Unimodular Binary Hierarchical Models

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Example

- Let T be the following $3 \times 2 \times 2$ table

$$\begin{array}{cc} \text{front} & \text{back} \\ \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 0 & 2 \end{pmatrix} \end{array}$$

- If we sum entries going down, we get the 2-way margin below. If we sum entries going left and back, we get the 1-way margin below.

$$\begin{pmatrix} 3 & 6 \\ 6 & 2 \end{pmatrix} \qquad \begin{pmatrix} 5 \\ 6 \\ 6 \end{pmatrix}$$

- We are interested in the matrix that maps tables to margins

Main Definition

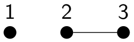
- $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is an integer vector, $d_i \geq 2$
- $\mathcal{C}$ denotes a simplicial complex on $[n]$
- $\text{facet}(\mathcal{C})$ denotes the inclusion-maximal faces of $\mathcal{C}$

Definition

Let $\mathcal{A}_{\mathcal{C},\mathbf{d}}$ be the matrix defined as follows:

- Columns are indexed by elements of $\bigoplus_{i=1}^{n} [d_i]$
- Rows are indexed by $\bigoplus_{F \in \text{facet}(\mathcal{C})} \bigoplus_{j \in F} [d_j]$
- Entry in row $(F, (j_1, \ldots, j_k))$ and column $(i_1, \ldots, i_n)$ is 1 if $i|_F = (j_1, \ldots, j_k)$
- All other entries are 0

Example

- Let $n = 3$ with $d_1 = 3, d_2 = 2, d_3 = 2$
- Let $\mathcal{C}$ be the complex 

- Then $\mathcal{A}_{\mathcal{C},\mathbf{d}}$ is the following matrix:

$$
\begin{array}{rc}
 & \begin{array}{cccccccccccc} \substack{1\\1\\1} & \substack{1\\1\\2} & \substack{1\\2\\1} & \substack{1\\2\\2} & \substack{2\\1\\1} & \substack{2\\1\\2} & \substack{2\\2\\1} & \substack{2\\2\\2} & \substack{3\\1\\1} & \substack{3\\1\\2} & \substack{3\\2\\1} & \substack{3\\2\\2} \end{array} \\
\begin{array}{r} \{1\},1 \\ \{1\},2 \\ \{1\},3 \\ \hline \{2,3\},11 \\ \{2,3\},12 \\ \{2,3\},21 \\ \{2,3\},22 \end{array} &
\left(\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right)
\end{array}
$$

Unimodularity

Definition

Let $A \in \mathbb{Z}^{n \times d}$ be an integral matrix with rank n. We say that A is **unimodular** if for every $b \in \mathbb{N}A$, the following polyhedron has integral vertices

$$P_{A,b} := \{x \in \mathbb{R}^d : Ax = b, x \geq 0\}.$$

Motivating Question

Question

Given a simplicial complex $\mathcal{C}$ on $[n]$ and an integer vector $\mathbf{d} = (d_1, \ldots, d_n)$ with $d_i \geq 2$, is $\mathcal{A}_{\mathcal{C},\mathbf{d}}$ unimodular?

Proposition

If $\mathcal{A}_{\mathcal{C},\mathbf{d}}$ is unimodular, then for all $\mathbf{d}'$ with $\mathbf{d}' \leq \mathbf{d}$ componentwise, $\mathcal{A}_{\mathcal{C},\mathbf{d}}$ is also unimodular.

Therefore we restrict our attention to the binary case; i.e. where $d_1 = d_2 = \cdots = d_n = 2$.

Motivating Question

Definition

$\mathcal{A}_{\mathcal{C}} := \mathcal{A}_{\mathcal{C},(2,\ldots,2)}$.

Question

Given a simplicial complex $\mathcal{C}$ on $[n]$, is $\mathcal{A}_{\mathcal{C}}$ unimodular?

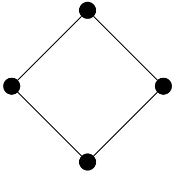
Unimodularity-Preserving Operations

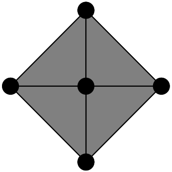
Definition (Cone Vertices)

If $\mathcal{C}$ is a simplicial complex on $[n]$, define cone($\mathcal{C}$) to be the complex on $[n+1]$ with facets

$$\mathrm{facet}(\mathrm{cone}(\mathcal{C})) = \{F \cup \{n+1\} : F \in \mathrm{facet}(\mathcal{C})\}.$$

We say that $n+1$ a *cone vertex*.





Unimodularity-Preserving Operations

Definition (Ghost Vertices)

If $\mathcal{C}$ is a simplicial complex on $[n]$, define $G(\mathcal{C})$ to be the simplicial complex on $[n+1]$ that has exactly the same faces as $\mathcal{C}$. We say that $n+1$ is a *ghost vertex*.

