Linear Systems (lec-06)

system $T()$ is linear if it satisfies the following properties

Additivity: $x_1(t), x_2(t)$

$$T(x_1(t)) = y_1(t)$$

$$T(x_2(t)) = y_2(t)$$

$$\Rightarrow T(x_1(t) + x_2(t)) = y_1(t) + y_2(t)$$

True $\forall$ $x_1(t), x_2(t)$

$\Rightarrow$ System satisfies additivity property

HOMOGENEITY

$$T(x(t)) = y(t)$$

$$T(\alpha x(t)) = \alpha y(t)$$

Any scalar quantity

$\Rightarrow$ any number

Homogeneity property

Linear system

$\Rightarrow$ Additivity + homogeneity

Linear system satisfies both properties.

Otherwise it is a non-linear system.

eg $y(t) = \dfrac{dx(t)}{dt}$

Differentiator = Linear.

$y(t) = 2x^2(t)$ $\rightarrow$ Non-linear system

TIME INVARIANT

$$T(x(t)) = y(t)$$

$$\Rightarrow T(x(t-t_0)) = y(t-t_0)$$

For all t_0

for all $x(t)$

Shifted input

Shifted output

Such a system is time invariant
↳ else time varying system.

eg $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

$\tilde{y}(t) = \int_{-\infty}^{t} x(\tau - t_0)\,d\tau$

$\tau - t_0 = \tilde{\tau}$

$d\tau = d\tilde{\tau}$

$= \int_{-\infty}^{t-t_0} x(\tilde{\tau})\,d\tilde{\tau}$

$y(t-t_0)$

Hence, time invariant system, since $x(t-t_0)$ yields o/p $y(t-t_0)$

$y(t) = T(x(t))$ → Time variant system.

$= t \cdot x(t)$

↳ Not time invariant

LINEAR TIME INVARIANT

(LTI) System

↳ systems that are Linear + Time Invariant.

LTI systems.

↳ satisfy linearity and time invariance properties.

Important class of systems that we frequently encounter

STABLE SYSTEMS.

stability of practical systems is important

→ BIBO → Bounded i/p, bounded o/p criterion for stability

$|x(t)| \le c$

$|x[n]| \le c$

constants

→ Input is bounded.

$\Rightarrow |y(T)| = |T(x(t))| \le K$

$|y[n]| = |T\{x[n]\}| \le K$

→ Bounded outputs

$T(x(t)) = y(t)$

$|x(t)| \le C$

$\Rightarrow |y(t)| \le K$

Bounded input

$\Rightarrow$ Bounded output

for every bounded input

BIBO Stable.

eg $y(t) = \int_{t-1}^{t} x(\tau)\, d\tau$

$|x(\tau)| \le c$

$|y(t)| = \left| \int_{t-T}^{t} x(\tau)\, d\tau \right|$

$\le \int_{t-T}^{t} |x(\tau)|\, d\tau \le C$

$\le \int_{t-T}^{t} c\, d\tau$

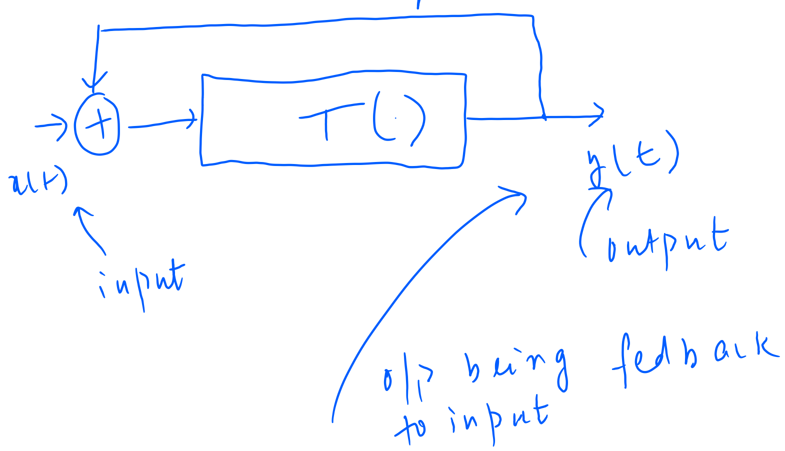
$= \underbrace{CT}_{K}$

$|x(t)| \leq C \Rightarrow |y(t)| \leq K = CT$

$\Rightarrow$ system is BIBO stable

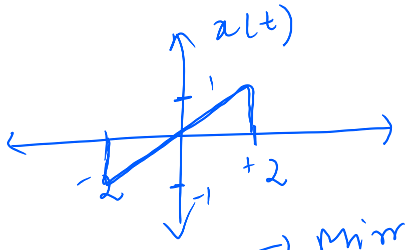
FEEDBACK SYSTEMS

Feedback loop.



