Geometric Combinatorics

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Contents

CHAPTER 1

Convexity and polytopes

1. Convexity basics

A linear combination of elements in a set $S \subseteq \mathbb{R}^d$ is an expression of the form

$$\sum_{i=1}^{n} t_i x_i$$

where each $x_i \in S$. Such a linear combination is called an ***affine combination*** if $\sum_i t_i = 1$, a ***conic combination*** if $t_i \geq 0$ for each i, and a ***convex combination*** if it is conic and affine. The set of all linear, affine, conic, and convex combinations of a set S will be denoted $\mathbb{R}S, \mathrm{Aff}(S), \mathbb{R}_{\geq 0}(S)$, and $\mathrm{Conv}(S)$. In words, we will refer to them as the linear span, the affine hull, the conic hull, and the convex hull of S.

Let us explore the geometric significance of these concepts when $S = \{x, y\}$ consists of two distinct points. When neither x nor y is the origin, the linear hull of S is the unique plane containing x, y and the origin. The affine hull of S is the unique line in $\mathbb{R}^d$ containing x and y and the convex hull of S is the line segment between x and y. The conic hull of S is the union of all rays from the origin through a point in the convex hull of x and y.

One says that $S \subseteq \mathbb{R}^d$ is a ***linear subspace*** when $S = \mathbb{R}(S)$, an ***affine subspace*** when $S = \mathrm{Aff}(S)$, a ***cone*** when $S = \mathbb{R}_{\geq 0}(S)$, and ***convex*** when $S = \mathrm{Conv}(S)$.

A ***V-polytope*** is the convex hull of a finite set of points, i.e. a set of the form $\mathrm{Conv}(\{x_1, \ldots, x_n\})$.

FIGURE 1. The square is a polytope. The disc is convex, but not a polytope.

A ***halfspace*** is a set of the form $\{x \in \mathbb{R}^d : ax \leq c\}$ where $a \in (\mathbb{R}^d)^*$ and $c \in \mathbb{R}$. An ***H-polyhedron*** is an intersection of finitely many halfspaces. An ***H-polytope*** is a bounded H-polyhedron. We will see in Section 3 that every V-polytope is an H-polytope and vice versa.

We now give several examples of polytopes

(1) **Standard simplices:** Fix an integer $d \geq 1$ and for each $1 \leq i \leq n$ define $e_i \in \mathbb{R}^d$ to be the i^{th} standard basis vector in $\mathbb{R}^d$. We define the standard simplices as follows

$$\begin{aligned}\Delta_{d-1} &:= \mathrm{Conv}\{e_1, \ldots, e_d\} \\ &= \{x \in \mathbb{R}^d : \sum_{i=1}^{d} x_i = 1; 0 \leq x_i \forall i = 1, \ldots, d\}.\end{aligned}$$

FIGURE 2. The first two polytopes are affinely isomorphic to each other, but not to the third. This is because affine functions preserve parallel lines and the first two have two sets of parallel lines, but the last one has none.

(2) **Cubes:**

$$\begin{aligned} C_d :&= \text{Conv}(\{+1,-1\}^d) \\ &= \{x \in \mathbb{R}^d : -1 \leq x_i \leq 1 \forall i = 1, \ldots, d\}. \end{aligned}$$

(3) **Cross polytopes**

$$\begin{aligned} C_d^* :&= \text{Conv}\{e_1, -e_1, e_2, -e_2, \ldots, e_d, -e_d\} \\ &= \{x \in \mathbb{R}^d : \sum_{i=1}^{d} |x_i| \leq 1\}. \end{aligned}$$

Perhaps the most fundamental quantity one can associate to a geometric object is its dimension. We would like a precise way to quantify the dimension of a polytope that is easy to work with. Linear spaces are just about the only thing in mathematics that have an obvious definition of their dimension, which is the size of a basis. The following lemma says that each affine space is uniquely associated to a linear space. We will then define the dimension of an affine space to be the dimension of the associated linear space. Then, with this at our disposal, we will define the dimension of a polytope to be the dimension of its affine hull.

