Week 10 L-56

Discrete time Fourier Transform (DTFT)

↳ Fourier analysis of discrete time aperiodic signals.

Consider a discrete time Aperiodic Sequence $x[n]$

The DTFT of $x(n)$ is given as,

$$\boxed{\underbrace{X(\Omega)}_{\text{DTFT}} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}}$$

Inverse DTFT,

$$\int_{-\pi}^{\pi} X(\Omega) e^{j\Omega k} d\Omega$$

$$= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} e^{j\Omega k} d\Omega$$

$$= \sum_{n=-\infty}^{\infty} x(n) \int_{-\pi}^{\pi} \underbrace{e^{j\Omega(k-n)}}_{} d\Omega$$

↳ Complex exponential
Fundamental period $= 2\pi$

$$= \begin{cases} 0, & \text{if } k \neq n \\ 2\pi, & \text{if } k = n \end{cases}$$

$$= 2\pi\, x(k)$$

$$2\pi x(k) = \int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega k}\, d\Omega$$

$$\Rightarrow \boxed{x(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega k}\, d\Omega}$$

→ inverse DFT.

Spectrum:

$X(\Omega)$ = complex.

$$X(\Omega) = |X(\Omega)|\, e^{j\phi(\Omega)}$$

$|X(\Omega)|$ = Magnitude spectrum

$\phi(\Omega)$ = Phase spectrum.

If $x(n)$ is real,

$$\Rightarrow X(\Omega) = X^*(-\Omega)$$

$$\Rightarrow |X(\Omega)| = |X(-\Omega)|$$ → magnitude spectrum = even.

$$\phi(\Omega) = -\phi(-\Omega)$$ → Phase spectrum = odd.

Convergence of $X(\Omega)$:

$X(\Omega)$ converges if

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$ → Finite quantity.

$\Rightarrow$ Signal is absolutely summable.

Relation between DTFT and Z-Transform

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\hookrightarrow z = e^{j\Omega}$$

$$X(z)\Big|_{z=e^{j\Omega}} = X(\Omega)$$

→ z - transform evaluated at $z = e^{j\Omega}$.

→ However only possible if $z = e^{j\Omega}$ in ROC of $X(z)$ implies ROC has to contain the unit circle.

$$\left| \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x(n)| \underbrace{|e^{-j\Omega n}|}_{1}$$

$$= \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

→ If signal is absolutely summable

→ = ROC contains unit circle.

→ not generally true if sequence is NOT absolutely summable

DTFT of common signals:

Impulse:

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$\text{DTFT} = \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\Omega n}$$

$$= 1\, e^{-j\Omega 0}$$

$$= 1.$$

$$\boxed{\delta(n) \longleftrightarrow 1}$$

Exponential signal

$$x(n) = a^n u(n)$$

↳ real

$|a| < 1$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n e^{-j\Omega n}$$

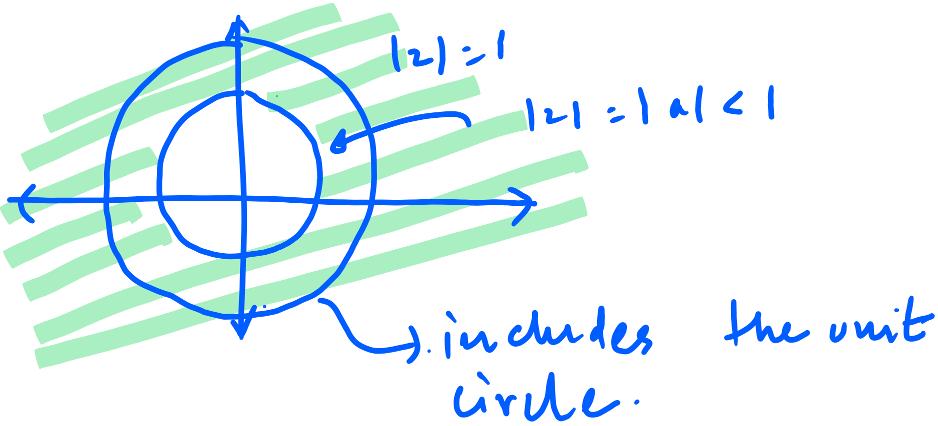
$$= \sum_{n=0}^{\infty} \left(a e^{-j\Omega}\right)^n$$

$$\boxed{X(\Omega) = \frac{1}{1 - a e^{-j\Omega}}}$$

$$a^n u(n) = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}.$$

ROC: $|z| > |a|$.

If $|a| < 1$ → ROC includes unit circle.



$$\left.\frac{1}{1 - a z^{-1}}\right|_{z = e^{j\Omega}}$$

$$= \frac{1}{1 - a e^{-j\Omega}} = X(\Omega)$$

Since ROC contains unit circle.