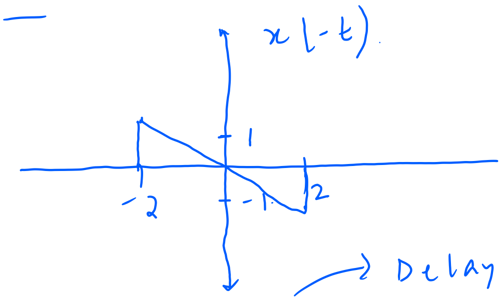
Examples & Problems.
Signal and system properties.

eg. Consider $x(t)$ below.
Plot $x(2-t)$



$\tilde{x}(t) = x(-t)$ → mirror image about 0

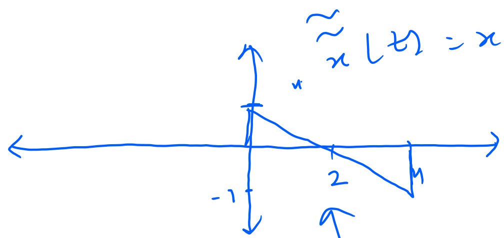
$\tilde{x}(t) = x(-t)$



$\tilde{\tilde{x}}(t) = \tilde{x}(t-2)$ → Delaying $\tilde{x}(t)$ by 2 seconds ie $t_0 = 2$

$= x(-(t-2))$

$= x(2-t)$



solution $= x(2-t)$

eg. $x(t) = e^{\sigma t}$
→ Find even and odd components of $x(t)$

ie Express $x(t) = x_e(t) + x_o(t)$

even odd

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$x_e(-t) = \frac{x(-t) + x(t)}{2}$$

$$= x_e(t)$$

even signal

$$x_e(t) = \frac{e^{\sigma t} + e^{-\sigma t}}{2}$$

$$= \cosh t$$

hyperbolic cosine

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

$$x_o(-t) = \frac{x(-t) - x(t)}{2}$$

odd signal

$$= -x_o(t)$$

$$x_o(t) = \frac{e^{\sigma t} - e^{-\sigma t}}{2}$$

$$= \sinh t$$

$$x(t) = e^{\sigma t} = x_e(t) + x_o(t)$$

$$= \cosh(t) + \sinh(t)$$

Solution.

even component

odd component.

eg : consider $x(t) = e^{j\frac{5\pi}{8}t}$

Sampled with sampling interval $T_s = 2/3$

Is the resulting signal periodic? If so, what is its period?

Solution $x(t) = e^{j 5\pi/8 \, t}$

$T_s = 2/3$

$$\tilde{x}(n) = x(n T_s)$$
$$= x(n \cdot 2/3)$$
$$= e^{j \frac{5\pi}{8} \cdot 2/3 \, n}$$
$$\underbrace{\tilde{x}(n) = e^{j 5\pi/12 \, n}}_{\text{Resulting Discrete Time Signal.}}$$

Periodic if there exist M such that,

$$\tilde{x}(n+M) = \tilde{x}(n)$$

$$\Rightarrow e^{j 5\pi/12 (n+M)} = \underbrace{e^{j 5\pi/12 \, n}}_{\tilde{x}(n)} \cdot \underbrace{e^{j \frac{5\pi}{12} M}}_{1}$$

$$\Rightarrow \underbrace{\frac{5\pi}{12}}\; M = 2k\pi$$

↳ integer Multiple of 2π

$$M = \frac{24 \times k}{5}$$

integer $= 24k$ is divisible by 5

Smallest k for which this holds $k = 5$

$$\Rightarrow M = \frac{24 \times 5}{5} = 24$$

↳ Fundamental period.

$\tilde{x}(n)$ is periodic

Fundamental period

$m = 24$.

Solution.

Lec - 8 Examples

↳ properties & Classification of signals and systems.

eg Consider DT signal

↳ Discrete time signal

$x[n] = \alpha^n u(n)$ → $= 1, n \geq 0$; $= 0, n < 0$

energy signal?

