Quantum Computation for High-Energy Physics

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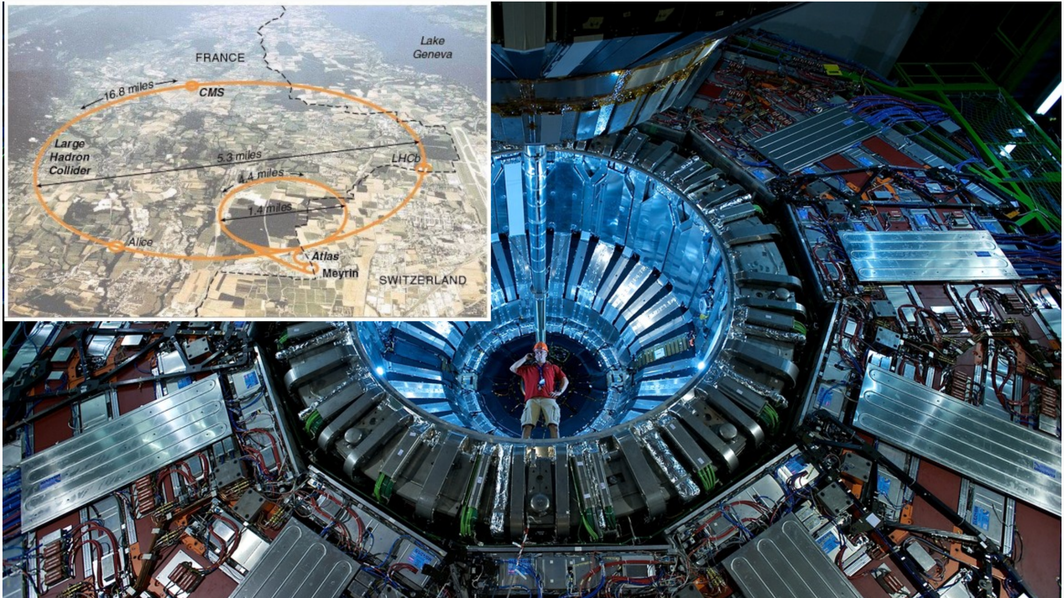
Introduction

High-energy physics

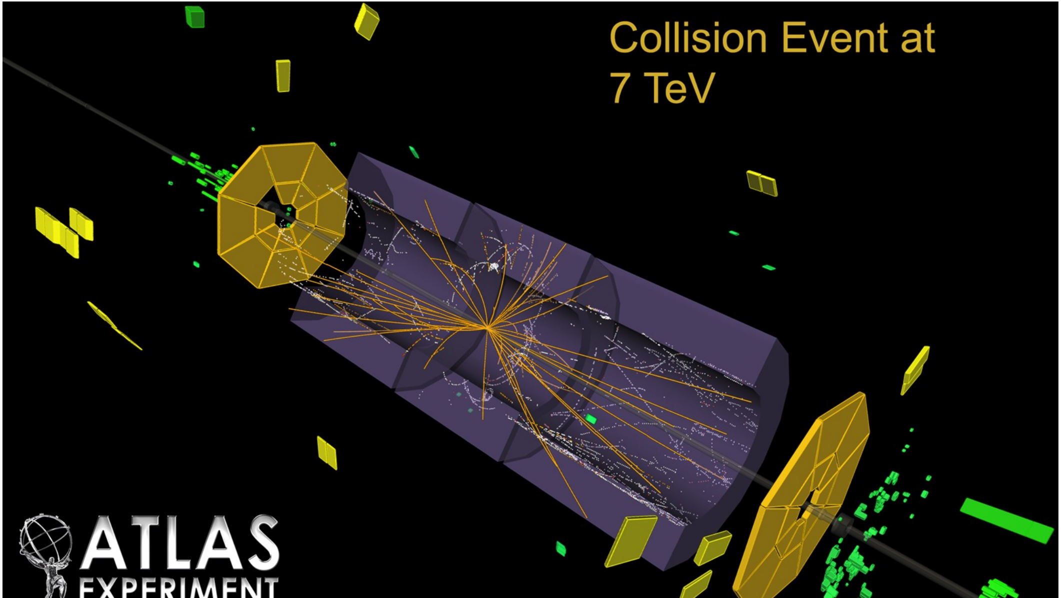
What it's all about:

