III. Particle in an electromagnetic plane wave

$restart : with(Physics) : with(Physics[Vectors]) : with(LinearAlgebra) : Setup(mathematicalnotation = true) :$

Making some assumptions

$Setup\left(realobjects = \left\{c, e, m, \varepsilon_0\right\}\right) :$
$Setup\left(realobjects = \left\{\text{xi, tau}, \Omega(\xi), \eta(\xi), w(\xi), v_x(\xi), v_y(\xi)\right\}\right);$

$$\left[realobjects = \left\{\hat{i}, \hat{j}, \hat{k}, \hat{\phi}, \hat{r}, \hat{\rho}, \hat{\theta}, c, e, m, \phi, r, \rho, \tau, \theta, x, \xi, y, z, \varepsilon_0, \Omega(\xi), \eta(\xi), v_x(\xi), v_y(\xi), w(\xi)\right\}\right] \quad \textbf{(1)}$$

Cliford conjugation

$CliffordConj := (X) \rightarrow \langle\langle X[2,2] | -X[1,2]\rangle, \langle -X[2,1] | X[1,1]\rangle\rangle :$

Define a contravariant vector v from a paravector P

$DefContrVect := (v, P) \rightarrow Define\left(v_{\sim\mathrm{mu}} = Simplify\left(Array\left(1..4, (l) \rightarrow \frac{1}{2} \cdot Trace(P.Psigma[l])\right)\right)\right) :$

Extract the 3D vector from a paravector P

$ExtractVector := (P) \rightarrow \frac{1}{2} \cdot Simplify(Vector(3, (l) \rightarrow Trace(Psigma[l].P)).Vector([\_i, \_j, \_k])) :$

Extract the electric field vector from the Faraday matrix

$GetElectricField := (F) \rightarrow Simplify(\mathrm{Re}(ExtractVector(F))) :$

Extract the magnetic field vector from the Faraday matrix

$GetMagneticField := (F) \rightarrow Simplify\left(\frac{\mathrm{Im}(ExtractVector(F))}{c}\right) :$

Extracting the scalar and vector parts

$ScalarPart := (L) \rightarrow \frac{(L + CliffordConj(L))}{2} :$

$$VectorPart := (L) \rightarrow \frac{(L - CliffordConj(L))}{2} :$$

Extract the Poynting vector

$$PoyntingCliford := (F) \rightarrow \frac{c \cdot \varepsilon_0}{2} \cdot VectorPart(\ F.Dagger(F)) :$$

$$PoyntingVector := (F) \rightarrow ExtractVector(PoyntingCliford(F)) :$$

The radiated energy per unit of time is given by Larmor's formula

$$PowerEmitted := (U) \rightarrow \frac{-e^2}{6 \cdot c \cdot \mathrm{Pi} \cdot \varepsilon_0} \cdot \frac{w(\xi)^2}{2} \cdot Trace\left(diff(U, \xi).ClifordConj\left(diff(U, \xi)\right)\right) :$$

Specifying the eigenspinor

$$\Lambda := MatrixExponential\left(\frac{\eta(\xi)}{2} \cdot Matrix(Psigma[3])\right).Matrix\left(\left[\left[1, \frac{v_x(\xi)}{c} - \frac{I \cdot v_y(\xi)}{c}\right], [0, 1]\right]\right)$$
$$.MatrixExponential\left(-I \cdot \frac{\Omega(\xi)}{2} \cdot Matrix(Psigma[3])\right);$$

$$\Lambda := \left[\left[e^{\frac{\eta(\xi)}{2}} \left(\cos\left(\frac{\Omega(\xi)}{2}\right) - \mathrm{I} \sin\left(\frac{\Omega(\xi)}{2}\right)\right), e^{\frac{\eta(\xi)}{2}} \left(\frac{v_x(\xi)}{c} - \frac{\mathrm{I}\, v_y(\xi)}{c}\right) \left(\cos\left(\frac{\Omega(\xi)}{2}\right) + \mathrm{I} \sin\left(\frac{\Omega(\xi)}{2}\right)\right)\right], \left[0, e^{-\frac{\eta(\xi)}{2}} \left(\cos\left(\frac{\Omega(\xi)}{2}\right) + \mathrm{I} \sin\left(\frac{\Omega(\xi)}{2}\right)\right)\right]\right] \tag{2}$$