Unit-step signal

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\sum_{n=-\infty}^{\infty} |u(n)| = \underbrace{\sum_{n=0}^{\infty} 1}_{} \not< \infty$$

NOT a finite quantity

$a^n u[n]$ with $a = 1$.

$$X(z) = \frac{1}{1 - z^{-1}}$$

ROC : $|z| > 1$ → does not include unit circle.

$$X(z)\Big|_{z=e^{j\Omega}} = \frac{1}{1-e^{-j\Omega}}$$

$$\neq X(\Omega)$$

$$\boxed{X(\Omega) = \pi\delta(\Omega) + \frac{1}{1-e^{-j\Omega}}}$$

DTFT of unit step signal.

L.57 Discrete time Fourier transform (DTFT)

Properties of DTFT

DTFT is periodic

$$X(\Omega + 2\pi) = X(\Omega)$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{j\Omega n}$$

$$X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x(n) e^{j(\Omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{j\Omega n} \underbrace{e^{j2\pi n}}_{1}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$= X(\Omega)$$

$$\boxed{X(\Omega + 2\pi) = X(\Omega)}$$

Linearity:

$$x_1(n) \longleftrightarrow X_1(\Omega)$$
$$x_2(n) \longleftrightarrow X_2(\Omega)$$
$$a_1 x_1(n) + a_2 x_2(n) \longleftrightarrow a_1 X_1(\Omega) + a_2 X_2(\Omega).$$

Time shifting:

$$x(n) \longleftrightarrow X(\Omega)$$
$$x(n-n_0) \longleftrightarrow ?$$

$$\tilde{X}(\Omega) = \sum_{n=-\infty}^{\infty} x(n-n_0)\, e^{-j\Omega n}$$

$$n - n_0 = m$$
$$n = m + n_0$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\Omega(m+n_0)}$$

$$= e^{-j\Omega n_0} \underbrace{\sum_{m=-\infty}^{\infty} x(m)\, e^{-j\Omega m}}_{X(\Omega)}$$

$$\tilde{X}(\Omega) = \underbrace{X(\Omega)\, e^{-j\Omega n_0}}_{\text{Modulation in frequency.}}$$

$$\boxed{x(n-n_0) \longleftrightarrow X(\Omega)\, e^{-j\Omega n_0}}$$

Frequency shifting

$$x(n) \longleftrightarrow X(\Omega)$$

$$\underbrace{e^{j\Omega_0 n} x(n)}_{\text{Modulation in time}} \longleftrightarrow X(\Omega - \Omega_0)$$

$\Rightarrow$ Shift in frequency.

<u>conjugation</u>:

$$x(n) \longleftrightarrow X(\Omega)$$

$$x^*(n) \longleftrightarrow X^*(-\Omega)$$

$\Rightarrow$ If $x(n) =$ Real

$$x(n) = x^*(n)$$

$$\Rightarrow \boxed{X(\Omega) = X^*(-\Omega)}$$

<u>Time Reversal</u>:

$$x(n) \longleftrightarrow X(\Omega)$$

$$x(-n) \longleftrightarrow ?$$

$$\tilde{X}(\Omega) = \sum_{n=-\infty}^{\infty} x(-n) e^{-j\Omega n} \qquad m = -n$$

$$= \sum_{m=-\infty}^{\infty} x(m) e^{j\Omega m}$$

$$\tilde{X}(-\Omega) = \underbrace{\sum_{m=-\infty}^{\infty} x(m) e^{-j\Omega m}}_{X(\Omega)}$$

$$\tilde{X}(-\Omega) = X(\Omega)$$

$$\tilde{X}(\Omega) = X(-\Omega)$$

$$\boxed{x(-n) \longleftrightarrow X(-\Omega)}$$

Time scaling :

Note: Not similar to continuous time signals

$$x(t) \longleftrightarrow X(\omega)$$

$$\boxed{x(at) \longleftrightarrow \frac{1}{|a|} X(\omega/a)}$$

NOT TRUE for discrete time signals in general!!