1. Accelerate particles to 0.99999c
2. Smash particles together
3. Watch debris fly off in all directions
4. Deduce fundamental laws of physics

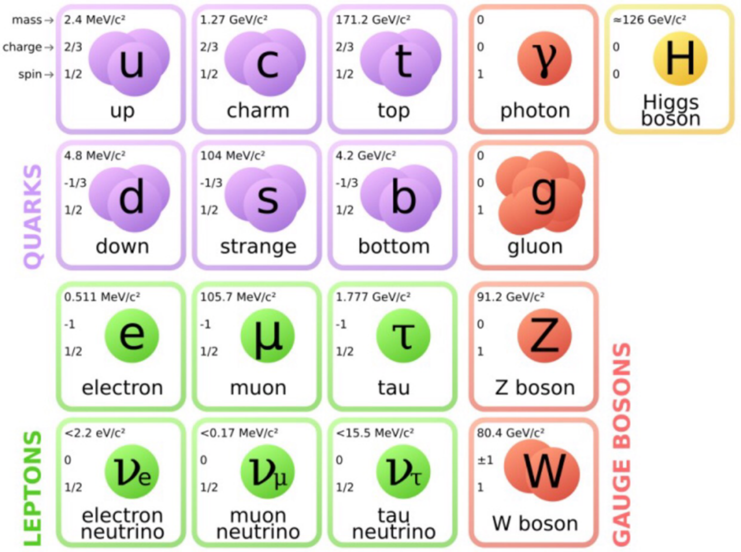
High-energy physics



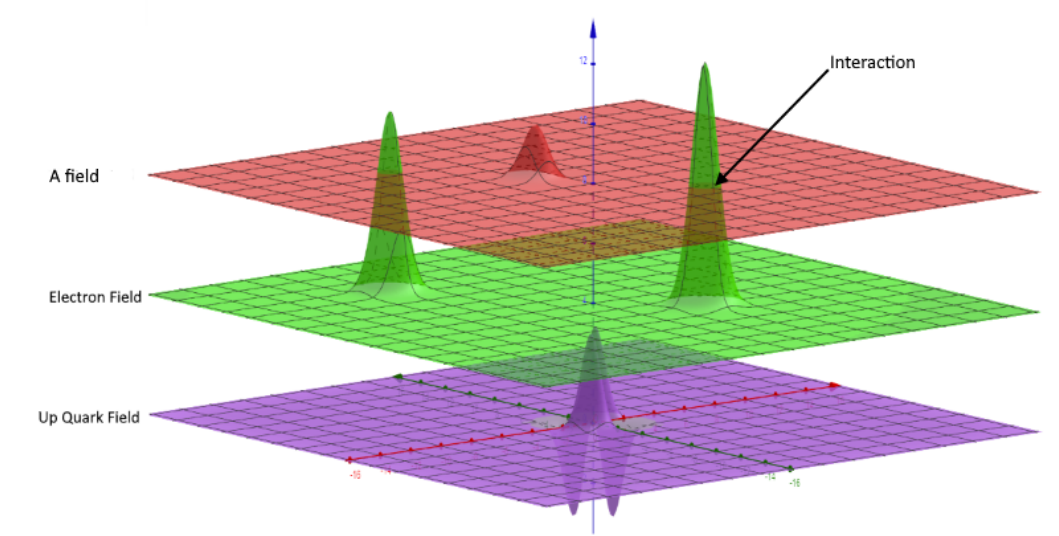
High-energy physics



High-energy physics



The Standard Model is a quantum field theory



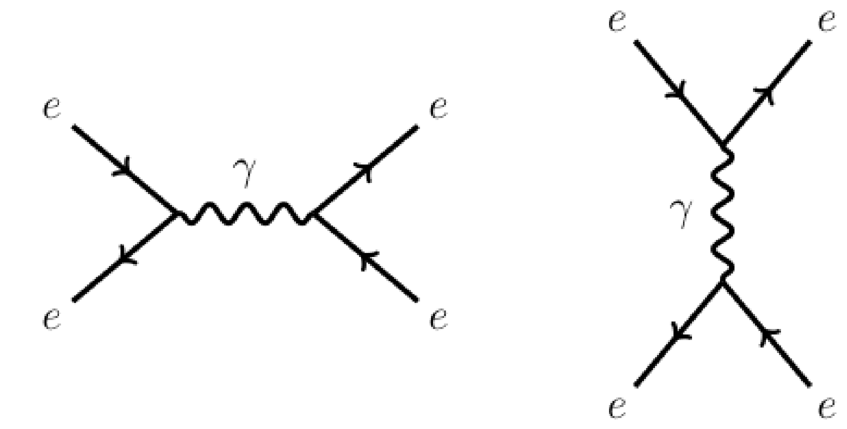
Quantum Field Theory

Scattering predictions are made by successive approximations (perturbation theory)

- Fix a physical process (i.e. incoming + outgoing particles)
- Draw all diagrams connecting incoming to outgoing particles
- Associate an equation to each diagram
- Sum up all contributions
