Dykhne Adiabatic Approximation for Pair-creation

Making some assumptions

$restart : with(Physics) : with(Physics[Vectors]) : with(LinearAlgebra) : Setup(mathematicalnotation = true) :$

$Setup\left(realobjects = \left\{e, m, c, \varepsilon_0\right\}\right) :$
$Setup\left(realobjects = \left\{\mathrm{t}, V_x(\mathrm{t}), V_y(t), \eta(t), \Omega(t)\right\}\right);$

$$\left[realobjects = \left\{\hat{i}, \hat{j}, \hat{k}, \hat{\phi}, \hat{r}, \hat{\rho}, \hat{\theta}, c, e, m, \phi, r, \rho, t, \theta, x, y, z, \varepsilon_0, \Omega(t), V_x(t), V_y(t), \eta(t)\right\}\right] \quad (1.1)$$

$alias\left(V_x = V_x(t), V_y = V_y(t), \eta = \eta(t), \Omega = \Omega(t)\right)$

$$V_x, V_y, \eta, \Omega \quad (1.2)$$

cRDI mathematical formalism

Cliford conjugation

$CliforConj := (X) \rightarrow \langle\langle X[2,2] | -X[1,2]\rangle, \langle -X[2,1] | X[1,1]\rangle\rangle :$

Define a contravariant vector v from a paravector P

$DefContrVect := (v, P) \rightarrow Define\left(v_{\sim\mathrm{mu}} = Simplify\left(Array\left(1..4, (l) \rightarrow \frac{1}{2} \cdot Trace(P.Psigma[l])\right)\right)\right) :$

Extract the 3D vector from a paravector P

$ExtractVector := (P) \rightarrow \frac{1}{2} \cdot Simplify(Vector(3, (l) \rightarrow Trace(Psigma[l].P)).Vector([\_i, \_j, \_k])) :$

Get the Faraday matrix from the eigenspinor

$GetFaraday := (\Lambda, W) \rightarrow w \cdot \frac{2 \cdot m \cdot c}{e} \cdot \left(diff(\Lambda, t).CliforConj(\Lambda) + \Lambda.diff(W, t).CliforConj(W).CliforConj(\Lambda)\right) :$

Extract the electric field vector from the Faraday matrix

$GetElectricField := (F) \rightarrow \frac{1}{2} \cdot Simplify(Vector(3, (l) \rightarrow \mathrm{Re}(Trace(Psigma[l].F))).Vector([\_i, \_j,$

$\_k])):$

Extract the magnetic field vector from the Faraday matrix

$$GetMagneticField := (F) \rightarrow \frac{1}{2 \cdot c} \cdot Simplify(Vector(3, (l) \rightarrow \mathrm{Im}(Trace(Psigma[l].F))).Vector([\_i, \_j, \_k])):$$

Extracting the scalar and vector parts

$$ScalarPart := (L) \rightarrow \frac{(L + CliffordConj(L))}{2}:$$

$$VectorPart := (L) \rightarrow \frac{(L - CliffordConj(L))}{2}:$$

Extract the Poynting vector

$$PoyntingClifford := (F) \rightarrow \frac{c \cdot \varepsilon_0}{2} \cdot VectorPart(\ F.Dagger(F)):$$

$$PoyntingVector := (F) \rightarrow ExtractVector(PoyntingClifford(F)):$$

The radiated energy per unit of time is given by Larmor’s formula

$$PowerEmitted := (U) \rightarrow \frac{-e^2}{6 \cdot c \cdot \mathrm{Pi} \cdot \varepsilon_0} \cdot \frac{w^2}{2} \cdot Trace(diff(U, t).CliffordConj(diff(U, t))):$$

Particle in electromagnetic plane wave combined with homogeneous axial electric fields

