Neural Networks, Vol. 6, pp. 1089–1094, 1993
Printed in the USA. 

0893-6080/93 \$6.00 + .00


# A Universal Mapping for Kolmogorov's Superposition Theorem

DAVID A. SPRECHER

University of California

(*Received* 13 *July* 1992; *revised and accepted* 10 *March* 1993)

**Abstract**—*Based on constructions of Kolmogorov and an earlier refinement of the author, we use a sequence of integrally independent positive numbers to construct a continuous function $\psi(x)$ having the following property: Every real-valued uniformly continuous function $f(x_1, \ldots, x_n)$ of $n \geq 2$ variables can be obtained as a superposition of continuous functions of one variable based on weighted sums of translates of the fixed function $\psi(x)$ that is independent of the number of variables n. From this is obtained a stronger version of the Hecht–Nielsen three-layer feedforward neural network for implementing $f(x_1, \ldots, x_n)$.*

**Keywords**—Superpositions, Kolmogorov, Representations of continuous functions of several variables, Feedforward neural networks, Processing elements, Weighted sums, Uniform continuity.

In what follows, $\mathcal{R}$ designates the real line, $\mathcal{E} = [0, 1]$, $\mathcal{D}_0 = [0, (1/5!)]$, $\mathcal{D}_1 = [0, 1 + (1/5!)]$, and $\mathcal{R}^n$ and $\mathcal{E}^n$ designate the respective $n$-fold Cartesian products of $\mathcal{R}$ and $\mathcal{E}$. Continuity and convergence are uniform in the Euclidean metric.

The Superposition Theorem of Kolmogorov (1957) establishes that for each integer $n \geq 2$ there are $n \times (2n + 1)$ continuous monotonically increasing functions $\psi_{pq}$ with the following property: For every real-valued continuous function $f: \mathcal{E}^n \to \mathcal{R}$ there are continuous functions $\Phi_q$ such that

$$f(x_1, \ldots, x_n) = \sum_{q=0}^{2n} \Phi_q\left[\sum_{p=1}^{n} \psi_{pq}(x_p)\right]. \tag{1}$$

In this paper, we established the existence of such superpositions with weighted sums utilizing a single continuous function $\psi$ that is independent of the number of variables $n$.

THEOREM 1. *Let $\{\lambda_k\}$ be a sequence of positive integrally independent numbers. There exists a continuous monotonically increasing function $\psi: \mathcal{D}_1 \to \mathcal{D}_1$ having the following property: For every real-valued continuous function $f: \mathcal{E}^n \to \mathcal{R}$ with $n \geq 2$ there are continuous functions $\Phi_q$ such that*

$$f(x_1, \ldots, x_n) = \sum_{q=0}^{2n} \Phi_q\left[\sum_{p=1}^{n} \lambda_p \psi(x_p + qa_n)\right], \tag{2}$$

*for a suitable constant $a_n$.*

The sequence $\{\lambda_k\}$ is **integrally independent** if $\sum t_p \lambda_p \neq 0$ for any finite selection of integers $t_p$ for which $\sum |t_p| \neq 0$. It is convenient to select $\lambda_1$ to be rational and $\lambda_k$ to be transcendental for $k > 1$. Theorem 1 modifies the cited theorem of Kolmogorov, and it also improves an earlier theorem of Sprecher (1965) in which the function $\psi$ depends on $n$. The constructions below are such that instead of constants $a_n$ depending on $n$ one could select a single constant for all values of $n$ by extending the definition of $\psi$ to the half-line $[0, \infty)$. Also, it is possible to replace the $2n + 1$ continuous functions $\Phi_q$ with a single continuous function $\Phi$ by adding a suitable constant to each of the arguments $\xi_q = \sum \lambda_p \psi[x_p + qa_n)]$ in formula (2).