Extracting proper velocity

$$U := simplify(\Lambda.Dagger(\Lambda));$$

$$U := \begin{bmatrix} \frac{e^{\eta(\xi)} \left(v_y(\xi)^2 + v_x(\xi)^2 + c^2\right)}{c^2} & \frac{-\mathrm{I}\, v_y(\xi) + v_x(\xi)}{c} \\ \frac{\mathrm{I}\, v_y(\xi) + v_x(\xi)}{c} & e^{-\eta(\xi)} \end{bmatrix} \tag{3}$$

Define contravariant vector of the velocity

$$DefContrVect(u, U);$$

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, \sigma_{\mu}, \partial_{\mu}, g_{\mu, \nu}, u^{\mu}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu}\right\} \tag{4}$$

combine(*expand*(*u*[~3]));

$$\frac{e^{\eta(\xi)} v_x(\xi)^2}{2 c^2} + \frac{e^{\eta(\xi)} v_y(\xi)^2}{2 c^2} + \frac{e^{\eta(\xi)}}{2} - \frac{e^{-\eta(\xi)}}{2} \tag{5}$$

The electromagnetic field (w is a parametric measure)

$$\mathrm{F} := \mathrm{w}(\xi) \cdot \frac{2 \cdot m \cdot c}{e} \cdot \mathit{diff}(\Lambda, \xi).\mathit{ClifordConj}(\Lambda) :$$

Extract the electromagnetic fields from the matrix F

*E_* := *GetElectricField*(*F*);

$$\vec{E} := \frac{1}{e}\left(e^{\frac{\eta(\xi)}{2}} w(\xi)\, m \left(v_y(\xi)\, e^{\frac{\eta(\xi)}{2}} \left(\frac{d}{d\xi} \Omega(\xi)\right) \hat{i} - v_x(\xi)\, e^{\frac{\eta(\xi)}{2}} \left(\frac{d}{d\xi} \Omega(\xi)\right) \hat{j} + e^{-\frac{\eta(\xi)}{2}} \left(\frac{d}{d\xi} \eta(\xi)\right) \hat{k}\, c + e^{\frac{\eta(\xi)}{2}} \left(\frac{d}{d\xi} v_x(\xi)\right) \hat{i} + e^{\frac{\eta(\xi)}{2}} \left(\frac{d}{d\xi} v_y(\xi)\right) \hat{j}\right)\right)$$

Simplify(*Component*($\vec{E}$, 1));

$$\frac{\left(v_y(\xi) \left(\frac{d}{d\xi} \Omega(\xi)\right) + \frac{d}{d\xi} v_x(\xi)\right) e^{\eta(\xi)} w(\xi)\, m}{e} \tag{7}$$

Simplify(*Component*($\vec{E}$, 2));

$$-\frac{\left(v_x(\xi) \left(\frac{d}{d\xi} \Omega(\xi)\right) - \left(\frac{d}{d\xi} v_y(\xi)\right)\right) e^{\eta(\xi)} w(\xi)\, m}{e} \tag{8}$$

combine(*Component*($\vec{E}$, 3));

$$\frac{w(\xi)\, m \left(\frac{d}{d\xi} \eta(\xi)\right) c}{e} \tag{9}$$

*B_* := *GetMagneticField*(*F*);

$$\vec{B} := -\frac{1}{e\,c}\left(e^{\frac{\eta(\xi)}{2}}\,w(\xi)\,m\left(v_x(\xi)\,e^{\frac{\eta(\xi)}{2}}\left(\frac{d}{d\xi}\,\Omega(\xi)\right)\hat{i} - e^{-\frac{\eta(\xi)}{2}}\left(\frac{d}{d\xi}\,\Omega(\xi)\right)\hat{k}\,c + v_y(\xi)\,e^{\frac{\eta(\xi)}{2}}\left(\frac{d}{d\xi}\,\Omega(\xi)\right)\hat{j} + e^{\frac{\eta(\xi)}{2}}\left(\frac{d}{d\xi}\,v_x(\xi)\right)\hat{j} - e^{\frac{\eta(\xi)}{2}}\left(\frac{d}{d\xi}\,v_y(\xi)\right)\hat{i}\right)\right) \quad \textbf{(10)}$$