The eigenspinor employed

$$\Lambda := MatrixExponential\left(\frac{\eta(t) \cdot Matrix(Psigma[3])}{2}\right) \cdot \begin{bmatrix} 1 & \frac{V_x(t)}{c} - \frac{I \cdot V_y(t)}{c} \\ 0 & 1 \end{bmatrix}$$

$$\Lambda := \begin{bmatrix} e^{\frac{\eta(t)}{2}} & e^{\frac{\eta(t)}{2}} \left(\frac{V_x(t)}{c} - \frac{\mathrm{I}\, V_y(t)}{c}\right) \\ 0 & e^{-\frac{\eta(t)}{2}} \end{bmatrix} \tag{3.1}$$

Proper velocity

$$U := simplify(\Lambda.Dagger(\Lambda))$$

$$U := \begin{bmatrix} \frac{e^{\eta(t)} \left(V_y(t)^2 + V_x(t)^2 + c^2\right)}{c^2} & \frac{-\mathrm{I}\, V_y(t) + V_x(t)}{c} \\ \frac{\mathrm{I}\, V_y(t) + V_x(t)}{c} & e^{-\eta(t)} \end{bmatrix} \tag{3.2}$$

$DefContrVect(u, U)$

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, \sigma_{\mu}, \partial_{\mu}, g_{\mu, \nu}, u^{\mu}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu}\right\} \tag{3.3}$$

Print proper velocity components

$'u[\sim 0]' = expand(u[\sim 0])$

$$u^{0} = \frac{1}{2\,e^{\eta}} + \frac{e^{\eta}\,V_{x}^{2}}{2\,c^{2}} + \frac{e^{\eta}\,V_{y}^{2}}{2\,c^{2}} + \frac{e^{\eta}}{2} \tag{3.4}$$

$'u[\sim 1]' = expand(u[\sim 1])$

$$u^{1} = \frac{V_{x}}{c} \tag{3.5}$$

$'u[\sim 2]' = expand(u[\sim 2])$

$$u^{2} = \frac{V_{y}}{c} \tag{3.6}$$

$'u[\sim 3]' = expand(u[\sim 3])$

$$u^{3} = \frac{e^{\eta}\,V_{y}^{2}}{2\,c^{2}} + \frac{e^{\eta}\,V_{x}^{2}}{2\,c^{2}} + \frac{e^{\eta}}{2} - \frac{1}{2\,e^{\eta}} \tag{3.7}$$

Enforce the laboratory time parametrization (w must be fixed as shown below for t to become the laboratory time)

$w := expand\left(\frac{1}{2} \cdot Trace(U)\right)$

$$w := \frac{1}{2\,e^{\eta}} + \frac{e^{\eta}\,V_{x}^{2}}{2\,c^{2}} + \frac{e^{\eta}\,V_{y}^{2}}{2\,c^{2}} + \frac{e^{\eta}}{2} \tag{3.8}$$

$expand(u[\sim 0] - w);$

$$0 \tag{3.9}$$

Circular trajectories in plane waves

$CircTrajParams := \left\{V_{x} = diff(r \cdot \cos(\omega \cdot t), t), V_{y} = diff(r \cdot \sin(\omega \cdot t), t), \eta = -\ln\left(\mathrm{sqrt}\left(1 + \left(\frac{r \cdot \omega}{c}\right)^{2}\right)\right)\right\};$

$$CircTrajParams := \left\{V_{x} = -r\,\omega \sin(\omega\,t), V_{y} = r\,\omega \cos(\omega\,t), \eta = -\frac{\ln\left(1 + \frac{r^{2}\,\omega^{2}}{c^{2}}\right)}{2}\right\} \tag{4.1}$$

Find trajectroy

$dsolve(\ subs(CircTrajParams, w \cdot diff(x(t), t) = u[\sim 1]))$assuming $c > 0;$

$$x(t) = \frac{r\cos(\omega t)}{\sqrt{r^2\omega^2 + c^2}} + \_C1 \tag{4.2}$$

dsolve(*subs*(*CircTrajParams*, $w \cdot$ *diff*($y(t), t$) = $u[\sim 2]$)) assuming $c > 0$ **and** omega > 0;

$$y(t) = \frac{r\sin(\omega t)}{\sqrt{r^2\omega^2 + c^2}} + \_C1 \tag{4.3}$$

dsolve(*subs*(*CircTrajParams*, $w \cdot$ *diff*($z(t), t$) = $u[\sim 3]$));

$$z(t) = \_C1 \tag{4.4}$$

Notice that the equation for critical time does not depend on t; thus, there will be no pari creation.

