Summary TA session

Part I - Summary of course.

→ **signal**

→ A fxn that conveys information about some phenomenon

eg $x(t) : \mathbb{R} \longrightarrow [-1, 1]$ given by
(time)

$x(t) = \sin(t)$, $-\infty < t < \infty$

→ 2 types based on time
Continuous time signal : t is continuous.
Discrete time signal : t is discrete

→ 2 types based on amplitude

Analog signal :- Amp is continuous
Digital signal :- Amp is discrete

→ Even signal $x(-t) = x(t)$
→ odd signal $x(-t) = -x(t)$
→ Periodic signal
$x(t+T) = x(t)$; $T \to$ period
$1/T \to$ frequency

→ Energy signal if $\underbrace{\int_{-\infty}^{\infty} |x(t)|^2 dt}_{\text{energy of } x(t)} < \infty$

→ Power signal if $\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt < \infty$ → Power of signal.

→ <u>systems</u>. Processes an i/p signal and o/ps a desired signal

$$x(t) \longrightarrow \boxed{s/s} \longrightarrow y(t)$$

→ <u>Memoryless</u> → if o/p depends only on current i/p

→ <u>Causal</u> → o/p does not depend on future i/ps.

→ <u>stable</u> → Bounded i/p gives bounded o/p (BIBO)

→ <u>Linear</u> obeys superposition principle

ie if $x_1(t) \longrightarrow y_1(t)$;

$x_2(t) \longrightarrow y_2(t)$

Then (i) $x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$

(ii) $a\,x(t) \longrightarrow a\,y(t)$ for any a.

<u>Time invariant</u> If $x(t) \longrightarrow y(t)$

$\Rightarrow x(t-\tau) \longrightarrow y(t-\tau)$ for any τ.

→ any LTI system is characterised by impulse response

→ o/p of LTI system is convolution of i/p and the impulse response

→ <u>Laplace Transform</u> → computed over LT domain signals.

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} \, dt$$

ROC (Region of Convergence) where $X(s)$ converges.

→ see notes for properties of L.T

→ Transfer fxn of a system

$$x(t) * h(t) = y(t)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \qquad X(s)H(s) = Y(s)$$

$$H(s) \triangleq \frac{Y(s)}{X(s)}.$$

→ z transform → performed over discrete time signal

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

ROC: region where $X(z)$ converges.

→ see notes for properties of $X(z)$

→ <u>System Function</u>

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$H(z) \triangleq \frac{Y(z)}{X(z)}.$$

→ <u>Fourier Series</u>: Defined for continuous time periodic signals.

(CE) $$x(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{j2\pi k f_0 t} \quad ; \quad f_0 = \text{fundamental frequency.}$$

(Trig) $$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2\pi k f_0 t) + \sum_{k=1}^{\infty} b_k \sin(2\pi k f_0 t).$$

DC term.

→ see notes for properties of F.S.

→ <u>Fourier transforms</u>: Defined for C.T aperiodic signals also.

$$X(w) \triangleq \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt.$$

<u>IFT</u> : $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w)\, e^{jwt}\, dw.$$

→ See notes for properties of F.T

→ LT evaluated at $s = jw$ is a F.T. → ROC must have jw axis.

→ <u>Frequency Response</u>

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t).$$

$$H(w) = Y(w) / X(w) \quad ; \quad e^{jw_0 t}$$ is eigen fxn of an LTI system.

→ Distortionless response

$$y(t) = k\,x(t-t_0)$$

→ Filters - lowpass, highpass, bandpass, band stop.

→ Sampling - obtaining discrete-time signal from a continuous time signal.

→ Fourier spectrum of a sampled signal is the periodic version of FT of unsampled signal with period = f_s → sampling frequency.

→ Nyquist sampling rate : Any signal can be uniquely recovered from sampled signal w/o any distortion if

$$f_s \geq 2 f_{max}.$$

f_s → sampling freq.

f_{max} → max freq content.

→ Discrete Fourier Series → computed for discrete time periodic signals.

$$x(n) = \sum_{k=0}^{N_0-1} c_k\, e^{jk 2\pi/N_0\, n} \quad ; \quad N_0 \to \text{fundamental period.}$$

→ See note for DFS properties.

→ Discrete Time Fourier Transform

computed for discrete-time aperiodic signals also.

$$\text{DTFT} \quad X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}.$$

IDTFT $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\Omega$.

→ See notes for DTFT properties.

→ <u>Freq response of a discrete time system</u>

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$\Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)}.$$

time discrete → freq also discrete

→ Discrete Fourier Transform

* used for finite length time sequences.
* the frequency domain signal is also discrete.

