Math 7760 – Homework 3 – Due: September 14, 2022

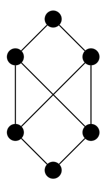
Practice Problems:

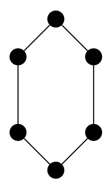
Problem 1. Show that the standard cube and the standard cross polytope are polar duals of each other.

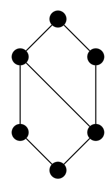
Problem 2. Show that least upper bounds and greatest lower bounds in a poset are unique.

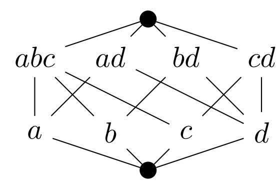
Problem 3. Prove that every finite lattice has a $\hat{0}$ and $\hat{1}$.

Problem 4. For each of the following posets, determine which are lattices. Among those that are, determine which are atomic and/or coatomic.









Problem 5. An *algebraic lattice* consists of a set S and two binary operations $\vee$ and $\wedge$ satisfying the following two axioms:

(1) $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ for all $x, y, z \in S$ (associativity)
(2) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y)$ for all $x, y \in S$ (absorption).

Show that if $(S, \leq)$ is a lattice with join and meet operations $\vee$ and $\wedge$, then $(S, \vee, \wedge)$ is an algebraic lattice. Then, show that if $(S, \vee, \wedge)$ is an algebraic lattice, then there exists a partial order $\leq$ on S that is a lattice with meet and join operations $\vee$ and $\wedge$.

Problems to write up:

Problem 6. Prove each of the following statements.

(1) The intersection of two polytopes is a polytope.
(2) The sum of two polytopes is a polytope.
(3) Every face of a polytope is exposed.

Problem 7. Define a partial order $\prec$ on $\mathbb{N}$ by $x \prec y$ if and only if every prime divisor of x is a prime divisor of y.

(1) Show that $(\mathbb{N}, \prec)$ is a lattice. What are the more familiar names for the meet and join operations?
(2) Does $(\mathbb{N}, \prec)$ have a $\hat{0}$ and/or a $\hat{1}$? If applicable, determine its atoms/coatoms.
(3) Is $(\mathbb{N}, \prec)$ atomic and/or coatomic?
(4) Show that $(\mathbb{N}, \prec)$ is isomorphic to the poset $(S, \subseteq)$ where S is the set of all finite multisets with elements in $\mathbb{N}$, partially ordered by inclusion.

Problem 8. Show that every polytope is affinely isomorphic to a bounded intersection of an orthant with an affine space.