$\boxed{\text{W-11 TA Session}}$ By Yashvanth L.

Topics covered DFT

Part I. Summary of week XI's lecture.

→ Problems related to DFS | DTFT | DFT

→ Discrete Fourier Transforms.

→ 1. Finite length sequences $x[n]$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{k} kn} ; \quad k = 0, 1, \ldots N-1$$

Twiddle factor

$$W_N = e^{-j 2\pi/N}.$$

- IDFT

$$x(m) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \, e^{+j\frac{2\pi}{k} km}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \, W_N^{-km}.$$

→ Both $x(n)$ & $X(k)$ have N samples.

→ Relation b/w DFS & DFT. If periodic version of $x[n]$ is considered then

$$\boxed{N c_k = X(k)}$$

→ Relation b/w DTFT & DFT

DFT = DTFT evaluated at $\Omega = \frac{2\pi k}{N}$

$\rightarrow$ sampled version of DTFT = DFT

Properties of DFT Ref: Oppenheim (Table 8.2)

Length N		N-point DFT
$x(n)$		$X(k)$
$x_1(n),\ x_2(n)$		$X_1(k)\ X_2(k)$
$a x_1(n) + b x_2(n)$	linearity	$a X_1(k) + b X_2(k)$
$X(n)$	Duality	$N x[(-k) \bmod N]$ $N x((-k))_N$.
$x((n-m) \bmod N)$	Time Shifting.	$W_N^{km} X(k)$.
$W_N^{-ln} x(n)$	freq shift	$X((k-l) \bmod N)$
$\sum_{m=0}^{N-1} x_1(m)\, x_2[((n-m))_N]$	convolution.	$X_1(k) X_2(k)$
$x_1(n) x_2(n)$	Multiplication	$\frac{1}{N} \sum_{l=0}^{N-1} X_1(l)\, X_2((k-l) \bmod N)$
$x^*(n)$	Conjugation.	$X^*((-k) \bmod N)$
$x^*((-n))_N$		$X^*(k)$
$\mathrm{Re}(x(n))$		$X_{ep}(k) = \frac{1}{2}\left\{X(((k))_N) + X^*[((-k))_N]\right\}$
$j\,\mathrm{Im}\{x(n)\}$.		$X_{op}(k) = \frac{1}{2}\left\{X((k))_N - X^*(((-k))_N)\right\}$

if $x(n)$ is real

$$X(k) = X^*[((-k))_N]$$

$((\ ))_N \downarrow$ mod N.

$$Re(X(k)) = Re(X(((-k))_N))$$

$$Im\{X(k)\} = -Im\{X[((-k))_N]\}$$

$$|X(k)| = |X[((-k))_N]|$$

$$\measuredangle X(k) = -\measuredangle\{X[((-k))_N]\}.$$

$$x_{ep}(n) = \frac{1}{2}\{x(n) + x[((-n))_N]\}$$

$$\downarrow$$

$$Re\{X(k)\}$$

$$x_{op}(n) = \frac{1}{2}\{x(n) - x[((-n))_N]\} \longrightarrow j\,Im\{X(k)\}$$

19. $x(n) \longrightarrow X(k)$
$X(n) \longrightarrow N\,x(-k \bmod N)$ } Duality property

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |X(k)|^2 \rightarrow \text{Energy in time domain} = \text{Energy in freq domain.}$$

Properties of DFT Matrix: $\rightarrow$ hermitian symmetry

$$\Rightarrow H^H H = I = H H^H.$$

Part II Tutorials

(1) Let $x[n] = \sin\left(\frac{\pi n}{2}\right)$ for $n = 0, 1, 2, 3$. Let $h[n] = \{\underset{\uparrow}{-1}, 2, -3, 4\}$. $h(0)$

Perform circular convolution of $x[n]$ & $h[n]$.

Solution

$$x(n) = \left\{ \sin 0, \sin \pi/2, \sin \frac{2\pi}{2}, \sin \frac{3\pi}{2} \right\}$$

$$= \{0, 1, 0, -1\}$$

$$h(n) = \{-1, 2, -3, 4\}$$

$$y(n) = \sum_{m=0}^{N-1} x(n)\, h\left[(n-m)_N\right]$$

$N = 4$

$N - 1 = 3$

$$= \sum_{m=0}^{3} x(n)\, h\left[(n-m) \bmod 4\right]$$

$$= x(0)\, h\left[n \bmod 4\right]$$
$$+ x(1)\, h\left[(n-1) \bmod 4\right]$$
$$+ x(2)\, h\left[(n-2) \bmod 4\right]$$
$$+ x(3)\, h\left[(n-3) \bmod 4\right]$$

$n = 0$

$$y(0) = x(0)\, h(0 \bmod 4) + x(1)\, h((-1) \bmod 4)$$

$+ x(2)\, h((0-2) \bmod 4) + x(3)\,(h(0-3) \bmod 4)$.