With Theorem 1, the following somewhat stronger version of an existence theorem for neural networks (Hecht–Nielsen, 1987a, b) is obtained (see also Kůrková, 1992):

THEOREM 2. *Every continuous function $f: \mathcal{E}^n \to \mathcal{R}$ with $n \geq 2$ can be implemented by a three-layer Feedforward neural network having n fanout processing elements in the x-input layer, $2n + 1$ processing elements in the second layer, and $2n + 1$ processing elements in the output layer. Furthermore, the first and second layers utilize a fixed processing element that is independent of n, and only the output layer depends on f.*

The application of Theorem 1 to the implementation of a function $f: \mathcal{E}^n \to \mathcal{R}$ with a neural network is depicted in Figure 1. In the $x$-input layer, each of the input variables is distributed to $2n + 1$ processing elements that are translates of the fixed element $\psi$, and these are transferred to weighted sum elements in the middle layer. The $2n + 1$ processing elements $\Phi_q$ in the output layer are determined by the function $f$, and these are generally highly nonlinear.

Acknowledgement: I would like to express my appreciation to Hidefumi Katsuura for his contribution to an improved exposition and some details of construction of this paper.

Requests for reprints should be sent to David A. Sprecher, Department of Mathematics, University of California, Santa Barbara, CA 93106.

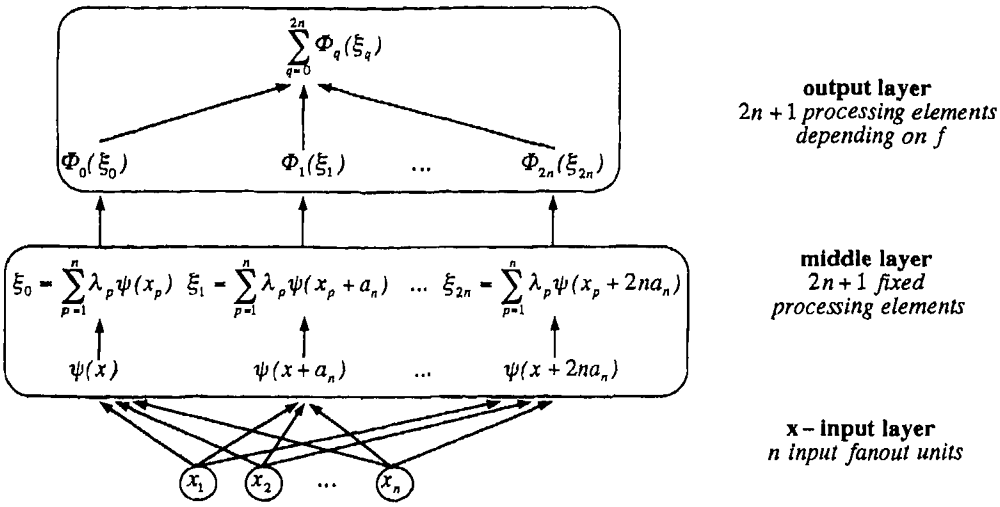


**FIGURE 1. Schematic representation of superposition formula (2).**

The technique developed by Kolmogorov for the proof of formula (1) utilizes $2n + 1$ systems of disjoint $n$-dimensional cubes of decreasing diameters covering $\mathscr{E}^n$ in a prescribed manner. These systems of cubes are obtained as the $n$-fold Cartesian products of systems of intervals constructed for a predetermined integer $n \geq 2$, and the construction of the functions $\psi_{pq}$ in Kolmogorov (1957) and the function $\psi$ in Sprecher (1963) are based on these intervals. The present paper follows a different procedure in which the function $\psi$ is constructed pointwise, and we then deduce the existence of an appropriate system of intervals with decreasing diameters and the following essential property: for any integer $n \geq 2$, we can generate from their $n$-fold Cartesian products $2n + 1$ families of disjoint cubes with the required properties. The function $\psi$ is such that the weighted sums $\xi_q = \sum \lambda_p \psi[x_p + qa_n)]$ map each of these families of cubes onto disjoint image intervals for $q = 0, 1, 2, \ldots, 2n$.

