$\boxed{\text{Week 12}}$ TA session Yashwanth L

I. Summary of Week 12's lecture. (Filter Delays and realizations structures)

(1) Phase delay.

$$\underset{x(t)}{A\cos\omega_0 t} \longrightarrow \boxed{LTI}^{\,h(t)} \longrightarrow A\,|H(\omega_0)|\cos(\omega_0 t + \underbrace{\phi(\omega_0)}_{\text{phase offset}})$$

Phase delay $\tau_p := -\dfrac{\phi(\omega_0)}{\omega_0}$.

delay of carrier input to LTI system.

2) Group delay : In addition to a carrier, if we pass a 'low frequency tone' or message

$$A\underbrace{\cos(\omega_m t)}_{\text{message}}\underbrace{\cos(\omega_0 t)}_{\text{carrier}}$$

$$A\,|H(\omega_0)|\cos\big(\omega_m(t-\phi'(\omega_0))\big) \times \cos\Big(\omega_0\Big(t+\underbrace{\frac{\phi(\omega_0)}{\omega_0}}_{\text{Phase delay}}\Big)\Big)$$

$$x(t) \longrightarrow \boxed{LT I} \longrightarrow y(t)$$

$\omega_m << \omega_0$.

group delay

$$\phi'(\omega_0) \triangleq -\left.\frac{d\phi(\omega)}{d\omega}\right|_{\omega=\omega_0}$$

$\phi(\omega)$ is the phase response of the system.

→ Message is delayed by group delay

→ carrier is delayed by phase delay.

→ if $\phi(w) = kw$ ie linear phase, then group delay = phase delay.

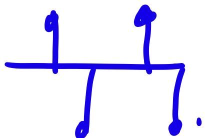
② Realization of IIR filter.

(i) direct form I realization.

IIR → infinite impulse response filter.

$h(n)$ is not time limited.

eg $h(n) = (-1)^n$



$$H(z) = \frac{P(z)}{D(z)}.$$

$$\text{let } H(z) = \frac{p_0 + p_1 z^{-1} + \cdots + p_3 z^{-3}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}$$

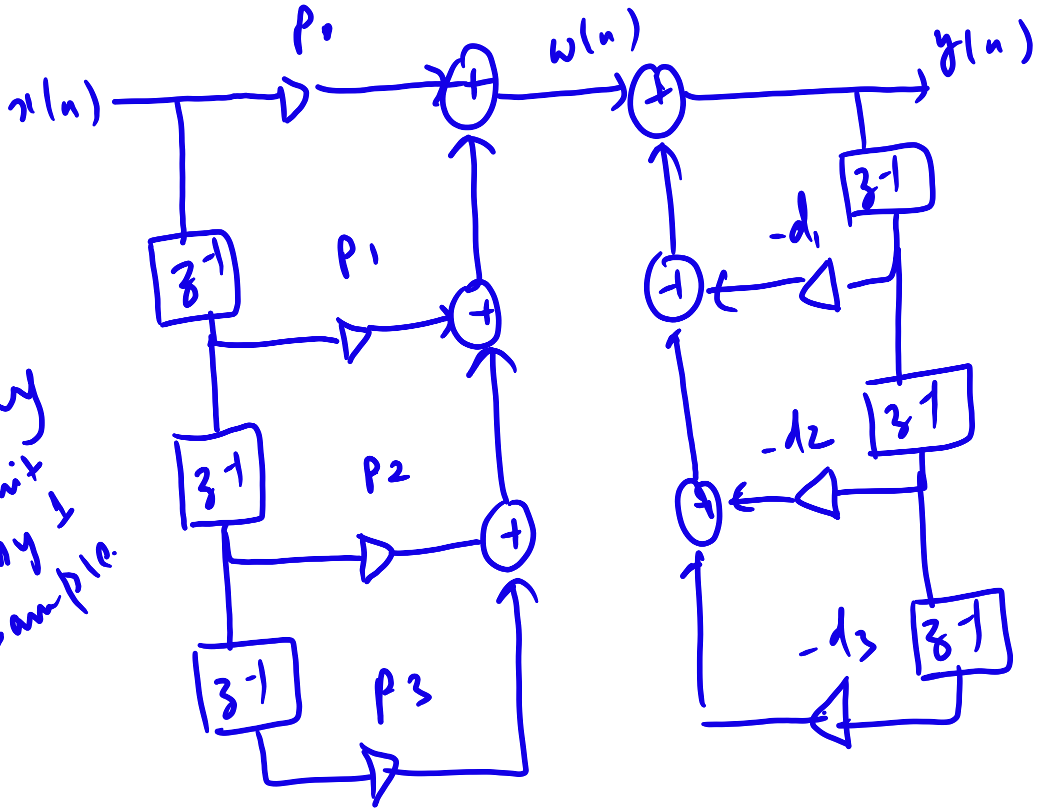
$$= \frac{P(z)}{D(z)}$$

$$H(z) = P(z) \cdot \frac{1}{D(z)}.$$

$$W(z) = X(z)\, P(z)$$

$$Y(z) = W(z)\, \frac{1}{D(z)}$$

delay unit by 1 sample



Direct form I realization

$$X(z) \longrightarrow \boxed{P(z)} \longrightarrow \boxed{\tfrac{1}{D(z)}} \longrightarrow Y(z).$$

; $Q(z) = 1/D(z)$

(ie) $y(n) = x(n) * p(n) * q(n)$

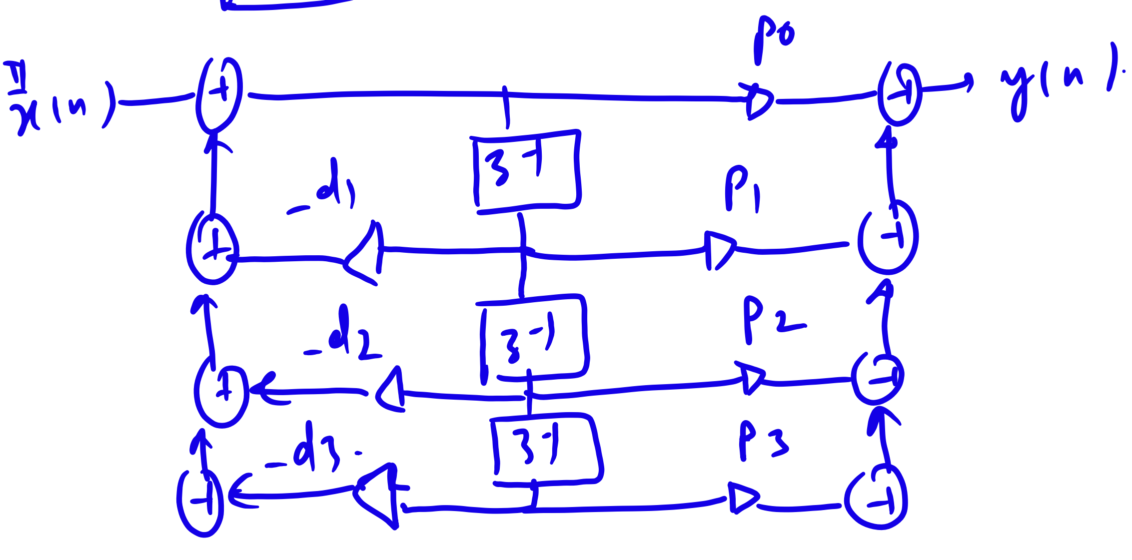
$= x(n) * q(n) * p(n)$

[convolution is commutative]

$$\Rightarrow \quad Y(z) = X(z) \cdot \frac{1}{D(z)} \cdot P(z).$$

$$\Rightarrow \quad X(z) \longrightarrow \boxed{\tfrac{1}{D(z)}} \longrightarrow \boxed{P(z)} \longrightarrow Y(z)$$

