Particle in an electromagnetic plane wave

Suggested notations: Use capital latter to denote matrices and small to denote co- (cotra-)variant vectors

$restart : with(Physics) : with(Physics[Vectors]) : Setup(mathematicalnotation = true) :$

Making some assumptions

$Setup(realobjects = \{c, e, m\}) :$

$Setup\left(realobjects = \{\text{eta, xi, tau}, w, v_x(\text{xi}), v_y(\text{xi})\}\right);$

$$\left[realobjects = \left\{\hat{i}, \hat{j}, \hat{k}, \hat{\phi}, \hat{r}, \hat{\rho}, \hat{\theta}, c, e, \eta, m, \phi, r, \rho, \tau, \theta, w, x, \xi, y, z, v_x(\xi), v_y(\xi)\right\}\right] \tag{1}$$

$PDEtools[declare]\left(\left(v_x, v_y\right)(\text{xi})\right);$

$$v_x(\xi) \textit{ will now be displayed as } v_x$$
$$v_y(\xi) \textit{ will now be displayed as } v_y \tag{2}$$

Cliford conjugation

$CliforConj := (X) \rightarrow \langle\langle X[2,2] | -X[1,2]\rangle, \langle -X[2,1] | X[1,1]\rangle\rangle :$

Define a contravariant vector v from a paravector P

$DefContrVect := (v, P) \rightarrow Define\left(v_{\sim mu} = Simplify\left(Array\left(1..4, (l) \rightarrow \frac{1}{2} \cdot Trace(P.Psigma[l])\right)\right)\right) :$

Extract the electric field vector from the Faraday matrix

$GetElectricField := (F) \rightarrow Simplify\left(Vector\left(3, (l) \rightarrow \frac{1}{2} \operatorname{Re}(Trace(Psigma[l].F))\right).Vector([\_i, \_j, \_k])\right) :$

Extract the magnetic field vector from the Faraday matrix

$GetMagneticField := (F) \rightarrow Simplify\left(Vector\left(3, (l) \rightarrow \frac{1}{2 \cdot c} \operatorname{Im}(Trace(Psigma[l].F))\right).Vector([\_i,$

$\_j, \_k])\Big):$

Specifying the eigenspinor

$\Lambda := Matrix\left(\left[\left[\exp\left(\frac{\text{eta}}{2}\right), 0\right], \left[0, \exp\left(-\frac{\text{eta}}{2}\right)\right]\right]\right).Matrix([[1, v_x(\text{xi}) - \text{I}\cdot v_y(\text{xi})], [0, 1]]);$

$$\Lambda := \begin{bmatrix} e^{\frac{\eta}{2}} & e^{\frac{\eta}{2}}\left(v_x - \mathrm{I}\, v_y\right) \\ 0 & e^{-\frac{\eta}{2}} \end{bmatrix} \tag{3}$$

The electromagnetic field (w is a parametric measure)

$\mathrm{F} := \mathrm{w}\cdot\frac{2\cdot m\cdot c}{e}\cdot diff(\Lambda, \text{xi}).CliffordConj(\Lambda);$

$$F := \begin{bmatrix} 0 & \frac{2\, w\, m\, c\left(e^{\frac{\eta}{2}}\right)^2 \left(v_{x_\xi} - \mathrm{I}\, v_{y_\xi}\right)}{e} \\ 0 & 0 \end{bmatrix} \tag{4}$$

$$\tag{5}$$

Extract the electromagnetic fields from the matrix F

$GetElectricField(F);$

$$\frac{\left(v_{x_\xi}\,\hat{i} + v_{y_\xi}\,\hat{j}\right) e^{\eta}\, c\, m\, w}{e} \tag{6}$$

$GetMagneticField(F);$

$$\frac{\left(v_{x_\xi}\,\hat{j} - v_{y_\xi}\,\hat{i}\right) e^{\eta}\, m\, w}{e} \tag{7}$$

Fix the parametric weight w

$$\tag{8}$$

$simplify(subs(w = c\cdot\exp(-\text{eta}), GetElectricField(F)));$

$$\frac{\left(v_{x_{\xi}}\,\widehat{i}+v_{y_{\xi}}\,\widehat{j}\right)c^2\,m}{e} \tag{9}$$

simplify(*subs*($w = c\cdot\exp(-\mathrm{eta})$, *GetMagneticField*($F$)));

$$\frac{\left(v_{x_{\xi}}\,\widehat{j}-v_{y_{\xi}}\,\widehat{i}\right)m\,c}{e} \tag{10}$$

$w := c\cdot\exp(-\mathrm{eta})$;

$$w := c\,\mathrm{e}^{-\eta} \tag{11}$$

Extracting proper velocity

$U :=$ *simplify*(Λ.*Dagger*(Λ));

$$U := \begin{bmatrix} \mathrm{e}^{\eta}\left(v_x^{\;2}+v_y^{\;2}+1\right) & v_x - \mathrm{I}\,v_y \\ v_x + \mathrm{I}\,v_y & \mathrm{e}^{-\eta} \end{bmatrix} \tag{12}$$

Extract trajectory (as a contravariant vector)

$$Define\left(trajectory_{\sim\mathrm{mu}} = Array\left(1..4, (l) \rightarrow Int\left(\frac{c}{2\cdot w}\cdot Trace(U.Psigma[1]), \mathrm{xi}\right)\right)\right);$$

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, \sigma_{\mu}, \partial_{\mu}, g_{\mu,\nu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}, trajectory^{\mu}\right\} \tag{13}$$

the parameter ξ corresponds to the forward ligh-cone variable

simplify(*combine*(*trajectory*[~0] − *trajectory*[~3]));

$$\int 1\,\mathrm{d}\xi \tag{14}$$

Energy

$$\frac{m\cdot c^2}{2}\cdot Trace(U.Psigma[0]);$$

$$\frac{m\,c^2\left(\mathrm{e}^{\eta}\left(v_x^{\;2}+v_y^{\;2}+1\right)+\mathrm{e}^{-\eta}\right)}{2} \tag{15}$$

Define contravariant vector of the velocity

DefContrVect(u, U);

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, \sigma_{\mu}, \partial_{\mu}, g_{\mu, \nu}, u^{\mu}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu}, \mathit{trajectory}^{\mu}\right\} \tag{16}$$

u[~2];

$$v_y \tag{17}$$