The Solow Model

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The Solow Model

- **Production Function:** A function that describes the relationship between the quantity of inputs used in production and the quantity of output.
- We will assume that real GDP (Y) is a function of capital, labor, and technology.
- Thus, we can write $Y = A \cdot F(K, L)$, where A reflects the available production technology and is a measure of productivity.

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- Assumptions:
  1. Increasing returns to inputs: More K or L leads to greater Y.
  2. Diminishing marginal returns: As K or L increase, their marginal product decreases.
  3. Constant returns to scale: $F(\lambda K, \lambda L) = \lambda F(K, L)$.

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- For now, we will assume that technology and labor are constant so that we can write our production function as $Y = F(K)$.
- Using the property of constant returns, we can write the output per worker, or real GDP per capita, as

$$y = Y/L = 1/L \cdot F(K, L) = F(K/L, L/L) = F(K/L) \equiv f(k)$$

where $k \equiv K/L$ is the capital-labor ratio.

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Example

Suppose output per worker in an economy is described by $y = \sqrt{k}$. If each worker has 100 units of capital, how much output can each produce? What if each had 200 units? 300? What property does this production function exhibit?

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$y = \sqrt{100} = 10$

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$y = \sqrt{200} = 14.14$

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$y = \sqrt{100} = 10$
$y = \sqrt{200} = 14.14$
$y = \sqrt{300} = 17.32$

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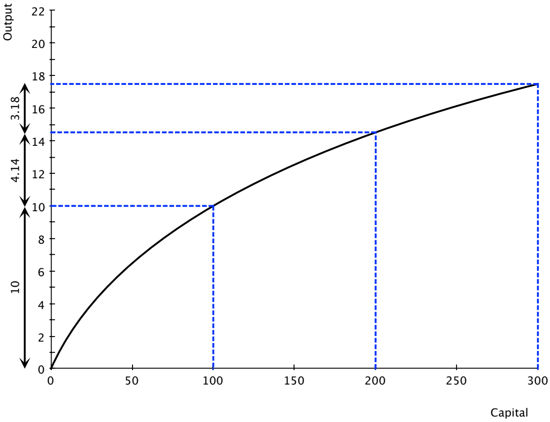

Figure: Returns to Capital

The Solow Model

Example

The United States and China both have this same production function. However, the US currently has 400 units of capital per worker, while China only has 225. If both countries receive another 100 units of capital worker this year, what will be the growth rate of real GDP per capita in each country?

$y_0^{us} = \sqrt{400} = 20.$

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$$\hat{y}^{us} = (22.36 - 20)/20 \times 100\% = 11.8\%.$$

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$y_0^{china} = \sqrt{225} = 15$
$y_1^{china} = \sqrt{325} = 18.03$

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$$y_0^{china} = \sqrt{225} = 15$$
$$y_1^{china} = \sqrt{325} = 18.03$$
$$\hat{y}^{us} = (18.03 - 15)/15 \times 100\% = 20.2\%.$$

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- Due to diminishing returns, countries with small capital stocks grow more rapidly than those with large capital stocks. However, as capital accumulates, growth slows.

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- We will assume that output can be used in one of two ways:
  1. Investment - used to produce more capital goods
  2. Consumption
- We assume that individuals save and invest a constant fraction of output, so we can write investment per worker in period t as $i = sy$, where s is the savings rate.
- Since output is either invested or consumed, we can write the consumption per worker in the economy in time t as $c = y - i = y - sy = (1 - s)y$.

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- Each period, we assume a constant fraction of the initial capital stock depreciates and is unusable in the next period.
- Capital depreciation per worker in time t is given as $d = \delta k$, where δ is the depreciation rate.
- Finally, we can write the **law of motion** for capital per worker as

$$k_{t+1} = k_t + i_t - d_t$$

The Solow Model

Example

A country has the production function $y = \sqrt{k}$ and an initial capital stock per worker of $k_0 = 25$. Assume capital depreciates 5% every period and the country invests 20% of their output for next period. How much output will they produce this period? How much will they invest? What will be the level of the capital stock, output, and real per capita GDP growth in the next three periods?

t	k_t	$y_t = \sqrt{k_t}$	$d_t = .05k_t$	$i_t = .20y_t$	$\hat{y}_t$
0	25	5	1.250	1	—
1	24.750	4.975	1.238	.995	−.501%
2	24.507	4.951	1.225	.990	−.491%
3	24.272	4.927	1.214	.985	−.481%

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- When depreciation is greater than investment, the capital stock is diminishing. Thus, output is also decreasing.
- Conversely, when depreciation is lower than investment, the capital stock and output are both increasing.

