TA session 06

Part I · Summary of week 6 lectures.

(1) examples on z transform.

(2) Inverse z transform using contour integration Method.

$X(z) \leftarrow$ z transform then

$$x(n) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1} dz$$

$\Gamma \rightarrow$ closed contour in anti clockwise direction enclosing all poles of $X(z) z^{n-1}$

$$\therefore x(n) = \sum_{i=1}^{P} \underset{z \to p_i}{\text{Res}} \left[X(z) z^{n-1} \right]$$

= sum of residuals evaluated at each pole of $X(z) z^{n-1}$.

(3) Fourier series of continuous time signals

→ Representation of the continuous time domain signal in the frequency or spectral domain.

→ For periodic signals.

(i) Complex exponential Representation

$x(t) \rightarrow$ periodic in T_0

then

$$x(t) = \sum_{n=-\infty}^{\infty} c_k e^{jk\omega_0 t} \; ; \; \omega_0 = 2\pi / T_0$$

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt$$

Fourier coefficients

$$c_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,dt$$ ← avg of $x(t)$ over a period
↓
DC component

(*) For real signals

$c_k^* = c_{-k}$ [conjugate symmetry]

& $|c_k| = |c_{-k}|$ ← magnitude of Fourier coefficients have even symmetry.

$\angle c_{-k} = -\angle c_k$ ← phase of Fourier coefficients have odd symmetry.

(2) Trigonometric Representation.

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

$$a_k = \frac{2}{T_0}\int_{T_0} x(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)\,dt.$$

Relation b/w Trigonometric & Complex Exp representation

$$\boxed{\begin{aligned} a_0 &= 2c_0 \\ a_k &= c_k + c_{-k} \\ b_k &= j(c_k - c_{-k}) \end{aligned}}$$

⇓

$$\boxed{c_0 = a_0/2,\quad c_k = \frac{a_k - jb_k}{2},\quad c_{-k} = \frac{a_k + jb_k}{2}}$$

If $x(t)$ is real, a_k & b_k are real!

$$a_k = 2\,\mathrm{Re}\{c_k\}$$

$$b_k = -\mathrm{Im}\{c_k\}.$$

(*) for even signals.

ie $x(-t) = x(t)$

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)\right)$$

$$x(-t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos k\omega_0 t - b_k\sin k\omega_0 t\right)$$

$$\neq x(t)$$

$\therefore$ $b_k = 0$ {as they are coefficient of $\sin(k\omega_0 t)$ }.

odd signal

(*) For odd signals

$a_k = 0$

(3) <u>Conditions for existence of Fourier Series</u>

(i) $x(t)$ is absolutely integrable over T_0 $\int_{T_0} |x(t)|\,dt < \infty$

(ii) $x(t)$ has finite # of maxima/minima in any finite intervals

iii) $x(t)$ has finite # of discontinuities & these should be finite

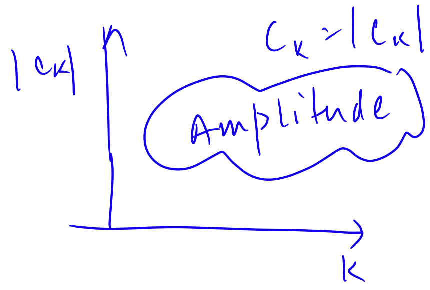
Dirichlet Conditions.

$\Downarrow$

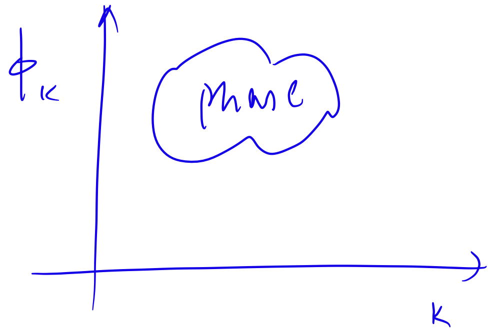
They are "sufficient" for existence of Fourier Series but not necessary

