Particle in an electromagnetic plane wave

Suggested notations: Use capital latter to denote matrices and small to denote co- (cotra-)variant vectors

$restart : with(Physics) : with(Physics[Vectors]) : Setup(mathematicalnotation = true) :$

Making some assumptions

$Setup\left(realobjects = \left\{c, e, m, \varepsilon_0\right\}\right) :$

$Setup\left(realobjects = \left\{\text{eta, xi, tau}, w, v_x(\text{xi}), v_y(\text{xi})\right\}\right);$

$$\left[realobjects = \left\{\hat{i}, \hat{j}, \hat{k}, \hat{\phi}, \hat{r}, \hat{\rho}, \hat{\theta}, c, e, \eta, m, \phi, r, \rho, \tau, \theta, w, x, \xi, y, z, \varepsilon_0, v_x(\xi), v_y(\xi)\right\}\right] \tag{1}$$

$\#PDEtools[declare]\left(\left(v_x, v_y\right)(\text{xi})\right);$

Cliford conjugation

$CliffordConj := (X) \rightarrow \langle\langle X[2,2] | -X[1,2]\rangle, \langle -X[2,1] | X[1,1]\rangle\rangle :$

Define a contravariant vector v from a paravector P

$DefContrVect := (v, P) \rightarrow Define\left(v_{\sim\text{mu}} = Simplify\left(Array\left(1..4, (l) \rightarrow \frac{1}{2} \cdot Trace(P.Psigma[l])\right)\right)\right) :$

Extract the 3D vector from a paravector P

$ExtractVector := (P) \rightarrow \frac{1}{2} \cdot Simplify(Vector(3, (l) \rightarrow Trace(Psigma[l].P)).Vector([\_i, \_j, \_k])) :$

Extract the electric field vector from the Faraday matrix

$GetElectricField := (F) \rightarrow Simplify(\mathrm{Re}(ExtractVector(F))) :$

Extract the magnetic field vector from the Faraday matrix

$GetMagneticField := (F) \rightarrow Simplify\left(\frac{\mathrm{Im}(ExtractVector(F))}{c}\right) :$

Extracting the scalar and vector parts

$$ScalarPart := (L) \rightarrow \frac{(L + CliffordConj(L))}{2} :$$

$$VectorPart := (L) \rightarrow \frac{(L - CliffordConj(L))}{2} :$$

Extract the Poynting vector

$$PoyntingCliford := (F) \rightarrow \frac{c \cdot \varepsilon_0}{2} \cdot VectorPart(\ F.Dagger(F)\ ) :$$

$$PoyntingVector := (F) \rightarrow ExtractVector(PoyntingCliford(F)) :$$

The radiated energy per unit of time is given by Larmor's formula

$$PowerEmitted := (U) \rightarrow \frac{-e^2}{6 \cdot c \cdot \mathrm{Pi} \cdot \varepsilon_0} \cdot \frac{w^2}{2} \cdot Trace(diff(U, \xi).CliffordConj(diff(U, \xi))) :$$

Specifying the eigenspinor

$$\Lambda := Matrix\left(\left[\left[\exp\left(\frac{\mathrm{eta}}{2}\right), 0\right], \left[0, \exp\left(-\frac{\mathrm{eta}}{2}\right)\right]\right]\right).Matrix([[1, v_x(\mathrm{xi}) - \mathrm{I} \cdot v_y(\mathrm{xi})], [0, 1]]);$$

The electromagnetic field (w is a parametric measure)

$$\mathrm{F} := \mathrm{w} \cdot \frac{2 \cdot m \cdot c}{e} \cdot diff(\Lambda, \mathrm{xi}).CliffordConj(\Lambda);$$

$$F := \begin{bmatrix} 0 & \dfrac{2\, w\, m\, c \left(e^{\frac{\eta}{2}}\right)^2 \left(\frac{d}{d\xi} v_x(\xi) - \mathrm{I}\left(\frac{d}{d\xi} v_y(\xi)\right)\right)}{e} \\ 0 & 0 \end{bmatrix} \qquad \mathbf{(2)}$$

