Week 4 TA session

Part I - Summary of week IV lecture

① Laplace transform

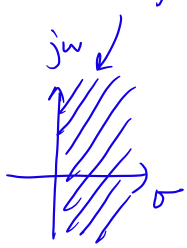
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Representation of a signal $x(t)$ in "Laplace Domain".

② Properties. Ref: Oppenheim.

	Signal		ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$a x_1(t) + b x_2(t)$	$a X_1(s) + b X_2(s)$	at least $R_1 \cap R_2$
Time shifting	$x(t-t_0)$	$e^{-st_0} X(s)$	R
Shifting in s domain.	$e^{s_0 t} x(t)$.	$X(s-s_0)$	shifted version of R (i.e. s is in the ROC if $s-s_0$ is in R)
Time Scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X(s/a)$	Scaled ROC (s is in ROC if s/a in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Diff in time Domain	$\frac{d\, x(t)}{dt}$	$s X(s)$	at least R
differentiation in s domain	$-t x(t)$	$\frac{d X(s)}{ds}$	R
Integration in time domain	$\int_{-\infty}^{t} x(\tau)\, d(\tau)$	$\frac{1}{s} X(s)$	at least $R \cap (Re\{s\} > 0)$

$$y(t) = x_1(t) * x_2(t)$$
$$Y(s) = X_1(s) X_2(s)$$



3. Laplace transform of some standard signals

Signal	Transform	ROC
$\delta(t)$	1	all s
$u(t)$	$1/s$	$Re(s)>0$
$-u(-t)$	$1/s$	$Re(s)<0$
$\dfrac{t^{n-1}}{(n-1)!}u(t)$	$\dfrac{1}{s^n}$	$Re(s)>0$
$\dfrac{-t^{n-1}}{(n-1)!}u(t)$	$1/s^n$	$Re(s)<0$
* $e^{-\alpha t}u(t)$	$\dfrac{1}{s+\alpha}$	$Re(s)>-\alpha$
* $-e^{-\alpha t}u(t)$	$\dfrac{1}{s+\alpha}$	$Re(s)<-\alpha$
$\dfrac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\dfrac{1}{(s+\alpha)^n}$	$Re(s)>-\alpha$
$-\dfrac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\dfrac{1}{(s+\alpha)^n}$	$Re(s)<-\alpha$
$\delta(t-T)$	e^{-sT}	all s
$\cos\omega_0 t\, u(t)$	$\dfrac{s}{s^2+\omega_0^2}$	$Re(s)>0$
$\sin\omega_0(t)\, u(t)$	$\dfrac{\omega_0}{s^2+\omega_0^2}$	$Re(s)>0$
$u_n(t)=\dfrac{d^n\delta(t)}{dt^n}$	s^n	all s
$u_{-n}(t)=\underbrace{u(t)*\cdots u(t)}_{n}$	$1/s^n$	$Re(s)>0$

$$u_{-n}(t)\rightarrow U_{-n}(s)=U(s)\,U(s)\cdots U(s)$$
$$=1/s\cdots 1/s \ (n \text{ times})$$
$$=\frac{1}{s^n}.$$

4. Poles / zeros.

let $X(s)=\dfrac{N(s)}{D(s)}$ $\rightarrow$ deg $=m$ (for $N(s)$), deg $=n$ (for $D(s)$).

$Z \triangleq \{s: N(s)=0\}$ is set of zeros of $X(s)$

$P \triangleq \{s: D(s)=0\}$ is set of poles of $X(s)$.

eg $X(s) = \frac{2s-3}{s+1}$

$N(s) = 2s - 3$

$D(s) = s + 1$

$N(s) = 0$

$s = 3/2 \rightarrow$ zero of $X(s)$

$D(s) = 0$

$s = -1$

$\hookrightarrow$ pole of $X(s)$

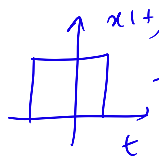
If $m \geq n \Rightarrow$ Improper

$m < n \Rightarrow$ Proper.

