TA Session -03.

Part I- Summary of week 3 lectures.

① Differential equation description of LTI systems.

(a) i/p - o/p relation represented as a differential equation

$$\sum_{m=0}^{M} a_m \frac{d^m y(t)}{dt^m} = \sum_{m=0}^{N} b_m \frac{d^m x(t)}{dt^m}$$

$\underline{\underline{sol^n}}$ $y(t) = y_p(t) + y_h(t)$.

$y_p(t)$ → Particular solution. $y_h(t)$ → homogeneous solution.

1) Linear system - all initial conditions = 0

2) Time invariant : system has to be at INITIAL REST

(i.e) if $x(t) = 0$ for $t \leq t_0$
→ $y(t) = 0$ for $t \leq t_0$

② Discrete LTI systems. (LSI) linear shift invariant

$x[n] \rightarrow \boxed{T(\cdot)} \rightarrow y[n]$

$\delta[n] \rightarrow \boxed{T(\cdot)} \rightarrow h[n]$

$h[n]$ → Discrete time impulse response.

$y[n] = x[n] * h[n]$ ← convolution.

$$= \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$$

(*) Memoryless.
if $h[n] = k\,\delta[n]$

(*) causality
if $h[n] = 0$ for $n < 0$

(*) BIBO stability
if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ [absolutely summable]

$\int |h(t)|\, dt < \infty$

(*) $x(n) = z^n$ is the eigen fn of discrete time LTI system.

(*) Difference equation description:

$$\sum_{m=0}^{M} a_m y(n-m) = \sum_{m=0}^{N} b_m x(n-m)$$

$$\left.\frac{d y^m(t)}{dt^n} \to y(n-m)\right] \checkmark$$

(3) Laplace Transform

↳ Represent a time domain signal in the "Laplace" Domain

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$$

a complex number $s = \sigma + j\omega$.

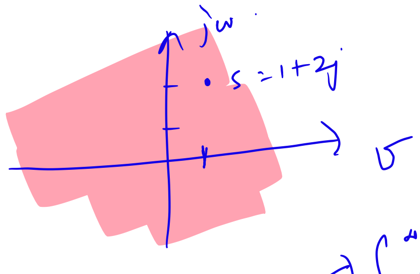
$$X(s) \triangleq \mathcal{L}[x(t)], \quad x(t) \xrightarrow{\mathcal{L}} X(s)$$

(*) The transform is defined in terms of integral ⇒ Integration should be convergent

∴ L.T is also characterised by its ROC

ROC : Region of s plane for which LT converges.

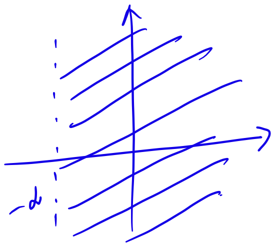
s plane



eg (1) $x(t) = e^{-\alpha t} u(t)$ $\to \int_{-\infty}^{\infty} e^{-\alpha t} u(t) e^{-st}\, dt$ $= \int_{0}^{\infty} e^{-at} e^{-st}\, dt$

$$X(s) = \frac{1}{s+\alpha} : Re(s) > -\alpha$$

$$\sigma > -\alpha.$$



(*) Two different signals can have same LT, but their ROCs can be different

(*) Standard LTs

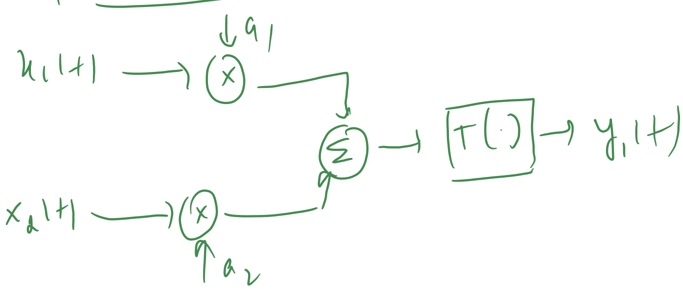
a) $x(t) = \delta(t)$
$X(s) = 1$, ROC everywhere