Lemma 1: *Let $A \subseteq \mathbb{R}^d$ be an affine subspace. Then there exists a unique linear subspace $L \subseteq \mathbb{R}^d$ such that $A = \{x + b : x \in L\}$ where b is an arbitrary element of A.*

PROOF. Let $b \in A$ and define $L := \{x - b : x \in A\}$. We must show that L is indeed a linear subspace and that it does not depend on our choice of b. Indeed, let $x_1, x_2 \in A$ so that $x_1 - b$ and $x_2 - b$ are arbitrary elements of L. Their sum is $(x_1 + x_2 - b) - b$, which is also an element of L as $x_1 + x_2 - b$ is an affine combination of elements in A and is therefore in A itself. Now, let $x \in A$ and $t \in R$. Then $t(x - b) = tx + (1-t)b - b$ which is in L since $tx + (1-t)b \in A$. Our choice of L does not depend on b since $x - c \in L$ for any $c \in A$ since $x - c = (x - c + b) - b$ and $x - c + b \in A$. □

The ***dimension*** of an affine subspace $A \subseteq \mathbb{R}^d$ is the size of a basis of the linear space $\{x - b \in \mathbb{R}^d : x \in A\}$ where $b \in A$. We denote this by $\dim(A)$. The dimension of a convex set $C \subseteq \mathbb{R}^d$ is $\dim(\text{Aff}(C))$. Two convex sets $P \in \mathbb{R}^d$ and $Q \in \mathbb{R}^e$ are ***affinely isomorphic*** if there exists an affine function $f : \mathbb{R}^d \to \mathbb{R}^e$ that is a bijection between P and Q. The polytopes in Figure 2 suggest that the notion of affine isomorphism is too strong for combinatorics since all three polytopes are, in a sense that we will make precise soon, "combinatorially equivalent" in the sense that they both have four edges and four vertices.

2. The relative boundary of a convex set

The interesting combinatorics of a convex set happens on its relative boundary, a topological notion we will recall soon. In particular, the relative boundary of a convex set is made up of lower-dimensional convex sets, called *faces*, that form a partially ordered set under inclusion. Since we are focusing on relative boundaries, we will often restrict our consideration to closed convex sets.

We now recall the relevant topological notions. Let $S \subseteq \mathbb{R}^d$ be a set. The ***interior*** of S is the union of all open sets contained in S and the ***closure*** of S is the intersection of all closed sets containing S. The ***boundary*** of S is the relative complement of the interior of S in the closure of S. The ***relative interior*** (resp. closure, boundary) of a convex set $C \subseteq \mathbb{R}^d$ is the interior (resp. closure, boundary) of C in the induced topology on $\mathrm{Aff}(C)$.

Definition 2: Let $C \subseteq \mathbb{R}^d$ be closed and convex. A subset $F \subseteq C$ is a ***face*** of C if

(1) F is closed,
(2) F is convex, and
(3) given $x, y \in C$, if $\mathrm{ri(Conv(x,y)) \cap F} \neq \emptyset$, then $x, y \in F$.

A face is called an ***extreme point*** if it has dimension 0, an ***edge*** if it has dimension 1 (or sometimes, in the case of cones, an extreme ray), and a ***facet*** if it has dimension $\dim(C) - 1$. A face is ***proper*** if it is neither C nor $\emptyset$.

Example 3: The set of proper faces of a polygon consists of its vertices and its edges. Every point on the boundary of a ball in any dimension is a proper face, and these are all the proper faces. An affine space has no proper faces. The only proper face of the halfspace $\{x : ax \geq c\}$ is its boundary hyperplane, namely $\{x : ax = c\}$.

