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THE ALGEBRAIC GEOMETRY OF STRESSES IN FRAMEWORKS*

NEIL L. WHITE[†] AND WALTER WHITELEY[‡]

Abstract. A bar-and-joint framework, with rigid bars and flexible joints, is said to be generically isostatic if it has just enough bars to be infinitesimally rigid in some realization in Euclidean n -space. We determine the equation that must be satisfied by the coordinates of the joints in a given realization in order to have a nonzero stress, and hence an infinitesimal motion, in the framework. This equation, called the pure condition, is expressed in terms of certain determinants, called brackets. The pure condition is obtained by choosing a way to tie down the framework to eliminate the Euclidean motions, computing a bracket expression by a method due to Rosenberg and then factoring out part of the expression related to the tie-down. A major portion of this paper is devoted to proving that the resulting pure condition is independent of the tie-down chosen. We then catalog a number of small examples and their pure conditions, along with the geometric conditions for the existence of a stress which are equivalent to the algebraic pure conditions. We also explain our methods for calculating these conditions and determining their factorization. We then use the pure conditions to investigate stresses and tension-compression splits in 1-overbraced frameworks. Finally we touch briefly upon some of the problems arising when multiple factors occur in the pure condition.

The statics of bar-and-joint frameworks have been studied by mathematicians and engineers for over a century. Two divergent traditions of analysis have evolved: a) direct arithmetic calculations based on the specific positions of the joints and the bars and b) general synthetic geometric algorithms. With the rise of the computer and the decline of geometry, the arithmetic calculations became the dominant method of understanding static behavior in frameworks.

However, there has been a recent revival of interest in underlying projective geometric patterns of points and lines which "explain" the behavior of all the particular arithmetic examples based on the same underlying graph [2], [4], [21], [22]. Numerous types of graphs in space have been analyzed using synthetic geometry and certain more abstract patterns also emerge in the form of these geometric explanations.

Summarized in naive geometric terms, a count of "conditions" and "choices" is made, based only on the number of joints and bars in certain subgraphs. For example, in the plane a framework with V joints and E bars has, if $k = E - (2V - 2)$, at least a $(k + 1)$ -dimensional space of static stresses (if $k \geq 0$) for any position of the joints, i.e., at least k choices of a stress, up to scalar multiple, or at most $-k$ conditions on the positions of the joints for a stress to exist (if $k < 0$) [4]. Up until now such "meta-theorems" remained intuitive guidelines rather than precise theorems. One of our major goals in this paper is to give precision of form and of proof to these statements in the cases $k = -1$ and $k = 0$. Other cases will be investigated in later papers.

Because of the weakness of our geometric traditions, as well as the expectation that two approaches are better than one, it is helpful to develop the middle ground between synthetic geometry and the arithmetic algorithms—the algebraic geometry and geometric algebra of frameworks.

As suggested by the projective invariance of static stresses and infinitesimal motions [12, Thm. 5.10], the algebraic language which works best is the language of projective geometric invariant theory—the language of brackets (§ 2). For some of the more geometric discussions we extend this language to the Grassmann algebra or Cayley algebra, languages including brackets along with algebraic operations roughly corresponding to the geometric intersection and join.

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The methods we use are primarily algebraic, and invariant theoretic, but the questions and motivations are geometric—based on problems arising with frameworks. In addition to the pleasures of a clean, more easily communicated foundation for some standard practices in the study of frameworks, our motivation in undertaking this study was an unsolved problem in the theory of tensegrity frameworks. For a variety of reasons, in the theory of both the infinitesimal and the finite rigidity of frameworks with cables it is important to know how the form (and signs) of the static stress changes in the frameworks as the positions of the vertices are varied continuously through space [3], [12]. This prior preoccupation may explain some of the topics studied in §§ 5 and 6.

These geometric questions, and the algebra which results, are not unique to the study of engineering frameworks. The same geometry (and algebra) has arisen in the fields of scene analysis and of satellite geodesy. In scene analysis, the basic problem is to recognize the correct projections of 3-dimensional polyhedral objects—a task which is equivalent to the detection of stresses in planar frameworks [13], [14], [21]. In satellite geodesy, the basic problem is to take certain earth-satellite measurements and then calculate all the additional distances in the configuration—a calculation which breaks down if this configuration, formed as a framework with bars for the measured lengths, has even an infinitesimal motion [1]. The analysis given here also extends and clarifies the current mathematical analysis of critical configurations in geodesy [15].

As we have pursued the basic questions arising in these fields into algebraic geometry over the reals, we have encountered a steadily growing array of interesting problems. Thus there is much more work to be done.

1. Preliminaries on frameworks. Our work is motivated by the study of one essential property of bar-and-joint frameworks. This property can be described in two equivalent ways—as infinitesimal rigidity (the absence of velocities assigned to the joints which infinitesimally deform the structure) or as static rigidity (the ability of the framework to absorb all suitable external forces). The essential information for both concepts is condensed in a single rigidity matrix for the framework. However, to study the algebraic geometry of this matrix, we must step back to a more abstract level of the underlying graph and related polynomial domains.

A *graph* G is a finite set $V = \{a, b, \dots, f\}$ of *vertices* together with a collection E of two element subsets of V called *edges*.

A *bar-and-joint* framework in dimension n is a *coordinatization* of a graph G by a function $\alpha : a \rightarrow (a_1, \dots, a_n, 1)$ for every $a \in V$, where a_1, \dots, a_n are elements of a polynomial domain $R = k[x_1, \dots, x_r]$. (For most applications $k = \mathbb{R}$.) In a coordinatization the edges are called *bars* and the points $\alpha(a)$ are called *joints* of the frameworks.

The coordinates $(a_1, a_2, \dots, a_n, 1)$ may be regarded as a vector in the vector space R^{n+1} , or as special homogeneous coordinates in $PG(R, n)$ a projective space of dimension n , and we will frequently alternate between these points of view. It is no problem that we employ vector spaces and projective spaces over an integral domain R instead of a field; the process is essentially equivalent to working over the field of fractions of R but we use the integral domain to allow nontrivial homomorphisms of R . While most authors use simple Euclidean coordinates for the joints of a framework, the underlying geometry is projective [21]. Thus projective coordinates are essential to the algebraic geometry we will study.

A *real framework* or a *realization* of a graph G in $\dim n$ is a coordinatization of G with $R = \mathbb{R}$.

The rigidity matrix $M(G(\alpha))$ of a framework $G(\alpha)$ in dimension n is an $|E| \times n|V|$ matrix:

$$\begin{array}{c} \text{vertex } a \quad \text{vertex } b \quad \text{vertex } f \\ \hline \text{edge } \{a, b\} \quad \left[\begin{array}{cccccc} a_1 - b_1, \dots, a_b - b_n & b_1 - a_1, \dots, b_n - a_n & \cdots & & 0, \dots, 0 \\ \text{edge } \{a, f\} \quad \left[\begin{array}{cccccc} a_1 - f_1, \dots, a_n - f_n & 0, \dots, 0 & \cdots & f_1 - a_1, \dots, f_n - a_n \\ \cdots & & & \end{array} \right. \\ \text{edge } \{e, f\} \quad \left[\begin{array}{cccccc} 0, \dots, 0 & 0, \dots, 0 & \cdots & f_1 - e_1, \dots, f_n - e_n \end{array} \right] \end{array} \right. \end{array}$$

Thus for edge $\{d, f\}$ the matrix has the row with $d_1 - f_1, \dots, d_n - f_n$ in the columns of $d, f_1 - d_1, \dots, f_n - d_n$ in the columns of f and 0 elsewhere.

In the vocabulary of infinitesimal kinematics a solution to the homogeneous system of equations $M(G(\alpha))X = 0$ is called an infinitesimal or *instantaneous motion*. Such a motion is viewed as an n -vector for each joint $(m(a), m(b), \dots, m(f))$ where the equation for bar $\{a, b\}$ becomes

$$(m(a) - m(b)) \cdot (a_1 - b_1, \dots, a_n - b_n) = 0$$

a record of the condition that the velocities preserve the length $(a_1 - b_1)^2 + \dots + (a_n - b_n)^2 = \text{constant}$. This system of equations always has a nontrivial solution space since the rigid motions of space (rotations, translations and their combinations) always provide the *trivial motions*. A framework is *infinitesimally rigid* if the space of instantaneous motions is exactly the space of trivial motions of the joints. If the joints of the framework span at least an affine space of dimension $n - 1$ (a *full framework*), then the trivial motions form a space of dimension $\binom{n+1}{2}$. For such full frameworks infinitesimal rigidity is equivalent to the statement that

$$\text{rank}(M(G(\alpha))) = n|V| - \binom{n+1}{2}.$$

For example, in dimension 2 we need

$$\text{rank}(M) = 2|V| - 3.$$

Thus a triangle ($V = 2, E = 3$) is infinitesimally rigid provided the rows of the matrix are independent—a requirement which translates to the geometric statement that the triangle is not collinear. If the triangle is collinear then a nontrivial instantaneous motion exists, with a zero velocity at two of the joints, while the third joint has a velocity orthogonal to the line of the triangle.

In dimension 3, the condition for infinitesimal rigidity is $\text{rank}(M) = 3|V| - 6$. For example, a tetrahedron ($V = 4, E = 6$) is infinitesimally rigid if and only if the rows of the matrix are independent—or equivalently if and only if the joints are not coplanar. By a similar count, any triangulated sphere has $E = 3|V| - 6$ and will be infinitesimally rigid if and only if the rows of the rigidity matrix are independent [23].

In the vocabulary of statics, we directly investigate the row space of the rigidity matrix. We write F_{ab} for the row corresponding to a bar $\{a, b\} \in E$, or F_{cd} for the corresponding vector for any pair of joints $\{c, d\}$ (even if $\{c, d\}$ is not a bar). These latter vectors are read as special *static loads*—forces or n -vectors assigned to the joints of the framework. If we define the *equilibrium loads* on a framework as the space of vectors orthogonal to the rigid or trivial motions, then a framework is *statically rigid* if and only if the row space of the rigidity matrix (the space of the $F_{ab}, \{a, b\} \in E$) coincides with the space of equilibrium loads. An equivalent characterization [23],

Thm. 2] says that: a framework is statically rigid in dimension n if and only if either there are $|V| \leq n$ joints which span an affine space of $\dim |V| - 1$ and the framework coordinatizes a complete graph or there are $|V| > n$ joints which affinely span the n -space and all loads F_{ca} lie in the row space of the rigidity matrix.

As noted above, the trivial motions of a full framework form a space of $\dim \binom{n+1}{2}$ so the equilibrium loads form a subspace of dimension $n|V| - \binom{n+1}{2}$. Static rigidity, for a full framework, is equivalent to the statement that $\text{rank}(M) = n|V| - \binom{n+1}{2}$. Static and infinitesimal rigidity are clearly equivalent for full frameworks, and this equivalence also holds for smaller frameworks [12, Thm. 4.3].

Still within the language of statics, a linear dependence of a set of rows is called a stress.

$$\sum \lambda_{ab} F_{ab} = 0 \quad (\text{sum over bars}).$$

These scalars give a set of tensions ($\lambda_{ab} < 0$) and compressions ($\lambda_{ab} > 0$) in the bars, and the equations, rewritten for each joint, describe a static equilibrium of the corresponding forces,

$$\sum \lambda_{ab} (a - b) = 0 \quad (\text{fixed } a, \text{ sum over } \{a, b\} \in E).$$

A minimal statically rigid framework on a set of joints—a statically rigid framework with no static stresses—is called *isostatic*. For an isostatic framework the rows of the rigidity matrix form a basis for the equilibrium loads on the joints, and these frameworks are the basic objects of study in the next three chapters.

Given a framework $F = (G, \alpha)$, the *coordinatization matrix* A has a row $\alpha(a)$ for each joint $a \in V$,

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n & 1 \\ b_1 & b_2 & \cdots & b_n & 1 \\ \dots \\ e_1 & e_2 & \cdots & e_n & 1 \end{bmatrix}.$$

If the entries $\{a_1, a_2, \dots, e_n\}$ are distinct algebraically independent elements of R (in which case we simply regard them to be distinct indeterminates over k), the framework F is a *generic coordinatization* of the graph G . If this generic coordinatization is an isostatic framework, we say that the graph G is *generically isostatic in dimension n*.

The small or flat frameworks (ones for which the joints do not even span an $(n - 1)$ -dimensional affine space) are well understood, as mentioned previously. However such flat frameworks would constantly clutter up the rest of our algebra in this paper, so we will *assume for the rest of the paper that the frameworks have $|V| \geq n$* which implies, in the generic case, that the framework is full.

A full framework is isostatic if and only if it has $E = n|V| - \binom{n+1}{2}$ bars, and there is no static stress (the bars are independent). The traditional way to check that the rows of a matrix form an appropriate basis is by taking determinants, but this is easier when the matrix is square. Our first task is to extend the rigidity matrix to a square matrix by adding $\binom{n+1}{2}$ independent rows—called a *tie-down*—so that the framework is isostatic if and only if this extended matrix has determinant $\neq 0$. There are many possible arrangements for the tie-down—but our objective is to introduce these rows in a natural format, as additional bars, and to later factor this extension out of the algebra (§ 3).