DFT II



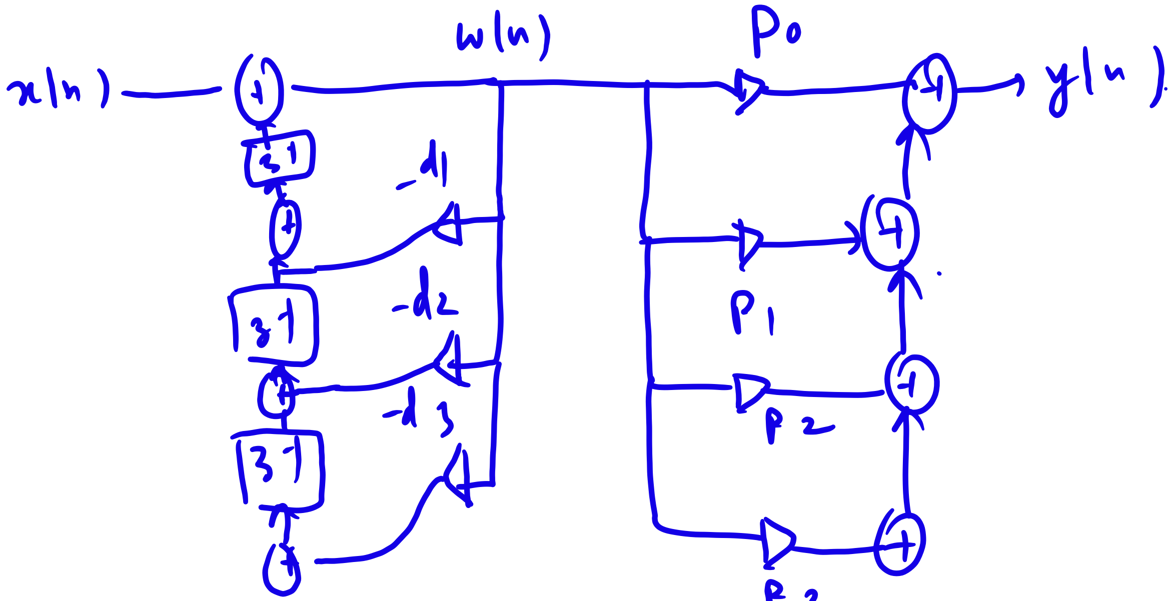
advantage over DF-I →
It requires only half the # delay units

Transpose forms.

$$Y(z) = X(z)H(z)$$
$$= X(z) \cdot \underbrace{\frac{1}{D(z)}}_{W(z)} P(z)$$

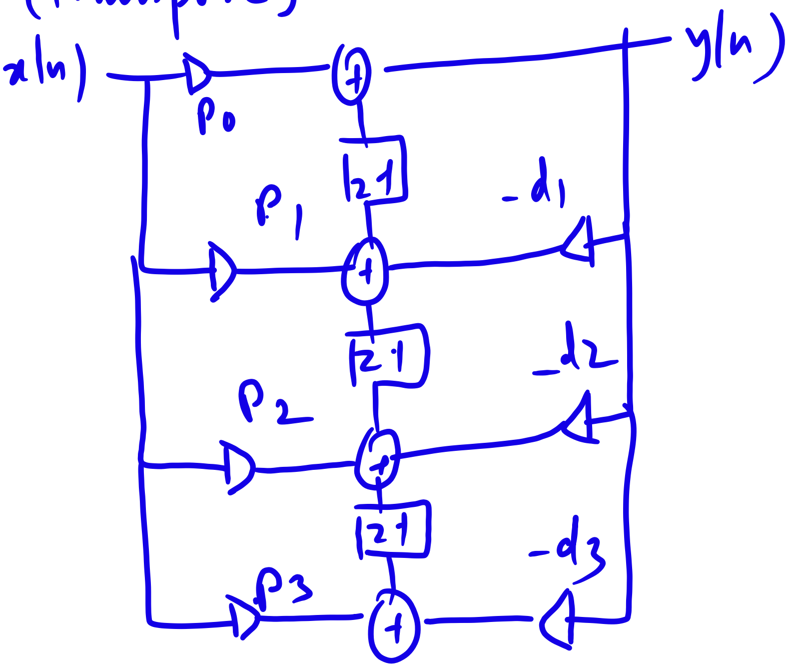
$\rightarrow X(z) = W(z) D(z)$

& $Y(z) = W(z) P(z)$



D.F. II (transpose)

→ DF2 - transpose

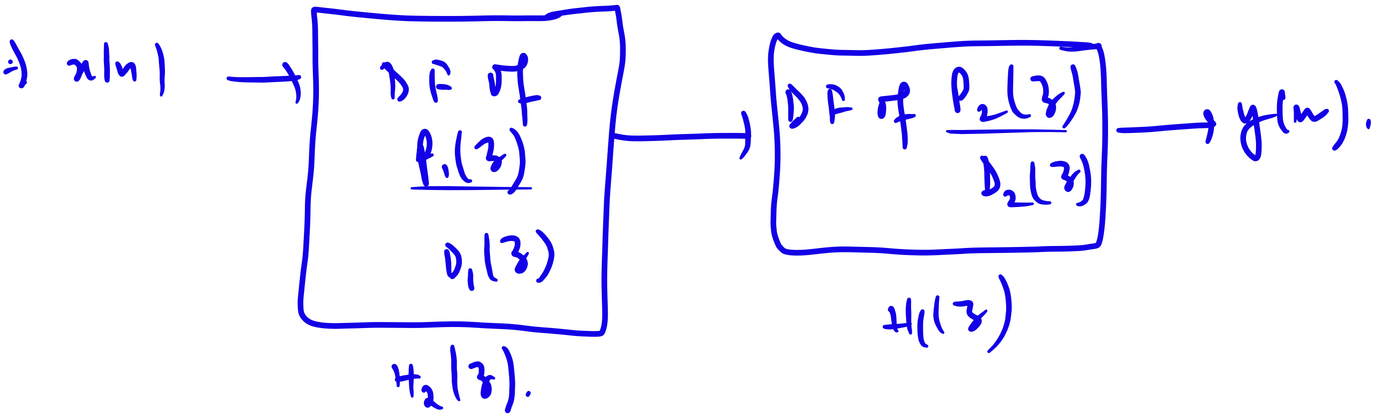


→ Cascade form

$$H(z) = \frac{P(z)}{D(z)} = \frac{P_1(z)}{D_1(z)} \cdot \frac{P_2(z)}{D_2(z)}$$

either 1st or 2nd order functions.