(3)

Extract the electromagnetic fields from the matrix F

$$GetElectricField(F);$$

$$\frac{e^{\eta}\, c\, m\, w \left(\left(\frac{d}{d\xi} v_x(\xi)\right) \hat{i} + \left(\frac{d}{d\xi} v_y(\xi)\right) \hat{j}\right)}{e} \qquad \mathbf{(4)}$$

$$GetMagneticField(F);$$

$$-\frac{e^{\eta}\, m\, w \left(\left(\frac{d}{d\xi} v_x(\xi) \right) \hat{j} - \left(\frac{d}{d\xi} v_y(\xi) \right) \hat{i} \right)}{e} \tag{5}$$

Fix the parametric weight w

(6)

$simplify(subs(w = c \cdot \exp(-\mathrm{eta}), GetElectricField(F)));$

$$\frac{c^2\, m \left(\left(\frac{d}{d\xi} v_x(\xi) \right) \hat{i} + \left(\frac{d}{d\xi} v_y(\xi) \right) \hat{j} \right)}{e} \tag{7}$$

$simplify(subs(w = c \cdot \exp(-\mathrm{eta}), GetMagneticField(F)));$

$$-\frac{m\, c \left(\left(\frac{d}{d\xi} v_x(\xi) \right) \hat{j} - \left(\frac{d}{d\xi} v_y(\xi) \right) \hat{i} \right)}{e} \tag{8}$$

$w := c \cdot \exp(-\mathrm{eta});$

$$w := c\, e^{-\eta} \tag{9}$$

Extracting proper velocity

$U := simplify(\Lambda.Dagger(\Lambda));$

$$U := \begin{bmatrix} e^{\eta} \left(v_x(\xi)^2 + v_y(\xi)^2 + 1 \right) & v_x(\xi) - \mathrm{I}\, v_y(\xi) \\ v_x(\xi) + \mathrm{I}\, v_y(\xi) & e^{-\eta} \end{bmatrix} \tag{10}$$

Extract trajectory (as a contravariant vector)

$Define\left(trajectory_{\sim \mathrm{mu}} = Array\left(1..4, (l) \to Int\left(\frac{c}{2 \cdot w} \cdot Trace(U.Psigma[1]), \mathrm{xi} \right) \right) \right);$

Defined objects with tensor properties

$$\left\{ \gamma_{\mu}, \sigma_{\mu}, \partial_{\mu}, g_{\mu,\nu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\beta,\mu,\nu}, trajectory^{\mu} \right\} \tag{11}$$

the parameter ξ corresponds to the forward ligh-cone variable

$simplify(combine(trajectory[\sim 0] - trajectory[\sim 3]));$

$$\int 1\, d\xi \tag{12}$$

Energy

$$\frac{m \cdot c^2}{2} \cdot Trace(U.Psigma[0]);$$

$$\frac{m\, c^2 \left(\mathrm{e}^{\eta} \left(v_x(\xi)^2 + v_y(\xi)^2 + 1\right) + \mathrm{e}^{-\eta}\right)}{2} \tag{13}$$

Define contravariant vector of the velocity

$DefContrVect(u, U);$

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, \sigma_{\mu}, \partial_{\mu}, g_{\mu, \nu}, u^{\mu}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu}, trajectory^{\mu}\right\} \tag{14}$$

$u[\sim 2];$

$$v_y(\xi) \tag{15}$$

$PoyntingVector(F);$

$$\frac{\varepsilon_0\, c^5\, \hat{k}\, m^2 \left(\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\, v_x(\xi)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}\xi}\, v_y(\xi)\right)^2\right)}{e^2} \tag{16}$$

Power emitted in terms of the norm of the Poynting vector

$$\frac{'P'}{'|P\_|'} = simplify\left(\frac{PowerEmitted(U)}{Component(PoyntingVector(F), 3)}\right);$$

$$\frac{P}{|\vec{P}|} = \frac{\mathrm{e}^{-2\,\eta}\, e^4}{6\, m^2\, \varepsilon_0^{\,2}\, \pi\, c^4} \tag{17}$$