W-11

Example problems of DFT:

(6) IDTFT of $2\pi\delta(\Omega-\Omega_0) = ?$

$$\frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi}\cancel{2\pi}\delta(\Omega-\Omega_0)e^{j\Omega n}\,d\Omega$$

$$= \int_{-\pi}^{\pi}\delta(\Omega-\Omega_0)\,e^{j\Omega n}\,d\Omega$$

$$= e^{j\Omega n}\Big|_{\Omega=\Omega_0}$$

$$= e^{j\Omega_0 n}$$

$$\boxed{e^{j\Omega_0 n} \longleftrightarrow 2\pi\delta(\Omega-\Omega_0)}$$

(7) DTFT of $\cos(\Omega_0 n) = ?$

$$\cos(\Omega_0 n) = \frac{e^{j\Omega_0 n} + e^{-j\Omega_0 n}}{2}$$

$$\longleftrightarrow \frac{1}{2}\cdot 2\pi\,\delta(\Omega-\Omega_0) + \frac{1}{2}\,2\pi\,\delta(\Omega+\Omega_0)$$

$$= \pi\delta(\Omega-\Omega_0) + \pi\delta(\Omega+\Omega_0)$$

$$\cos(\Omega_0 n) \longleftrightarrow \pi\delta(\Omega-\Omega_0) + \pi\delta(\Omega+\Omega_0)$$

(8) IDTFT of $\frac{1}{(1-ae^{-j\Omega})^2}$; $|a| < 1$.

$$x(n) = a^n u(n)$$

$$X(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

$$\frac{dX(\Omega)}{d\Omega} = \frac{(1-ae^{-j\Omega})\cdot 0 - \frac{d}{d\Omega}(1-ae^{-j\Omega})\cdot 1}{(1-ae^{-j\Omega})^2}$$

$$= -\frac{(-ae^{-j\Omega}(-j))}{(1-ae^{-j\Omega})^2}$$

$$\frac{dX(\Omega)}{d\Omega} = \frac{-jae^{-j\Omega}}{(1-ae^{-j\Omega})^2}$$

$$\Rightarrow \boxed{\frac{j\,dX(\Omega)}{d\Omega} = \frac{ae^{-j\Omega}}{(1-ae^{-j\Omega})^2}}$$

$$n\,x(n) \qquad n a^n u(n).$$

$$na^n u(n) \longrightarrow \frac{ae^{-j\Omega}}{(1-ae^{-j\Omega})^2}$$

$$\Rightarrow -na^n u(n) \longleftrightarrow \frac{-ae^{-j\Omega}}{(1-ae^{-j\Omega})^2}$$

$$= \frac{(1-ae^{-j\Omega}) - 1}{(1-ae^{-j\Omega})^2}$$

$$= \frac{1}{1-ae^{-j\Omega}} - \frac{1}{(1-ae^{-j\Omega})^2}$$

$$a^n u(n) - \tilde{x}(n)$$

$$\Rightarrow -n a^n u(n) = a^n u(n) - \tilde{x}(n)$$

$$\Rightarrow \tilde{x}(n) = a^n u(n) + n a^n u(n)$$

$$= a^n u(n) [1+n]$$

$$\boxed{(n+1) a^n u(n) \longleftrightarrow \frac{1}{(1-ae^{-j\Omega})^2}}$$

$\to$ IDTFT of given quantity

$$\frac{1}{(1-ae^{-j\Omega})^2} = \frac{1}{(1-ae^{-j\Omega})} \times \frac{1}{(1-ae^{-j\Omega})}$$

$$\longleftrightarrow a^n u(n) * a^n u(n)$$

$$= \sum_{k=-\infty}^{\infty} a^k u(k) a^{n-k} u(n-k)$$

$$= a^n \sum_{k=-\infty}^{\infty} u(k) u(n-k)$$

$$= a^n \sum_{k=0}^{n} 1 \quad \text{for } n \geq 0$$

$$= \underline{(n+1) a^n u(n) \text{ For } n \geq 0}$$

IDTFT of given signal.

⑨ DTFT of $u(n) = ?$

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$= x(n) + y(n)$$

$$x(n) = \frac{1}{2} \; \forall \; n,$$

$$y(n) = \frac{1}{2}, \; n \geq 0$$

$$= -\tfrac{1}{2}, \; n < 0$$

$$x(n) = \tfrac{1}{2}, \quad 1 \longleftrightarrow 2\pi\delta(\Omega)$$

$$\Rightarrow \tfrac{1}{2} \longleftrightarrow \pi\delta(\Omega)$$

$$\Rightarrow X(\Omega) = \pi\delta(\Omega)$$

$$y[n] - y[n-1] = \delta(n).$$

$$Y(\Omega)(1 - e^{-j\Omega}) = 1$$

$$\Rightarrow Y(\Omega) = \frac{1}{1 - e^{-j\Omega}}$$

$$u(n) = x(n) + y(n)$$

$$\Rightarrow U(\Omega) = X(\Omega) + Y(\Omega)$$

$$\Rightarrow \boxed{U(\Omega) = \pi\delta(\Omega) + \frac{1}{1 - e^{-j\Omega}}}$$