$Simplify(Component(\vec{B}, 1));$

$$-\frac{\left(v_x(\xi)\left(\frac{d}{d\xi}\,\Omega(\xi)\right) - \left(\frac{d}{d\xi}\,v_y(\xi)\right)\right)e^{\eta(\xi)}\,w(\xi)\,m}{e\,c} \quad \textbf{(11)}$$

$Simplify(Component(\vec{B}, 2));$

$$-\frac{\left(v_y(\xi)\left(\frac{d}{d\xi}\,\Omega(\xi)\right) + \frac{d}{d\xi}\,v_x(\xi)\right)e^{\eta(\xi)}\,w(\xi)\,m}{e\,c} \quad \textbf{(12)}$$

$combine(Component(\vec{B}, 3));$

$$\frac{w(\xi)\,m\left(\frac{d}{d\xi}\,\Omega(\xi)\right)}{e} \quad \textbf{(13)}$$

Energy

$Simplify\left(\frac{m \cdot c^2}{2} \cdot Trace(U.Psigma[0])\right);$

$$\frac{m\left(e^{\eta(\xi)}\left(v_y(\xi)^2 + v_x(\xi)^2 + c^2\right) + e^{-\eta(\xi)}\,c^2\right)}{2} \quad \textbf{(14)}$$

The case of circular trajectory

Describe a circular trajectory

$$CircularTrajectory := \left\{v_x(\xi) = diff\left(r \cdot \cos\left(\frac{\omega \cdot \xi}{\mathrm{sqrt}\left(1 - \left(\frac{r \cdot \omega}{c}\right)^2\right)}\right), \xi\right), v_y(\xi) = diff\left(r \cdot \sin\left(\frac{\omega \cdot \xi}{\mathrm{sqrt}\left(1 - \left(\frac{r \cdot \omega}{c}\right)^2\right)}\right), \xi\right)\right\}:$$

Characterize the newly enter parameters as real

$Setup(realobjects = \{\omega, r\});$

$$\left[realobjects = \left\{\hat{i}, \hat{j}, \hat{k}, \hat{\phi}, \hat{r}, \hat{\rho}, \hat{\theta}, c, e, m, \omega, \phi, r, \rho, \tau, \theta, x, \xi, y, z, \varepsilon_0, \Omega(\xi), \eta(\xi), v_x(\xi), v_y(\xi), w(\xi)\right\}\right] \tag{15}$$

Find $\eta(\xi)$ that eliminates the drift along the z-axis

$CircularTrajectory := CircularTrajectory\ \mathbf{union}\ \{\eta(\xi) = solve(Simplify(subs(CircularTrajectory, u[\sim 3])), \eta(\xi))[1]\};$

$$CircularTrajectory := \left\{\eta(\xi) = \ln\left(\frac{\sqrt{-r^2\,\omega^2 + c^2}}{c}\right), v_x(\xi) = -\frac{r\,\omega \sin\left(\frac{\omega\,\xi}{\sqrt{1 - \frac{r^2\,\omega^2}{c^2}}}\right)}{\sqrt{1 - \frac{r^2\,\omega^2}{c^2}}}, v_y(\xi) = \frac{r\,\omega \cos\left(\frac{\omega\,\xi}{\sqrt{1 - \frac{r^2\,\omega^2}{c^2}}}\right)}{\sqrt{1 - \frac{r^2\,\omega^2}{c^2}}}\right\} \tag{16}$$

$'u[\sim 0]' = Simplify(subs(CircularTrajectory, u[\sim 0]));$

$$u^0 = \frac{c}{\sqrt{-r^2\,\omega^2 + c^2}} \tag{17}$$

$CircularTrajectory := CircularTrajectory\ \mathbf{union}\ \{w(\xi) = 1\}:$

Get the final trajectory

$Simplify\left(subs\left(CircularTrajectory, Array\left(1..4, (l) \rightarrow int\left(\frac{c}{2 \cdot w(\xi)} \cdot Trace(U.Psigma[1]), \xi\right)\right)\right)\right);$

$$\left[\begin{array}{cccc} r\cos\left(\frac{\omega\,\xi}{\sqrt{\frac{-r^2\,\omega^2 + c^2}{c^2}}}\right) & r\sin\left(\frac{\omega\,\xi}{\sqrt{\frac{-r^2\,\omega^2 + c^2}{c^2}}}\right) & 0 & \frac{\xi\,c^2}{\sqrt{-r^2\,\omega^2 + c^2}} \end{array}\right] \tag{18}$$