④ Amplitude and phase spectrum of $x(t)$

$$c_k = |c_k| e^{j\phi_k}$$

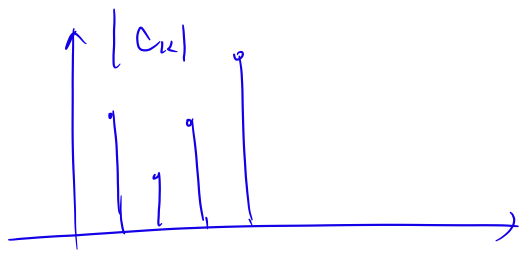


$(\omega = k\omega_0)$



$(\omega = k\omega_0)$

Discrete →



"Line spectrum"

⑤ Parseval's Theorem.

$$\lim_{T_0 \to \infty} \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Power of C.T signal.

Power of the signal remains equal in both time domain and frequency domain representations

c_k as a fxn of k is a discrete sequence.

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$\{\ldots, c_{-1}, c_0, c_1, c_2 \ldots\}$

$\hookrightarrow \sum_{i=-\infty}^{\infty} |c_i|^2 =$ Power of discrete seq.

$x(t)$ is a real signal:

$$c_k^* = c_{-k}$$

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$c_k^* = \left(1/T_0 \int x(t) e^{-jk\omega_0 t}\, dt \right)^*$$

$$= 1/T_0 \int \left(x(t)\, e^{-jk\omega_0 t}\right)^* dt$$

$$= \frac{1}{T_0}\int x^*(t)\, e^{jk\omega_0 t}\, d\theta$$

$$= 1/T_0 \int \underbrace{x(t)}_{\text{Real signal}} e^{jk\omega_0 t}\, dt$$

$$= 1/T_0 \int x(t)\, e^{-j(-k)\omega_0 t}\, dt = c_{-k}.$$

$\Rightarrow c_k^* = c_{-k}$. $|c_k| = |c_k^*| = |c_{-k}|$.

$\angle c_k^* = -\angle c_k$
$\hookrightarrow \angle c_k$
$\Rightarrow \angle c_{-k} = -\angle c_k$.

Part II Tutorials:

1) Determine the inverse z transform of $X(z) = \dfrac{1}{1 - 0.8z^{-1} + 0.12z^{-2}}$

a) if ROC is $|z| > 0.6$ {ROC is ext part of circle}

b) if ROC is $|z| < 0.2$

c) if ROC is $0.2 < |z| < 0.6$.

{ROC is interior part of circle}

{ROC is an annular region}.

$\underline{\underline{Sol^n}}$ Do the partial fraction expansion of $X(z)/z$

$$X(z) = \frac{1}{1-0.8z^{-1}+0.12z^{-2}}$$

$$= \frac{1}{z^{-2}(z^2-0.8z+0.12)}$$

$$= \frac{z^2}{(z^2-0.8z+0.12)}$$

$$= \frac{z^2}{(z-0.6)(z-0.2)}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{z}{(z-0.6)(z-0.2)} \quad ; P_1 = 0.6$$

$P_2 = 0.2$

simple poles.

$$\frac{X(z)}{z} = \frac{A_1}{z-0.6} + \frac{A_2}{z-0.2}$$

$$A_1 = (z-0.6)\frac{X(z)}{z}\Big|_{z=0.6}$$

$$= \frac{z}{z-0.2}\Big|_{z=0.6}$$

$$= \frac{0.6}{0.4} = 1.5$$

$$A_2 = (z-0.2)\frac{X(z)}{z}\Big|_{z=0.2}$$

$$= -0.5$$

$$\frac{X(z)}{z} = \frac{1.5}{z-0.6} - \frac{0.5}{z-0.2}$$

$$X(z) = 1.5\frac{z}{z-0.6} - \frac{0.5z}{z-0.2}$$

(i) ROC $|z| > 0.6$ $\rightarrow$ exterior part of circle $|z| = 0.6$

$\therefore x(n)$ should be right handed signal

$$x(n) = 1.5(0.6)^n u(n) - 0.5(0.2)^n u(n)$$

(ii) ROC $|z| < 0.2$ $\rightarrow$ Interior part of circle $|z| = 0.2$

$\Rightarrow x(n)$ should be left sided !

$$x(n) = -1.5(0.6)^n u(-n-1) + 0.5(0.2)^n u(-n-1)$$

(iii) ROC $0.2 < |z| < 0.6 \Rightarrow$ annular region.