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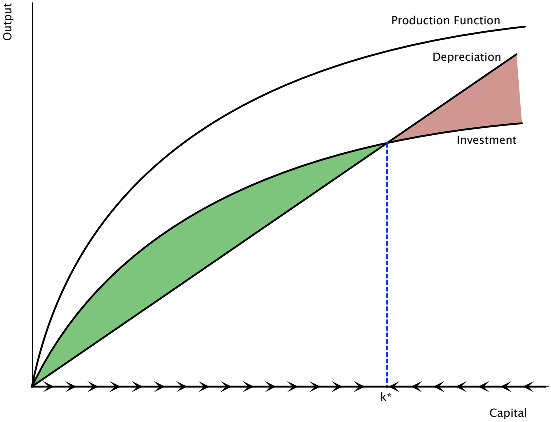

Figure: Capital Accumulation

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- An economy's **steady state** is an equilibrium path in which $k_t = k^*$ for all t.
  - If $i > d$, then k and y are increasing.
  - If $i < d$, then k and y are decreasing.
  - Finally, if $i = d$, then k, y, i, c, d are all constant.
- Thus, the condition that must hold is that investment and depreciation must be equal.

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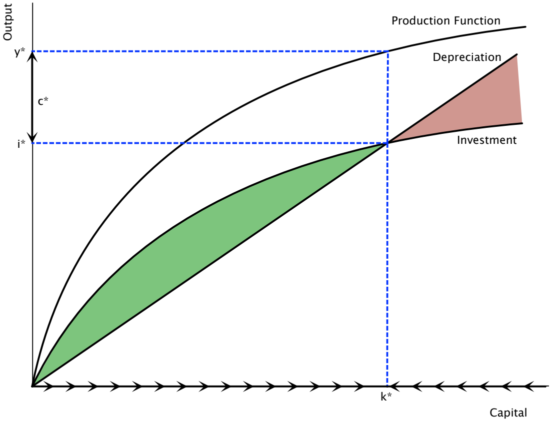

Figure: The Steady State

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- As a country approaches its steady state, the growth rate of capital and output slows due to diminishing returns.
- In the steady state, real GDP is constant each period and thus economic growth is zero. Because of diminishing returns, capital accumulation alone cannot lead to sustained long-run growth.

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Example

A country has the production function $y = \sqrt{k}$. Assume capital depreciates 10% every period and the country invests 25% of their output for next period. What is the steady state level of capital per worker? The steady state level of output per worker?

$d = .10k$, $i = .25\sqrt{k}$

Step 1: Set $i = d$

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$d = .10k$, $i = .25\sqrt{k}$
Step 1: Set $i = d \Rightarrow .25\sqrt{k} = .10k$
Step 2: Solve for k^*:

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$d = .10k$, $i = .25\sqrt{k}$

Step 1: Set $i = d \Rightarrow .25\sqrt{k} = .10k$

Step 2: Solve for k^*: $\Rightarrow (.10k)^2 = (.25\sqrt{k})^2 \Rightarrow .01k^2 = .0625k \Rightarrow$
$k^* = .0625/.01 = 6.25$

Step 3: Find y^*

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Step 2: Solve for k^*: $\Rightarrow (.10k)^2 = (.25\sqrt{k})^2 \Rightarrow .01k^2 = .0625k \Rightarrow$
$k^* = .0625/.01 = 6.25$

Step 3: Find $y^* \Rightarrow y^* = \sqrt{6.25} = 2.5$.

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Example

Suppose this country has reached its steady state. How much capital per worker depreciates each period? How much output is invested? How much output is consumed?

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$$d^* = \delta k^* = .10(6.25) = .625$$

The Solow Model

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Suppose this country has reached its steady state. How much capital per worker depreciates each period? How much output is invested? How much output is consumed?

$d^* = \delta k^* = .10(6.25) = .625$

$i^* = sy^* = .25(2.5) = .625$

The Solow Model

Example

Suppose this country has reached its steady state. How much capital per worker depreciates each period? How much output is invested? How much output is consumed?

$$d^* = \delta k^* = .10(6.25) = .625$$
$$i^* = sy^* = .25(2.5) = .625$$
$$c^* = y^* - i^* = (1 - s)y^* = .75(2.5) = 1.875.$$

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- Comparative statics: What happens to the steady state when we change exogenous variables in the model?

