Math 7120 – Homework 3 – Due: February 16, 2022

Practice problems:

Problem 1. Dummit and Foot 10.4 numbers 2,3,4

Problem 2. Prove that tensor product commutes with direct sum, i.e. if M, M', N are R-modules, then
$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N).$$
Conclude that if S is an $\mathbb{R}$-algebra, then $S \otimes_R R^n \cong S^n$ as R-algebras.

Test prep:

Problem 3. Simplify the following tensor products:

(1) $\mathbb{Z}_4 \otimes_{\mathbb{Z}} \mathbb{Z}_2$
(2) $\mathbb{Z}_4 \otimes_{\mathbb{Z}} \mathbb{Z}$
(3) $\mathbb{Z}_4 \otimes_{\mathbb{Z}} \mathbb{Q}$

Type solutions to the following problems in LaTeX, and email the tex and PDF files to me at `dbernstein1@tulane.edu` by 10am on the indicated date. Please title them as [lastname].tex and [lastname].pdf. When preparing your solutions, you must follow the rules as laid out in the course syllabus.

Graded Problems:

Problem 4. Let R be a commutative ring with a 1 and let I, J be ideals of R.

(1) Let $I = (2, x)$ be the ideal of the ring $R = \mathbb{Z}[x]$. Show that $2 \otimes 2 + x \otimes x$ is not a simple tensor in $I \otimes_R I$, i.e. cannot be written in the form $a \otimes b$ for some $a, b \in I$.
(2) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \bmod I) \otimes r \bmod J$
(3) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r \bmod I) \otimes (s \bmod J)$ to $rs \bmod (I + J)$.
(4) Suppose R is an integral domain and I is a principal ideal. Prove that $I \otimes_R I$ has no nonzero torsion elements.

Problem 5. Prove the part of the splitting lemma left as an exercise in class, i.e. that if $0 \to A \xrightarrow{\psi} B \xrightarrow{\phi} C \to 0$ is a short exact sequence of R-modules and there exists $f : B \to A$ such that $f \circ \psi = \mathrm{id}_A$, then the sequence splits.