Homework 2 – Timiș Diana (917)

1. Prove using the ε-definition that $\lim\limits_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$.

$$\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2} \overset{\varepsilon\text{-def}}{\Longleftrightarrow} \forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} : \left|\frac{n+1}{2n+3} - \frac{1}{2}\right| < \varepsilon,\ \forall n \geq N_\varepsilon$$

$$\left.\begin{array}{l} \left|\frac{n+1}{2n+3} - \frac{1}{2}\right| = \left|\frac{2(n+1)-(2n+3)}{2(2n+3)}\right| = \left|\frac{2n+2-2n-3}{4n+6}\right| = \left|\frac{-1}{4n+6}\right| = \frac{1}{4n+6} \\ \left|\frac{n+1}{2n+3} - \frac{1}{2}\right| < \varepsilon \end{array}\right\} \Longleftrightarrow$$

$$\Longleftrightarrow \frac{1}{4n+6} < \varepsilon \Longleftrightarrow \frac{1}{\varepsilon} < 4n+6 \Longleftrightarrow \frac{1}{\varepsilon} - 6 < 4n \Longleftrightarrow \frac{1}{4\varepsilon} - \frac{3}{2} < n$$

If we consider $N_\varepsilon = \left[\frac{1}{4\varepsilon} - \frac{3}{2}\right] + 1$, then for $\forall n \geq N_\varepsilon$ we know that $\frac{1}{4\varepsilon} - \frac{3}{2} < n$ is true $\Longrightarrow \lim\limits_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$

2. Find the limit for each of the following sequences:

a)
$$\left.\begin{array}{l} \lim\limits_{n\to\infty} (1+2+\dots+n)^{\frac{1}{n}} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{n(n+1)}{2}} \\ \text{We know that: } \lim\limits_{n\to\infty} \frac{\frac{(n+1)(n+2)}{2}}{\frac{n(n+1)}{2}} = \lim\limits_{n\to\infty} \frac{2(n+1)(n+2)}{2n(n+1)} = \lim\limits_{n\to\infty} \frac{n+2}{n} = \frac{1}{1} = 1 \in \overline{\mathbb{R}} \end{array}\right\} \overset{\text{Root Test}}{\Longrightarrow}$$

$$\Longrightarrow \lim_{n\to\infty} \sqrt[n]{\frac{n(n+1)}{2}} = 1 \Longleftrightarrow \lim_{n\to\infty} (1+2+\dots+n)^{\frac{1}{n}} = 1$$

b)
$$\lim_{n\to\infty} \left(\frac{\ln(n+1)}{\ln n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{\ln(n+1)}{\ln n} - 1\right)^n = \lim_{n\to\infty} \left(1 + \frac{\ln(n+1) - \ln n}{\ln n}\right)^n =$$

$$= \lim_{n\to\infty} \left(1 + \frac{\ln\left(\frac{n+1}{n}\right)}{\ln n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right)^n = \lim_{n\to\infty} \left[\left(1 + \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right)^{\frac{\ln n}{\ln\left(1+\frac{1}{n}\right)}}\right]^{n \cdot \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}} =$$

$$= e^{\lim\limits_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)^n}{\ln n}} = e^{\frac{\lim\limits_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n}{\lim\limits_{n\to\infty} \ln n}} = e^{\frac{\ln\left(\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)}{\lim\limits_{n\to\infty} \ln n}} = e^{\frac{e}{\infty}} = e^0 = 1$$

c)
$$\lim_{n\to\infty} \frac{n^n}{1+2^2+3^3+\dots+n^n} \overset{\text{Stolz-Cesàro}\left(\frac{\infty}{\infty}\right)}{=} \lim_{n\to\infty} \frac{(n+1)^{n+1} - n^n}{(1+2^2+\dots+n^n+(n+1)^{n+1}) - (1+2^2+\dots+n^n)} =$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1} - n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \left(1 - \frac{n^n}{(n+1)^{n+1}}\right) = 1 - \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} = 1 - \lim_{n\to\infty} \frac{\left(\frac{n}{n+1}\right)^n}{n+1} =$$

$$= 1 - \lim_{n\to\infty} \frac{\left(1 - \frac{1}{n+1}\right)^n}{n+1} = 1 - \frac{\lim\limits_{n\to\infty}\left[\left(1-\frac{1}{n+1}\right)^{-(n+1)}\right]^{\frac{n}{-(n+1)}}}{\lim\limits_{n\to\infty}(n+1)} = 1 - \frac{e^{\lim\limits_{n\to\infty} \frac{-n}{n+1}}}{\lim\limits_{n\to\infty}(n+1)} =$$

$$= 1 - \frac{e^{-\frac{1}{1}}}{\infty} = 1 - \frac{e^{-1}}{\infty} = 1 - 0 = 1$$

3. For $x_n = \frac{\sin(n)}{n}$ study if (x_n) is bounded, monotone and convergent. Find its limit.

We know that $-1 \le \sin n \le 1$, $\forall n \in \mathbb{N}$ $\mid \cdot \frac{1}{n}$

$\Rightarrow -\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$, $\forall n \in \mathbb{N} \Leftrightarrow -\frac{1}{n} \le x_n \le \frac{1}{n}$, $\forall n \in \mathbb{N}$

We know that $\lim\limits_{n\to\infty} \frac{1}{n} = \lim\limits_{n\to\infty}\left(-\frac{1}{n}\right) = 0$

$\xRightarrow{\text{Sandwich Theorem}}$

$\Rightarrow \lim\limits_{n\to\infty} x_n = 0 \Rightarrow (x_n)$ convergent $\Rightarrow (x_n)$ bounded

But (x_n) is not monotone.

