Chapter 1

Time Series

1.1 Introduction

A random variable is a $(\Omega, \mathscr{F}) \setminus (\mathbb{R}^m, \mathscr{B})$ measure function. A time series is a sequence of random variables $(y_1(\omega), y_2(\omega), \ldots, y_n(\omega)) \in \mathbb{R}^{m \times n}$, and it can be extended to a doubly infinite sequence $(..., y_{t-1}, y_t, y_{t+1}, ...) \in \mathbb{R}^{m \times \infty}$. We consider discrete time series (instead of the continuous time series). For each fixed ω, the sequence is a deterministic vector $(\omega) \in \mathbb{R}^{m \times n}$; for each fixed t, $y_t(\omega)$ is a common random vector in $\mathbb{R}^m$.

1.2 Stationarity

In reality, we have only one realized sequence, but statistics needs repeated observations. We introduce the concept *stationarity* to produce "repeated" observations.

Definition 1.1. (y_t) is **covariance stationarity** or **weakly stationarity** if the mean $\mu = E[y_t]$, covariance $\Sigma = E\left[(y_t - \mu)(y_t - \mu)'\right]$ and autocovariance $\Gamma(\ell) = E\left[(y_t - \mu)(y_{t-\ell} - \mu)'\right]$ are independent of t.

- For a vector-valued weakly stationarity time series, $\Sigma = \Gamma(0)$ is a positive-definite symmetric matrix. The autovariance $\Gamma(\ell), \ell \neq 0$ is not symmetric in general, and

$$\Gamma(-\ell) = E\left[(y_t - \mu)(y_{t+\ell} - \mu)'\right] = E\left[(y_{t-\ell} - \mu)(y_t - \mu)'\right] = \Gamma(\ell)'.$$

- When $m = 1$ (scalar time series), we use $\gamma(0), \gamma(1), ...,$ for the autocovariance, and we define *autocorrelation* as $\rho(\ell) = \gamma(\ell)/\gamma(0)$. By the Cauchy-Schwarz inequality $\rho(\ell) \in [-1, 1]$.

Definition 1.2. (y_t) is **strictly stationarity**, if for every $\ell \in \mathbb{Z}^+$, the joint distribution of $(y_t, y_{t+1}, \ldots, y_{t+\ell})$ is independent of t.

When one mentions "stationarity" without referring to a quantifier, in econometrics it means strictly stationarity by default.

- If (y_t) is i.i.d, it is a special case of strict stationarity.
- If (y_t) is strictly stationary, its transformation $x_t \in \phi(y_t, y_{t-1}, ...) \in \mathbb{R}^q$ is also strictly stationary. In other words, strict stationarity is preserved by transformation.

The infinite series x_t is **convergent** if the partial sum $\sum_{j=1}^{N} a_j y_{t-j}$ has a finite limit as $N \to \infty$ *almost surely.*

- If y_t is strictly stationarity, $E\|y\| < \infty$ and $\sum_{j=0}^{N} |a_j| < \infty$ (absolutely summable), then x_t is convergent and strictly stationarity.

1.3 Ergodicity

A time series (y_t) is **ergodic** if all invariant events are trivial.

Definition 1.3. Formal definitions
Let $\widetilde{y}_t = (..., y_{t-1}, y_t, y_{t+1}, ...)$, and the ℓ-th time shift is $\widetilde{y}_{t+\ell} = (..., y_{t-1+\ell}, y_{t+\ell}, y_{t+\ell+1}, ...)$.
Let an event $D \in \{\widetilde{y}_t \in G\}$ for some $G \subseteq \mathbb{R}^{m\times\infty}$, and a time shift of the event is $D_\ell \in \{\widetilde{y}_{t+\ell} \in G\}$.
An event is **invariant** if $D_\ell = D$ for all $\ell \in \mathbb{Z}$. An event is **trivial** if $P(D) = 0$ or $P(D) = 1$.

