A Lending Value (LV) model for illiquid Lombard financing

Deriving liquidity adjusted LV

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Applied Credit Risk Modeling

Table of contents

1. Intro

2. Background

   Lombard Risk

   Lending Values

3. Model

   Data

   Parameter Estimation

   Empirical Results

   Demo

4. Conclusion

Intro

Mandate

Develop a Lending Value (LV) model for illiquid Lombard financing:

- Understand the credit risk inherent in loans collateralized by liquid assets (e.g. stocks, bonds, funds).
- Investigate the liquidity (size) effects on the riskiness of the transaction.

Lombard lending

Secured loans where the collateral consists of liquid assets, such as publicly traded stocks, bonds, etc.

- **Lending value**: Variable credit limit that is determined as a percentage of the collateral's market value.
- **Haircut**: Difference between the collateral's value and the loan amount. If running haircut drops below a predetermined threshold, the bank has the right to liquidate the collateral to protect its loan exposure.
- **Margin call**: Demand by the bank for the obligor to add more collateral or pay back part of the loan to maintain an agreed-upon level of equity in the borrowing account.

Example Lombard loan

Imagine a client has assets worth 100,000 CHF and wants to take out a Lombard loan.

- Lending value (80%): 80,000 CHF
- Initial haircut (20%): 20,000 CHF
- Asset value drops to: 96,000 CHF
- Running haircut: 16,000 CHF
- Haircut erosion (20%): 4,000 CHF

Warning Stage: If the haircut erosion lies btw. 0 - 25% of the req. margin, the client's positions enter a monitoring stage, but no immediate action is taken.

Example Lombard loan cont.

What if the asset value drops to 94,000 CHF?

- Running haircut: 14,000 CHF
- Haircut erosion (30%): 6,000 CHF

Margin Call Stage: When the haircut erosion exceeds 25% of the req. margin, a margin call is triggered. The client then needs to reestablish the req. margin (typically within 10 business days).

Liquidation: If the client does not respond to the margin call/if the market does not move favorably to automatically restore the req. haircut, the bank may begin liquidating the assets.

Lombard credit risk

- Market risk: Risk of loss due to changes in the market value of the collateral.
- Obligor-specific risk: Risk that the borrower will not respond to margin calls, which could lead to a loss for the bank.~~Obligor-specific risk: Risk that the borrower will not respond to margin calls, which could lead to a loss for the bank.~~

Bank's loss at the closeout period (time between last margin call and liquidation) resembles a market default event combined with a client default.

Loss becomes the payoff of a put option on the collateral with a stochastic strike price.

Who are the clients?

Very-/ultra-HNW clients (who may not want to have an in-depth assessment of their creditworthiness[1]) that wish to

- secure liquidity/bridge shortfalls,
- diversify,
- and/or increase return potential.

[1]Banks generally focus on the collateral quality than on an individual's creditworthiness when issuing Lombard loans.

Background

Lombard loan components

Lombard loans are comprised of three components:

1. **Bank's exposure to the client**: The client's utilization of the lending limit.
2. **Default triggers**: Definition of default events.
3. **Market value of pledged assets**: A valuation model of the collateral.

Notation

Let $(\Omega, \mathcal{G}, P)$ be a probability space equipped with natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.

- Market value of the collateral over time t:

$$V = (V_t)_{t \geq 0}.$$

- Exposure to the obligor ($\mathbb{F}$-adapted):

$$X = (X_t)_{t \geq 0}.$$

- Lending value (fixed through time):

$$\lambda \in (0, 1].$$

Notation cont.

- At time $t = 0$,

$$\text{loan amount } = \lambda V_0,$$

where V_0 is the initial MV of the collateral, and

$$\text{initial/req. haircut } = (1 - \lambda).$$

- At time t,

$$\text{running haircut } = \frac{(V_t - X_t)}{V_t}, \text{ and}$$
$$\text{running maximum of } V = V^*_{0,t}, \text{ over time interval } (0, t].$$

Bank's exposure to the client

Appendix

- Margin call policy: Controlled by

$$\frac{X}{V} > \frac{\lambda}{\beta}.$$

- Margin call trigger: Defined for $\alpha \in (0, 1)$, resulting in

$$\beta := 1 - (1 - \lambda)\alpha > \lambda.$$

- Stopping times (η_n): Instances of margin calls and recoveries, with η_1 being the first critical point.

