

Math 7760 – Homework 3 – Due: September 14, 2022

Practice Problems:

Problem 1. Show that the standard cube and the standard cross polytope are polar duals of each other.

Problem 2. Prove directly that the polar dual of a standard simplex is a simplex.

Problems to write up:

Problem 3. Prove each of the following statements.

- (1) The intersection of two polytopes is a polytope.
- (2) The sum of two polytopes is a polytope.
- (3) Every face of a polytope is exposed.

Problem 4. Given $x, y \in \mathbb{R}^d$, we use the shorthand $x \geq y$ to mean that $x_i \geq y_i$ for all i . Use the hyperplane separation theorem to prove (one of many versions of) the *Farkas Lemma*:

Lemma 1. Let $A \in \mathbb{R}^{m \times d}$ and let $z \in \mathbb{R}^m$. Then there exists a point $x \in \mathbb{R}^d$ with $Ax = 0$ and $x \geq 0$, or there exists a row vector $c \in (\mathbb{R}^m)^*$ with $cA \geq 0$ and $cz < 0$, but not both.

There are two other versions of the Farkas lemma in Ziegler's book (the one above also appears as version 2). Convince yourself that they are indeed equivalent to the one above (no need to write anything up for it).

Problem 5. Show that every polytope is affinely isomorphic to a bounded intersection of an orthant with an affine space.