$= x(0)h(0) + x(1)h(3) + x(2)h(2) + x(3)h(1)$.

$-1 \bmod 4$
$= (-1+4) \bmod 4$
$= 3 \bmod 4$
$= 3$.

$$4 \overline{)\,3\,}\; 0$$
$$\underline{0}$$
$$3$$

$= 0 + 4 + 0 + (-2)$
$= 4 - 2$
$= 2$

$y(0) = 2$

$|||^{ry}$ $y(1) = x(0)h(1) + x(1)h(0) + x(2)h(3) + x(3)h(2)$

$= 0 - 1 + 0 + 3 = 2$.
$\therefore y(1) = 2$.

$|||^{ry}$ $y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) + x(3)h(3)$
$= 0 + 2 + 0 + (-4)$
$= -2$

$|||^{ry}$ $y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) + x(3)h(0)$

$= 0 - 3 + 0 + 1$
$= -2$

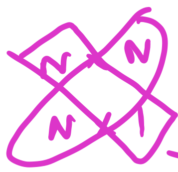
$\therefore y(n) = \{2, 3, -3, -2\}$
$\uparrow$

Easy evaluation of circular convolution b/w 2 N-length sequences $x[n]$ & $h[n]$

$$y(n) = x(n) \circledast h(n)$$

$\hookrightarrow$ circular convolution!

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}}_{N\times 1} = \underbrace{\begin{bmatrix} h(0) & h(N-1) & h(N-2) & \cdots & h(1) \\ h(1) & h(0) & h(N-1) & & h(2) \\ \vdots & h(1) & h(0) & & \vdots \\ \vdots & \vdots & \vdots & & h(N-1) \\ h(N-1) & h(N-2) & h(N-3) & & h(0) \end{bmatrix}}_{N\times N} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}}_{N\times 1}$$



$\longrightarrow$ result $= N \times 1$

② Let $x(n) = \{\underset{\uparrow}{4}, 5, 4, 5, 4, 5, 4, 5\}$

Find DFT of $x(n)$.

Solⁿ
$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N} k n}$$

$N = 8$

$$X(k) = \sum_{n=0}^{7} x(n)\, e^{-j 2\pi/8\, k n} \quad ; k = 0, 1, \ldots 7$$

$$= 4 + 5e^{-j\pi/4} + 4e^{-j2\pi/4 n} + 5e^{-j3\pi/4 n} + 4e^{-j4\pi/4 n} + 5e^{-j5\pi/4 n} + 4e^{-j6\pi/4 n} + 5e^{-j7\pi/4 n}$$

