Homework 3 – Timiș Diana (917)

1. Find the sum for each of the following series:

a) $\sum_{n\ge 2} \ln\left(1-\frac{1}{n^2}\right)$

$$S_n = \sum_{k=2}^{n} \ln\left(1-\frac{1}{k^2}\right) = \sum_{k=2}^{n} \ln\frac{k^2-1}{k^2} = \sum_{k=2}^{n} \ln\frac{(k-1)(k+1)}{k^2} = \sum_{k=2}^{n}\left(\ln\frac{k-1}{k} + \ln\frac{k+1}{k}\right) =$$

$$= \sum_{k=2}^{n}\left(\ln\frac{k-1}{k} - \ln\frac{k}{k+1}\right) = \sum_{k=2}^{n}\ln\frac{k-1}{k} - \sum_{k=2}^{n}\ln\frac{k}{k+1} =$$

$$= \sum_{k=2}^{n}\ln\frac{k-1}{k} - \sum_{k=3}^{n+1}\ln\frac{k-1}{k} = \ln\left(\frac{1}{2}\right) - \ln\frac{n}{n+1} = -\ln 2 - \ln\frac{n}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(-\ln 2 - \ln\frac{n}{n+1}\right) = -\ln 2 - \lim_{n\to\infty}\ln\frac{n}{n+1} = -\ln 2 - \ln\left(\lim_{n\to\infty}\frac{n}{n+1}\right) =$$

$$= -\ln 2 - \ln 1 = -\ln 2 - 0 = -\ln 2 \implies$$

$$\implies \sum_{n\ge 2}\ln\left(1-\frac{1}{n^2}\right) = -\ln 2$$

b) $$S = \sum_{n\ge 1}\frac{n+1}{3^n} = \sum_{n\ge 1}\frac{n+2-1}{3^n} = \sum_{n\ge 1}\frac{n+2}{3^n} - \sum_{n\ge 1}\frac{1}{3^n} = 3\cdot\sum_{n\ge 1}\frac{n+2}{3^{n+1}} - \overbrace{\sum_{n\ge 1}\frac{1}{3^n}}^{\text{geometric progression}} =$$

$$= 3\left(\sum_{n\ge 0}\frac{n+2}{3^{n+1}} - \frac{2}{3}\right) - \left(\frac{1}{1-\frac{1}{3}} - 1\right) = 3\left(\sum_{n\ge 1}\frac{n+1}{3^n} - \frac{2}{3}\right) - \left(\frac{1}{\frac{2}{3}} - 1\right) =$$

$$= 3\left(S - \frac{2}{3}\right) - \left(\frac{3}{2} - 1\right) = 3S - 2 - \frac{1}{2} = 3S - \frac{5}{2} \iff$$

$$\iff S = 3S - \frac{5}{2} \iff 2S = \frac{5}{2} \iff S = \frac{5}{4} \iff \sum_{n\ge 1}\frac{n+1}{3^n} = \frac{5}{4}$$

c) $$\sum_{n\ge 1}\frac{n}{n^4+n^2+1} = \sum_{n\ge 1}\frac{n}{(n^4+2n^2+1)-n^2} = \sum_{n\ge 1}\frac{n}{(n^2+1)^2-n^2} = \sum_{n\ge 1}\frac{n}{(n^2-n+1)(n^2+n+1)} =$$

$$= \frac{1}{2}\cdot\sum_{n\ge 1}\frac{(n^2+n+1)-(n^2-n+1)}{(n^2-n+1)(n^2+n+1)} = \frac{1}{2}\cdot\sum_{n\ge 1}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right) = \frac{1}{2}\cdot\left(\sum_{n\ge 1}\frac{1}{n^2-n+1} - \sum_{n\ge 1}\frac{1}{n^2+n+1}\right) =$$

$$= \frac{1}{2}\cdot\left(1 + \sum_{n\ge 2}\frac{1}{n^2-n+1} - \sum_{n\ge 1}\frac{1}{n^2+n+1}\right) = \frac{1}{2}\cdot\left(1 + \sum_{n\ge 1}\frac{1}{n^2+n+1} - \sum_{n\ge 1}\frac{1}{n^2+n+1}\right) = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

2. Study the convergence of the following series:

a) $\sum_{n\ge 1}\frac{x^n}{n^p}$, $x > 0$, $p\in\mathbb{N}$

Case I: $x < 1$

We apply the ratio test:

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\left(\frac{x^{n+1}}{(n+1)^p}\cdot\frac{n^p}{x^n}\right) = \lim_{n\to\infty}\left(x\cdot\left(\frac{n}{n+1}\right)^p\right) = x\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p = x\cdot 1 = x < 1 \overset{\text{Ratio Test}}{\implies}$$

$$\implies \sum_{n\ge 1}\frac{x^n}{n^p} \text{ converges for } x<1,\ p\in\mathbb{N}$$