Lemma 4: *Let $C \subseteq \mathbb{R}^d$ be convex of dimension at least 1. Then $\mathrm{ri(C)}$ is nonempty.*

Proof. Let k denote the dimension of C and let $x_1, \ldots, x_{k+1}$ affinely span $\mathrm{Aff}(C)$. Consider the function $f : \Delta_k \to C$ given by

$$\sum_{i=1}^{k+1} t_i e_i \mapsto \sum_{i=1}^{k+1} t_i x_i.$$

Then f is continuous and injective. Since Δ_k is compact, $f(\Delta_k)$ is homeomorphic to Δ_k [1, Theorem 26.6]. Since $\frac{1}{k}\sum_{i=1}^{k+1} e_i \in \mathrm{ri(\Delta_k)}$, $f(\Delta_k)$, and therefore C, has nonempty relative interior. $\square$

Proposition 5: *If $C \subseteq \mathbb{R}^d$ is closed and convex and $F \subset C$ is a proper face, then $F \subseteq \mathrm{rb(C)}$.*

Proof. Let $y \in F$ and let $x \in C \setminus F$. For $n = 1, 2, \ldots,$ define

$$S_n := \{ty + (1-t)x : 0 \leq t \leq 1 + 1/n\}.$$

Since F is a face of C and S_n is a line segment whose interior intersects F, there exists a point $y_n \in S_n \setminus C$. Then, $y_n \to y$ as $n \to \infty$. But this implies $y \in \mathrm{rb(C)}$ because $y_n \in \mathrm{Aff}(C)$ as $S_n \subseteq \mathrm{Aff}(C)$. $\square$

Proposition 6: *Let $C \subseteq \mathbb{R}^d$ be closed and convex. If F is a proper face of C, then $\dim(F) < \dim(C)$.*

Proof. Since $F \subseteq C$, $\dim(F) \leq \dim(C)$. Assume for the sake of contradiction that $\dim(F) = \dim(C)$. Since $\text{Aff}(F) \subseteq \text{Aff}(C)$, this implies that $\text{Aff}(F) = \text{Aff}(C)$. Passing to this affine hull if necessary, we may assume without loss of generality that $\dim(C) = d$. Since $\dim(F) = d$, Lemma 4 implies that ri(F) is a nonempty open subset of $\mathbb{R}^d$. Therefore ri(F) $\subseteq$ ri(C) implies ri(F) $\subseteq$ ri(C). But this contradicts Lemma 5. $\square$

Proposition 7: *Let C be closed and convex and let $F \subseteq C$ be a face. Then:*

(1) *every face of F is a face of C, and*
(2) *every face of C contained in F is a face of C.*

Proof. Let F' be a face of F. Let $x, y \in C$ be such that $\mathrm{ri}(\mathrm{Conv(x,y)}) \cap \mathrm{F'} \neq \emptyset$. Then $x, y \in F$ since F is a face of C. This implies $x, y \in F'$ as F' is a face of F. So F' is a face of C.

Now let F' be a face of C contained in F. Let $x, y \in F$ with $F' \cap \mathrm{ri}(\mathrm{Conv(x,y)}) \neq \emptyset$. Since F' is a face of C, this implies $x, y \in F'$. So F' is a face of C. $\square$

Lemma 8: *Let $C \subseteq \mathbb{R}^d$ be closed and convex, let $a \in (\mathbb{R}^d)^*$, let $c \in \mathbb{R}$, and assume $ax \leq c$ for all $x \in C$. The set*

$$F_{a,c} := \{x \in C : ax = c\}$$

is a face of C.

Proof. Let $x, y \in C$ and assume that there exists $z \in F_{a,c} \cap \mathrm{ri}(\mathrm{Conv(x,y)})$. Let $t \in [0,1]$ such that $z = tx + (1-t)y$. Then

$$c = az = tax + (1-t)ay.$$

Since $ax \geq c$ and $ay \geq c$, this implies $ax = ay = c$, i.e. that $x, y \in F_{a,c}$. $\square$

A face F of a closed convex set $C \subseteq \mathbb{R}^d$ is called ***exposed*** if $F = F_{a,c}$ as in Lemma 8. The geometric interpretation of an exposed face is as follows. If $ax \leq c$ for all $x \in C$, then the hyperplane $\{x \in \mathbb{R}^d : ax = c\}$ lies tangent to C. The intersection of this hyperplane with C is the face $F_{a,c}$. A convex set may have faces that are not exposed - see Figure 3, for example. That said, we will eventually see that all faces of a polytope are exposed so we will not spend much time talking about non-exposed faces.