(*) Poles cannot be in ROC!

(*) for a finite Duration signal

ROC = entire s-plane.

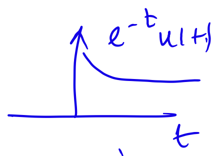


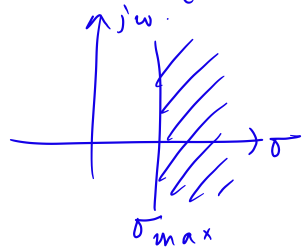
$\rightarrow$ ROC entire s plane.

(*) for Right sided signal

ROC $= Re\{s\} \geq \sigma_{max}$

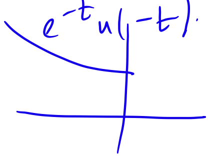
Real part of max pole





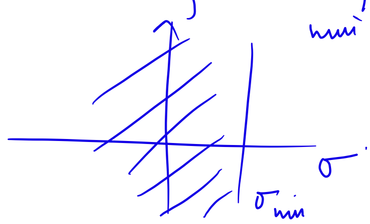
(*) for a left sided signal

eg $e^{-t}u(-t)$.

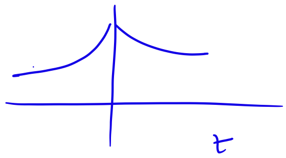


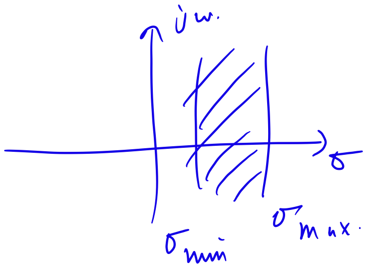
ROC $= Re\{s\} < \sigma_{min}$

Real part of min pole.



(x) Two sided signal





(5) Partial fraction expansion of Laplace Transform

— useful in finding inverse LT.

$P_1, P_2, \ldots, P_n$ distinct poles.

$$X(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \cdots + \frac{c_n}{s-p_n}$$

$$c_k = (s-p_k)\,X(s)\Big|_{s=p_k}$$

If Multiple poles (repeating poles)

$$X(s) = \frac{\lambda_1}{s-p_i} + \frac{\lambda_2}{(s-p_i)^2} + \cdots + \frac{\lambda_r}{(s-p_i)^r}$$

if p_i repeats 'r' times

+ term for other poles.

$$\lambda_{r-k} = \frac{1}{k!}\frac{d^k}{ds^k}\left((s-p_i)^r X(s)\right)\Big|_{s=p_i}$$

(6) LTI system and Laplace transforms.

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

$X(s) \rightarrow \boxed{H(s)} \rightarrow Y(s)$

$$y(t) = x(t) * h(t)$$

$$\Rightarrow Y(s) = X(s)\,H(s)$$

$H(s) \leftarrow$ transfer fxn of system.

Causality

Recall $h(t) = 0$ for $t < 0$
$\Rightarrow$ right sided signal.

$\Rightarrow$ $H(s)$ is s.t

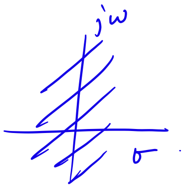
$Re(s) > \sigma_{max}$ [ROC]

BIBO stability

$$\int_{-\infty}^{\infty} |h(t)| < dt$$

$\Rightarrow$ ROC of $H(s)$ contain

$j\omega$ axis.



(7) Transfer function

$\rightarrow$ any LTI system can be represented as a linear constant coefficient differential equation.

ie $$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$\rightarrow$ using L.T property

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}.$$

(8) One sided LT / unilateral LT

$$X_I(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt$$

ROC : $Re(s) > \sigma_{max}$.

$$x(t) \leftrightarrow X_I(s)$$

$$\frac{d\,x(t)}{dt} \leftrightarrow s X_I(s) - x(0^-)$$

$||^{ly}$ $$\int_{0^-}^{\tau} x(\tau) d\tau \leftrightarrow \frac{X_I(s)}{s}.$$

$$\int_{-\infty}^{\tau} x(\tau) d\tau \leftrightarrow \frac{X_I(s)}{s} + \frac{1}{s}\int_{-\infty}^{0^-} x(\tau) d\tau$$

