III. Particle in an electromagnetic plane wave

$restart : with(Physics) : with(Physics[Vectors]) : with(LinearAlgebra) : Setup(mathematicalnotation = true) :$

Making some assumptions

$Setup(realobjects = \{c, e, m\}) :$
$Setup\left(realobjects = \left\{\mathrm{xi}, \mathrm{tau}, \Omega(\xi), \eta(\xi), w(\xi), v_x(\xi), v_y(\xi)\right\}\right);$

$$\left[realobjects = \left\{\hat{i}, \hat{j}, \hat{k}, \hat{\phi}, \hat{r}, \hat{\rho}, \hat{\theta}, c, e, m, \phi, r, \rho, \tau, \theta, x, \xi, y, z, \Omega(\xi), \eta(\xi), v_x(\xi), v_y(\xi), w(\xi)\right\}\right] \tag{1}$$

Cliford conjugation

$CliforConj := (X) \rightarrow \langle\langle X[2, 2] | -X[1, 2]\rangle, \langle -X[2, 1] | X[1, 1]\rangle\rangle :$

Define a contravariant vector v from a paravector P

$DefContrVect := (v, P) \rightarrow Define\left(v_{\sim\mathrm{mu}} = Simplify\left(Array\left(1..4, (l) \rightarrow \frac{1}{2} \cdot Trace(P.Psigma[l])\right)\right)\right) :$

Extract the electric field vector from the Faraday matrix

$GetElectricField := (F) \rightarrow Simplify\left(Vector\left(3, (l) \rightarrow \frac{1}{2} \mathrm{Re}(Trace(Psigma[l].F))\right).Vector([\_i, \_j, \_k])\right) :$

Extract the magnetic field vector from the Faraday matrix

$GetMagneticField := (F) \rightarrow Simplify\left(Vector\left(3, (l) \rightarrow \frac{1}{2 \cdot c} \mathrm{Im}(Trace(Psigma[l].F))\right).Vector([\_i, \_j, \_k])\right) :$

The radiated energy per unit of time is given by Larmor's formula

$$PowerEmitted := (U) \to \frac{e^2}{6 \cdot c \cdot \mathrm{Pi} \cdot \varepsilon_0} \cdot \frac{w(\xi)^2}{2} \cdot Trace(diff(U, \xi).CliffordConj(diff(U, \xi))) :$$

Specifying the eigenspinor

$$\Lambda := MatrixExponential\left(\frac{\eta(\xi)}{2} \cdot Matrix(Psigma[3])\right).Matrix\left(\left[\left[1, \frac{v_x(\xi)}{c} - \frac{I \cdot v_y(\xi)}{c}\right], [0, 1]\right]\right) .MatrixExponential\left(-I \cdot \frac{\Omega(\xi)}{2} \cdot Matrix(Psigma[3])\right);$$

$$\Lambda := \left[\left[e^{\frac{\eta(\xi)}{2}} \left(\cos\left(\frac{\Omega(\xi)}{2}\right) - \mathrm{I}\sin\left(\frac{\Omega(\xi)}{2}\right)\right), e^{\frac{\eta(\xi)}{2}} \left(\frac{v_x(\xi)}{c} - \frac{\mathrm{I}\, v_y(\xi)}{c}\right) \left(\cos\left(\frac{\Omega(\xi)}{2}\right) + \mathrm{I}\sin\left(\frac{\Omega(\xi)}{2}\right)\right)\right], \left[0, e^{-\frac{\eta(\xi)}{2}} \left(\cos\left(\frac{\Omega(\xi)}{2}\right) + \mathrm{I}\sin\left(\frac{\Omega(\xi)}{2}\right)\right)\right]\right] \tag{2}$$

Extracting proper velocity

$$U := simplify(\Lambda.Dagger(\Lambda));$$

$$U := \begin{bmatrix} \frac{e^{\eta(\xi)} \left(v_y(\xi)^2 + v_x(\xi)^2 + c^2\right)}{c^2} & \frac{-\mathrm{I}\, v_y(\xi) + v_x(\xi)}{c} \\ \frac{\mathrm{I}\, v_y(\xi) + v_x(\xi)}{c} & e^{-\eta(\xi)} \end{bmatrix} \tag{3}$$