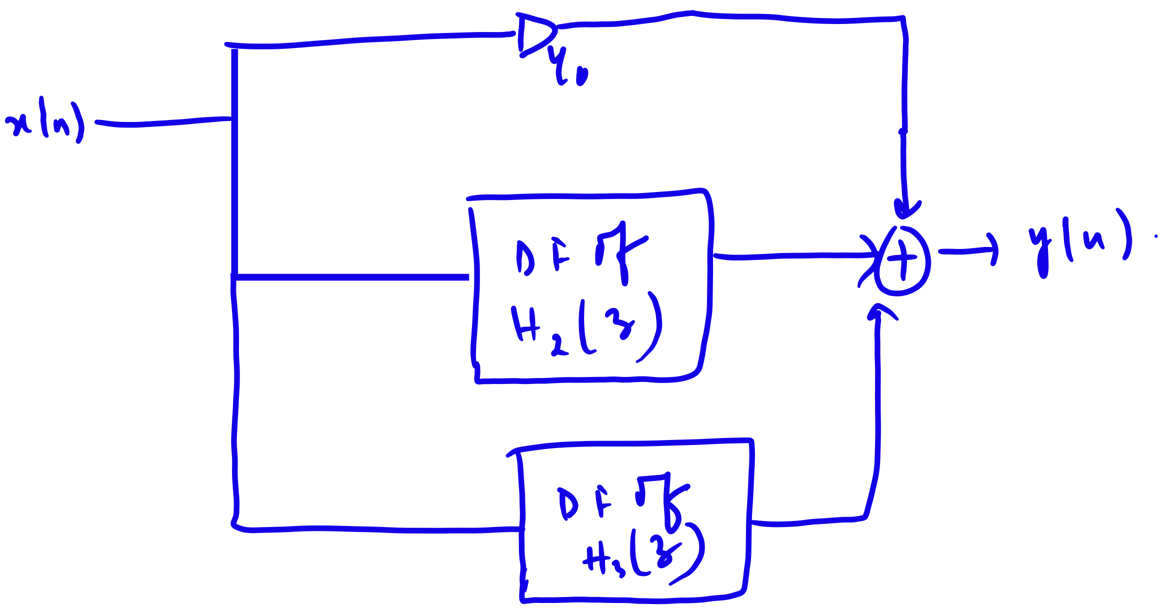
$$\Rightarrow Y(z) = \underbrace{\frac{P_2(z)}{D_2(z)}}_{H_1(z)} \cdot \underbrace{\frac{P_1(z)}{D_1(z)}}_{H_2(z)} \cdot X(z)$$



→ Parallel form I

Do partial fraction expansion of $H(z) = \frac{P(z)}{D(z)}$.

$$H(z) = \underbrace{\gamma_0}_{H_1(z)} + \underbrace{\frac{\gamma_{01}}{1+\alpha_{11} z^{-1}}}_{H_2(z)} + \underbrace{\frac{\gamma_{02} + \gamma_{12} z^{-1}}{1+\alpha_{12} z^{-1} + \alpha_{22} z^{-2}}}_{H_3(z)}$$



Parallel form 2

→ same realization as P.F I except that consider implementing for partial fraction expressed in "z" instead of "z^{-1}"

Part II Tutorials.

① The input to a channel is a band pass signal. It is obtained by linearly modulating sinusoidal carrier with a single tone signal. The o/p obtained by passing it through an LTI system is

$$y(t) = \frac{1}{100}\cos(100t - 10^{-6})\cos(10^6 t - 1.56).$$

Find the group and phase delay.

Sol^n Recall the general o/p :-

$$y(t) = A|H(w_0)|\cos(w_m t - \phi_m)\cos(w_0 t - \phi_0).$$

w_m ↑ msg freq

w_0 ↑ carrier freq.

$w_m << w_0$ ∴ $w_m = 100$ (msg freq)

$w_0 = 10^6$ (carrier freq).

$\phi_m = 10^{-6}$

$\phi_0 = 1.56$

$$y(t) = \frac{1}{100}\cos(100(t - 10^{-8}))\cos(10^6(t - 1.56\times10^{-6}))$$

$10^{-8} \rightarrow$ G.D

$1.56 \times 10^{-6} \rightarrow$ P.D.

$\therefore$ Group delay $= 10^{-8} = \tau_g$

Phase delay $\Rightarrow \tau_p = 1.56 \times 10^{-6}$ s.

② Find the group & phase delay for the following systems

(a) $h(t) = \delta(t - t_0)$

(b) $h(t) = e^{-t/RC} u(t)$.

Sol^n a) $h(t) = \delta(t - t_0)$.

$$H(\omega) = e^{-j\omega t_0}$$ (Time shift property).

$$\angle H(\omega) = \phi(\omega) = -\omega t_0.$$

(i) $$\tau_g(\omega_0) = -\frac{d\phi(\omega)}{d\omega}\bigg|_{\omega = \omega_0}.$$

$$= +\frac{d(\omega t_0)}{d\omega}\bigg|_{\omega = \omega_0}$$

$$= t_0 \bigg|_{\omega = \omega_0}$$

$$= t_0$$

(ii) $$\tau_p(\omega_0) = -\frac{\phi(\omega_0)}{\omega_0} = -\frac{(-\omega_0 t_0)}{\omega_0} = t_0.$$

$$e^{-at} u(t) \xleftrightarrow{FT} \frac{1}{j\omega + a}.$$

(b) $h(t) = e^{-t/RC} u(t)$.

$$H(\omega) = \frac{1}{j\omega + 1/RC}$$

$$\therefore \angle H(w) = \angle 1 - \left(\angle jw + 1/RC\right) = 0 - \tan^{-1}\left(\frac{w}{1/RC}\right)$$

$$\angle H(w) = -\tan^{-1}(wRC)$$
$$= \phi(w)$$

$$\therefore \tau_p(w_0) = \frac{-\phi(w_0)}{w_0} = \frac{\tan^{-1}(w_0 RC)}{w_0}.$$

$$\tau_g(w_0) = -\left.\frac{d\phi(w)}{dw}\right|_{w=w_0}$$
$$= \left.\frac{-d}{dw}\left(-\tan^{-1}(wRC)\right)\right|_{w=w_0}$$
$$= \left.\frac{d}{dw}\left(\tan^{-1}(wRC)\right)\right|_{w=w_0}$$
$$= \left.\frac{RC}{1+w^2R^2C^2}\right|_{w=w_0}$$

$$\boxed{\tau_g(w_0) = \frac{RC}{1+w_0^2R^2C^2}} \leftarrow \text{group delay.}$$

3 let $H(z) = \dfrac{1+2z^{-1}}{1-3z^{-1}+2z^{-2}}$

Find direct form I and II realizations.