b) $x(t) = u(t)$
$X(s) = 1/s$, ROC $Re\{s\} > 0$

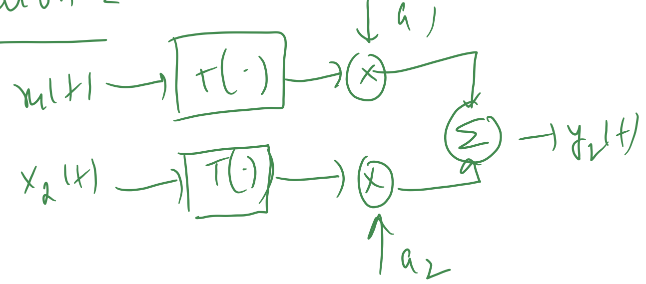
Part II Tutorials

① Check whether $y(t) = 2x(t) - 5$ is a valid o/p of a linear system

Soln Operation 1

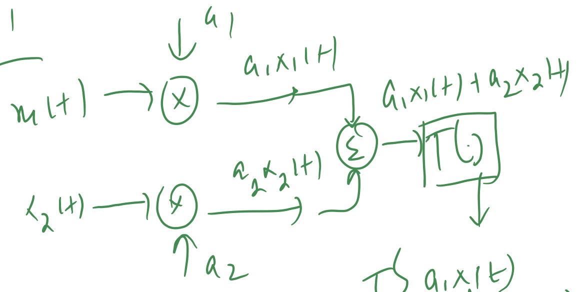


Operation 2



if $y_1(t) = y_2(t)$
$T(\cdot)$ is a linear system

op 1

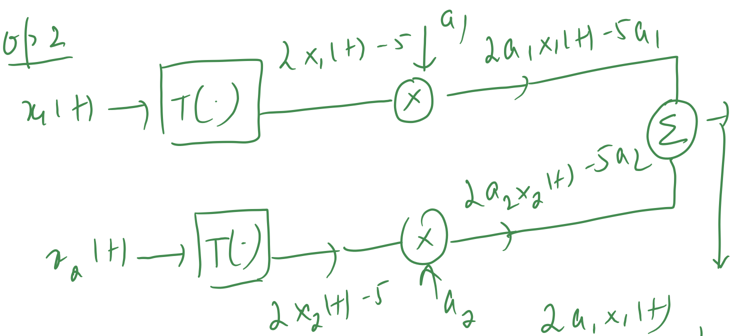


$$T\{a_1x_1(t) + a_2x_2(t)\}$$

$$= 2\{a_1x_1(t) + a_2x_2(t)\} - 5$$

$$= 2a_1x_1(t) + 2a_2x_2(t) - 5$$

op 2



$$2a_1x_1(t) + 2a_2x_2(t) - 5(a_1+a_2)$$

$y_1(t) \neq y_2(t)$

Not linear

8) Perform convolution of the following signals

$$x_1(t) = e^{-2t}u(t)$$
$$x_2(t) = e^{-5t}u(t)$$

Sol^n $x(t) \triangleq x_1(t) * x_2(t)$

$$= \int_{\tau=-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2t} u(\tau)\, e^{-5(t-\tau)} u(t-\tau)$$

$u(\tau) \geq 0$ for $\tau \geq 0$

$u(t-\tau) \geq 0$ for $t-\tau \geq 0$, $t \geq \tau$, $\tau \leq t$

$$= \int_0^t e^{-2\tau} e^{-5(t-\tau)}\, d\tau$$

$$= \int_0^t e^{-2\tau} e^{-5t} e^{5\tau}\, d\tau$$

$$= e^{-5t}\int_0^t e^{-2\tau} e^{5\tau}\, d\tau$$

$$= e^{-5t}\int_0^t e^{3\tau}\, d\tau$$

$$= e^{-5t} \left. \frac{e^{3\tau}}{3} \right|_0^t$$

$$= e^{-5t} \frac{(e^{3t} - 1)}{3}$$

$$x_3(t) = \begin{cases} \frac{e^{-2t} - e^{-5t}}{3}, & t \geq 0 \\ 0, & \text{else} \end{cases}$$

$$= e^{-5t}\left(\frac{e^{3t}}{3} - 1/3\right) u(t)$$

(3) Convolve

$x_1(t) = \cos t \; u(t)$

$x_2(t) = t\, u(t)$

Sol^n $x_3(t) = x_1(t) * x_2(t)$

$$= \int_{\tau=-\infty}^{\infty} \cos(\tau)\, u(\tau)\, (t-\tau)\, u(t-\tau)\, d\tau$$