Case II: $x=1$

$\sum_{n\geq 1}\frac{1}{n^p}$ converges for $p>1$ and diverges for $p=1$ $(p\in\mathbb{N})$

Case III: $x>1$

We apply the ratio test:

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\left(\frac{x^{n+1}}{(n+1)^p}\cdot\frac{n^p}{x^n}\right)=\lim_{n\to\infty}\left(x\cdot\left(\frac{n}{n+1}\right)^p\right)=x\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p=x\cdot 1=x>1\xRightarrow[\text{Test}]{\text{Ratio}}$$

$\Longrightarrow\sum_{n\geq 1}\frac{x^n}{n^p}$ diverges for $x>1$, $p\in\mathbb{N}$

b) $\sum_{n\geq 2}\frac{1}{(\ln n)^{\ln n}}:=\sum_{n\geq 2}x_n$ $\quad((x_n)$ decreasing with $x_n>0$, $\forall n\geq 2)$

Consider the series:

$$\sum_{n\geq 1}2^n x_{2^n}=\sum_{n\geq 1}\frac{2^n}{(\ln 2^n)^{\ln 2^n}}=\sum_{n\geq 1}\frac{2^n}{(n\ln 2)^{n\ln 2}}=\sum_{n\geq 1}\frac{2^n}{((n\ln 2)^{\ln 2})^n}$$

We have: $\lim_{n\to\infty}(n\ln 2)^{\ln 2}=\infty\Longrightarrow\exists N\in\mathbb{N}$ such that $(n\ln 2)^{\ln 2}\geq 4$, $\forall n\geq N\Longrightarrow$

$\Longrightarrow((n\ln 2)^{\ln 2})^n\geq 4^n$, $\forall n\geq N\Longrightarrow\frac{2^n}{((n\ln 2)^{\ln 2})^n}\leq\frac{2^n}{4^n}=\frac{1}{2^n}$, $\forall n\geq N\Longrightarrow$

$\Longrightarrow\sum_{n\geq N}\frac{2^n}{((n\ln 2)^{\ln 2})^n}\leq\sum_{n\geq N}\frac{1}{2^n}=\frac{1}{1-\frac{1}{2}}-\left(1+\frac{1}{2}+\dots+\frac{1}{2^{N-1}}\right)\leq\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2\Longrightarrow$

$\Longrightarrow\sum_{n\geq N}\frac{2^n}{((n\ln 2)^{\ln 2})^n}$ convergent $\Longrightarrow\sum_{n\geq 1}\frac{2^n}{((n\ln 2)^{\ln 2})^n}$ convergent $\Longleftrightarrow$

$\Longleftrightarrow\sum_{n\geq 1}2^n x_{2^n}$ convergent $\xLongrightarrow[\text{Test}]{\text{Cauchy Condensation}}\sum_{n\geq 2}x_n$ convergent $\Longleftrightarrow$

$\Longleftrightarrow\sum_{n\geq 1}\frac{1}{(\ln n)^{\ln n}}$ convergent

c) $\sum_{n\geq 1}(\sqrt[n]{n}-1)$

We have:

$$\left.\begin{array}{l}\sqrt[n]{n}-1=n^{\frac{1}{n}}-1=e^{\ln n^{\frac{1}{n}}}-1=e^{\frac{\ln n}{n}}-1,\quad\forall n\in\mathbb{N}\\ \text{We know that } e^x\geq x+1,\ \forall x\in\mathbb{R}.\text{ For } x=\frac{\ln n}{n}\text{ we will have:}\\ e^{\frac{\ln n}{n}}\geq\frac{\ln n}{n}+1\Longleftrightarrow e^{\frac{\ln n}{n}}-1\geq\frac{\ln n}{n},\quad\forall n\in\mathbb{N}\end{array}\right\}\Longrightarrow$$

$$\left.\begin{array}{l}\Longrightarrow\sqrt[n]{n}-1\geq\frac{\ln n}{n},\quad\forall n\in\mathbb{N}\\ \text{We know that } 3\geq e\Longrightarrow\ln 3\geq\ln e=1\Longrightarrow\ln n\geq\ln 3\geq 1,\ \forall n\geq 3\end{array}\right\}\Longrightarrow$$

$$\left.\begin{array}{l}\Longrightarrow\sqrt[n]{n}-1\geq\frac{1}{n},\ \forall n\geq 3\Longrightarrow\sum_{n\geq 3}(\sqrt[n]{n}-1)\geq\sum_{n\geq 3}\frac{1}{n}\\ \text{But }\sum_{n\geq 1}\frac{1}{n}\text{ diverges (harmonic series)}\Longrightarrow\sum_{n\geq 3}\frac{1}{n}\text{ diverges}\end{array}\right\}\xRightarrow[\text{Test}]{\text{Comparison}}$$

$$\Longrightarrow \sum_{n\geq 3}(\sqrt[n]{n}-1) \text{ diverges} \Longrightarrow \sum_{n=1}(\sqrt[n]{n}-1) \text{ diverges}$$

