Homework 1 - Timiș Diana

1. Let $a, b \in \mathbb{R}$ with $a > 0$. If $S \subset \mathbb{R}$ is nonempty and bounded above, prove that $\sup\limits_{x \in S}(ax+b) = a\sup(S) + b$.

$S \subset \mathbb{R}$ is nonempty and bounded above $\Rightarrow$

$\Rightarrow ub(S) \neq \emptyset \Rightarrow \exists \sup(S) = L \in \mathbb{R} \Rightarrow$

$\Rightarrow x \leq L,\ \forall x \in S \quad \Big| \cdot a \quad (a \in \mathbb{R},\ a > 0)$

$ax \leq aL,\ \forall x \in S \quad \Big| + b$

$ax + b \leq aL + b,\ \forall x \in S \Rightarrow$

$\Rightarrow aL + b \in \underset{x \in S}{ub}(ax+b) \iff a\sup(S) + b \in \underset{x \in S}{ub}(ax+b)$

We know that $\sup\limits_{x \in S}(ax+b)$ is the least upper bound of $\{ax+b \mid x \in S\}$ $\Bigg\} \Rightarrow$

$\Rightarrow \sup\limits_{x \in S}(ax+b) \leq a\sup(S) + b \quad (1)$

For $\forall x \in S$, we have:

$$x = \frac{ax+b-b}{a} = \frac{1}{a}\cdot(ax+b) - \frac{b}{a} \leq \frac{1}{a}\cdot\sup_{x \in S}(ax+b) - \frac{b}{a} = \frac{\sup\limits_{x \in S}(ax+b) - b}{a} \Rightarrow$$

$\Rightarrow \dfrac{\sup\limits_{x \in S}(ax+b) - b}{a} \in ub(S)$

We know that $\sup(S)$ is the least upper bound of S $\Bigg\} \Rightarrow$

$\Rightarrow \sup(S) \leq \dfrac{\sup\limits_{x \in S}(ax+b) - b}{a} \quad \Big| \cdot a$

$a\sup(S) \leq \sup\limits_{x \in S}(ax+b) - b \quad \Big| + b$

$a\sup(S) + b \leq \sup\limits_{x \in S}(ax+b) \quad (2)$

$(1), (2) \Rightarrow \sup\limits_{x \in S}(ax+b) = a\sup(S) + b$

2. Let $a, b \in \mathbb{R}$. Prove that there exist neighborhoods $U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ such that $U \cap V = \emptyset$.

(1) Let $a < b$.

$U \in \mathcal{V}(a) \Rightarrow \exists \varepsilon_1 > 0$ such that $(a - \varepsilon_1, a + \varepsilon_1) \subseteq U$

$V \in \mathcal{V}(b) \Rightarrow \exists \varepsilon_2 > 0$ such that $(b - \varepsilon_2, b + \varepsilon_2) \subseteq V$

For $\varepsilon_1 = \varepsilon_2 = \frac{b-a}{2}$ (>0) and $U = (a - \varepsilon_1, a + \varepsilon_1)$ and $V = (b - \varepsilon_2, b + \varepsilon_2)$, we have:

$$\left.\begin{cases} U = \left(a - \frac{b-a}{2}, a + \frac{b-a}{2}\right) = \left(\frac{3a-b}{2}, \frac{a+b}{2}\right) \in \mathcal{V}(a) \\ V = \left(b - \frac{b-a}{2}, b + \frac{b-a}{2}\right) = \left(\frac{a+b}{2}, \frac{3b-a}{2}\right) \in \mathcal{V}(b) \end{cases}\right\} \Rightarrow$$

$$\Rightarrow U \cap V = \left(\frac{3a-b}{2}, \frac{a+b}{2}\right) \cap \left(\frac{a+b}{2}, \frac{3b-a}{2}\right) = \emptyset$$

(2) Let $a > b$.

$U \in \mathcal{V}(a) \Rightarrow \exists \varepsilon_1 > 0$ such that $(a - \varepsilon_1, a + \varepsilon_1) \subseteq U$

$V \in \mathcal{V}(b) \Rightarrow \exists \varepsilon_2 > 0$ such that $(b - \varepsilon_2, b + \varepsilon_2) \subseteq V$

For $\varepsilon_1 = \varepsilon_2 = \frac{a-b}{2}$ (>0) and $U = (a - \varepsilon_1, a + \varepsilon_1)$ and $V = (b - \varepsilon_2, b + \varepsilon_2)$, we have:

$$\left.\begin{cases} U = \left(a - \frac{a-b}{2}, a + \frac{a-b}{2}\right) = \left(\frac{a+b}{2}, \frac{3a-b}{2}\right) \in \mathcal{V}(a) \\ V = \left(b - \frac{a-b}{2}, b + \frac{a-b}{2}\right) = \left(\frac{3b-a}{2}, \frac{a+b}{2}\right) \in \mathcal{V}(b) \end{cases}\right\} \Rightarrow$$

$$\Rightarrow U \cap V = \left(\frac{a+b}{2}, \frac{3a-b}{2}\right) \cap \left(\frac{3b-a}{2}, \frac{a+b}{2}\right) = \emptyset$$

(1), (2) $\Rightarrow$ for $a, b \in \mathbb{R}$, $a \neq b$, $\exists\, U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ such that $U \cap V = \emptyset$