A *tie-down* of a framework $G(\alpha)$ in dimension n is a set of $\binom{n+1}{2}$ tie-down bars $\{a, x\}$, $a \in V$, $x \notin V$, where x has $m(x) = 0$ for any infinitesimal motion and adds the

row (nonzero only in the columns of a)

$$(a_1 - x_1, \dots, a_n - x_n, 0, \dots, 0)$$

to the extended rigidity matrix $M(G(\alpha), T)$.

We anticipate that, for an isostatic framework in dimension n , some correctly chosen set of $(n+1 \choose 2)$ tie-down bars will give an invertible matrix $M(G(\alpha), T)$. In the vocabulary of kinematics, these bars must block the trivial motions or, in the language of statics, they must generate the nonequilibrium loads. We begin with a simple proof that such tie-downs exist.

PROPOSITION 1.1. *A full framework in dimension n is isostatic if and only if there is a tie-down T of $(n+1 \choose 2)$ bars which produces an invertible rigidity matrix.*

Proof. Assume the tied-down framework has an invertible matrix—and hence no nonzero motions (solutions to the homogeneous system). Removing the $(n+1 \choose 2)$ tie-downs will introduce a space of infinitesimal motions of dimension $(n+1 \choose 2)$. Since this removal also introduces the rigid motions (a space of dimension $(n+1 \choose 2)$), there are no additional infinitesimal motions and the smaller framework is isostatic.

Conversely, assume that the full framework is isostatic in n -space. The rows of the rigidity matrix form an independent set of $n|V| - (n+1 \choose 2)$ vectors in the vector space $\mathbb{R}^{n|V|}$. We can extend this independent set with $(n+1 \choose 2)$ vectors from the standard basis $(1, 0, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ to form a basis for the entire space and an invertible matrix.

For each standard vector chosen we define a tie-down as follows: if the standard vector has 1 in the column for b_i , then the tie-down bar is $\{b, y\}$ and $\alpha(y) = (b_1, \dots, b_i - 1, \dots, b_n, 1)$. This choice gives the desired standard vector as a row of the extended rigidity matrix and thus is a “correct” tie-down. Q.E.D.

For any two isostatic frameworks on the same joints, the row spaces are the same, and thus the correct tie-downs will be the same. There is a geometric characterization of the correct tie-downs: when $(n+1 \choose 2)$ bars connect two infinitesimally rigid objects in n space (e.g., the framework and the ground), then the new unit is infinitesimally rigid if and only if the lines of the bars are independent as line segments of projective n -space (or equivalently, the Plucker coordinates for the lines are linearly dependent) [9, p. 659] or [21, Thm. 5.1, Corollary 5.3]. We will build this observation into the following proposition about the form of a static stress (row dependence) in the extended matrix.

PROPOSITION 1.2. *Given a framework $G(\alpha)$ in dimension n , with $|E| = n|V| - (n+1 \choose 2)$ bars, and tie-down T of $(n+1 \choose 2)$ bars, then the extended rigidity matrix $M(G(\alpha), T)$ has a row dependence if and only if either the tie-down bars lie on dependent lines on the projective space or there is a row dependence omitting the tie-down bars (a nontrivial stress on the original framework).*

Proof. Assume there is a nonzero motion in the tied-down framework (i.e., a row dependence in the square matrix). Either this motion is a rigid motion of the framework (excluding the ground) or the original framework is not infinitesimally rigid.

In the first case the tie-down bars did not block all rigid motions and this remaining rigid motion requires the dependence of the tie-down bars in projective space.

In the second case the framework has more than an $(n+1 \choose 2)$ -dimensional space of infinitesimal motions and the lower rank for the rigidity matrix without tie-downs gives the desired row dependence.

Conversely, if we assume a row dependence omitting the tie-downs, then $M(G, T)$ has a row dependence. If we assume the tie-down bars are dependent line segments in projective space, then the original framework has a rigid motion relative to the

ground which does not alter any of the tie-down bars (instantaneously). The square matrix $M(G, T)$ has a nonzero determinant—and a row dependence. Q.E.D.

Remark. In § 3, we will give a new combinatorial characterization of correct generic tie-downs. Proposition 3.5 gives the details about these combinatorially good arrangements.

2. The bracket ring and Cayley algebra. While we know that a framework in dimension n , with $|E| = n|V| - \binom{n+1}{2}$ bars, is infinitesimally rigid if and only if for some tie-down T , $\det(M(G(\alpha), T)) \neq 0$, we also recognize that this rigidity was determined by the rigidity matrix $M(G(\alpha))$. Our essential problem is to extract from $\det(M(G(\alpha), T)) \neq 0$, for some T , an algebraic condition which is independent of T (§ 3).

First, however, we must introduce the language of brackets, the classical language of projective geometric invariants, which is the most suitable for efficient expression and manipulation of $\det(M(G(\alpha), T))$. This language has been employed in the projective theory of frameworks [18], [21] and has reappeared in several nonprojective studies of the rigidity matrix [11], [15].

For example, in [11], Rosenberg gives a direct combinatorial decomposition of $\det(M(G, T))$ in the case $n = 2$. When generalized to n dimensions the basic units of his formulae are the *brackets* $[a, b, \dots, d]$ which represent the volume of the n simplex with $n + 1$ vertices a, b, \dots, d , a volume which is equivalent to an $(n + 1) \times (n + 1)$ determinant using the affine coordinates of the points as rows of a square matrix (i.e., an $(n + 1) \times (n + 1)$ subdeterminant of A). In Rosenberg's expansion the determinant of $M(G(\alpha), T)$ is the sum of products of such brackets in the joints. In each product of the sum, a joint occurs exactly (the valence of the vertex) + 1 - n times, so the expansions belong to the language of brackets, homogeneous in occurrences of symbols for the joints.

If a, b, \dots, d are $n + 1$ joints in V , the element of R obtained as the determinant of the corresponding $n + 1$ rows of A is a *bracket* $[a, b, \dots, d]$. The brackets satisfy the following well-known relations, called *syzygies*.

- 1) $[x_0, x_1, \dots, x_n] = 0$ if $x_1 = x_i$ for some i, j with $i \neq j$, or if x_0, x_1, \dots, x_n are affinely dependent.
- 2) $[x_0, x_1, \dots, x_n] = \text{sign } (\sigma)[x_{\sigma 0}, x_{\sigma 1}, \dots, x_{\sigma n}]$ for any permutation σ of $0, 1, \dots, n$.
- 3) $[x_0, x_1, \dots, x_n][y_0, y_1, \dots, y_n] = \sum_{i=0}^n [y_i, x_1, \dots, x_n] \\ \times [y_0, y_1, \dots, y_{i-1}, x_0, y_{i+1}, \dots, y_n].$

Let B be the subring of R generated by all $(n + 1) \times (n + 1)$ determinants of A . B is called the *bracket ring* on V . If A is a generic coordinatization, then B is isomorphic to the bracket ring of the uniform matroid of rank $n + 1$ on V , as defined in [17], according to Hodge and Pedoe [10, p. 315, Thm. 1].

The commutative ring B is clearly an integral domain, since it is a subring of the integral domain R . We now wish to show that the generic bracket ring B has certain unique factorization properties. We first need the following result on factorization of invariants in R . Let A be generic. An element $f(a_1, \dots, e_n)$ of R is called an *invariant* if there exists an integer $s \geq 0$ such that for each nonsingular linear transformation S of the row space of A to itself, if $S(x)$ denotes the image of x , normalized by a scalar multiple so that $S(x)_{n+1} = 1$, then $f(S(a)_1, S(a)_2, \dots, S(e)_n) = (\det S)^s f(a_1, a_2, \dots, e_n)$. The integer s is the *degree* of the invariant f . Now the invariants in R are precisely the elements of B which are homogeneous in total degree, by the first fundamental theorem of invariant theory [7, Thm. 1].

Remark. Although we are working with a generic coordinatization of G , any coordinatization of G may be realized as a specialization of the generic one, by assigning values to the indeterminates. The question which will concern us in the following sections is, for a given graph G , which specializations of the generic coordinatization induce a stress or stresses in the framework.

We now adjoint generic vertices z_1, z_2, \dots, z_{n+1} distinct from a, b, \dots, e , letting $V' = V \cup \{z_1, z_2, \dots, z_{n+1}\}$. Letting A' be the matrix for V' analogous to A , R' the polynomial ring and B' the bracket ring, we note that R and B are subrings of R' and B' .

THEOREM 2.1. *Let f be an invariant element of R , where A is generic. Then any polynomial which is a factor of f in R is also invariant.*

Proof. We define a linear transformation S on the row space of A' by

$$S(a) = ([a, z_2, \dots, z_{n+1}], [z_1, a, z_3, \dots, z_{n+1}], \dots, [z_1, z_2, \dots, z_n, a]).$$

We note that $\det S = [z_1, \dots, z_{n+1}]^n$, since $\det(S(z_1), S(z_2), \dots, S(z_{n+1})) = [z_1, \dots, z_{n+1}]^{n+1}$. Now let f be an invariant element of R , and suppose that f factors in R as

$$f(a, b, \dots, e) = \prod_{j=1}^l g_j(a, b, \dots, e)^{r_j},$$

where the $g_j(a, b, \dots, e)$ are irreducible in R . By the definition of invariant, f is also invariant in R' and

$$f(Sa, Sb, \dots, Se) = [z_1, \dots, z_{n+1}]^{ns} f(a, b, \dots, e)$$

or

$$(*) \quad \prod_{j=1}^l g_j(Sa, Sb, \dots, Se)^{r_j} = [z_1, \dots, z_{n+1}]^{ns} \prod_{j=1}^l g_j(a, b, \dots, e)^{r_j}.$$

Now, $g_j(Sa, Sb, \dots, Se)$ is a polynomial in the coordinates of Sa, Sb, \dots, Se , but each such coordinate is a bracket, by our choice of S . Thus $g_j(Sa, Sb, \dots, Se)$ is a polynomial in B' , hence, is invariant. Furthermore, $[z_1, z_2, \dots, z_{n+1}]$ is an irreducible polynomial in R' , as is well known (see [6, Lemma A]), where the same argument works even though the last row of our matrix consists of 1's). Thus from $(*)$ we see

$$g_j(Sa, Sb, \dots, Se) = [z_1, z_2, \dots, z_{n+1}]^k h_j(a, b, \dots, e),$$

where $h_j(a, b, \dots, e)$ has no occurrences of any of the coordinates z_{ij} , $1 \leq i \leq n$, $1 \leq j \leq n+1$, that is, $h_j(a, b, \dots, e) \in R$.

Now $h_j(a, b, \dots, e) = g_j(Sa, Sb, \dots, Se)/[z_1, z_2, \dots, z_{n+1}]^{k_j}$ is a nonconstant invariant, as may be seen by applying an arbitrary linear transformation S' , hence h_j is in B . Now

$$f(a, b, \dots, e) = \prod_{j=1}^l g_j(a, b, \dots, e)^{r_j} = \prod_{j=1}^l h_j(a, b, \dots, e)^{r_j}$$

provides two factorizations of f , each involving the same number of nontrivial factors, with the $g_j(a, b, \dots, e)$ irreducible in R . But R is a unique factorization domain, hence for every j , $g_j(a, b, \dots, e) = \alpha_j h_i(a, b, \dots, e)$ for some i and some scalar α_j in k . Thus each irreducible factor of f is invariant. Q.E.D.

COROLLARY 2.2. *B is an integral domain in which each homogeneous element has a unique factorization into irreducible elements, and furthermore, the irreducible elements involved are homogeneous.*

Another algebraic structure to which we will frequently refer is the Cayley algebra. We will give here a brief and very informal introduction to this algebra, referring the reader to [7] for details.

We begin with a vector space U , which for our purpose we will take to be the row space of the matrix A considered earlier. The Cayley algebra is an extension of U with the usual operations of addition and scalar multiplication and two additional operations, join and meet, denoted \vee and \wedge . If u, v, \dots, w are m vectors, $m \leq n+1$, then $u \vee v \vee \dots \vee w$, also denoted $uv \cdots w$, is called an *extensor of step m*. Computationally, $uv \cdots w$ may be identified with the vector of Plücker coordinates of the subspace $\text{span}(u, v, \dots, w)$, that is, the sequence of $m \times m$ minors of the $m \times (n+1)$ matrix whose rows are u, v, \dots, w . The m -dimensional subspace $\text{span}(u, v, \dots, w)$ is also called the *support* of the extensor $uv \cdots w$, assuming that u, v, \dots, w are linearly independent. An extensor $uv \cdots w$ of step $n+1$ is denoted as a bracket $[u, v, \dots, w]$, and may be identified with the brackets discussed previously. The meet of an extensor E of step m with an extensor E' of step l is an extensor of step $m+l-n-1$, provided $n+1 \leq m+l$, and its support is the intersection of the supports of E and E' , provided the union of those supports spans U . Thus the join and meet in the Cayley algebra correspond to the lattice operations on subspaces of U , provided the subspaces are independent in the case of join or are sufficiently large in the case of meet.

The condition in Proposition 1.2 that the tie-down bars $\{a, x\}, \{b, y\}, \dots, \{c, z\}$ be on dependent lines in projective space may now be restated as the condition that the 2-extensors ax, by, \dots, cz are linearly dependent in the Cayley algebra.