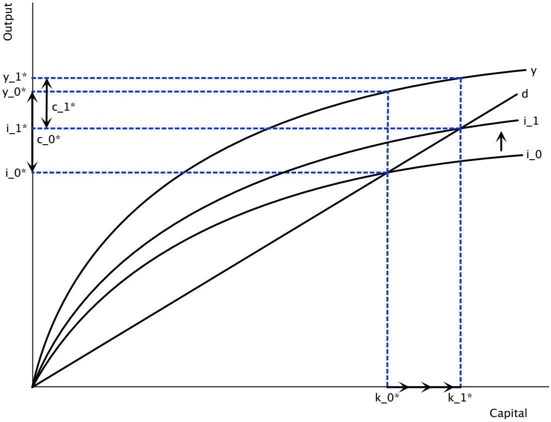

Figure: An Increase in the Savings Rate

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- An increase in the savings rate will:
  1. Immediately increase investment and decrease consumption.
  2. Lead to a higher steady-state level of capital, output, and investment.
  3. The steady-state level of consumption may increase or decrease depending on the savings rate.
- A decrease in the savings rate will have the opposite effect on 1 & 2, and the same effect on 3.

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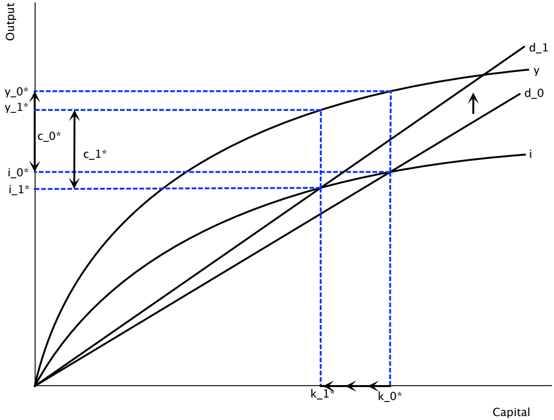

Figure: A Change in the Depreciation Rate

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- An increase in the depreciation rate will:
  1. Have no impact on current investment and consumption.
  2. Lead to a lower steady-state level of capital, output, investment, and consumption
- A decrease in the depreciation rate will have the opposite effect on 2, and the same effect on 1.

The Solow Model

- We model available production technology in the Solow Model by writing $y = A \times f(k)$.
- A is a measure of productivity and has a multiplicative effect.
- Technological advances allow countries to sustain growth in the long run. This type of growth is referred to as cutting-edge growth.

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A country has the production function $y = 2\sqrt{k}$. Assume capital depreciates 10% every period and the country invests 25% of their output for next period. What is the steady state level of capital per worker? The steady state level of output?

The Solow Model

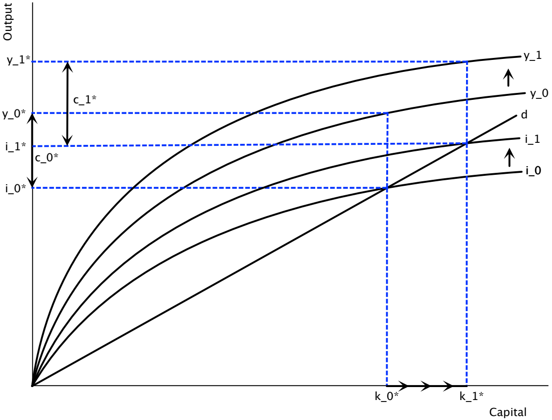

Figure: A Change in Technology

The Solow Model

- Assumption: Population and the labor force grow at a constant rate n.
- Now, capital per worker decreases each period not only through depreciation, but also because it is spread out over more workers. This is referred to as capital dilution.
- Thus, the depreciation of capital per worker every period is now written as $d_t = (n + \delta)k_t$.

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Example

A country has the production function $y = \sqrt{k}$. Assume capital depreciates 10% every period and the country invests 25% of their output for next period. Moreover, the population grows 2% every year. What is the steady state level of capital per worker? The steady state level of output?

Readings and Assignments

- Today: Mankiw Ch. 26
- Next time: Mankiw Ch. 28
- Problem Set 5, section 3