d) $\displaystyle\sum_{n\geq 1}\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot(2n)}\cdot\frac{1}{2n+1}$

$$\frac{x_n}{x_{n+1}}=\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot(2n)}\cdot\frac{1}{2n+1}\cdot\frac{2\cdot 4\cdot\ldots\cdot(2n)\cdot(2n+2)}{1\cdot 3\cdot\ldots\cdot(2n-1)(2n+1)}\cdot(2n+3)=$$

$$=\frac{(2n+2)(2n+3)}{(2n+1)^2}=\frac{4n^2+10n+6}{4n^2+4n+1}$$

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right)=\lim_{n\to\infty} n\cdot\frac{4n^2+10n+6-4n^2-4n-1}{4n^2+4n+1}=$$

$$=\lim_{n\to\infty} n\cdot\frac{6n+5}{4n^2+4n+1}=\lim_{n\to\infty}\frac{6n^2+5n}{4n^2+4n+1}=\frac{6}{4}\overset{(2}{=}\frac{3}{2}>1 \xrightarrow{\text{Raabe-Duhamel}}$$

$$\Longrightarrow \sum_{n\geq 1}\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot(2n)}\cdot\frac{1}{2n+1} \text{ converges}$$

3. Start with an equilateral triangle of side 1. For each side, remove the middle third and add there another equilateral triangle. Repeat this process at each iteration. How many sides are there at iteration n? What is the limit of the perimeter and the area?

Number of sides at a certain iteration (N)

$i=0: N_0=3$ sides

$i=1: N_1=12=3\cdot 4$ sides

$i=2: N_2=48=3\cdot 4^2$ sides

...

$i=n: N_n=3\cdot 4^n$ sides

At each iteration each side is divided in 3 parts, but only 2 of them are used as sides in the new geometric figure. The other 2 sides added come from the added triangle. So, at each iteration, each side transformes 4 new sides, resulting that $N_n=4\cdot N_{n-1}=\ldots=4^n\cdot N_0=3\cdot 4^n$.

The length of a side at a certain iteration (l)

$i=0: l_0=1$ $\qquad l_3=\frac{1}{3^3}$

$i=1: l_1=\frac{1}{3}$ $\qquad \ldots$

$i=2: l_2=\frac{1}{3^2}$ $\qquad l_n=\frac{1}{3^n}$

At a certain iteration, the perimeter is:

$$P_n = N_n \cdot l_n = 3 \cdot 4^n \cdot \frac{1}{3^n} = 3 \cdot \left(\frac{4}{3}\right)^n$$

So, we have: $\lim\limits_{n\to\infty} P_n = \lim\limits_{n\to\infty} 3 \cdot \left(\frac{4}{3}\right)^n = 3 \cdot \lim\limits_{n\to\infty} \left(\frac{4}{3}\right)^n = 3 \cdot \infty = \infty$

At a certain iteration, the area is:

$i=0: \quad A_0 = \frac{\sqrt{3}}{4} \cdot l_0^2 = \frac{\sqrt{3}}{4} \cdot 1^2 = \frac{\sqrt{3}}{4}$

$i=1: \quad A_1 = A_0 + N_1 \cdot \frac{\sqrt{3}}{4} \cdot l_1^2 = \frac{\sqrt{3}}{4} + 3 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot 3 \cdot \left(\frac{1}{3}\right)^2$

$i=2: \quad A_2 = A_1 + N_2 \cdot \frac{\sqrt{3}}{4} \cdot l_2^2 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{1}{3} + \frac{\sqrt{3}}{4} \cdot 3 \cdot 4 \cdot \left(\frac{1}{3^2}\right)^2$

...

$i=n: \quad A_n = A_{n-1} + N_n \cdot \frac{\sqrt{3}}{4} \cdot l_n^2 =$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \sum_{k=0}^{n-1} 3 \cdot 4^k \cdot \left(\frac{1}{3^{k+1}}\right)^2 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \sum_{k=0}^{n-1} \frac{3 \cdot 4^k}{3^{2k+2}} =$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{1}{3} \cdot \sum_{k=0}^{n-1} \left(\frac{4}{9}\right)^k = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{1}{3} \cdot \frac{\left(\frac{4}{9}\right)^n - 1}{\frac{4}{9} - 1} =$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{1}{3} \cdot \frac{1 - \left(\frac{4}{9}\right)^n}{1 - \frac{4}{9}} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{1}{3} \cdot \frac{9}{5} \cdot \left(1 - \left(\frac{4}{9}\right)^n\right) =$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{3}{5} \left(1 - \left(\frac{4}{9}\right)^n\right)$$

So, we have:

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{3}{5}\left(1 - \left(\frac{4}{9}\right)^n\right)\right) = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{3}{5} \cdot (1 - 0) =$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{3}{5} = \frac{5\sqrt{3} + 3\sqrt{3}}{20} = \frac{8\sqrt{3}}{20}^{(4} = \frac{2\sqrt{3}}{5}$$