We will denote by $U^{(m)}$ the subspace of the Cayley algebra spanned by all extensors of step m from U . It also makes sense to use a Cayley algebra over a projective space $PG(R, n)$, by regarding it as the Cayley algebra over the corresponding vector space of dimension $n+1$ over R . Again, $PG(R, n)^{(m)}$ denotes the space of step- m extensors in this case.

3. The pure condition for a stress in an isostatic framework. It follows immediately from the discussion in § 1 that for a generically isostatic graph G with an independent set T of $(n+1) \choose 2$ tie-down bars specified, the condition for the existence of a stress in a specialization α of the generic coordinatization is that the $n|V| \times n|V|$ rigidity matrix $M(G(\alpha), T)$ has determinant equal to zero.

Let T be a tie-down consisting of ax, by, \dots, cz , where $a, b, \dots, c \in V$ and are not necessarily distinct and $x, y, \dots, z \notin V$ are distinct. Let $x_1, x_2, \dots, x_n, y_1, \dots, z_n$ the coordinates of x, y, \dots, z , be distinct indeterminants not involved in the coordinatization of V . Then we say that T is a *generic tie-down*.

LEMMA 3.1. *If G is generically isostatic with a set T of $(n+1) \choose 2$ independent tie-down bars, then the determinant of the rigidity matrix $M(G, T)$ equals an element $C(G, T)$ of the bracket ring B on the set of vertices of $G \cup T$.*

Proof. Assume that G is given a generic coordinatization and that T is also generic. Since the entries of $M(C, T)$ are linear combinations of the coefficients of the vertices, its determinant is an element of $R = k[a_1, a_2, \dots, a_n, b_1, \dots, e_n, w_1, w_2, \dots, w_n, x_1, \dots, y_n]$, where a, b, \dots, e are the vertices of G and w, x, \dots, y the vertices of T which are not vertices of G . We wish to show that $\det M$ is an invariant of degree v , where $v = |V|$, by induction on v .

Let N be an arbitrary $n \times n$ minor in the first n columns of $M(G, T)$. If $\det N \neq 0$, each row of N corresponds to a bar incident to a . Let us denote the bars involved as

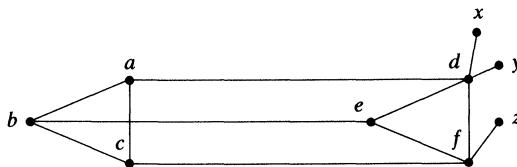
ap, aq, \dots, ar , which may be either bars of $G(\alpha)$ or tie-down bars. Then

$$\begin{aligned} \det N &= \det \begin{bmatrix} a_1 - p_1 & \cdots & a_n - p_n \\ a_1 - q_1 & \cdots & a_n - q_n \\ \cdots & & \cdots \\ a_1 - r_1 & \cdots & a_n - r_n \end{bmatrix} = \det \begin{bmatrix} a_1 - p_1 & \cdots & a_n - p_n & 0 \\ a_1 - q_1 & \cdots & a_n - q_n & 0 \\ \cdots & & \cdots & \cdots \\ a_1 - r_1 & \cdots & a_n - r_n & 0 \\ a_1 & \cdots & a_n & 1 \end{bmatrix} \\ &= \pm \det \begin{bmatrix} p_1 & \cdots & p_n & 1 \\ q_1 & \cdots & q_n & 1 \\ \cdots & & \cdots & \cdots \\ r_1 & \cdots & r_n & 1 \\ a_1 & \cdots & a_n & 1 \end{bmatrix} = \pm [p, q, \dots, r, a]. \end{aligned}$$

Let S be an arbitrary nonsingular transformation applied to $a, b, \dots, e, w, x, \dots, y$. If we now compute the determinant of $SM(G, T)$ by a Laplace expansion by the first n columns, each term in the expansion is an $n \times n$ minor times an $n(v-1) \times n(v-1)$ minor, with S applied to each entry of both minors. Thus, compared to the corresponding term in the expansion of $\det M(G, T)$ the $n \times n$ minor has been multiplied by $\det S$, since such a minor is a bracket by the preceding paragraph. By the induction hypothesis, the $n(v-1) \times n(v-1)$ minor is multiplied by $(\det S)^{v-1}$. Thus $C(G, T) = \det M(G, T)$ is an invariant of degree v and hence an element of B . The lemma now follows by specializing the generic coordinatization and the tie-down. Q.E.D.

The remainder of this section is devoted to showing that the bracket condition $C(G, T)$ factors as $C(G, T) = C(G)C(T)$ in B , where $C(G)$ is independent of the choice of the tie-down bars T and $\alpha C(G)$ is zero in a given specialization α of the coordinatization of V if and only if $G(\alpha)$ has a stress. We call $C(G)$ the *pure condition* for G . We prove these facts via a series of lemmas, after illustrating with an example.

Example.



Let G be the generically isostatic graph shown with 6 vertices and 9 edges in the plane ($n = 2$). Let us adjoin tie-down bars dx, dy, fz . We have chosen a tie-down that will give a particularly simple form to $C(T)$ (see Lemma 4.1). By Rosenberg's method [11] we may compute

$$C(G, T) = \pm [dxy][dfz][abc][def]([adb][ecf] - [ade][bcf]).$$

The sign of $C(G, T)$ depends upon the order in which we list the bars of $G \cup T$ to index the rows of $M(G, T)$. We note parenthetically that computation of $C(G, T)$ can lead to other bracket expressions which are equivalent to the given one via the syzygies. We also note that this is a polynomial of degree 12 in 18 variables, thus we avoid expanded notation whenever possible.

Now we note that this tied-down framework has a trivial motion whenever dxy or dfz is collinear. Thus the irreducible polynomials $[dxy]$ and $[dfz]$ should be factors of $C(T)$, and indeed it can be shown that $C(T) = [dxy][dfz]$.

Thus $C(G) = \pm[abc][def]([adb][ecf] - [ade][bcf])$. From this we can recognize that $G(\alpha)$ has a stress in any coordinatization α in which abc is collinear or def is collinear. The third factor corresponds to the Cayley algebra expression $ad \wedge be \wedge cf$ and is 0 whenever the lines ad , be and cf are concurrent or parallel. We will consider factoring of the pure condition and the corresponding geometric meaning in § 4.

LEMMA 3.2. *Let T be a generic tie-down of $(n+1) \choose 2$ bars. Then the dependence of the step-two extensors ax, by, \dots, cz is a bracket condition $C(T) = 0$, where $C(T)$ is a factor of $C(G, T)$.*

Proof. Each of the step-two extensors ax, by, \dots, cz may be represented by its vector of Plücker coordinates, that is, by the sequence of 2×2 minors of the $(n+1) \times 2$ matrix with columns a and x in the case of ax , etc. Each vector of Plücker coordinates is of length $(n+1) \choose 2$, but we have exactly $|T| = (n+1) \choose 2$ of them, hence we have a square matrix N , and the dependence of the step-two extensors in $PG(R, n)^{(2)}$ is equivalent to $\det N = 0$.

The entries of N are all polynomials in the coordinates of $G \cup T$. It can be shown directly by applying elementary linear transformations that $\det N$ is an invariant, hence an element of B .

Let $C(T) = \det N$. Let \bar{k} be the algebraic closure of k , and let us temporarily work in $\bar{k}[a_1, a_2, \dots, z_n]$. We know from Proposition 1.2 that whenever the step-two extensors ax, by, \dots, cz are dependent in $PG(R, n)^{(2)}$ then $M(G, T)$ has dependent rows. Thus $\alpha C(G, T) = 0$ for any specialization α for which $\alpha C(T) = 0$. Then, by Hilbert's Nullstellensatz, $C(T)|(C(G, T))^r$ for some integer r . But $C(T)$ is at most linear in each coefficient of the vectors x, y, \dots, z , hence $C(T)$ has no multiple factors, hence $C(T)|C(G, T)$.

Since $C(T)$ and $C(G, T)$ both have integer coefficients, we must also have $C(T)|C(G, T)$ in the polynomial ring $k[a_1, a_2, \dots, z_n]$. Q.E.D.

We must now characterize the independent generic tie-downs. This is done in Proposition 3.5, after some preliminary lemmas.

LEMMA 3.3. *Let w_1, \dots, w_k be distinct vectors in a vector space or projective space W of dimension n over $R = k[x_1, \dots, x_r]$, where we assume a standard basis e_1, \dots, e_n of W has been fixed. If $\Phi: R \rightarrow R'$ is a k -algebra homomorphism, then we denote by $\tilde{\Phi}: W \rightarrow W'$ the map obtained by applying Φ coordinatewise. Then w_1, \dots, w_k is linearly independent if and only if there exists Φ such that $\tilde{\Phi}w_1, \dots, \tilde{\Phi}w_k$ is linearly independent with distinct elements.*

Proof. The “only if” statement is trivial. Now we suppose that $\tilde{\Phi}w_1, \dots, \tilde{\Phi}w_k$ is linearly independent with distinct elements, and we extend it to a basis $B' = \tilde{\Phi}w_1, \dots, \tilde{\Phi}w_k, e'_{k+1}, \dots, e'_n$ where (after reindexing) e'_i is from the standard basis of W' . If we reindex e_1, \dots, e_n similarly so that $e'_i = \tilde{\Phi}e_i$, then $B = w_1, \dots, w_k, e_{k+1}, \dots, e_n$ is a preimage of B' . But $\Phi(\det B) = \det B' \neq 0$, hence $\det B \neq 0$ and w_1, \dots, w_k is linearly independent. Q.E.D.

The maps Φ and $\tilde{\Phi}$ of the previous lemma are what we have previously referred to as “specialization” maps.

LEMMA 3.4. *Let v_1, \dots, v_{n+1} be any distinct linearly independent points of $PG(R, n)$. Then the $(n+1) \choose 2$ lines v_iv_j , $i \neq j$, are a basis of $PG(R, n)^{(2)}$.*

We will now show that certain generic tie-downs are independent by applying a specialization map from the tie-down to the $(n+1) \choose 2$ lines of Lemma 3.4. This specialization is no longer a tie-down in the usual sense since we are using only vertices of

the original framework, but this is an easy configuration to use as a standard specialization.

PROPOSITION 3.5. *Let $F = (G, \alpha)$ be a generic framework with $|V| = m \geq n$ and T a generic tie-down of $\binom{n+1}{2}$ bars. For $v_i \in V$, let α_i be the number of tie-down bars incident to v_i , and assume that we have reindexed so that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m$. Then $C(T) \neq 0$ if and only if*

$$(*) \quad \sum_{i=1}^k \alpha_i \leq nk - \binom{k}{2} \quad \text{for all } k, \quad 1 \leq k \leq n-1.$$

Proof. We know $C(T) \neq 0$ if and only if the step-two extensors of the bars in T are linearly independent in $PG(R, n)^{(2)}$. Let K be the subspace of $PG(R, n)$ spanned by v_1, v_2, \dots, v_k . Since P is generic, $\dim K = k-1$. It is well known (and easy to verify) that the subspace of $PG(R, n)^{(2)}$ consisting of all 2-extensors corresponding to lines which intersect K has dimension $n + (n-1) + (n-2) + \dots + (n-k+1) = nk - \binom{k}{2}$. Thus $\sum_{i=1}^k \alpha_i \leq nk - \binom{k}{2}$ for all j_1, j_2, \dots, j_k distinct, and in particular, $\sum_{i=1}^k \alpha_i \leq nk - \binom{k}{2}$. We note that $\dim PG(R, n)^{(2)} = \binom{n+1}{2}$ and $\sum_{i=1}^m \alpha_i = \binom{n+1}{2} = n^2 - \binom{n}{2}$.

Conversely, suppose that the α_i 's satisfy (*). To prove that the 2-extensors of T are linearly independent, it suffices to find a specialization in which they are linearly independent, by Lemma 3.3. Since the vertices in V as well as the tie-down vertices (i.e., vertices on the ‘‘ground’’) all have distinct indeterminants as coordinates, we may specialize so that any pair of vertices that we wish is identified, by specifying a homomorphism Φ that maps the coordinates of one to the coordinates of the other. First we will identify various vertices in V so that exactly $n+1$ distinct vertices remain. The fact that new stresses are thus introduced in F need not concern us, since we are presently concerned only with the independence of the tie-down bars. If we started with $|V| = n$, we add a dummy vertex to V , having no bars, so again $|V| = n+1$. We then identify each tie-down vertex with a vertex of V , hence identifying each tie-down bar with a line between two vertices in V . We will show that this can be done in such a way that distinct tie-down bars are identified with distinct such lines, and hence by Lemma 3.4, the specialized bars are independent, and we are done.

Now suppose that $m > n+1$. We identify v_{n+2} with v_{n+1} to form a new vertex with $\alpha_{n+1} + \alpha_{n+2}$ incident tie-down bars. We choose l such that $\alpha_{l-1} > \alpha_{n+1} + \alpha_{n+2} \geq \alpha_l$, $1 \leq l \leq n+1$. Let $\alpha'_i = \alpha_i$ if $i < l$, $\alpha'_i = \alpha_{i-1}$ if $l < i \leq n+1$, $\alpha'_i = \alpha_{i+1}$ if $n+2 \leq i \leq m-1$ and $\alpha'_i = \alpha_{n+1} + \alpha_{n+2}$. Then α'_i is the number of the tie-down bars incident to the i th vertex after the identification, correctly indexed so that $\alpha'_1 \geq \alpha'_2 \geq \dots \geq \alpha'_{m-1}$. If we show that the α'_i satisfy (*), then by induction we may assume that $m = n+1$ and that (*) is still satisfied. It suffices to consider k such that $l \leq k \leq n$.