Define contravariant vector of the velocity

$$DefContrVect(u, U);$$

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, \sigma_{\mu}, \partial_{\mu}, g_{\mu,\nu}, u^{\mu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}\right\} \tag{4}$$

$$combine(expand(u[\sim 3]));$$

$$\frac{e^{\eta(\xi)}\, v_y(\xi)^2}{2\, c^2} + \frac{e^{\eta(\xi)}\, v_x(\xi)^2}{2\, c^2} + \frac{e^{\eta(\xi)}}{2} - \frac{e^{-\eta(\xi)}}{2} \tag{5}$$

The electromagnetic field (w is a parametric measure)

$$F := w(\xi)\cdot\frac{2\cdot m\cdot c}{e}\cdot diff(\Lambda,\xi).CliffordConj(\Lambda) :$$

Extract the electromagnetic fields from the matrix F

$E\_ := GetElectricField(F);$

$$\vec{E} := \frac{1}{e}\left(\left(\left(v_y(\xi)\,e^{\frac{\eta(\xi)}{2}}\left(\frac{d}{d\xi}\Omega(\xi)\right)\hat{i} - v_x(\xi)\,e^{\frac{\eta(\xi)}{2}}\left(\frac{d}{d\xi}\Omega(\xi)\right)\hat{j} + c\,e^{-\frac{\eta(\xi)}{2}}\left(\frac{d}{d\xi}\eta(\xi)\right)\hat{k} + \left(\frac{d}{d\xi}v_y(\xi)\right)e^{\frac{\eta(\xi)}{2}}\hat{j} + \left(\frac{d}{d\xi}v_x(\xi)\right)e^{\frac{\eta(\xi)}{2}}\hat{i}\right)e^{\frac{\eta(\xi)}{2}}w(\xi)\,m\right)\right)$$

$Simplify(Component(\vec{E}, 1));$

$$\frac{\left(v_y(\xi)\left(\frac{d}{d\xi}\Omega(\xi)\right) + \frac{d}{d\xi}v_x(\xi)\right)e^{\eta(\xi)}w(\xi)\,m}{e} \tag{7}$$

$Simplify(Component(\vec{E}, 2));$

$$-\frac{\left(v_x(\xi)\left(\frac{d}{d\xi}\Omega(\xi)\right) - \left(\frac{d}{d\xi}v_y(\xi)\right)\right)e^{\eta(\xi)}w(\xi)\,m}{e} \tag{8}$$

$combine(Component(\vec{E}, 3));$

$$\frac{w(\xi)\,m\,c\left(\frac{d}{d\xi}\eta(\xi)\right)}{e} \tag{9}$$

$B\_ := GetMagneticField(F);$

$$\vec{B} := \frac{1}{e\,c}\left(\left(\left(v_y(\xi)\,e^{\frac{\eta(\xi)}{2}}\left(\frac{d}{d\xi}\Omega(\xi)\right)\hat{j} + v_x(\xi)\,e^{\frac{\eta(\xi)}{2}}\left(\frac{d}{d\xi}\Omega(\xi)\right)\hat{i} - c\,e^{-\frac{\eta(\xi)}{2}}\left(\frac{d}{d\xi}\Omega(\xi)\right)\hat{k} - \left(\frac{d}{d\xi}v_y(\xi)\right)e^{\frac{\eta(\xi)}{2}}\hat{i} + \left(\frac{d}{d\xi}v_x(\xi)\right)e^{\frac{\eta(\xi)}{2}}\hat{j}\right)e^{\frac{\eta(\xi)}{2}}w(\xi)\,m\right)\right) \tag{10}$$

$Simplify(Component(\vec{B}, 1));$

$$\frac{\left(v_x(\xi)\left(\frac{d}{d\xi}\Omega(\xi)\right)-\left(\frac{d}{d\xi}v_y(\xi)\right)\right)e^{\eta(\xi)}w(\xi)\,m}{e\,c} \tag{11}$$