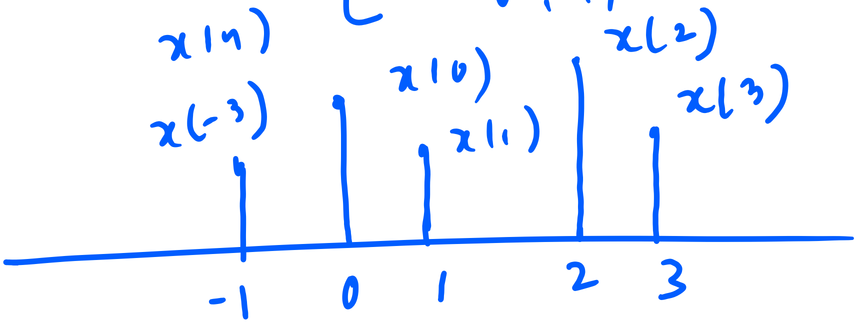
$$x(n) \longleftrightarrow X(\Omega)$$

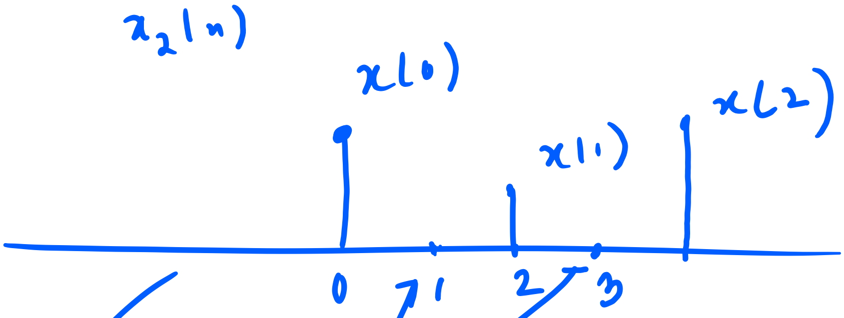
$$\boxed{x(2n) \not\longleftrightarrow \frac{1}{2} X\left(\frac{\Omega}{2}\right)}$$

Not true in general.

$$x_m(n) = \begin{cases} x(n/m), & \text{if } n = km \\ 0, & n \neq km \end{cases}$$

$$x_2(n) = \begin{cases} x(n/2) & \text{if } n = \text{even} \\ 0, & \text{if } n = \text{odd} \end{cases}$$



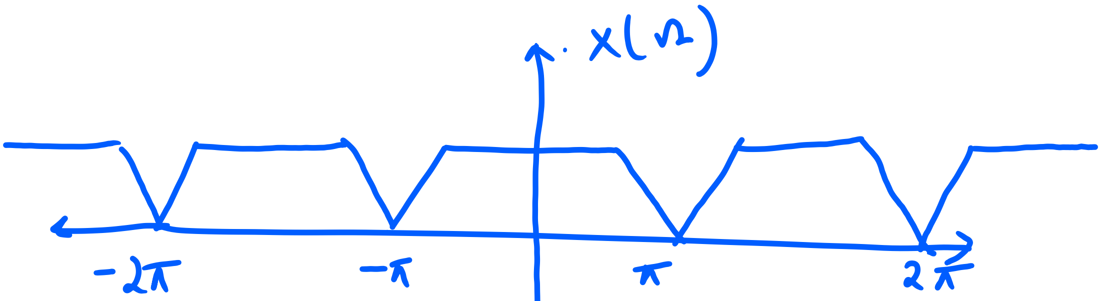


Inserting zeros in between

upsampling = by a factor of 2.

$x(n) \longleftrightarrow X(\Omega)$

$\hookrightarrow \boxed{x_m(n) \longleftrightarrow X(m\Omega)}$



$x_2(n) \longleftrightarrow X(2\Omega)$

$\hookrightarrow X(2\pi)$

Replicas of original spectrum

Duality:

$x(t) \longleftrightarrow X(\omega)$

$X(t) \longleftrightarrow 2\pi\, x(-\omega)$

$x(n) \longleftrightarrow X(\Omega)$

$X(t) \longleftrightarrow ?$

Periodic signal

$$\boxed{X(t+2\pi) = X(t)}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$X(t) = \sum_{n=-\infty}^{\infty} x(n) e^{-jnt}$$

$m = -n$

$$= \sum_{m=-\infty}^{\infty} x(-m) e^{jmt}$$

$T_0 = 2\pi$

$\omega_0 = 2\pi / T_0 = 1$

$$X(t) = \sum_{m=-\infty}^{\infty} x(-m) e^{jm\omega_0 t}$$

$$\boxed{X(t) = \underline{\sum_{m=-\infty}^{\infty} x(-m) e^{jm\omega_0 t}}}$$

Complex exponential Fourier Series.

$x(-m)$ = coefficient of $e^{jm\omega_0 t}$ in CEFS.

$$\boxed{X(t) \overset{CEFS}{\longleftrightarrow} x(-m)}$$

$$\boxed{\begin{array}{c} x[n] \xleftrightarrow{DTFT} X(\Omega) \\ X(t) \xleftrightarrow{CEFS} x(-n) \end{array}}$$

Duality of DTFT

$T_0 = 2\pi$

$f_0 = 1/2\pi$

$\omega_0 = 1$ rad/s.