Bank's continued exposure

- Time constraints: Client given $\delta > 0$ time units to meet calls; loan maturity at $T > 0$.
- Assumptions:
  - Assumption 2.1: Extension of T if margin call occurs near maturity.
  - Assumption 2.2: Client's cooperation until time τ_C.
  - Assumption 2.3: Maximizing borrowing within loan and collateral limits.

Critical margin call times

- $\widetilde{\tau}_n$: Critical times for margin calls, defined for $n \geq 1$.
- Exposure process (X_t): Exposure at time t, dependent on $\widetilde{\tau}_n$ and client's cooperation,

$$X_t = \lambda \sum_{n=1}^{\infty} V^*_{\widetilde{\tau}_{n-1}} 1_{\{\widetilde{\tau}_{n-1} \leq t < \widetilde{\tau}_n\}} \text{ on } \{\tau_C > t\}.$$

Default triggers

Appendix

- Default time (τ): The earliest time a default can occur, defined by certain financial conditions. It measures the potential for collateral failure in covering the loan, where

$$\tau := \inf\left\{t \geq 0 \mid \widetilde{N}_t \geq 1\right\} = \inf\left\{\widetilde{\tau}_n \mid \widetilde{\tau}_n \geq \tau_C\right\}.$$

- Incurred loss (L): The loss calculated at default time, reflecting the shortfall in collateral value compared to the loan exposure,

$$L = \left(X_\tau - V_\tau\right)^+ 1_{\{\tau \leq T+\delta\}}.$$

Market value of pledged assets

Appendix

- Asset value process (V): The evolution of the collateral's market value and solution of the SDE

$$dV_t = V_t\left(\mu dt + \sigma dB_t\right), \quad t \geq 0;$$
$$V_0 = v_0.$$

- By Itô's Lemma: $V_t = v_0 \exp\left((\mu - \sigma^2/2)t + \sigma B_t\right), \quad t \geq 0.$

Lending values

Appendix

- Lombard risk definition: Define LV as largest number in $(0,1)$ such that the probability of the collateral's value being less than the loan exposure at any default time plus δ remains below a threshold ϵ,

$$P[V_{\tau_n+\delta} \leq X_{\tau_n}] \leq \epsilon.$$

- Lending value: The maximum proportion of the collateral that can be borrowed, considering the risk tolerance level ϵ, drift μ, and volatility σ;

$$\lambda \leq \beta \exp\left((\mu - \frac{\sigma^2}{2})\delta + \sigma\sqrt{\delta}\Phi^{-1}(\epsilon)\right).$$

Liquidity adjusted lending values

- Price adjustment: Adjusts the price of the collateral to include liquidity costs[2]

$$V_t(x) = e^{\gamma x} V_t.$$

- Liquidity adj. LV: Reflects liquidity costs at the time of liquidation, with dependence on transaction size and market conditions,

$$\lambda \leq \beta \exp\left(\boxed{-\gamma x} + (\mu - \frac{\sigma^2}{2})\delta + \sigma\sqrt{\delta}\Phi^{-1}(\epsilon)\right).$$

[2] Modeled as an exponential function of the transaction size.

Liquidity adjusted lending values cont.

$$\lambda \leq \frac{(1-\alpha)\exp\left(\boxed{-\gamma x} + (\mu - \frac{\sigma^2}{2})\delta + \sigma\sqrt{\delta}\Phi^{-1}(\epsilon)\right)}{1 - \alpha\exp\left(\boxed{-\gamma x} + (\mu - \frac{\sigma^2}{2})\delta + \sigma\sqrt{\delta}\Phi^{-1}(\epsilon)\right)}$$

where

$$\sigma \equiv \text{Historical volatility (1-month)},$$

$$\mu \equiv \text{Drift of underlying GBM} := \frac{\sigma^2}{2},$$

$$\delta \equiv \text{Response time period} := \frac{10}{250},$$

$$\epsilon \equiv \text{Risk tolerance level} := 0.01,$$

$$\alpha \equiv \text{Margin call threshold} := 0.25.$$

Model

Data

- LSEG (Refinitiv) database.
- Cohort of 15 SWX stocks.
- Sample period 21.12.2023 to 18.03.2024.
- Intraday tick data.