Sol^n DF-I

$$H(z) = \frac{1+2z^{-1}}{1-3z^{-1}+2z^{-2}} = \frac{P(z)}{D(z)}.$$

DF I.

$$X(z) \to \boxed{P(z)} \xrightarrow{W(z)} \boxed{1/D(z)} \to Y(z)$$

$$\boxed{X(z)\cdot P(z) = W(z)} \longrightarrow \boxed{W(z)\cdot \frac{1}{D(z)} = Y(z)}$$

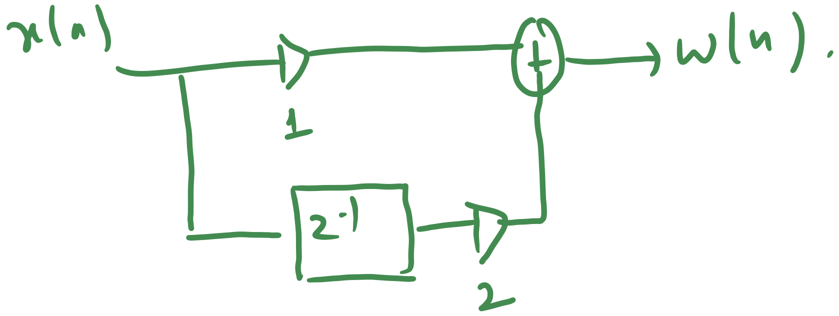
A B

A :

$$X(z)\left(1+2z^{-1}\right) = W(z)$$

$$\Rightarrow X(z) + 2X(z)z^{-1} = W(z)$$

$$\Rightarrow x(n) + 2x(n-1) = w(n).$$



B

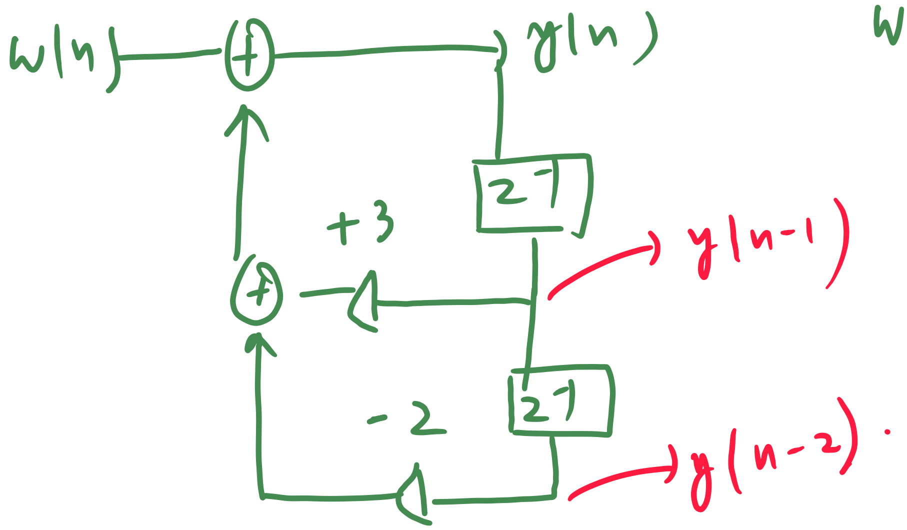
$$W(z)\cdot \frac{1}{D(z)}$$

$$\Rightarrow W(z)\left(\frac{1}{1-3z^{-1}+2z^{-2}}\right) = Y(z)$$

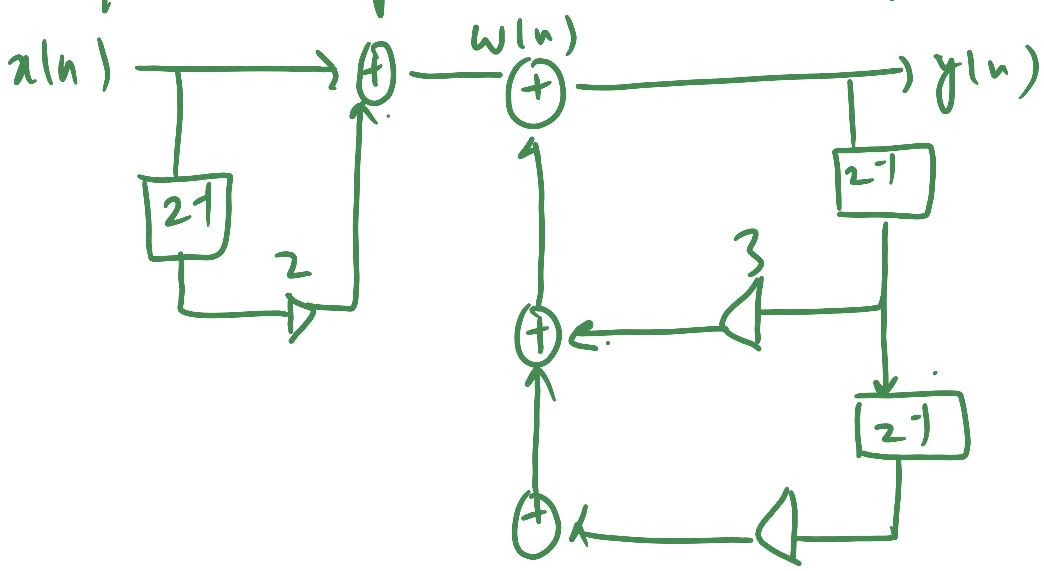
$$\Rightarrow W(z) = Y(z) - 3z^{-1}Y(z) + 2z^{-2}Y(z)$$

$$w(n) = y(n) - 3y(n-1) + 2y(n-2)$$

$$W(z) + 3z^{-1}Y(z) - 2z^{-2}Y(z) = Y(z)$$

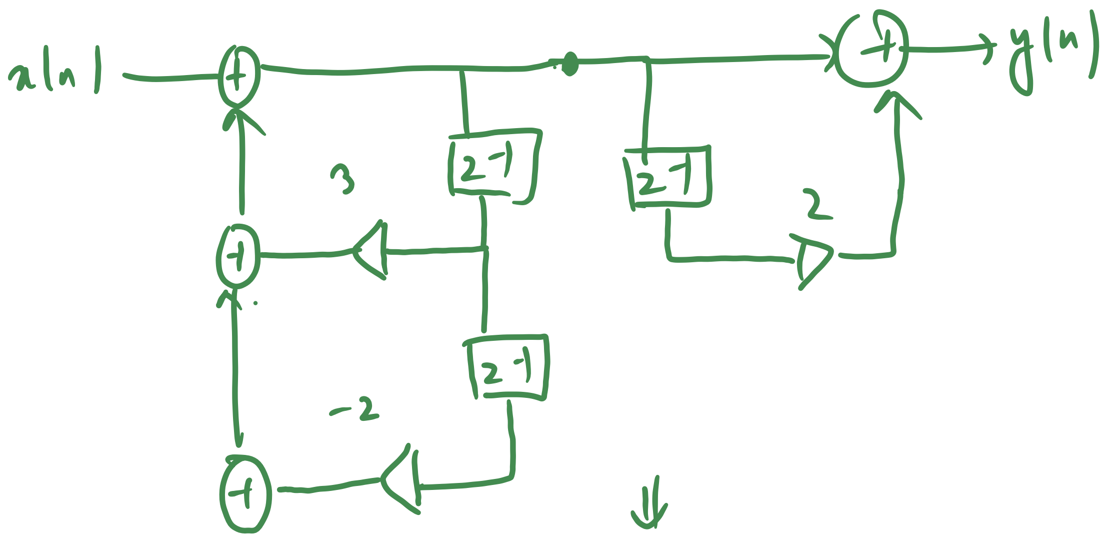


By connecting both. (DF-I)