DTFT of unit step signal

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

accumulation property of DTFT

Similar to integrator for CT signals.

$$x(n) \longleftrightarrow X(\Omega)$$

$$y(n) \longleftrightarrow ?$$

$$y(n) = \sum_{k=0}^{n} x(k)$$

$$= x(n) * u(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, u(n-k)$$

$$= \sum_{k=-\infty}^{n} x(n)$$

$$Y(\Omega) = X(\Omega)\, U(\Omega)$$

$$= X(\Omega) \left\{ \pi\delta(\Omega) + \frac{1}{1-e^{-j\Omega}} \right\}$$

$$= \pi\, X(\Omega)\, \delta(\Omega) + \frac{X(\Omega)}{1-e^{-j\Omega}}$$

$$\boxed{Y(\Omega) = \pi\, X(0)\, \delta(\Omega) + \frac{X(\Omega)}{1-e^{-j\Omega}}.}$$

DTFT of accumulator.

L-64. examples problems DTFT:

(10) Impulse response:

$$y(n) - \tfrac{7}{6}\, y(n-1) + \tfrac{1}{6}\, y(n-2) = x(n).$$

Difference equation

Frequency response = ?

$h(n) = ?$

Taking DTFT on both sides

$$y(\Omega) - \tfrac{5}{6} y(\Omega) e^{-j\Omega} + \tfrac{1}{6} y(\Omega) e^{-j2\Omega} = x(\Omega)$$

$$y(\Omega)\left(1 - \tfrac{5}{6} e^{-j\Omega} + \tfrac{1}{6} e^{-j2\Omega}\right) = x(\Omega)$$

$$\Rightarrow \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - \tfrac{5}{6} e^{-j\Omega} + \tfrac{1}{6} e^{-j2\Omega}} = H(\Omega)$$

Freq response

$$H(\Omega) = \frac{1}{\left(1 - \tfrac{1}{3} e^{-j\Omega}\right)\left(1 - \tfrac{1}{2} e^{-j\Omega}\right)}$$

PF expansion

$$= 6\left\{ \frac{1/2}{1 - \tfrac{1}{2} e^{-j\Omega}} - \frac{1/3}{1 - \tfrac{1}{3} e^{-j\Omega}} \right\}$$

$$\frac{1}{1 - a e^{-j\Omega}} \xrightarrow{\text{IDFT}} a^n u[n].$$

$$6\left\{ \tfrac{1}{2}\left(\tfrac{1}{2}\right)^n u[n] - \tfrac{1}{3}\left(\tfrac{1}{3}\right)^n u[n] \right\}$$

$$\boxed{h[n] = 3\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{3}\right)^n u[n]}$$

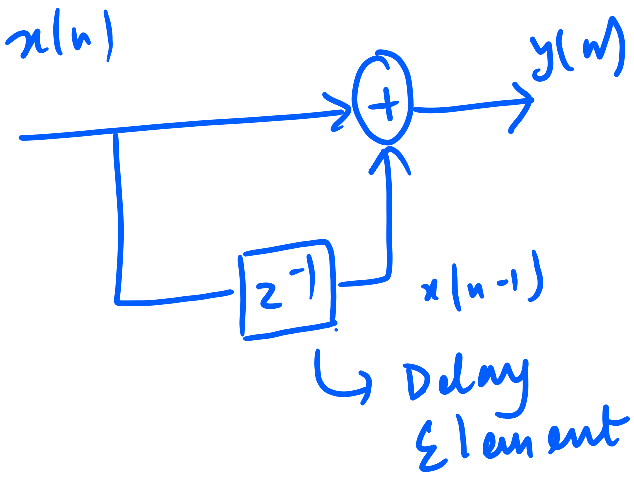
impulse Response of LTI system described by D.E.

(ii) Consider LTI system in Figure

$H(\Omega) = ?$

$h[n] = ?$

3 dB Frequency : ?



$$\boxed{y(n) = x(n) + x(n-1)}$$

DE for above system.

To find h(n), set $x(n) = \delta(n)$

$$h(n) = \delta(n) + \delta(n-1)$$

$$h(0) = \delta(0) + \delta(-1)$$

$$= 1$$

$$h(1) = 1$$

$$h(n) = 0 \quad \text{for } n \neq 0 \text{ or } 1$$

$$h(n) = \begin{cases} 1, & 0 \le n \le 1 \\ 0, & \text{o/w.} \end{cases}$$

impulse response.