If so, what is its energy?

$$x[n] = \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} (\alpha^n)^2$$

$$= \sum_{n=-\infty}^{\infty} \alpha^{2n}$$

$$= 1 + \alpha^2 + \alpha^4 + \cdots$$

infinite geometric progression.

$$\text{Sum} = \frac{1}{1-\alpha^2} \text{ if } |\alpha| < 1$$

$$= \infty, \text{ if } |\alpha| \geq 1$$

$\Rightarrow \alpha^n u(n) \rightarrow$ energy signal if $|\alpha| < 1$

$$\text{Energy} = \frac{1}{1-\alpha^2}$$

Problem:

evaluate $\delta(at)$, $a>0$

$$\int_{-\infty}^{\infty} x(t)\,\delta(t) = x(t)\Big|_{t=0} = x(0)$$

$$\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt$$

$at = \tau$

$dt = \frac{d\tau}{a}$

$$= \int_{-\infty}^{\infty} x(\tau/a)\cdot\delta(\tau)\frac{d\tau}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau/a)\,\delta(\tau)\,d\tau$$

$$= 1/a\; x(\tau/a)\Big|_{\tau=0}$$

$$= 1/a\; x(0)$$

$$\underbrace{\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \int_{-\infty}^{\infty} x(t)\frac{\delta(t)}{a}\,dt}_{\text{Holds for any arbitrary } x(t)}$$

$$\Rightarrow \boxed{\delta(at) = \frac{1}{a}\delta(t) \quad \text{for } a>0}$$

$$\int_{-\infty}^{\infty}\delta(at)\,dt = \int_{-\infty}^{\infty}\frac{1}{a}\delta(t) = 1/a,\; a>0$$

Problem evaluate

$$\int_{-\infty}^{\infty}\phi(t)\,\delta'(t)\,dt$$

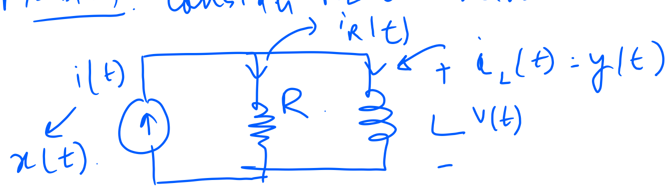
$\phi(\infty) = \phi(-\infty) = 0$

$$\int_{-\infty}^{\infty}\phi(t)\,\delta'(t)\,dt = \phi(t)\,\delta(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\phi'(t)\,\delta(t)\,dt$$

$$= 0 - \phi'(t)\Big|_{t=0} = -\phi'(0)$$

PROPERTIES AND CLASSIFICATION OF SYSTEMS

Problem: Consider RL ckt below.



find i/p o/p relation for RL circuit.

$$i(t) = i_R(t) + i_L(t)$$

$$= \frac{v(t)}{R} + i_L(t)$$

Ohm's law.

From property of inductor.

$$v(t) = L\frac{di_L(t)}{dt}$$

$$\Rightarrow i(t) = \frac{L}{R}\frac{di_L(t)}{dt} + i_L(t)$$

$$x(t) = \frac{L}{R}\frac{dy(t)}{dt} + y(t)$$

$$\boxed{L/R\,\frac{dy(t)}{dt} + y(t) = x(t)}$$

input output relation expressed as a differential equation.

Problem $T(x(t)) = x(t)\cos(2\pi f_c t)$.

$y(t)$

Modulation.

very important system in a larger communication system.

Carrier frequency.

Is this an LTI system?

LTI systems problems.

$y(t) = x(t)\cos(2\pi f_c t) \rightarrow$ LTI?

Linearity?

additivity $T(x_1(t)) = \underbrace{x_1(t)\cos(2\pi f_c t)}_{y_1(t)}$

$T(x_2(t)) = \underbrace{x_2(t)\cos(2\pi f_c t)}_{y_2(t)}$

$T(x_1(t) + x_2(t)) = (x_1(t) + x_2(t))\cos 2\pi f_c t$

$= \underbrace{x_1(t)\cos 2\pi f_c t}_{y_1(t)} + \underbrace{x_2(t)\cos 2\pi f_c t}_{y_2(t)}$

$= y_1(t) + y_2(t)$

$= T(x_1(t)) + T(x_2(t))$

↳ Therefore satisfies additivity property.

Homogeneity:

$T(x(t)) = \underbrace{x(t)\cos(2\pi f_c t)}_{y(t)}$

$\Rightarrow T(\alpha x(t)) = \alpha x(t) \cos 2\pi f_c t$

$= \alpha y(t)$

$= \alpha T(x(t))$

So, system satisfies homogeneity.

Since, it satisfies additivity + homogeneity, system is LINEAR.

Time Invariance?

$T(x(t)) = x(t)\cos(2\pi f_c t)$

$\swarrow$

$y(t)$.