L-58 Discrete time Fourier Transform

Differentiation in freq:

$$x(n) \longleftrightarrow X(\Omega)$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{d\Omega} e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-jn)\, e^{-j\Omega n}$$

$$\frac{dX(\Omega)}{d\Omega} = -j \underbrace{\sum_{n=-\infty}^{\infty} n\, x(n)\, e^{-j\Omega n}}_{\text{DTFT of } n\,x(n)}$$

$$j\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} n\, x(n)\, e^{j\Omega n}$$

$$\boxed{n\,x(n) \longleftrightarrow j\frac{dX(\Omega)}{d\Omega}}$$

Difference in time:

$$x(n) \longleftrightarrow X(\Omega)$$

$$x(n) - x(n-1) \longleftrightarrow ?$$

$$\Updownarrow$$

$$X(\Omega) - e^{-j\Omega} X(\Omega) = \tilde{X}(\Omega)$$

$$\Rightarrow \tilde{X}(\Omega) = \left(1 - e^{-j\Omega}\right) X(\Omega)$$

$$\boxed{x(n) - x(n-1) = \left(1 - e^{-j\Omega}\right) X(\Omega)}$$

Accumulation

$$y(n) = \sum_{k=-\infty}^{n} x(k) \qquad \int_{-\infty}^{t} x(t)\,dt$$

similar to integrator for CT signals.

$$\sum_{k=-\infty}^{n} x(k) \longleftrightarrow \pi X(0)\,\delta(\Omega) + \frac{X(\Omega)}{1 - e^{-j\Omega}}$$

Convolution :-

$$y(n) = x_1(n) * x_2(n)$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)$$

$\updownarrow$

2

$x_1(n) \longleftrightarrow X_1(\Omega)$

$x_2(n) \longleftrightarrow X_2(\Omega)$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)\, e^{-j\Omega n}.$$

$$= \sum_{m=-\infty}^{\infty} x_1(m) \underbrace{\sum_{n=-\infty}^{\infty} x_2(n-m)\, e^{-j\Omega n}}_{\text{DTFT of } x_2(n-m)}$$

$X_2(\Omega)\, e^{-j\Omega m}$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\, e^{-j\Omega m} \times X_2(\Omega)$$

$$= X_2(\Omega) \sum_{m=-\infty}^{\infty} x_1(m)\, e^{-j\Omega m}$$

$X_1(\Omega$

$$\boxed{Y(\Omega) = X_1(\Omega)\, X_2(\Omega)}$$

$$\boxed{x_1(n) * x_2(n) \longleftrightarrow X_1(\Omega)\, X_2(\Omega)}$$

Multiplication in frequency domain.

convolution time

MULTIPLICATION:

$$x_1(n)\, x_2(n) \longleftrightarrow \frac{1}{2\pi} X_1(\Omega) \circledast X_2(\Omega)$$

Periodic convolution

$$= \frac{1}{2\pi} \int_{2\pi} X_1(\theta)\, X_2(\Omega-\theta)\, d\theta$$

Any period 2π

If $x(n)$ = Real.

$$x(n) = \underbrace{x_e(n)}_{\text{even component}} + \underbrace{x_o(n)}_{\text{odd component.}}$$

$$x(n) \longleftrightarrow A(\Omega) + jB(\Omega)$$

$$\Rightarrow x_e(n) \longleftrightarrow A(\Omega)$$

= Real part of DTFT $X(\Omega)$

$$x_o(n) \longleftrightarrow \underbrace{jB(\Omega)}_{\text{purely imaginary}}$$

$$\underbrace{X^*(-\Omega) = X(\Omega)}_{\text{For any real signal } x(n)}$$

$$\Rightarrow \underbrace{\left(A(\Omega) + jB(\Omega)\right)}_{X(\Omega)} = \left(A(-\Omega) + jB(-\Omega)\right)^*$$

$$\Rightarrow A(\Omega) + jB(-\Omega)$$
$$= A(-\Omega) - jB(-\Omega)$$

Equating real and imaginary parts,

$$\underbrace{A(\Omega) = A(-\Omega)}_{\text{even function}}$$

$$\underbrace{B(\Omega) = -B(-\Omega)}_{\text{odd function}}$$

$\Rightarrow x(n) =$ real + even

$\Rightarrow x(n) = x_e(n)$, $x_o(n) = 0$

$\Rightarrow X(\Omega) = A(\Omega)$

Real + even

$\Rightarrow$ if $x(n)$ is real and odd.

$\Rightarrow x(n) = x_o(n)$

$x_e(n) = 0$

$\Rightarrow X(\Omega) = X_o(\Omega)$

$= jB(\Omega)$

↳ Purely imaginary + odd.