## CONSTRUCTION OF THE FUNCTION $\psi$

According to formula (2), $\psi$ must be defined over each interval $[0, 1 + 2na_n]$ for any integer $n \geq 2$. We prove the theorem for the specific constant $a_n = \sum_{r=2n+2}^{\infty} (1/r!)$ [see eqn (21)]. It is readily verified that $2na_n < (1/5!)$ for all integers $n \geq 2$, so that it suffices to define $\psi$ over the domain $\mathscr{D}_1$. We shall actually construct $\psi$ on the interval $\mathscr{D}_0$ only, and then extend its definition to $\mathscr{D}_1$ through the translations

$$\psi(x) = \psi\left(x - \frac{i}{5!}\right) + \frac{i}{5!}, \quad x \in \left[\frac{i}{5!}, \frac{i+1}{5!}\right], \quad i = 0, 1, 2, \ldots, 5! \quad (3)$$

along the diagonal of $\mathscr{D}_1$. The construction is carried out by induction on $k$ through a selection of an everywhere dense set of points $(i/k!, \psi(i/k!))$ on its graph. We begin with the value $k = 5$ because smaller values of $k$ are not useful in the proof of Theorem 1.

Let $\{\rho_k\}$ be a sequence of rational numbers such that

$$\rho_5 < \frac{1}{5!} \quad \text{and} \quad 0 < \rho_{k+1} < \frac{\rho_k}{k} \quad \text{for} \quad k \geq 5. \quad (4)$$

For $k = 5$ we set $\psi(0) = 0$ and $\psi(1/5!) = 1/5!$; for $k = 6$, we set $j = 6i + t$, where $t = 0, 1, 2, \ldots, 5$ when $i = 0$, and $t = 0$ when $i = 1$, and define

$$\psi\left(\frac{j}{6!}\right) = \psi\left(\frac{i}{5!} + \frac{t}{6!}\right) = \begin{cases} \psi\left(\frac{i}{5!}\right) + \frac{t}{5}\rho_5 & \text{for} \quad t = 0, 1, 2, 3, 4 \\ \frac{1}{2}\left[\psi\left(\frac{i}{6!}\right) + \psi\left(\frac{i+1}{6!}\right)\right] & \text{for} \quad t = 5 \end{cases}. \quad (5)$$

This gives the following values for $\psi(j/6!)$:

$$0, \frac{1}{5}\rho_5, \frac{2}{5}\rho_5, \frac{3}{5}\rho_5, \frac{4}{5}\rho_5, \frac{1}{2 \cdot 5!}, \frac{1}{5!}$$

(see Figure 2). Suppose that the values $\psi(i/(k-1)!)$ are already determined. Set $j = ki + t$, where $t = 0, 1, 2, \ldots, k - 1$ when $i = 0, 1, 2, \ldots, (k! - 1)/5!$, and $t = 0$ when $i = k!/5!$, and we define

$$\psi\left(\frac{j}{k!}\right) = \psi\left(\frac{i}{(k-1)!} + \frac{t}{k!}\right) = \begin{cases} \psi\left(\frac{i}{(k-1)!}\right) + \frac{t}{k-1}\rho_{k-1} \\ \quad \text{for} \quad t = 0, 1, 2, \ldots, k-2 \\ \frac{1}{2}\left\{\psi\left(\frac{i}{(k-1)!}\right) + \psi\left(\frac{i+1}{(k-1)!}\right)\right\} \\ \quad \text{for} \quad t = k-1 \end{cases}. \quad (6)$$