simplify(*subs*(*CircTrajParams*, $u[\sim 0]$));

$$\frac{\sqrt{r^2\omega^2 + c^2}}{|c|} \tag{4.5}$$

Pair creation in the homogeneous electric field

EParams := $\{V_x = 0, V_y = 0\}$;

$$EParams := \{V_x = 0, V_y = 0\} \tag{5.1}$$

Calculate the fields

F := *simplify*(*subs*(*EParams*, (*GetFarayday*(*subs*(*EParams*, Λ), *IdentityMatrix*(2))))) assuming $c > 0$ **and** $m > 0$;

$$F := \begin{bmatrix} \frac{\left(e^{-\eta(t)} + e^{\eta(t)}\right) m\, c\, \dot{\eta}(t)}{2\, e} & 0 \\ 0 & -\frac{\left(e^{-\eta(t)} + e^{\eta(t)}\right) m\, c\, \dot{\eta}(t)}{2\, e} \end{bmatrix} \tag{5.2}$$

GetMagneticField(F);

$$0 \tag{5.3}$$

GetElectricField(F)

$$\frac{\left(e^{-\eta} + e^{\eta}\right) m\, c \left(\frac{\partial}{\partial t}\eta\right) \hat{k}}{2\, e} \tag{5.4}$$

Find η to make electric homogeneous time independent

odetest$\left(\eta = \operatorname{arcsinh}\left(\frac{E_0\, e\, t}{m \cdot c}\right), \textit{coeff}(\textit{GetElectricField}(F), \_k) = E_0\right)$;

$$0 \tag{5.5}$$

Update params

EParams := *EParams* **union** $\left\{\eta = \operatorname{arcsinh}\left(\frac{E_0\, e\, t}{m \cdot c}\right)\right\}$;

$$EParams := \left\{V_x = 0, V_y = 0, \eta = \operatorname{arcsinh}\left(\frac{E_0\, e\, t}{m\, c}\right)\right\} \quad \textbf{(5.6)}$$

Find crytical time

$t0 := solve(subs(EParams, w), t);$

$$t0 := \frac{\mathrm{I}\, m\, c}{E_0\, e} \quad \textbf{(5.7)}$$

Calculate the pair creation rate notice $2 \cdot mc^2$ in front of w

$\exp\left(-\frac{2 \cdot \mathrm{Im}\left(int\left(subs\left(EParams, 2 \cdot m \cdot c^2 \cdot w\right), t = 0..t0\right)\right)}{\hbar}\right);$

$$\mathrm{e}^{-\frac{\pi\, m^2\, c^3\, \Re\left(\frac{1}{E_0}\right)}{e\, \hbar}} \quad \textbf{(5.8)}$$

Pair creation in two-color laser field (ω + 3ω)

$LaserParams := \left\{V_x = \frac{c}{\operatorname{sqrt}(g)} \cdot \cos(\omega \cdot t), V_y = 0, \eta = 0\right\}$

$$LaserParams := \left\{V_x = \frac{c \cos(\omega\, t)}{\sqrt{g}}, V_y = 0, \eta = 0\right\} \quad \textbf{(6.1)}$$

Calculate the fields

$F := simplify(subs(LaserParams, (GetFarayday(subs(LaserParams, \Lambda), IdentityMatrix(2)))))$ assuming $c > 0$ **and** $m > 0$;

$$F := \begin{bmatrix} 0 & -\frac{\left(c^2 \cos(\omega\, t)^2 + 2\, c^2\, g\right) m\, \omega \sin(\omega\, t)}{e\, c\, g^{3/2}} \\ 0 & 0 \end{bmatrix} \quad \textbf{(6.2)}$$