Tick data from LSEG (Refinitiv)

	A	B	C	D	E	F	G	H	I	J
1	**UBS Group AG, ABB Ltd, Swiss Re AG, Nestle SA, Kudelski SA | Time And Sales 19-Mar-2024 13:48:58**									
2										
3										
4	UBSG.S, ABBN.S, SRENH.S, NESN.S, KUD.S									
5	Display: Time and Sales, Time Range: Trading Day, # of Days: 1, Days Back: 63									
6										
7	**RIC**	**Timestamp**	**Tick**	**Last Trade**	**Volume**	**VWAP**	**Bid**	**Ask**	**Flow**	**Calc VWAP**
8	ABBN.S	21-Dec-2023 17:40:16.203	DOWN ↓	37.1	1				-227,353,886.3	36.9
9	ABBN.S	21-Dec-2023 17:40:02.423							-227,353,849.2	36.9
10	UBSG.S	21-Dec-2023 17:40:02.422							-8,666,999,569.9	26.3
11	KUD.S	21-Dec-2023 17:40:02.420							6,720.4	1.2
12	SRENH.S	21-Dec-2023 17:40:02.419							1,430,147,386.1	94.2
13	NESN.S	21-Dec-2023 17:40:02.419							-296,998,197.8	97.4
14	SRENH.S	21-Dec-2023 17:40:00.274							1,430,147,386.1	94.2
15	UBSG.S	21-Dec-2023 17:40:00.266							-8,666,999,569.9	26.3
16	UBSG.S	21-Dec-2023 17:40:00.000	UP ↑	26.4	3				-8,666,999,569.9	26.3
17	SRENH.S	21-Dec-2023 17:40:00.000	UP ↑	95.7	4				1,430,147,386.1	94.2
18	NESN.S	21-Dec-2023 17:39:40.640	UP ↑	97.0	14	96.8	96.9	97.0	-296,998,197.8	97.4
19	NESN.S	21-Dec-2023 17:39:21.922	UP ↑	97.0	50	96.8	96.9	97.0	-296,999,555.7	97.4
20	NESN.S	21-Dec-2023 17:37:35.490	UP ↑	97.0	200	96.8	96.9	97.0	-297,004,405.2	97.4
21	NESN.S	21-Dec-2023 17:33:37.333	UP ↑	97.0	90	96.8	96.9	97.0	-297,023,803.2	97.4
22	NESN.S	21-Dec-2023 17:32:41.659	UP ↑	97.0	500	96.8	96.9	97.0	-297,032,532.3	97.4
23	NESN.S	21-Dec-2023 17:32:38.565	UP ↑	97.0	30	96.8	96.9	97.0	-297,081,027.3	97.4
24	SRENH.S	21-Dec-2023 17:30:26.198							1,430,147,003.2	94.2
25	KUD.S	21-Dec-2023 17:30:26.193							6,720.4	1.2
26	UBSG.S	21-Dec-2023 17:30:26.192							-8,666,999,649.0	26.3
27	ABBN.S	21-Dec-2023 17:30:26.192							-227,353,849.2	36.9

UBSG.S, ABBN.S, SRENH.S, NESN.S

List of SWX stocks

Ticker	Company Name	Avg MC (M)	Avg Close	ADTV	Vola
UBSG	UBS Group AG	88680.02	25.53	6549272.90	0.0149
NESN	Nestle SA	258004.01	96.60	3676266.76	0.0119
ABBN	Abb Ltd	72241.62	38.44	3117397.42	0.0111
CLN	Clariant AG	3790.86	11.34	977708.88	0.0124
SRENH	Swiss Re AG	32139.75	101.65	819664.63	0.0100
SIKA	Sika AG	40535.89	250.84	294424.68	0.0142
LONN	Lonza Group AG	30918.84	418.98	239353.20	0.0224
UHR	Swatch Group AG	6146.65	211.26	165203.39	0.0145
SCMN	Swisscom AG	26445.64	510.24	86109.05	0.0091
KUD	Kudelski SA	65.85	1.29	66002.85	0.0524
SCHN	Schindler Holding AG	14073.83	210.45	23833.59	0.0100
GIVN	Givaudan SA	33318.68	3613.24	18690.14	0.0141
DOKA	Dormakaba Holding AG	1870.05	444.39	4409.93	0.0134
VLRT	Valartis Group AG	37.85	12.09	236.66	0.0197
LISN	Chocoladefabriken Lindt & Spruengli AG	14530.10	108298.31	102.63	0.0127