$Simplify(Component(\vec{B}, 2));$

$$\frac{\left(v_y(\xi)\left(\frac{d}{d\xi}\Omega(\xi)\right)+\frac{d}{d\xi}v_x(\xi)\right)e^{\eta(\xi)}w(\xi)\,m}{e\,c} \tag{12}$$

$combine(Component(\vec{B}, 3));$

$$-\frac{w(\xi)\,m\left(\frac{d}{d\xi}\Omega(\xi)\right)}{e} \tag{13}$$

Energy

$Simplify\left(\frac{m\cdot c^2}{2}\cdot Trace(U.Psigma[0])\right);$

$$\frac{m\left(e^{\eta(\xi)}\left(v_y(\xi)^2+v_x(\xi)^2+c^2\right)+e^{-\eta(\xi)}c^2\right)}{2} \tag{14}$$

The case of circular trajectory

Describe a circular trajectory

$$CircularTrajectory := \left\{v_x(\xi) = diff\left(r\cdot\cos\left(\frac{\omega\cdot\xi}{\mathrm{sqrt}\left(1-\left(\frac{r\cdot\omega}{c}\right)^2\right)}\right), \xi\right), v_y(\xi) = diff\left(r\cdot\sin\left(\frac{\omega\cdot\xi}{\mathrm{sqrt}\left(1-\left(\frac{r\cdot\omega}{c}\right)^2\right)}\right), \xi\right)\right\}:$$

Characterize the newly enter parameters as real

$Setup(realobjects = \{\omega, r\});$

$$\left[realobjects = \left\{\hat{i}, \hat{j}, \hat{k}, \hat{\phi}, \hat{r}, \hat{\rho}, \hat{\theta}, c, e, m, \omega, \phi, r, \rho, \tau, \theta, x, \xi, y, z, \Omega(\xi), \eta(\xi), v_x(\xi), v_y(\xi), w(\xi)\right\}\right] \tag{15}$$

Find $\eta(\xi)$ that eliminates the drift along the z-axis

$solve(Simplify(subs(CircularTrajectory, u[\sim 3])), \eta(\xi));$

$CircularTrajectory := CircularTrajectory \ \mathbf{union} \ \{\eta(\xi) = solve(Simplify(subs(CircularTrajectory, u[\sim 3])), \eta(\xi))[1]\};$

$$CircularTrajectory := \left\{\eta(\xi) = \ln\left(\frac{\sqrt{-r^2\omega^2 + c^2}}{c}\right), v_x(\xi) = -\frac{r\,\omega\sin\left(\frac{\omega\,\xi}{\sqrt{1 - \frac{r^2\omega^2}{c^2}}}\right)}{\sqrt{1 - \frac{r^2\omega^2}{c^2}}}, v_y(\xi) = \frac{r\,\omega\cos\left(\frac{\omega\,\xi}{\sqrt{1 - \frac{r^2\omega^2}{c^2}}}\right)}{\sqrt{1 - \frac{r^2\omega^2}{c^2}}}\right\} \quad (16)$$

$'u[\sim 0]' = Simplify(subs(CircularTrajectory, u[\sim 0]));$

$$u^0 = \frac{c}{\sqrt{-r^2\omega^2 + c^2}} \quad (17)$$

$CircularTrajectory := CircularTrajectory \ \mathbf{union} \ \{w(\xi) = 1\}:$

Get the final trajectory

$Simplify\left(subs\left(CircularTrajectory, Array\left(1..4, (l) \rightarrow int\left(\frac{c}{2\cdot w(\xi)}\cdot Trace(U.Psigma[1]), \xi\right)\right)\right)\right);$

$$\left[\begin{array}{cccc} r\cos\left(\frac{\omega\,\xi}{\sqrt{\frac{-r^2\omega^2 + c^2}{c^2}}}\right) & r\sin\left(\frac{\omega\,\xi}{\sqrt{\frac{-r^2\omega^2 + c^2}{c^2}}}\right) & 0 & \frac{\xi\,c^2}{\sqrt{-r^2\omega^2 + c^2}} \end{array}\right] \quad (18)$$