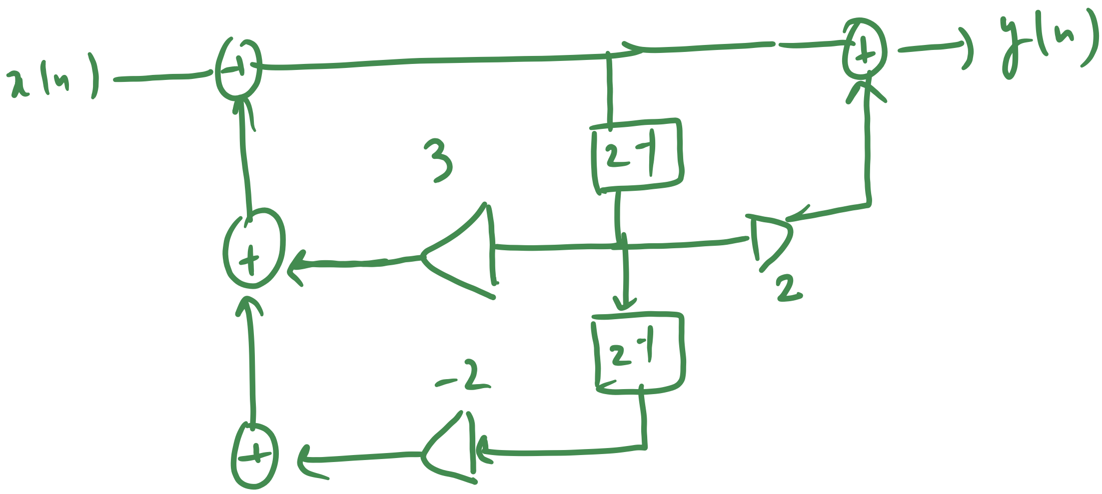


To obtain DF-2:

interchange $\boxed{B}$ & $\boxed{A}$ blocks.



⇓



(4) For the same system in Q-3; obtain the cascaded

realization of the system

Sol^n $H(z) = \dfrac{1+2z^{-1}}{1-3z^{-1}+2z^{-2}}$

As much as possible, realise with 1st order cascaded realization.

$$= \frac{1+2z^{-1}}{1-z^{-1}-2z^{-1}+2z^{-2}}$$

$$= \frac{1+2z^{-1}}{(1-z^{-1})-2z^{-1}(1-z^{-1})}$$

$$= \frac{(1+2z^{-1})}{(1-z^{-1})(1-2z^{-1})}$$

$$H(z) = \underbrace{\frac{1+2z^{-1}}{(1-2z^{-1})}}_{H_1(z)} \underbrace{\left(\frac{1}{1-z^{-1}}\right)}_{H_2(z)}.$$

Implementation of $H_1(z)$

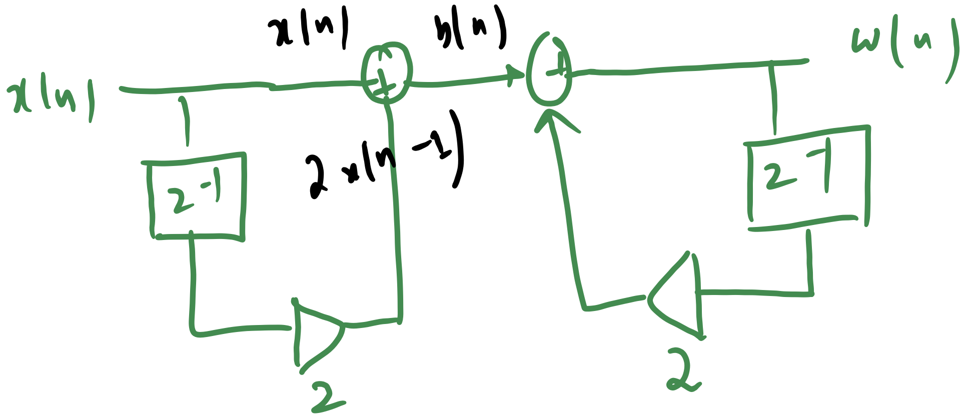
$H_1(z) = \dfrac{1+2z^{-1}}{1-2z^{-1}}$; $w(n) = x(n) * h_1(n)$

$\Rightarrow W(z) = X(z)\, H(z)$

$$\Rightarrow W(z) = X(z)\, \frac{1+2z^{-1}}{1-2z^{-1}}$$

$$(1-2z^{-1})\, W(z) = (1+2z^{-1})\, X(z)$$

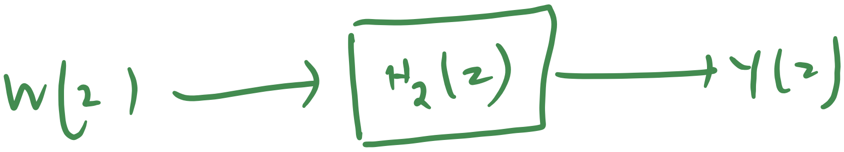
$$\Rightarrow \underbrace{W(z) - 2W(z)z^{-1}}_{B(z)} = \underbrace{X(z) + 2X(z)z^{-1}}$$



$$b(n) + 2w(n-1) = w(n).$$

$$\Rightarrow b(n) = w(n) - 2w(n-1)$$

$$B(z) = W(z) - 2W(z)z^{-1}$$

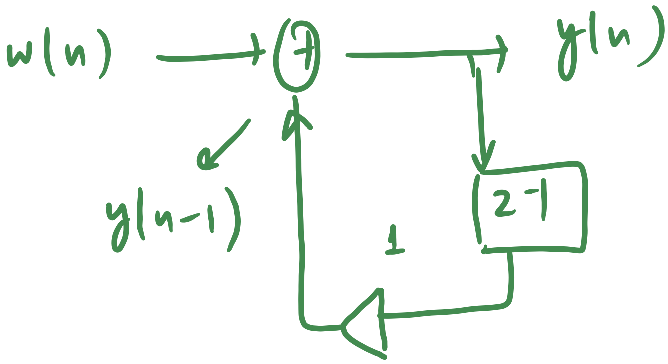


$$Y(z) = W(z)\,H_2(z)$$

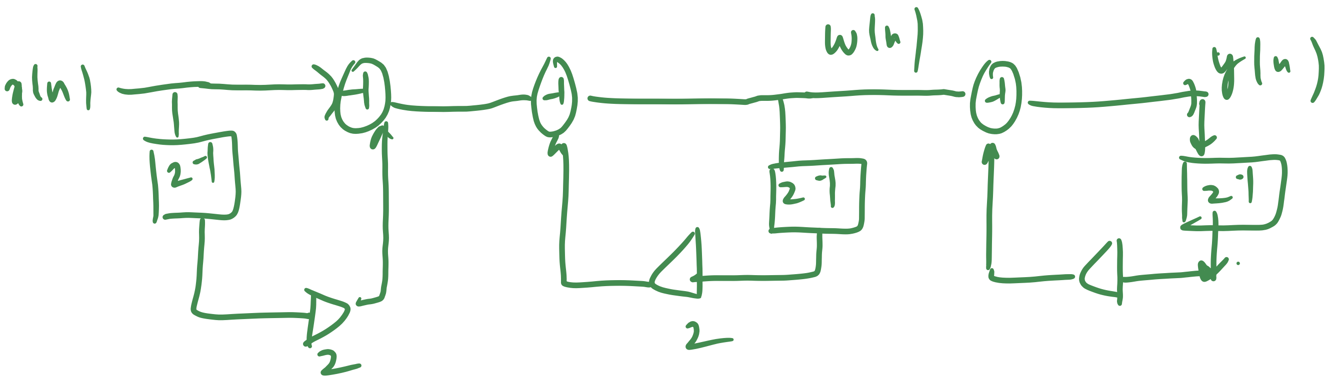
$$= W(z)\left(\frac{1}{1-z^{-1}}\right)$$