Case 1. If $\sum_{i=1}^k \alpha_i < nk - \binom{k}{2} - \alpha_{n+2}$, we are done, for $\sum_{i=1}^k \alpha'_i = (\sum_{i=1}^{k-1} \alpha_i) + \alpha_{n+1} + \alpha_{n+2} \leq (\sum_{i=1}^k \alpha_i) + \alpha_{n+2} < nk - \binom{k}{2}$.

Case 2. $\sum_{i=1}^k \alpha_i \geq nk - \binom{k}{2} - \alpha_{n+2}$. Then $\sum_{i=k+1}^{n+2} \alpha_i = \sum_{i=1}^{n+2} \alpha_i - \sum_{i=1}^k \alpha_i \leq \binom{n+1}{2} - nk + \binom{k}{2} + \alpha_{n+2}$. Since the α_i are decreasing, $\alpha_{n+1} + \alpha_{n+2} \leq 2$ (average $\{\alpha_i\}_{i=k+1}^{n+2}$), hence

$$(n-k+2)(\alpha_{n+1} + \alpha_{n+2}) \leq 2 \left[\binom{n+1}{2} - nk + \binom{k}{2} + \alpha_{n+2} \right],$$

$$\begin{aligned} (n-k+1)(\alpha_{n+1} + \alpha_{n+2}) &\leq (n-k+2)\alpha_{n+1} + (n-k)\alpha_{n+2} \leq 2 \left[\binom{n+1}{2} - nk + \binom{k}{2} \right] \\ &= (n-k+1)(n-k), \end{aligned}$$

so $\alpha_{n+1} + \alpha_{n+2} \leq n-k$. Now $\sum_{i=1}^k \alpha'_i = (\sum_{i=1}^{k-1} \alpha_i) + \alpha_{n+1} + \alpha_{n+2} \leq n(k-1) - \binom{k-1}{2} + n-k = nk - \binom{k}{2} - 1$. Thus we may assume that $m = n+1$.

Let us now form a bipartite graph on the set B consisting of copies of v_i for all i , $B = \{v_{11}, v_{12}, \dots, v_{1\alpha_1}, v_{21}, \dots, v_{n+1,\alpha_{n+1}}\}$, and the set E of pairs v_iv_j , $i < j$, by letting v_{ik} be adjacent to v_iv_j and v_iv_i for all i, j, k . We will now show by Hall's marriage theorem (see, for example, [8, Thm. 5.1.11]) that a complete matching of B into E exists, that is, that it is possible to assign to each v_{ik} an adjacent v_iv_j in one-to-one fashion. We will then be done, for we may specialize the k th tie-down bar at v_i to the line v_iv_j , obtaining the independent specialization required.

If $U \subseteq B$, let $R(U) = \{e \in E : \text{for some } u \in U, e \text{ is adjacent to } u\}$. By Hall's marriage theorem, it suffices to show that $|U| \leq |R(U)|$ for all $U \subseteq B$. For $U \subseteq B$, let $I = \{i : 1 \leq i \leq n+1, \text{ and for some } j, v_{ij} \in U\}$ and $U^* = \{v_{ik} : i \in I\}$. Then $|U| \leq |U^*| = \sum_{i \in I} \alpha_i \leq n|I| - \binom{|I|}{2} = |R(U^*)| = |R(U)|$, completing the proof. Q.E.D.

Let us say that the tie-down T is *saturated at v_k* if $\sum_{i=1}^k \alpha_i = nk - \binom{k}{2}$ and *unsaturated* if it is unsaturated at v_k for all k , $1 \leq k \leq n-1$. In showing that we could collapse down to the case $m = n+1$, we actually showed that if we started with an unsaturated tie-down, then it remains unsaturated after the collapse to $m = n+1$. By a virtually identical proof we could have actually collapsed one step further to $m = n$. However, we then automatically get a tie-down which is saturated at v_n and which can easily be saturated at other v_k as well, even if the original tie-down was unsaturated. We did not need this further collapse for the remainder of the proof, but neither do we need the information about unsaturation. However, we will consider unsaturated tie-downs further in Proposition 3.12.

COROLLARY 3.6. *The dependence or independence of the bars of a generic tie-down of a generic framework is determined solely by the unordered list (with repetition) of the $\binom{n+1}{2}$ vertices to which the tie-down bars are incident.*

COROLLARY 3.7. *If $n = 2$, for the generic tie-down T and the generic framework $F = (G, \alpha)$, $C(T) = 0$ if and only if all three bars of T are incident to the same vertex of G . If $n = 3$, $C(T) = 0$ if and only if, of the six bars of T , at least 4 are incident to one vertex, or 3 are incident to each of two vertices.*

Remark. The situation is much more complicated if we take a nongeneric tie-down. For example, in dimension 3, four bars which determine lines on a common regulus are dependent. The indeterminate (generic) endpoints of our tie-down bars prevent any such special position from occurring. However, we may still state the following for nongeneric tie-downs.

COROLLARY 3.8. *Condition (*) of Proposition 3.5 is a necessary condition for an arbitrary tie-down T to satisfy $C(T) \neq 0$.*

LEMMA 3.9. *Let $I \subseteq \{1, 2, \dots, n\}$, $I \neq \emptyset$ and $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{n+1}$ nonnegative integers such that $\sum_{i=1}^k \alpha_i \leq nk - \binom{k}{2}$ for all k , $1 \leq k \leq n+1$. Let p be minimal such that $p \notin I$, $1 \leq p \leq n+1$ and let q be maximal such that $q \in I$, $1 \leq q \leq n+1$. If $\sum_{i \in I} \alpha_i = n|I| - \binom{|I|}{2}$, then $\alpha_p < \alpha_q$. Furthermore, $q = p-1$ and $I = \{1, 2, \dots, q\}$.*

Proof.

$$\begin{aligned} \alpha_p &= \sum_{i \in I \cup \{p\}} \alpha_i - \sum_{i \in I} \alpha_i \leq \sum_{i=1}^{|I|+1} \alpha_i - \sum_{i \in I} \alpha_i \\ &\leq n(|I|+1) - \binom{|I|+1}{2} - n|I| + \binom{|I|}{2} = n - |I|, \end{aligned}$$

and

$$\begin{aligned} \alpha_q &= \sum_{i \in I} \alpha_i - \sum_{i \in I - \{q\}} \alpha_i \geq \sum_{i \in I} \alpha_i - \sum_{i=1}^{|I|-1} \alpha_i \\ &\geq n|I| - \binom{|I|}{2} - n(|I|-1) + \binom{|I|-1}{2} = n - |I| + 1 \end{aligned}$$

hence $\alpha_p < \alpha_q$, and the rest follows immediately. Q.E.D.

LEMMA 3.10. *Let T be a generic tie-down of $(n+1)/2$ bars of a generic framework F , with α_i tie-down bars incident to the vertex v_i , $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m$, $m = |V| \geq n$. Let T^* be the tie-down obtained from T by specializing one tie-down bar $\{v_j, x\}$ at v_j to the line $v_j v_l$, for $j < l$. Let Φ denote the specialization map such that $\tilde{\Phi}: x \rightarrow v_l$, where $\tilde{\Phi}$ fixes all vertices other than x . Then $\Phi C(T) \neq 0$ if and only if $C(T) \neq 0$.*

Proof. If $\Phi C(T) \neq 0$ then $C(T) \neq 0$ by Lemma 3.3. If $C(T) \neq 0$, then $\sum_{i=1}^k \alpha_i \leq nk - \binom{k}{2}$ for all k . To prove that $\Phi C(T) \neq 0$, we proceed as in the proof of Lemma 3.5, until we form the bipartite graph. Since $\{v_j, x\}$ has already been assigned the line $v_j v_l$, we eliminate $\{v_j, x\}$ from T to obtain a tie-down with incidence numbers $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_j - 1 \geq \dots \geq \alpha_m$. We also eliminate v_{j1} from B and $v_j v_l$ from E in our bipartite graph, letting $B' = B - \{v_{j1}\}$, $E' = E - \{v_j v_l\}$. Now if $U \subseteq B'$ and $R(U)$, I and U^* are defined as before but in terms of the new graph, then if $j \notin I$ and $l \notin I$, then $|U| < |R(U)|$ as before. If $j \in I$, then

$$|U| \leq |U^*| = \left(\sum_{i \in I} \alpha_i \right) - 1 \leq n|I| - \binom{|I|}{2} - 1 = |R(U)|.$$

Finally, if $l \in I$ and $j \notin I$, then by Lemma 3.9, since $j < l$, $\sum_{i \in I} \alpha_i < n|I| - \binom{|I|}{2}$. Thus

$$|U| \leq |U^*| = \sum_{i \in I} \alpha_i \leq n|I| - \binom{|I|}{2} - 1 = |R(U)|,$$

and we are done. Q.E.D.

We will call a tie-down T *nondegenerate* if T has $(n+1)/2$ bars and $C(T) \neq 0$.

Let us return now to the problem of factoring $C(G, T)$ for G generically isostatic. By Lemma 3.2, $C(T)$ is a factor, so we obtain a factorization $C(G, T) = C(T)C_T(G)$. It remains to be shown that $C_T(G)$ is independent of T and is therefore the pure condition $C(G)$ which we seek. Let $F = (G, \alpha)$ be generic, with $|V| \geq n$.

LEMMA 3.11. *Let T' be a generic tie-down of F obtained from the generic tie-down T by replacing a bar ax by dx , where ad is an edge of G . Assume T and T' are nondegenerate. Then $C_T(G) = C_{T'}(G)$.*

Proof. Let us first specialize x to x^* , a point in general position on the line ad (e.g., $x_i^* = \beta a_i + (1-\beta)d_i$ for $i = 1, 2, \dots, n$, where β is an indeterminate). Let T^* and T'^* denote the sets of tie-down bars obtained from T and T' (resp.) by specializing x to x^* , and Φ the specialization map $\tilde{\Phi}: x \rightarrow x^*$. Since both $C(T) \neq 0$ and $C(T') \neq 0$, Lemma 3.10 implies that either $\Phi C(T) \neq 0$ or $\Phi C(T') \neq 0$, depending on which of a or d has more incident tie-down bars. But $\Phi C(T) \neq 0$ if and only if $\Phi C(T') \neq 0$, since T^* determines the same set of lines as T'^* . Thus $\Phi C(T) \neq 0$ and $\Phi C(T') \neq 0$.

Now let us examine the rows of the rigidity matrix for $G \cup T^*$ corresponding to the bars ad and ax^* . These rows have nonzero entries only in columns corresponding to the vertices a and d , namely:

$$\begin{array}{ccc} & a & d \\ ad & \left(a_1 - d_1, a_2 - d_2, \dots, a_n - d_n, \quad d_1 - a_1, d_2 - a_2, \dots, d_n - a_n \right) \\ ax^* & \left(a_1 - x_1^*, a_2 - x_2^*, \dots, a_n - x_n^*, \quad 0, \quad 0, \dots, 0 \right) \end{array}$$

Since a , d and x^* are collinear, the scalar $1-\beta$ satisfies $1-\beta = (a_i - x_i^*)/(a_i - d_i)$ for $i = 1, 2, \dots, n$. If we subtract $1-\beta$ times the row ad from the row ax^* , we have

$$\begin{array}{ccc} ad & \left(a_1 - d_1, a_2 - d_2, \dots, a_n - d_n, \quad d_1 - a_1, d_2 - a_2, \dots, d_n - a_n \right) \\ ax^* & \left(0, \quad 0, \dots, 0, \quad a_1 - x_1^*, a_2 - x_2^*, \dots, a_n - x_n^* \right) \end{array}$$

If $\lambda = (1 - \beta)/(-\beta)$, $\lambda = (a_i - x_i^*)/(d_i - x_i^*)$ for $i = 1, 2, \dots, n$, so we now have

$$\begin{matrix} ad \\ ax^* \end{matrix} \begin{pmatrix} a_1 - d_1, a_2 - d_2, \dots, a_n - d_n, & d_1 - a_1, & d_2 - a_2, & \dots, & d_n - a_n \\ 0, & 0, \dots, & 0, & \lambda(d_1 - x_1^*), \lambda(d_2 - x_2^*), & \dots, \lambda(d_n - x_n^*) \end{pmatrix}$$

and we have used only row operations which leave the determinant of the rigidity matrix unchanged. But these are the rows corresponding to the bars ad and dx^* , except for the factor λ in the rigidity matrix for $G \cup T''$. Thus $\Phi C(G, T) = \lambda \Phi C(G, T')$. Now the vector of Plücker coordinates for ax^* is also λ times the vector for dx^* , as may be easily verified. Hence $\Phi C(T) = \lambda \Phi C(T')$.