- Compare with experimental data[1]

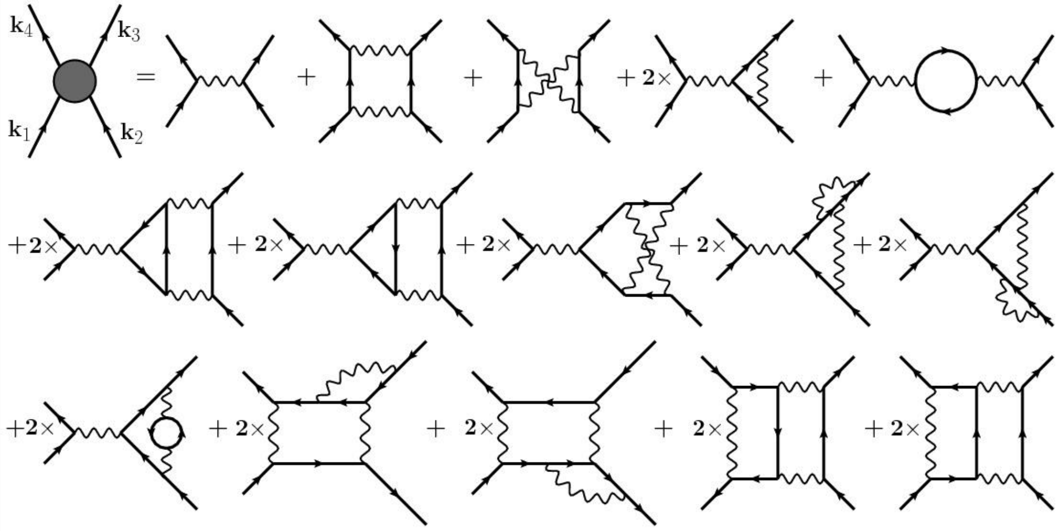
[1] After some more complicated maths...

Bhabha scattering ($e^+e^- \longrightarrow e^+e^-$)



$$\mathcal{M} = \quad e^2\bar{v}_k\gamma^\nu u_p \frac{1}{(k+p)^2}\bar{u}_{p'}\gamma_\nu v_{k'} \quad - e^2\bar{v}_k\gamma^\mu v_{k'}\frac{1}{(k-k')^2}\bar{u}_{p'}\gamma_\mu u_p$$

Want more accuracy? Add more diagrams!



Problem:

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Making accurate high-energy physics predictions is *extremely inefficient*.

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Making accurate high-energy physics predictions is *extremely inefficient*.

Solution: let a quantum computer handle it!

