Angel and Shreiner: Interactive Computer Graphics, Seventh Edition

Chapter 4 Solutions

4.1 If the scaling matrix is uniform then

$$\mathbf{RS} = \mathbf{RS}(\alpha, \alpha, \alpha) = \alpha\mathbf{R} = \mathbf{SR}$$

Consider $\mathbf{R}_x(\theta)$, if we multiply and use the standard trigonometric identities for the sine and cosine of the sum of two angles, we find

$$\mathbf{R}_x(\theta)\mathbf{R}_x(\phi) = \mathbf{R}_x(\theta + \phi)$$

By simply multiplying the matrices we find

$$\mathbf{T}(x_1, y_1, z_1)\mathbf{T}(x_2, y_2, z_2) = \mathbf{T}(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

4.4 Translation

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling

$$\mathbf{S} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Shear

$$\mathbf{H} = \begin{bmatrix} 1 & \cot\phi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The general form of a homogeneous coordinate transformation matrix for working with two dimensional graphics is

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

and has six degrees of freedom.

4.5 There are 12 degrees of freedom in the three–dimensional affine transformation. Consider a point $\mathbf{p} = [x, y, z, 1]^T$ that is transformed to $\mathbf{p}' = [x'y', z', 1]^T$ by the matrix $\mathbf{M}$. Hence we have the relationship $\mathbf{p}' = \mathbf{Mp}$ where $\mathbf{M}$ has 12 unknown coefficients but $\mathbf{p}$ and $\mathbf{p}'$ are known. Thus we have 3 equations in 12 unknowns (the fourth equation is simply the identity 1=1). If we have 4 such pairs of points we will have 12 equations in 12 unknowns which could be solved for the elements of $\mathbf{M}$. Thus if we know how a quadrilateral is transformed we can determine the affine transformation.
In two dimensions, there are 6 degrees of freedom in $\mathbf{M}$ but $\mathbf{p}$ and $\mathbf{p}'$ have only x and y components. Hence if we know 3 points both before and after transformation, we will have 6 equations in 6 unknowns and thus in two dimensions if we know how a triangle is transformed we can determine the affine transformation.

4.6 The signs on the sine terms in the rotation matrices must all be changed. You can check this result by noting that a positive 90 degree rotation about z in a right–handed system brings the positive x axis to the positive z axis, while in a left–handed system a positive 90 degree rotation about the z axis brings the positive y axis to the positive x axis. Similar results hold for rotations about the x and y axes.

4.7 It is easy to show by simply multiplying the matrices that the concatenation of two rotations yields a rotation and that the concatenation of two translations yields a translation. If we look at the product of a rotation and a translation, we find that the left three columns of $\mathbf{RT}$ are the left three columns of $\mathbf{R}$ and the right column of $\mathbf{RT}$ is the right column of the translation matrix. If we now consider $\mathbf{RTR}'$ where $\mathbf{R}'$ is a rotation matrix, the left three columns are exactly the same as the left three columns of $\mathbf{RR}'$ and the and right column still has 1 as its bottom element. Thus, the form is the same as $\mathbf{RT}$ with an altered rotation (which is the concatenation of the two rotations) and an altered translation.

Inductively, we can see that any further concatenations with rotations and translations do not alter this form.

4.8 A simple sequence can be obtained by considering a square centered at the origin. We rotate it by 45 degrees and then scale it *nonuniformly*, thus creating a non–right parallelogram. We then rotate it back (-45 degrees) leaving a sheared square.

4.9 If we do a translation by -h we convert the problem to reflection about a line passing through the origin. From m we can find an angle by which we can rotate so the line is aligned with either the x or y axis. Now reflect about the x or y axis. Finally we undo the rotation and translation so the sequence is of the form $\mathbf{T}^{-1}\mathbf{R}^{-1}\mathbf{SRT}$.