DFT : $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi/N\, nk}$; $k = 0, 1, \dots N-1$.

IDFT $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi/N\, nk}$

- DTFT at $\omega = \frac{2\pi k}{N}$ → DFT : $X(k)$
- See notes for DFT properties.

→ <u>Phase Delay</u>

$$\tau_p = -\frac{\phi(\omega_0)}{\omega_0}$$ → delay in carrier i/p

→ <u>Group Delay</u>

$$\tau_g = -\frac{d\phi(\omega)}{d\omega}\Bigg|_{\omega=\omega_0}$$ → delay in msg i/p.

→ Practically, IIR filters are realized in various forms

1) & I, II, transpose, cascade, parallel, I, II.

Part II Gate questions.

(1) In what range should $Re\{s\}$ remain so that the Laplace Transform of the function

$$x(t) = e^{(a+2)t+5} u(t) \text{ exists ?}$$

a) $Re\{s\} > a+2$
b) $Re\{s\} > a+7$
c) $Re\{s\} < 2$
d) $Re\{s\} > a+5$

Soln

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt.$$

$$= \int_{0}^{\infty} e^{(a+2)t+5} e^{-st} dt$$

$$= e^{5} \int_{0}^{\infty} e^{-(s-a-2)t} dt$$

$$= \frac{e^{5}}{a+2-s} e^{-(s-a-2)t} \Big|_{0}^{\infty}$$

c

$$= c\{e^{-\underbrace{(s-a-2)}\infty} - \cancel{e^{0}}^{1}\}$$

↓
converge!
↓
Go to zero.

$\Rightarrow Re\{s-a-2\} > 0$

$\Rightarrow Re\{s\} > a+2.$

(2) A continuous time LTI system is described by:

$$\frac{d^2y(t)}{dt^2} + 4\frac{d\,y(t)}{dt} + 3y(t) = 2\frac{d}{dt}x(t) + 4x(t)$$

Assuming zero initial conditions, the response $y(t)$ of the above system for the input $x(t) = e^{-2t}u(t)$ is given by [Gate 2010]

a) $(e^{t} - e^{3t})u(t)$ c) $(e^{t} + e^{-3t})u(t)$

b) $(e^{-t} - e^{-3t})u(t)$ d) $(e^{t} + e^{3t})u(t)$.

Sol^n It is given that

$$\frac{d^2y(t)}{dt^2} + \frac{4dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + 4x(t)$$

Take Laplace transform on both sides:

$$s^2Y(s) + 4sY(s) + 3Y(s) = 2sX(s) + 4X(s)$$

$$\Rightarrow Y(s)(s^2+4s+3) = X(s)(2s+4)$$

$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{2s+4}{s^2+4s+3} = H(s)$$

$$= \frac{2s+4}{s^2+s+3s+3} = \frac{2s+4}{s(s+1)+3(s+1)}$$

$$= \frac{2s+4}{(s+3)(s+1)}$$

Also $x(t) = e^{-2t}u(t)$

$$X(s) = \frac{1}{s+2} \quad \text{—(2)}$$

Put (2) in (1)

$$Y(s) = \frac{2s+4}{s^2+4s+3} \quad X(s) = \frac{2\cancel{(s+2)}}{s^2+4s+3} \times \frac{1}{\cancel{s+2}}$$

$$= \frac{2}{(s+1)(s+3)}$$

Partial fraction $= \frac{A}{s+1} + \frac{B}{s+3}$

$$A = \left.\frac{2}{s+3}\right|_{s=-1} = \frac{2}{2} = 1$$

$$B = \left.\frac{2}{s+1}\right|_{s=-3} = \frac{2}{-2} = -1$$

$$Y(s) = \frac{1}{s+1} \quad \frac{-1}{s+3}.$$

$$y(t) = (e^{-t} - e^{-3t})\,u(t).$$ [option b]

(3) If the unit step response of a network is $(1-e^{-\alpha t})$, then what is its unit impulse response?

a) $\alpha e^{-\alpha t}$ b) $\alpha^{-1} e^{-\alpha t}$ c) $(1-\alpha^{-1}) e^{-\alpha t}$

d) $(1-\alpha) e^{\alpha t}$ [Gate 2011]

Sol^n

$$u(t) \longrightarrow \boxed{h(t)} \longrightarrow y_u(t) = 1-e^{\alpha t}$$

$y_{u(t)}$

what is $h(t)$?