It then follows, since $\Phi C(T) \neq 0$ and $\Phi C(T') \neq 0$, that $C_{T''}(G) = C_{T'}(G)$. Thus $C_T(G)$ and $C_{T'}(G)$ are elements of the polynomial ring $k[a_1, \dots, z_n]$ whose images are equal under the specialization map $\Phi: x_i \rightarrow x_i^*, i = 1, 2, \dots, n$. But x_1, x_2, \dots, x_n do not appear in $C_T(G)$ or $C_{T'}(G)$, hence $C_T(G) = C_{T'}(G)$. Q.E.D.

PROPOSITION 3.12. *Let G be generically isostatic in dimension n , with $|V| = m \geq n$. Then any two nondegenerate generic tie-downs T' and T'' satisfy $C_{T'}(G) = C_{T''}(G)$.*

Proof. By Lemma 3.11, we may move a tie-down bar of a nondegenerate generic tie-down T from any vertex to an adjacent vertex, keeping $C_T(G)$ fixed, provided we begin and end such a move with a nondegenerate generic tie-down. We will call such a move an *edge move*, and we will show that we can transform T' to T'' by a sequence of edge moves.

We know that a generic tie-down T with incidence numbers $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m$ is nondegenerate if and only if

$$\sum_{i=1}^k \alpha_i \leq kn - \binom{k}{2},$$

for all k , $1 \leq k \leq n-1$, and T has $\binom{n+1}{2}$ bars. Thus T is unsaturated if and only if T is nondegenerate, and if a single tie-down bar is moved from any vertex to any other vertex, the resulting tie-down is also nondegenerate.

Now we show that if T is nondegenerate, then T may be made unsaturated by a sequence of edge moves. Since G is generically isostatic, G is connected. (In fact, it is not difficult to show that G must be n -connected.) Suppose that r is maximal so that $\alpha_1 = \alpha_2 = \dots = \alpha_r$. Then it is possible to find a path P from some $v \in \{v_1, v_2, \dots, v_r\}$ to v_n such that the path contains no other vertex besides v from $\{v_1, v_2, \dots, v_r\}$. We may reindex so that $v = v_r$. We now successively do an edge move along each edge of the path P . We must show that all the intermediate tie-downs are nondegenerate. But at each step along P , the net effect is to have moved a single tie-down bar from v_r to some vertex v_s , where $s > r$. But the sequence $\alpha_1, \alpha_2, \dots, \alpha_r - 1, \dots, \alpha_s + 1, \dots, \alpha_m$ satisfies $(*)$ if $\alpha_1, \dots, \alpha_n$ does. Thus the intermediate tie-downs are nondegenerate, and by valid edge moves, we have transformed T to a tie-down T_1 with incidence numbers $\beta_1 \geq \dots \geq \beta_m$, where $\beta_r = \alpha_r - 1$, $\beta_t = \alpha_n + 1$ for some $t \leq n$ and $\beta_i = \alpha_i$ otherwise. But $t < n$ only if $\alpha_t = \alpha_{t+1} = \dots = \alpha_n$, whence by Lemma 3.9, $\sum_{i=1}^k \beta_i = \sum_{i=1}^k \alpha_i < nk - \binom{k}{2}$ for $t \leq k \leq n-1$ and similarly for $1 \leq k \leq t-1$. For $r \leq k \leq t-1$, $\sum_{i=1}^k \beta_i < \sum_{i=1}^k \alpha_i$, and so T_1 is unsaturated.

Thus we may assume that T' and T'' are unsaturated. Let $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m$ be the incidence numbers of T' and $\beta_1, \beta_2, \dots, \beta_m$ the incidence numbers of T'' . Thus the vertex v_i has α_i incident tie-down bars in T' and β_i in T'' . We may assume that if $\alpha_i = \alpha_{i+1}$, then $\beta_i \geq \beta_{i+1}$. We proceed now by induction on $\sum_{i=1}^m |\beta_i - \alpha_i| = q$. If $q = 0$, $\alpha_i = \beta_i$ for all i and T' is isomorphic to T'' and $C_{T'}(G) = C_{T''}(G)$. If $q > 0$, we consider two cases.

Case 1. There exist $j < l$ such that $\beta_j < \alpha_j > \alpha_l < \beta_l$. Then we choose any path in G from v_j to v_l and do a sequence of edge moves along that path for the tie-down T' . Since T' is unsaturated, at each step along the path the resulting tie-down is nondegenerate, and the result is a tie-down T''' with incidence numbers $\alpha_1, \dots, \alpha_j - 1, \dots, \alpha_l + 1, \dots, \alpha_m$, where we may have to reindex to keep decreasing order, say $\alpha'_1 \geq \dots \geq \alpha'_m$ but if so, we reindex the β_i in the same way.

Now we must check that T''' is also unsaturated. If $\alpha_j - 1 \geq \alpha_l + 1$, we cannot have increased the partial sums $\sum_{i=1}^k \alpha'_i$. If $\alpha_j - 1 < \alpha_l + 1$, then $\alpha_j = \alpha_l + 1$ and $\alpha_i = \alpha'_i$ for all i , although α_i and α'_i need not refer to the same vertex. Therefore the partial sums $\sum_{i=1}^k \alpha_i$ remain unchanged, and T''' is unsaturated since T' is. But now T'' and T''' satisfy $C_{T''}(G) = C_{T'''}(G)$ by the induction hypothesis, hence $C_{T'}(G) = C_{T''}(G)$.

Case 2. There exists p such that $\beta_i \geq \alpha_i$ for all $i < p$ and $\beta_i \leq \alpha_i$ for all $i \geq p$. Thus $\beta_i \geq \alpha_i \geq \alpha_j \geq \beta_j$ if $i < p \leq j$. Let us now reindex so that $\beta_1 \geq \beta_2 \geq \dots \geq \beta_m$ and reindex the α_i in the same way. Note that the original $\beta_1, \dots, \beta_{p-1}$ remain the first $p-1$ entries in perhaps different order. Since $q > 0$, and $\sum_{i=1}^m \alpha_i = \sum_{i=1}^m \beta_i = {n+1 \choose 2}$, there exist j, l such that $\alpha_j < \beta_j$ and $\alpha_l > \beta_l$. But then $j < p$ and $l \geq p$ and since $\alpha_j \geq \alpha_l, \beta_j > \beta_l$. Thus $\alpha_j < \beta_j > \beta_l < \alpha_l$ and we apply the argument of Case 1 with α and β interchanged. Q.E.D.

THEOREM 3.13. *If G is a generically isostatic graph in dimension n with $|V| \geq n$, then there exists an element $C(G)$ of the bracket ring on the vertices of G such that for any specialization α of the generic coordinatization of G given by a specialization map Φ , $G(\alpha)$ has a stress if and only if $\Phi C(G) = 0$.*

Proof. Choose a nondegenerate generic tie-down T for $G(\alpha)$. Then by Proposition 1.2, $G(\alpha)$ has a stress if and only if $G(\alpha) \cup T$ has a stress, if and only if $\Phi C(G, T) = 0$, if and only if $\Phi C_T(G) = 0$, since $C(T) \neq 0$. Since $C_T(G)$ is independent of the choice of T , we take $C(G) = C_T(G)$. Q.E.D.

COROLLARY 3.14. *Let G be an isostatic framework with $|V| > 3$ and $n = 2$ or 3. With the generic coordinatization on G , let T^* be an arbitrary tie-down. Then if $C(G, T^*)$ is the determinant of the rigidity matrix and $C(T^*)$ the bracket condition for the dependence of the tie-down bars (now specialized to T^* instead of a generic tie-down), we still have $C(G, T^*) = C(T^*)C(G)$.*

Proof. To the polynomial ring $k[a_1, a_2, \dots, a_n, \dots, z_n]$ apply the specialization map Φ (a ring homomorphism) taking $x_1, x_2, \dots, x_n, \dots, z_n$, to the coordinates of the corresponding endpoints of the bars in T^* . The equation $C(G, T) = C(T)C(G)$ for a generic tie-down T is preserved under the homomorphism but $C(G)$ is independent of $x_1, x_2, \dots, x_n, \dots, z_n$, hence the result. Q.E.D.

There are a number of unsolved problems regarding the pure condition. Can two distinct isostatic frameworks on the same set of vertices have identical pure conditions? Given a bracket expression, what frameworks have a pure condition with the given expression as a factor? Other problems relating to the factoring of the pure condition are discussed in the following sections.

4. Factoring pure conditions. We know that, for a graph with the correct count $|E| = n|V| - {n+1 \choose 2}$, the infinitesimal rigidity of the framework in n -space is equivalent to the pure condition being nonzero at that coordinatization. What do these pure conditions actually look like? Since there is no generally available collection of graphs and their pure conditions, we begin with two tables of examples—one for the plane and one for 3-space. Since the pure condition may change sign if the order of the edges is changed, all conditions are given up to a global sign.

These tables require some explanation and raise certain obvious questions: How does one determine these pure conditions? Do algebraic patterns, such as the factoring,

reflect underlying patterns in the graph? How do we know that the factors shown are irreducible?

The answers to all these questions are tied up together, since knowledge of the factoring is often used to determine the pure condition given in the tables. We will summarize the techniques we used under four headings. The first three subsections will derive pure conditions, using certain patterns in the graph and in the factoring of the conditions, while the fourth subsection outlines techniques to show that the given factors are irreducible. Along the way we will explain the conditions given in Tables 1 and 2.

TABLE 1
Plane frameworks

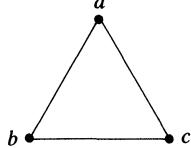
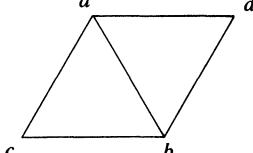
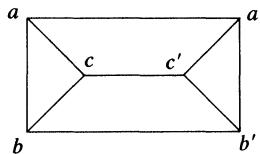
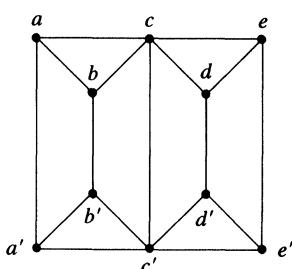
Name and graph	Pure condition and geometric condition
1.1. Triangle	Points a, b, c , collinear $[abc]$
	
1.2. Two triangles	One of the triangles abc or abd is collinear $[abc][abd]$
	
1.3. Triangular prism	Triangles $abc, a'b'c'$ are perspective from a line \equiv Triangle abc or $a'b'c'$ is collinear or the two triangles are perspective from a point $[abc][a'b'c']([abb'][a'c'c] - [a'bb'][ac'c'])$
	
1.4. Edge linked prisms	Either one of the triangles is collinear or one of the triples aa' , bb' , cc' or cc' , dd' , ee' is concurrent $[abc][a'b'c'][cde][c'd'e']$ · $([abb'][a'c'c] - [a'bb'][ac'c'])$ · $([cdd'][c'e'e] - [c'dd'][ce'e])$
	

TABLE 1 (cont.)

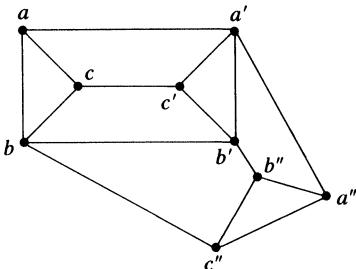
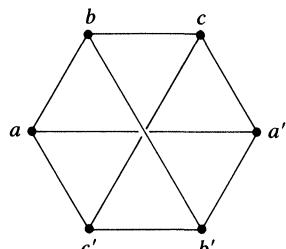
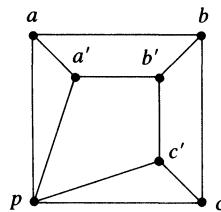
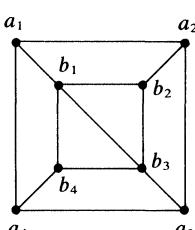
Name and graph	Pure condition and geometric condition
1.5. 3 vertex linked prisms	<p>Either one of the triangles is collinear or one of the triples aa', bb', cc' or $a'a''$, $b'b''$, bc'' is concurrent</p>  <p>$[abc][a'b'c'][a''b''c'']$ $\cdot ([abb'][a'c'c] - [a'bb'][ac'c])$ $\cdot ([a'b'b''][a''c''b] - [a''b'b''][a'c''b])$</p>
1.6. $K_{3,3}$	<p>The six joints lie on a plane conic</p>  <p>$[abc][ab'c'][a'b'c][a'bc']$ $-[a'bc][a'b'c'][ab'c][abc']$</p>
1.7.	<p>Either one of the triangles is collinear or the three points $ab \wedge a'b'$, $bc \wedge b'c'$ and p are collinear</p>  <p>$[paa'][pcc'][[aba'][[bcb'][b'c'p] - [abb'][bcb'][a'c'p]$ $+ [abb'][bcc'][a'b'p])$</p>
1.8. Cube with 1 bar	<p>Either one of the triangles is collinear or the three points $a_1b_1 \wedge a_2b_2$, $a_3b_3 \wedge a_4b_4$ and $a_1a_4 \wedge a_2a_3$ are collinear</p>  <p>$[b_1b_2b_3][b_1b_3b_4][a_1a_2b_1][a_2a_3b_2][a_3a_4b_3][a_4a_1b_4]$ $- [a_1a_2b_2][a_2a_3b_3][a_3a_4b_4][a_4a_1b_1])$</p>