Estimation of the liquidity parameter

We obtain daily estimates of γ:

$$\log\left(\frac{v_{i+1}}{v_t}\right) = \log\left(\frac{V_{t_i+1}(x_{i+1})}{V_{t_i}(x_i)}\right)$$

$$= \gamma(x_{i+1} - x_i) + (\mu - \frac{\sigma^2}{2})(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} \cdot \epsilon_i,$$

using tick data with trading sequence (t_i, x_i, v_i), $i = 1, ..., n$; i.e. the first trade in that day occurs at time t_1 with size x_1 and price v_1, etc.

$$y_i := \frac{\log(v_{i+1}/v_i)}{\sqrt{t_{i+1} - t_i}} = \gamma \cdot \frac{x_{i+1} - x_i}{\sqrt{v_{i+1} - t_i}} + \left(\mu - \frac{\sigma^2}{2}\right)\sqrt{t_{i+1} - t_i} + \sigma\epsilon_i,$$

i.e. the parameter γ can be estimated by means of a linear regression[3] for the response variables y_i and the predictors

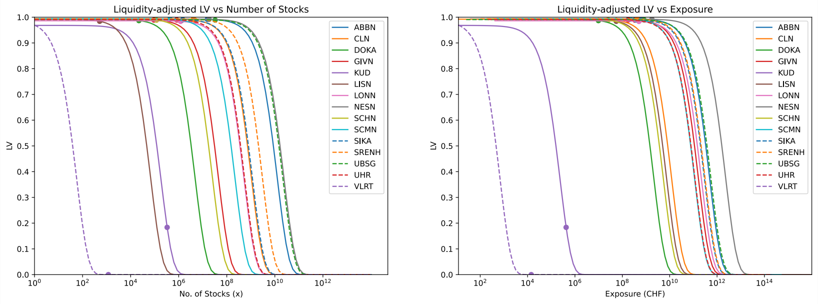
$$w_i := (x_{i+1} - x_i)/\sqrt{t_{i+1} - t_i} \quad \text{and}$$

$$z_i := \sqrt{t_{i+1} - t_i}, \quad i = 1, \ldots, n-1.$$

[3] Observe that the regression implicitly yields an estimate for μ through an estimate for σ.

Parameter estimation results

Lending value curves



Implementation

We can approximate γ from the ADTV with the following linear relationship:

$$\widehat{\gamma} \approx 10^{\widehat{a}} \cdot \text{ADTV}^{\widehat{b}}.$$

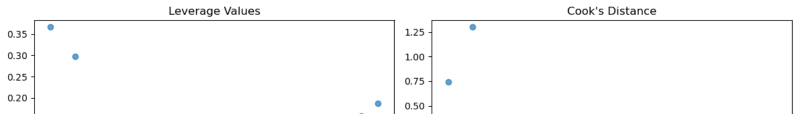
ADTV regression

```
                            OLS Regression Results
==============================================================================
Dep. Variable:              log_gamma   R-squared:                       0.810
Model:                            OLS   Adj. R-squared:                  0.795
Method:                 Least Squares   F-statistic:                     55.26
Date:                Mon, 25 Mar 2024   Prob (F-statistic):           4.95e-06
Time:                        08:36:25   Log-Likelihood:                -21.353
No. Observations:                  15   AIC:                             46.71
Df Residuals:                      13   BIC:                             48.12
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept     -0.5429      1.034     -0.525      0.608      -2.777       1.691
log_adtv      -1.4950      0.201     -7.434      0.000      -1.929      -1.060
==============================================================================
Omnibus:                        8.997   Durbin-Watson:                   2.364
Prob(Omnibus):                  0.011   Jarque-Bera (JB):                5.492
Skew:                           1.377   Prob(JB):                       0.0642
Kurtosis:                       4.097   Cond. No.                         19.8
==============================================================================
```

Finding influential data points

We analyze the influence of each observation in the model with the following:

- Leverage: How far an observation's independent variable values are from those of other observations. Observations with high leverage have a larger impact on the determination of the regression line.
- Studentized Residuals: Residuals adjusted for their standard deviation. Observations with large absolute studentized residuals are potential outliers.
- Cook's Distance: Influence of each observation on the fitted values. A large Cook's distance indicates that the observation has a large influence on the regression coefficients.



Model Demo

Welcome to SVB (Swiss Valais Bank)! Today we will introduce the new LV calculator for Lombard loans.