— Eqn ①

$$e^{j5\pi/4 n} = e^{-j\pi \; j\pi/4 n} = -e^{-j\pi/4 n}$$

$$e^{j3\pi/4 n} = e^{-(j\pi + j\pi/4)n} = e^{-j\pi/4 n}$$

$$e^{-j7\pi/4 n} = e^{-j2\pi + j\pi/4 n} = e^{j\pi/4 n}$$

$$e^{-j6\pi/4 n} = e^{-j2\pi + j2\pi/4 n} = e^{j\pi/2 n}$$

Using these in eqn 1

$$x(n) = 4 + (5-5)e^{-j\pi/4 n} + 4\left(e^{-j\pi/2 n} + e^{j\pi/2 n}\right)$$

$$= 4 + 4 \times 2 \times \left(\frac{e^{-j\pi/2 n} + e^{j\pi/2 n}}{2}\right)$$

$$+ 4e^{-j\pi nk}$$

$$X(k) = 4 + 8\cos\left(\pi/2\, nk\right) + 4(-1)^{nk}$$

3) Compute the N-point DFT of

a) $x_1(n) = \delta(n)$

b) $x_2(n) = \delta(n - n_0)$. $0 \le n \le N$

c) $x_3(n) = \alpha^n$; $0 \le n \le N$

d) $x_4(n) = u(n) - u(n - N_0)$, $0 < n < N$

Sol^n: $x_1(n) = \delta(n)$

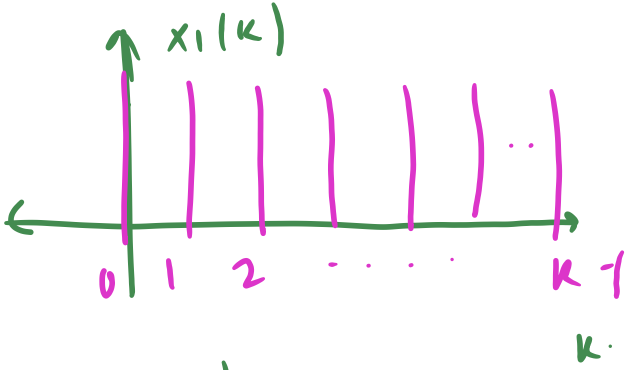
$$X_1(k) = \sum_{n=0}^{N-1} \delta(n) e^{-j\frac{2\pi}{N}nk}$$

$$= \delta(0)\, e^{j2\pi/N \cdot 0 \cdot n} = \delta(0)$$

$$= 1$$

Sifting property of $\delta(n)$.

$\therefore \forall k$, $X_1(k) = 1$



b) $x_2(n) = \delta(n-n_0)$

$$X_2(k) = \sum_{n=0}^{N-1} \delta(n-n_0)\, e^{-j2\pi/N\, nk}$$

$$= \delta(n-n_0)\, e^{-j2\pi/N\, kn_0}$$

$$= e^{-j2\pi/N\, kn_0}$$

$$X_2(k) = W_N^{n_0 k} \; ; \; k = 0, 1, \dots K-1$$

c) $x_3(n) = \alpha^n$

$$X_3(k) = \sum_{n=0}^{N-1} \alpha^n e^{-j2\pi/N\, n.k}$$

$$X_3(k) = \sum_{n=0}^{N-1} \alpha^n W_N^{nk} = \sum_{n=0}^{N-1} (\alpha W_N^k)^n$$

Geo. series.

$$= \frac{1-(\alpha W_N^k)^N}{1-\alpha W_N^k} \; ; \; k = 0, 1, \dots N-1$$

d) $x_4(n) = u(n) - u(n-n_0) = \begin{cases} 1, & 0 \le n \le n_0 - 1 \\ 0, & \text{else.} \end{cases}$

$$X_4(k) = \sum_{n=0}^{N_0-1} 1 \cdot W_N^{kn}$$

$$= \frac{1-W_N^{kn_0}}{1-W_N^k}.$$

Take $W_N^{kn_0/2}$ common in numerator and $W_N^{k/2}$ common in denominator.

$$X_u[k] = \frac{W_N^{kn_0/2}\left(W_N^{-kn_0/2} - W_N^{kn_0/2}\right)}{W_N^{k/2}\left(W_N^{-k/2} - W_N^{k/2}\right)}$$

$$\boxed{X_u(k) = W_N^{k/2(n_0-1)} \frac{\sin\left(\frac{n_0 \pi k}{N}\right)}{\sin(\pi k/N)} \quad k = 0, 1, \ldots, N}$$

④ find the 10 point IDFT of

$$x(k) = \begin{cases} 3 & \text{if } k = 0 \\ 1, & 1 \le k \le 9 \end{cases}$$

Solution

$x(k) = 1 + 2\delta(k)$; $0 \le k \le 9$.

$\therefore$ $x(n) = \text{IDFT}(1) + \text{IDFT}(2\delta(k))$

$\text{IDFT}[1]$: $= \delta(n)$ [From Q3 a) above]

$\text{IDFT}[2\delta(n)] =$ (use of duality property)

$$\delta(n) \longrightarrow 1$$

Let $y(n) = \delta(n) \Rightarrow Y(k) = 1$

$$Y(n) = 1 \xrightarrow{DFT} N\,y(-k)_N = N\delta(-k)_N$$
$$= N\delta(k)$$

$$1 \xrightarrow{DFT} N\delta(k)$$

$$1/N \xleftrightarrow{DFT} \delta(n)$$

$$2/N \xleftrightarrow{DFT} 2\delta(k) \Rightarrow IDFT(2\delta(k))$$
$$= 2/N.$$

Sol^n

$$\boxed{x(n) = \delta(n) + 1/5}$$

(5) Find N point DFT of

$$x(n) = 4 + \cos^2\left(\frac{2\pi n}{N}\right) \quad \text{in } n = 0, 1, \ldots N-1$$