Part 2 Tutorials

① Using the various Laplace Transform properties, find the L.T of following signals from the L.T of $u(t)$

a) $\delta(t)$

b) $\delta'(t) = \frac{d}{dt}\delta(t)$

c) $t\,u(t)$

d) $e^{-at}u(t)$

e) $\cos w_0 t \; u(t)$

f) $e^{-at}\cos(w_0 t)\,u(t)$

Soln

a) $\delta(t) = \frac{d}{dt}u(t)$.

$L(u(t)) = 1/s$.

$x(t) \longleftrightarrow X(s)$

$\frac{d\,x(t)}{dt} \longrightarrow s\,X(s)$

$x(t) \longrightarrow u(t)$

$X(s) \longrightarrow 1/s$

$\therefore \frac{d\,x(t)}{dt} = \frac{d}{dt}u(t) \xleftrightarrow{L} s \cdot 1/s = 1$

$\delta(t) \xleftrightarrow{L} 1 \;;\; \forall s$.

b) $\delta'(t) = \frac{d}{dt}\delta(t)$

$= \frac{d^2}{dt}u(t)$

$\frac{d^2 x(t)}{dt^2} \xleftrightarrow{L} s^2 X(s)$

$u(t) = u(t)$

$X(s) = 1/s$

$\delta'(t) \xleftrightarrow{L} s \quad \forall s \quad (ROC)$

) $t\,u(t)$

$x(t) \xleftrightarrow{L} X(s)$

$t\,x(t) \xleftrightarrow{L} -\frac{d}{ds} X(s)$

$x(t) = u(t)$

$t\,u(t) \xleftrightarrow{L} -\frac{d}{ds}\left(1/s\right).$

$= -\left(-1/s^2\right)$

$= 1/s^2 \quad Re\{s\} > 0$

d) $e^{-at} u(t).$

$x(t) \xrightarrow{L} X(s)$

$e^{s_0 (t)} x(t) \xrightarrow{L} X(s - s_0)$

Here $x(t) = u(t)$, $s_0 = -a$

$X(s) = 1/s.$

$e^{-at} x(t) \rightarrow X(s+a)$

$= \frac{1}{s+a} \quad . Re\{s\} > -a$

e) $\cos(w_0 t)\, u(t)$

$= \frac{1}{2}\{e^{jw_0 t} + e^{-jw_0 t}\}\, u(t)$

$= \underbrace{\frac{1}{2} e^{jw_0 t} u(t)}_{\text{I}} + \underbrace{\frac{1}{2} e^{-jw_0 t} u(t)}_{\text{II}}$

LT of I :

$s_0 = jw_0.$

$LT = \frac{1}{2} X(s - jw_0) = \frac{1}{2}\left(\frac{1}{s - jw_0}\right)$

LT of II

$s_0 = -jw_0$

$LT = \frac{1}{2} X(s + jw_0)$

$= \frac{1}{2} \frac{1}{(s + jw_0)}$

adding up

$$-\frac{1}{2}\left\{\frac{1}{s-j\omega_0}\right\} + \frac{1}{2}\left\{\frac{1}{s+j\omega_0}\right\}$$

$$= \frac{s}{s^2+\omega_0^2} \quad : Re\{s\} > 0.$$

f) $e^{-at} \sin \omega_0 t \; u(t)$

$$e^{-a(t)} x(t) \longrightarrow X(s+a)$$

$$= \frac{(s+a)}{(s+a)^2 + \omega_0^2}$$

$\swarrow f.$

② $x(t) = e^{-3t} u(t-2) - e^{5t} u(4-t)$

find Laplace Transform.

