Extra homework 4 – Timiș Diana (917)

2\. (F_n) is the Fibonacci sequence, with $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$; $\sum_{n=0}^{\infty} F_n x^n$

$F_n = F_{n-1} + F_{n-2}, \ \forall n \geq 2 \iff F_{n+2} = F_{n+1} + F_n, \ \forall n \geq 0 \implies$

$\implies F_{n+2} x^{n+2} = F_{n+1} x^{n+2} + F_n x^{n+2} \quad \forall n \geq 0, \ \forall x \in \mathbb{R} \implies$

$\implies \sum_{n=0}^{\infty} F_{n+2} x^{n+2} = \sum_{n=0}^{\infty} F_{n+1} x^{n+2} + \sum_{n=0}^{\infty} F_n x^{n+2}$

$$\sum_{n=2}^{\infty} F_n x^n = x \cdot \sum_{n=1}^{\infty} F_n x^n + x^2 \cdot \sum_{n=0}^{\infty} F_n x^n$$

$$\sum_{n=0}^{\infty} F_n x^n - F_0 \cdot x^0 - F_1 \cdot x = x\left(\sum_{n=0}^{\infty} F_n x^n - F_0 \cdot x^0\right) + x^2 \cdot \sum_{n=0}^{\infty} F_n x^n$$

$$\sum_{n=0}^{\infty} F_n x^n - 1 - x = x \cdot \sum_{n=0}^{\infty} F_n x^n - x + x^2 \cdot \sum_{n=0}^{\infty} F_n x^n$$

$$(1 - x - x^2) \cdot \sum_{n=0}^{\infty} F_n x^n = 1$$

We find the solutions of the equation: $1 - x - x^2 = 0$

$$\Delta = (-1)^2 - 4 \cdot (-1) \cdot 1 = 1 + 4 = 5$$

$$x_{1,2} = \frac{1 \pm \sqrt{5}}{-2} \implies \begin{cases} x_1 = -\frac{1 - \sqrt{5}}{2} = \frac{\sqrt{5} - 1}{2} \\ x_2 = -\frac{1 + \sqrt{5}}{2} \end{cases}$$

I. For $x \in \mathbb{R} \setminus \left\{-\frac{1+\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}\right\}$ we have:

$$\sum_{n=0}^{\infty} F_n x^n = \frac{1}{1 - x - x^2}$$, which converges

II. For $x \in \left\{-\frac{1+\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}\right\}$ we have that $0 \cdot \sum_{n=0}^{\infty} F_n x^n = 1$, which is impossible in $\mathbb{R}$. $\implies$ The series $\sum_{n=0}^{\infty} F_n x^n$ diverges.

3. C_n is the number of full binary trees with $n+1$ leaves (the Catalan number)

a) a recurrence relation for C_n

b) $f(x) = \sum_{n=0}^{\infty} C_n x^n$; prove that $C_n = \frac{1}{n+1}\binom{2n}{n}$

a) Recurrence relation:

$$C_0 = 1, \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \ \forall n \geq 0 \Leftrightarrow C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}, \ \forall n \geq 1$$

b)
$$f(x) = \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n =$$
$$= C_0 \cdot C_0 \cdot x^0 + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n =$$
$$= 1 + x \cdot \sum_{n=1}^{\infty} \sum_{k=0}^{n} C_k C_{n-k} x^n = 1 + x \cdot \left(\sum_{n=0}^{\infty} C_n x^n\right)^2 = 1 + x \cdot (f(x))^2 \Longleftrightarrow$$
$$\Longleftrightarrow x \cdot (f(x))^2 - f(x) + 1 = 0 \Longrightarrow$$
$$\Longrightarrow f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Since $\lim_{x \to 0} f(x) = \lim_{x \to 0} \sum_{n=0}^{\infty} C_n x^n = C_0 = 1$ $\Longrightarrow$

$\Longrightarrow$ We must choose the negative square root, so: $c(x) = \frac{1-\sqrt{1-4x}}{2x}$

We have:

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n = 1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2} \cdot \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \cdot \ldots \cdot \left(-\frac{2n-3}{2}\right)}{n!} \cdot (-4x)^n =$$
$$= 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{(2n-3)!!}{2^n \cdot n!} \cdot (-4x)^n = 1 - \sum_{n=1}^{\infty} \frac{2^n (2n-3)!!}{n!} \cdot x^n =$$
$$= 1 - 2 \sum_{n=1}^{\infty} \frac{2^{n-1} (n-1)! \cdot \prod_{k=1}^{n-1} (2k-1)}{n \cdot ((n-1)!)^2} \cdot x^n = 1 - 2 \cdot \sum_{n=1}^{\infty} \frac{(2n-2)!}{n \cdot ((n-1)!)^2} \cdot x^n =$$
$$= 1 - 2 \sum_{n=1}^{\infty} \frac{1}{n} \binom{2(n-1)}{n-1} x^n$$

So, we obtain:

$$f(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{1 - 1 + 2\sum_{n=1}^{\infty} \frac{1}{n}\binom{2(n-1)}{n-1} x^n}{2x} = \sum_{n=1}^{\infty} \frac{1}{n} \binom{2(n-1)}{n-1} x^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n$$

But we know that $f(x) = \sum_{n=0}^{\infty} C_n x^n$ $\Longrightarrow$

$$\Longrightarrow \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n \Longrightarrow C_n = \frac{1}{n+1} \binom{2n}{n}, \ \forall n \geq 0$$