Sol^n

$$x(n) = 4 + \left(\frac{e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}}}{2}\right)^2$$

$$= 4 + 1/4\left(e^{j4\pi n/N} + e^{-j4\pi n/N} + 2\right)$$

$$= 1/4\, e^{j4\pi/N\, n} + 1/4\, e^{-j\frac{4\pi n}{N}} + 9/2$$

$$e^{-j4\pi/N\, n} = e^{-j4\pi/N\, n}\, e^{j2\pi} = e^{j2\pi/N(N-2)n}$$

$$= 9/2 + 1/4\, e^{j2\pi/N(2n)} + 1/4\, e^{j2\pi/N(N-2)n} \quad \text{— (1)}$$

$$x(n) = 1/N \sum_{k=0}^{N-1} X(k)\, e^{j2\pi/N\, nk} \quad \text{— (2)}$$

Compare (1) & (2)

for $k=0$

$$X(k)/N = 9/2$$

$$\Rightarrow X(k) = 9/2\, N.$$

for $k=2$

$$X(k)/N = 1/4$$

$$X(k) = 1/4 \cdot N$$

for $k = N-2$

$$\frac{X(k)}{N} = 1/4 \Rightarrow X(k) = N/4.$$

for any other k $\quad \frac{X(k)}{N} = 0$

$$X(k) = \begin{cases} 9N/2, & \text{if } k=0 \\ N/4 & \text{if } k=2 \text{ or } k=N-2 \\ 0, & o/w. \end{cases}$$

6) Consider the finite length sequence

$$x(n) = \delta(n) + 2\delta(n-5)$$

a) find 10 point DFT of $x(n)$

b) find the sequence that has a DFT:

$$Y(k) = e^{j 2\pi/10 \, 2k} X(k)$$

Sol:

$$x(n) = \delta(n) + 2\delta(n-5)$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j 2\pi/N \, kn}$$

$$= \sum_{n=0}^{N-1} \delta(n) e^{-j 2\pi/N \, nk} + 2 \sum_{n=0}^{N-1} \delta(n-5) e^{-j 2\pi/N \, kn}$$

$$= 1 + 2e^{-j 2\pi/N \, 5k}$$

$$= 1 + 2e^{-j \pi/\cancel{5} \, \cancel{5}k}$$

$$= 1 + 2e^{-j\pi k}$$

$$= 1 + 2(-1)^k.$$

b) By Time Shifting property:-

$$y(n) = x(n) + 2 \qquad n - n_0 \rightarrow e^{-j 2\pi/N \, k n_0} X(k)$$

$= \delta\left(\left((n+2)\right)_{10}\right) + 2\,\delta\left((n+2-5)_{10}\right)$

$= \delta(n-8) + 2\delta(n-3)$

$n=0,1\ldots 9$

$n=-2$

$-2 \bmod 10$

$8 \bmod 10$

7 Let $x(n) = 4\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)$

Let $X(k)$ be the 6 point DFT of $x(n)$.

a) Find the finite length sequence $y(n)$ that has a 6 point DFT:

$$Y(k) = e^{-j2\pi/6\,(4k)} X(k)$$

b) Find the finite length sequence $w(n)$ that has 6 point DFT = Real part of $X(k)$.