Quantum Simulation of Quantum Field Theory

Discretise space and introduce periodic boundary conditions:

- Lattice spacing a
- Field period $L = \ell a$

$$\phi(\mathsf{x} + L\mathsf{e}_j) = \phi(\mathsf{x})$$

- Fundamental cell Ω (with $N = \ell^d$ lattice sites)

Discretise space and introduce periodic boundary conditions:

- Lattice spacing a
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- Fundamental cell Ω (with $N = \ell^d$ lattice sites)
- Dual lattice spacing $\kappa = 2\pi/a$
- Dual fundamental cell Γ

Lattice Hamiltonian:

$$H = a^d \sum_{\mathbf{x}\in\Omega} \left[\frac{1}{2}\pi(\mathbf{x})^2 + \frac{1}{2}D_a\phi(\mathbf{x})^2 + \frac{1}{2}m^2\phi(\mathbf{x})^2 + \frac{\lambda}{4!}\phi(\mathbf{x})^4\right]$$

$$\pi(\mathbf{x}) = \partial_t\phi(\mathbf{x}), \qquad D_a\phi(\mathbf{x})^2 = \sum_{j=1}^{d} \frac{(\phi(\mathbf{x}+a\mathbf{e}_j) - \phi(\mathbf{x}))^2}{a^2}.$$

Canonical quantisation!

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = \frac{i}{a^d}\delta(\mathbf{x}-\mathbf{y})$$

Field Representation

At any given time, the state of the quantum field can be written

$$|\Psi\rangle = \int \Psi(\phi_1, \ldots, \phi_N)\, |\phi_1, \ldots, \phi_N\rangle \,\mathrm{d}^N\phi\,,$$

where we have introduced the field eigenstates

$$\phi(\mathbf{x})\, |\phi_1, \ldots, \phi_N\rangle = \phi_i\, |\phi_1, \ldots, \phi_N\rangle\,,$$

and the lattice sites are labeled in the lexicographic order

$$\mathbf{x} \in \Omega \longleftrightarrow i = 1, \ldots, N$$

There's also the closely related *conjugate* field eigenstates

$$\pi(\mathsf{x}) \left|\pi_1, \ldots, \pi_N\right\rangle = \pi_i \left|\pi_1, \ldots, \pi_N\right\rangle ,$$

We can pass from one to the other via Fourier transform

$$\left|\pi_1, \ldots, \pi_N\right\rangle = \frac{1}{(2\pi)^{\frac{N}{2}}} \int e^{-i(\phi_1\pi_1 + \cdots + \phi_N\pi_N)} \left|\phi_1, \ldots, \phi_N\right\rangle \mathrm{d}^N\phi \,,$$

$$\left|\phi_1, \ldots, \phi_N\right\rangle = \frac{1}{(2\pi)^{\frac{N}{2}}} \int e^{i(\phi_1\pi_1 + \cdots + \phi_N\pi_N)} \left|\pi_1, \ldots, \pi_N\right\rangle \mathrm{d}^N\pi \,.$$

Algorithm Outline

We are going to allocate m qubits per lattice site to represent the eigenstate $|\phi_i\rangle$. The algorithm is comprised of four stages:

1. Building the vacuum state of the free theory ($\lambda = 0$)
2. Building the discretised wavepacket
3. Simulating time evolution
4. Simulating a scattering measurement (energy/momenta)

Quantum Fourier Transform

The m-qubit Quantum Fourier Transform is uniquely determined by its action on the computational basis

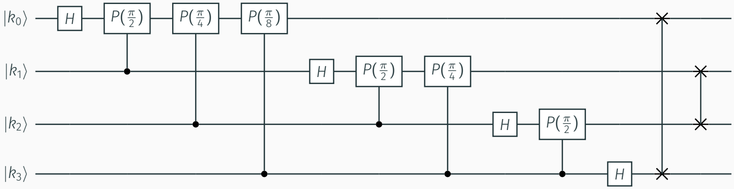
$$\mathcal{F}_m |k\rangle = \frac{1}{\sqrt{2^m}} \sum_{h=0}^{2^m-1} e^{-\frac{2\pi i}{2^m} kh} |h\rangle ,$$

with its inverse transformation being simply given by

$$\mathcal{F}_m^{-1} |k\rangle = \frac{1}{\sqrt{2^m}} \sum_{h=0}^{2^m-1} e^{\frac{2\pi i}{2^m} kh} |h\rangle .$$

This subroutine is used everywhere throughout the simulation.