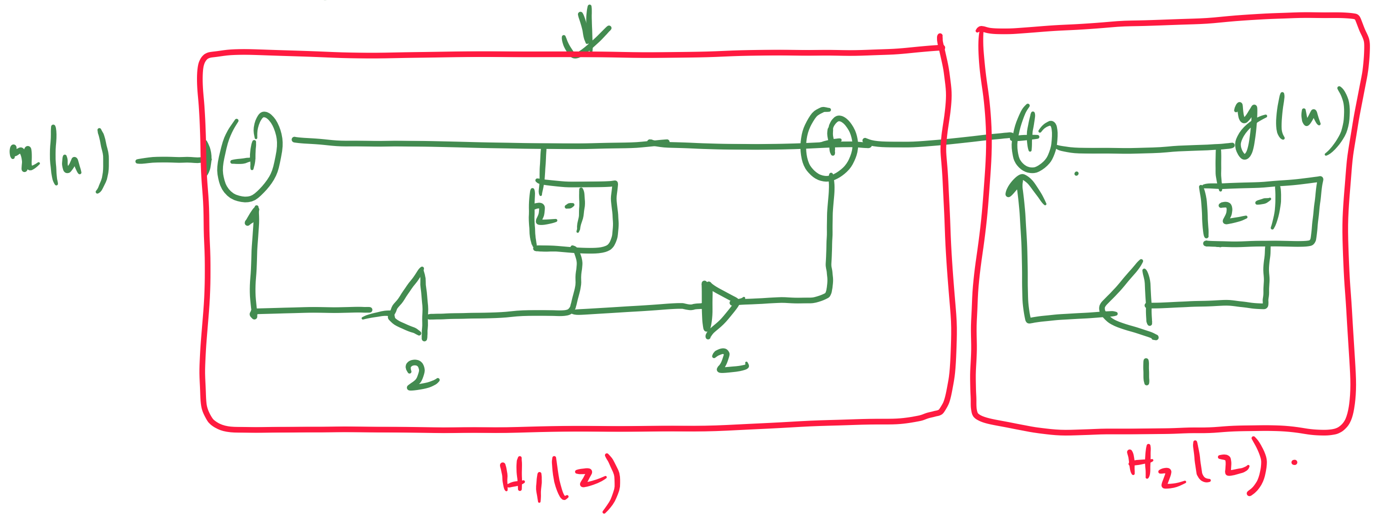
$$\Rightarrow Y(z) - z^{-1}Y(z) = W(z).$$

$$\Rightarrow y(n) - y(n-1) = w(n)$$



<u>Commuting them.</u>





(5) For the same system in Q. 3, obtain parallel forms I & II realizations.

Sol^n: $H(z) = \dfrac{1+2z^{-1}}{1-3z^{-1}+2z^{-2}}$

Parallel form I

$$H(z) = \frac{1+2z^{-1}}{1-z^{-1}-2z^{-1}+2z^{-2}}$$

$$= \frac{1+2z^{-1}}{(1-z^{-1}) - 2z^{-1}(1-z^{-1})}$$

$$= \frac{1+2z^{-1}}{(1-2z^{-1})(1-z^{-1})}$$

$$\Rightarrow H(z) = \frac{A}{1-2z^{-1}} + \frac{B}{1-z^{-1}}$$

A: $H(z)(1-2z^{-1})\Big|$

$(1-2z^{-1})=0 \Rightarrow 1-\frac{2}{z}=0$

$\Rightarrow z=2.$

$$H(z)(1-2z^{-1})\Big|_{z=2} = \frac{1+2z^{-1}}{(1-z^{-1})(1-2z^{-1})}(1-2z^{-1})\Bigg|_{z=2}$$

$$= \frac{1+2/z}{1-1/z}\Bigg|_{z=2} = \frac{1+2/2}{1-1/2} = \frac{2}{1/2}$$

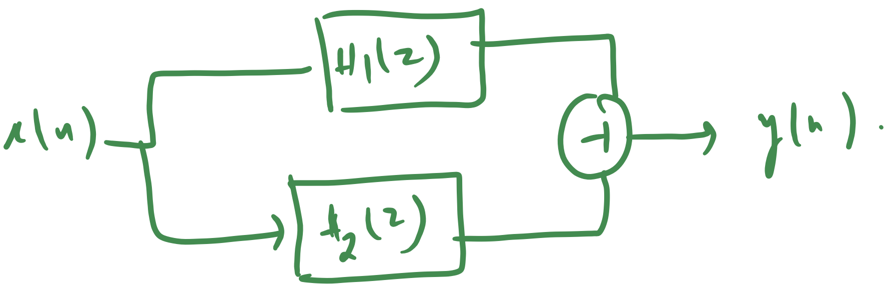
$$= \underline{\underline{4}}$$

B: $H(z)(1-z^{-1})\Big|$

$(1-z^{-1})=0 \Rightarrow z=1$

$$\Rightarrow \frac{1+2z^{-1}}{(1-z^{-1})(1-2z^{-1})}(1-z^{-1})\Bigg|_{z=1} = \frac{1+2}{1-2} = -3.$$

$$H(z) = \frac{4}{1-2z^{-1}} - \frac{3}{1-z^{-1}} \Rightarrow H(z) = H_1(z)+H_2(z)$$



