Extra homework 1 - Timiș Diana

1. • Any union of open sets is open.

Let $A_1, A_2, A_3, \dots, A_n$, $n \in \mathbb{N}$, be open sets.

We know that $\bigcup_{i=1}^{n} A_i \neq \emptyset$, so let $x \in \bigcup_{i=1}^{n} A_i \Longrightarrow$

$$\left.\begin{array}{l} \Rightarrow \exists\, k \in \{1, 2, 3, \dots, n\} \text{ such that } x \in A_k \\ A_k \text{ is an open set } \Longleftrightarrow A_k = \operatorname{int}(A_k) \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{l} \Rightarrow \exists\, \varepsilon > 0 \text{ such that } (x-\varepsilon, x+\varepsilon) \subseteq A_k \\ A_k \subseteq \bigcup_{i=1}^{n} A_i \end{array}\right\} \Longrightarrow$$

$\Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq \bigcup_{i=1}^{n} A_i \Longrightarrow \bigcup_{i=1}^{n} A_i$ is an open set

• Any intersection of closed sets is closed.

Let $A_1, A_2, A_3, \dots, A_n$, $n \in \mathbb{N}$, be open sets.

$$\left.\begin{array}{l} \text{We previously proved that } \bigcup_{i=1}^{n} A_i \text{ is an open set} \Longrightarrow \\ \Rightarrow \left(\bigcup_{i=1}^{n} A_i\right)^c \text{ is closed} \\ \text{But } \left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} (A_i)^c \text{ and } (A_i)^c \text{ is closed (because it is the} \\ \text{complement of an open set)} \end{array}\right\} \Longrightarrow$$

$\Rightarrow$ any intersection of closed sets is closed

• Any finite intersection of open sets is open.

Let $A_1, A_2, A_3, \dots, A_n$, $n \in \mathbb{N}$, be open sets.

$$\left.\begin{array}{l} \text{Let } x \in \bigcap_{i=1}^{n} A_i \Longrightarrow x \in A_k,\ \forall\, k \in \{1, 2, 3, \dots, n\} \\ A_k \text{ is an open set } \Longleftrightarrow A_k = \operatorname{int}(A_k) \end{array}\right\} \Longrightarrow$$

$\Rightarrow x \in \operatorname{int}(A_k)$, $\forall\, k \in \{1, 2, \dots, n\} \Longrightarrow$

$$\left.\begin{array}{l} \Rightarrow x \in \operatorname{int}\left(\bigcap_{i=1}^{n} A_i\right) \\ \text{But } x \text{ is an arbitrary element of } \bigcap_{i=1}^{n} A_i \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{l} \Rightarrow \bigcap_{i=1}^{n} A_i \subseteq \operatorname{int}\left(\bigcap_{i=1}^{n} A_i\right) \\ \text{But } \operatorname{int}\left(\bigcap_{i=1}^{n} A_i\right) \subseteq \bigcap_{i=1}^{n} A_i \text{ (by definition)} \end{array}\right\} \Longrightarrow$$

$\Rightarrow \bigcap_{i=1}^{n} A_i = \operatorname{int}\left(\bigcap_{i=1}^{n} A_i\right) \Longrightarrow \bigcap_{i=1}^{n} A_i$ is an open set

• Any finite union of closed sets is closed.

Let $A_1, A_2, A_3, \dots, A_n$, $n \in \mathbb{N}$, be open sets.

We previously proved that $\bigcap_{i=1}^{n} A_i$ is an open set. $\Longrightarrow$

$\Longrightarrow \left(\bigcap_{i=1}^{n} A_i\right)^c$ is closed

But $\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} (A_i)^c$ and $(A_i)^c$ is closed (because it is the complement of an open set) $\Longrightarrow$

$\Longrightarrow$ any finite union of closed sets is closed.