w.r.t. $y_{u(t)}(t) = u(t) * h(t)$

$$Y_{u(t)}(s) = U(s)\ H(s)$$

$$\underset{u(t)}{Y(s)} = \frac{1}{s} H(s)$$

$$\Rightarrow H(s) = s\underset{u(t)}{Y(s)}$$

$$\boxed{h(t) = \frac{d}{dt} y_{u(t)}(t)}$$

(By diff property of Laplace Transform)

$$= \frac{d}{dt}(1 - e^{-\alpha t})$$

$$= \alpha e^{-\alpha t} \qquad \therefore \text{option (a)}$$

Q) Impulse response of a system is $h(t) = t\,u(t)$. For an input $u(t-1)$, what is the output?

a) $\frac{t^2}{2} u(t)$ b) $\frac{t(t-1)}{2} u(t-1)$ c) $\frac{(t-1)^2}{2} u(t-1)$

d) $\frac{t^2-1}{2} u(t-1)$ [Gate 2013]

Sol^n

$$\left.\begin{aligned} h(t) &= t\,u(t) \\ x(t) &= u(t-1) \end{aligned}\right\} y(t) = ?$$

$\therefore$ LT of $h(t) = H(s) = 1/s^2$.

LT of $x(t) = \frac{e^{-s}}{s}$.

$\therefore$ By convolution property,

$$Y(s) = X(s)\,H(s).$$

$$= \frac{e^{-s}}{s} \cdot 1/s^2 = \frac{e^{-s}}{s^3}.$$

$$\frac{e^{-s}}{s^3} \Rightarrow 1/s^3 \Rightarrow \frac{e^{-s}}{s^3}$$

$\Rightarrow \frac{t^2}{2}u(t) \Rightarrow \frac{(t-1)^2}{2}u(t-1)$. (option c)

5) The Fourier series expansion of a real periodic signal with fundamental freq f_0 is given by

$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

It is given that $c_3 = 3+j5$. What is c_{-3}?

a) $5+j3$ c) $-5+j3$

b) $-3-j5$ d) $3-j5$

[Gate 2003]

Soln For a real signal

$$c_{-k} = c_k^*$$

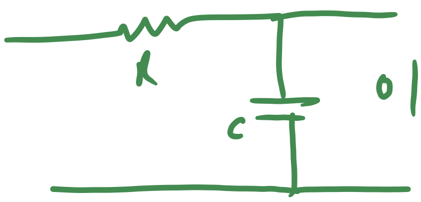
$$\therefore c_{-3} = c_3^* = \underline{3-j5}$$

6) Let $H(f)$ denote the freq. response of RC low pass filter. Let f_1 be the highest frequency s.t

$$0 \le |f| \le f_1 \quad \frac{|H(f_1)|}{H(0)} \ge 0.95$$

Let $R = 1\,k\Omega$, $C = 1\,\mu F$, what is f_1.

Soln.



$$H(f) = \frac{1/j\omega C}{1/j\omega C + R}$$

$$= \frac{1}{1+j\omega RC}$$

$$= \frac{1}{1+j2\pi f RC}$$

for RC LPF $H(0) = 1$.

$$\frac{|H(f_1)|}{H(0)} = \frac{1}{\sqrt{1+4\pi^2 f^2 R^2 C^2}} \geq 0.95$$

$$\Rightarrow 1.108 \geq 1 + 4\pi^2 f_1^2 R^2 C^2$$

Substituting for R & C

$$f_1 \leq \frac{0.329}{2\pi \times 10^{-3}} = 52.2\,Hz.$$

$$\underline{f_1 = 52.2\,Hz}$$

(9) Let $\tau_g(f)$ be the group delay function of the given RC-LPF and $f_0 = 100\,Hz$. Then what $\tau_g(f)$ in milliseconds?

Soln.

$$H(\omega) = \frac{1}{1+j\omega RC}.$$

$$\angle H(\omega) = \angle 1 - \tan^{-1}\omega RC$$

$$= -\tan^{-1}\omega RC$$

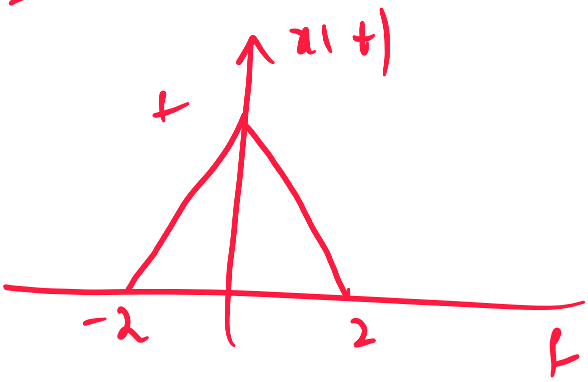
$$\tau_g(f) = -\frac{d\angle H(\omega)}{d\omega}\Big|_{\omega=f}$$