Example 1.1. If $x_t = \sum_{j=0}^{\infty} a_j y_{t-j}$ if convergent and (y_t) is erdogic, then x_t is also erdogic.

Ergodicty is preserved by transformation. If (y_t) is stationarity and erdogic, the same is for $x_t \in \phi(y_t, y_{t-1}, ...)$ (function with infinite terms).

Fact 1.1 (Cesaro mean). *If $a_j \to a$ as $j \to \infty$, then $\frac{1}{n}\sum_{j=0}^{\infty} a_j \to a$ as $n \to \infty$.*

Theorem 1.1. *If $y_t \in \mathbb{R}^m$ is stationarity and erdogic, and $var(y_t) < \infty$, then $\frac{1}{n}\sum_{\ell=1}^{n} cov(y_t, y_{t+\ell}) \to 0$ as $n \to \infty$*

A stationarity (y_t) is erdogic if for all events A and B,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} P(A_\ell \cap B) = P(A) P(B)$$

Let $B = A$, and then we solve $P(A) = [P(A)]^2 \Rightarrow P(A) = 0$ or 1.
A "sufficient" condition for ergodicity is $P(A_\ell \cap B) \to P(A) P(B)$ as $\ell \to \infty$, according to Cesaro means. This sufficient condition is called "mixing".

- Mixing says that separate events (any A and B) are asymptotically independent when one of the event, say A, is shifted to A_ℓ as $\ell \to \infty$.

- Ergodicity is slightly weaker than mixing (weak dependence), in the sense that the independence is "on average" in the form of $\frac{1}{n}\sum_{\ell=1}^{n} P(A_\ell \cap B)$.

Theorem 1.2 (Ergodic Theorem). *If $y_t \in \mathbb{R}^m$ is stationarity, erdogic, and $E\|y\| < \infty$, then $E\|\bar{y} - \mu\| \to 0$ and $\bar{y} \xrightarrow{p} \mu$.*

This is a version of LLN for time series.

1.4 Information Set

- for a univariate time series, definite $E_{t-1}[y_t] = E[y_t \mid y_{t-1}, y_{t-2}, ...]$ as the condition expectation of y_t given the past history $(y_{t-1}, y_{t-2}, ...)$

- More generally, we write $\mathscr{F}_t$ as the σ-field generated by the information up to time t. $\mathscr{F}_t$ is called an **information set**. We can write $E_{t-1}[y_t] = E[y_t \mid \mathscr{F}_{t-1}]$.

- Information sets are nested: $\cdots \subseteq \mathscr{F}_{t-1} \subseteq \mathscr{F}_t \subseteq \mathscr{F}_{t+1} \subseteq ...$

- Information sets associate with the generating variables may differ. For example, in general $\sigma(y_t, y_{t-1}, ...) \neq \sigma(y_t, x_t, y_{t-1}, x_{t-1}, ...)$. The former is the information set for (y_t), whereas the latter is the information set for (y_t, x_t).

1.5 Martingale Difference Sequence (MDS)

- Let (e_t) be a time series, and $\mathscr{F}_t$ be an information set. We say (e_t) is **adapted** to $\mathscr{F}_t$ if $E[e_t \mid \mathscr{F}_t] = e_t$. It means that $\mathscr{F}_t$ contain the complete information of e_t. A **natural filtration** is $\mathscr{F}_t = \sigma(e_t, e_{t-1}, ...)$; it is the smallest information set to which (e_t) is adapted.

Definition 1.4 (MDS)**.** A process $\{e_t, \mathscr{F}_t\}$ is MDS if

1. e_t is adapted to $\mathscr{F}_t$
2. $E|e_t| < \infty$
3. $E[e_t \mid \mathscr{F}_{t-1}] = 0$

Interpretation: e_t is unforeseeable given the information $\mathscr{F}_{t-1}$. The definition of mds is about the mean independence. It does not rule our predictability in other moments.