Taking DTFT

$$x(\Omega) + e^{-j\Omega} x(\Omega) = Y(\Omega)$$

$$x(\Omega)\left(1 + e^{-j\Omega}\right) = Y(\Omega)$$

$$\Rightarrow H(\Omega) = \frac{Y(\Omega)}{x(\Omega)} = 1 + e^{-j\Omega}$$

$$= e^{-j\Omega/2}\left\{e^{j\Omega/2} + e^{-j\Omega/2}\right\}$$

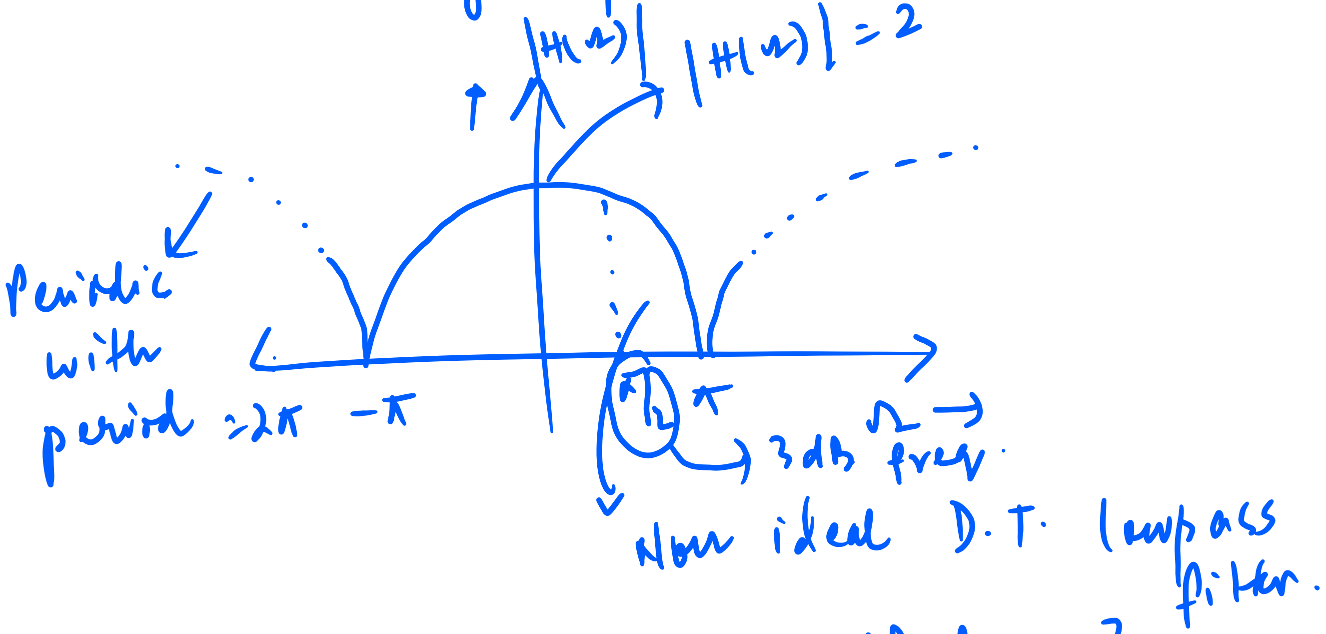
$$\boxed{H(\Omega) = e^{-j\Omega/2}\, 2\cos(\Omega/2)}$$

Freq response of given LTI system.

Periodic with period $= 2\pi$

$$|H(\Omega)| = 2\left|\cos\left(\frac{\Omega}{2}\right)\right|$$

Mag response



3 dB freq = ?

$$\max |H(\Omega)| = |H(0)|$$
$$= 2|\cos 0| = 2$$

$$H(\Omega_0) = \frac{1}{\sqrt{2}} \max |H(\Omega)|$$
$$= 1/\sqrt{2} \times 2 = \sqrt{2}$$

$$2\cos(\Omega_0/2) = \sqrt{2}$$
$$\cos(\Omega_0/2) = 1/\sqrt{2}$$
$$\Rightarrow \frac{\Omega_0}{2} = \pi/4$$
$$\Rightarrow \Omega_0 = \pi/2$$

3 dB freq $= \pi/2$

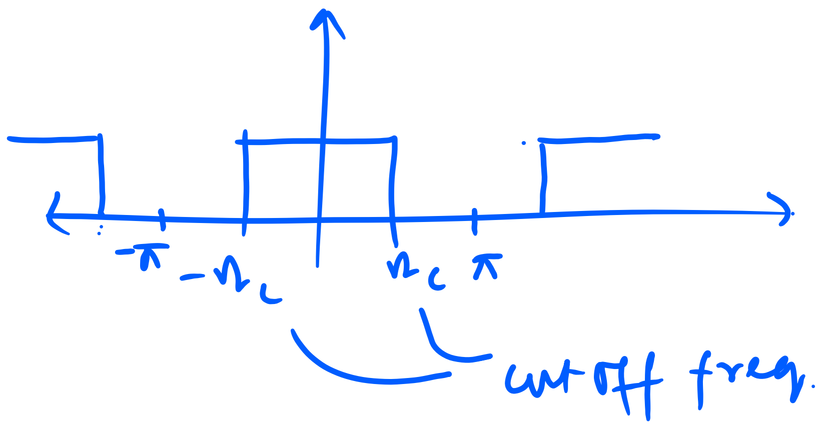
$$\boxed{3 \text{ dB BW} = \pi/2}$$

(12) HPF from LPF :

