Math 7120 – Homework 6 – Due: March 11, 2022

The punchline of this worksheet is that the exterior algebra of a vector space is, in a certain sense, an algebra of linear subspaces. **This homework will not be graded.**

Problem 1. Let $\mathbb{K}$ be a field and let V, W be finite dimensional $\mathbb{K}$-vector spaces. Let $T : V \to W$ be a linear transformation. Suppose that $\{e_i\}$ and $\{f_j\}$ are bases of V and W and let A denote the matrix of T map with respect to these bases. Let $A_{i_1,\dots,i_M;j_1,\dots,j_M}$ denote the determinant of the $M \times M$ matrix obtained from A by restricting to the rows $i_1, \dots, i_M$ and columns $j_1, \dots, j_M$. Define a linear map

$$\bigwedge T : \bigwedge V \to \bigwedge W$$

by extending the following function linearly to all of $\bigwedge V$

$$e_{j_1} \wedge \cdots \wedge e_{j_M} \mapsto \sum_{i_1 < \cdots < i_M} A_{i_1,\dots,i_M;j_1,\dots,j_M} f_{i_1} \wedge \cdots \wedge f_{i_M}.$$

Verify that $\bigwedge T$ is a $\mathbb{K}$-linear ring homomorphism. Show that if $v_1, \dots, v_M \in V$ are linearly independent and $w_1, \dots, w_M \in \operatorname{span}\{v_1, \dots, v_M\}$ with $w_i = \sum_{j=1}^M a_{ij} v_j$, then, letting A be the matrix whose ij entry is a_{ij}, the following holds

$$w_1 \wedge \cdots \wedge w_M = \det(A) v_1 \wedge \cdots \wedge v_M.$$

Definition 2. Let $\omega \in \bigwedge^m V$. Then

(1) ω is *completely decomposable* if there exist $v_1, \dots, v_m$ such that $\omega = v_1 \wedge \cdots \wedge v_m$, and
(2) ω is *partially decomposable* if $\omega = v \wedge \eta$ for some $v \in V$ and $\eta \in \bigwedge^{m-1} V$.

Note that if w is completely decomposable, it is also partially decomposable.

Problem 3. Let $\omega \in \bigwedge^m V$ and define $\phi_\omega : V \to \bigwedge^{m+1} V$ by $v \mapsto v \wedge \omega$. Prove the following.

(1) If ω is partially decomposable, then $\omega \wedge \omega = 0$.
(2) ω is partially decomposable if and only if ϕ_ω has nontrivial kernel.
(3) If $\{v_1, \dots, v_M\}$ is a basis for the kernel of ϕ_ω, then there exists $\eta \in \bigwedge^{m-M} V$ such that

$$\omega = v_1 \wedge \cdots \wedge v_M \wedge \eta.$$

(4) ω is completely decomposable if and only if the kernel of ϕ_ω has dimension m.

Definition 4. Given elements u, v of an $\mathbb{K}$-vector space W, we say that u, v are *protectively equivalent* if there exists a nonzero scalar $\lambda \in \mathbb{K}$ such that $u = \lambda v$.

Problem 5. Let W be a vector space of dimension N and let $M \le N$. For each vector subspace V of W of dimension M, define $j(V)$ to be the projective equivalence class of $v_1 \wedge \cdots \wedge v_M$ where $v_1, \dots, v_M$ is a basis of V. Prove the following:

(1) j is a well-defined map (i.e. does not depend on choice of basis),
(2) j is a bijection between linear subspaces of W of dimension M, and projective equivalence classes of completely decomposable elements of $\bigwedge^M W$, and
(3) if V_1, V_2 are subspaces of W such that $V_1 \cap V_2 = \{0\}$, then $j(V_1 + V_2) = j(V_1) \wedge j(V_2)$.