MDS implies that the series is a white noise (zero autocovariance at all orders), because

$$cov(e_t, e_{t-\ell}) = E[e_t e_{t-\ell}] = E[E[e_t e_{t-\ell} \mid \mathscr{F}_{t-1}]] = E[e_{t-\ell} E[e_t \mid \mathscr{F}_{t-1}]] = 0.$$

Example 1.2. Suppose $e_t = u_t u_{t-1}$, where $u_t \sim i.i.d.\, N(0,1)$. In this case, e_t is MDS. Consider the filtration $\mathscr{F}_t = \sigma(u_t, u_{t-1}, ...)$, which subsumes $\sigma(e_t, e_{t-1}, ...)$.

$$E[e_t|\mathscr{F}_{t-1}] = E[u_t u_{t-1}|\mathscr{F}_{t-1}] = u_{t-1} E[u_t|\mathscr{F}_{t-1}] = u_{t-1} \cdot 0 = 0.$$

On the other hand, the covariance of e_t^2 and e_{t-1}^2 is not 0 as

$$cov\left(e_t^2, e_{t-1}^2\right) = E\left[u_t^2 u_{t-2}^2 u_{t-1}^4\right] - E\left[u_t^2 u_{t-1}^2\right] E\left[u_{t-1}^2 u_{t-2}^2\right] = 3 \times 1 \times 1 - (1 \times 1)^2 = 2$$

as the kurtosis of $N(0,1)$ is 3. Therefore, (e_t) is an mds but not iid.

A MDS $(e_t, \mathscr{F}_t)$ is a **conditional homoskedastic** if $E\left[e_t^2 \mid \mathscr{F}_{t-1}\right] = \sigma^2$. In the above example, $e_t = u_t u_{t-1}$ is MDS, but conditional heteroskedastic because

$$E\left[e_t^2|\mathscr{F}_{t-1}\right] = E\left[u_t^2 u_{t-1}^2|\mathscr{F}_{t-1}\right] = u_{t-1}^2 E\left[u_t^2|\mathscr{F}_{t-1}\right] = \sigma^2 u_{t-1}^2$$

varies over time.

In the real world, mds is a good model for the stock return. Indeed, mds is implied by the efficient market hypothesis. On the other hand, empirical evidence shows that the conditional variance of stock return is very predictable. There are many models about conditional volatility, for example the well-known ARCH and GARCH models.

Theorem 1.3 (CLT for MDS)**.** *If (e_t) is strictly stationary, ergodic and MDS, then*

$$S_n = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e_t \xrightarrow{d} N(0, \Sigma)$$

where $\Sigma = E[e_t e_t']$. There is the time series counterpart of the Lindeberg-Lévy CLT.

In the above theorem, because (e_t) is strict stationary, its variance Σ must be a constant matrix. It does not rule out u_t being conditional heteroskedastic.

1.6 Mixing

MDS is useful, but too restrictive in that it rules out serial correlation. If we are reluctant to impose MDS, we will need stronger assumption on the dependence than ergodicity to establish large sample results.

We introduce some more definitions. The **alpha-coefficient for two events** is defined as $\alpha(A,B) = |P(AB) - P(A)P(B)|$. Denote two σ-fields be $\mathscr{F}_{-\infty}^{t} = \sigma(..., y_{t-1}, y_t)$ and $\mathscr{F}_t^{\infty} = \sigma(y_t, y_{t+1}, ...)$. The **strong mixing coefficient** (alpha-coefficient) is defined as

$$\alpha(\ell) = \sup_t \sup_{A \in \mathscr{F}_{-\infty}^{t-\ell}, B \in \mathscr{F}_t^{\infty}} \alpha(A,B)$$

We say (y_t) is **strong mixing** (alpha mixing) if $\alpha(\ell) \to 0$ as $\ell \to \infty$.