let $h(n)$ = impulse response of an ideal LPF cutoff $= w_c$

$$(-1)^n h(n) = \tilde{h}(n)$$

$$\tilde{H}(\Omega) = ?$$



$$\tilde{h}(n) = (-1)^n h(n)$$

$$= (e^{j\pi})^n h(n)$$

$$= e^{j\pi n} h(n)$$

$$e^{j\Omega_0 n} h(n)$$

$e^{j\Omega_0 n} \rightarrow$ modulation property

$$h(n) \longleftrightarrow H(\Omega)$$

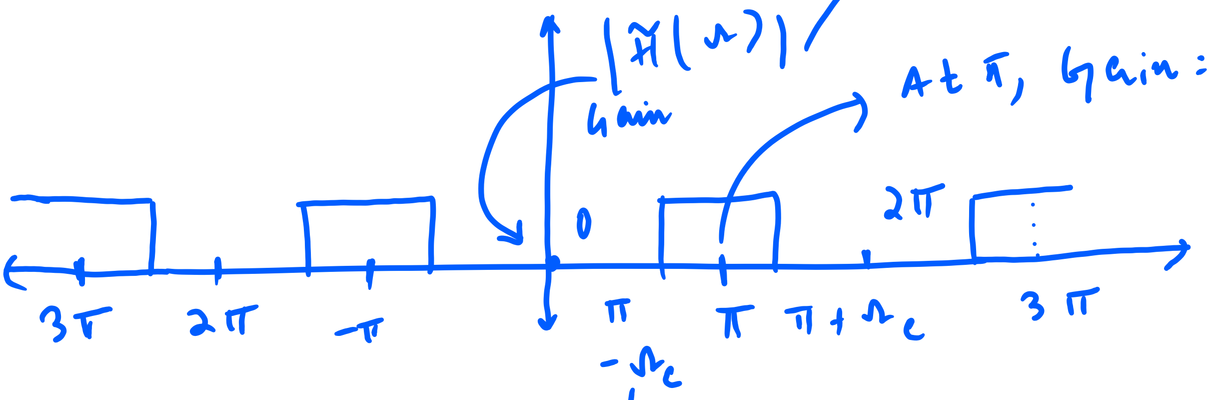
$$e^{j\Omega_0 n} \longleftrightarrow H(\Omega - \Omega_0)$$

$$\boxed{\text{DTFT} = H(\Omega - \pi)}$$

$$\boxed{\tilde{H}(\Omega) = H(\Omega - \pi)}$$

Shifted LPF

high pass filter.



cutoff frequency

$$\boxed{(-1)^n h(n) = \tilde{h}(n) \longleftrightarrow \tilde{H}(\Omega) = H(\Omega - \pi)}$$

ideal HPF cutoff $= \pi - \Omega_c$

L-65 Example problems (DTFT)

(13) $$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

DE for LTI system LPF

what is corresponding DE for HPF?

Taking DTFT,

$$Y(\Omega) + \sum_{k=1}^{N} a_k e^{jk\Omega} Y(\Omega) = \sum_{k=1}^{M} b_k e^{-jk\Omega} X(\Omega)$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

Transfer function

$$= \frac{\sum_{k=1}^{M} b_k e^{-jk\Omega}}{1 + \sum_{k=}^{N} a_k e^{-jk\Omega}}$$

To get HPF, shift by π

$$\Rightarrow \Omega \longrightarrow \Omega - \pi$$

$$\tilde{H}(\Omega) = \frac{\sum_{k=1}^{M} b_k e^{jk(\Omega-\pi))}}{1 + \sum_{k=1}^{N} a_k e^{-jk(\Omega-\pi)}}$$

$(-1)^k$

$$= \frac{\sum_{k=1}^{M} b_k (e^{j\pi})^k e^{-jk\Omega}}{1 + \sum_{k=1}^{N} a_k (e^{j\pi})^k e^{-jk\Omega}}$$

$$\tilde{H}(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{k=1}^{M} (-1)^k a_k e^{-jk\Omega}}{1 + \sum_{k=1}^{N} (-1)^k b_k e^{-jk\Omega}}$$

$$\Rightarrow Y(\Omega)\left(1+\sum_{k=1}^{N}(-1)^k b_k e^{-jk\Omega}\right) = X(\Omega)\sum_{k=1}^{M} a_k(-1)^k e^{-jk\Omega}$$

inverse DTFT

$$\Rightarrow \boxed{y(n) = -\sum_{k=1}^{N}(-1)^k b_k\, y(n-k) + \sum_{k=1}^{M}(-1)^k a_k\, x(n-k)}$$

$\hookrightarrow$ Difference equation of HPF.

