Math 7110 – Homework 3 – Due: September 29, 2021

Practice Problems:

Definition. Let H and K be subgroups of a group and define

$$HK = \{hk : h \in H, k \in K\}.$$

The following problem walks you through the results that are proved at the end of Section 3.2 in the Dummit and Foote. Feel free to consult the text.

Problem 1. Let H and K be subgroups of a group G.

(1) Is HK always a subgroup of G? Prove or give a counterexample (spoiler alert: answer is below).
(2) Assume H and K are finite and prove the following

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

(3) Prove that HK is a subgroup of G if and only if $HK = KH$.
(4) Does $HK = KH$ mean that every element of H commutes with every element of K? (Hint: find subgroups H and K of D_8 such that $D_8 = HK$).

Problem 2. Read the statement and proofs of the second and third isomorphism theorems in Dummit and Foote.

Type solutions to the following problems in LaTeX, and email the tex and PDF files to me at `dbernstein1@tulane.edu` by 10am on the due date. Please title them as [lastname].tex and [lastname].pdf. When preparing your solutions, you must follow the rules as laid out in the course syllabus.

Graded Problems:

Problem 3. In this problem, you will prove the Jordan-Hölder Theorem. Let G be a finite nontrivial group.

(1) Prove that G has a composition series.
(2) Assume that G has two composition series

$$1 = N_0 \trianglelefteq \cdots \trianglelefteq N_r = G \qquad \text{and} \qquad 1 = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G.$$

Show that $r = 2$ and that the list of composition factors is the same (use the second isomorphism theorem).
(3) Prove the following by induction on $\min\{r, s\}$: If

$$1 = N_0 \trianglelefteq \cdots \trianglelefteq N_r = G \qquad \text{and} \qquad 1 = M_0 \trianglelefteq \cdots \trianglelefteq M_s = G$$

are composition series for G, then $r = s$ and there is some permutation π of $\{1, \ldots, r\}$ such that

$$M_{\pi(i)}/M_{\pi(i-1)} \cong N_i/N_{i-1} \qquad \text{for} \qquad 1 \leq i \leq r$$

(hint: apply the induction hypothesis to $H := N_{r-1} \cap M_{s-1}$).

Problem 4. Solve the following problems.

(1) Find all finite groups with exactly two conjugacy classes.
(2) Let n be odd. Show that the set of n-cycles consists of two equally-sized conjugacy classes of A_n.