If the time series is strictly stationary, the definition of the alpha-coefficient can be simplified as $\alpha(\ell) = \sup_{A \in \mathscr{F}_{-\infty}^{t-\ell}, B \in \mathscr{F}_t^{\infty}} \alpha(A,B)$.

Fact 1.2. *An α-mixing process is ergodic.*

To use the α-mixing process, we usually need **rate conditions** (for example $\alpha(\ell) = O(\ell^{-r})$ gives the speed of decay) or **summation restriction** (for example $\sum_{\ell=0}^{\infty} [\alpha(\ell)]^r < \infty$ or $\sum_{\ell=0}^{\infty} \ell^s \alpha(\ell)^r < \infty$.)

Strong mixing is preserved by finite transformation.

Fact 1.3. *Suppose y_t has mixing coefficients $\alpha_y(\ell)$, and $x_t = \phi(y_t, y_{t-1}, ..., y_{t-q})$ is a finite transformation of (y_t). Then Then $\alpha_x(\ell) < \alpha_y(\ell - q)$ for $\ell \geq q$.The α-coefficients satisfy the same rate and summation properties.*

Another widely used measurement of dependence is **absolute regularity** (beta-coefficient)

$$\beta(\ell) = \sup_t \sup_{A \in \mathscr{F}_t^{\infty}} \left| P\left(A \mid \mathscr{F}_{-\infty}^{t-\ell}\right) - P(A) \right|.$$

β mixing is stronger than α mixing in that $\beta(\ell) \to 0$ implies $\alpha(\ell) \to 0$.

1.7 CLT for Correlated Variables

The scaled partial sum of a scalar time series $(y_t)_{t=1}^{n}$ has variance

$$\begin{aligned}
var(S_n) &= var\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} y_t\right) \\
&= \frac{1}{n} \mathbf{I}_n' E\left[YY'\right] \mathbf{I}_n \\
&= \frac{1}{n} \mathbf{I}_n' \begin{bmatrix} \sigma^2 & \gamma(1) & \gamma(2) & & \gamma(n-1) \\ \gamma(1) & \sigma^2 & \gamma(1) & & \\ \gamma(2) & \gamma(1) & \sigma^2 & & \\ \vdots & & & \ddots & \vdots \\ \gamma(n-1) & & & & \sigma^2 \end{bmatrix} \mathbf{I}_n \\
&= \frac{1}{n}\left(n\sigma^2 + 2(n-1)\gamma(1) + 2(n-2)\gamma(2) + ... + 2\gamma(n-1) + 2 \times 0 \times \gamma(n)\right) \\
&= \sigma^2 + 2\sum_{\ell=1}^{n}\left(1 - \frac{\ell}{n}\right)\gamma(\ell)
\end{aligned}$$

Since $\gamma(-\ell) = \gamma(\ell)$, it is equivalent to write $var(S_n) = \sum_{\ell=-n}^{n} \left(1 - \frac{|\ell|}{n}\right) \gamma(\ell)$. If y_t is a vector time series, similarly

$$var(S_n) = \Gamma(0) + \sum_{\ell=1}^{n} \left(1 - \frac{\ell}{n}\right) \left(\gamma(\ell) + \gamma(\ell)'\right) = \sum_{\ell=-n}^{n} \left(1 - \frac{|\ell|}{n}\right) \gamma(\ell)$$

For any CLT to work, $var(S_n)$ must be convergent in the limit. The sum of the autocovariances is

$$\begin{aligned} \sum_{\ell=1}^{n} \left(1 - \frac{\ell}{n}\right) \gamma(\ell) &= \frac{1}{n} \sum_{\ell=1}^{n} (n - \ell) \gamma(\ell) \\ &= \frac{1}{n} \sum_{\ell=1}^{n-1} \sum_{j=1}^{\ell} \gamma(j) \to \sum_{j=1}^{\infty} \gamma(j) \end{aligned}$$

by the Theorem of Cesaro means if $\sum_{\ell=1}^{\infty} \gamma(\ell)$ is convergent.