$B\_ := simplify(GetMagneticField(F))$ assuming $\omega > 0$ **and** $g > 1$;

$$\vec{B} := -\frac{\sin(\omega\, t)\, \omega\, \hat{j}\, m\, (\cos(2\, \omega\, t) + 1 + 4\, g)}{4\, e\, g^{3/2}} \quad \textbf{(6.3)}$$

$E\_ := simplify(GetElectricField(F))$ assuming $\omega > 0$ **and** $g > 1$;

$$\vec{E} := -\frac{\sin(\omega\, t)\, \omega\, \hat{i}\, c\, m\, (\cos(2\, \omega\, t) + 1 + 4\, g)}{4\, e\, g^{3/2}} \quad \textbf{(6.4)}$$

$E\_.B\_;$

$$0 \quad \textbf{(6.5)}$$

Find crytical time

$solve(subs(LaserParams, w), t)$ assuming $g > 0$;

$$\frac{\frac{\pi}{2} - \mathrm{I}\,\mathrm{arcsinh}\left(\sqrt{2}\,\sqrt{g}\right)}{\omega}, \frac{\frac{\pi}{2} + \mathrm{I}\,\mathrm{arcsinh}\left(\sqrt{2}\,\sqrt{g}\right)}{\omega} \tag{6.6}$$

$t0 := convert(combine(solve(subs(LaserParams, w), t)[2]), \mathrm{arcsinh})$ assuming $g > 0$;

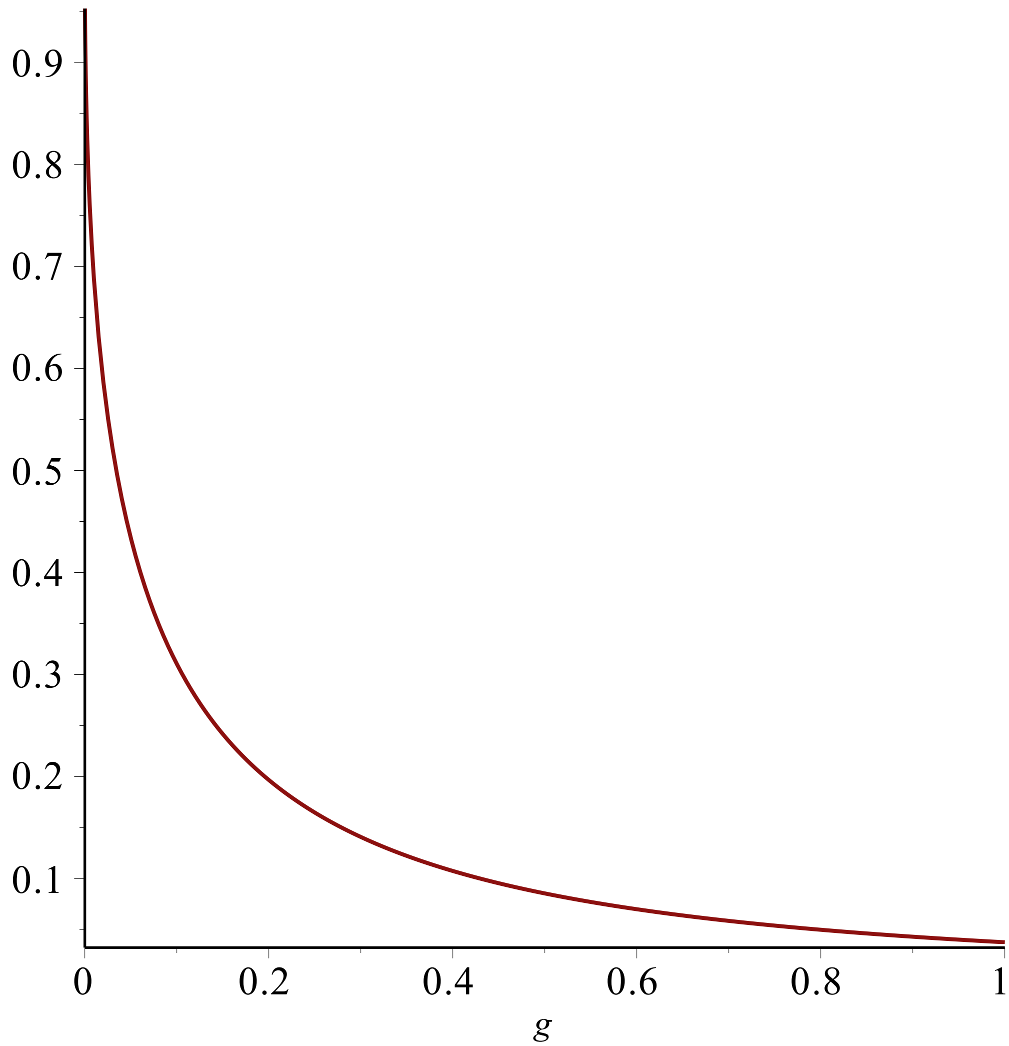
$$t0 := \frac{\frac{\pi}{2} - \mathrm{I}\,\mathrm{arcsinh}\left(\mathrm{I}\sqrt{-2\,g}\right)}{\omega} \tag{6.7}$$