TABLE 2
Spatial frameworks

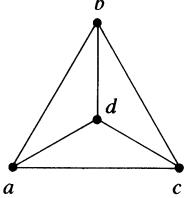
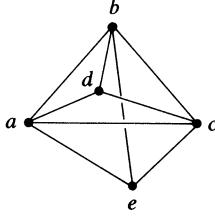
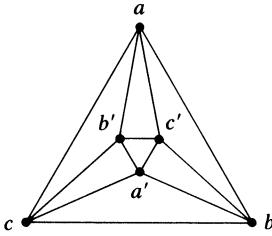
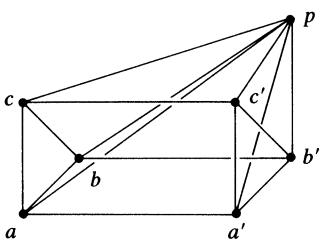
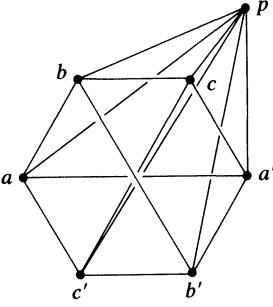
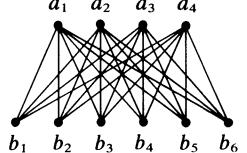
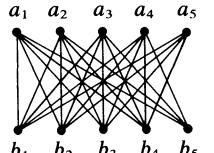
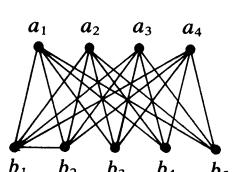
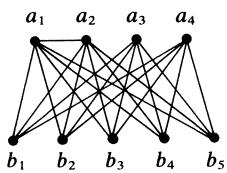
Name and graph	Pure condition and geometric condition
2.1. Tetrahedron	The four points are coplanar [abcd]
	
2.2. Triangular bipyramid	One of the quadruples abcd or abce is coplanar [abcd][abce]
	
2.3. Octahedron	Four alternate face planes abc, ab'c', a'bc', a'b'c are concurrent in a point [abc'a'][bca'b'][cab'c']+ [abc'b'][bca'c'][cab'a']
	
2.4. 1-point cone on the prism	Projected from p onto a plane, the prism appears perspective from a line [abcp][a'b'c'p][(abb'p)[a'cc'p]-[a'bb'p][acc'p])]
	

TABLE 2 (cont.)

Name and graph	Pure condition and geometric condition
2.5. 1-point cone on $K_{3,3}$	Projected from p onto a plane, the $K_{3,3}$ appears to lie on a conic $[abcp][ab'c'p][a'b'cp][a'bc'p]$ $-[a'bcp][a'b'c'p][ab'cp][abc'p]$
	
2.6. $K_{4,6}$	Either $a_1a_2a_3a_4$ are coplanar or the 10 points lie on a quadric surface $[a_1a_2a_3a_4]^2Q(a_1 \cdots a_4, b_1 \cdots b_6)$
	
2.7. $K_{55} - \{a_5, b_5\}$	Either $a_1a_2a_3a_4$ or $b_1b_2b_3b_4$ are coplanar or the ten points lie on a quadric surface $[a_1a_2a_3a_4][b_1b_2b_3b_4]Q(a_1 \cdots a_5, b_1 \cdots b_5)$
	
2.8. $K_{4,5} + 1$ edge	Either $a_1a_2a_3a_4$ are coplanar or the nine points $a_1 \cdots a_4b_1 \cdots b_5$ and the line of the added edge lie on a quadric surface $[a_1a_2a_3a_4]Q(a_1 \cdots a_4, b_1 \cdots b_5, (b_1+b_2)/2)$ $[a_1a_2a_3a_4]Q(a_1 \cdots a_4, b_1 \cdots b_5, (a_1+a_2)/2)$
	
	

4.1. Direct calculation of pure conditions. It is possible to directly decompose $\det(M(G, T))$ as a bracket expression, using a Laplace expansion which we will return to at the end of this section. In such a direct calculation, it is desirable to have the tie-down condition $C(T)$ appear as an immediate factor.

LEMMA 4.1. *If an n -isostatic graph includes the vertices v_1, \dots, v_n and the generic tie-down T is given with $\alpha_1 = n, \alpha_2 = n - 1, \dots, \alpha_n = 1$ and tie-down bars $\{v_i, x_{i,j}\}, i < j \leq n + 1$, then the tie-down factor is*

$$C(T) = [v_1 x_{1,2} \cdots x_{1,n+1}] [v_1 v_2 x_{2,3} \cdots x_{2,n+1}] \cdots [v_1 v_2 \cdots v_n x_{n,n+1}].$$

Proof. The tie-down factor is independent of the n -isostatic graph which includes the joints v_1, \dots, v_n . For convenience we will use the complete graph on these n joints and reshuffle the edges to give the order

$$\{v_1, x_{1,2}\}, \dots, \{v_1, x_{1,n+1}\}, \{v_1, v_2\}, \{v_2, x_{2,3}\}, \dots, \{v_{n-1}, v_n\}, \{v_n, x_{n,n+1}\}.$$

We now do a Laplace expansion on the last n -columns—the columns of v_n : The only nonzero term uses the last n -rows and is the bracket $[v_1, \dots, v_n, x_{n,n+1}]$ times the corresponding minor. We now do a Laplace expansion of this minor by the n -columns for v_{n-1} —giving only one nonzero term with the bracket $[v_1, \dots, v_{n-1}, x_{n-1,n}, x_{n-1,n+1}]$. We continue this process, finding the last factor (using the n columns for v_1) $[v_1, x_{1,2}, \dots, x_{1,n+1}]$. Thus $C(G, T) = [v_1, x_{1,2}, \dots, x_{1,n+1}] \cdots [v_1, \dots, v_n, x_{n,n+1}] = C(T)$.

The graph G has the pure condition 1 because such a complete graph on n joints on n -space is isostatic if and only if the joints span an affine $n - 1$ space—and this is true whenever $C(T) \neq 0$.

We conclude that $C(T)$ has the desired form. Q.E.D.

In any reasonable decomposition of $\det M(G, T)$, using this standard tie-down, the given brackets will appear as factors of each monomial of the decomposition. Thus no energy or ingenuity need be expended in pulling out this tie-down factor, and we always choose this tie-down in actual calculations of pure conditions.

The proof of Lemma 4.1 gives the following corollary. An n -simplex is the complete graph on $n + 1$ vertices.

COROLLARY 4.2. *The pure condition for an $n - 1$ simplex in n -space is 1.*

The pure condition for an n simplex in n -space is $[v_1, \dots, v_{n+1}]$.

Proof. The $n - 1$ simplex was directly given in the proof of Lemma 4.1.

To obtain the n simplex from the $n - 1$ simplex we add one vertex, v_{n+1} , and n edges $\{v_i, v_{n+1}\}, 1 \leq i \leq n$. We add n columns for v_{n+1} , and these n rows at the bottom of the matrix used in Lemma 4.1. A Laplace expansion by the last n columns gives the brackets $(\pm)[v_1 \cdots v_{n+1}]$ times the cofactor which is $C(T)$. Q.E.D.

Remark. It is clear from this analysis that if any graph G' is built from an n -isostatic graph G by adding a new n -valent vertex p with edges $\{p, a_i\}, 1 \leq i \leq n$, then the pure condition has the form $C(G') = [p, a_1, \dots, a_n]C(G)$.

In general $\det(M, T)$ can always be decomposed by taking a series of Laplace expansions on the n columns for each vertex in turn (Rosenberg, [11]). Such an expression will produce a sum of monomials, each of which has the form

$$\prod_i [a_i, b_{i,1}, \dots, b_{i,n}],$$

where the rows $\{a_i, b_{ij}\}, 1 \leq j \leq n$ were used for the columns of a_i in this term. The following useful property follows from this expression by a simple counting argument (using the simple tie-down).

LEMMA 4.3. *The pure condition for an n -isostatic graph G is homogeneous of degree $k+1-n$ in each vertex of valence k in the graph.*

4.2. Factors determined by decomposition of the graph. A second source of pure conditions lies in certain decompositions of the graph. We begin with the simplest result of this type.

PROPOSITION 4.4. *If G is an n -isostatic graph and H is an n -isostatic subgraph with at least $n+1$ vertices, then $C(G) = C(H) \cdot C'$ for some factor C' .*

Proof. We attach the simple tie-down T to n vertices in H . The tie-down rows, plus the rows corresponding to edges in H now give a square submatrix, with all other entries in these rows zero, and a simple Laplace expansion, using these rows as a block, gives

$$C(G, T) = C(H, T) \cdot C' = C(T) \cdot C(H) \cdot C'. \quad \text{Q.E.D.}$$

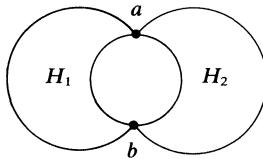
This result explains the illustrated factoring for examples such as the prism (Table 1, 1.3), the combinations of prisms (Table 1, 1.4 and 1.5), other planar graphs with triangles (Table 1, 1.7, 1.8) and the 1-point cone on a prism (Table 2, 2.4).

The form of C' depends on the pattern of the rest of the graph. When the number of edges or vertices of attachment to H is small, then we can give more details about C' . A number of such examples are illustrated in Table 3.

TABLE 3a
Plane

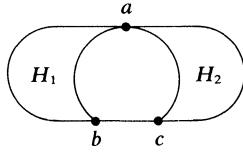
3.1

If H_1 is 2-isostatic then $C(G) = C(H_1) \cdot C(H_2 + \{a, b\})$. If neither H_1 nor H_2 is 2-isostatic $C(G) \equiv 0$



3.2

If H_1 and H_2 are 2-isostatic then $C(G) = C(H_1) \cdot [abc] \cdot C(H_2)$. Otherwise $C(G) \equiv 0$



3.3

If H_1 and H_2 are 2-isostatic then $C(G) = C(H_1) \cdot ([a_1b_1a_2] \cdot a_3b_3b_2 - [a_1b_1a_2][a_3b_3a_2]) \cdot C(H_2)$. Otherwise $C(G) \equiv 0$

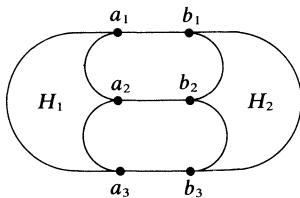
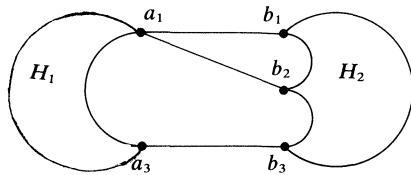


TABLE 3a (cont.)

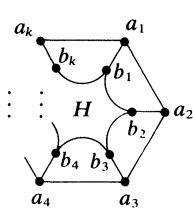
3.4

If H_1 and H_2 are 2-isostatic then $C(G) = C(H_1) \cdot [a_1 b_1 b_2] \cdot [a_3 a_1 b_3] \cdot C(H_2)$



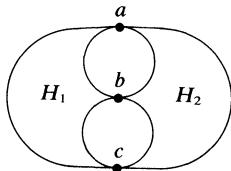
3.5

If H is 2-isostatic $C(G) = C(H) \cdot ([a_1 a_2 b_1] \cdots [a_k a_1 b_k] + (-1)^{k+1} [a_1 a_2 b_2] \cdots [a_k a_1 b_1])$

TABLE 3b
Space

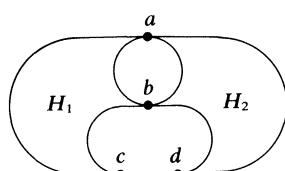
3.6

If H_1 is 3-isostatic then $C(G) = C(H_1) \cdot C(H_2 \cup \{a, b\}, \{b, c\}, \{c, a\})$



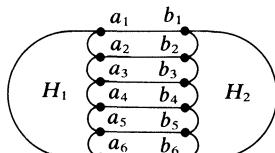
3.7

If H_1 is 3-isostatic then $C(G) = C(H_1) \cdot [abcd] \cdot C(H_2 \cup \{a, b\})$. If neither H_1 nor H_2 is 3-isostatic then $C(G) = 0$



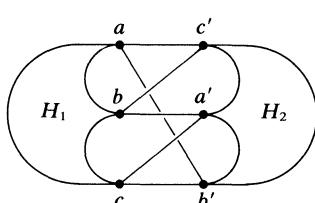
3.8

If H_1 and H_2 are 3-isostatic then $C(G) = C(H_1) \cdot C(H_2) \cdot C(S)$, where $C(S)$ is the "tie down" factor for $a_1 b_1, \dots, a_6 b_6$.



3.9

If H_1 and H_2 are 3-isostatic then $C(G) = C(H_1) \cdot C(H_2) \cdot ([bca'b'][cab'c'][abc'a'] + [cab'a'][abc'b'][bca'c'])$. If one of H_1, H_2 is not 3-isostatic then $C(G) = 0$.



The simplest examples in Table 3 cover two n -isostatic subgraphs tied together by $(n+1)_2$ edges (Table 3, 3.3, 3.4, 3.8, 3.9). Such a tie-together S gives a factoring

$$C(G) = C(H_1) \cdot C(S) \cdot C(H_2),$$

where $C(S)$ is a specialization of the tie-down factor due to any identification of the ends of the tie-together edges. If we tie down H_1 , then H_1 functions as a “ground” while S gives a tie-down for H_2 .