A necessary condition for $\sum_{\ell=1}^{\infty} \gamma(\ell)$ to be convergent is that $\gamma(\ell) \to 0$ as $\ell \to \infty$; and a sufficient condition is Sufficient: $\sum_{\ell=1}^{\infty} |\gamma(\ell)| < \infty$.

Theorem 1.4. *(CLT) Suppose u_t is strictly stationarity with $E[u_t] = 0$, and its α-mixing coefficient satisfies $\sum_{\ell=0}^{\infty} \alpha(\ell)^{1-2/r} < \infty$ and $E\|u_t\|^r < \infty$ for some $\gamma > 2$. Then $S_n \xrightarrow{d} N(0, \Omega)$ where $\Omega = \sum_{\ell=-\infty}^{\infty} \Gamma(\ell)$ is the long-run variance.*

1.8 Linear Projection

- In regression problems, $\mathscr{P}(y \mid X) = X\beta^* = X' \left(E[XX']\right)^{-1} E[Xy]$
- Extend to a projection to the infinite past history $\tilde{y}_{t-1} = (y_{t-1}, y_{t-2}, ...)$

Denote $\mathscr{P}_{t-1}(y_t) = \mathscr{P}[y_t \mid \tilde{y}_{t-1}]$, and the projection error $e_t = y_t - \mathscr{P}_{t-1}(y_t)$

Theorem 1.5. *Projection Theorem:*
If $y_t \in \mathbb{R}$ is covariance stationarity, then the projection error statistics
(1) $E[e_t] = 0$
(2) $\sigma^2 = E\left[e_t^2\right] \leq E\left[y_t^2\right]$
(3) $E[e_t e_{t-j}] = 0$ for all $j \geq 1$.
*In other words, $\{e_t\}$ is a **white noise**.*

- If $\{y_t\}$ is strictly stationarity, then $\{e_t\}$ is strictly stationarity.

Definition 1.5. A time series is a white noise if it is covariance stationarity with 0 autocovariance.

It is helpful to imagine the projection as a linear combination

$$y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + ... + e_t$$

The nature of projection ensures e_t is uncorrelated with all regressions.
e_{t-j} is a linear combination $y_{t-j} - \alpha_1 y_{t-j-1} - \alpha_2 y_{t-j-2} - ....$
Then e_t is uncorrelated with e_{t-j}.

1.9 Wold Decomposition

- If y_t is covariance stationarity, and the linear projection error has $\sigma^2 > 0$, then $y_t = u_t + \sum_{j=0}^{\infty} b_j e_{t-j}$, $b_0 = 1$, and $u_t = \lim_{m\to\infty} \mathscr{P}_{t-m}(y_t)$

Project y_t onto the orthogonal elements $e_t, e_{t-1,} e_{t-2}, ...$ For simplicity, we can consider the case $\mu_t = \mu$.

Definition 1.6. Lag operator: $\mathrm{L}y_t = y_{t-1}$, $\mathrm{L}^2 y_t = \mathrm{L}(\mathrm{L}y_t) = \mathrm{L}y_{t-1} = y_{t-2}$, and so on.

$$
\begin{aligned}
y_t &= \mu + \sum_{j=0}^{\infty} b_j e_{t-j} \\
&= \mu + \left(b_0 + b_1 \mathrm{L} + b_2 \mathrm{L}^2 + ...\right) e_t \\
&= \mu + b(\mathrm{L}) e_t
\end{aligned}
$$

where $b(\mathrm{L})$ is an infinite-order polynomial.

- Autoregressive Wold Representation: If y_t is covariance stationarity with $y_t = u_t + b(\mathrm{L}) e_t$, then with some additional technical restrictions, $y_t = \mu + \sum_{j=1}^{\infty} a_j y_{t-j} + e_j$.

Zhentao Shi. March 21, 2024