• Example of an intersection of open sets that is not open: $\bigcap_{i=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$

• Example of an union of closed sets that is not closed: $\bigcup_{i=1}^{\infty}\left[\frac{1}{n}, 1-\frac{1}{n}\right] = (0, 1)$

2. $\mathcal{L} \in \mathbb{R} \setminus \mathbb{Q}$, $S_{\mathcal{L}} := \{\{n\mathcal{L}\} \mid n \in \mathbb{N}\}$

a) $S_{\mathcal{L}}$ is dense in $[0, 1]$

b) $\{\{n\mathcal{L}\} + m \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$

a) We can divide $[0, 1]$ into k intervals of length $\frac{1}{k}$:

$$[0, 1] = \left[0, \frac{1}{k}\right] \cup \left[\frac{1}{k}, \frac{2}{k}\right] \cup \left[\frac{2}{k}, \frac{3}{k}\right] \cup \dots \cup \left[1-\frac{1}{k}, 1\right].$$

Using Dirichlet's principle, we know that there are two natural numbers a, b, $a < b$, such that $\{a\mathcal{L}\}, \{b\mathcal{L}\}$ are in the same interval.

$a < b$ and $a, b \in \mathbb{N} \Longrightarrow b - a > 0$ and (in this case) $(b-a) \in \mathbb{N}$

I. $\{a\mathcal{L}\} < \{b\mathcal{L}\} \Longrightarrow \ <\{(b-a)\mathcal{L}\} < \frac{1}{k} \Longrightarrow \{(b-a)\mathcal{L}\} \in \left[0, \frac{1}{k}\right]$

II. $\{a\mathcal{L}\} > \{b\mathcal{L}\} \Longrightarrow 1 > \{(b-a)\mathcal{L}\} > 1 - \frac{1}{k} \Longrightarrow \{(b-a)\mathcal{L}\} \in \left[1-\frac{1}{k}, 1\right]$ $\Longrightarrow$

$\Longrightarrow \{(b-a)\mathcal{L}\} \in \left[0, \frac{1}{k}\right]$ or $\{(b-a)\mathcal{L}\} \in \left[1-\frac{1}{k}, 1\right]$ (and $\{(b-a)\mathcal{L}\} \neq 0$, because $(b-a)\mathcal{L} \notin \mathbb{Z}$, $b - a \neq 0$ and $\mathcal{L} \in \mathbb{R} \setminus \mathbb{Q}$)

If we take all multiples of $(b-a)\mathcal{L}$, $p(b-a)\mathcal{L}$ with $p \in \mathbb{N}$, then we obtain that in each of the k intervals must be at least one of the values $\{p(b-a)\mathcal{L}\}$ (because we either go "upwards", from $\left[0, \frac{1}{k}\right]$, or "downwards", from $\left[1-\frac{1}{k}, 1\right]$, but we can never skip an interval) $\Longrightarrow$

$\Longrightarrow S_{\mathcal{L}}$ is dense in $[0, 1]$

b) $\{\{n\mathcal{L}\} + m \mid n, m \in \mathbb{Z}\} = \{n \cdot \mathcal{L} - [n\mathcal{L}] + m \mid n, m \in \mathbb{Z}\} =$

$= \{n \cdot \mathcal{L} + (m - [n\mathcal{L}]) \mid n, m \in \mathbb{Z}\}$

$$\left.\begin{array}{l} n \in \mathbb{Z} \\ m \in \mathbb{Z} \text{ and } [n\mathcal{L}] \in \mathbb{Z} \text{ (using the definition of the integer part)} \Rightarrow \\ \quad \Rightarrow (m - [n\mathcal{L}]) \in \mathbb{Z} \\ \mathcal{L} \in \mathbb{R} \setminus \mathbb{Q} \end{array}\right\} \Rightarrow$$

$\xRightarrow[\text{Theorem}]{\text{Kronecker's}} \{n \cdot \mathcal{L} + (m - [n\mathcal{L}]) \mid n, m \in \mathbb{Z}\} = \{\{n\mathcal{L}\} + m \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$