

# **Geometric Combinatorics**

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## CHAPTER 1

# Convexity and polytopes

### 1. Convexity basics

A linear combination of elements in a set  $S \subseteq \mathbb{R}^d$  is an expression of the form

$$\sum_{i=1}^n t_i x_i$$

where each  $x_i \in S$ . Such a linear combination is called an **affine combination** if  $\sum_i t_i = 1$ , a **conic combination** if  $t_i \geq 0$  for each  $i$ , and a **convex combination** if it is conic and affine. The set of all linear, affine, conic, and convex combinations of a set  $S$  will be denoted  $\mathbb{R}S$ ,  $\text{Aff}(S)$ ,  $\mathbb{R}_{\geq 0}(S)$ , and  $\text{Conv}(S)$ . In words, we will refer to them as the linear span, the affine hull, the conic hull, and the convex hull of  $S$ .

Let us explore the geometric significance of these concepts when  $S = \{x, y\}$  consists of two distinct points. When neither  $x$  nor  $y$  is the origin, the linear hull of  $S$  is the unique plane containing  $x, y$  and the origin. The affine hull of  $S$  is the unique line in  $\mathbb{R}^d$  containing  $x$  and  $y$  and the convex hull of  $S$  is the line segment between  $x$  and  $y$ . The conic hull of  $S$  is the union of all rays from the origin through a point in the convex hull of  $x$  and  $y$ .

One says that  $S \subseteq \mathbb{R}^d$  is a **linear subspace** when  $S = \mathbb{R}(S)$ , an **affine subspace** when  $S = \text{Aff}(S)$ , a **cone** when  $S = \mathbb{R}_{\geq 0}(S)$ , and **convex** when  $S = \text{Conv}(S)$ .

A **V-polytope** is the convex hull of a finite set of points, i.e. a set of the form  $\text{Conv}(\{x_1, \dots, x_n\})$ .



FIGURE 1. The square is a polytope. The disc is convex, but not a polytope.

A **halfspace** is a set of the form  $\{x \in \mathbb{R}^d : ax \leq c\}$  where  $a \in (\mathbb{R}^d)^*$  and  $c \in \mathbb{R}$ . An **H-polyhedron** is an intersection of finitely many halfspaces. An **H-polytope** is a bounded H-polyhedron. We will see in Section 3 that every V-polytope is an H-polytope and vice versa.

We now give several examples of polytopes

- (1) **Standard simplices:** Fix an integer  $d \geq 1$  and for each  $1 \leq i \leq d$  define  $e_i \in \mathbb{R}^d$  to be the  $i^{\text{th}}$  standard basis vector in  $\mathbb{R}^d$ . We define the standard simplices as follows

$$\begin{aligned} \Delta_{d-1} &:= \text{Conv}\{e_1, \dots, e_d\} \\ &= \{x \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1; 0 \leq x_i \forall i = 1, \dots, d\}. \end{aligned}$$



FIGURE 2. The first two polytopes are affinely isomorphic to each other, but not to the third. This is because affine functions preserve parallel lines and the first two have two sets of parallel lines, but the last one has none.

(2) **Cubes:**

$$\begin{aligned} C_d &:= \text{Conv}(\{+1, -1\}^d) \\ &= \{x \in \mathbb{R}^d : -1 \leq x_i \leq 1 \forall i = 1, \dots, d\}. \end{aligned}$$

(3) **Cross polytopes**

$$\begin{aligned} C_d^* &:= \text{Conv}\{e_1, -e_1, e_2, -e_2, \dots, e_d, -e_d\} \\ &= \{x \in \mathbb{R}^d : \sum_{i=1}^d |x_i| \leq 1\}. \end{aligned}$$

Perhaps the most fundamental quantity one can associate to a geometric object is its dimension. We would like a precise way to quantify the dimension of a polytope that is easy to work with. Linear spaces are just about the only thing in mathematics that have an obvious definition of their dimension, which is the size of a basis. The following lemma says that each affine space is uniquely associated to a linear space. We will then define the dimension of an affine space to be the dimension of the associated linear space. Then, with this at our disposal, we will define the dimension of a polytope to be the dimension of its affine hull.