$T(x(t-t_0)) = x(t-t_0)\cos(2\pi f_c t)$

$\neq y(t-t_0)$

$= x(t-t_0)$

$\cos(2\pi f_c(t-t_0))$.

$\underbrace{T(x(t-t_0)) \neq y(t-t_0).}_{}$

So not Time Invariant.

Hence, the modulation is NOT an LTI system.

eg. Let $T(\cdot)$ represent LTI system, show

$$T(e^{j2\pi f_0 t}) = c e^{j2\pi f_0 t}$$

input = complex sinusoid.

output is complex sinusoid scaled by c

c = constant.

$$T(e^{j2\pi f_0 t}) = y(t)$$

Since LTI, $\Rightarrow T(e^{j2\pi f(t-t_0)}) = y(t-t_0)$.

$$= T(e^{j2\pi f t} e^{-j2\pi f_0 t_0}) = y(t-t_0).$$

scaling factor α.

$$\Rightarrow e^{-j2\pi f_0 t_0} T(e^{j2\pi f_0 t}) = y(t-t_0).$$

$y(t)$.

$$\Rightarrow y(t) e^{-j2\pi f_0 t_0} = y(t-t_0).$$

$\forall t, t_0$.

Set $t=0$. and $t_0 = -t$.

$$\Rightarrow y(0) e^{-j2\pi f_0 t} = y(t).$$

C.

$$\Rightarrow y(t) = c e^{j2\pi f_0 t}$$

if input $= e^{j2\pi f_0 t}$

o/p = constant × input

$e^{j2\pi f_0 t}$ is an eigen function of LTI system.

Analysis of LTI systems.

linear time invariant system.

Deeper look at properties and analysis of LTI systems.

IMPULSE response

$$\delta(t) \rightarrow \boxed{T(\cdot)} \rightarrow h(t)$$

$$h(t) = T(\delta(t))$$

impulse response

output of LTI system corresponding to impulse.

impulse

Response of LTI system to arbitrary input $x(t)$.

$$x(t) \rightarrow \boxed{T(\cdot)} \rightarrow$$

$h(t)$ = impulse response

$$T(\delta(t)) = h(t)$$

Sifting property

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

o/p of LTI system $y(t)$

$$y(t) = T(x(t))$$

$$= T\left(\int_{-\infty}^{\infty} \underbrace{x(\tau)}_{\text{weight}}\ \underbrace{\delta(t-\tau)}_{\text{signals}}\ d\tau\right)$$

From linearity

$$\simeq \sum x(\tau)\,\delta(t-\tau)$$

o/p of linear combination
= linear combination of o/ps.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, T\big(\delta(t-\tau)\big)\, d\tau.$$

From Time Invariance.

since $T(\delta(t)) = h(t)$

$T(\delta(t-\tau)) = h(t-\tau)$

follows from time invariance.

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau}$$

convolution

Describes o/p $y(t)$ for any arbitrary input $x(t)$.

Impulse response $h(t)$ completely determines output $y(t)$ for any arbitrary input $x(t)$ for an LTI system.

$x(t)$ convolved with $h(t)$.

$$\boxed{y(t) = x(t) * h(t)}$$

CONVOLUTION operation

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

PROPERTIES OF CONVOLUTION INTEGRAL

① Commutative

$\Rightarrow x(t) * h(t) = h(t) * x(t).$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau .$$

$$t-\tau = \tilde{\tau}$$
$$-d\tau = d\tilde{\tau}$$

$$= \int_{\infty}^{-\infty} x(t-\tilde{\tau})\, h(\tilde{\tau})(-d\tilde{\tau})$$

change order of limits

$$= \int_{-\infty}^{\infty} x(t-\tilde{\tau})\, h(\tilde{\tau})\, d\tilde{\tau}$$

$$= \int_{-\infty}^{\infty} h(\tilde{\tau})\, x(t-\tilde{\tau})\, d\tilde{\tau}$$

$$= h(t) * x(t)$$

② Associativity

Convolution is associative

$$\Rightarrow (x(t) * h_1(t)) * h_2(t)$$
$$= x(t) * (h_1(t) * h_2(t))$$

③ Distributive

Convolution is Distributive

$$x(t) * (h_1(t) + h_2(t))$$
$$= x(t) * h_1(t) + x(t) * h_2(t)$$

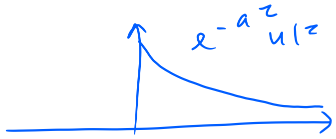
Distributive property

GRAPHICAL REPRESENTATION OF CONVOLUTION

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

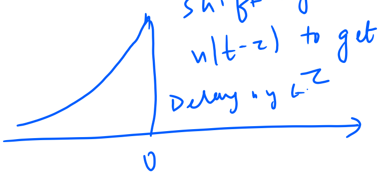
$$h(t) = e^{-at}\, u(t)$$
$$h(\tau) = e^{-a\tau}\, u(\tau).$$

