Extra homework 3 - Timiș Diana (917)

1. ? $\exists$ a bijective map $\tilde{\pi}: \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} \frac{\tilde{\pi}(n)}{n^2} < \infty$

Let (S_n) be a sequence of positive numbers given by $S_n = \sum_{k=1}^{n} \frac{\tilde{\pi}(k)}{k^2}$, where $\tilde{\pi}: \mathbb{N} \to \mathbb{N}$ is a bijective map.

Suppose that $\sum_{n=1}^{\infty} \frac{\tilde{\pi}(n)}{n^2}$ is convergent $\Rightarrow$ (S_n) is convergent $\Leftrightarrow$
$\Leftrightarrow$ (S_n) is Cauchy $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|S_m - S_n| < \varepsilon$, $\forall m, n \geq N$

If we consider $m = 2n$, where $n \geq N$, we will have that $|S_{2n} - S_n| < \varepsilon$

$$|S_{2n} - S_n| = \left|\sum_{k=1}^{2n} \frac{\tilde{\pi}(k)}{k^2} - \sum_{k=1}^{n} \frac{\tilde{\pi}(k)}{k^2}\right| = \left|\sum_{k=n+1}^{2n} \frac{\tilde{\pi}(k)}{k^2}\right| = \sum_{k=n+1}^{2n} \frac{\tilde{\pi}(k)}{k^2} \geq \sum_{k=n+1}^{2n} \frac{\tilde{\pi}(k)}{(2n)^2} = \frac{1}{(2n)^2} \cdot \sum_{k=n+1}^{2n} \tilde{\pi}(k)$$

$$\Rightarrow \frac{1}{(2n)^2} \cdot \sum_{k=n+1}^{2n} \tilde{\pi}(k) < \varepsilon \qquad (1)$$

Because $\tilde{\pi}$ is a bijective map $\Rightarrow$

$$\Rightarrow \sum_{k=n+1}^{2n} \tilde{\pi}(k) \geq \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$ (the smallest sum of n different natural numbers) $\Rightarrow$

$$\Rightarrow \frac{1}{(2n)^2} \cdot \sum_{k=n+1}^{2n} \tilde{\pi}(k) \geq \frac{1}{(2n)^2} \cdot \frac{n(n+1)}{2} = \frac{n^2+n}{8n^2} = \frac{n^2}{8n^2} + \frac{n}{8n^2} = \frac{1}{8} + \frac{1}{8n} > \frac{1}{8}$$

$$(1), (2) \Rightarrow \frac{1}{8} < \frac{1}{(2n)^2} \cdot \sum_{k=n+1}^{2n} \tilde{\pi}(k) < \varepsilon \Leftrightarrow \frac{1}{8} < \varepsilon \text{ contradiction} \Rightarrow$$

$\Rightarrow (S_n)$ is not Cauchy $\Rightarrow$ (S_n) is not convergent $\Rightarrow$
$\Rightarrow \sum_{n=1}^{\infty} \frac{\tilde{\pi}(n)}{n^2}$ divergent $\Rightarrow$ $\nexists$ a bijective map $\tilde{\pi}: \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} \frac{\tilde{\pi}(n)}{n} < \infty$

2. (x_n) decreasing, $x_n > 0$, $\forall n \in \mathbb{N}$. If $\sum_{n \geq 1} x_n$ converges, then $\lim_{n \to \infty} n x_n = 0$?

Suppose (S_n) be a sequence of positive numbers given by $S_n = \sum_{k=1}^{n} x_k$.

$\sum_{n\geq 1} x_n$ converges $\Leftrightarrow$ (S_n) is convergent $\Leftrightarrow$
$\Leftrightarrow$ (S_n) is Cauchy $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|S_n - S_m| < \varepsilon$, $\forall n, m \geq N$

If we consider $m = N$ and $n \geq N$, we will have:

$$|S_n - S_N| = \left|\sum_{k=1}^{n} x_k - \sum_{k=1}^{N} x_k\right| = \left|\sum_{k=N+1}^{n} x_k\right| = \sum_{k=N+1}^{n} x_k = x_{N+1} + x_{N+2} + \dots + x_n < \varepsilon \quad (1)$$

We know that (x_n) is decreasing $\Rightarrow$ $\forall i, j \in \mathbb{N}$, $i \leq j$, $x_i \geq x_j$ $\Rightarrow$
$\Rightarrow$ We have:

$$x_{N+1} + x_{N+2} + \dots + x_n \geq \underbrace{x_n + x_n + \dots + x_n}_{(n-N) \text{ times}} = (n-N)\, x_n = (n-N) \cdot |x_n| =$$
$$= |(n-N) x_n| \qquad (2)$$

$(1), (2) \Longrightarrow |(n-N)x_n| < \varepsilon \Longleftrightarrow |(n-N)x_n - 0| < \varepsilon \Longrightarrow$

$\Longrightarrow \lim_{n\to\infty}(n-N)x_n = 0 \Longleftrightarrow \lim_{n\to\infty} n x_n - \lim_{n\to\infty} N x_n = 0 \Longleftrightarrow \lim_{n\to\infty} n x_n = \lim_{n\to\infty} N x_n$ (3)

We know that $\sum_{n\geq 1} x_n$ converges $\Longrightarrow \lim_{n\to\infty} x_n = 0 \Longrightarrow$

$\Longrightarrow \lim_{n\to\infty} N x_n = N \cdot \lim_{n\to\infty} x_n = N \cdot 0 = 0$ (4)

$(3), (4) \Longrightarrow \lim_{n\to\infty} n x_n = 0$

3. The Cantor Set (C)

a) The length of the removed interval:

$$S = \frac{1}{3} + 2\cdot\frac{1}{9} + 4\cdot\frac{1}{27} + \dots = \frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \dots = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} =$$

$$= \frac{1}{2}\cdot\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{2}\cdot\left(\underbrace{\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n}_{\text{geometric progression}} - 1\right) = \frac{1}{2}\cdot\left(\frac{1}{1-\frac{2}{3}} - 1\right) = \frac{1}{2}\left(\frac{1}{\frac{1}{3}} - 1\right) = \frac{1}{2}(3-1) = \frac{1}{2}\cdot 2 = 1$$

b) $x \in C \iff x = \sum_{n=1}^{\infty} a_n 3^{-n}$, with $a_n \in \{0, 2\}$?

The n-th iteration removes the open segments consisting of all numbers with a 1 in the n-th place of the ternary expansion. Thus, the numbers remaining after n iterations will have only 0's and 2's in the first n places. So the numbers remaining at the end are precisely those with only 0's and 2's in all places. (Some numbers don't have an unique representation, namely those that have a representation that terminates. For those numbers, we choose the infinitely repeating representation instead; if it consists of all 0's and 2's, it is in the Cantor Set. This works because we remove an open interval each time and the numbers with terminating representations are the endpoints of one of the removed intervals.)

c) By constructing a surjective function from C to [0, 1], prove that C is uncountable.

The Cantor-Lebesgue function is defined on the Cantor Set by writing x's ternary expansion in 0's and 2's, switching 2's to 1's and reinterpreting as a binary expansion. It is continuous and surjective onto [0, 1].

So the Cantor Set and [0, 1] have the same cardinality.

But we know that [0, 1] is uncountable.

$\Rightarrow$ The Cantor Set (C) is uncountable.