**Lemma 1:** *Let  $A \subseteq \mathbb{R}^d$  be an affine subspace. Then there exists a unique linear subspace  $L \subseteq \mathbb{R}^d$  such that  $A = \{x + b : x \in L\}$  where  $b$  is an arbitrary element of  $A$ .*

PROOF. Let  $b \in A$  and define  $L := \{x - b : x \in A\}$ . We must show that  $L$  is indeed a linear subspace and that it does not depend on our choice of  $b$ . Indeed, let  $x_1, x_2 \in A$  so that  $x_1 - b$  and  $x_2 - b$  are arbitrary elements of  $L$ . Their sum is  $(x_1 + x_2 - b) - b$ , which is also an element of  $L$  as  $x_1 + x_2 - b$  is an affine combination of elements in  $A$  and is therefore in  $A$  itself. Now, let  $x \in A$  and  $t \in \mathbb{R}$ . Then  $t(x - b) = tx + (1 - t)b - b$  which is in  $L$  since  $tx + (1 - t)b \in A$ . Our choice of  $L$  does not depend on  $b$  since  $x - c \in L$  for any  $c \in A$  since  $x - c = (x - c + b) - b$  and  $x - c + b \in A$ .  $\square$

The **dimension** of an affine subspace  $A \subseteq \mathbb{R}^d$  is the size of a basis of the linear space  $\{x - b \in \mathbb{R}^d : x \in A\}$  where  $b \in A$ . We denote this by  $\dim(A)$ . The dimension of a convex set  $C \subseteq \mathbb{R}^d$  is  $\dim(\text{Aff}(C))$ .

Two polytopes  $P \in \mathbb{R}^d$  and  $Q \in \mathbb{R}^e$  are **affinely isomorphic** if there exists an affine function  $f : \mathbb{R}^d \rightarrow \mathbb{R}^e$  that is a bijection between  $P$  and  $Q$ .

The polytopes in Figure 2 suggest that the notion of affine isomorphism is too strong for combinatorics since all three polytopes are, in a sense that we will make precise soon, “combinatorially equivalent” in the sense that they both have four edges and four vertices.

## 2. Faces of Polytopes

We begin by recalling some topological preliminaries. Let  $S \subseteq \mathbb{R}^d$  be a set. The **interior** of  $S$  is the union of all open sets contained in  $S$  and the **closure** of  $S$  is the intersection of all closed sets containing  $S$ . The **boundary** of  $S$  is the relative complement of the interior of  $S$  in the closure of  $S$ . The **relative interior** (resp. closure, boundary) of a convex set  $C \subseteq \mathbb{R}^d$  is the interior (resp. closure, boundary) of  $C$  in the induced topology on  $\text{Aff}(C)$ .

The interesting combinatorics of a convex set takes place on its relative boundary. For this reason, we will often restrict our focus to **closed** convex sets. We will need the notion of a **face** of a convex set, defined below.

**Definition 2:** Let  $C \subseteq \mathbb{R}^d$  be closed and convex. A subset  $F \subseteq C$  is a **face** of  $C$  if

- (1)  $F$  is closed,
- (2)  $F$  is convex, and
- (3) given  $x, y \in C$ , if  $\text{ri}(\text{Conv}(x, y)) \cap F \neq \emptyset$ , then  $x, y \in F$ .

A face is called an **extreme point** if it has dimension 0, an **edge** if it has dimension 1 (or sometimes, in the case of cones, an extreme ray), and a **facet** if it has dimension  $\dim(C) - 1$ . A face is **proper** if it is neither  $C$  nor  $\emptyset$ .

**Example 3:** The set of proper faces of a polygon consists of its vertices and its edges. Every point on the boundary of a ball in any dimension is a proper face, and these are all the proper faces. An affine space has no proper faces. The only proper face of the halfspace  $\{x : ax \geq c\}$  is its boundary hyperplane, namely  $\{x : ax = c\}$ .