$\Rightarrow x(n)$ should be two sided !

$$x(n) = x_1(n) + x_2(n)$$

where $x_1(n) = IZT\left(\frac{1.5 z}{z-0.6}\right)$

$$x_2(n) = IZT\left(\frac{-0.5 z}{z-0.2}\right).$$

for $\frac{1.5 z}{z-0.6}$; $|z| < 0.6$

$$\Rightarrow x_1(n) = -1.5(0.6)^n u(-n-1)$$

for $\frac{-1.5 z}{z-0.2}$; $|z| > 0.2$

$$x_2(n) = -0.5(0.2)^n u(n)$$

$$\therefore x(n) = -1.5(0.6)^n u(-n-1) - 0.5(0.2)^n u(n)$$

② Determine the complex exponential Fourier series representation for each of the following signals :

a) $x(t) = \cos(w_0 t)$

b) $x(t) = \sin(w_0 t)$

c) $x(t) = \cos(2t + \pi/4)$

d) $x(t) = \cos(4t) + \cos(6t)$

e) $x(t) = \sin^2(t)$

Solution $x(t) = \cos(w_0 t)$.

$$X(t) = \sum_{k=-\infty}^{\infty} C_k e^{jkw_0 t} \quad \text{—(1)}$$

$$= \frac{e^{jw_0 t} + e^{-jw_0 t}}{2} \quad \text{—(2)}$$

[Euler's formula].

Equate (1) & (2)

$$\frac{e^{jw_0t}+e^{-jw_0t}}{2} = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0t}$$

Comparing the coefficients for every term:

$$c_1 = 1/2, \; c_{-1} = 1/2, \; c_k = 0 \; |k| \neq 1$$

$$\frac{1}{2}e^{jw_0t} + \frac{1}{2}e^{-jw_0t}$$

$$= \dots + c_{-3}e^{-j3w_0t} + c_{-2}e^{-j2w_0t} + c_{-1}e^{-jw_0t} + c_0 + c_1 e^{jw_0t} + c_2 e^{j2w_0t} + \dots$$

$e^{jw_0t} = 2$ $\quad c_1 = 1/2, \; c_2 = 0,$

$c_{-1} = 1/2.$

(b) $x(t) = \sin(w_0t)$

$$= \frac{1}{2j}\left(e^{jw_0t} - e^{-jw_0t}\right) \rightarrow \text{Euler's Formula.}$$

$\therefore$ similar to part (a)

$$c_1 = \frac{1}{2j}$$

$$c_{-1} = -1/2j \qquad c_k = 0 \; |k| \neq 1.$$

c) $x(t) = \cos(2t + \pi/4)$.

$$w_0 = 2$$

$$\therefore x(t) = \sum_{n=-\infty}^{\infty} c_k e^{j2kt} \quad \text{—— (1)}$$

$$x(t) = \cos(2t + \pi/4) = \frac{e^{j(2t+\pi/4)} + e^{-j(2t+\pi/4)}}{2}$$

$$= \frac{1}{2} e^{j2t} e^{j\pi/4} + \frac{1}{2} e^{j2t} e^{-j\pi/4}$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) e^{j2t} + \frac{1}{2}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right) e^{-j2t} \quad \text{— (2)}$$