(14) Consider impulse response $h(n)$

$h(n)$ = real

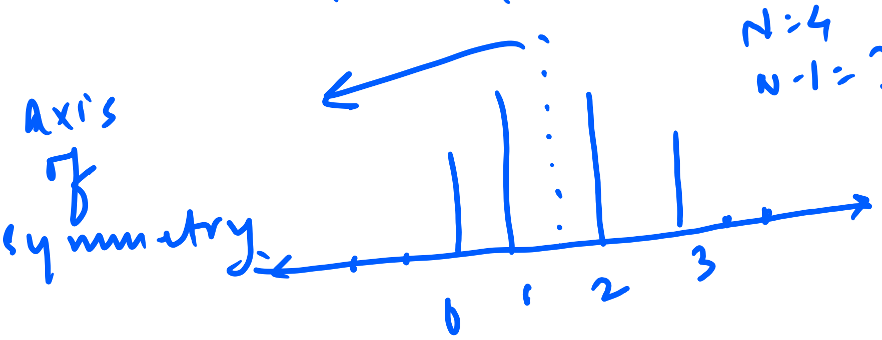
$h(n) = 0$, $n < 0$ or $n \geq N$

Non-zero only for $0 \leq n \leq N-1$.

$h(n) = h(N-1-n)$

$N = 4$

$N-1 = 3$



$h(0) = h(3-0)$
$= h(3)$

$h(1) = h(3-1) = h(2)$

Find phase response of $\theta(\Omega)$

$= ?$

$h(n) = h(N-1-n)$

↳ observe the flipping & delay by $N-1$ = same filter.

$\tilde{h}(n) = h(-n)$

$\Rightarrow \tilde{H}(\Omega) = H(-\Omega)$ → Time reversal property

Delay by $N-1$

$\tilde{\tilde{h}}(n) = \tilde{h}(n-(N-1))$

$= h(N-1-n)$

$= h(n)$

$\tilde{\tilde{H}}(\Omega) = \tilde{H}(\Omega)\, e^{-j(N-1)\Omega}$

$= H(-\Omega)\, e^{-j(N-1)\Omega}$

$= H^*(\Omega)\, e^{-j(N-1)\Omega}$

1 ← $= |H(\Omega)|\, e^{-j\theta(\Omega)}\, e^{-j(N-1)\Omega}$

from symmetry of filter → $= H(\Omega)$ $= \boxed{|H(\Omega)|\, e^{j\theta(\Omega)}}$ — 2

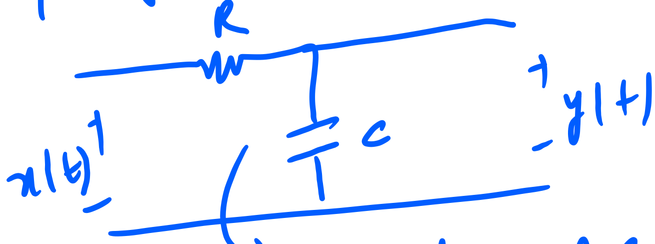
Equating phases of both terms, we have

$-\theta(\Omega) - (N-1)\Omega = \theta(\Omega)$

$\Rightarrow 2\theta(\Omega) = -(N-1)\Omega$

linear phase ← $\Rightarrow \boxed{\theta(\Omega) = -\frac{1}{2}(N-1)\Omega}$ → Linear phase constraint.

(15) Sampling: C.T → D.T



Consider $RC = 2$

DT filter obtained by sampling impulse response with sampling interval T_s.

→ impulse response of Transfer function.

$$y(t) + RC\frac{dy(t)}{dt} = x(t)$$

$$\Rightarrow Y(s) + 2sY(s) = X(s) \quad \Big\} \text{ given } RC = 2$$

$$\Rightarrow (1+2s)Y(s) = X(s)$$

$$\Rightarrow \underbrace{\frac{Y(s)}{X(s)}}_{H(s)} = \frac{1}{1+2s} = \frac{1}{2(s+1/2)}$$

$$\Rightarrow h(t) = \frac{1}{2}e^{-t/2}u(t).$$

→ taking inverse

$h(n)$ = sampled version of $h(t)$.

$$\tilde{h}(n) = h(nT_s)$$

$$= \frac{1}{2}e^{-nT_s/2}u(nT_s)$$

$$= \frac{1}{2} e^{-nT_s/2} u(n)$$

$$\tilde{h}(n) = \frac{1}{2}\left(e^{-T_s/2}\right)^n u(n) = \frac{1}{2} a^n u(n)$$

impulse response of equivalent DT system

z-transform

$$\tilde{H}(z) = \frac{1}{1 - e^{-T_s/2} z^{-1}}$$

Transfer function of equivalent DT system.

$$\tilde{H}(\Omega) = \frac{1}{1 - e^{-T_s/2} e^{-j\Omega}}$$

$$\frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - e^{-T_s/2} e^{-j\Omega}}$$

$$\Rightarrow Y(\Omega)\left(1 - e^{-T_s/2} e^{-j\Omega}\right) = X(\Omega)$$

$$\Rightarrow \boxed{y(n) - e^{-T_s/2} y(n-1) = x(n)}$$

DE of DT system equivalent DT system.

L-66 Example problems.