**Lemma 4:** Let  $C \subseteq \mathbb{R}^d$  be convex of dimension at least 1. Then  $\text{ri}(C)$  is nonempty.

PROOF. Let  $k$  denote the dimension of  $C$  and let  $x_1, \dots, x_{k+1}$  affinely span  $\text{Aff}(C)$ . Consider the function  $f : \Delta_k \rightarrow C$  given by

$$\sum_{i=1}^{k+1} t_i e_i \mapsto \sum_{i=1}^{k+1} t_i x_i.$$

Then  $f$  is continuous and injective. Since  $\Delta_k$  is compact,  $f(\Delta_k)$  is homeomorphic to  $\Delta_k$  [1, Theorem 26.6]. Since  $\frac{1}{k} \sum_{i=1}^{k+1} e_i \in \text{ri}(\Delta_k)$ ,  $f(\Delta_k)$ , and therefore  $C$ , has nonempty relative interior.  $\square$

**Proposition 5:** If  $C \subseteq \mathbb{R}^d$  is closed and convex and  $F \subset C$  is a proper face, then  $F \subseteq \text{rb}(C)$ .

PROOF. Let  $y \in F$  and let  $x \in C \setminus F$ . For  $n = 1, 2, \dots$ , define

$$S_n := \{ty + (1-t)x : 0 \leq t \leq 1 + 1/n\}.$$

Since  $F$  is a face of  $C$  and  $S_n$  is a line segment whose interior intersects  $F$ , there exists a point  $y_n \in S_n \setminus C$ . Then,  $y_n \rightarrow y$  as  $n \rightarrow \infty$ . But this implies  $y \in \text{rb}(C)$  because  $y_n \in \text{Aff}(C)$  as  $S_n \subseteq \text{Aff}(C)$ .  $\square$

**Proposition 6:** Let  $C \subseteq \mathbb{R}^d$  be closed and convex. If  $F$  is a proper face of  $C$ , then  $\dim(F) < \dim(C)$ .

PROOF. Since  $F \subseteq C$ ,  $\dim(F) \leq \dim(C)$ . Assume for the sake of contradiction that  $\dim(F) = \dim(C)$ . Since  $\text{Aff}(F) \subseteq \text{Aff}(C)$ , this implies that  $\text{Aff}(F) = \text{Aff}(C)$ . Passing to this affine hull if necessary, we may assume without loss of generality that  $\dim(C) = d$ .

Since  $\dim(F) = d$ , Lemma 4 implies that  $\text{ri}(F)$  is a nonempty open subset of  $\mathbb{R}^d$ . Therefore  $\text{ri}(F) \subseteq \text{ri}(C)$  implies  $\text{ri}(F) \subseteq \text{ri}(C)$ . But this contradicts Lemma 5.  $\square$

**Proposition 7:** *Let  $C$  be closed and convex and let  $F \subseteq C$  be a face. Then:*

- (1) *every face of  $F$  is a face of  $X$ , and*
- (2) *every face of  $X$  contained in  $F$  is a face of  $X$ .*

PROOF. Let  $F'$  be a face of  $F$ . Let  $x, y \in C$  be such that  $\text{ri}(\text{Conv}(x, y)) \cap F' \neq \emptyset$ . Then  $x, y \in F$  since  $F$  is a face of  $X$ . This implies  $x, y \in F'$  as  $F'$  is a face of  $F$ . So  $F'$  is a face of  $C$ .

Now let  $F'$  be a face of  $C$  contained in  $F$ . Let  $x, y \in F$  with  $F' \cap \text{ri}(\text{Conv}(x, y)) \neq \emptyset$ . Since  $F'$  is a face of  $X$ , this implies  $x, y \in F'$ . So  $F'$  is a face of  $X$ .  $\square$

