot\frac{1-(\mathcal{L}-1)^n}{2-\mathcal{L}}\right) = x_1 + \frac{x_2 - x_1}{2-\mathcal{L}}\cdot\lim_{n\to\infty}\left(1-(\mathcal{L}-1)^n\right) =$$

$$= x_1 + \frac{x_2-x_1}{2-\mathcal{L}}\cdot\left(1 - \lim_{n\to\infty}(\mathcal{L}-1)^n\right) = x_1 + \frac{x_2-x_1}{2-\mathcal{L}}\cdot(1-0) =$$

$$= x_1 + \frac{x_2-x_1}{2-\mathcal{L}} = \frac{x_1(2-\mathcal{L}) + x_2 - x_1}{2-\mathcal{L}} = \frac{(1-\mathcal{L})x_1 + x_2}{2-\mathcal{L}} \iff$$

$$\iff \lim_{n\to\infty} x_n = \frac{(1-\mathcal{L})x_1 + x_2}{2-\mathcal{L}}$$

We know that $x_1, x_2, \mathcal{L}, (1-\mathcal{L}), (2-\mathcal{L}) \in \mathbb{R}$ and $2-\mathcal{L} \neq 0$ $(\mathcal{L}\in(0,1))$ $\Rightarrow$

$$\Rightarrow \lim_{n\to\infty} x_n = \frac{(1-\mathcal{L})x_1 + x_2}{2-\mathcal{L}} \in \mathbb{R} \Rightarrow$$

$\Rightarrow (x_n)$ is convergent

3. Example of a sequence having the set of its limit points equal to $[0,1]$ and justification

We consider the sequence (x_n) given by $x_n = \sin^2 n$, $\forall n\in\mathbb{N}$, and the set $A = \{x_n \mid n\in\mathbb{N}\} = \{\sin^2 n \mid n\in\mathbb{N}\}$.

$A \subseteq [0,1]$, because $\sin^2 n \in [0,1]$, $\forall n\in\mathbb{N}$

Let $b\in[0,1]$ arbitrary. Then $\exists\, a\in\mathbb{R}$ such that $\sin^2 a = b$.

According to Kronecker's Theorem, the set $B = \{-2\pi\cdot n + m \mid n, m\in\mathbb{R}\}$ is dense in $\mathbb{R}$. $\Rightarrow \exists\,(b_n)$, $b_n\in B$, $\forall n\in\mathbb{N}$, given by $b_n = -2\pi\cdot u_n + v_n$, $u_n, v_n\in\mathbb{N}, \forall n\in\mathbb{N}$, such that $\lim_{n\to\infty} b_n = a$.

We know that $(\sin^2)$ is a continuous function. So, we have:

$$b = \sin^2 a = \sin^2\left(\lim_{n\to\infty} b_n\right) = \lim_{n\to\infty}\left(\sin^2 b_n\right) = \lim_{n\to\infty}\sin^2(-2\pi\cdot u_n + v_n) = \lim_{n\to\infty}\left(\sin^2 v_n\right)$$

We know that $v_n\in\mathbb{N}$, $\forall n\in\mathbb{N} \Rightarrow \sin^2 v_n \in A$, $\forall n\in\mathbb{N}$ $\Rightarrow$

$\Rightarrow b$, an arbitrary number from $[0,1]$, is a limit of a sequence of elements from $A \Rightarrow A$ is dense in $[0,1]$ $\iff$

$\iff \{x_n \mid n\in\mathbb{N}\}$ is dense in $[0,1]$ $\Rightarrow$

$\Rightarrow$ The sequence (x_n) given by $x_n = \sin^2 n$, $\forall n\in\mathbb{N}$, has the set of limit points equal to $[0,1]$.