16 consider the C.T system with

$$H(s) = \frac{1}{(s+2)(s+3)}$$

sample with duration $= T_s$

What is $H_d(\Omega)$ DTFT of resulting system?

$$H(s) = \frac{1}{(s+2)(s+3)}$$

$$= \frac{1}{s+2} \quad \frac{-1}{s+3}$$

$$\hookrightarrow e^{-2t}u(t) - e^{-3t}u(t)$$

$$h(t) = \left(e^{-2t} - e^{-3t}\right)u(t)$$

$\hookrightarrow$ impulse response

$$h(nT_s) = h_d(n)$$

$\to$ DT impulse response

$$= \left(e^{-2nT_s} - e^{-3nT_s}\right)u(nT_s)$$

$$= e^{-2nT_s}u(n) - e^{-3nT_s}u(n)$$

$$= \left(e^{-2T_s}\right)^n u(n) - \left(e^{-3T_s}\right)^n u(n)$$

$$H_d(z) = \frac{1}{1-e^{-2T_s}z^{-1}} - \frac{1}{1-e^{-3T_s}z^{-1}}$$

Substitute $z = e^{j\Omega}$

$$H_d(\Omega) = \frac{1}{1-e^{-2T_s}e^{-j\Omega}} - \frac{1}{1-e^{-3T_s}e^{-j\Omega}}$$

$\hookrightarrow$ Frequency response of corresponding DT system.

(17) Consider LTI system with IIR $h(n)$
$\downarrow$
Infinite impulse response

Find FIR (finite impulse response) approximation s.t.

$\tilde{h}(n) = 0$, $n < 0$
$n > N-1$

non-zero only for $0 \le n \le N-1$

such that $\int_{-\pi}^{\pi} |H(\Omega) - \tilde{H}(\Omega)|^2 d\Omega$

Best FIR approximation to given IIR

Observe

$$h(n) \longleftrightarrow H(\Omega)$$
$$\tilde{h}(n) \longleftrightarrow \tilde{H}(\Omega)$$
$$\Rightarrow h(n) - \tilde{h}(n) \longleftrightarrow H(\Omega) - \tilde{H}(\Omega)$$

from Linearity

Parseval's theorem

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\Omega) - \tilde{H}(\Omega)|^2 d\Omega$$

$$= \sum_{n=-\infty}^{\infty} |h(n) - \tilde{h}(n)|^2$$

$\tilde{h}(n) = 0$

Do not depend on $\tilde{h}(n)$ $= \sum_{n=-\infty}^{-1} |h(n)|^2 + \sum_{n=N}^{\infty} |h(n)|^2 = c$

$$+\sum_{n=0}^{N-1} \left| h(n) - \tilde{h}(n) \right|^2$$

To minimise error

minimise this to minimise original error.

always ≥ 0

min occurs when $\boxed{h(n) = \tilde{h}(n)}$

$0 \leq n \leq N-1$

Best FIR approximation to $h(n)$ given IIR filter.

$$\tilde{h}(n) = \begin{cases} 0, & n < 0 \\ & n > N-1 \\ h(n) & 0 \leq n \leq N-1 \end{cases}$$

This minimises the squared error of the frequency response.