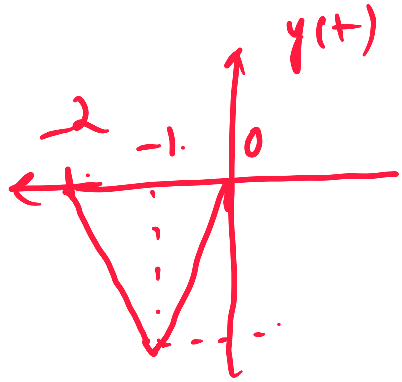
$$= \frac{RC}{1+(2\pi RCf)^2}$$

Substitute for R, C & $f = 100$ Hz

$= 0.717$ ms.

8) Let $x(t)$ and $y(t)$ [with Fourier Transforms $X(f)$ and $Y(f)$] be related as shown below.





Then $Y(f)$ is

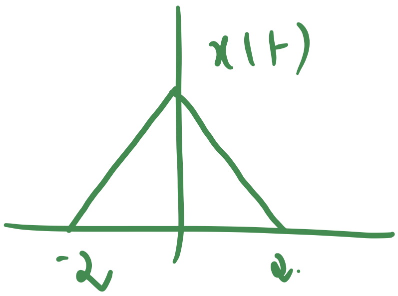
a) $-\frac{1}{2} X(f/2)\, e^{-j2\pi f}$

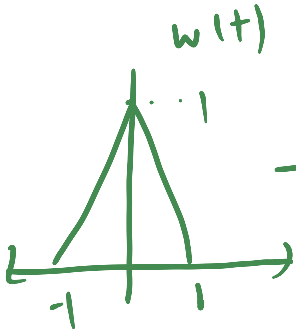
b) $-\frac{1}{2} X(f/2)\, e^{j2\pi f}$ ✓

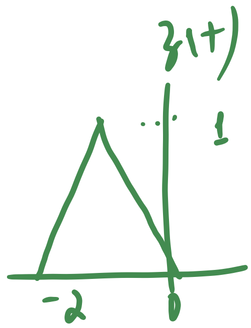
c) $-X\left(\frac{f}{2}\right) e^{j2\pi f}$

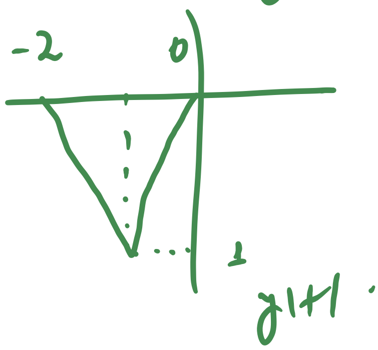
d) $-X(f/2)\, e^{-j2\pi f}$

Sol^n:









$w(t) \rightarrow x(2t)$

$W(f) \rightarrow \frac{1}{2} X(f/2)$

$z(t) \rightarrow w(t+1)$

$Z(f) \rightarrow W(f)\, e^{j2\pi f}$

$= \frac{1}{2} X(f/2)\, e^{j2\pi f}$

$y(t) = -z(t) \rightarrow Y(f) = -\frac{1}{2} X(f/2)\, e^{j2\pi f}$

option (b)

(9) The 3 dB bandwidth of low pass signal $e^{-t}u(t)$, where $u(t)$ is the unit step function; is ______.

a) $\frac{1}{2\pi}$ Hz b) $\frac{1}{2\pi}\sqrt{\sqrt{2}-1}$ Hz c) ∞ d) 1 Hz.

Sol:

Sol^n: Laplace transform of $e^{-t}u(t)$

$$\frac{1}{s+1}.$$

$\therefore$ Magnitude at 3 dB freq. is $1/\sqrt{2}$.

$$\Rightarrow \left| \frac{1}{s+1} \right| = \frac{1}{\sqrt{2}} \qquad s = j\omega$$

$$\Rightarrow \frac{1}{\sqrt{1+\omega^2}} = 1/\sqrt{2} \Rightarrow \omega = 1 \text{ rad.}$$

$$\boxed{f = 1/2\pi \text{ Hz}}$$

(10) A 5-point sequence $x[n]$ is given as
$x[-3] = 1$, $x[-2] = 1$, $x[-1] = 0$, $x[0] = 5$
$x[1] = 1$. Let $X(\omega)$ denote the DTFT of $x(n)$. What is $\int_{-\pi}^{\pi} X(\omega)\, d\omega$?