Parseval's relation

$$y(n) = x_1(n)\, x_2(n)$$

$$Y(\Omega) = \frac{1}{2\pi} \int_{2\pi} X_1(\theta)\, X_2(\Omega - \theta)\, d\theta$$

$$\underbrace{\sum_{n=-\infty}^{\infty} y(n) e^{-j\Omega n} = Y(\Omega)}_{\text{Set } \Omega = 0}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} y(n) = Y(0)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n) = Y(0)$$

$$= \frac{1}{2\pi} \int_{2\pi} X_1(\theta)\, X_2(-\theta)\, d\theta$$

$$\underbrace{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n) = \frac{1}{2\pi} \int_{2\pi} X_1(\Omega)\, X_2(-\Omega)\, d\Omega}$$

General parseval's theorem.

$$x_2(n) = x_1^*(n)$$

$$\Rightarrow X_2(\Omega) = X_1^*(-\Omega)$$

$$\Rightarrow \boxed{X_2(-\Omega) = X_1^*(\Omega)}$$

$$\boxed{\sum_{n=-\infty}^{\infty} |x_1(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X_1(\Omega)|^2 d\Omega}$$

Parseval's theorem for DTFT.

<u>L-59.</u>

DTFT : <u>Discrete time LTI systems</u>

↳ Frequency response

$$\underset{\text{input}}{\xrightarrow{x(n)}} \boxed{h(n)} \xrightarrow{\text{output}} y(n)$$

$h(n)$ = impulse response
LTI system.

$$y(n) = x(n) * h(n)$$

DTFT

$$\Rightarrow Y(\Omega) = X(\Omega)\, H(\Omega)$$

$$\Rightarrow \boxed{H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}}$$

Frequency response
Transfer function.

$x(n)$: Periodic
$y(n) = ?$ Given $H(\Omega)$

$$x(n) = e^{j\Omega_0 n}$$

$$y(n) = h(n) * x(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, e^{j\Omega_0 (n-k)}$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, e^{j\Omega_0 n}\, e^{-j\Omega_0 k}$$

$$= e^{j\Omega_0 n} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{-j\Omega_0 k}}_{H(\Omega_0)}$$

$$y(n) = e^{j\Omega_0 n}\, H(\Omega_0)$$

$x(n) = e^{j\Omega_0 n} \longrightarrow$ eigen function.

$$y(n) = H(\Omega_0)\, e^{j\Omega_0 n}$$

If $x(n)$: Periodic

$$x(n) = \sum_{k=0}^{N_0 - 1} c_k\, e^{jk\Omega_0 n}$$

$c_k \rightarrow$ DFS of $x(n)$

, $\Omega_0 = 2\pi/N_0$ → fundamental freq.

Input $= e^{jk\Omega_0 n}$

output $= H(k\Omega_0)\, e^{j\Omega_0 n k}$

$\Rightarrow c_k e^{jk\Omega_0 n} \xrightarrow[\text{output}]{} c_k H(k\Omega_0) e^{j\Omega_0 n k}$

$$x(n) = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n}$$

from linearity

$$\text{output} = \underbrace{\sum_{k=0}^{N_0-1} c_k H(k\Omega_0) e^{jk\Omega_0 n}}_{\text{output for periodic signal}}$$

$x(n)$ period $= N_0$

$= 2\pi/\Omega_0$.

LTI system characterised by difference equations

$$\underbrace{\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)}_{\text{Difference equation}}$$

$$\sum_{k=0}^{N} a_k Y(\Omega) e^{-jk\Omega} = \sum_{k=0}^{M} b_k X(\Omega) e^{-jk\Omega}$$

$$Y(\Omega) \sum_{k=0}^{N} a_k e^{-jk\Omega} = X(\Omega) \sum_{k=0}^{M} b_k e^{-jk\Omega}$$

M

$$\boxed{H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \sum_{k=0} b_k e^{-jk\Omega} \Big/ \sum_{k=0}^{N} a_k e^{-jk\Omega}.}$$

Taking inverse DTFT gives impulse response

L-60 Discrete Fourier transform:

used for finite length sequence

$x(n) \quad 0 \le n \le N-1$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$0 \le k \le N-1$

$$X(k) = \sum_{n=0}^{N-1} x(n) \underbrace{e^{-(j2\pi/N)kn}}_{W_N}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{kn}.$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\Rightarrow W_N^{lN} = e^{-j2\pi/N \, lN}$$

$$= e^{-j2\pi l}$$

$$= 1$$

For IDFT Note

$$\sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$$

$$= \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}\, e^{j2\pi km/N}$$

$$= \sum_{n=0}^{N-1} x(n) \underbrace{\sum_{k=0}^{N-1} e^{j2\pi k/N\,(m-n)}}$$

$= 0$, if $m \neq n$

$= N$, if $m = n$

$$= N\, x(m)$$

$$N\, x(m) = \sum_{k=0}^{N-1} X(k)\, e^{j2\pi km/N}$$

$$\Rightarrow x(m) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi k\frac{m}{N}}$$

$$\boxed{x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-km}}$$

expression for IDFT inverse Discrete Fourier Transform.