4-qubit circuit implementation:



Here $|k\rangle = |k_0\rangle\,|k_1\rangle\,|k_2\rangle\,|k_3\rangle$ is the binary decomposition of k.

Time Evolution

Trotter formula

If the lattice Hamiltonian didn't have the first term...

$$H_\phi = a^d \sum_{\mathrm{x}\in\Omega} \left[\frac{1}{2} D_a \phi(\mathrm{x})^2 + \frac{1}{2} m^2 \phi(\mathrm{x})^2 + \frac{\lambda}{4!}\phi(\mathrm{x})^4\right]$$

...time evolution would be quite simple!

$$e^{-iH_\phi t} |\Psi\rangle = \int \Psi(\phi_1, \ldots, \phi_N) e^{-i\Theta(\phi_1,\ldots,\phi_N)t} |\phi_1, \ldots, \phi_N\rangle \, \mathrm{d}^N\phi\,,$$

where the phase function reads

$$\Theta(\phi_1, \ldots, \phi_N) = a^d \sum_{i=1}^{N} \left[\frac{1}{2} \sum_{j=1}^{d} \frac{(\phi_{i+\ell^{j-1}} - \phi_i)^2}{a^2} + \frac{1}{2} m^2 \phi_i^2 + \frac{\lambda}{4!}\phi_i^4\right].$$

Trotter formula

As for the remaining Hamiltonian term...

$$H_\pi = a^d \sum_{\mathrm{x}\in\Omega} \frac{1}{2}\pi(\mathrm{x})^2$$

...it acts nicely on conjugate field eigenstates

$$e^{-iH_\pi t}\left|\pi_1,\ldots,\pi_N\right\rangle = e^{-i\Phi(\pi_1,\ldots,\pi_N)t}\left|\pi_1,\ldots,\pi_N\right\rangle,$$

where this time the phase function is given by

$$\Phi(\pi_1,\ldots,\pi_N) = a^d \sum_{i=1}^{N} \frac{1}{2}\pi_i^2.$$

Trotter formula

For a general field state, first apply a Fourier transform

$$|\Psi\rangle = \frac{1}{(2\pi)^{\frac{N}{2}}} \int \Psi(\phi_1, \ldots, \phi_N) e^{i(\phi_1 \pi_1 + \cdots + \phi_N \pi_N)} |\pi_1, \ldots, \pi_N\rangle \, \mathrm{d}^N\phi \, \mathrm{d}^N\pi \, .$$

Now time evolution is easy!

$$e^{-iH_\pi t} |\Psi\rangle = \frac{1}{(2\pi)^{\frac{N}{2}}} \int \Psi(\phi_1, \ldots, \phi_N) e^{i(\phi_1 \pi_1 + \cdots + \phi_N \pi_N)} \times \\ \times e^{-i\Phi(\pi_1, \ldots, \pi_N)t} |\pi_1, \ldots, \pi_N\rangle \, \mathrm{d}^N\phi \, \mathrm{d}^N\pi \, ,$$

and then we Fourier transform back

$$e^{-iH_\pi t} |\Psi\rangle = \frac{1}{(2\pi)^N} \int \Psi(\varphi_1, \ldots, \varphi_N) e^{i(\varphi_1 \pi_1 + \cdots + \varphi_N \pi_N - \Phi(\pi_1, \ldots, \pi_N)t)} \times \\ \times e^{-i(\phi_1 \pi_1 + \cdots + \phi_N \pi_N)} |\phi_1, \ldots, \phi_N\rangle \, \mathrm{d}^N\phi \, \mathrm{d}^N\pi \, \mathrm{d}^N\varphi \, .$$

Trotter formula

Unfortunately

$$e^{-iH_\phi t}e^{-iH_\pi t} \neq e^{-i(H_\phi + H_\pi)t},$$

but we can use the Trotter-Suzuki formula!

$$\lim_{n\to\infty} (e^{-iH_\phi t/n}e^{-iH_\pi t/n})^n = e^{-i(H_\phi + H_\pi)t}$$

With a sufficiently large number of steps n, we can approximate e^{-iHt} with arbitrary precision.