(19) DFT

Discrete Fourier Transform

$x(n) = \sin\left(\pi/_2\, n\right) \quad 0 \leq n \leq 3$

$h(n) = 2^n \quad 0 \leq n \leq 3.$

$N = 4$

length $= 4$

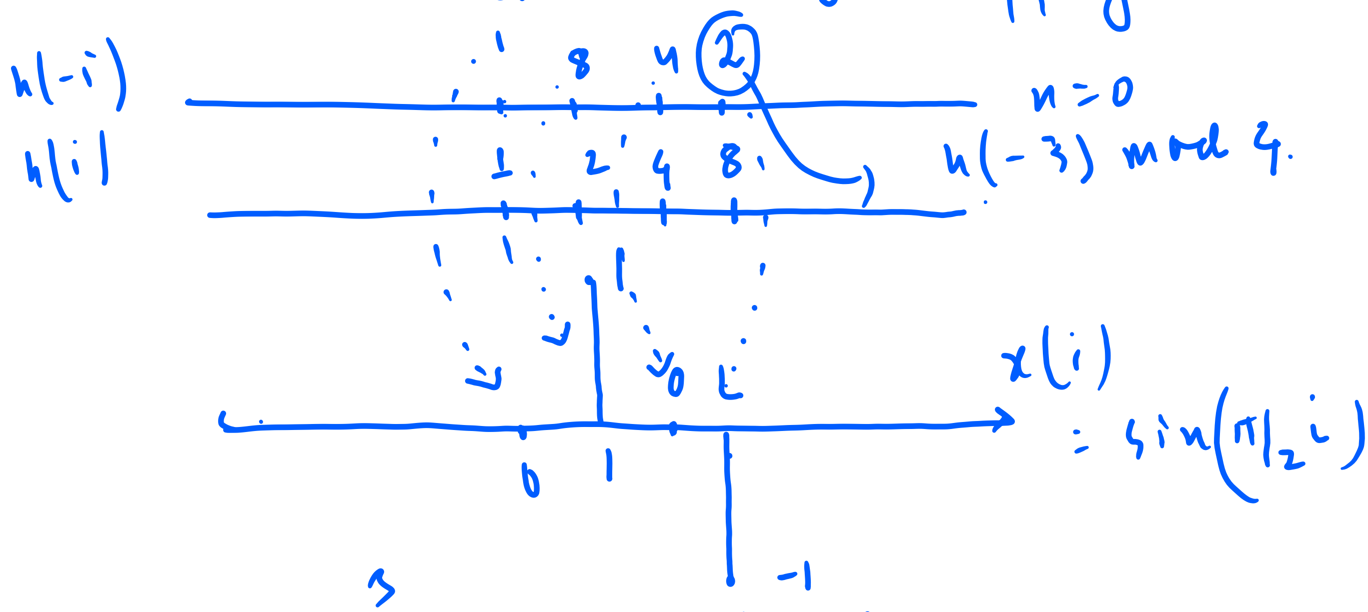
$y(n) = x(n) \circledast h(n)$

circular convolution

$$\boxed{y(n) = \sum_{i=0}^{N-1} x(i)\, h(n-i) \bmod N.}$$

circular convolution

convolution by wrapping around



$$y(0) = \sum_{i=0}^{3} x(i)\, h(-i) \bmod 4$$

$= x(0)\,h(0) + x(1)\,h(-1) \bmod 4$

$+ x(2)\,h(-2) \bmod 4 + x(3)\,h(-3) \bmod 4$

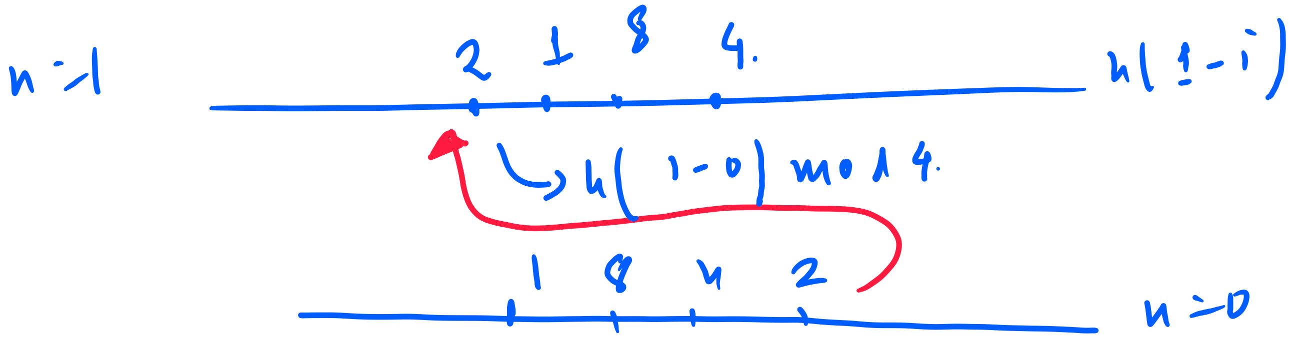
$= x(0)\,h(0) + x(1)\,h(3) + x(2)\,h(2)$

$+ x(3)\,h(1)$

$= 0 \times 1 + 1 \times 8 + 0 \times 4 + (-1 \times 2)$

$= 1 \times 8 - 1 \times 2$

$= 6$



$$y(1) = x(0)\,h(1)_{\bmod 4} + x(1)\,h(0) \bmod 4$$
$$+\ x(2)\,h(-1) \bmod 4$$
$$+\ x(3)\,h(-2)$$

$$= x(0)\,h(1) + x(1)\,h(0)$$
$$+\ x(2)\,h(3) + x(3)\,h(2)$$
$$= 1\times 1 - 4\times 1$$
$$= -3$$

8 4 2 1 — $n = 3$

4 2 1 8 — $n = 2$

$$y(2) = 2\times 1 + 8\times(-1)$$
$$= 2 - 8 = -6$$

$$y(3) = 4\times 1 + 1\times(-1)$$
$$= 4 - 1$$
$$= 3.$$

$$y(n) = \underbrace{6, -3, -6, 3}_{0 \le n \le 3.}$$

→ o/p of circular conv of given signals.

L-67. Example problems (DFT):

$$x(n) = \sin\left(\tfrac{\pi}{2} n\right) \quad 0 \le n \le 3.$$

$$= 0, 1, 0, -1$$

$$h(n) = 1, 2, 4, 8.$$

$$y(n) = x(n) \circledast h(n)$$

→ circular convolution

Evaluate circular convolution using DFT

DFT

$$W_N = W_4 = e^{-j2\pi/N} = e^{-j2\pi/4} = e^{-j\pi/2} = -j$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$= \underbrace{x(0)}_{=0} + x(1) W_N^{k} + \underbrace{x(2)}_{=0} W_N^{2k} + x(3) W_N^{3k}$$