$u(\tau) \geq 0$ for $\tau \geq 0$

$u(t-\tau) \geq 0$ for $t - \tau \geq 0$

$t \geq \tau$

or $\tau \leq t$

$$= \int_0^{\tau} \cos(\tau)(t-\tau)\, d\tau$$

$$= t\int_0^t \cos\tau\, d\tau - \int_0^t \tau \cos\tau\, d\tau$$

$$= t \sin\tau \Big|_0^t - \int_0^t \tau\cos\tau\, d\tau$$

$$= t\{\sin t - \sin 0\} - \int_0^t \tau\cos\tau\, d\tau$$

$$= t \sin t$$

$$\int_0^t \tau \cos\tau \, d\tau \rightarrow \text{Integrate by parts}$$

$$= \tau \int \cos\tau \, d\tau \Big|_0^t - \int_0^t \left(\frac{d}{d\tau} \tau \int \cos\tau \, d\tau \right) d\tau$$

$$= t\sin t - \int_0^t \sin\tau \, d\tau$$

$$= t\sin t + \cos\tau \Big|_0^t$$

$$= t\sin t + (\cos t - 1)$$

$t \geq 0$

$$x_3(t) = t\sin t - \{t\sin t + \cos t - 1\}$$

$$= 1 - \cos t, \quad t \geq 0$$

$$\Rightarrow \boxed{x_3(t) = (1-\cos t)\,u(t)}$$

Another interpretation

$$y(t) = \int_{\tau=-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

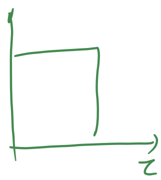
$x_2(\tau) \rightarrow x_2(-\tau)$

$\downarrow$

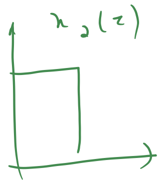
$x_2(-\tau+t)$

$x_2(t-\tau)$.

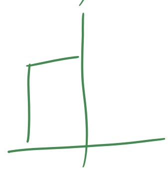
$x_1(\tau)$



$x_2(\tau)$



$x_2(-\tau)$



$x_2(t-\tau)$

$t \geq 0$.

$x_1(\tau)$

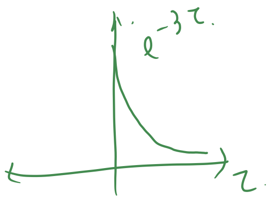
$t \geq 0$

$\downarrow$

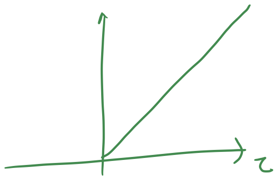
$y(t)$

④ Convolve $x_1(t) = e^{-3t}u(t)$; $x_2(t) = t\,u(t)$ using graphical Method.

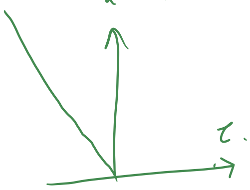
$x_1(\tau)$



$x_2(\tau)$

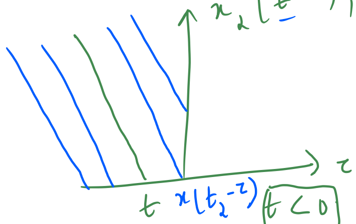


$x_2(-\tau)$



↓ shift by t

$x_2(t-\tau)$, $t<0$

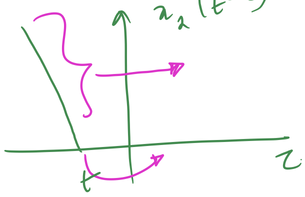


$x(t_2-\tau)$ $\boxed{t<0}$

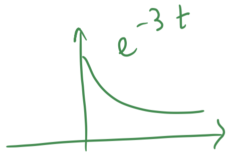
$y(t)\ \forall\ t$

Case 1 when $t<0$

$x_2(t-\tau)$

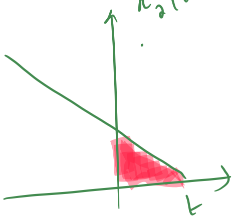


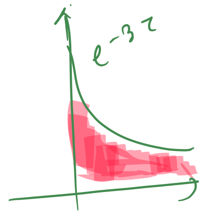
$x_1(\tau)$



$$\therefore y(t) = \int x_1(\tau)\, x_2(t-\tau) = 0 \quad t<0$$

$\boxed{\text{Let } t\geq 0}$ Case 2





$$y(t) = \int x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_0^t e^{-3\tau}\, \underline{(t-\tau)}\, d\tau$$