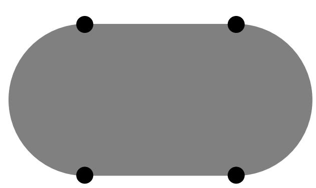

Figure 3. All of the faces of the above convex set $C \subset \mathbb{R}^2$ are exposed, aside from the four extreme points indicated by black dots. To see this, note that the tangent line to C at any one of these points will intersect along the entire edge that it lies on.

We now come to the first big theorem in convexity theory: the hyperplane separation theorem. There are various similar theorems that go by the same name and we will stick with the one that has the exact level of generality we need. The second homework guides you through a proof of this theorem. A proof will be added to these notes after that assignment is turned in.

Theorem 9 (Hyperplane separation theorem)**:** *Given a convex $C \subset \mathbb{R}^n$ and a point $y \in \mathbb{R}^d \setminus C$, there exists $a \in (\mathbb{R}^n)^*$ and $b \in \mathbb{R}$ such that $ax \leq b$ for all $x \in C$ and $ay \geq b$.*

The geometric content of Theorem 9 is as follows: it says that given a convex set $C \subseteq \mathbb{R}^d$ and a point $y \notin C$, there exists a hyperplane H that contains C in one of its two half-spaces and contains y in the other. When $y \notin \mathrm{rb(C)}$, this hyperplane H can be chosen so that neither C nor $\{y\}$ intersects H, and C and y lie on opposite sides of this hyperplane. When $y \in \mathrm{rb(C)}$, H will contain y and be tangent to C. The hyperplane separation theorem allows us to close an important circle of ideas that will allow us to move away from topological considerations. In particular, we have the following theorem.

Theorem 10: *Let $C \subseteq \mathbb{R}^d$ be closed and convex. Then* $\mathrm{rb(C)}$ *is the union of its proper faces.*

Proof. Proposition 5 implies that the union of the proper faces of C is contained in $\mathrm{rb(C)}$. It therefore suffices to let $x \in \mathrm{rb(C)}$ and find a proper face of C containing x. By restricting to $\mathrm{Aff}(C)$ if necessary, we may assume that $\dim(C) = d$. Since $\mathrm{ri(C)}$ is convex, Theorem 9 implies that there exists $a \in (\mathbb{R}^d)^*$ and $c \in \mathbb{R}$ such that $ax = c$ and $ay \leq c$ for all $y \in C$. Lemma 8 implies that $F_{a,c} := \{y \in C : ay = c\}$ is a face of C and it is clear that this contains x. The dimension of $F_{a,c}$ is at most $d-1$ and since $\dim(C) = d$ and $F_{a,c} \neq \emptyset$, $F_{a,c}$ is proper. □

Theorem 11: *Let $C \subset \mathbb{R}^d$ be a compact, convex set of dimension k. Then for each $x \in C$, there exist extreme points $x_1, \ldots, x_{k+1}$, not necessarily distinct, such that $x \in \mathrm{Conv}(x_1, \ldots, x_{k+1})$. Moreover, one such x_i may be chosen arbitrarily.*

Proof. We induct on k. When $k = 0$, C is a single point and the theorem follows. Now assume $k \geq 1$ and let $x \in C$. If $x \in \mathrm{rb(C)}$, then Theorem 10 implies that there exists a face F such that $x \in F$. Proposition 6 implies that $\dim(F) < k$ so we are done by induction.

