

# ALGEBRAIC MATRIX COMPLETION PROBLEMS

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This documents contains several open problems (as of the time of writing) related to the algebraic geometry of low-rank matrix completion problems. They are to be discussed (and hopefully solved!) at the ICERM working group on matrix completion. For more background, consult the references listed at the end. A list of the working group participants can be found here:

[https://docs.google.com/spreadsheets/d/1\\_2RAPcDqnVM1\\_2YxYW6z-MnqUbpipMbWXGYWa0QUCbo/edit#gid=0](https://docs.google.com/spreadsheets/d/1_2RAPcDqnVM1_2YxYW6z-MnqUbpipMbWXGYWa0QUCbo/edit#gid=0)

## 1. TYPICAL RANKS OF NON-SYMMETRIC PARTIAL MATRICES

**Problem 1.** What is the maximum typical rank of the graph corresponding to non-symmetric partial matrices of the following form? Question marks correspond to unknown entries and stars correspond to known entries.

$$\begin{pmatrix} ? & * & * & * & * \\ * & ? & * & * & * \\ * & * & ? & * & * \\ * & * & * & ? & * \\ * & * & * & * & ? \end{pmatrix}.$$

What about the case of arbitrary  $n \times n$  partial matrices with unknown diagonal?

**Problem 2.** Let  $G$  be a planar bipartite graph. If  $G$  is not a tree, then  $G$  has generic completion rank 2. Therefore the greatest possible typical rank that  $G$  can have is 3. Under what conditions on  $G$  is 3 obtained as a typical rank? More generally, when does a given graph of generic completion rank 2 also have 3 as a typical rank? One necessary condition here is that  $G$  have a non-empty 2-core. Is this necessary condition sufficient?

**Problem 3.** Develop (numerical) techniques for computing the typical ranks of a bipartite graph.

**Problem 4.** Find classes of graphs whose generic completion rank is predicted by a dimension count.

## 2. TYPICAL RANKS OF SYMMETRIC PARTIAL MATRICES

**Problem 5.** Characterize the semi-simple graphs on  $n$  vertices that have  $n$  as a typical rank.

**Problem 6.** Develop (numerical) techniques for computing the typical ranks of a semisimple graph.

**Problem 7.** Find classes of graphs whose generic completion rank is predicted by a dimension count.

### 3. GENERIC COMPLETION RANK 2, NON-SYMMETRIC CASE

**Problem 8.** Let  $G = ([m], [n], E)$  be a bipartite graph and consider the projection  $\pi_E : \text{Mat}_2^{m \times n} \rightarrow \mathbb{C}^E$  of the variety of  $m \times n$  matrices of rank at most 2 onto the coordinates indexed by  $E$ . When is the corresponding elimination ideal generated by  $3 \times 3$  minors? *Problem suggested by Elizabeth Gross.*

Before attempting the next two problems, one might wish to familiarize themselves with the characterization of the independent sets in the algebraic matroid underlying the variety of  $m \times n$  matrices of rank at most two given in [1].

**Problem 9.** Does there exist a polynomial-time algorithm to check that a given bipartite graph has generic completion rank 2? Exhibit such an algorithm, or show that this problem is NP-hard.

**Problem 10.** Given a basis of the algebraic matroid underlying the variety of  $m \times n$  matrices of rank at most 2, what is the degree of the corresponding projection map?

**Problem 11.** Find a nice characterization of the circuits of the algebraic matroid underlying the variety of  $m \times n$  matrices of rank at most 2.

Following a solution to Problem 11, one might attempt the following.

**Problem 12.** Find a combinatorial algorithm that takes one of these circuits as input, and outputs the corresponding circuit polynomial.

### 4. GENERIC COMPLETION RANK 2, SYMMETRIC CASE

Unlike in the non-symmetric (and skew-symmetric) case, there is no known combinatorial characterization of the algebraic matroid underlying the variety of  $n \times n$  symmetric matrices of rank at most 2. Hence the only problem listed here is quite broad.

**Problem 13.** Find a combinatorial description of the algebraic matroid underlying the variety of symmetric matrices of rank at most 2.

### 5. MAXIMUM LIKELIHOOD THRESHOLD

**Problem 14.** For which planar graphs is the maximum likelihood threshold four?

**Problem 15.** Develop (numerical) techniques for computing maximum likelihood thresholds.

### REFERENCES

- [1] Daniel Irving Bernstein. Completion of tree metrics and rank 2 matrices. *Linear Algebra and its Applications*, 533:1–13, 2017. arXiv:1612.06797.
- [2] Daniel Irving Bernstein, Grigoriy Blekherman, and Rainer Sinn. Typical and generic ranks in matrix completion. *arXiv preprint*, 2018. arXiv:1802.09513.
- [3] Grigoriy Blekherman and Rainer Sinn. Maximum likelihood threshold and generic completion rank of graphs. *Discrete & Computational Geometry*, pages 1–22, 2017. arXiv:1703.07849.
- [4] Elizabeth Gross and Seth Sullivant. The maximum likelihood threshold of a graph. *Bernoulli*, 24(1):386–407, 2018. arXiv:1404.6989.