Math 7120 – Homework 7 – Due: March 21, 2022

Maybe: Dummit and Foote, 12.1 1-6 and 20

Practice problems:

Problem 1. Dummit and Foote, 12.2 problems 3,4,8

Test prep:

Problem 2. Construct examples of each of the following, or briefly explain why it doesn't exist:

(1) An R-module of rank one that is not free, where R is an integral domain
(2) A nonzero free $F[t]$-module M where F is a field and M is finite-dimensional as an F-vector space
(3) A non-Noetherian ring
(4) An infinitely-generated submodule of a finitely-generated R-module

Problem 3. Determine all possible rational canonical forms for a linear transformation $T : \mathbb{R}^6 \to \mathbb{R}^6$ with characteristic polynomial $x^2(x^2+1)^2$.

Problem 4. Determine all possible Jordan canonical forms for a linear transformation $T : \mathbb{C}^5 \to \mathbb{C}^5$ with characteristic polynomial $(x-2)^3(x-3)^2$.

Type solutions to the following problems in LaTeX, and email the tex and PDF files to me at `dbernstein1@tulane.edu` by 10am on the indicated date. Please title them as [lastname].tex and [lastname].pdf. When preparing your solutions, you must follow the rules as laid out in the course syllabus.

Graded Problems:

Recall that an R-module M is *torsion* if for each $x \in M$, there is $r \in R \setminus \{0\}$ such that $rx = 0$.

Problem 5. Let R be an integral domain.

(1) Prove that an R-module M has rank n if and only if M contains a free submodule N of rank n such that M/N is torsion.
(2) Prove that if A and B are R-modules with ranks m and n, then $A \oplus B$ has rank $m+n$.