4.10 We can start with a rotation about any of the axes. The next rotation can be about either of the other two axes and the third can be either about the third axis or the first. Hence there are 3*2*2 = 12 possible orders: xyz, xyx, xzy, xzx, yxz, yxy, yzx, yzy, zyx, zyz, zxy, zxz.

4.11 The most sensible place to put the shear is second so that the instance transformation becomes $\mathbf{I} = \mathbf{TRHS}$. We can see that this order makes sense if we consider a cube centered at the origin whose sides are aligned with the axes. The scale gives us the desired size and proportions. The shear then converts the right parallelepiped to a general parallelepiped. Finally we can orient this parallelepiped with a rotation and place it where desired with a translation. Note that the order $\mathbf{I} = \mathbf{TRSH}$ will work too.

4.12 A plane can be described by the equation $ax + by + cz + d = 0$. If we define the two homogeneous coordinate column matrices $\mathbf{p} = \begin{bmatrix} x & y & z & 1 \end{bmatrix}^T$ and $\mathbf{n} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$ then the equation of the plane becomes $\mathbf{p} \cdot \mathbf{n} = 0$.

4.13 A vertex in a three-dimensional system is a location. It has no other properties but its location.

4.14

$$\mathbf{R} = \mathbf{R}_z(\theta_z)\mathbf{R}_y(\theta_y)\mathbf{R}_x(\theta_x) =$$

$$\begin{bmatrix} \cos\theta_y\cos\theta_z & \cos\theta_z\sin\theta_x\sin\theta_y - \cos\theta_x\sin\theta_z & \cos\theta_x\cos\theta_z\sin\theta_y + \sin\theta_x\sin\theta_z & 0 \\ \cos\theta_y\sin\theta_z & \cos\theta_x\cos\theta_z + \sin\theta_x\sin\theta_y\sin\theta_z & -\cos\theta_z\sin\theta_x + \cos\theta_x\sin\theta_y\sin\theta_z & 0 \\ -\sin\theta_y & \cos\theta_y\sin\theta_x & \cos\theta_x\cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.15 Points, vectors and scalars.

4.16 Consider the homogeneous recurrence

$$y(k) + a_{n-1}y(k-1) + ... + a_0 y(k-n) = 0,$$

We know there are n linearly independent solutions $\{z_i(k)\}i = 1, .., n$ and any solution can be written as

$$y(k) = \sum_{i=0}^{n} c_i z_i(k)$$

where the constants $\{c_i\}$ are determined by the initial conditions. A virtually identical result holds for linear differential equations. We can regard the sequences $\{z_i(k)\}i = 1, .., n$ as basis vectors and thus the solution of the homogeneous equation is a point in the vector space spanned by this basis.
For the inhomogeneous equation

$$y(k) + a_{n-1}y(k-1) + ... + a_0 y(k-n) = x(k),$$

where $\{x(k)\}$ is a known sequence, if $\{u(k)\}$ is *any* solution of this recurrence, any other solution can be written as

$$y(k) = u(k) + \sum_{i=0}^{n} c_i z_i(k).$$

Thus $\{u(k)\}$ acts as a *point* and any solution is another point that is obtained by adding a vector to it.

4.18 It would seem that the matrix $\mathbf{R} = \mathbf{R}_x(45)\mathbf{R}_y(45)$ would be correct but it is not. After the first rotation by 45 degrees, the resulting side view is not symmetric and the required angle of rotation has a cosine of $\frac{\sqrt{6}}{3}$. The angle is approximately 35.36 degrees. We consider this problem in Section 5.3.

4.19 One test is to use the first three vertices to find the equation of the plane $ax + by + cz + d = 0$. Although there are four coefficients in the equation only three are independent so we can select one arbitrarily or normalize so that $a^2 + b^2 + c^2 = 1$. Then we can successively evaluate

$ax + bc + cz + d$ for the other vertices. A vertex will be on the plane if we evaluate to zero. An equivalent test is to form the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}$$

for each $i = 4, ...$ If the determinant of this matrix is zero the ith vertex is in the plane determined by the first three.

