Homework 11

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Questions 5 - 6
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Question 5:

a) Theorem: For any positive integer n, 3 divides $n^3 + 2n$

Proof:

Base Case:
$n = 1$
$n^3 + 2n = (1)^3 + 2(1) = 1 + 2 = 3$
3 is evenly divisible by 3. Base case is true.

Inductive Step:
For any $k \geq 2$, if $k^3 + 2k = 3m$, for some integer m, then $(k+1)^3 + 2(k+1) = 3m$.

$$\begin{aligned}
3m &= (k+1)^3 + 2(k+1) \\
&= k(k^2 + 2k + 1) + (k^2 + 2k + 1) + 2k + 2 && \text{Algebra} \\
&= k^3 + 3k^2 + 3k + 2k + 3 && \text{Algebra} \\
&= (k^3 + 2k) + 3k^2 + 3k + 3 && \text{Algebra} \\
&= (3j) + 3k^2 + 3k + 3 && \text{Inductive hypothesis} \\
&= 3(j + k^2 + k + 1) && \text{Algebra}
\end{aligned}$$

Since j and k are integers, then $(j+k^2+k+1)$ is also an integer, therfore $3(j+k^2+k+1)$ is evenly divisible by 3. ■

b)Theorem: For any positive integer $n \geq 2$, n can be written as a product of primes.

<u>Proof:</u>

Base Case:
$n = 2$
n can be written as $2 \cdot 1$, and is therefore a product of a prime number, as requested.

Inductive Step:
Assuming that $k \geq 2$, any integer j in the range from 2 through k can be expressed as a product of prime numbers. We will show that k + 1 can be expressed as a product of prime numbers.

We have two cases, which are when $(k + 1)$ is a prime number, and when $(k + 1)$ is a composite number. In the case that $(k + 1)$ is a prime number, then it is a product of one prime number, $(k + 1)$. If $k + 1$ is composite, it can be expressed as the product of two integers, a and b such that $a \geq 2$ and $b \geq 2$.

Now we need to show that both a and b fall within the range $[j, k]$Since $(k+1) = a \cdot b$, we can say that $a = \frac{(k+1)}{b}$. Since $b \geq 2$, we know that $a \leq k$ and is within the range $[j, k]$. The same can be said for b knowing that $b = \frac{(k+1)}{a}$ and $a \geq 2$. So, by the inductive hypothesis we can also say that a and b can be written as the product of a prime number, and therefore $(k + 1)$ can be written as the product of a prime number. ■

Question 6:

Exercise 7.4.1:

a)

$$P(n) = \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$
$$P(3) = (1)^2 \cdot (2)^2 \cdot (3)^2 = \frac{(3)(3+1)(2(3)+1)}{6}$$
$$(1) + (4) + (9) = \frac{(3)(4)(7)}{6}$$
$$14 = 14$$

P(3) is true.

b)

$$P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

c)

$$P(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

d)

What must be proven is that $n = 1$ is true.

e)

The inductive step is would state that for any positive integer k, if P(k) is true, then P(k+1) is also true.

f)

The inductive hypothesis in the inductive step from my previous answer is $P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$.

g)

Theorem: For every positive integer k, $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

Proof:

Base Case: $P(1) = (1)^2 = \frac{(1)((1)+1)(2(1)+1)}{6}$ $1 = \frac{1(2)(3)}{6} = \frac{6}{6}$ $1 = 1$ $n = 1$ is true.

Inductive Step: For every positive integer k, assuming $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$, we will show that $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$.

Preliminary calculation (simplifying):

$$\begin{aligned} \frac{(k+1)(k+2)(2k+3)}{6} &= \frac{(k)(2k^2+3k+4k+6)+(2k^2+3k+4k+6)}{6} \\ &= \frac{2k^3+3k^2+4k^2+6k+2k^2+3k+4k+6}{6} \\ &= \frac{2k^3+9k^2+13k+6}{6} \end{aligned}$$

$$\begin{aligned} &\text{1. Algebra} & \sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2 \\ &\text{2. Inductive Hypothesis} & &= (\frac{k(k+1)(2k+1)}{6}) + \frac{6((k+1)^2)}{6} \\ &\text{3. Algebra} & &= \frac{k(2k^2+k+2k+1)+6k^2+12k+6}{6} \\ &\text{4. Algebra} & &= \frac{2k^3+k^2+2k^2+k+6k^2+12k+6}{6} \\ &\text{5. Algebra} & &= \frac{2k^3+9k^2+13k+6}{6} \quad \blacksquare \end{aligned}$$

I simplified P(k+1) as extra insurance since I wouldn't have been able to factor out (k+1)(k+2) after everything is simplified.

Exercise 7.4.3:

c)

Theorem: For $n \geq 1$, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Proof:

Base Case:

$$\text{For} n = 1,\ \sum_{j=1}^{1} \frac{1}{j^2} = \frac{1}{(1)^2} \leq 2 - \frac{1}{(1)}$$

$$1 \leq 1$$

$n = 1$ is true.

Inductive Step:

Assuming $\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$ for any $k \geq 1$, we will show that $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$

$$\sum_{j=1}^{k+1} \frac{1}{j^2} = \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2} = (2 - \frac{1}{k}) + \frac{1}{(k+1)^2}$$

$$\text{Since } k \geq 1,\ \frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)} : \quad 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k)(k+1)}$$

$$(\sum_{j=1}^{k+1} \frac{1}{j^2}) \leq 2 - \frac{1}{k} + \frac{1}{(k)(k+1)}$$

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{(k+1)}{(k)(k+1)} + \frac{1}{(k)(k+1)}$$

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 + \frac{1-(k+1)}{(k)(k+1)}$$

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 + \frac{1-k-1}{(k)(k+1)}$$

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 + \frac{-k}{(k)(k+1)}$$

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 + \frac{-1}{(k+1)}$$

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$$

Therefore, $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$. $\blacksquare$

Exercise 7.5.1:

a)
Base Case: Inductive Step: Assuming that for any $n \geq 1$, $3^{2k} - 1 = 4m$ for some integer m, we will show that $3^{2(k+1)} - 1 = 4m$.

$$4m = 3^{2k} - 1$$
$$4m + 1 = 3^{2k}$$

$$\begin{aligned}
\text{1. Algebra} \quad & 3^{2(k+1)} - 1 = 3^{2k+2} - 1 \\
\text{2. Algebra} \quad & = 3^2 \cdot 3^{2k} - 1 \\
\text{3. Inductive Hypothesis} \quad & = 9 \cdot (4m + 1) - 1 \\
\text{4. Algebra} \quad & = 36m + 8 \\
\text{5. Algebra} \quad & = 4(9m + 2) \quad \blacksquare
\end{aligned}$$

Since m is an integer, $(9m+2)$ is also an integer. Since $3^{2(k+1)} - 1$ can be represented as the product of 4 and some integer, it is evenly divisible by 4.