$\underline{\underline{Sol^n}}$ $x(t) = e^{-3t} u(t-2) - e^{5t} u(4-t)$

$$= e^{-6} e^{-3(t-2)} u(t-2)$$

$$+ \left(-e^{5t} u(4-t)\right)$$

$$u(t) \longrightarrow e^{-3t} u(t)$$

$$\longrightarrow e^{-3(t-t_0)} u(t-t_0)$$

$$1/s \rightarrow \frac{1}{s+3} \longrightarrow \frac{e^{-st_0}}{s+3}$$

$\therefore$ $t_0 = 2$ (1st term)

$$= e^{-6} \frac{e^{-2s}}{s+3}$$

2nd term $= -e^{5t} u(4-t)$

$$= e^{20} e^{5(t-4)} u(4-t)$$

$$e^{a(t-t_0)} u(4-t).$$

$$= -e^{20} e^{5(t-4)} u(-(t-4))$$

$$u(t) \longrightarrow -e^{5t} u(-t)$$

$$\longrightarrow e^{-5(t-4)} u(-(t-4))$$

$$1/s \rightarrow \frac{1}{s-5} \longrightarrow e^{-4s} / s-5$$

$\hookrightarrow Re\{s\} < 5$

LT of 1st term

$$e^{-6}\frac{e^{-2s}}{s+3} \quad : Re\{s\} > -3$$

LT of 2nd term

$$e^{20}\frac{e^{-4s}}{s-5} \quad : Re\{s\} < 5$$

$$\therefore \boxed{X(s) = e^{-6}\frac{e^{-2s}}{s+3} + e^{20}\frac{e^{-4s}}{s-5}} ; \quad -3 < Re\{s\} < 5$$

(3) What is the LT of $te^{-at}u(-t)$

Ans

$$-e^{-at}u(-t) \longleftrightarrow \frac{1}{s+a}$$

$$\Rightarrow \underbrace{e^{-at}u(-t)}_{x(t)} \longleftrightarrow \underbrace{\frac{-1}{s+a}}_{X(s)}$$

$$t\,x(t) \longleftrightarrow -\frac{d}{ds}X(s)$$

$$= -\frac{d}{ds}\left(-\frac{1}{s+a}\right)$$

$$te^{-at}u(-t) \quad = -\frac{d}{ds}\left(\frac{-1}{s+a}\right)$$

$$= \frac{d}{ds}\left(\frac{1}{s+a}\right)$$

$$= \frac{-1}{(s+a)^2}$$

(4) Find the Laplace Transform of

$$y(t) = t\,x(-t)$$

Soln

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

Let $z(t) = x(-t)$

$$Z(s) = \int_{-\infty}^{\infty} x(-t)e^{-st}\,dt$$

$$t \rightarrow -t : \Rightarrow -\int_{\infty}^{-\infty} x(t)e^{st}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{+st}\,dt = X(-s)$$

$$x(t) \longleftrightarrow X(s)$$

$$x(-t) \longleftrightarrow X(-s)$$

$$\tilde{x}(t) \triangleq x(-t)$$

$$\Rightarrow y(t) = t\,\tilde{x}(t)$$

$$t\,\tilde{x}(t) \longleftrightarrow -\frac{d}{ds}\tilde{X}(s)$$

$$= -\frac{d}{ds}X(-s).$$

$$\therefore\ t\,x(-t) \xleftrightarrow{\mathcal{L}} -\frac{d}{ds}X(-s)$$

(5) Find the inverse Laplace transform of $X(s) = \dfrac{8s^2+11s}{(s+1)^3(s+2)}$

$Re\{s\} > -1$

Sol^n: Poles of $X(s)$: $s=-2$ ← distinct

$s=-1$ ← repeats 3 times.

$$X(s) = \frac{K_1}{s+2} + \frac{K_2}{(s+1)^3} + \frac{K_3}{(s+1)^2} + \frac{K_4}{(s+1)}$$