Calculate the pair creation rate notice $2 \cdot mc^2$ in front of w

$transition := \exp\left(-\frac{2 \cdot \mathrm{Im}\left(int\left(subs\left(LaserParams, 2 \cdot m \cdot c^2 \cdot w\right), t = 0 \,..\, t0\right)\right)}{\hbar}\right)$ assuming $r > 0$ **and** $c > 0$ **and** $\omega > 0$ **and** $g > 1$;

$$transition := \mathrm{e}^{-\frac{m\,c^2\left(-2\sqrt{2}\,\sqrt{1+2\,g}\,g + 8\,\mathrm{arcsinh}\left(\sqrt{2}\,\sqrt{g}\right) g^{3/2} + 2\,\mathrm{arcsinh}\left(\sqrt{2}\,\sqrt{g}\right)\sqrt{g}\right)}{2\,\omega\,g^{3/2}\,\hbar}} \tag{6.8}$$

$plot(subs(m = 1, c = 1, \mathrm{omega} = 1, \hbar = 1, (transition)), g = 0 \,..\, 1,);$



Get the asymptotic expression for large g

$$asympt\left(-\frac{m\,c^2\left(-2\sqrt{2}\sqrt{1+2\,g}\;g+8\,\mathrm{arcsinh}\left(\sqrt{2}\sqrt{g}\right)g^{3/2}+2\,\mathrm{arcsinh}\left(\sqrt{2}\sqrt{g}\right)\sqrt{g}\right)}{2\,\omega\,g^{3/2}\,\hbar},\ g, 2\right);$$

$$-\frac{m\,c^2\left(-4+8\ln\left(2\sqrt{2}\right)+4\ln(g)\right)}{2\,\omega\,\hbar}-\frac{m\,c^2\left(2\ln\left(2\sqrt{2}\right)+\ln(g)\right)}{2\,\omega\,\hbar\,g}+\mathrm{O}\left(\frac{1}{g^2}\right) \qquad \textbf{(6.9)}$$

$$simplify\left(\exp\left(-\frac{m\,c^2\,(4\ln(g))}{2\,\omega\,\hbar}\right)\right);$$

$$g^{-\frac{2\,m\,c^2}{\omega\,\hbar}} \qquad \textbf{(6.10)}$$

$$simplify\left(\exp\left(-\frac{m\,c^2\left(-4+8\ln\left(2\sqrt{2}\right)\right)}{2\,\omega\,\hbar}\right)\right);$$

$$\mathrm{e}^{-\frac{2\,m\,c^2\,(-1+3\ln(2))}{\omega\,\hbar}} \qquad \textbf{(6.11)}$$

$$evalf(-1+3\ln(2));$$

$$1.079441542 \qquad \textbf{(6.12)}$$