$$= t\int_0^t e^{-3\tau}\, d\tau - \int_0^t e^{-3\tau}\, \tau\, d\tau$$

$$x_3(t) = \begin{cases} t/3 + \frac{e^{-3t}}{9} - 1/9, & t\geq 0 \\ 0, & t\leq 0 \end{cases}$$

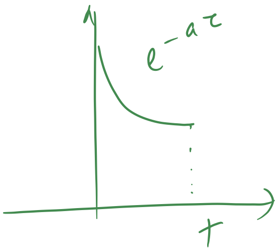
$$x_3(t) = \left(t/3 + \frac{e^{-3t}}{9} - 1/9\right) u(t).$$

(5) Convolve

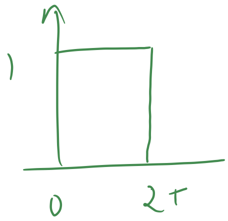
$x_1(t) = e^{-at}$; $0 \le t \le T$

$x_2(t) = 1$; $0 \le t \le 2T$

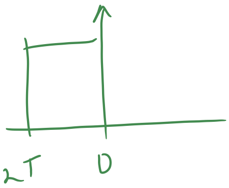
$x_1(\tau)$



$x_2(\tau)$

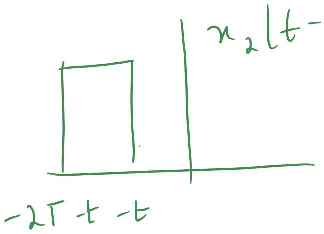


$x_2(-\tau)$



Shifting.

$x_2(t-\tau)$ $t<0$

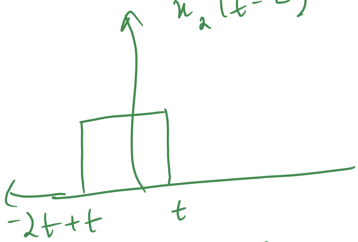


Case 1. when $t<0$,

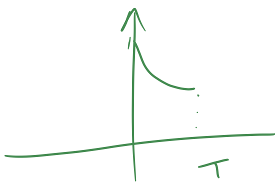
$y(t) = 0$, as there is no overlap.

Case 2 when $t>0$.

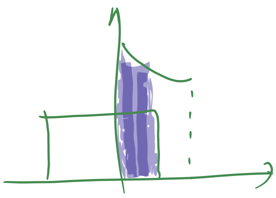
$x_2(t-\tau)$



$x_1(\tau)$



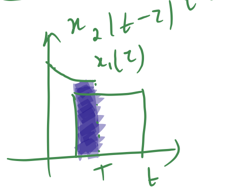
? overlap

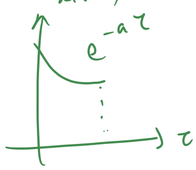


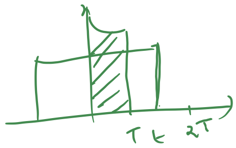
$$y(t) = \int_0^t e^{-a\tau} \cdot 1 \cdot d\tau$$

$$= 1/a\,(1 - e^{-at})$$

Case 3 $2T > t > T$

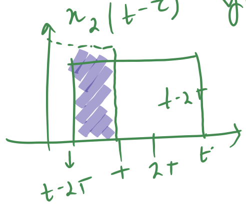






$$y(t) = \int_0^T e^{-a\tau} \, 1 \, d\tau = \frac{1}{a}(1-e^{aT})$$

Case 4 $t > 2T$

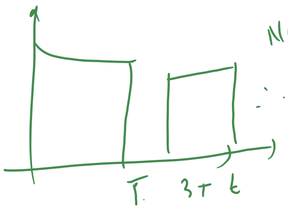


$$y(t) = \int_{t-2T}^{T} e^{-a\tau} \cdot 1 \cdot d\tau = \frac{1}{a}\left\{ e^{-a(t-2T)} - e^{-aT} \right\}$$

Case 5:

$t - 2T > T$

$t > 3T$

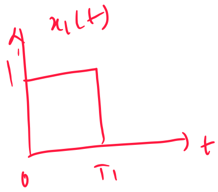


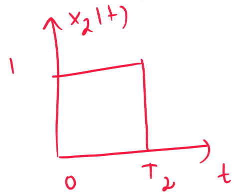
No overlap.