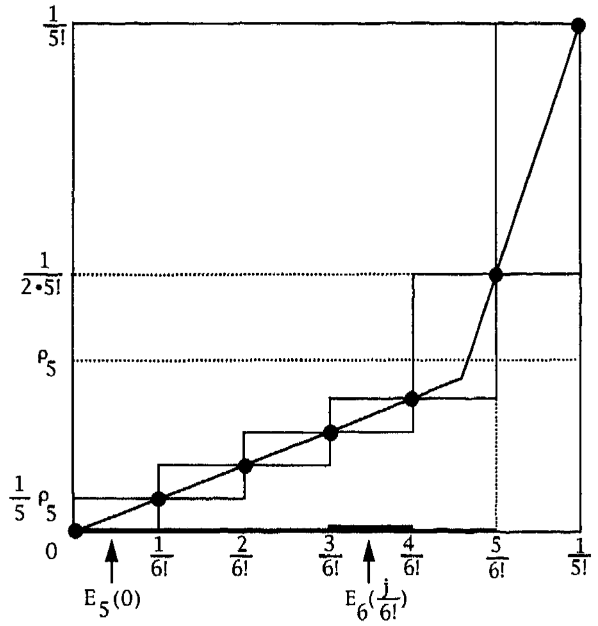


**FIGURE 2. The values of $\psi(j/6!)$ for $j = 0, 1, 2, \ldots, 6$.**

We observe that this construction can be described by means of a staircase pattern: a pattern similar to that depicted in Figure 2 is replicated for $k = 7, 8, 9, \ldots$ for each closed interval $[i/k!, (i+1)/k!]$.

It is clear from (3) and (6) that

$$\left|\psi\left(\frac{i+1}{k!}\right) - \psi\left(\frac{i}{k!}\right)\right| \leq \frac{1}{2^{k-5}5!}, \tag{7}$$

for $0 \leq i \leq (k! - 1)/5!$ and $k \geq 5$.

This inductive procedure therefore defines $\psi$ on an everywhere dense set of points in the interval $[0, 1/5!]$. The correspondence $i/k! \to \psi(i/k!)$ is monotonically increasing: if $i/k! < j/r!$ then $\psi(i/k!) < \psi(j/r!)$. From (7) we can therefore conclude that there is a unique continuous monotonically increasing function $\psi(x)$ with domain $\mathcal{D}_0$ whose graph contains the points $(i/k!, \psi(i/k!))$. With the help of (3) we extend this function to a continuous monotonic increasing function on the interval $\mathcal{D}_1$, and we designate also the extended function by $\psi$. Because the numbers $\rho_k$ are rational, it follows that the function values $\psi(i/k!)$ defined above are likewise rational for all integers $i \in I_k$.

The function $\psi$ has the following property that is essential in the proof of Theorem 1.

LEMMA 1. *Let*

$$\delta_k = \frac{1}{k!} - \sum_{r=k+1}^{\infty} \frac{1}{r!}, \tag{8}$$

$$\nu_k = \sum_{r=k}^{\infty} \frac{r-1}{r} \rho_r. \tag{9}$$

*Then*

$$\psi\left(\frac{i}{k!} + \delta_k\right) = \psi\left(\frac{i}{k!}\right) + \nu_k. \tag{10}$$

*Proof.* For an arbitrary fixed value $i/k!$ and integer $N \geq k$ consider the point

$$\xi_N = \frac{i}{k!} + \sum_{r=k+1}^{N} \frac{r-2}{r!}. \tag{11}$$

An iteration of (6) shows that the value $\psi(\xi_N)$ corresponding to this point is

$$\begin{aligned}\psi(\xi_N) &= \psi\left(\frac{i}{k!}\right) + \sum_{r=k+1}^{N} \frac{r-2}{r-1} \rho_{r-1} \\ &= \psi\left(\frac{i}{k!}\right) + \sum_{r=k}^{N-1} \frac{r-1}{r} \rho_r\end{aligned} \tag{12}$$