A second property of a graph which is reflected in the pure condition concerns 1 point cones (Table 2, 2.4, 2.5).

DEFINITION 4.1. Given a graph $G = \langle V, E \rangle$ the 1-point cone $G * p$ is the graph with vertices $V \cup \{p\}$ and edges $E \cup \{\{p, v_i\} | v_i \in V\}$.

PROPOSITION 4.5. If G is an n -isostatic graph with condition $C(G)$, then the 1 point cone $G * p$ is an $(n+1)$ -isostatic graph with pure condition $C(G * p) = C(G) * p$, where $(L) * p$ means extending each bracket in L by inserting an $(n+1)$ st entry p .

Proof. We take the standard tie-down T_n on G , at vertices a_1, \dots, a_n , and obtain $M(G, T_n)$ and take the standard tie-down T_{n+1} of $G * p$, at vertices p, a_1, \dots, a_n to obtain $M(G * p, T_{n+1})$.

We obtain $\det M(G, T_n) = C(T_n) \cdot C(G)$ by a series of Laplace expansions, each on the n columns of a vertex of G , and similarly $\det M(G * p, T_{n+1}) = C(T_{n+1}) \cdot C(G * p)$ by a series of Laplace expansions each on the $n+1$ columns of a vertex of $G * p$. The term for the columns of p is part of the tie-down factor, $[py_1 \cdots y_{n+1}]$, and for each other vertex v_i the nonzero terms will involve n rows of G , plus the row for $\{p, v_i\}$. Otherwise, some term has no occurrence of p —another term has two occurrences of p —and we have the zero term. Since these two expansions give the form $C(G * p, T_{n+1}) = [py_i \cdots y_{n+1}] \cdot (C(G, T_n) * p)$ and $C(T_{n+1}) = [py_i \cdots y_{n+1}] \cdot (C(T_n) * p)$ we have the desired result. Q.E.D.

Remark. This proposition reflects the geometric theorem that a framework realizing a 1-point cone in $(n+1)$ -space with apex p has a static stress if and only if the projection of the framework from p into an n -space has a stress in the n -space [19, § 10]. The geometric process of projection is expressed in the brackets as a reduction by p —placing p in each bracket as the last entry—and then deleting the p 's, thus moving from brackets of length $n+2$ to brackets of length $n+1$.

4.3. Factors and pure conditions by geometry. Other direct analyses of the infinitesimal and static behavior of frameworks have produced projective geometric statements of sufficient (and necessary) conditions for nontrivial motions or stresses in frameworks with various graphs [19], [20], [21], [22]. Either by the direct presentation, or by a simple translation, such projective conditions for n -isostatic graphs reduce to a single polynomial $F(G)$ such that $F(G) = 0$ is sufficient for a realization of G to not be isostatic.

LEMMA 4.6. If a polynomial F in the vertices of an n -isostatic graph G has the property that $F(G) = 0 \Rightarrow C(G) = 0$, then each irreducible factor of F is a bracket expression which is a factor of $C(G)$.

Proof. The analysis of $C(G)$, and of the sufficient conditions $F(G) = 0$, are done over the complex numbers. By Hilbert's Nullstellensatz we have

$$(F(G) = 0 \Rightarrow C(G) = 0) \Rightarrow A \cdot F(G) = (C(G))^r,$$

where A is some bracket expression and r is a positive integer. Clearly each irreducible factor of F is a factor of $(C(G))^r$ —and thus of $C(G)$. Q.E.D.

Lemma 4.6 can give us some factors of $C(G)$, based on our geometric analyses. When combined with Lemma 4.3, which limits the total occurrences of each vertex in $C(G)$, it is possible to count how many, if any, occurrences of joints remain after this factoring. In many cases these two lemmas give a complete description of the pure condition.

For example, the factors used in the condition for the prism (Table 1, 1.3), the $K_{3,3}$ (Table 1, 1.5), and other plane examples (Table 1, 1.7, 1.8), as well as the octahedron in space (Table 2, 2.3) were originally calculated by a direct geometric analysis. In each case these known factors contain all available occurrences of the joints—so they must be the pure condition, provided they are irreducible. In § 4.4 we will verify this irreducibility.

A more surprising class of examples includes the bipartite framework $K_{4,6}$ in 3-space (Table 2, 2.6). We recall that a *bipartite graph* $K_{m,n}$ has vertices $V = \{a_1, \dots, a_m, b_1, \dots, b_n\}$ and edges $\{a_i, b_j\}$, $1 \leq i \leq m$, $1 \leq j \leq n$. It is a simple counting argument to check that $K_{n+1,m}$, where $m = \binom{n+1}{2}$, counts to be an n -isostatic graph—(or at least give a square matrix $M(G, T)$).

PROPOSITION 4.7. *The pure condition in n -space for the bipartite graph $K_{n+1,m}$, where $m = \binom{n+1}{2}$, is*

$$[a_1 \cdots a_{n+1}]^d \cdot Q(a_1, \dots, a_{n+1}, b_1, \dots, b_m),$$

where $d = (n+1)(n-2)/2$ and $Q(a_1, \dots, b_m)$ is the bracket expression for all the joints to lie on a quadric surface in n -space.

Proof. By the analysis of [22] if the joints lie on a quadric surface, then there is an infinitesimal motion. The expression for the joints to be on a quadric surface is a projectively invariant equation of degree 2 in each joint, since picking values for all but 1 joint must leave a general quadric equation. (This expression is irreducible, as we will see in Proposition 4.9.) When this factor Q is removed from the pure condition, we are only left with the vertices a_1, \dots, a_{n+1} , which still occur to degree $d = \binom{n+1}{2} - (n+1)$. The only possible nonzero n -bracket formula with $n+1$ vertices is $[a_1 \cdots a_{n+1}]$, so this occurs d times. (The factor is nonzero, since the graph does have n -isostatic realizations.) *Q.E.D.*

Remark 1. The factor $[a_1 \cdots a_{n+1}]$ ($n \geq 3$) was also predicted by the geometric analysis, since [22, Thm. 1.1] guarantees a nontrivial infinitesimal motion whenever all points a_1, \dots, a_{n+1} lie in a quadric surface of a hyperplane in n -space. In fact $[a_1 \cdots a_{n+1}] = 0$ guarantees that these joints lie on d such quadric surfaces of the hyperplane, thus giving d motions and d stresses to match the d identical factors. We return to this “coincidence” in Chapter 6.

Remark 2. Since, for example, $K_{4,6}$ has 2 stresses when $[a_1 \cdots a_4] = 0$, we also know that removing one bar in that case will still leave at least 1 stress, regardless of the position of the b_i . Removing $\{a_4, b_6\}$ also leaves b_6 as a 3-valent joint, which will not participate in the dependence unless b_6 is in the plane of $a_1a_2a_3$. Thus the condition $[a_1 \cdots a_4] = 0$ actually is sufficient for a stress in the 1-underbraced frameworks realizing $K_{4,5}$. As a result, any graph G containing this $K_{4,5}$ as a subgraph must have $[a_1 \cdots a_4]$ as a factor of its pure condition. By a similar argument, we find $K_{n+1,n+2}$ can be a strongly $(d-1)$ underbraced graph for $n > 3$ which still induces a factor $[a_1 \cdots a_{n+1}]$ in the pure condition of any n -isostatic graph containing it.

Remark 3. We have previously observed (§ 4.2) that the presence of an n -simplex gives the factor $[a_1 \cdots a_{n+1}]$. We have now found that a bipartite framework with *none* of these edges present can give the same factor. There is no simple correlation between factors and subgraphs. Similar factors give a similarity in the *geometric*

conditions under which a framework is critical (not-isostatic). However we conjecture that if two graphs give the same pure condition, then the frameworks are the same.

By an argument similar to the proof of Proposition 4.7, we conclude that the graph $K_{5,5} - \{a_5, b_5\}$ has the pure condition given in Table 2, 2.7. The simplest known bracket expression for $Q(a_1, \dots, b_5)$ is given in [16, p. 266] as the sum of 240 bracket monomials! The same paper gives expressions for a quadric through 9 points and 1 line (e.g., $Q(a_1, \dots, a_4, b_1, \dots, b_5, (a_1+a_2)/2)$ in Table 2, 2.8) as the sum of 6 monomials. The pure condition for this last example follows from the geometric analysis of [22, Thm. 4.1] by an analogous argument.

4.4. Irreducibility of factors of the pure condition. In the tables and the preceding discussion we have offered many “irreducible” factors. However these were often enormous polynomials in, say 40 variables (for $K_{4,6}$), and it requires some proof to see that such expressions are irreducible.

At present we have two main tools for proving this irreducibility—symmetries of the graph, or the corresponding geometric conditions, which would impose symmetry on the factoring, and specializations of the coordinatization which reduce the expression to a simpler form, which is either known to be irreducible or else factors only in a way which is incompatible with the original symmetries or geometry of the condition. Without being exhaustive, we illustrate these techniques on the simple examples given in the tables.

PROPOSITION 4.8. *The tie-down factor $C(T)$ for $T = \{\{a_i, x_i\} \mid 1 \leq i \leq \binom{n+1}{2}\}$ is irreducible if a_i, x_i are distinct.*

Proof. We view T as a tie-down of a rigid body—and recognize that this factor is independent of the n -isostatic graph used to connect the a_i . We take a specialization Φ which identifies the a_i with appropriate joints b_j of the standard tie-down and gives a homomorphism of the polynomial

$$\Phi(C(T)) = [b_1 y_{1,2} \cdots y_{1,n+1}] [b_1 b_2 y_{2,3} \cdots y_{2,n+1}] [b_1 \cdots b_n y_{n,n+1}].$$

From this factoring of $\Phi(C(T))$ we conclude, for example that $\Phi^{-1}(y_{1,2}), \dots, \Phi^{-1}(y_{1,n+1})$ (some of the x_i) must be in the same factor of $C(T)$. But this identification was arbitrary—so all the x_i lie in the same factor. Similarly, $\Phi^{-1}(b_n)$ and $\Phi^{-1}(y_{n,n+1})$ must be in the same factor of $C(T)$. However $\Phi^{-1}(b_n)$ is an arbitrary a_i —so we conclude that all a_i and x_i are in the same factor. Since $C(T)$ is of first degree in all a_i and x_i , this requires that $C(T)$ is irreducible. Q.E.D.

If we apply this result to the tie-together of 2 bodies, we have shown the irreducibility of the factors in the examples in Table 1, 1.3, 1.4, 1.5. The result for 1-point cones also gives the factoring of $C(G * p)$, so we have also explained the factors of Table 2, 2.4.

PROPOSITION 4.9. *The bracket condition Q_n for $\binom{n+2}{2}$ points to lie on a quadric surface in n -space is irreducible.*

Proof. Assume that the condition Q_n factors as $f \cdot g$.

Case 1. f is of first degree in some point a . Since Q is of degree 2, g is also of first degree in a . By a suitable choice of real position for all the other points (which define, in general, a unique quadric) the condition Q can be specialized to $Q(a) = a_1^2 + a_2^2 = f(a) \cdot g(a)$. Since $a_1^2 + a_2^2$ is irreducible, we have a contradiction.

Case 2. F is of degree 2 in some set of points a, \dots, c , while g is of degree 2 in the remaining points d, \dots, f . However the geometric condition is symmetric in all points, so if a and c share a factor, so must a and d , etc. We conclude that all points appear in the factor f , and $g = 1$. Q.E.D.

If the factor being examined is small (a sum of monomials with ≤ 3 brackets), then any further factoring must include a factor which is a sum of single brackets. However the factors are homogeneous, so there must be a single bracket as a factor—and thus some set of $n+1$ joints which, if coplanar, would induce a stress. In many cases this possibility can be eliminated by a direct inspection of the geometry.

Consider example Table 1, 1.7. The third factor is reducible if and only if some *triple of joints in this factor* being collinear causes a stress. Since we know the geometric condition (see the table), it is simple to check that any such triple can be collinear while the factor is $\neq 0$. We conclude that this third factor is irreducible.

The octahedron (Table 2, 2.3) also gives an irreducible factor. By Cauchy's theorem and its extensions [23] any set of 4 joints such as $abcc'$ or $aba'b'$ can be made coplanar without making the condition $=0$. By the symmetries of the graph the same is true for all quadruples, so the condition is an irreducible polynomial.

If we take any planar graph ($|V|-|E|+|F|=2$) with $|E|=2|V|-3$, $|V|\geq 3$, a simple counting argument shows that there must be at least one triangle. Thus a planar graph with more than 3 vertices must have factors in its pure condition. Only nonplanar graphs, such as $K_{3,3}$, can have irreducible pure conditions.

The situation for 3-isostatic graphs is much more complicated. While the construction of general 3-isostatic graphs is an unsolved problem, some classes, such as triangulated spheres, are well known [23].

5. Overbraced frameworks. So far we have been considering isostatic frameworks, which may be described as maximal frameworks which have no stress in generic position. Let $I(v)=nv-{n+1 \choose 2}$. Then we know that an isostatic framework has exactly $I(v)$ bars. We now consider *overbraced* frameworks, that is, frameworks with more than $I(v)$ bars. Such frameworks always have a stress. Let G be such a framework. Then every subframework G' of G having exactly $I(v)$ bars is either generically isostatic, in which case we have the pure bracket condition $C(G')$ for the existence of a stress of G' , or else G' generically has a stress, in which case the rigidity matrix of G' has dependent rows, and we may set $C(G')=0$.