K_1: $K_1 = X(s)(s+2)\Big|_{s=-2}$

$$= \frac{8s^2+11s}{(s+1)^3}\Bigg|_{s=-2}$$

$$= \frac{8\times4+11\times(-2)}{(-2)^3}$$

$$= -10$$

K_2: $K_2 = X(s)(s+1)^3\Big|_{s=-1}$

$$= \frac{8s^2+11s}{(s+2)}\Bigg|_{s=-1}$$

$$= -3.$$

$$k_3 = \frac{d}{ds}\left[X(s)(s+1)^3\right]\Bigg|_{s=-1}$$

$$= \frac{d}{ds}\left\{\frac{8s^2+11s}{(s+2)}\right\}\Bigg|_{s=-1}$$

$$= \frac{(16s+11)(s+2) - (8s^2+11s)\times 1}{(s+2)^2}\Bigg|_{s=-1}$$

$$= -2$$

$$k_4 = \frac{1}{2!}\frac{d^2}{ds^2}\left[X(s)(s+1)^3\right]\Bigg|_{s=-1}$$

$$= 10$$

$$\therefore X(s) = \frac{-10}{s+2} - \frac{3}{(s+1)^3} - \frac{2}{(s+1)^2} + \frac{10}{(s+1)}$$

$$= -10e^{-2t}u(t)$$

$$+10e^{-t}u(t)$$

2nd term

$$L^{-1}\left(\frac{1}{(s+1)^3}\right) = \frac{1}{2!}t^2e^{-t}u(t)$$

3rd term

$$L^{-1}\left(\frac{1}{(s+1)^2}\right) = te^{-t}u(t).$$

$\therefore$ Combining

$$x(t) = (10 - 2t - 1.5t^2)e^{-t}u(t)$$
$$-10e^{-2t}u(t).$$

6) Find the inverse Laplace Transform of $X(s) = \frac{4}{(s+2)(s+4)}$

if the ROC is

1) $-2 > Re\{s\} > -4$

2) $Re\{s\} < -4$

3) $Re\{s\} > -2.$

Sol^n Poles are

$s = -2, -4 \leftarrow$ distinct poles.

$$\therefore X(s) = \frac{k_1}{s+2} + \frac{k_2}{s+4}.$$

$\underline{K_1}$: $= X(s)(s+2)\Big|_{s=-2}$

$= \dfrac{4}{s+4}\Big|_{s=-2} = 2$

$\underline{K_2}$ $K_2 = X(s)(s+4)\Big|_{s=-4}$

$= \dfrac{4}{s+2}\Big|_{s=-4}$

$= -2$

$\therefore X(s) = \dfrac{2}{s+2} - \dfrac{2}{s+4}$.

(i) $-4 < Re\{s\} < -2$

For $2/_{s+2}$:- ROC : $Re\{s\} < -2$

$\hookrightarrow$ ILT

$-2e^{-2t}u(-t)$.

for $\dfrac{-2}{s+4}$: $Re\{s\} > -4$.

$\hookrightarrow -2e^{-4t}u(t)$.

ILT $-2e^{-2t}u(-t)$

$-2e^{-4t}u(t)$

(ii) $Re\{s\} < -4$

$\Rightarrow Re\{s\} < -2$

For $2/_{s+2}$: ILT $=$

$-2e^{-2t}u(-t)$

For $-2/_{s+4}$: ILT :

$-(-2e^{-4t}u(-t))$

Adding up:-

$2\{e^{-4t} - e^{-2t}\}u(-t)$. ALSS

$\hookrightarrow$ Anticausal signal

iii) $Re\{s\} > -2$

$\rightarrow Re\{s\} > -4$.

For $\frac{2}{s+2}$: ILT : $2e^{-2t}u(t)$.

for $\frac{-2}{s+4}$: ILT : $-2e^{-4t}u(t)$.

$\therefore$ adding up.

ILT $2\left[e^{-2t} - e^{-4t}\right]u(t)$

$\downarrow$

Right sided signal.

(7) Using Laplace Transform, find the impulse response of a causal system whose i/p-o/p relation is:

$$\frac{dy(t)}{dt} + a\,y(t) = x(t).$$

Sol^n Taking L.T on Both sides.

$$sY(s) + aY(s) = X(s)$$

T.F $H(s) = \frac{Y(s)}{X(s)}$

$$Y(s)(s+a) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = 1/s+a \quad \text{LT}$$

$$\boxed{Re\{s\} > a}$$

$h(t) = e^{-at}u(t)$

$\hookrightarrow$ Impulse response!

