Where Does the "Rule of 70" Come From?

ECON 101

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You hopefully know from class that for a variable (say, real GDP) that grows at a constant rate g every period, the approximate time it takes for that variable to double in size is

$$T_2 \approx \frac{70}{g}, \tag{1}$$

where g is expressed in percent form. This is the so-called "Rule of 70."

For example, if real GDP grows at 5% each year in country A, then country A's real GDP would double in approximately $70/5 = 14$ years.

Here, we will see where this "rule" comes from.

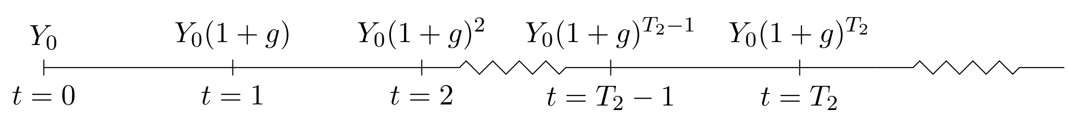
Compound Growth

To make life easy, let's focus on the growth of real GDP, Y. We start at time $t = 0$ with some stock of real GDP, Y_0. Real GDP grows at rate g each period and we want to know how long it takes real GDP to double, i.e., reach a real GDP level of $2 \times Y_0$. Let's label this time T_2.

If real GDP is Y_0 in period 0, then in period 1 it will be $Y_1 = Y_0(1 + g)$, where here g is expressed in decimal form (e.g., $g = .05$). In period 2, real GDP is $Y_2 = Y_1(1 + g) = Y_0(1 + g)^2$. Continuing this, real GDP in any period t is given by

$$Y_t = Y_0(1 + g)^t. \tag{2}$$

Here is a pretty timeline to help see what's going on:



Real GDP at T_2 is $Y_{T_2} = Y_0(1+g)^{T_2}$. But if real GDP has doubled at T_2, then $Y_{T_2} = 2Y_0$. Equating the two expressions, we have

$$2Y_0 = Y_0(1 + g)^{T_2}. \tag{3}$$

*Any mistakes in this document are my own. Not for circulation.

Teeny Bit of Algebra (and Calc)

From Equation (3), we can solve for T_2:

$$2Y_0 = Y_0(1+g)^{T_2} \Rightarrow 2 = (1+g)^{T_2} \Rightarrow \ln(2) = T_2 \times \ln(1+g) \Rightarrow T_2 = \frac{\ln(2)}{\ln(1+g)}.$$

Woo, we have the formula for the *actual* doubling time!

Thus, the true doubling time for country A's real GDP is $\ln(2)/\ln(1.05) = 14.207$ years. The approximation is pretty close, eh?

Now, notice that $\ln(2) \approx .693$. To simplify further, let's go with $\ln(2) \approx .70$.

A linear approximation of $\ln(1+g)$ gives us that $\ln(1+g) \approx g$ for g close to 0.[1]

Putting those together, we have where our approximation for doubling times comes from:

$$T_2 \approx \frac{.70}{\underbrace{g}_{\text{in decimal form}}} = \frac{.70}{g} \times \frac{100}{100} = \frac{70}{100g} = \frac{70}{\underbrace{g}_{\text{in percent form}}}.$$

For You

What is the true formula for the quadruple time, T_4, and the eight-fold time, T_8? What could be the "Rule" for each?

Quadruple time: Use the equation for T_2 above - replace "2" with "4." $T_4 = \ln(4)/\ln(1+g)$. $\ln(4) \approx 1.39$. So, let's go with "Rule of 140."

Eight-fold time: $\ln(8) \approx 208$. "Rule of 208" works for me.

What is the quadrupling time expressed in terms of T_2 (i.e., write T_4 as a function of T_2)? The eight-fold time? **Hint:** A property of logs is that $\ln(x^a) = a \times \ln(x)$.

$T_4 = \frac{\ln(4)}{\ln(1+g)} = \frac{\ln(2^2)}{\ln(1+g)} = \frac{2 \cdot \ln(2)}{\ln(1+g)} = 2 \cdot T_2$. This gives a "Rule of 140," equal to the answer above.

$T_8 = \frac{\ln(8)}{\ln(1+g)} = \frac{\ln(2^3)}{\ln(1+g)} = \frac{3 \cdot \ln(2)}{\ln(1+g)} = 3 \cdot T_2$. This gives a "Rule of 210," close to the answer above.

This makes sense. Quadrupling means you are doubling real GDP twice. Eight-fold means you double real GDP three times. Since we only discussed the "Rule of 70" in class, use the concept in this part for any future questions you might see.

[1] This comes from a Taylor series expansion of $\ln(1+x)$ around $x=0$. Remember from calculus that a Taylor series of $f(x)$ around $x=a$ is given by $f(a) + \frac{f'(a)}{1!} \cdot (x-a) + \frac{f''(a)}{2!} \cdot (x-a)^2 + \frac{f'''(a)}{3!} \cdot (x-a)^3 + \ldots$. So, a linear approximation around $x=0$ is given by $f(x) \approx f(0) + \frac{f'(0)}{1!} \cdot x$. For $f(x) = \ln(1+x)$, we have $f(0) = \ln(1) = 0$, $f'(x) = 1/(1+x)$, and $f'(0) = 1/(1+0) = 1$. Thus, $\ln(1+x) \approx x$.