Phase Kick-back

In addition to m qubits per lattice site, let's allocate an extra register

$$|\phi_1, \dots, \phi_N\rangle \, |k\rangle$$

Suppose we have a quantum quantum gate U_f such that

$$U_f \, |\phi_1, \dots, \phi_N\rangle \, |k\rangle = |\phi_1, \dots, \phi_N\rangle \, |k + f(\phi_1, \dots, \phi_N) \bmod 2^m\rangle$$

for a given function $f(\phi_1, \dots, \phi_N)$. We'll worry about how to construct it later...

Phase Kick-back

If we prepare the extra register in the equal weight superposition

$$|F_m\rangle = \frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} e^{-\frac{2\pi i}{2^m}k} |k\rangle ,$$

Then U_f 'kicks back' $f(\phi_1, \ldots, \phi_N)$ into the phase

$$U_f |\phi_1, \ldots, \phi_N\rangle |F_m\rangle = e^{-\frac{2\pi i}{2^m}f(\phi_1,\ldots,\phi_N)} |\phi_1, \ldots, \phi_N\rangle |F_m\rangle$$

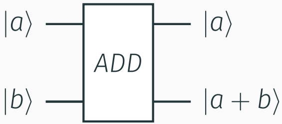
We can efficiently prepare $|F_m\rangle$ with the Quantum Fourier Transform

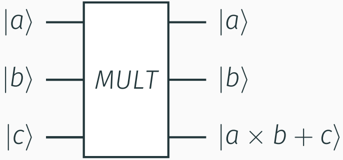
$$\mathcal{F}_m |1\rangle = \frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} e^{-\frac{2\pi i}{2^m}k} |k\rangle .$$

Quantum Arithmetic

The functions $\Theta(\phi_1, \ldots, \phi_N)$ and $\Phi(\pi_1, \ldots, \pi_N)$ are polynomials.

We just need to build the modular arithmetic gates





These too can be efficiently

implemented using the Quantum Fourier Transform!

Measurement Simulation

Ladder operator

Introducing the dispersion relation

$$\omega(\mathbf{p}) = \sqrt{m^2 + \frac{4}{a^2}\sum_{j=1}^{d}\sin^2\left(\frac{\mathbf{p}\cdot\mathbf{e}_j}{2}\right)},$$

we can define the ladder operators

$$a_{\mathbf{p}} = a^d \sum_{\mathbf{x}\in\Omega} e^{-i\mathbf{p}\cdot\mathbf{x}}\left[\sqrt{\frac{\omega(\mathbf{p})}{2}}\phi(\mathbf{x}) + i\sqrt{\frac{1}{2\omega(\mathbf{p})}}\pi(\mathbf{x})\right],$$

$$a_{\mathbf{p}}^{\dagger} = a^d \sum_{\mathbf{x}\in\Omega} e^{i\mathbf{p}\cdot\mathbf{x}}\left[\sqrt{\frac{\omega(\mathbf{p})}{2}}\phi(\mathbf{x}) - i\sqrt{\frac{1}{2\omega(\mathbf{p})}}\pi(\mathbf{x})\right].$$

Momentum Occupation Numbers

They obey the commutation relations

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}] = L^{d}\delta(\mathbf{p} - \mathbf{q}),$$

it follows the positive operators

$$N_{\mathbf{p}} = \frac{1}{L^{d}} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}},$$

have integer eigenvalues $n_{\mathbf{p}} = 0, 1, 2, 3, \ldots$ which count the number of particles with momentum $\mathbf{p}$.