(8) Consider a continuous time LTI system for which the input $x(t)$ and output $y(t)$ are related by

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} - 2y(t) = x(t).$$

(i) Find the system function / T.F $H(s)$

(2) Determine the impulse response $h(t)$ for the following 3 cases.

a) System is causal

b) system is stable

c) system is neither causal nor stable.

Solution (i) Take Laplace Transform on Both sides.

$$s^2Y(s) + sY(s) - 2Y(s) = X(s)$$

$$\Rightarrow (s^2+s-2)\,Y(s) = X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+s-2}$$

$$= \frac{1}{(s+2)(s-1)}.$$

$$H(s) = \frac{k_1}{s+2} + \frac{k_2}{s-1}$$

k_1: $H(s)(s+2)\Big|_{s=-2} = \frac{1}{s-1}\Big|_{s=-2} = \frac{1}{-3}$

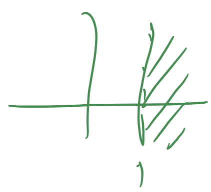
k_2: $H(s)(s-1)\Big|_{s=1} = \frac{1}{s+2}\Big|_{s=1} = 1/3$

$$H(s) = -\frac{1}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s-1} \checkmark$$

(ii) System is causal.

$\therefore$ ROC should be right sided.

$$Re\{s\} > \sigma_{max}$$



poles $= 1, -2$ $\quad \sigma_{max} = 1$.

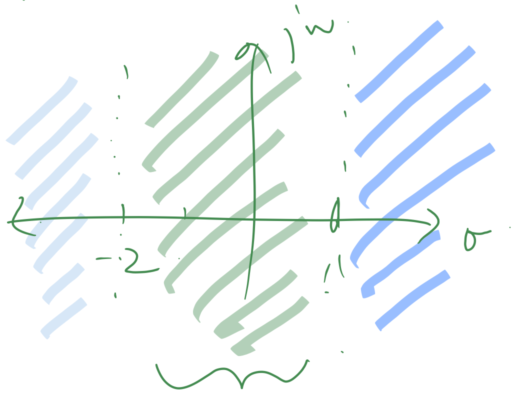
$\therefore Re\{s\} > 1 \checkmark$

(L) $Re\{s\} > -2$.

$$\therefore h(t) = -\frac{1}{3} e^{-2t} u(t) + \frac{1}{3} e^{t} u(t)$$

ii) System is stable

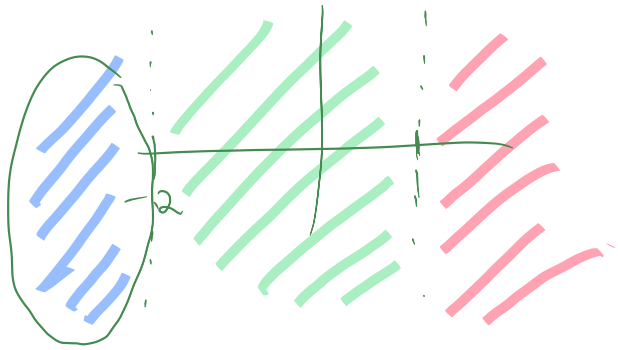
jω axis should be included in ROC!



$1 > Re\{s\} > -2 \;\}$ ROC

$h(t) = -\frac{1}{3}e^{-2t} - \frac{1}{3}e^{t}u(-t)$

iii) System → neither causal nor stable.



No stability → jω axis should not be in ROC

No causality ⇒ ROC should not be right sided.

→ valid ROC.

$Re\{s\} < -2 \rightarrow Re\{s\} < 1$

ROC.

$\therefore$ ILT of $H(s) = \frac{1}{3}e^{-2t}u(-t) - \frac{1}{3}e^{t}u(t)$.