4.20 If they are collinear, one vertex is a linear combination of the other two and the determinant of the matrix

$$\begin{bmatrix} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

will be zero.

4.21 Although we will have the same number of degrees of freedom in the objects we produce, the class of objects will be very different. For example if we rotate a square before we apply a nonuniform scale, we will shear the square, something we cannot do if we scale then rotate.

4.23 The vector $a = u \times v$ is orthogonal to u and v. The vector $b = u \times a$ is orthogonal to u and a. Hence, u, a and b form an orthogonal coordinate system.

4.24 The determinant of the matrix is $1 + \theta^2$. Repeated multiplications by this matrix increase the determinant so the resulting operation becomes further and further from a rotation matrix causing the point to become further and further from the origin. One remedy is to use the matrix

$$\mathbf{R} = \begin{bmatrix} 1 & -\theta & 0 & 0 \\ \theta & 1-\theta^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which has a determinant of 1.

4.25 Using $r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{v}$, with $\theta = 90$ and $\mathbf{v} = (1, 0, 0)$, we find for rotation about the x-axis

$$r = \frac{\sqrt{2}}{2}(1, 1, 0, 0).$$

Likewise, for rotation about the y axis

$$r = \frac{\sqrt{2}}{2}(1, 0, 1, 0).$$

4.26 Using the notation $\sin\theta_x = s_x$, $\cos\theta_x = c_x$, and likewise for θ_y and θ_z, we find

$$\mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z) = \begin{bmatrix} c_y c_z & -c_y s_z & s_y & 0 \\ -c_z s_x s_y + c_x s_z & -s_x s_y s_z + c_z c_z & -s_x c_y & 0 \\ -c_x s_y c_z + s_x s_z & c_x s_y s_z + s_x c_z & c_x c_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using quaternions, we form the three unit quaternions

$$r_x = \cos\frac{\theta_x}{2} + \sin\frac{\theta_x}{2}(1, 0, 0),$$
$$r_x = \cos\frac{\theta_y}{2} + \sin\frac{\theta_y}{2}(0, 1, 0),$$
$$r_x = \cos\frac{\theta_z}{2} + \sin\frac{\theta_z}{2}(0, 0, 1).$$

We can now compute $r_x r_y r_z$ using quaternion multiplication and obtain a resulting quaternion whose elements correspond to those of the matrix.

4.30 Possible reasons include (1) object-oriented systems are slower, (2) users are often comfortable working in world coordinates with higher-level objects and do not need the flexibility offered by a coordinate-free approach, (3) even a system that provides scalars, vectors, and points would have to have an underlying frame to use for the implementation.

4.31 (a) ¿1 (b) ¿0 and ¡1

4.32 Expanding repeatedly as in the hint

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \ldots = \alpha_1 P_1 + (\alpha_2 + \alpha_1 - \alpha_1)P_2 + \ldots$$
$$= \alpha_1(P_1 - P_2) + (\alpha_1 + \alpha_2)P_2 + \alpha_3 P_3 + \ldots$$
$$= \alpha_1(P_1 - P_2) + (\alpha_1 + \alpha_2)P_2 + (\alpha_1 + \alpha_2 + \alpha_3 - \alpha_1 - \alpha_2)P_3 + \ldots$$

$$= \alpha_1(P_1 - P_2) + (\alpha_1 + \alpha_2)(P_2 - P_3) + (\alpha_1 + \alpha_2 + \alpha_3)P_3 + \ldots$$

until eventually we have

$$P = \alpha_1(P_1 - P_2) + (\alpha_1 + \alpha_2)(P_2 - P_3) + \ldots +$$

$$(\alpha_1 + \ldots + \alpha_{N-1})(P_N - P_{N-1}) + (\alpha_1 + \alpha_2 + \ldots + \alpha_N)P_N$$

If

$$\alpha_1 + \alpha_2 + \ldots + \alpha_N = 1$$

then, since $1 \times P_N = P_N$, we have the sum of a point and $N - 1$ vectors which yields a point.