3. Let $A = (0, 1) \cap \mathbb{Q}$. Show rigorously that $\inf(A) = 0$, $\sup(A) = 1$, $\operatorname{int}(A) = \emptyset$ and $\operatorname{cl}(A) = [0, 1]$.

$$\left.\begin{array}{l} A = (0,1) \cap \mathbb{Q} \Rightarrow lb(A) = (-\infty, 0] \\ \text{We know that } \inf(A) \text{ is the greatest lower bound of } A \end{array}\right\} \Rightarrow$$

$\Rightarrow \inf(A) = 0$

$$\left.\begin{array}{l} A = (0,1) \cap \mathbb{Q} \Rightarrow ub(A) = [1, \infty) \\ \text{We know that } \sup(A) \text{ is the least upper bound of } A \end{array}\right\} \Rightarrow$$

$\Rightarrow \sup(A) = 1$

Let $x \in \mathbb{R}$. Suppose $x \in int(A) \Rightarrow \exists V \in \mathcal{V}(x)$ such that $V \subseteq A$
If $V \in \mathcal{V}(x) \Rightarrow \exists \varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq V$ $\Big\} \Rightarrow$

$\Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq A$
We know that for $\forall x, \varepsilon \in \mathbb{R}, \varepsilon > 0, (x-\varepsilon, x+\varepsilon) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$ $\Big\} \Rightarrow$

$\Rightarrow A \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset \Leftrightarrow (0,1) \cap \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$, contradiction $\Rightarrow$

$\Rightarrow int(A) = \emptyset$

Let $x \in (-\infty, 0) \Rightarrow (x-\varepsilon, 0) \in \mathcal{V}(x), \forall \varepsilon > 0$
$(x-\varepsilon, 0) \cap A = (x-\varepsilon, 0) \cap (0,1) \cap \mathbb{Q} = \emptyset, \forall \varepsilon > 0$
$x \in cl(A) \Leftrightarrow \forall V \in \mathcal{V}(x), V \cap A \neq \emptyset$ $\Bigg\} \Rightarrow$

$\Rightarrow cl(A) \cap (-\infty, 0) = \emptyset$ (1)

Let $x \in (1, \infty) \Rightarrow (1, x+\varepsilon) \in \mathcal{V}(x), \forall \varepsilon > 0$
$(1, x+\varepsilon) \cap A = (1, x+\varepsilon) \cap (0,1) \cap \mathbb{Q} = \emptyset, \forall \varepsilon > 0$
$x \in cl(A) \Leftrightarrow \forall V \in \mathcal{V}(x), V \cap A \neq \emptyset$ $\Bigg\} \Rightarrow$

$\Rightarrow cl(A) \cap (1, \infty) = \emptyset$ (2)

(1), (2) $\Rightarrow cl(A) \subseteq [0, 1]$ (3)

Let $x \in (0,1)$ and $V \in \mathcal{V}(x) \Rightarrow \exists \varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq V$

$x \in (0,1) \Rightarrow$ We can choose ε such that $0 < x-\varepsilon < x+\varepsilon < 1$
Suppose $A \cap V = \emptyset$
$(x-\varepsilon, x+\varepsilon) \subseteq V$ $\Big\} \Rightarrow A \cap (x-\varepsilon, x+\varepsilon) = \emptyset \Leftrightarrow (x-\varepsilon, x+\varepsilon) \cap (0,1) \cap \mathbb{Q} = \emptyset$ $\Bigg\} \Rightarrow$

$\Rightarrow (x-\varepsilon, x+\varepsilon) \cap \mathbb{Q} = \emptyset$, contradiction
V arbitrary $\Big\} \Rightarrow \forall V \in \mathcal{V}(x), A \cap V \neq \emptyset, \forall x \in (0,1) \Rightarrow$

$\Rightarrow x \in cl(A), \forall x \in (0,1) \Leftrightarrow (0,1) \subseteq cl(A)$ (4)

For $\forall U \in \mathcal{V}(0), \exists \varepsilon_0 > 0$ such that $(-\varepsilon_0, \varepsilon_0) \subseteq U \Rightarrow$

$\Rightarrow U \cap A \supseteq (-\varepsilon_0, \varepsilon_0) \cap A = (-\varepsilon_0, \varepsilon_0) \cap (0,1) \cap \mathbb{Q} = (0, \min\{\varepsilon_0, 1\}) \cap \mathbb{Q} \neq \emptyset \Rightarrow$

$\Rightarrow U \cap A \neq \emptyset, \forall U \in \mathcal{V}(0) \Rightarrow 0 \in cl(A)$ (5)

For $\forall U \in \mathcal{V}(1), \exists \varepsilon_1 > 0$ such that $(1-\varepsilon_1, 1+\varepsilon_1) \subseteq U \Rightarrow$

$\Rightarrow U \cap A \supseteq (1-\varepsilon_1, 1+\varepsilon_1) \cap A = (1-\varepsilon_1, 1+\varepsilon_1) \cap (0,1) \cap \mathbb{Q} = (\max\{1-\varepsilon_1, 0\}, 1) \cap \mathbb{Q} \neq \emptyset \Rightarrow$

$\Rightarrow U \cap A \neq \emptyset, \forall U \in \mathcal{V}(1) \Rightarrow 1 \in cl(A)$ (6)

(3), (4), (5), (6) $\Rightarrow cl(A) = [0, 1]$