Math 7110 – Homework 1 – Due: September 1, 2021

Practice Problems:

Problem 1. Show that an element in S_n has order two if and only if it is a product of commuting cycles.

Problem 2. Show that $\mathrm{GL}_n(F)$ is non-abelian if $n \geq 2$.

Problem 3. Show that the following group is isomorphic to the dihedral group D_4

$$\langle x, y | x^2 = y^2 = (xy)^2 = 1 \rangle.$$

Problem 4. Make an appointment to visit me in my office sometime before 5 PM on Thursday, September 2. I will pass around a signup sheet for meetings on Wednesday, August 25th.

Be prepared to answer the following questions:

(1) Why are you taking this class and what do you hope to get out of it?
(2) What other math classes have you taken?
(3) What are your career goals after you graduate?

Type solutions to the following problems in LaTeX, and email the tex and PDF files to me at `dbernstein1@tulane.edu` by 10am on the indicated date. Please title them as [lastname].tex and [lastname].pdf. When preparing your solutions, you must follow the rules as laid out in the course syllabus.

Graded Problems:

Problem 5. Answer the following questions:

(1) If $\phi : G \to H$ is an isomorphism, prove that $|\phi(x)| = |x|$ for all $x \in G$.
(2) Is the result true if ϕ is just a homomorphism? Prove or give a counterexample.
(3) Given $x \in S_n$, show that $|x|$ is the least common multiple of the lengths of the cycles in its cycle decomposition.
(4) Are S_4 and D_{24} isomorphic? Why or why not?

Problem 6. Let G be a group and let $\mathrm{Aut}(G)$ be the set of all isomorphisms from G onto G.

(1) Prove that $\mathrm{Aut}(G)$ is a group under function composition.
(2) Let G be a finite group and let $\sigma \in \mathrm{Aut}(G)$ be such that $\sigma(g) = g$ if and only if $g = 1$. If σ^2 is the identity map, prove that G is abelian.