$\therefore y(t) = 0$

$$y(t) = \begin{cases} 0, & \text{if } t<0 \\ \frac{1}{a}(1-e^{-at}) & \text{if } 0<t<T \\ \frac{1}{a}(1-e^{-aT}) & \text{if } T<t<2T \\ \frac{1}{a}\left(e^{-a(t-2T)} - e^{-at}\right), & \text{if } 2T<t<3T \\ 0, & \text{if } t>3T \end{cases}$$

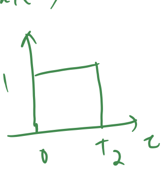
6) Convolve the following signals.

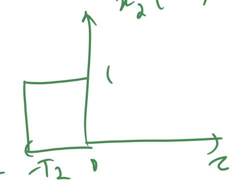




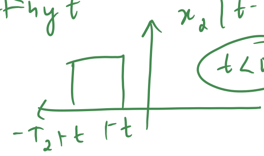
$\boxed{T_1 \geq T_2}$

Solution:



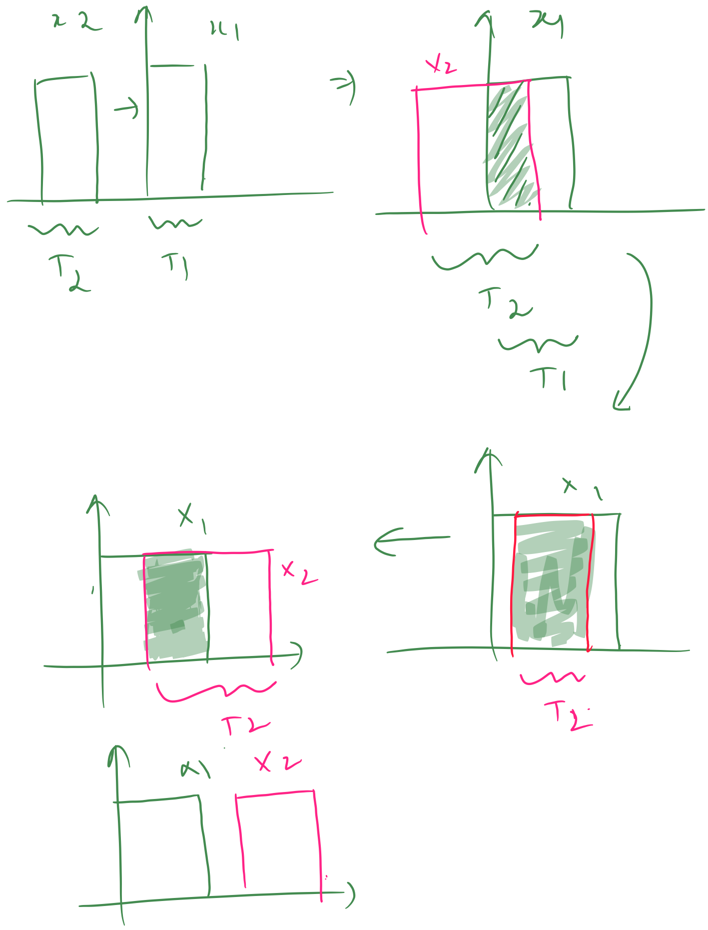


Shifting by t



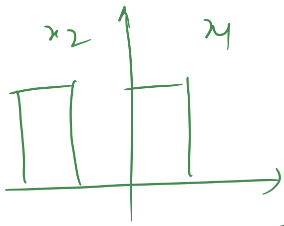
$t<0$

Sliding Scenarios:-



Case 01

$t < 0$

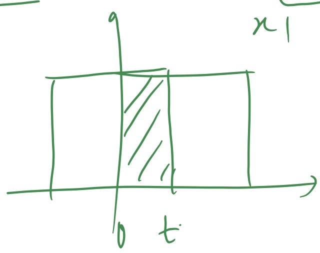


No overlap

$y(t) = 0$

Case 02

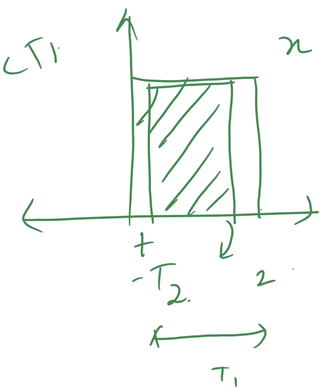
$\boxed{0 < t < T_2}$



$$y(t) = \int_0^t 1 \cdot 1 \, d\tau$$
$$= 1$$

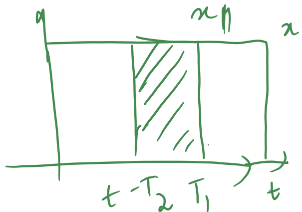
Case - 03

$T_2 < t < T_1$



$$y(t) = \int_{t-T_2}^{t} 1 \cdot 1 \, d\tau$$
$$= t - t + T_2$$
$$= T_2$$

Case 04



$$= \int_{t-T_2}^{T_1} 1 \cdot 1 \, d\tau$$

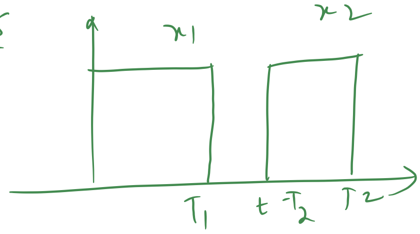
$$= T_1 - t + T_2$$

$T_1 < t < T_1 + T_2$

$t - T_2 < T_1$

$t < T_1 + T_2$

Case 05



$= 0 \cdot 0$

$= 0$

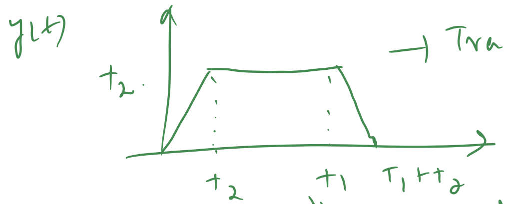
$y(t) = 0$

$t - T_2 > T_1$

$\Rightarrow t > T_1 + T_2$

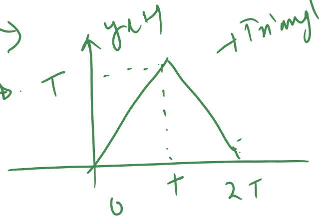