(see also Figure 2). A direct calculation gives

$$\sum_{r=k+1}^{N} \frac{r-2}{r!} = \frac{1}{k!} - \sum_{r=k+1}^{N} \frac{1}{r!} - \frac{1}{N!}$$

and because all the series in the above equations converge absolutely, we have

$$\lim_{N\to\infty} \sum_{r=k+1}^{N} \frac{r-2}{r!} = \frac{1}{k!} - \lim_{N\to\infty} \left(\sum_{r=k+1}^{N} \frac{1}{r!} - \frac{1}{N!}\right) = \delta_k$$

so that (11) gives

$$\lim_{N\to\infty} \xi_N = \frac{i}{k!} + \delta_k,$$

and (12) gives

$$\lim_{N\to\infty} \psi(\xi_N) = \psi\left(\frac{i}{k!}\right) + \lim_{N\to\infty} \sum_{r=k}^{N-1} \frac{r-1}{r} \rho_r = \psi\left(\frac{i}{k!}\right) + \nu_k.$$

This concludes the proof of the Lemma. ∎

For each positive integer $k \geq 5$, consider now the system of closed intervals

$$E_k(i) = \left[\frac{i}{k!}, \frac{i}{k!} + \delta_k\right], \quad i \in I_k. \tag{13}$$

For each value of $k$, these intervals are separated by gaps of diameter

$$e_k = \frac{1}{k!} - \delta_k = \sum_{r=k+1}^{\infty} \frac{1}{r!}, \tag{14}$$

and a direct calculation shows that these are related to the diameters of the intervals $E_k(i)$ through the inequality

$$\delta_k > (k-1)e_k. \tag{15}$$

Except for these gaps, however, the intervals $E_k(i)$ cover the interval $\mathcal{D}_1$. It is readily verified that

$$E_k(j) \subset E_{k-1}(i) \quad \text{iff} \quad j = ki + t, \quad t = 0, 1, 2, \ldots, k-2. \tag{16}$$

Furthermore, when $j$ and $i$ are related as in (16), the initial points of $E_k(j)$ and $E_{k-1}(i)$ coincide when $t = 0$, and their endpoints coincide when $t = k - 2$. When $t = k - 1$, $E_k(j)$ lies in the gap separating the intervals $E_{k-1}(i)$ and $E_{k-1}(i+1)$ (see Figure 3).

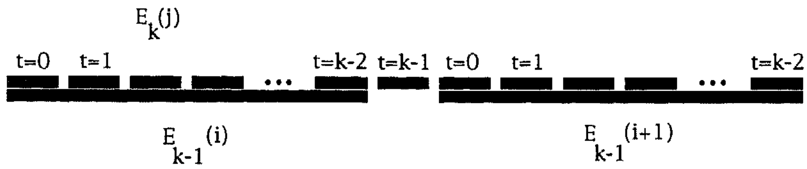


**FIGURE 3. The intersection property of the intervals $E_k(j)$ and $E_{k-1}(i)$ when $j = ki + t$ and $t = 0, 1, 2, \ldots, k-1$.**

We can formally express the following relationship between the intervals (13) and their image intervals under $\psi$:

$$\psi\left[\frac{i}{k!}, \frac{i}{k!} + \delta_k\right] = \left[\psi\left(\frac{i}{k!}\right), \psi\left(\frac{i}{k!}\right) + \nu_k\right]. \qquad (17)$$

The monotonicity of the function $\psi$ guarantees that the images of the intervals $E_k(i)$ are pairwise disjoint for each value $k \geq 5$. In the proof of Theorem 1, however, the weighted sums are required to map also the $n$-fold Cartesian products of the intervals $E_k(i)$ onto disjoint image intervals. This imposes a more stringent condition than that in (4) on the selection of the rational numbers $\{\rho_k\}$ used in the construction, and this is the subject of the following.