If G is a 1-overbraced framework, that is, a framework with exact $I(v)+1$ bars, we can actually compute the coefficients of a stress of G , using brackets.

THEOREM 5.1. *Let G be a 1-overbraced framework with bars E_0, E_1, \dots, E_I such that for some j , $G-E_j$ is isostatic. Then there exists a nonzero stress on G whose value on the bar E_i is $(-1)^i C(G-E_i)$, where the bars of $G-E_i$ are taken in order of subscript in computing $C(G-E_i)$.*

Proof. Choose a nondegenerate generic tie-down T , and let M be the generic rigidity matrix of $G \cup T$. Since M is an $(nv+1) \times nv$ matrix, its rows are linearly dependent and the coefficients of any such dependence are the values of a stress. Let F_i be the i th row of M , $i = 0, 1, \dots, nv+1$, and M_i the matrix M with the row F_i deleted. Then, by Cramer's rule, $\sum_{i=0}^{nv+1} (-1)^i (\det M_i) F_i = 0$. If $i > I+1$, so that F_i is a row corresponding to one of the tie-down bars, then since the framework $G \cup T$ with one tie-down bar removed is clearly stressed, $M_i = 0$. Thus we have

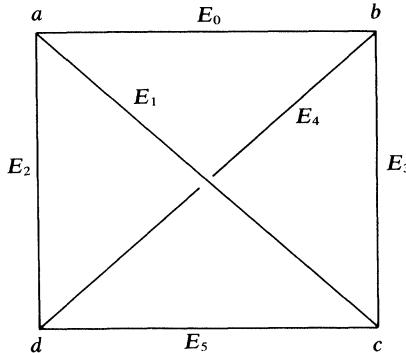
$$\sum_{i=0}^{I+1} (-1)^i C(G-E_i, T) F_i = 0.$$

Now each term has the same tie-down factor $C(T)$, and we chose T to be nondegenerate, so $C(T) \neq 0$. Thus $C(T)$ may be factored out, leaving

$$\sum_{i=0}^{I+1} (-1)^i C(G-E_i) F_i = 0.$$

Finally, we note that since $G - E_i$ is isostatic by hypothesis, the coefficient $C(G - E_i)$ is generically nonzero, so we have the desired stress. Q.E.D.

Example 5.2.



The tetrahedral framework in the plane is a 1-overbraced framework. The stress coefficients $(-1)^i C(G - E_i)$ are tabulated below.

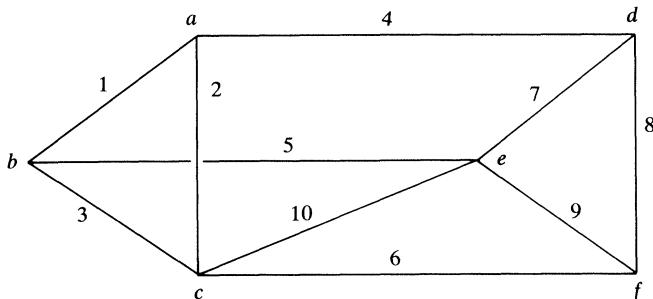
E_0	$[acd][bcd]$	E_3	$[abd][acd]$
E_1	$-[abd][bcd]$	E_4	$-[abc][acd]$
E_2	$[abc][bcd]$	E_5	$[abc][abd]$

The determination of the signs of the coefficients directly from the definition of $C(G - E_i)$ is tricky; in any case the best we can do is determine the relative signs of two coefficients. One way to determine the signs is to recall that by the definition of a stress, if S_{ab} is the value of a stress on a bar ab , then for each $a \in V$, $\sum_{b \in V, ab \in G} S_{ab} \cdot \mathbf{ab} = 0$, where \mathbf{ab} is the vector $a - b$. This is equivalent to the Cayley algebra equation $\sum_{b \in V, ab \in G} (-1)^i C(G - E_i)ab = 0$, where ab here is a step-two extensor. But this equation must be a syzygy in the Cayley algebra, which can be determined directly. For example,

$$[acd][bcd]ab - [abd][bcd]ac + [abc][bcd]ad = 0$$

is a syzygy in the Cayley algebra, and this gives the correct relative signs for E_0 , E_1 and E_2 . Alternatively, we may specialize coordinates to any particular coordinatization whose stress coefficients have known signs in order to determine the correct signs generically.

Example 5.3.



This framework G is the triangular prism with one additional bar ce , in the plane. The stress coefficient $(-1)^i C(G - E_i)$ is given for each bar E_i , now denoted simply i .

- 1 $[acd][bce][cef][def]$
- 2 $-[abd][bce][cef][def]$
- 3 $[abe][acd][cef][def]$
- 4 $[abc][bce][cef][def]$
- 5 $-[abc][acd][cef][def]$
- 6 $[abc][ade][bce][def]$
- 7 $[abc][adf][bce][cef]$
- 8 $-[abc][ade][bce][cef]$
- 9 $[abc][ade][bce][cdf]$
- 10 $[abc][def]([abd][cef] - [ade][bcf]).$

Now that we have a bracket formulation for a stress on a 1-overbraced framework in generic position, the stress for any particular coordinatization (or realization) of the framework in real n -space may be computed by simply plugging in the values for the brackets for that particular coordinatization. One especially important piece of information which can be obtained from the stress values is the split (or partition) of tension members versus compression members. This is obtained simply by observing which bars have a positive value in the stress, and which have negative. The fact that we have determined only the relative signs of the stress coefficients is no problem, since the split between tension and compression members is reversible.

Example 5.3 (continued). Let us fix the positions of a, c, d, e and f in the framework G of Example 5.3 and think of b as moving around. The irreducible factors which involve b and which occur in the stress coefficients are $[abc]$, $[abd]$, $[abe]$, $[bce]$ and $([abd][cef] - [ade][bcf])$. The locus of points which makes each of these factors 0 is a curve; in this case each is a line, namely, ac , ad , ae , ce and $(ad \cap cf)e$, respectively (note that the last factor listed is equivalent to the Cayley algebra expression $ad \wedge be \wedge cf$). We will call these curves *switching curves for b* (or *switching surfaces for b* for examples in 3-space).

We could more generally consider the 12-dimensional space of all affine realizations of $\{a, b, c, d, e, f\}$ and consider all of the irreducible factors of the stress coefficients, obtaining various *switching hypersurfaces*.

For an arbitrary framework, if a vertex b lies on one of the switching surfaces, then some of the stress coefficients are zero and the support of the stress is a proper subframework of G . If we move b from one side of the switching surface to the other, the corresponding factor switches sign (assuming that we are doing our factoring over \mathbb{R}), thus all members having that factor to an odd power switch from tension to compression, or vice-versa, while the remaining members do not switch. Thus a crucial question is when an irreducible factor of a pure condition can occur to a higher power than one, or, more precisely, what are the relative powers of a given factor in the stress coefficients.

We illustrate in Fig. 1 the switching curves for the vertex b in one realization of G . We also illustrate the tension compression split for several of the plane regions determined by the switching curves. Once the split has been determined in one region,

it may quickly be determined in all others by successively crossing switching curves, one at a time and switching members accordingly. We show switching curves by dotted lines, tension members by dashed lines and compression members by cross-hatched lines.

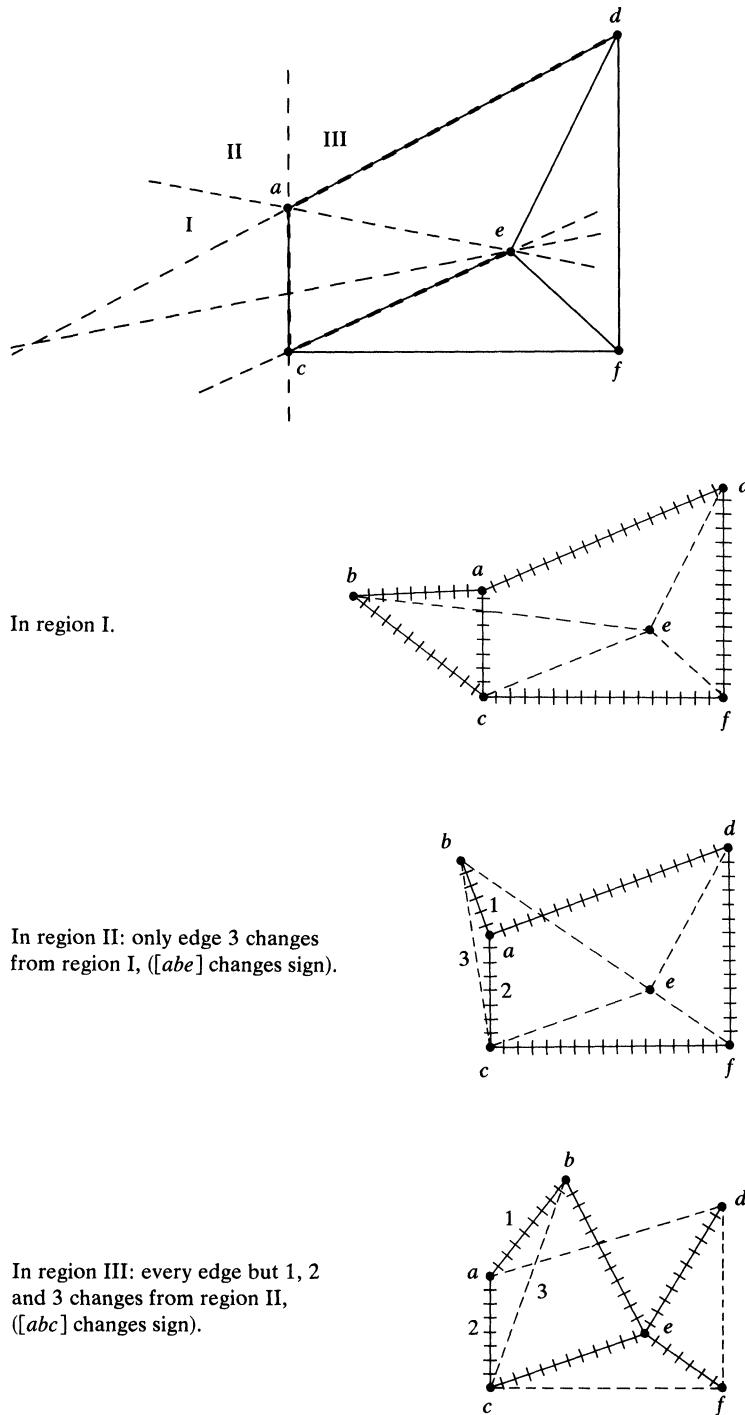


FIG. 1

6. Multiple stresses. In our current list of examples, we have no example where a factor occurs to a k th power in one coefficient of a stress and a power $< k - 1$ in some other coefficient. This would seem to indicate that every factor of a pure condition gives a switching surface for every 1-overbraced framework containing this subgraph, unless this factor occurs to the same power in all coefficients (for example, $K_{5,5}$).

The only examples we have of a factor occurring to higher than the first power are in the bipartite frameworks. In these examples the multiple occurrences of a factor are associated with a multiple stress. The following result generalizes this observation.

PROPOSITION 6.1. *If, for some irreducible factor H of the pure condition of an n -isostatic graph G , all coordinatizations α with $H(\alpha(G)) = 0$ give at least r stresses, then H' is a factor of $C(G)$.*

Proof. For $r = 1$, this is Lemma 4.6. We proceed by induction on r , with the additional assumption that some α with $H(\alpha(G)) = 0$ give joints which span the space. If, on the contrary, all such α gave flat coordinatizations, then $H = 0$ implies $[a \cdots d] = 0$ for each $(n + 1)$ -tuple a, \dots, d , so H would be precisely such a single bracket. This, together with an over-all flatness in G requires that G is the n -simplex, where we know $C(G) = [a \cdots d]$ and $r = 1$.

Since $\alpha(G)$ is not flat for a generic coordinatization with $H(\alpha(G)) = 0$, and the complete graph is statically rigid in such coordinatizations, we can find an extra bar E which is independent in $\alpha(G + E)$. We now examine the stress equation for the 1-overbraced framework $G + E$:

$$\sum (-1)^i C(G + E - E_i) F_i + C(G) F_E = 0.$$

We assume $H(\alpha(G)) = 0$ gives an $(r + 1)$ -tuple stress. This must also give a r -tuple stress in $\alpha(G + E - E_i)$ and by our induction hypothesis H is an r -fold factor of all these coefficients. We factor H' out of the stress equation, leaving:

$$\sum (C_i) F_i + C' F_E = 0.$$

Since E is independent for the generic α with $H(\alpha(G)) = 0$, we have $H = 0$ implies $C' = 0$. Over the complex numbers this gives, via Hilbert's Nullstellensatz, $AH = (C')^s$ for some integer s . Since H is irreducible, H divides C' .

We conclude that H is an $(r + 1)$ -fold factor of $C(G)$. Q.E.D.

Is the converse of this proposition true? If an irreducible factor H is an r -fold factor of $C(G)$ do all coordinatizations α with $H(\alpha(G)) = 0$ give r stresses? This problem remains a basic block to a good analysis of the behavior of a stress at a “switching surface”.

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