

Math 7120 – Homework 6 – Due: March 11, 2022

The punchline of this worksheet is that the exterior algebra of a vector space is, in a certain sense, an algebra of linear subspaces. **This homework will not be graded.**

Problem 1. Let \mathbb{K} be a field and let V, W be finite dimensional \mathbb{K} -vector spaces. Let $T : V \rightarrow W$ be a linear transformation. Suppose that $\{e_i\}$ and $\{f_j\}$ are bases of V and W and let A denote the matrix of T map with respect to these bases. Let $A_{i_1, \dots, i_M; j_1, \dots, j_M}$ denote the determinant of the $M \times M$ matrix obtained from A by restricting to the rows i_1, \dots, i_M and columns j_1, \dots, j_M . Define a linear map

$$\bigwedge T : \bigwedge V \rightarrow \bigwedge W$$

by extending the following function linearly to all of $\bigwedge V$

$$e_{j_1} \wedge \cdots \wedge e_{j_M} \mapsto \sum_{i_1 < \cdots < i_M} A_{i_1, \dots, i_M; j_1, \dots, j_M} f_{i_1} \wedge \cdots \wedge f_{i_M}.$$

Verify that $\bigwedge T$ is a \mathbb{K} -linear ring homomorphism. Show that if $v_1, \dots, v_M \in V$ are linearly independent and $w_1, \dots, w_M \in \text{span}\{v_1, \dots, v_M\}$ with $w_i = \sum_{j=1}^M a_{ij} v_j$, then, letting A be the matrix whose ij entry is a_{ij} , the following holds

$$w_1 \wedge \cdots \wedge w_M = \det(A) v_1 \wedge \cdots \wedge v_M.$$

Definition 2. Let $\omega \in \bigwedge^m V$. Then

- (1) ω is *completely decomposable* if there exist v_1, \dots, v_m such that $\omega = v_1 \wedge \cdots \wedge v_m$, and
- (2) ω is *partially decomposable* if $\omega = v \wedge \eta$ for some $v \in V$ and $\eta \in \bigwedge^{m-1} V$.

Note that if w is completely decomposable, it is also partially decomposable.

Problem 3. Let $\omega \in \bigwedge^m V$ and define $\phi_\omega : V \rightarrow \bigwedge^{m+1} V$ by $v \mapsto v \wedge \omega$. Prove the following.

- (1) If ω is partially decomposable, then $\omega \wedge \omega = 0$.
- (2) ω is partially decomposable if and only if ϕ_ω has nontrivial kernel.
- (3) If $\{v_1, \dots, v_M\}$ is a basis for the kernel of ϕ_ω , then there exists $\eta \in \bigwedge^{m-M} V$ such that

$$\omega = v_1 \wedge \cdots \wedge v_M \wedge \eta.$$

- (4) ω is completely decomposable if and only if the kernel of ϕ_ω has dimension m .

Definition 4. Given elements u, v of an \mathbb{K} -vector space W , we say that u, v are *protectively equivalent* if there exists a nonzero scalar $\lambda \in \mathbb{K}$ such that $u = \lambda v$.

Problem 5. Let W be a vector space of dimension N and let $M \leq N$. For each vector subspace V of W of dimension M , define $j(V)$ to be the projective equivalence class of $v_1 \wedge \cdots \wedge v_M$ where v_1, \dots, v_M is a basis of V . Prove the following:

- (1) j is a well-defined map (i.e. does not depend on choice of basis),
- (2) j is a bijection between linear subspaces of W of dimension M , and projective equivalence classes of completely decomposable elements of $\bigwedge^M W$, and
- (3) if V_1, V_2 are subspaces of W such that $V_1 \cap V_2 = \{0\}$, then $j(V_1 + V_2) = j(V_1) \wedge j(V_2)$.