Please run ***app.py*** and open the following link:

http://127.0.0.1:5000/

svb Swiss Valais Bank

Conclusion

Limitations of the model

- Limited amount of intraday tick data (time-frame).
- Small and geographically limited cohort.
- LV calculations with multi-asset collateral types assume independence.

Ongoing work

Appendix

- Include more data.
- Fix the button.
- Integrate correlations through covariance matrices[4].
- More stressed scenario valuations.

[4] An attempt to bypass this issue would be to consider other supply curves (that would also req. just as strong, if not stronger, assumptions).

Questions?

References i

Jarrow, Robert and Philip Protter (2005). **"Liquidity risk and risk measure computation".** In: Review of Futures Markets 11.1, pp. 27–39.

Juri, Alessandro (2014). **"Lending Values and Liquidity Risk".** In: Journal of Applied Finance & Banking 4.1, pp. 173–221.

Why borrow if you are already wealthy (2021). Accessed: 2024-03-07. URL: *https://t.ly/8Z97v*.

Backup Slides

More math

The definitions found in the following slides are taken from Juri 2014 and Jarrow and Protter 2005.

Bank's exposure to the client

Back

The bank sets a policy that involves margin calls[5] to control the loan.

- Margin call trigger, for fixed threshold $\alpha \in (0, 1)$:

$$\beta := 1 - (1 - \lambda)\alpha > \lambda.$$

- Stopping times (when calls occur) are given by $(\eta_n)_{n \geq 1}$, $n_1 := \inf\{t > 0 \mid V_t/X_t < \beta/\lambda\}$ and, for $n > 1$:

$$\eta_{2n} := \inf\{t > \eta_{2n} \mid V_t/X_t > \beta/\lambda\},$$
$$\eta_{2n+1} := \inf\{t > \eta_{2n} \mid V_t/X_t < \beta/\lambda\}.$$

[5] Occurring when $\frac{X}{V} > \frac{\lambda}{\beta}$.

Bank's exposure to the client cont.

Let $\delta > 0$ be the stipulated time allotted to client to meet the call and $T > 0$ the loan maturity.

- Assumption 2.1: When the margin call occurs within $(T - \delta, T]$, T is extended to give client δ time units.
- Assumption 2.2: Client's willingness to adjust his exposure is defined by time τ_C where up to τ_C, the client is cooperative in reducing exposure.
- Assumption 2.3: The client aims to maximize his borrowing within the bounds of the loan terms and the value of his collateral.

Bank's exposure to the client cont.

We define critical margin call times, random times $\widetilde{\tau}_n, n \geq 1$ as

$$\widetilde{\tau}_n := \inf\left\{t > \widetilde{\tau}_{n-1} + \delta \mid V^*_{t-\delta,t} < \beta V^*_{\widetilde{\tau}_{n-1},t}\right\}, n \geq 1,$$

where $\widetilde{\tau}_0 := 0$.

Exposure process $(X_t)_{t\geq 0}$ is then:

$$X_t = \lambda \sum_{n=1}^{\infty} V^*_{\widetilde{\tau}_{n-1}} 1_{\{\widetilde{\tau}_{n-1} \leq t < \widetilde{\tau}_n\}} \text{ on } \{\tau_C > t\}.$$

Default triggers

Back

Default time τ is given as

$$\tau := \inf \left\{ t \geq 0 \mid \widetilde{N}_t \geq 1 \right\} = \inf \left\{ \widetilde{\tau}_n \mid \widetilde{\tau}_n \geq \tau_C n \geq 1 \right\}.$$

We assume that immediate liquidation is possible, so incurred loss L becomes

$$L = (X_\tau - V_\tau)^+ \, 1_{\{\tau \leq T+\delta\}} = \left(\lambda \beta^{-1} V_{\tau-\delta} - V_\tau \right)^+ 1_{\{\tau \leq T+\delta\}}.$$

Default triggers cont.

For simplicity, we assume that the client never reacts on margin calls, such that τ occurs δ time units after the first critical margin call time $\widetilde{\tau}_1$,

$$\tau = \widetilde{\tau}_1 + \delta = \inf\left\{ t \geq \delta \mid V^*_{\tau-\delta,t} < \beta V^*_{0,t} \right\} + \delta.$$

Market value of pledged assets

Back

Defining our pledged assets as a single stock portfolio, market value process V becomes the solution of the SDE[6]

$$\begin{aligned} dV_t &= V_t\left(\mu\, dt + \sigma\, dB_t\right), \quad t \geq 0, \\ V_0 &= v_0; \end{aligned}$$

where B is a standard Brownian motion, $(\mu, \sigma) \in \mathbb{R} \times (0, \infty)$, $v_0 > 0$, and constants μ, σ, and v_0 denote the drift, volatility, and initial MV, respectively.

