Homework 7 - Timiș Diana (917)

1. a) $\lim\limits_{n\to\infty} \frac{\sqrt[n]{e} + 2\sqrt[n]{e^2} + \ldots + n\sqrt[n]{e^n}}{n^2} = \lim\limits_{n\to\infty} \frac{\sum_{k=1}^{n} k \cdot e^{\frac{k}{n}}}{n^2} = \lim\limits_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n} \cdot e^{\frac{k}{n}}$ Riemann Sum

$$= \int_0^1 x e^x \, dx = x e^x \Big|_0^1 - \int_0^1 e^x \, dx = x e^x \Big|_0^1 - e^x \Big|_0^1 = (x-1) e^x \Big|_0^1 = 0 - (-1) = 1$$

b) $\lim\limits_{n\to\infty} \sqrt[n]{\sin\frac{\pi}{2n} \sin\frac{2\pi}{2n} \cdot \ldots \cdot \sin\frac{(n-1)\pi}{2n}} = \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln\left(\sin\frac{\pi}{2n} \cdot \ldots \cdot \sin\frac{(n-1)\pi}{2n}\right)} =$

$$= e^{\lim\limits_{n\to\infty} \frac{1}{n} \ln\left(\prod_{k=1}^{n-1} \sin\frac{k\pi}{2n}\right)} = e^{\lim\limits_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \ln\left(\sin\frac{k\pi}{2n}\right)} \overset{RS}{=} e^{\int_0^1 \ln\left(\sin\frac{x\pi}{2}\right) dx} \overset{(2)}{=} e^{-\ln 2} = 2^{-1} = \frac{1}{2}$$

$$\int_0^1 \ln\left(\sin\frac{x\pi}{2}\right) dx \overset{u = \frac{x\pi}{2}}{=} \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \ln(\sin u) \, du \overset{(1)}{=} \frac{2}{\pi} \cdot \left(-\frac{\pi}{2} \ln 2\right) = -\ln 2 \quad (2)$$

$$\int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx \overset{t = x - \frac{\pi}{2}}{=} \int_0^{\frac{\pi}{2}} \ln(\cos t) \, dt \ ; \ \int_{\frac{\pi}{2}}^{\pi} \ln(\sin x) \, dx \overset{t = x - \frac{\pi}{2}}{=} \int_0^{\frac{\pi}{2}} \ln(\cos t) \, dt$$

$$2\int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx + \int_0^{\frac{\pi}{2}} \ln(\cos x) \, dx = \int_0^{\frac{\pi}{2}} \ln(\sin 2x) \, dx - \frac{\pi}{2} \ln 2 =$$

$$= \frac{1}{2} \int_0^{\pi} \ln(\sin x) \, dx - \frac{\pi}{2} \ln 2 =$$

$$= \frac{1}{2} \left(\int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx + \int_{\frac{\pi}{2}}^{\pi} \ln(\sin x) \, dx \right) - \frac{\pi}{2} \ln 2 =$$

$$= \frac{1}{2} \cdot 2 \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx - \frac{\pi}{2} \ln 2 = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx - \frac{\pi}{2} \ln 2 \Rightarrow$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx = -\frac{\pi}{2} \ln 2 \quad (1)$$

2. $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx \ , \ \alpha > 0$

$$\Gamma(\alpha+1) = \int_0^{\infty} x^{\alpha} e^{-x} \, dx = \underbrace{-x^{\alpha} e^{-x} \Big|_0^{\infty}}_{=0} - \int_0^{\infty} \alpha x^{\alpha-1} (-e^{-x}) \, dx = 0 + \alpha \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx =$$

$$= \alpha \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx = \alpha \Gamma(\alpha) \ , \ \forall \alpha > 0$$

For $n \in \mathbb{N}^*$: $\Gamma(n) = (n-1)\,\Gamma(n-1)$

$\Gamma(n-1) = (n-2)\,\Gamma(n-2)$

$\ldots$

$\Gamma(2) = 2\,\Gamma(1)$

$$\Gamma(1) = \int_0^{\infty} x^{1-1} e^{-x} \, dx = \int_0^{\infty} e^{-x} \, dx = \lim_{t\to\infty} \int_0^{t} e^{-x} \, dx =$$

$$= \lim_{t\to\infty} -e^{-x} \Big|_0^{t} = -\lim_{t\to\infty} \left(e^{-t} - e^0\right) = -0 + e^0 = e^0 = 1$$

$\Rightarrow$

$$\Rightarrow \Gamma(n) = (n-1)(n-2)(n-3) \cdot \ldots \cdot 3 \cdot 2 \cdot \Gamma(1) = (n-1)! \cdot \Gamma(1) = (n-1)! \cdot 1 = (n-1)! \Rightarrow$$

$$\Rightarrow \Gamma(n) = (n-1)! \ , \ \forall n \in \mathbb{N}^*$$

```
import numpy as np

1 usage
def f(x):
    return np.exp(-x**2)

1 usage
def trapezoidal_rule(func, a, b, n):
    h = (b - a) / n
    sum = func(a) + func(b)

    for i in range(1, n):
        sum += 2 * func(a + i * h)

    return (h / 2) * sum


for a in range(1, 21):
    result = trapezoidal_rule(f, -a, a,  n: 1000)
    print(f"For a={a}, integral value is approximately {result} and we know that sqrt(pi) is approximately {np.sqrt(np.pi)}")
```

```
For a=1, integral value is approximately 1.4936477751188677 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=2, integral value is approximately 1.764162586158551 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=3, integral value is approximately 1.7724146920763713 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=4, integral value is approximately 1.7724538235695357 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=5, integral value is approximately 1.77245385090279 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=6, integral value is approximately 1.7724538509055126 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=7, integral value is approximately 1.7724538509055157 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=8, integral value is approximately 1.772453850905515 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=9, integral value is approximately 1.772453850905515 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=10, integral value is approximately 1.7724538509055165 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=11, integral value is approximately 1.7724538509055152 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=12, integral value is approximately 1.7724538509055126 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=13, integral value is approximately 1.7724538509055154 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=14, integral value is approximately 1.7724538509055165 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=15, integral value is approximately 1.7724538509055152 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=16, integral value is approximately 1.7724538509055157 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=17, integral value is approximately 1.7724538509055154 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=18, integral value is approximately 1.772453850905516 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=19, integral value is approximately 1.7724538509055165 and we know that sqrt(pi) is approximately 1.7724538509055159
For a=20, integral value is approximately 1.7724538509055152 and we know that sqrt(pi) is approximately 1.7724538509055159
```