Comparing coefficients b/w (1) & (2)

$$C_1 = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$$

$$C_{-1} = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - j\,1/\sqrt{2}\right)$$

$$C_k = 0 \quad , |k| \neq 1$$

d) $x(t) = \cos 4t + \cos 6t$

T_1 : period of $\cos 4t$ $= 2\pi/4$
$= \pi/2$

$$\cos 4(t+T) = \cos(4t + 4T) = \cos 4t$$

$$4T = n2\pi$$

$$T = n \cdot 2\pi/4 \quad ; n \in \mathbb{Z}$$

$n = 1$; Fundamental period
$= 2\pi/4$

T_2 : period of $6t$ $= 2\pi/6 =$

$$\text{LCM}\left(2\pi/4, 2\pi/6\right)$$

$$= \frac{1}{12} \text{ LCM of } \left(\frac{2\pi}{\cancel{4}} \times \cancel{12}^{3}, \frac{2\pi}{\cancel{6}} \times \cancel{12}^{2}\right)$$

$$= \frac{1}{12} \text{ LCM of } (6\pi, 4\pi).$$

$$= \frac{1}{12} \pi \text{ LCM of } (6, 4).$$

$$= \frac{1}{\cancel{12}} \cdot \cancel{12}\pi$$

$$= \pi$$

$\therefore$ Fundamental Period of $x(t)$
$= \cos 4t + \cos 6t$
$= \underline{\underline{\pi}}$

$\therefore$ Fundamental freq $= w_0$

angular freq $\leftarrow$ $= 2\pi f_0$

$= 2\pi / T_0$

$= 2\pi / \pi$

$= 2$

$w_0 = 2$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2kt} \quad \text{--- (1)}$$

$$x(t) = \cos 4t + \cos 6t$$

$$= \frac{e^{j4t} + e^{-j4t}}{2} + \frac{e^{j6t} + e^{-j6t}}{2}$$

$$= \frac{e^{-j6t}}{2} + \frac{e^{-j4t}}{2} + \frac{e^{j4t}}{2} + \frac{e^{j6t}}{2} \quad \text{--- (2)}$$

Comparing (1) & (2) —

$$\left\{ \begin{matrix} c_{-3} = 1/2, & c_{-2} = 1/2 \\ c_2 = 1/2, & c_3 = 1/2. \end{matrix} \right\}$$

e) $x(t) = \sin^2(t)$

$$\sin^2(t) = \frac{1 - \cos 2t}{2}$$

$\hookrightarrow$ $w_0 = 2$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Now $x(t) = \sin^2(t)$

$$= \left(\frac{e^{jt} - e^{-jt}}{2j}\right)^2$$

$$= \frac{e^{j2t} + e^{-j2t} - 2}{-4}$$

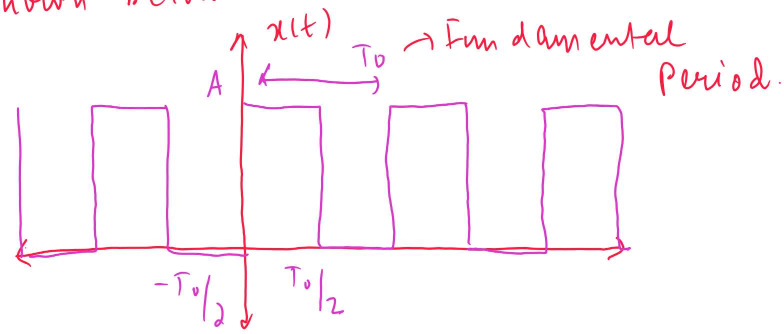
$$= -1/4\, e^{j2t} - 1/4\, e^{-j2t} + 1/2.$$

$c_1 = -1/4, \quad c_{-1} = -1/4$

$c_0 = 1/2.$

$c_k = 0 ;\ |k| \neq 0$

(3) Consider the periodic square wave as shown below.



a) Determine the complex exponential Fourier series of x(t)

b) Determine the trigonometric Fourier series of x(t).