LEMMA 2. *Let $\{\lambda_k\}$, $\lambda_k > 0$, be a sequence of integrally independent numbers. For each integer $k \geq 5$, let*

$$\mu_k\sigma_k = \min_{i_p, j_p \in I_k} \left| \sum_{p=1}^{k-3} \lambda_p \left[\psi\left(\frac{i_p}{k!}\right) - \psi\left(\frac{j_p}{k!}\right)\right]\right|, \quad \sum_{p=1}^{k-3} |i_p - j_p| \neq 0 \qquad (18)$$

*where*

$$\sigma_k = 1 + \sum_{p=1}^{k-3} \lambda_p. \qquad (19)$$

*It is possible to select the rational numbers $\rho_k$ in (4) such that $\nu_k < \mu_k$.*

*Proof.* We begin with the observation that because the numbers $\psi(i/k!)$ are rational, we are assured that $\mu_k \neq 0$ for all values of $k$. It is clear that the constructions in Lemma 1 remain valid if the rational numbers $\rho_k$ are replaced with rational numbers $0 < \tilde{\rho}_k < \rho_k$. Since $\rho_k < \rho_{k-1}/(k-1)$ according to (4), it is sufficient to show that rational numbers $\tilde{\rho}_k$ can be selected to satisfy the inequalities $0 < \tilde{\rho}_k < \min[\mu_k, \tilde{\rho}_{k-1}/(k-1)]$. This is accomplished with a boot-strapping procedure, as follows: Using the values $\psi(i/5!) = i/5!$ for $i \in I_5$ we define

$$\mu_5\sigma_5 = \min_{i_p, j_p \in I_5} \left| \sum_{p=1}^{2} \lambda_p \left[\psi\left(\frac{i_p}{5!}\right) - \psi\left(\frac{j_p}{5!}\right)\right]\right|, \quad \sum_{p=1}^{2} |i_p - j_p| \neq 0$$

where $\sigma_5 = 1 + \lambda_1 + \lambda_2$ and select a rational number $0 < \tilde{\rho}_5 < \mu_5$. Using this value in (5), we define

$$\mu_6\sigma_6 = \min_{i_p, j_p \in I_6} \left| \sum_{p=1}^{3} \lambda_p \left[\psi\left(\frac{i_p}{6!}\right) - \psi\left(\frac{j_p}{6!}\right)\right]\right|, \quad \sum_{p=1}^{3} |i_p - j_p| \neq 0,$$

where $\sigma_6 = 1 + \lambda_1 + \lambda_2 + \lambda_3$, and select a rational number $0 < \tilde{\rho}_6 < \min(\mu_6, \tilde{\rho}_5/5)$. Assuming that $\psi(i/(k-1)!)$ and $\rho_{k-1}$ are already determined, we use these values to define $\psi(i/k!)$ in (6), and then proceed to define (18) for $i \in I_k$. We follow this with the selection of a rational number $0 < \tilde{\rho}_k < \min[\mu_k, \tilde{\rho}_{k-1}/(k-1)]$. ■

We may now assume that the numbers $\rho_k$ in Lemma 1 satisfy the conditions of Lemma 2, and this completes the construction of $\psi$.

## PROOF OF THEOREM 1

We observe in passing that the intervals $[\psi(i/k!), \psi(i/k!) + \nu_k]$ are pairwise disjoint for fixed $k$, and nested for increasing $k$ whenever the intervals $E_k(i)$ are nested. Because of the one-to-one correspondence $E_k(i) \leftrightarrow [\psi(i/k!), \psi(i/k!) + \nu_k]$, there corresponds to every nonempty intersection of intervals $E_k(i)$ a unique nonempty intersection of intervals $[\psi(i/k!), \psi(i/k!) + \nu_k]$.