[6] By Itô's Lemma: $V_t = v_0 \exp\left((\mu - \sigma^2/2)t + \sigma B_t\right), \quad t \geq 0.$

Modeling lending values

Back

For Lombard risk, we define the LV as the largest number in $(0, 1)$ such that

$$P[V_{\tau_n+\delta} \leq X_{\tau_n}] = P[V_{\tau_n+\delta} \leq \lambda\beta^{-1} V_{\tau_n}] \leq \epsilon \quad \text{for all } n \geq 1.$$

With $V_{\tau_n+\delta} = V_{\tau_n} Z_\delta$ for r.v. $Z_\delta \sim$ Lognormal$((\mu - \sigma^2/2)\delta, \sigma^2\delta)$,

$$\begin{aligned} P[V_{\tau_n+\delta} \leq X_{\tau_n}] &= P[V_{\tau_n} Z_\delta \leq \lambda\beta^{-1} V_{\tau_n}] \\ &= P[Z_\delta \leq \lambda\beta^{-1}] \\ &= \Phi\left(\frac{\log(\frac{\lambda}{\beta}) - (\mu - \frac{\sigma^2}{2})\delta}{\sigma\sqrt{\delta}}\right). \end{aligned}$$

Modeling lending values cont.

Transforming the equality, lending value λ becomes:

$$\begin{aligned}\lambda &\leq \beta \exp\left((\mu - \frac{\sigma^2}{2})\delta + \sigma\sqrt{\delta}\Phi^{-1}(\epsilon)\right) \\ &\leq \frac{(1-\alpha)\exp\left((\mu - \frac{\sigma^2}{2})\delta + \sigma\sqrt{\delta}\Phi^{-1}(\epsilon)\right)}{1-\alpha\exp\left((\mu - \frac{\sigma^2}{2})\delta + \sigma\sqrt{\delta}\Phi^{-1}(\epsilon)\right)}.\end{aligned}$$

Modeling liquidity adj. lending values

We include liquidity costs through the supply curve taking account transaction size, $x \in \mathbb{R}$[7], such that the price per share is given by

$$V_t(x) = e^{\gamma x} V_t.$$

Position of size $x > 0$ where we want to liquidate $\theta \in [0,1]$ at time t is given by:

$$\begin{aligned} \widetilde{U}_t &= \underbrace{x V_t}_{\text{Classical value } U_t} - \underbrace{-\theta x (e^{-\gamma\theta x} - 1) V_t}_{\text{Liquidity cost } L_t} \\ &= (1 - \theta + \theta e^{-\gamma\theta x}) U_t \leq U_t. \end{aligned}$$

[7] Where x denotes the order flow bought ($x > 0$) or sold ($x < 0$).

Modeling liquidity adj. lending values cont.

At liquidation, θ must satisfy

$$\frac{X_\tau - \theta\, V_\tau}{(1-\theta)\, V_\theta} = \lambda,$$

$$\theta = \frac{\lambda}{1-\lambda}\left(\frac{1}{\beta Z_\delta} - 1\right) 1_{\{\lambda\beta^{-1} < Z_\delta \leq \beta^{-1}\}} + 1_{\{Z_\delta < \beta^{-1}\}}$$

Thus,

$$\begin{aligned}
\lambda &\leq \beta \exp\left(-\gamma x + (\mu - \frac{\sigma^2}{2})\delta + \sigma\sqrt{\delta}\Phi^{-1}(\epsilon)\right) \\
&\leq \frac{(1-\alpha)\exp\left(-\gamma x + (\mu - \frac{\sigma^2}{2})\delta + \sigma\sqrt{\delta}\Phi^{-1}(\epsilon)\right)}{1 - \alpha\exp\left(-\gamma x + (\mu - \frac{\sigma^2}{2})\delta + \sigma\sqrt{\delta}\Phi^{-1}(\epsilon)\right)}
\end{aligned}$$