Counter exemples:

$3 < \pi < 4$ and $\sin 3 > \sin \pi > \sin 4 \Leftrightarrow \sin 3 > 0 > \sin 4 \Rightarrow$

$\Rightarrow \frac{\sin 3}{3} > 0 > \frac{\sin 4}{4} \Leftrightarrow \frac{\sin 3}{3} > \frac{\sin 4}{4} \Leftrightarrow x_3 > x_4$ (1)

$6 < 2\pi < 7$ and $\sin 6 < \sin 2\pi < \sin 7 \Leftrightarrow \sin 6 < 0 < \sin 7 \Rightarrow$

$\Rightarrow \frac{\sin 6}{6} < 0 < \frac{\sin 7}{7} \Leftrightarrow \frac{\sin 6}{6} < \frac{\sin 7}{7} \Leftrightarrow x_6 < x_7$ (2)

(1), (2) confirm that (x_n) is not monotone

4. Prove that the sequence (x_n) given by $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$ is decreasing and bounded, hence convergent.

We know that the sequence (e_n), $e_n = \left(1+\frac{1}{n}\right)^n$ is strictly increasing and that it converges to e, so $\left(1+\frac{1}{n}\right)^n < e$, $\forall n \in \mathbb{N}$. (1)

We consider the sequence (a_n), $a_n = \left(1+\frac{1}{n}\right)^{n+1}$. We know that $\lim\limits_{n\to\infty} a_n = e$ and we want to prove that (a_n) is strictly decreasing.

We have:

$$\frac{a_n}{a_{n+1}} = \frac{\left(\frac{n+1}{n}\right)^{n+1}}{\left(\frac{n+2}{n+1}\right)^{n+2}} = \left(\frac{n^2+2n+1}{n^2+2n}\right)^{n+1} \cdot \frac{n+1}{n+2} = \left(1+\frac{1}{n^2+2n}\right)^{n+1} \cdot \frac{n+1}{n+2} \overset{\text{Bernoulli's ineq.}}{\ge}$$

$$\ge \left(1+\frac{n+1}{n^2+2n}\right)\cdot\frac{n+1}{n+2} = \frac{n^3+4n^2+4n+1}{n^3+4n^2+4n} > 1, \ \forall n \in \mathbb{N} \Rightarrow$$

$\Rightarrow a_n > a_{n+1}$, $\forall n \in \mathbb{N} \Rightarrow (a_n)$ strictly decreasing

$\lim\limits_{n\to\infty} a_n = e$

$\Rightarrow$

$\Rightarrow a_n > e$, $\forall n \in \mathbb{N}$ (2)

$$(1),(2) \Longrightarrow \left(\frac{n+1}{n}\right)^n < e < \left(\frac{n+1}{n}\right)^{n+1}, \ \forall n \in \mathbb{N} \xLeftrightarrow{\ln()}$$

$$\Longleftrightarrow n\left(\ln\left(\frac{n+1}{n}\right)\right) < 1 < (n+1)\left(\ln\left(\frac{n+1}{n}\right)\right)$$

$$n(\ln(n+1) - \ln n) < 1 < (n+1)(\ln(n+1) - \ln n) \Longrightarrow$$

$$\Longrightarrow \left\{\begin{array}{ll} \ln(n+1) - \ln n < \frac{1}{n}, & \forall n \in \mathbb{N} \\ \ln(n+1) - \ln n > \frac{1}{n+1}, & \forall n \in \mathbb{N} \end{array}\right\} \Longrightarrow$$

$$\Longrightarrow \frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}, \quad \forall n \in \mathbb{N} \qquad (3)$$

For the sequence (x_n) we have:

$$x_{n+1} - x_n = \left(1 + \frac{1}{2} + \dots + \frac{1}{n} + \frac{1}{n+1} - \ln(n+1)\right) - \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n\right) =$$

$$= \frac{1}{n+1} - \ln(n+1) + \ln n \overset{(3)}{<} 0, \quad \forall n \in \mathbb{N} \Longrightarrow$$

$\Longrightarrow (x_n)$ is strictly decreasing $\quad (4)$

$$(4) \Longrightarrow x_n < x_1 = 1 - \ln 1 = 1 - 0 = 1, \quad \forall n \in \mathbb{N} \qquad (5)$$

$$x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n \overset{(3)}{>} (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \dots + (\ln(n+1) - \ln n) - \ln n =$$

$$= \ln(n+1) - \ln n - \ln 1 = \ln(n+1) - \ln n - 0 = \ln(n+1) - \ln n > 0, \ \forall n \in \mathbb{N} \Longleftrightarrow$$

$$\Longleftrightarrow x_n > 0, \ \forall n \in \mathbb{N} \qquad (6)$$

$(5),(6) \Longrightarrow (x_n)$ is bounded $\quad (7)$

$(4),(7) \Longrightarrow (x_n)$ is convergent

(Its limit is typically denoted by γ and is known as the Euler-Mascheroni constant.)