The case of Laser Accelerator Scheme (LAS)

Get the final trajectory

$LASTrajectory := \{\eta(\xi) = 0, w(\xi) = c^2\};$

$$LASTrajectory := \{\eta(\xi) = 0, w(\xi) = c^2\} \quad (19)$$

The electromagentic fields

$'E\_' = Simplify(expand(subs(LASTrajectory, E\_)));$

$$\vec{E} = \frac{1}{e}\left(m\,c^2\left(v_y(\xi)\left(\frac{d}{d\xi}\,\Omega(\xi)\right)\hat{\imath} - v_x(\xi)\left(\frac{d}{d\xi}\,\Omega(\xi)\right)\hat{\jmath} + \left(\frac{d}{d\xi}\,v_y(\xi)\right)\hat{\jmath} + \left(\frac{d}{d\xi}\,v_x(\xi)\right)\hat{\imath}\right)\right) \quad (20)$$

$'B\_' = Simplify(expand(subs(LASTrajectory, B\_)));$

$$\vec{B} = \frac{1}{e}\left(m\,c\left(v_y(\xi)\left(\frac{d}{d\xi}\,\Omega(\xi)\right)\hat{\jmath} + v_x(\xi)\left(\frac{d}{d\xi}\,\Omega(\xi)\right)\hat{\imath} - c\left(\frac{d}{d\xi}\,\Omega(\xi)\right)\hat{k} - \left(\frac{d}{d\xi}\,v_y(\xi)\right)\hat{\imath} + \left(\frac{d}{d\xi}\,v_x(\xi)\right)\hat{\jmath}\right)\right) \quad (21)$$

Specify trajectory for LAS

$$LASTrajectory := LASTrajectory\ \mathbf{union}\left\{v_x(\xi) = diff\left(\frac{e\cdot E_0}{m\cdot c\cdot\omega}\cdot\xi\cdot\cos\left(\frac{\omega\cdot\xi}{c}\right), \xi\right), v_y(\xi) = diff\left(\frac{e\cdot E_0}{m\cdot c\cdot\omega}\cdot\xi\cdot\sin\left(\frac{\omega\cdot\xi}{c}\right), \xi\right), \Omega(\xi) = \frac{\omega\cdot\xi}{c}\right\}:$$

$'E\_' = Simplify(expand(subs(LASTrajectory, E\_)));$

$$\vec{E} = -\hat{\imath}\,E_0\sin\left(\frac{\omega\,\xi}{c}\right) + \hat{\jmath}\,E_0\cos\left(\frac{\omega\,\xi}{c}\right) \quad (22)$$

$'B\_' = Simplify(expand(subs(LASTrajectory, B\_)));$

$$\vec{B} = -\frac{m\,c^2\,\omega\,\hat{k} + \hat{\jmath}\,E_0\sin\left(\frac{\omega\,\xi}{c}\right)e + \hat{\imath}\,E_0\cos\left(\frac{\omega\,\xi}{c}\right)e}{e\,c} \quad (23)$$

This leads to monotonically increasing energy

$$Simplify\left(subs\left(LASTrajectory, m\cdot c^2\cdot u[\sim 0]\right) - m\cdot c^2\cdot\left(1 + \frac{E_0{}^2\cdot e^2}{2\cdot m^2\cdot c^6\cdot\omega^2}\cdot\left(c^2 + \xi^2\cdot\omega^2\right)\right)\right);$$

$$0 \quad (24)$$

$expand(Simplify(subs(LASTrajectory, u[\sim 0])));$

$$1 + \frac{E_0{}^2\,e^2\,\xi^2}{2\,m^2\,c^6} + \frac{E_0{}^2\,e^2}{2\,m^2\,c^4\,\omega^2} \quad (25)$$

$Simplify(expand(subs(LASTrajectory, PowerEmitted(U))));$

$$-\frac{\left(\xi^2\,\omega^2 + 4\,c^2\right)E_0{}^2\,e^4}{6\,\varepsilon_0\,\pi\,c^5\,m^2} \quad (26)$$