Summarising

$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t \le T_2 \\ T_2, & T_2 \le t \le T_1 \\ T_1 + T_2 - t & T_1 \le t \le T_1 + T_2 \\ 0, & t > T_1 + T_2 \end{cases}$$



$\rightarrow$ Trapezium

if $T_1 = T_2$
$= T \longrightarrow T$



$\rightarrow$ Triangle

Conclusion: convolution of 2 rectangular signals with

a) unequal widths gives a trapezium
b) equal widths gives a triangle.

7) Determine the range of values of "a" and "b" for the stability of LTI systems with impulse response

$$h(n) = \begin{cases} b^n, & n < 0 \\ a^n, & n \ge 0 \end{cases}$$

Solution: An LTI s/s is stable if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{-1} |b|^n + \sum_{n=0}^{\infty} |a|^n$$

$$= \sum_{n=1}^{\infty} |b|^{-n} + \sum_{n=0}^{\infty} |a|^n$$

$$= \sum_{n=1}^{\infty} |b|^{-n} + |b|^{-0} - |b|^{-0} + \sum_{n=0}^{\infty} |a|^n$$

$$= \sum_{n=0}^{\infty} |b|^{-n} \underbrace{-|b|^0}_{-1} + \sum_{n=0}^{\infty} |a|^n$$

$$= \sum_{n=0}^{\infty} (\underbrace{|b|^{-1}}_{c})^n - 1 + \sum_{n=0}^{\infty} |a|^n$$

$$= \sum_{n=0}^{\infty} c^n - 1 + \sum_{n=0}^{\infty} |a|^n$$

infinite GP series $|c|<1$

$0 < |b|^{-1} < 1$

$\Rightarrow |b| > 1$.

$$= \underbrace{\frac{1}{1-c} - 1 + \frac{1}{1-|a|}}_{\text{Finite number}}$$

$|a|<1$ ⇒ $|a|<1$.