Unimodularity-Preserving Operations

Definition (Alexander Duality)

If $\mathcal{C}$ is a simplicial complex on $[n]$, then the *Alexander dual* complex $\mathcal{C}^*$ is the simplicial complex on $[n]$ with facets

$$\text{facet}(\mathcal{C}^*) = \{[n] \setminus S : S \text{ is a minimal non-face of } \mathcal{C}\}.$$

Results: Constructive Classification

Definition

We say that a simplicial complex $\mathcal{C}$ is *nuclear* if it satisfies one of the following:

1. $\mathcal{C} = \Delta_k$ for some $k \geq -2$ (i.e. a simplex)
2. $\mathcal{C} = \Delta_m \sqcup \Delta_n$ (i.e. a disjoint union of simplices)
3. $\mathcal{C} = \text{cone}(\mathcal{D})$ where $\mathcal{D}$ is nuclear
4. $\mathcal{C} = G(\mathcal{D})$ where $\mathcal{D}$ is nuclear
5. $\mathcal{C}$ is the Alexander dual of a nuclear complex.

Theorem (B-Sullivant 2015)

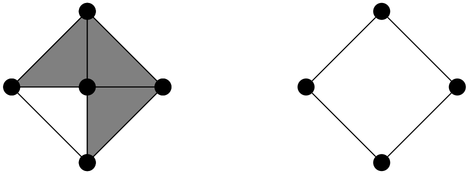
The matrix $\mathcal{A}_{\mathcal{C}}$ is unimodular if and only if $\mathcal{C}$ is nuclear.

Simplicial Complex Minors: Vertex Deletion

Let $\mathcal{C}$ be a simplicial complex on $[n]$.

Definition (Deletion)

Let $v \in [n]$ be a vertex of $\mathcal{C}$. Then $\mathcal{C} \setminus v$ denotes the induced simplicial complex on $[n] \setminus \{v\}$.

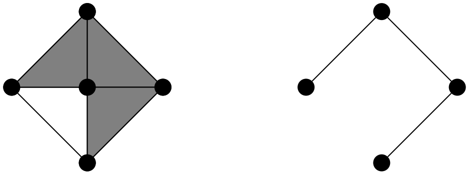


Simplicial Complex Minors: Links of Vertices

Definition (Link)

Let $v \in [n]$ be a vertex of $\mathcal{C}$. Then $\text{link}_v(\mathcal{C})$ denotes the simplicial complex on $[n] \setminus \{v\}$ with the following facets

$$\text{facet}(\text{link}_v(\mathcal{C}) = \{F \setminus \{v\} : F \text{ is a facet of } \mathcal{C} \text{ with } v \in F\}.$$



Definition (Simplicial Complex Minor)

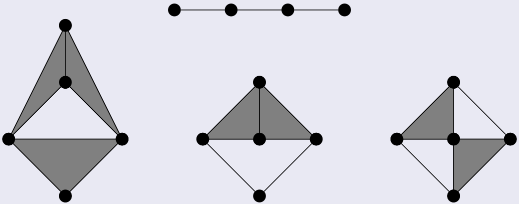
Let $\mathcal{C}, \mathcal{D}$ be simplicial complexes. If $\mathcal{D}$ can be obtained from $\mathcal{C}$ by taking links of vertices and deleting vertices, then we say that $\mathcal{D}$ is a *minor* of $\mathcal{C}$.

Results: Excluded Minor Classification

Theorem (B-Sullivant 2015)

The matrix $\mathcal{A}_{\mathcal{C}}$ is unimodular if and only if $\mathcal{C}$ has no simplicial complex minors isomorphic to any of the following

- $\partial\Delta_k \sqcup \{v\}$, *the disjoint union of the boundary of a simplex and an isolated vertex*
- O_6, *the boundary complex of an octahedron, or its Alexander dual O_6^**
- *The four simplicial complexes shown below*



Next Steps

Proposition

Let A be a matrix of full row rank. If we can row reduce A to get the matrix $[I_n|D]$, then A is unimodular if and only if D is totally unimodular.

Question

Can we use Seymour's decomposition for regular matroids [4] to prove our results?

Next Steps

Question

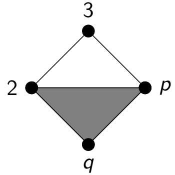
Given a simplicial complex $\mathcal{C}$ on $[n]$ and an integer vector $\mathbf{d} = (d_1, \ldots, d_n)$ with $d_i \geq 2$, is $\mathcal{A}_{\mathcal{C},\mathbf{d}}$ unimodular?

Corollary (B-Sullivant 2015)

If $\mathcal{A}_{\mathcal{C},\mathbf{d}}$ is unimodular then $\mathcal{C}$ is nuclear.

Question

Let $\mathcal{C}$ and $\mathbf{d}$ be specified by the figure below. For which values of p and q is $\mathcal{A}_{\mathcal{C},\mathbf{d}}$ unimodular?



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