Soln a) $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$;

$\omega_0 = 2\pi/T_0$

$c_k = 1/T_0 \int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$

$= 1/T_0 \int_0^{T_0/2} A\, e^{-jk\omega_0 t}\, dt$

$= \frac{A}{T_0} \int_0^{T_0/2} e^{-jk\omega_0 t}\, dt$

$= \frac{A}{T_0}\left[\frac{1}{-jk\omega_0} e^{-jk\omega_0 t} \Big|_0^{T/2} \right]$

$= \frac{A}{T_0}\left[\frac{-1}{jk\omega_0} \left(e^{-jk\omega_0 T/2} - 1 \right) \right]$

$= \frac{A}{T_0}\left[\frac{1 - e^{-jk\omega_0 T_0/2}}{jk\omega_0} \right]$

$\omega_0 = 2\pi/T_0 \Rightarrow \omega_0 T_0 = 2\pi$

$= \frac{A}{\cancel{T_0}}\left[\frac{1 - e^{-jk\, 2\pi/2}}{jk \frac{2\pi}{\cancel{T_0}}} \right]$

$$= \frac{A}{jk2\pi}\left[1 - e^{-jk\pi}\right]$$

$$= \frac{A}{jk2\pi}\left[1-(-1)^k\right] \quad \left(\because e^{-j\pi} = \cos(\pi) - j\sin\pi \right)$$

$$= -1+0$$

$$= -1$$

$\Rightarrow k \neq 0$:-

(i) k is even : $k = 2m;\ m \in \mathbb{Z}$

$$\Rightarrow (-1)^k = 1 \Rightarrow c_k = 0$$

(ii) k = odd $\Rightarrow k = 2m+1;\ m \in \mathbb{Z}$

$$(-1)^k = -1 \Rightarrow c_k = \frac{A}{jk\pi}$$

$\underline{\underline{k=0}}$

$$c_k = \frac{A}{jk2\pi}\left(1-(-1)^k\right)\Big|_{k=0}$$

$\left(\frac{0}{0}\right)$ form $\rightarrow$ not applicable

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt = \frac{1}{T_0}\int_0^{T_0/2} A\, dt$$

$$= \frac{1}{\cancel{T_0}} A \cdot \cancel{T_0}/2$$

$$= A/2$$

<u>Summarising</u>

$$c_0 = A/2, \quad c_{2m} = 0$$

$$c_{2m+1} = \frac{A}{j(2m+1)\pi}$$

$$x(t) = A/2 + \frac{A}{j\pi}\left\{\sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t}\right\}$$

where $\omega_0 = 2\pi/T$

b) w.r.t $x(t)$ is real

$$a_0 = 2c_0 = 2 \cdot A/2 = A$$

$a_{2m} = b_{2m} = 0 \; ; \; m \neq 0$

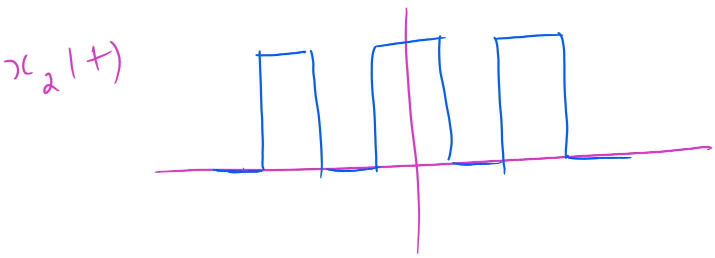
$a_{2m+1} = 2\,\mathrm{Re}\{c_{2m+1}\} = 0$

$b_{2m+1} = -2\,\mathrm{Im}\{c_{2m+1}\} = \dfrac{2A}{(2m+1)\pi}$

$$\therefore x(t) = A/2 + \frac{2A}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin\big((2m+1)\omega_0 t\big)$$

$$\Rightarrow x(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \cdots\right)$$

↳ odd square pulse train contains odd harmonics



squane pulse train contains odd harmonics.