∴ For BIBO stability

$|a|<1$ and $|b|>1$

(8) Show that

(i) $x(t) * \delta(t) = x(t)$.

(ii) $x(t) * \delta(t-t_0) = x(t-t_0)$

(iii) $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

(iv) $x(t) * u(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau)\,d\tau$

Sol^n $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

$= x(\tau)\Big|_{\tau = t}$

$= x(t)$

Sifting property

(ii) $x(t) * \delta(t-t_0) = \delta(t-t_0) * x(t)$

Commutativity of Convolution.

$= \int_{-\infty}^{\infty} \delta(\tau - t_0)\, x(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} x(t-\tau)\, \delta(\tau - t_0)\, d\tau$.

$= x(t-\tau)\Big|_{\tau = t_0}$.

Sifting property

(iii) $x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$.

$$u(t-\tau) = \begin{cases} 1, \text{ if } \tau < t \\ 0, \text{ if } \tau > t \end{cases}$$

iv) $x(t) * u(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau-t_0)\, d\tau$.

$$u(t-\tau-t_0) = \begin{cases} 1, \text{ if } t-\tau-t_0 \geq 0 \quad \tau \leq t - t_0 \\ 0, \text{ else} \end{cases}$$

$= \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$.

(9) Consider a continuous time LTI system described by

$$y(t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} x(\tau)\, d\tau.$$

a) Find and plot the impulse response $h(t)$ of the system.

b) Is the system causal?

Solution

a) Approach 1

Put $x(\tau) = \delta(\tau)$

$$y(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} \delta(\tau)\, d\tau$$

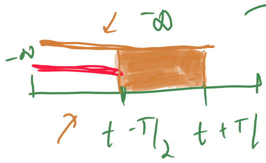
$$= 1/T\, u(t) \Big|_{t-T/2}^{t+T/2}$$

$$= \frac{1}{T}\left[u(t+T/2) - u(t-T/2)\right]$$

Approach 2

$$y(t) = 1/T \int_{t-T/2}^{t+T/2} x(\tau)\, d\tau$$

$$= \frac{1}{T}\int_{-\infty}^{t+T/2} x(\tau)\, d\tau - 1/T \int_{-\infty}^{t-T/2} x(\tau)\, d\tau$$

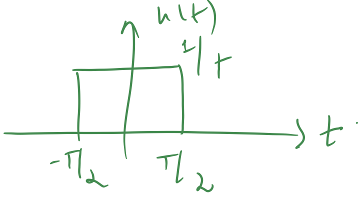


$$= \frac{1}{T}\, x(t) * u(t+T/2)$$

$$- \frac{1}{T}\, x(t) * u(t-T/2)$$

$$= \frac{1}{T}\, x(t) * \underbrace{\left\{ \begin{array}{l} u(t+T/2) \\ -u(t-T/2) \end{array} \right\}}_{\text{Impulse response}}$$

$$h(t) = \frac{1}{T}\left[u(t+T/2) - u(t-T/2)\right]$$



b) $h(t) \neq 0$, for $t<0$

$\Rightarrow$ It's a non causal system.

(10) Consider a C.T LTI systems with i/p - o/p relation

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}\, x(\tau)\, d\tau$$

a) find impulse response $h(t)$

b) Show that e^{st} is eigen fxn of s/s

c) Find eigenvalue of s/s corresponding to e^{st} using $h(t)$ obtained in (a)

Solution: $h(t) = \int_{-\infty}^{t} e^{-(t-\tau)} \delta(\tau)\, d\tau.$

$$= e^{(t-\tau)}\Big|_{\tau=0} \qquad t>0$$

$$= e^{-t},\ t\geq 0$$

$$h(t) = e^{-t} u(t).$$

b) $x(t) = e^{st}$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{t} e^{-(t-\tau)} e^{s\tau}\, d\tau$$

$$= e^{-t} \int_{-\infty}^{t} e^{(s+1)\tau}\, d\tau.$$

$$= \frac{1}{s+1} e^{st} \text{ if } Re\{s\} > -1.$$

$y(t) = \lambda\, x(t) \Rightarrow x(t) = e^{st}$ is eigen fxn!

Eigenvalue $\frac{1}{s+1} = \lambda$