Homework 6 - Tirris Diana (917)

1. a) $\lim\limits_{x\to 0} \dfrac{\sin x - x + \frac{x^3}{6}}{x^5}$

Using Taylor series, the function $\sin x$ can be written as:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \implies$$

$$\implies \sin x - x + \frac{x^3}{6} = \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots = \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$$

So, we have:

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0} \frac{\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}}{x^5} = \lim_{x\to 0} \frac{x^5 \cdot \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n-4}}{x^5} =$$

$$= \lim_{x\to 0} \left(\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n-4} \right) = \lim_{x\to 0} \left(\frac{1}{5!} + \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n-4} \right) =$$

$$= \lim_{x\to 0} \left(\frac{1}{5!} + x \cdot \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n-5} \right) = \frac{1}{5!} + 0 = \frac{1}{5!} = \frac{1}{120}$$

b) $\lim\limits_{x\to 0} \dfrac{e^{x^2} - \cos x - \frac{3x^2}{2}}{x^4}$

Using Taylor series, the functions e^{x^2} and $\cos x$ can be written as:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\implies e^{x^2} - \cos x - \frac{3x^2}{2} = -\frac{3x^2}{2} + 1 + x^2 + \sum_{n=2}^{\infty} \frac{x^{2n}}{n!} - \left(1 - \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n}\right) =$$

$$= \sum_{n=2}^{\infty} \frac{x^{2n}}{n!} - \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n} = \sum_{n=2}^{\infty} x^{2n} \cdot \left(\frac{1}{n!} - \frac{(-1)^n}{(2n)!} \right)$$

So, we have:

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x - \frac{3x^2}{2}}{x^4} = \lim_{x\to 0} \frac{\sum_{n=2}^{\infty} x^{2n}\left(\frac{1}{n!} - \frac{(-1)^n}{(2n)!}\right)}{x^4} = \lim_{x\to 0} \frac{x^4 \cdot \sum_{n=2}^{\infty} x^{2n-4}\left(\frac{1}{n!} - \frac{(-1)^n}{(2n)!}\right)}{x^4} =$$

$$= \lim_{x\to 0} \left(\sum_{n=2}^{\infty} x^{2n-4} \left(\frac{1}{n!} - \frac{(-1)^n}{(2n)!} \right) \right) = \lim_{x\to 0} \left(\frac{1}{2} - \frac{(-1)^2}{4!} + \sum_{n=3}^{\infty} x^{2n-4} \left(\frac{1}{n!} - \frac{(-1)^n}{(2n)!} \right) \right) =$$

$$= \lim_{x\to 0} \left(\frac{1}{2} - \frac{1}{24} + x \cdot \sum_{n=3}^{\infty} x^{2n-5} \left(\frac{1}{n!} - \frac{(-1)^n}{(2n)!} \right) \right) = {}^{12/}\frac{1}{2} - \frac{1}{24} + 0 = \frac{12-1}{24} = \frac{11}{24}$$

$$\Bigg(f(x) = e^{x^2},\ f(0) = 1;\ f'(x) = 2x \cdot e^{x^2},\ f'(0) = 0;\ f''(x) = e^{x^2}(4x^2+2),\ f''(0) = 2;$$
$$f^{(3)}(x) = e^{x^2}(8x^3 + 12x),\ f^{(3)}(0) = 0;\ f^{(4)}(x) = e^{x^2}(16x^4 + 48x^2 + 12),\ f^{(4)}(0) = 12;$$
$$\text{So: } e^{x^2} = f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = 1 + 0 + \frac{2x^2}{2!} + 0 + \frac{12x^4}{4!} + 0 = 1 + x^2 + \frac{x^4}{2} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \Bigg)$$

2. Prove that the Taylor series of $\ln(1+x)$ around 0 is $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$.

$f : (-1, \infty) \to \mathbb{R}, \quad f(x) = \ln(1+x)$

$f(0) = \ln(1+0) = \ln 1 = 0$

$f'(x) = \frac{1}{1+x} \; ; \; f'(0) = \frac{1}{1+0} = \frac{1}{1} = 1$

$f''(x) = -\frac{1}{(1+x)^2} \; ; \; f''(0) = -\frac{1}{(1+0)^2} = -\frac{1}{1} = -1$