Now suppose $x \in \mathrm{ri(C)}$ and let x_{k+1} be an extreme point of C. Compactness of C implies that $\mathrm{Aff}\{x, x_{k+1}\} \cap C = \mathrm{Conv}\{x_{k+1}, y\}$ where $y \in \mathrm{rb(C)}$. Theorem 10 implies that there exists a face F of C with $y \in F$ and Proposition 6 implies that $\dim(F) < k$. Since $x \in \mathrm{ri(C)}$, the definition of a face implies that $x_{k+1} \notin F$. The inductive hypothesis implies that $y \in \mathrm{Conv}\{x_1, \ldots, x_k\}$ for extreme points $x_1, \ldots, x_k$ of F. Proposition 7 implies that $x_1, \ldots, x_k$ are also extreme points of C. Since $x \in \mathrm{Conv}\{x_{k+1}, y\}$ and $y \in \mathrm{Conv}\{x_1, \ldots, x_k\}$, we have that $x \in \mathrm{Conv}\{x_1, \ldots, x_{k+1}\}$. □

3. The main theorem and duality

The goal of this section is to prove the main theorem of polytopes, i.e. that H-polytopes are V-polytopes and vice-versa. We will do this by first showing that every H-polytope is a V-polytope. Once we have this, we will will introduce convex duality which will enable us to prove the other direction.

Each hyperplane $H \subseteq \mathbb{R}^d$ defines two halfspaces which we will denote H^+ and H^-. There is a choice to be made as to which halfspace is which, but when H is given explicitly as

$$H := \{x \in \mathbb{R}^d : ax = c\},$$

we define

$$H^+ := \{x \in \mathbb{R}^d : ax \leq c\} \qquad \text{and} \qquad \{x \in \mathbb{R}^d : ax \geq c\}.$$

Using this notation, each H-polytope can be written as

$$\bigcap_{i=1}^{n} H_i^+ \tag{1}$$

for hyperplanes $H_1, \dots, H_n \subset \mathbb{R}^d$. The following lemma characterizes the extreme points of an H-polytope.

Lemma 12: *Let P be an H-polytope as in* (1)*, let $x \in P$, and define*

$$I := \{i \in \{1, \dots, n\} : x \in H_i\}.$$

Then x is an extreme point of P if and only if

$$\{x\} = \bigcap_{i \in I} H_i. \tag{2}$$

Proof. Let $a_1, \dots, a_n \in (\mathbb{R}^d)^*$ and $c_1, \dots, c_n \in \mathbb{R}$ such that $H_i^+ = \{a_i x \leq c_i\}$. Assume (2). Let $y, z \in \mathbb{R}^d$ with $y, z \neq x$ and $y \in P$ such that $x \in \mathrm{ri(Conv(y,z))}$. By our hypothesis, there exists $i \in I$ such that $a_i y < c_i$. Since $a_i x = c_i$ and $x \in \mathrm{ri(Conv(y,z))}$, it follows that $a_i z > c_i$ so $z \notin P$.

Now assume (2) fails and define $A := \bigcap_{i \in I} H_i$. Then $P \cap A$ is an H-polytope in A which we can write as $P \cap A = \{y \in A : a_i y \leq c_i \text{ for all } i \notin I\}$. We claim that $P \cap A$ is at least one-dimensional. Indeed, $P \cap A$ has the same dimension as A (which is at least one) since otherwise $P \cap A$ would lie in a hyperplane of A and so there would be some $i \notin I$ such that $a_i y = c_i$ for all $y \in P \cap A$. But $x \in P \cap A$, so this would imply $i \in I$, a contradiction. Since $a_i x < c_i$ for all $i \notin I$, $x \in \mathrm{ri(P \cap A)}$. Therefore, there exist $y, z \in P \cap A$ such that $x \in \mathrm{ri(Conv(y,z))}$. This implies that x is not an extreme point of P. □

Corollary 13: *Every H-polytope is a V-polytope.*

Proof. Let P be an H-polytope. Lemma 12 implies that P has finitely many extreme points $x_1, \dots, x_k$. Theorem 11 then implies that $P = \mathrm{Conv}\{x_1, \dots, x_k\}$. □

We now develop the theory of convex duality. This will enable us to use Corollary 13 in order to prove its converse.