$e^{-a\tau}u(\tau)$

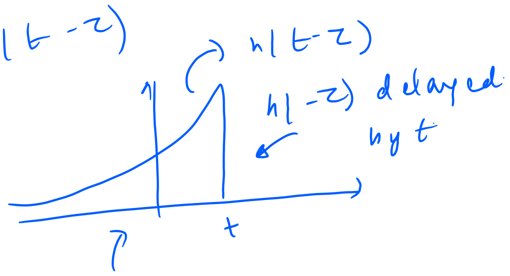


$$h(-\tau) = e^{a\tau}u(-\tau)$$

shift by
$h(t-\tau)$ to get
Delaying by t



$h(t-\tau)$



→ Multiply by $x(t)$ and followed by integration

→ Repeat for every value of t.

lec - 11

Properties and analysis of LTI systems.

Memoryless Systems.

$y(t) = T(x(t))$

$= k\,x(t)$.

Depends only on input at current time instant

→ $h(t) = k\,\delta(t)$.

(impulse response for a memoryless system.

$h(t) = 0$ if $t \neq 0$

if $h(t) \neq 0$ for $t \neq 0$,

then system is not Memoryless.

Causality

(Causal system.

for any LTI system.

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

For a Causal system,

$y(t)$ should depend only on $\underbrace{x(t-\tau) \text{ for } \tau \geq 0}_{\text{only past values of } x(t)}$

→ $h(\tau)$ Must be non zero only for $\tau \geq 0$.

→ $h(\tau) \neq 0$ only for $\tau \geq 0$.

$\boxed{h(\tau) = 0 \text{ for } \tau < 0}$

property of causal LTI system.

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau.$$

$$= \int_{0}^{\infty} x(t-\tau) \cdot h(\tau)\, d\tau.$$ since $h(\tau) = 0$ for $\tau < 0$.

Non causality

$h(\tau) \neq 0$ for $\tau < 0$

anticausal

$h(\tau) = 0$ for $\tau > 0$

$y(t)$ depends only future values of $x(t)$

STABILITY

BIBO : Bounded i/p Bounded o/p

important criterion for stability

BIBO stable

if $|x(t)| \leq c$ some finite value

$\Rightarrow |T(x(t))| \leq K$

$\forall t$

$|y(t)| \leq K$.

Bounded o/p.

System is BIBO stable if,

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

finite

Impulse response is absolutely integrable

Assume Bounded input

$$|x(t)| \le c$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \alpha$$

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau \right|.$$

$$\le \int_{-\infty}^{\infty} \underbrace{|x(t-\tau)|}_{\le c \;\to\; \text{Bounded input property}} |h(\tau)|\, d\tau$$

$$\le c \int_{-\infty}^{\infty} \underbrace{|h(t)|\, dt}_{\alpha}$$

$$= c\alpha$$

$$\underbrace{|y(t)|}_{\text{Bounded o/p}} \le \underbrace{c\alpha}_{K}$$

$$\boxed{\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty}$$

condition for LTI system to be stable

EIGEN FUNCTIONS OF LTI SYSTEMS

Consider $x(t) = e^{\alpha t}$

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{\alpha(t-\tau)}\, h(\tau)\, d\tau$$

$$= e^{\alpha t} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-\alpha \tau}\, d\tau}_{H(\alpha)}$$

$$\underbrace{T(e^{\alpha t})}_{\text{o/p of } e^{\alpha t}} = \underbrace{H(\alpha)}_{\text{Scaling factor.}} \underbrace{e^{\alpha t}}_{\text{input signal.}}$$

eigen value

o/p is scaled version of i/p

$A \bar{x} = \lambda \bar{x}$

A → Matrix

$\bar{x}$ → vector, eigenvector

λ → eigenvalue

$\lambda \bar{x}$ → scaled version of eigenvector

Hence $e^{\alpha t}$
= eigen fxn of any LTI system.

$$H(\alpha) = \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-\alpha \tau} d\tau}_{\text{Transform of impulse response } h(\tau)}$$

Transform play a very important role in understanding and analysis of LTI systems

⇒ Eigen functions are very important for the understanding of LTI systems.