$f^{(3)}(x) = \frac{2}{(1+x)^3} \; ; \; f^{(3)}(0) = \frac{2}{(1+0)^3} = \frac{2}{1} = 2$

$f^{(4)}(x) = -\frac{3!}{(1+x)^4} \; ; \; f^{(4)}(0) = -\frac{3!}{(1+0)^4} = -\frac{6}{1} = -6$

We observe that $f^{(n)}(x) = (-1)^{n+1} \cdot \frac{(n-1)!}{(1+x)^n}, \; \forall n \geq 1$, and $f^{(n)}(0) = (-1)^{n+1} \cdot (n-1)!, \; \forall n \geq 1$.

The Taylor series of $\ln(1+x)$ around 0 is:

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n =$$

$$= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (n-1)!}{n!} \cdot x^n = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n} \iff \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$$

3. (I) Let x and h be real numbers, $h \neq 0$.

Using the Taylor's formula with Lagrange's remainder, we have:

$$f(x+h) = \sum_{n=0}^{1} \frac{f^{(n)}(x)}{n!} \cdot (x+h-x)^n + \frac{f''(c)}{2!} \cdot (x+h-x)^2 =$$

$$= f(x) + f'(x) \cdot h + \frac{f''(c)}{2} \cdot h^2, \quad \text{where } c \in (x, x+h) \implies$$

$$\left.\begin{aligned} &\implies f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{f''(c) \cdot h}{2} \\ &\lim_{h \to 0} \frac{h}{\frac{f''(c) \cdot h}{2}} = \lim_{h \to 0} \frac{2}{f''(c)} = \frac{2}{f''(c)} \in (0, \infty) \end{aligned}\right\} \implies f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

(II) Let x and h be real numbers, $h \neq 0$.

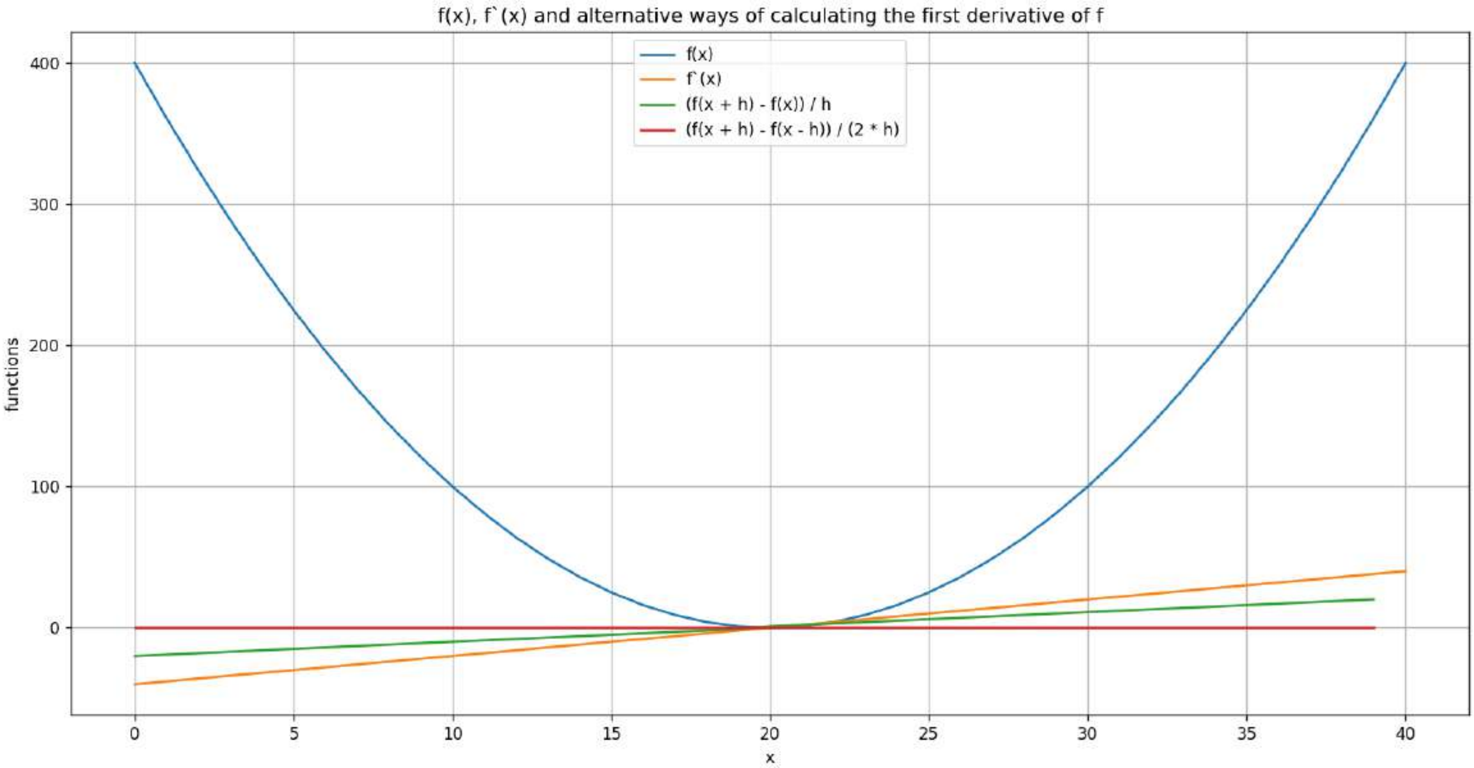
Using the Taylor's formula with Lagrange's remainder, we have:

$$\left.\begin{aligned} f(x+h) &= f(x) + f'(x) \cdot h + \frac{f''(c_1)}{2} \cdot h^2, \quad \text{where } c_1 \in (x, x+h) \\ f(x-h) &= f(x) - f'(x) \cdot h + \frac{f''(c_2)}{2} \cdot h^2 \quad \text{where } c_2 \in (x-h, x) \end{aligned}\right\} \implies$$

$$\implies f(x+h) - f(x-h) = 2h \cdot f'(x) + h^2 \cdot \frac{f''(c_1) - f''(c_2)}{2} \implies$$

$$\left.\begin{aligned} &\implies f'(x) = \frac{f(x+h) - f(x-h)}{2h} - h^2 \cdot \frac{f''(c_1) - f''(c_2)}{2} \\ &\lim_{h \to 0} \frac{h^2}{h^2 \cdot \frac{f''(c_1) - f''(c_2)}{2}} = \lim_{h \to 0} \frac{2}{f''(c_1) - f''(c_2)} = \frac{2}{f''(c_1) - f''(c_2)} \in (0, \infty) \end{aligned}\right\} \implies f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

```python
import matplotlib.pyplot as plt

5 usages
def f(x): return x**2

1 usage
def the_first_derivative_of_f(x): return 2 * x

1 usage
def alternative_formula_1_for_calculating_the_first_derivative_of_f(x, h): return (f(x + h) - f(x)) / h

1 usage
def alternative_formula_2_for_calculating_the_first_derivative_of_f(x, h): return (f(x + h) - f(x - h)) / (2 * h)


h = [i for i in range(-20, 21)]
h.remove(0)
value = 0
plt.plot( *args: [f(x) for x in range(-20, 21)], label="f(x)")
plt.plot( *args: [the_first_derivative_of_f(x) for x in range(-20, 21)], label="f`(x)")
plt.plot( *args: [alternative_formula_1_for_calculating_the_first_derivative_of_f(value, num) for num in h],
         label="(f(x + h) - f(x)) / h")
plt.plot( *args: [alternative_formula_2_for_calculating_the_first_derivative_of_f(value, num) for num in h],
         label="(f(x + h) - f(x - h)) / (2 * h)")

plt.xlabel("x")
plt.ylabel("functions")
plt.title("f(x), f`(x) and alternative ways of calculating the first derivative of f")
plt.legend()
plt.grid(True)
plt.show()
```



f(x), f`(x) and alternative ways of calculating the first derivative of f
f(x)
f`(x)
(f(x + h) - f(x)) / h
(f(x + h) - f(x - h)) / (2 * h)
functions
400
300
200
100
0
0
5
10
15
20
25
30
35
40
x