Where Does the “Rule of 70” Come From?

ECON 101

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You hopefully know from class that for a variable (say, real GDP) that grows at a constant rate g every period, the approximate time it takes for that variable to double in size is

$$T_2 \approx \frac{70}{g}, \tag{1}$$

where g is expressed in percent form. This is the so-called “Rule of 70.”

For example, if real GDP grows at 5% each year in country A, then country A’s real GDP would double in approximately $70/5 = 14$ years.

Here, we will see where this “rule” comes from.

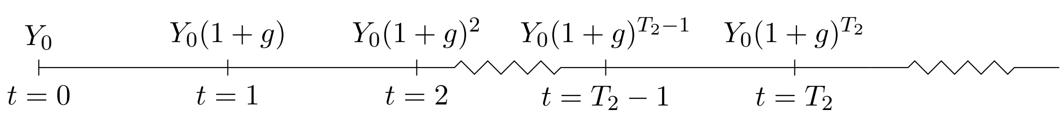
Compound Growth

To make life easy, let’s focus on the growth of real GDP, Y. We start at time $t = 0$ with some stock of real GDP, Y_0. Real GDP grows at rate g each period and we want to know how long it takes real GDP to double, i.e., reach a real GDP level of $2 \times Y_0$. Let’s label this time T_2.

If real GDP is Y_0 in period 0, then in period 1 it will be $Y_1 = Y_0(1+g)$, where here g is expressed in decimal form (e.g., $g = .05$). In period 2, real GDP is $Y_2 = Y_1(1+g) = Y_0(1+g)^2$. Continuing this, real GDP in any period t is given by

$$Y_t = Y_0(1+g)^t. \tag{2}$$

Here is a pretty timeline to help see what’s going on:



Real GDP at T_2 is $Y_{T_2} = Y_0(1+g)^{T_2}$. But if real GDP has doubled at T_2, then $Y_{T_2} = 2Y_0$. Equating the two expressions, we have

$$2Y_0 = Y_0(1+g)^{T_2}. \tag{3}$$

[*] Any mistakes in this document are my own. Not for circulation.

Teeny Bit of Algebra (and Calc)

From Equation (3), we can solve for T_2:

$$2Y_0 = Y_0(1+g)^{T_2} \Rightarrow 2 = (1+g)^{T_2} \Rightarrow \ln(2) = T_2 \times \ln(1+g) \Rightarrow T_2 = \frac{\ln(2)}{\ln(1+g)}.$$

Woo, we have the formula for the *actual* doubling time!

Thus, the true doubling time for country A's real GDP is $\ln(2)/\ln(1.05) = 14.207$ years. The approximation is pretty close, eh?

Now, notice that $\ln(2) \approx .693$. To simplify further, let's go with $\ln(2) \approx .70$.

A linear approximation of $\ln(1+g)$ gives us that $\ln(1+g) \approx g$ for g close to 0.[1]

Putting those together, we have where our approximation for doubling times comes from:

$$T_2 \approx \frac{.70}{\underbrace{g}_{\text{in decimal form}}} = \frac{.70}{g} \times \frac{100}{100} = \frac{70}{100g} = \frac{70}{\underbrace{g}_{\text{in percent form}}}.$$

For You

What is the true formula for the quadruple time, T_4, and the eight-fold time, T_8? What could be the "Rule" for each?

What is the quadrupling time expressed in terms of T_2 (i.e., write T_4 as a function of T_2)? The eight-fold time? **Hint:** A property of logs is that $\ln(x^a) = a \times \ln(x)$.

[1] This comes from a Taylor series expansion of $\ln(1+x)$ around $x = 0$. Remember from calculus that a Taylor series of $f(x)$ around $x = a$ is given by $f(a) + \frac{f'(a)}{1!} \cdot (x-a) + \frac{f''(a)}{2!} \cdot (x-a)^2 + \frac{f'''(a)}{3!} \cdot (x-a)^3 + \dots$. So, a linear approximation around $x = 0$ is given by $f(x) \approx f(0) + \frac{f'(0)}{1!} \cdot x$. For $f(x) = \ln(1+x)$, we have $f(0) = \ln(1) = 0$, $f'(x) = 1/(1+x)$, and $f'(0) = 1/(1+0) = 1$. Thus, $\ln(1+x) \approx x$.