The case of Laser Accelerator Scheme (LAS)

Get the final trajectory

$LASTrajectory := \{\eta(\xi) = 0, w(\xi) = c\};$

$$LASTrajectory := \{\eta(\xi) = 0, w(\xi) = c\} \tag{19}$$

The electromagentic fields

$'E\_' = Simplify(expand(subs(LASTrajectory, E\_)));$

$$\vec{E} = \frac{1}{e}\left(c\, m\left(v_y(\xi)\left(\frac{d}{d\xi}\Omega(\xi)\right)\hat{i} - v_x(\xi)\left(\frac{d}{d\xi}\Omega(\xi)\right)\hat{j} + \left(\frac{d}{d\xi}v_x(\xi)\right)\hat{i} + \left(\frac{d}{d\xi}v_y(\xi)\right)\hat{j}\right)\right) \tag{20}$$

$'B\_' = Simplify(expand(subs(LASTrajectory, B\_)));$

Specify trajectory for LAS

$LASTrajectory := LASTrajectory\ \mathbf{union}\left\{v_x(\xi) = diff\left(\frac{e\cdot E_0}{m\cdot c\cdot\omega}\cdot\xi\cdot\cos\left(\frac{\omega\cdot\xi}{c}\right), \xi\right), v_y(\xi) = diff\left(\frac{e\cdot E_0}{m\cdot c\cdot\omega}\cdot\xi\cdot\sin\left(\frac{\omega\cdot\xi}{c}\right), \xi\right), \Omega(\xi) = \frac{\omega\cdot\xi}{c}\right\}:$

$'E\_' = Simplify(expand(subs(LASTrajectory, E\_)));$

$$\vec{E} = \frac{-\hat{i}\, E_0 \sin\left(\frac{\omega\,\xi}{c}\right) + \hat{j}\, E_0 \cos\left(\frac{\omega\,\xi}{c}\right)}{c} \tag{21}$$

$'B\_' = Simplify(expand(subs(LASTrajectory, B\_)));$

$$\vec{B} = \frac{\hat{i}\, E_0 \cos\left(\frac{\omega\,\xi}{c}\right) e + m\,\omega\,\hat{k}\, c^2 + \hat{j}\, E_0 \sin\left(\frac{\omega\,\xi}{c}\right) e}{c^2\, e} \tag{22}$$

This leads to monotonically increasing energy

$Simplify\left(subs\left(LASTrajectory, m\cdot c^2\cdot u[\sim 0]\right) - m\cdot c^2\cdot\left(1 + \frac{E_0^{\ 2}\cdot e^2}{2\cdot m^2\cdot c^6\cdot\omega^2}\cdot\left(c^2 + \xi^2\cdot\omega^2\right)\right)\right);$

$$0 \tag{23}$$

$expand(Simplify(subs(LASTrajectory, u[\sim 0])));$

$$1 + \frac{E_0^{\ 2}\, e^2\, \xi^2}{2\, m^2\, c^6} + \frac{E_0^{\ 2}\, e^2}{2\, m^2\, c^4\, \omega^2} \tag{24}$$

$Simplify(expand(subs(LASTrajectory, PowerEmitted(U))));$

$$\frac{\left(\xi^2\,\omega^2 + 4\,c^2\right) E_0^{\,2}\,e^4}{6\,\varepsilon_0\,\pi\,c^7\,m^2} \tag{25}$$

simplify(*expand*(*subs*(*LASTrajectory*, *PoyntingVector*(*F*))));

$$\frac{E_0\,\varepsilon_0\left(-\sin\left(\frac{\omega\,\xi}{c}\right)\hat{j}\,c^2\,m\,\omega - \cos\left(\frac{\omega\,\xi}{c}\right)\hat{i}\,c^2\,m\,\omega + E_0\,\hat{k}\,e\right)}{e\,c} \tag{26}$$