$$W(k) = \mathrm{Re}\{X(k)\}.$$

Solution

a) By time shifting property

$y(n) = x((n-n_0) \bmod 6)$; $n_0 = 4$

$= x((n-4) \bmod 6)$

$= 4\delta((n-4) \bmod 6) + 3\delta((n-1-4) \bmod 6)$

$+ 2\delta((n-2-4) \bmod 6) \longrightarrow \delta(n-6 \bmod 6) = \delta(n)$

$+ \delta((n-3-4) \bmod 6) \longrightarrow \delta(n-7 \bmod 6) = \delta(n-1).$

$= 4\delta((n-4)) + 3\delta(n-5) + 2\delta(n)$

$+ \delta(n-1).$

(b) $Re\{X(k)\} = \dfrac{X(k) + X^*(k)}{2}$

$\therefore\ w(n) = \dfrac{IDFT(X(k)) + IDFT(X^*(k))}{2}$

$= \dfrac{x(n)}{2} + \dfrac{IDFT(X^*(k))}{2}$ —— ①

IDFT of $X^*(k)$

$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$

$\Rightarrow X^*(k) = \left(\sum_n x(n) e^{-j2\pi/N\, kn}\right)^*$

$= \sum_n \left(x^*(n) e^{+j\frac{2\pi}{N}kn}\right)$

$e^{-j2\pi} = \cos 2\pi - j\sin 2\pi = 1$

$= \sum_n x^*(n) e^{j2\pi/N\, kn} \underbrace{e^{-j2\pi k}}_{=1}$

$= \sum_n x^*(n) e^{-j\frac{2\pi}{N}(N-n)k}$

Put $n' = (N-n) \bmod N$

$= \sum_{n'=0}^{N-1} x^*((N-n') \bmod N)\, e^{-j2\pi/N\, n'k}$

$=$ DFT of $x^*((N-n) \bmod N)$

IDFT of $X^*(k) = x^*((N-n) \bmod N)$ —— ②

Put (2) in (1)

$\Rightarrow w(n) = \frac{x(n)}{2} + \frac{x^*((N-n) \bmod N)}{2}.$

$\underline{x^*((N-n) \bmod N)}$?

Recall $x(n) = \{4, 3, 2, 1\}$

$y(0) = x^*(6 \bmod 6) = x^*(0) = 4$

$y(1) = x^*(5 \bmod 6) = x^*(5) = 0$

$y(2) = x^*(4 \bmod 6) = x^*(4) = 0$

$y(3) = x^*(3 \bmod 6) = x^*(3) = 1$

$y(4) = x^*(2 \bmod 6) = x^*(2) = 2$

$y(5) = x^*(1 \bmod 6) = x^*(1) = 3.$

$x^*((N-n) \bmod N) = \{\underset{\uparrow}{4}, 0, 0, 1, 2, 3\}$

$w(n) = \frac{1}{2}\left[\{4, 3, 2, 1, 0, 0\} + \{4, 0, 0, 1, 2, 3\}\right]$

$= \{4, 3/2, 1, 1, 1, 3/2\}.$

(8) Let the 8 point sequence $x(n)$ have DFT

$X(k) = k+1; \quad 0 \leq k \leq 7$

Find $\sum_{n=0}^{3} x(2n)$? $\rightarrow x(0) + x(2) + x(4) + x(6).$

Sol^n

$X(k) = \sum_{n=0}^{N-1} x(n) W_8^{kn}.$

$$\therefore X(0) = \sum_{n=0}^{N-1} x(n) W_8^0$$

$$= \sum_{n=0}^{N-1} x(n)$$

$$X(4) = \sum_{n=0}^{N-1} x(n) W_8^{4n} = \sum_{n=0}^{N-1} (-1)^n x(n).$$

$$W_8^{4n} = e^{-j2\pi/8 \, 4n} = e^{-j\pi n} = (-1)^n.$$

$$X[0] = x(0) + \cancel{x(1)} + x(2) + \dots + x(6) + \cancel{x(7)}$$

$$+ \; X(4) = x(0) - \cancel{x(1)} + x(2) \dots\dots + x(6) - \cancel{x(7)}.$$

$$X(0) + X(4) = 2\left(x(0) + x(2) + x(4) + x(6)\right)$$

$$\Rightarrow x(0) + x(2) + x(4) + x(6) = \frac{X(0) + X(4)}{2}$$

$$\downarrow$$

$$\sum_{n=0}^{3} x(2n). \qquad = \frac{1+5}{2}$$

$$= 3.$$

10 Let $x(n) = \delta(n) + 2\delta(n-2) + \delta(n-3)$..

a) Find 4 point DFT of $x(n)$

b) if $y(n)$ is a 4 point circular convolution of $x(n)$ with itself, then find $y(n)$ & 4pt DFT $Y(k)$

Sol^n a) $x(n) = \delta(n) + 2\delta(n-2) + \delta(n-3)$

$$= 1 + 2W_4^{2k} + W_4^{3k}$$

h) $y(n) = x(n) \circledast x(n)$

$Y(k) = X(k)\,X(k)$ (By convolution property)

$$= \left(1 + 2W_4^{2k} + W_4^{3k}\right)^2$$

verify. $\downarrow$

$$= 5 + 4W_4^{k} + 5W_4^{2k} + 2W_4^{3k}$$

$$y(n) = 5\delta(n) + 4\delta(n-1) + 5\delta(n-2) + 2\delta(n-2).$$

$\blacksquare$