Math 7120 – Homework 2 – Due: February 9, 2022

Practice problems:

Problem 1. Dummit and Foote 10.2 problems 4 and 6.

Problem 2. Read the proof of the universal theorem for free modules (Theorem 6 in section 10.3).

Test prep:

Problem 3. Dummit and Foote 10.2 problems 3 and 5.

Type solutions to the following problems in LaTeX, and email the tex and PDF files to me at `dbernstein1@tulane.edu` by 10am on the indicated date. Please title them as [lastname].tex and [lastname].pdf. When preparing your solutions, you must follow the rules as laid out in the course syllabus.

Graded Problems:

Problem 4. An R-module M is a *torsion* module if for all $m \in M$ there exists a nonzero $r \in R$ such that $rm = 0$.

(1) Prove that every finite abelian group is a torsion $\mathbb{Z}$-module.
(2) If G is an infinite abelian group, is it necessarily true that G is *not* a torsion $\mathbb{Z}$-module?
(3) Let R be an integral domain. Prove that if M is a torsion R-module, then M has a nonzero annihilator (see previous HW for definition of annihilator).
(4) Give an example of a ring R and a torsion R-module M such that the annihilator of M is the zero ideal of R.

Problem 5. An R-module is *irreducible* if $M \neq 0$ and if 0 and M are the only submodules.

(1) Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as a generator.
(2) Prove the following fundamental result of representation theory, often known as *Schur's lemma*, which says that if M_1 and M_2 are irreducible R-modules, then any nonzero R-module homomorphism $M_1 \to M_2$ is an isomorphism.
(3) Show that if M is irreducible, then $\mathrm{Hom}_R(M, M)$ is a division ring.
(4) Assume R is commutative. Show that an R-module M is irreducible if and only if M is isomorphic (as an R-module) to R/I where I is a maximal ideal of R.
(5) Determine all irreducible $\mathbb{Z}$-modules.