**Lemma 8:** *Let  $C \subseteq \mathbb{R}^d$  be closed and convex, let  $a \in (\mathbb{R}^d)^*$ , let  $c \in \mathbb{R}$ , and assume  $ax \leq c$  for all  $x \in C$ . The set*

$$F_{a,c} := \{x \in C : ax = c\}$$

*is a face of  $C$ .*

PROOF. Let  $x, y \in C$  and assume that there exists  $z \in F_{a,c} \cap \text{ri}(\text{Conv}(x, y))$ . Let  $t \in [0, 1]$  such that  $z = tx + (1 - t)y$ . Then

$$c = az = tax + (1 - t)ay.$$

Since  $ax \geq c$  and  $ay \geq c$ , this implies  $ax = ay = c$ , i.e. that  $x, y \in F_{a,c}$ .  $\square$

A face  $F$  of a closed convex set  $C \subseteq \mathbb{R}^d$  is called **exposed** if  $F = F_{a,c}$  as in Lemma 8. The geometric interpretation of an exposed face is as follows. If  $ax \leq c$  for all  $x \in C$ , then the hyperplane  $\{x \in \mathbb{R}^d : ax = c\}$  lies tangent to  $C$ . The intersection of this hyperplane with  $C$  is the face  $F_{a,c}$ . A convex set may have faces that are not exposed - see Figure 3, for example. That said, we will eventually see that all faces of a polytope are exposed so we will not spend much time talking about non-exposed faces.



FIGURE 3. All of the faces of the above convex set  $C \subset \mathbb{R}^2$  are exposed, aside from the four extreme points indicated by black dots. To see this, note that the tangent line to  $C$  at any one of these points will intersect along the entire edge that it lies on.

We now come to the first big theorem in convexity theory: the hyperplane separation theorem. There are various similar theorems that go by the same name and we will stick with the one that has the exact level of generality we need. The second homework guides you through a proof of this theorem. A proof will be added to these notes after that assignment is turned in.

**Theorem 9 (Hyperplane separation theorem):** *Given a convex  $C \subset \mathbb{R}^n$  and a point  $y \in \mathbb{R}^d \setminus C$ , there exists  $a \in (\mathbb{R}^n)^*$  and  $b \in \mathbb{R}$  such that  $ax \leq b$  for all  $x \in C$  and  $ay \geq b$ .*

The geometric content of Theorem 9 is as follows: it says that given a convex set  $C \subseteq \mathbb{R}^d$  and a point  $y \notin C$ , there exists a hyperplane  $H$  that contains  $C$  in one of its two half-spaces and contains  $y$  in the other. When  $y \notin \text{rb}(C)$ , this hyperplane  $H$  can be chosen so that neither  $C$  nor  $\{y\}$  intersects  $H$ , and  $C$  and  $y$  lie on opposite sides of this hyperplane. When  $y \in \text{rb}(C)$ ,  $H$  will contain  $y$  and be tangent to  $C$ . The hyperplane separation theorem allows us to close an important circle of ideas that will allow us to move away from topological considerations. In particular, we have the following theorem.

**Theorem 10:** *Let  $C \subseteq \mathbb{R}^d$  be closed and convex. Then  $\text{rb}(C)$  is the union of its proper faces.*

PROOF. Proposition 5 implies that the union of the proper faces of  $C$  is contained in  $\text{rb}(C)$ . It therefore suffices to let  $x \in \text{rb}(C)$  and find a proper face of  $C$  containing  $x$ . By restricting to  $\text{Aff}(C)$  if necessary, we may assume that  $\dim(C) = d$ . Since  $\text{ri}(C)$  is convex, Theorem 9 implies that there exists  $a \in (\mathbb{R}^d)^*$  and  $c \in \mathbb{R}$  such that  $ax = c$  and  $ay \leq c$  for all  $y \in C$ . Lemma 8 implies that  $F_{a,c} := \{y \in C : ay = c\}$  is a face of  $C$  and it is clear that this contains  $x$ . The dimension of  $F_{a,c}$  is at most  $d - 1$  and since  $\dim(C) = d$  and  $F_{a,c} \neq \emptyset$ ,  $F_{a,c}$  is proper.  $\square$

### 3. The fundamental theorem of polytopes





## Bibliography

- [1] James R Munkres. *Topology*, volume 2. Prentice Hall Upper Saddle River, 2000.