Portfolio value determination

Back

Given supply curve[8]

$$S(t,x) = S(t,0)[1 + \alpha_c 1_c x + \alpha_n (1 - 1_c) x],$$

the value at the position at time T including liquidity costs, denoted V_T^L, is:

$$V * L_T \equiv Y_T + X_T S(T,0) = Y_0 + X_0 S(0, X_0) + \int_0^T X_{u-} \, dS(u,0) - L_T.$$

where

$$L_T = \sum_{0 \le u \le T} \Delta X_u [S(u, \Delta X_u) - S(u,0)] + \int_0^T \frac{\partial S}{\partial x}(u,0) \, d[X,X]_u^c.$$

Due to crisis at time T, we assume we liquidate $\theta \in [0,1]$ percent of holdings, so liquidity costs are

$$L_T = -\theta X_T [S(T, -\theta X_T) - S(T,0)].$$

[8] Where slope coefficients $\alpha_c \ge \alpha_n \ge 0$ are constants and 1_c is an indicator function.

Portfolio value determination cont.

Thus, $V * L_T$ is the classical value less the time T liquidation costs, i.e.

$$V_T^L = V_T - L_T = V_T + \theta X_T[S(T, -\theta X_T) - S(T, 0)] \leq V_T.$$

Liquidity costs from immediate liquidation shifts the entire distribution of the terminal value V_T to the left (with probability on).

Single asset portfolio

For a single asset portfolio[9]:

$$
\begin{aligned}
V_T^L &= X_T S(T,0) - L_T \\
&= X_T S(T,0)[1 - \alpha_c \theta^2 X_T] \\
&= V_T[1 - \alpha_c \theta^2 X_T] \le V_T.
\end{aligned}
$$

[9]The time T value including liquidity costs is equal to $[1 - \alpha_c \theta^2 X_T]$ times the classical time T value. This adjustment shifts the portfolio's distribution to the left, i.e. it reduces the portfolios value for all possible states of the economy (with probability one). Indeed, if $V_T > 0$, then $X_T > 0$ and $[1 - \alpha_c \theta^2 X_T] < 1$, implying that less dollars are received when selling shares. If $V_T < 0$, then $X_T < 0$ and $[1 - \alpha_c \theta^2 X_T] > 1$ implying more dollars are paid when buying back shares (covering short positions). Note that the decline in value is greater when the slope of the supply curve α_c is larger or when the percent of the position that is liquidated θ is larger.

Multi-asset portfolio

For a multi-asset portfolio containing N assets indexed by $i = 0, 1, \ldots, N$[10]:

$$\begin{aligned} V_T^L &= \sum_{i \geq 1} X_T^i S^i(T,0)[1 - \alpha_c^i (\theta^i)^2 X_T^i] + X_T^0 S^0(T,0) \\ &\leq V_T = \sum_{i \geq 1} X_T^i S^i(T,0) + X_T^0 S^0(T,0). \end{aligned}$$

[10]It indicates that one needs to multiply the final value of each asset by its liquidity discount $[1 - \alpha_c^i (\theta^i)^2 X_T^i]$. This value $[1 - \alpha_c^i (\theta^i)^2 X_T^i] < 1$ if $X_T^i > 0$ and shares are sold at liquidation, and $[1 - \alpha_c^i (\theta^i)^2 X_T^i] > 1$ if $X_T^i < 0$ and shares are purchased at liquidation. Liquidity costs shifts (with probability one) the value of the portfolio at liquidation to the left.

Assumptions

Assumption 2.1.

If a margin call occurs within $(T\delta, T]$*, then the maturity of the contract is artificially prolonged so that the client still has* δ *time units to react to that margin call.*

Assumption 2.2 (Client creditworthiness).

There is a non-negative random variable τ_c *such that, prior to* τ_c*, the obligor is willing to reduce its exposure if a margin call occurs whereas from* τ_c *onward he is not.*

Assumption 2.3 (Speculative client).

(i) An obligor always draws up to his limit as long as the market value of the collateral increases and he sticks to the current exposure otherwise.

(ii) If a margin occurs at the time η and over $[\eta, \eta + \delta)$ the required haircut is not reestablished by the movements of the collaterals market value itself, i.e. $V^*_{\eta,\eta+\delta} = V_\eta$, then the obligors exposure remains constant over $[\eta, \eta + \delta)$, i.e. $X_s = X_\eta$ for all $s \in [\eta, \eta + \delta)$.

(iii) If the obligor reacts on a margin call occurring at time η, then he reduces the exposure to exactly reestablish the required haircut δ time units after the margin call time, i.e. $X_{\eta+\delta} = \lambda V_{\eta+\delta}$.