Now let an integer $n \geq 2$ be given. For each integer $q = 0, 1, 2, \ldots, 2n$, we define systems of intervals

$$E_k^q(i) = \left[\frac{i}{k!} - qe_{2n+1}, \frac{i}{k!} + \delta_k - qe_{2n+1}\right] \qquad (20)$$

where $i \in I_k$ and $k \geq 5$ (see Figure 4). These intervals are translates of the intervals $E_k(i)$ through distances $-qe_{2n+1}$, and they have the property that for each value of $q$ and $k$ there are integers $i'$ and $i''$ such that $0 \in E_k^q(i')$ and $1 + (1/5!) \in E_k^q(i'')$. Whereas in the construction of $\psi$ the range of the index $i$ depends only on $k$, it is clear that now it depends on $q$ as well. These ranges are easily computed, however, and we shall assume that they are always appropriately chosen. We observe that definition (20) now specifies that the constant $a_n$ appearing in formula (2) has the value

$$a_n = e_{2n+1} = \sum_{r=2n+2}^{\infty} \frac{1}{r!}. \qquad (21)$$

LEMMA 3. *Let an integer $n \geq 2$ be given. Then for each positive integer $k \geq 2n+1$ and $q = 0, 1, 2, \ldots, 2n$ we have*

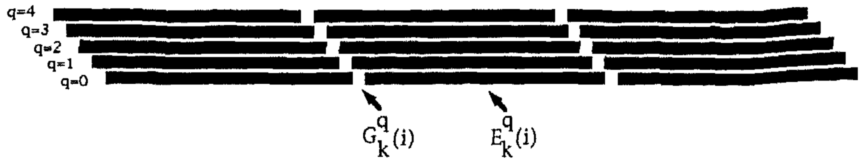


**FIGURE 4. The intervals $E_k^q(i)$ and gaps $G_k^q(i)$ for variable $q$.**

$$\mathrm{diam}[E_k^q(i)] \to 0 \quad as \quad k \to \infty \tag{22}$$

$$E_k^q(i) \cap E_k^q(j) = 0 \quad when \quad i \neq j \tag{23}$$

*For each point $x \in \mathcal{D}_1$ there are at least $2n$ values of $q$*

$$\text{such that } x \in E_k^q(i). \tag{24}$$

*Proof.* Essentially, this Lemma is a direct consequence of the properties of the intervals $E_k(i)$ and assertion (24) is the only one requiring some justification. We note that the intervals $E_k^q(i)$ are separated by open gaps

$$G_k^q(i) = \left(\frac{i}{k!} + \delta_k - qe_{2n+1}, \frac{i+1}{k!} - qe_{2n+1}\right) \tag{25}$$

(see Figure 4). Consider gaps $G_k^q(i')$ and $G_k^{q+1}(i'')$. A direct calculation shows that the initial point of $G_k^q(i')$ coincides with the terminal point of $G_k^{q+1}(i'')$ when the integral indices $i'$ and $i''$ are related through the equation $i' - i'' = k!(e_k - e_{2n+1}) - 1$. Hence, because of (14) and (15), none of the gaps $G_k^q(i)$ intersect for $q = 0, 1, 2, \ldots, 2n$ and fixed $k$. It follows that any point $x \in \mathcal{D}_1$ can belong to at most one gap $G_k^q(i)$, and the assertion follows from the fact that $x$ must be contained in an interval $E_k^q(i)$ or a gap $G_k^q(i)$ for each value of $q$. ∎

LEMMA 4. *Let an integer $n \geq 2$ be given, and let $k \geq 2n + 1$ and $q = 0, 1, 2, \ldots, 2n$. For each integer $1 \leq p \leq n$, let the families $\{E_k^q(i_p)\}$ be laid on the pth coordinate axis of the unit cube $\mathcal{E}^n$. Corresponding to the intervals $E_k^q(i_p)$, consider the closed intervals*

$$T_k^q(i_1, \ldots, i_n) = \left\{\sum_{p=1}^{n} \lambda_p \psi\left(\frac{i_p}{k!} + qe_{2n+1}\right), \right.$$
$$\left. \sum_{p=1}^{n} \lambda_p \left[\psi\left(\frac{i_p}{k!} + qe_{2n+1}\right) + \nu_k\right]\right\}. \tag{26}$$