Realization of $H_1(z)$

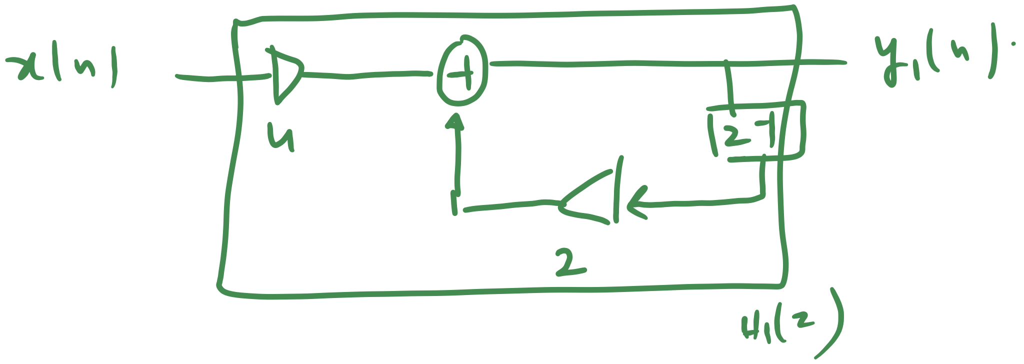
$$H_1(z) \quad \frac{4}{1-2z^{-1}}$$

$$\Rightarrow Y_1(z) = X(z)\,H_1(z)$$

$$\Rightarrow Y_1(z) = X(z)\,\frac{4}{1-2z^{-1}}$$

$$\Rightarrow Y_1(z) - 2z^{-1}Y_1(z) = 4X(z)$$

$$\Rightarrow 4X(z) + 2z^{-1}Y_1(z) = Y(z).$$



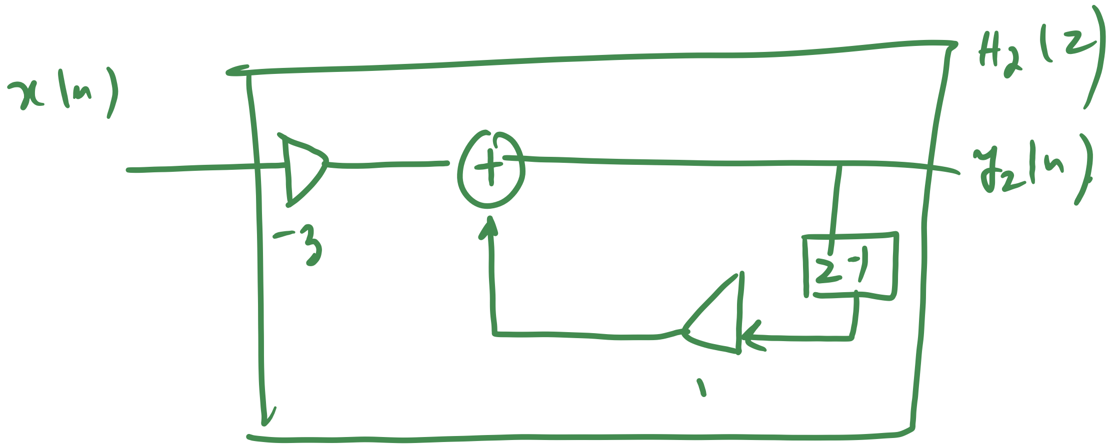
Realization of $H_2(z)$

$$Y_2(z) = X(z)H(z)$$

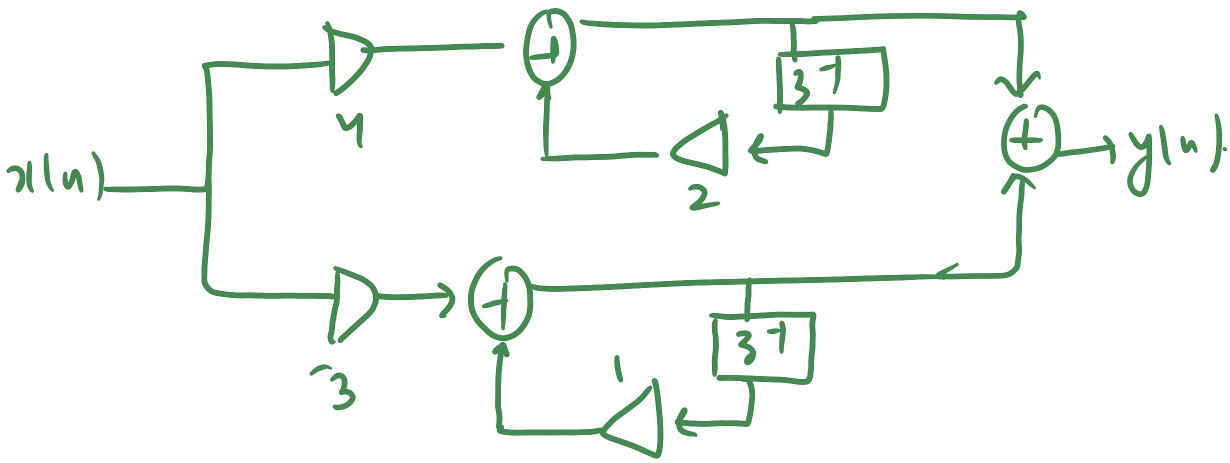
$$Y_2(z) = \frac{-3X(z)}{1 - z^{-1}}$$

$$\Rightarrow Y_2(z) - z^{-1}Y_2(z) = -3X(z)$$

$$\Rightarrow Y_2(z) = -3X(z) + z^{-1}Y_2(z)$$



Combining both of them.



P.F II.

$$H(z) = \frac{1+2z^{-1}}{1-3z^{-1}+2z^{-2}}$$

Multiply N^r & D^r by z^2

$$= \frac{z^2+2z}{z^2-3z+2}$$

$$= \frac{z(z+2)}{z^2-3z+2}$$

$$\begin{array}{r|l} & 1 \\ z^2-3z+2 & z^2+2z \\ & -\,(z^2-3z+2) \\ \hline & 5z-2 \end{array}$$

$$H(z) = 1 + \frac{5z-2}{(z-1)(z-2)}$$

↓ PF expansion.

$$\frac{5z-2}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

A : $\left.\frac{5z-2}{\cancel{(z-1)}(z-2)}\cancel{(z-1)}\right|_{z=1} = \frac{5-2}{1-2} = -3$

B $\left.\frac{5z-2}{(z-1)\cancel{(z-2)}}\cancel{(z-2)}\right|_{z=2} = \frac{10-2}{2-1} = 8$

$$\therefore H(z) = 1 - \frac{3}{z-1} + \frac{8}{z-2}$$

$$H(z) = \underbrace{1}_{H_1(z)} - \underbrace{\frac{3z^{-1}}{1-z^{-1}}}_{H_2(z)} + \underbrace{\frac{8z^{-1}}{1-2z^{-1}}}_{H_3(z)}$$

$H_1(z)$ $\quad x(n) \xrightarrow{\;1\;} y_1(n)$

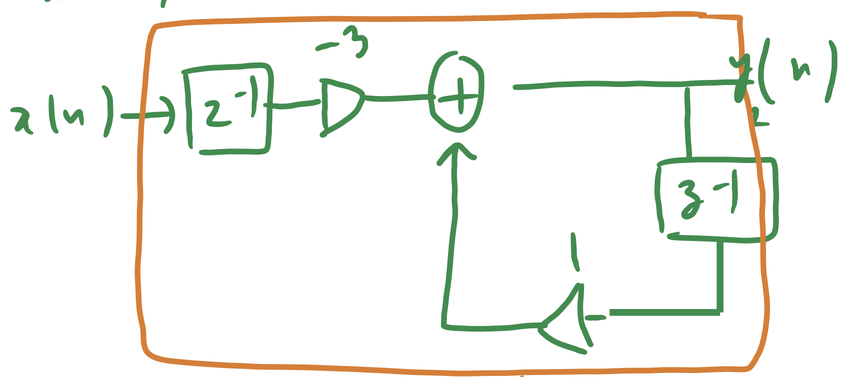
$$x(n) \longrightarrow \boxed{H_2(z)} \longrightarrow y_2(n)$$