$x(n) \longrightarrow X(k)$

$0 \le n \le N-1$ $\qquad$ $0 \le k \le N-1$

Both have N samples.

<u>Relation between DFT & DFS</u>

Consider periodic extension of $x(n)$.

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\Omega_0 k n}$$

$$\Rightarrow N c_k = \sum_{n=0}^{N-1} x(n)\, e^{-j\Omega_0 k n} \qquad \Omega_0 = 2\pi/N$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N} k n}$$

$$= X(k).$$

$$\Rightarrow \boxed{N c_k = X(k)}$$

k^{th} DFS coeff of periodic extension

k^{th} DFT coefficient of original signal.

Relation between DFT and DTFT

$$x(n) \quad 0 \le n \le N-1$$

$$0, \quad o/w.$$

$$X(\Omega) = \sum_{n=0}^{N-1} x(n)\, e^{-j\Omega n}$$

$$\Omega = \frac{2\pi k}{N}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k}{N} n}$$

$$= X(k)$$

$$\boxed{X(k) = X(\Omega)\Big|_{\Omega = \frac{2\pi k}{N}}}$$

Properties of DFT:

$x(n) \quad 0 \le n \le N-1$

$X(k) \quad 0 \le k \le N-1$

$\Rightarrow x(n-n_0) \equiv x((n-n_0) \bmod N)$

$\tilde{x}(n)$

$n=1, \quad N=4, \quad n_0=5$

$\tilde{x}(1) = x(1-5)$

$= x(-4)$

$= x(-4 \bmod 4)$

$= x(0)$

Properties of DTFT

Linearity

$x_1(n) \rightarrow X_1(k)$

$x_2(n) \rightarrow X_2(k)$

$a x_1(n) + b x_2(n) \rightarrow a X_1(k) + b X_2(k)$

Time shifting

$x(n) \longleftrightarrow X(k)$

$\underbrace{x(n-n_0)}_{\tilde{x}(n)} \longleftrightarrow ?$

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{kn}$$

$n - n_0 \mod N = m$

$n - n_0 = LN + m$

$n = n_0 + LN + m.$

$$\tilde{X}(k) = \sum_{n=0}^{N-1} x(n-n_0) W_N^{kn}$$

$$= \sum_{m=0}^{N-1} x(m) W_N^{k(n_0+LN+m)}$$

$$= \sum_{m=0}^{N-1} x(m) W_N^{kn_0} W_N^{kLN} W_N^{km}$$

$$= W_N^{kn_0} \underbrace{\sum_{m=0}^{N-1} x(m) W_N^{km}}_{X(k)}$$

$$= W_N^{kn_0} X(k)$$

$$\boxed{x[n-n_0] \longleftrightarrow W_N^{kn_0} X(k)}$$

L-61 <u>Discrete Fourier Transform</u> (DFT)

<u>Conjugation</u>:

$x(n) \longleftrightarrow X(k)$

$x^*(n) \longleftrightarrow X^*(-k)_{\bmod N}$

<u>Frequency shift property</u>

$x(n) \longleftrightarrow X(k)$

$W_N^{k_0 n} x(n) \longleftrightarrow X(k-k_0)$

Duality:

$$x(n) \longleftrightarrow X(k)$$

$$X(n) \longleftrightarrow ?$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

Interchange n, k

$$x(k) = \frac{1}{N} \sum_{n=0}^{N-1} X(n) W_N^{-nk}.$$

$$x(-k) = \frac{1}{N} \sum_{n=0}^{N-1} X(n) W_N^{nk}$$

$$N x(-k) = \underbrace{\sum_{n=0}^{N-1} X(n) W_N^{nk}}_{\text{DFT of } X(n)}$$

$$\boxed{X(n) \longleftrightarrow N x(-k)_{\bmod N}}$$

Circular convolution

$$x_1(n) \otimes x_2(n) = \sum_{i=0}^{N-1} x(i)\, x_2(n-i)_{\bmod N}$$

$$x_1(n) \otimes x_2(n) \longleftrightarrow ?$$

$$\sum_{n=0}^{N-1} \sum_{i=0}^{N-1} x_1(i)\, x_2(n-i)\, W_N^{kn}$$

$$= \sum_{i=0}^{N-1} x_1(i) \underbrace{\sum_{i=0}^{N-1} x_2(n-i)\, W_N^{kn}}_{W_N^{ki} X_2(k)}$$

$$= \sum_{i=0}^{N-1} \underbrace{x_1(i) W_N^{ki}}_{X_1(k)} X_2(k)$$

$$= X_1(k) X_2(k)$$

$$\boxed{x_1(n) \circledast x_2(n) \longrightarrow X_1(k) X_2(k)}$$

circular convolution

Multiplication in Time

$$x_1(n) \cdot x_2(n) \longleftrightarrow \frac{1}{N} X_1(k) \circledast X_2(k)$$