*Then*

$$\mathrm{diam}[T_k^q(i_1, \ldots, i_n)] \to 0 \quad as \quad k \to \infty \tag{27}$$

$$T_k^q(i_1, \ldots, i_n) \cap T_k^q(j_1, \ldots, j_n) = 0 \quad when$$
$$(i_1, \ldots, i_n) \neq (j_1, \ldots, j_n) \tag{28}$$

*Furthermore, let*

$$\xi_q = \sum_{p=1}^{n} \lambda_p \psi(x_p + qe_{2n+1}). \tag{29}$$

*Then for any point $(x_1, \ldots, x_n) \in \mathcal{E}^n$ there are at least $n + 1$ values of $q$ for which $\xi_q \in T_k^q(i_1, \ldots, i_n)$.*

*Proof.* Assertion (27) follows at once from definition (26). The validity of assertion (28) is seen from the following: The intervals $E_k(i)$ and $E_k^q(i)$ are related through the translation mappings $y^q = x + qe_{2n+1}$ so that $x \in E_k(i)$ iff $y^q \in E_k^q(i)$. Therefore, $\mu_k$ in (18) remains unchanged when we respectively replace $\psi(x)$ and $E_k(i)$ by $\psi(x + qe_{2n+1})$ and $E_k^q(i)$. According to Lemma 2, $\nu_k < \mu_k$, and using (26) we see that

$$\begin{aligned}
&\mathrm{diam}[T_k^q(i_1, \ldots, i_n)] \\
&= \nu_k \sum_{p=1}^{k-3} \lambda_p < \mu_k \sum_{p=1}^{k-3} \lambda_p \\
&= \frac{1}{\sigma_k}\left(\sum_{p=1}^{k-3} \lambda_p\right) \times \min_{i_p, j_p \in I_k} \left|\sum_{p=1}^{k-3} \lambda_p \left[\psi\left(\frac{i_p}{k!}\right) - \psi\left(\frac{j_p}{k!}\right)\right]\right| \\
&< \min_{i_p, j_p \in I_k} \left|\sum_{p=1}^{k-3} \lambda_p \left[\psi\left(\frac{i_p}{k!}\right) - \psi\left(\frac{j_p}{k!}\right)\right]\right|.
\end{aligned}$$

The last assertion of the lemma is an immediate consequence of Lemma 3: Let an integer $k$ and a point $(x_1, \ldots, x_n) \in \mathcal{E}^n$ be given. According to assertion (24) in Lemma 3, there is at most one value of $q$ for each value of $p$ for which $x_p \notin E_k^q(i_p)$ for some integer $i_p$. Therefore, there are at most $n$ values of $q$ for which $\xi_q \notin T_k^q(i_1, \ldots, i_n)$, where $\xi_q$ is related to $(x_1, \ldots, x_n)$ through (29). Hence, $\xi_q \in T_k^q(i_1, \ldots, i_n)$ for at least $n + 1$ values of $q$, as was to be shown. ∎

We now note that with the formal introduction of the Cartesian products

$$S_k^q(i_1, \ldots, i_n) = \prod_{p=1}^{n} E_k^q(i_p), \tag{30}$$

the proof of Theorem 1 can be completed with a direct application of the arguments in Kolmogorov (1957) or Sprecher (1963) to these families of cubes and their corresponding intervals in Lemma 4, as these are sufficient for the derivation of formula (2).

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## NOMENCLATURE

$\mathcal{R}$ real line
$\mathcal{R}^n$ $n$-dimensional Euclidean space
$\mathcal{E}$ interval $[0, 1]$
$\mathcal{E}^n$ $n$-dimensional unit cube
$\mathcal{D}_0$ interval $[0, (1/5!)]$
$\mathcal{D}_1$ interval $[0, 1 + (1/5!)]$
$I_k$ set of integers $0 \le i \le [1 + (1/5!)]k!$