$$Y_2(z) = X(z)\, H_2(z)$$

$$Y_2(z) = \frac{-3z^{-1}X(z)}{1-z^{-1}}$$

$$Y_2(z) - Y_2(z)\, z^{-1} = -3z^{-1}X(z)$$

$$Y_2(z) = -3z^{-1}X(z) + z^{-1}Y_2(z)$$



$H_2(z)$

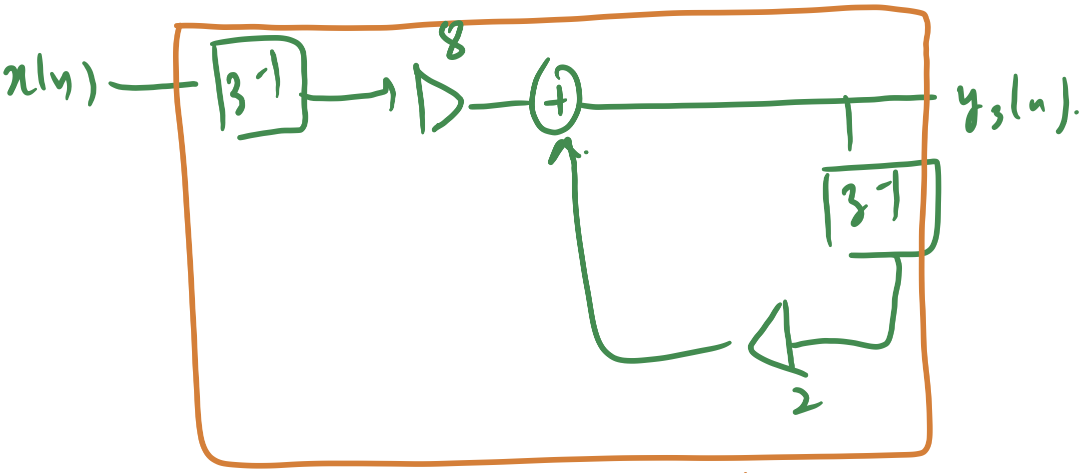
$\underline{H_3(z)}$

$$X(z)\, H_3(z) = Y_3(z)$$

$$X(z)\, \frac{8z^{-1}}{1-2z^{-1}} = Y_3(z)$$

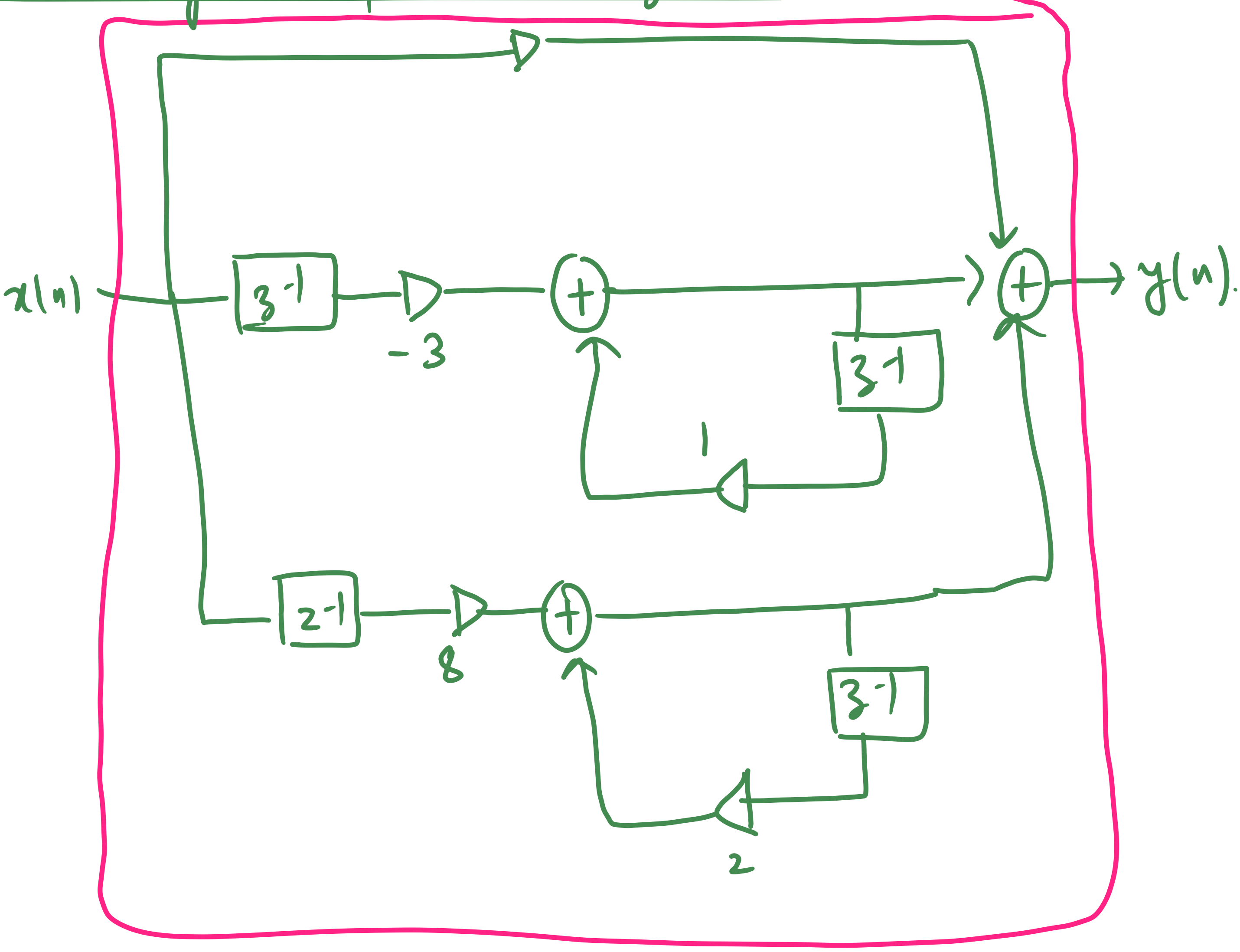
$$8z^{-1}X(z) = Y_3(z) - 2z^{-1}Y_3(z)$$

$$\Rightarrow Y_3(z) = +2z^{-1}Y_3(z) + 8z^{-1}X(z)$$



$H_3(z)$.

Combining all of them together



$H(z)$
PF · II

HW realise the following IIR filter by
(i) DF I & II ii) Cascaded iii) PF I & II.
structures

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 3/4\, z^{-1} + 1/8\, z^{-2}}$$

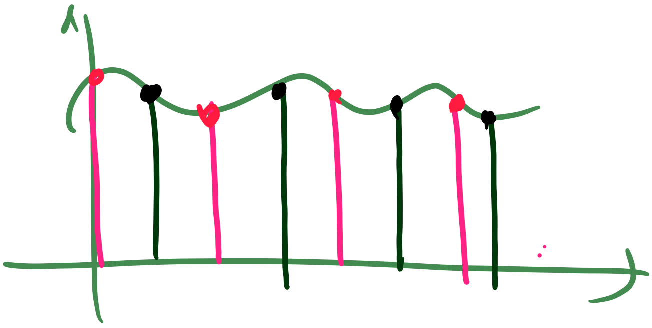
DTFT : $x(n) \longrightarrow X(\omega)$

continuous variable

<u>DFT</u> $x(n) \longrightarrow X(k)$

Discrete freq variable.

$x(n)$.



4-DFT

&

8-DFT