Parseval's Relation:

$$x_2(n) = x_1^*(n)$$

$$|x_1(n)|^2 \longleftrightarrow \frac{1}{N} X_1(k) \circledast X_1^*(-k)$$

$$= \frac{1}{N} \sum_{i=0}^{N-1} X_1(i) X_1^*(i-k)$$

$$\underbrace{\sum_{n=0}^{N-1} |x_1(n)|^2 W_N^{kn}}_{k^{th} \text{ DFT coefficient of } |x_1(n)|}$$

$$= \frac{1}{N} \sum_{i=0}^{N-1} X_1(i) X_1^*(i-k)$$

set $k=0$

$$\boxed{\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{i=0}^{N-1} |X_1(i)^2|}$$

Parseval's Relation for DFT

Example problems for Fourier analysis of Discrete Time signals

① Consider periodic signal N_0 = even

$$x(n) = \begin{cases} 1, & 0 \le n \le N_0/2 - 1 \\ 0, & \frac{N_0}{2} \le n \le N_0 - 1 \end{cases}$$

Description for a single period.

DFT of $x(n)$ = ?

$$\Omega_0 = 2\pi/N_0$$

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\Omega_0 k n}$$

$$= \frac{1}{N_0} \sum_{n=0}^{\frac{N_0}{2}-1} e^{-j\Omega_0 k n}$$

$$= \frac{1}{N_0} \frac{1 - e^{-j\Omega_0 k N_0/2}}{1 - e^{-j\Omega_0 k}}$$

$\Omega_0 = 2\pi/N_0$

$\Omega_0 N_0 = 2\pi$

$$c_k = \frac{1}{N_0} \frac{1 - e^{-j\pi k}}{1 - e^{-j\Omega_0 k}}$$

$$= \frac{1}{N_0} \frac{e^{-j\pi k/2}}{e^{-j\Omega_0 k/2}} \frac{e^{-j\pi k/2} - e^{-j\pi k/2}}{e^{+j\Omega_0 k/2} - e^{-j\Omega_0 k/2}}$$

$$\boxed{c_k = \frac{1}{N_0} e^{-jk/2(\pi - \Omega_0)} \frac{\sin(k\pi/2)}{\sin\left(\frac{\Omega_0 k}{2}\right)}}$$

DFS coefficient of original periodic signal $x(n)$.

L-62 - Example problems: for fourier analysis of DT signals. (DFS)

② Fourier series of:

$$x(n) = \cos\left(\frac{\pi}{2} n\right) + \sin\left(\pi/3\, n\right)$$

$$\cancel{\pi}/2\, N_1 = 2\cancel{\pi} \Rightarrow \boxed{N_1 = 4}$$

$$\cancel{\pi}/3\, N_2 = 2\cancel{\pi} \Rightarrow \boxed{N_2 = 6}$$

N_0 = Period of $x(n)$

= LCM(4, 6)

= 12.

$$\cos\left(\pi/2\, n\right) + \sin\left(\pi/3\, n\right)$$

$$= \frac{e^{j\pi/2 n} + e^{-j\pi/2 n}}{2} + \frac{e^{j\pi/3 n} - e^{-j\pi/3 n}}{2j}$$

$\Omega_0 = \frac{2\pi}{N_0} = \frac{2\pi}{12} = \pi/6$

$\pi/2 = 3\Omega_0$

$\pi/3 = 2\Omega_0$

$$= \left[\underbrace{\frac{1}{2} e^{j3\Omega_0 n}}_{c_3} + \underbrace{\frac{1}{2} e^{-j3\Omega_0 n}}_{c_{-3}}\right] + \left[\underbrace{\frac{1}{2j} e^{j2\Omega_0 n}}_{c_2} - \underbrace{\frac{1}{2j} e^{-j2\Omega_0 n}}_{c_{-2}}\right]$$

$$\Rightarrow c_{-3} = c_{-3+12} = c_9$$

$$c_{-2} = c_{-2+12} = c_{10}$$

$$c_3 = 1/2, \quad c_9 = 1/2, \quad c_2 = 1/2j, \quad c_{10} = -1/2j$$

DFS $x(n)$

$$= \frac{1}{2j} e^{j2\Omega_0 n} + \frac{1}{2} e^{j3\Omega_0 n} + \frac{1}{2} e^{j9\Omega_0 n} - \frac{1}{2j} e^{j10\Omega_0 n}$$

DFS Representation of signal $x(n)$.