Sol^n Recall IDTFT

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} d\omega.$$

Put $n=0$

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(w)\, dw.$$

$$\Rightarrow \int_{-\pi}^{\pi} X(w)\, dw = 2\pi\, x(0)$$

$$= 2\pi \times 5$$

$$= 10\pi$$

(11) The first 5 points of the 8 point DFT of a real valued sequence are $5,\ 1-j3,\ 0,\ 3-j4,\ 3+j4$, the last 2 points of the DFT are respectively

a) $0,\ 1-j3$ b) $0,\ 1+j3$ c) $1+j3,\ 5$ d) $1-j3,\ 5$

(Gate 2011)

Sol^n For a real valued sequence

$$X(k) = X^*(N-k).$$

$$X(-k)_{\bmod} = X_k^*$$

$N=8$ in our case)

$$\therefore X(k) = X^*(8-k).$$

$$X(6) = X^*(8-6) = X^*(2) = 0$$

$$X(7) = X^*(8-7) = X^*(1) = (1-j3)^*$$

$$= \underline{1+j3}.$$

option b

(12) A causal LTI system is described by the difference equation

$2y(n) = \alpha y(n-2) - 2x(n) + \beta x(n-1)$

The system is stable only if

a) $|\alpha| = 2$, $|\beta| < 2$ [Gate 2004]

b) $|\alpha| > 2$, $|\beta| > 2$

c) ✓ $|\alpha| < 2$, any value of β

d) $|\beta| < 2$, any value of α

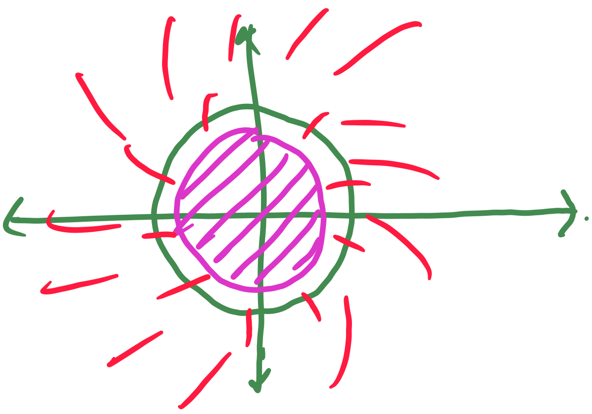
Sol^n $2y(n) = \alpha y(n-2) - 2x(n) + \beta x(n-1)$

Taking z-transform on both sides.

$$2Y(z) = \alpha z^{-2} Y(z) - 2X(z) + \beta z^{-1} X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{\beta z^{-1} - 2}{2 - \alpha z^{-2}} = H(z)$$

For a system to be stable



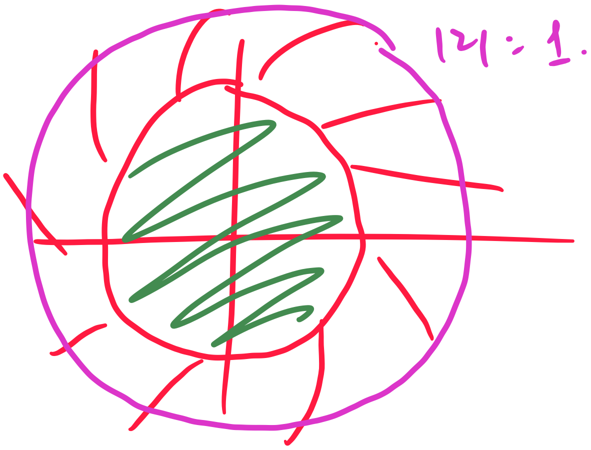
ROC outside unit circle.

→ As it is causal.

$\Rightarrow 2 - \alpha z^{-2} = 0 \Rightarrow 2 = \alpha / z^2$

$\alpha / \sqrt{2}$ is a pole $\Rightarrow z = \alpha / \sqrt{2}$

$\Rightarrow \alpha / \sqrt{2} > 1 \Rightarrow \alpha > \sqrt{2}$



→ Since the system is causal, ⇒ ROC is exterior part of a circle.

→ poles should be within unit circle.

poles $z - \alpha z^{-2} = 0 \Rightarrow z = \alpha / z^2$

$\Rightarrow z = \sqrt{\alpha / 2}$

Now $\sqrt{\alpha / 2} < 1 \Rightarrow \boxed{\alpha < 2}$