Definition 14: Let $C \subseteq \mathbb{R}^d$. The ***(polar) dual*** C^* of C is

$$C^* := \{a \in (\mathbb{R}^d)^* : ax \leq 1 \text{ for all } x \in C\}.$$

We pause to note two things about our definition of duality. In particular, C need not be convex, and C^* lives in the dual of the vector space that contains C. Using the natural isomorphism between a vector space and its double dual, we may view C and C^{**} as subsets of the same space.

Theorem 15: *Let $C \subseteq \mathbb{R}^d$. Then*

1. *C^* is closed and convex*
2. *If $D \subseteq \mathbb{R}^d$ and $C \subseteq D$, then $D^* \subseteq C^*$*
3. *$C \subseteq C^{**}$*
4. *$0 \in C^*$*
5. *If $0 \in \mathrm{ri(C)}$ then C^* is compact.*
6. *If $C \subseteq \mathbb{R}^d$ is convex, compact, and d-dimensional, then $C^{**} = C$.*

Proof. We will prove (6), leaving (1) through (5) as an exercise. We know from (3) that $C \subseteq C^{**}$, so it suffices to show that if $x \notin C$ then $x \notin C^{**}$. Theorem 9 implies that there exists $a \in (\mathbb{R}^d)^*$ and $c \in \mathbb{R}$ such that $ay \leq c$ for all $y \in C$ and $ax \geq c$. Since C is closed, we can choose a, c so that $ax > c$. We may also assume that $c \neq 0$, since if $c = 0$, then a, ε satisfy the desired conditions for small $\varepsilon > 0$. Compactness of C implies that the functional a achieves a maximum α on C and since $0 \in C$, we know $\alpha \geq 0$. This implies $c > 0$ and therefore that $\frac{1}{c}ax > 1$. But this shows that $x \notin C^{**}$ since $\frac{1}{c}ay \leq 1$ for all $y \in C$ (i.e. that $a \in C^*$). $\square$

Lemma 16: *Let $P \subset \mathbb{R}^d$ be a d-dimensional V-polytope with $0 \in \mathrm{ri(P)}$. Then P^* is a d-dimensional H-polytope and $0 \in \mathrm{ri(P^*)}$.*

Proof. We already know from Theorem 15 that P^* is compact so it suffices to show that P^* is an intersection of finitely many half-spaces and that $0 \in \mathrm{ri(P^*)}$. Assume $P = \mathrm{Conv}\{v_1, \ldots, v_k\}$. If $a \in P^*$ then $av_i \leq 1$ for $i = 1, \ldots, k$. Conversely, if $av_i \leq 1$ for all i and $x \in P$, then since $x = \sum_{i=1}^k t_i v_i$ with $\sum_{i=1}^k t_i = 1$, we have

$$ax = \sum_{i=1}^{k} t_i a v_i \leq \sum_{i=1}^{k} t_i = 1$$

and therefore $a \in P^*$.

Now we argue that P^* is full-dimensional with 0 in its interior. The inequalities $av_i \leq 1$ are satisfied strictly for $a = 0$ and therefore for all a in a small open neighborhood of 0. Thus 0 is in the interior of P^*. Since P^* has a nonempty (non-relative) interior, P^* is full-dimensional. $\square$

Theorem 17: *Every H-polytope is a V-polytope and vice versa.*

Proof. In light of Corollary 13, it suffices to show that every V-polytope is an H-polytope. Indeed, let $P \subset \mathbb{R}^d$ be a V-polytope. By passing to $\mathrm{Aff}(P)$ and translating if necessary, we may assume that P is full-dimensional and that $0 \in \mathrm{ri(P)}$. Now, Lemma 16 and Corollary 13 together tell us that P^* is a V-polytope. Applying Lemma 16 once more tells us that P^{**} is an H-polytope. Theorem 15 then tells us that $P = P^{**}$ so that P is an H-polytope as well. $\square$

Bibliography

[1] James R Munkres. *Topology*, volume 2. Prentice Hall Upper Saddle River, 2000.