③ Let $x(n)$ be a real periodic sequence

$x(n)$ = Real + Periodic

$c_k = a_k + j b_k$

DFS coefficient

Trigonometric Discrete Fourier series coefficients

N_0 : odd

$\Rightarrow N_0 - 1$: even

$$x[n] = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n}$$

$$= c_0 + \sum_{k=1}^{N_0-1} c_k e^{jk\Omega_0 n} + c_{N_0-k} e^{j(N_0-k)\Omega_0 n}$$

since $x(n)$ = real

$$\Rightarrow c_k = c_{-k}^*$$

$$= c_{N_0-k}^*$$

$$e^{j(N_0-k)\Omega_0 n}$$

$$= \underbrace{e^{j2\pi n}}_{1} e^{-jk\Omega_0 n}$$

$$= e^{-jk\Omega_0 n}.$$

$$x(n) = c_0 + \sum_{k=1}^{\frac{N_0-1}{2}} c_k e^{jk\Omega_0 n} + \underbrace{c_k^* e^{-jk\Omega_0 n}}_{\left(c_k e^{jk\Omega_0 n}\right)^*}$$

$$= c_0 + \sum_{k=1}^{(N_0-1)/2} 2\,\mathrm{Re}\left\{c_k e^{jk\Omega_0 n}\right\}$$

$$\mathrm{Re}\left\{c_k e^{jk\Omega_0 n}\right\} = \mathrm{Re}\left\{(a_k + jb_k)(\cos k\Omega_0 n + j\sin k\Omega_0 n)\right\}$$

$$= \underbrace{a_k \cos(k\Omega_0 n) - b_k \sin(k\Omega_0 n)}_{\text{Trigonometric DFS}}$$

N_0 : odd

$$x(n) = C_0 + 2\sum_{k=1}^{(N_0-1)/2}\left(a_k \cos k\Omega_0 n - b_k \sin k\Omega_0 n\right)$$

$$a_k = \text{Re}\{c_k\} \qquad b_k = \text{Im}\{c_k\}$$

Problems on DTFT

4. $x(n) = u(n) - u(n-N)$

$\rightarrow$ DTFT $= ?$

$$\Rightarrow x(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & o/w \end{cases}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$= \sum_{n=0}^{N-1} e^{-j\Omega n}$$

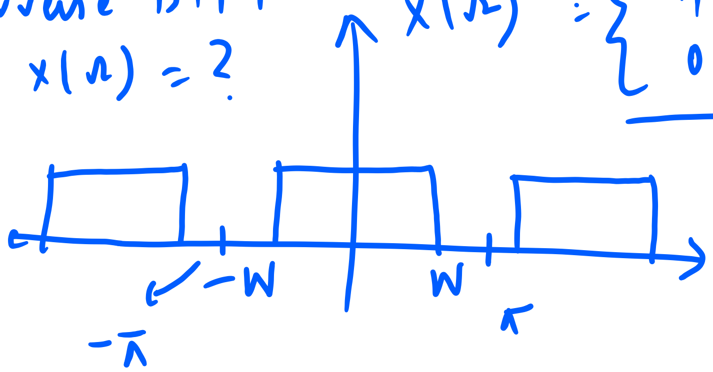
$$= \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}}$$

$$= \frac{e^{-j\Omega N/2}\left[e^{+j\Omega N/2} - e^{-j\Omega N/2}\right]}{e^{-j\Omega/2}\left[e^{j\Omega/2} - e^{-j\Omega/2}\right]}$$

$$= e^{-j\Omega\left(\frac{N-1}{2}\right)} \frac{\sin(\Omega N/2)}{\sin(\Omega/2)}$$

5) Inverse DTFT of $X(\Omega) = ?$

$$X(\Omega) = \begin{cases} 1, & |\Omega| \le W \\ 0, & W < |\Omega| \le \pi \end{cases}$$



For a single period.

Inverse DTFT

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\left. \frac{e^{j\Omega n}}{jn} \right|_{-W}^{W}$$

$$= \frac{1}{2\pi}\left[\frac{e^{jWn} - e^{-jWn}}{jn} \right]$$

$$= \frac{1}{\cancel{2}\pi}\left[\frac{\cancel{2}\cancel{j}\sin Wn}{\cancel{j}n} \right]$$

$$= \frac{\sin Wn}{\pi n}$$

$$\boxed{x(n) = \frac{\sin Wn}{\pi n}}$$

Inverse DTFT of given $X(\Omega)$.