$$\boxed{X(k) = W_N^{k} - W_N^{3k}}$$

$$H(k) = \sum_{n=0}^{N-1} h(n) W_N^{kn}$$

$$= 1 + 2W_N^{k} + 4W_N^{2k} + 8W_4^{3k}$$

$$Y(k) = X(k)H(k)$$

$$= \left(W_N^{k} - W_N^{3k}\right) \times \left(1 + 2W_N^{k} + 4W_N^{2k} + 8W_N^{3k}\right)$$

$$W_N^{4k} = \left(e^{j2\pi/4}\right)^{4k} = e^{j2\pi k} = 1$$

$$\boxed{W_N^{Nk} = 1}$$

N=4 in this example

$$Y(k) = \left(W_N^{k} - W_N^{3k}\right)\left(1 + 2W_N^{k} + 4W_N^{2k} + 8W_N^{3k}\right)$$

$$= W_N^k - W_N^{3k} + 2W_N^{2k} - 2 + 4W_N^{3k} - 4W_N^k + 8$$
$$-8W_N^{2k}$$

$$Y(k) = 6 - 3W_N^k - 6W_N^{2k} + 3W_N^{3k}$$
$$= y(0) + y(1)W_N^k + y(2)W_N^{2k} + y(3)W_N^{3k}$$

$y(0) = 6$
$y(1) = -3$
$y(2) = -6$
$y(3) = 3$

Time Domain evaluation.

(19) DFT of $e^{j\Omega_0 n}$ $\quad 0 \le n \le N-1$

$$\Omega_0 = \frac{2\pi k}{N}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$
$$= \sum_{n=0}^{N-1} e^{j\Omega_0 n} W_N^{kn}$$
$$= \sum_{n=0}^{N-1} \left(e^{j\Omega_0} W_N^k\right)^n$$
$$= \frac{1 - e^{j\Omega_0 N} W_N^{Nk}}{1 - e^{j\Omega_0} W_N^k}$$

$$= \frac{1 - e^{j\Omega_0 N} e^{-j\frac{2\pi}{N} k N}}{1 - e^{j\Omega_0} e^{-j2\pi/N \, k}}$$

$$W_N^k = e^{-j\frac{2\pi}{N} kn}$$

$$= \frac{1 - e^{j(\Omega_0 - 2\pi k/N) N}}{1 - e^{j(\Omega_0 - 2\pi k/N)}}$$

$$= \frac{e^{j(\Omega_0 - \frac{2\pi k}{N}) N/2}}{e^{j(\Omega_0 - \frac{2\pi k}{N}) 1/2}} \times \frac{e^{-j(\Omega_0 - \frac{2\pi k}{N}) N/2} - e^{j(\Omega_0 - \frac{2\pi k}{N}) N/2}}{e^{-j(\Omega_0 - \frac{2\pi k}{N}) 1/2} - e^{j(\Omega_0 - \frac{2\pi k}{N}) 1/2}}$$

$$X(k) = e^{j(\Omega_0 - \frac{2\pi k}{N}) \frac{(N-1)}{2}} \times \frac{\sin\left[\left(\Omega_0 - \frac{2\pi k}{N}\right)\frac{N}{2}\right]}{\sin\left(\left(\Omega_0 - \frac{2\pi k}{N}\right)\frac{1}{2}\right)}$$

$X(k) \rightarrow$ k^{th} DFT coefficient

L-68 Example problems for discrete Fourier transforms.

(20) DFT, IDFT in MATRIX form

$$\bar{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \rightarrow N \times 1 \text{ vector} = N \text{ samples}$$

$$\bar{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \longrightarrow \text{DFT coefficients}$$

Express: $\bar{X} = W_N \bar{x}$ → DFT Matrix

$\bar{x} = \tilde{W}_N \bar{X}$ → IDFT Matrix.

Linear Transformations.

DFT given as

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$X(0) = x(0) + x(1) + \cdots + x(N-1)$$

$$X(1) = x(0) + x(1) W_N + x(2) W_N^2 + \cdots + x(N-1) W_N^{N-1}$$

$$X(2) = x(0) + x(1) W_N^2 + x(2) W_N^4 + \cdots + x(N-1) W_N^{2(N-1)}$$

$$X(N-1) = x(0) + x(1) W_N^{N-1} + x(2) W_N^{2(N-1)} + \cdots + x(N-1) W_N^{(N-1)^2}$$

write in Matrix form

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 \cdots & W_N^{2(N-1)} \\ \vdots & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} \cdots & W_N^{(N-1)^2} \end{bmatrix}}_{\substack{N \times N \\ \text{matrix} = \text{DFT matrix}}} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\substack{N \times 1 \\ \text{signal vector}}}$$

$$W_N = e^{-j2\pi/N}$$

$$\boxed{\vec{X} = W_N \vec{x}}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$$

$$\underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\substack{N \times 1 \\ \text{signal vector}}} = \frac{1}{N} \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & & & & \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}}_{\tilde{W}_N = N \times N \text{ IDFT matrix}} \underbrace{\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}}_{\substack{N \times 1 \\ \text{DFT vector}}}$$

$$\boxed{\bar{x} = \tilde{W}_N \bar{X}} \rightarrow 2$$

From 1, 2

$$\bar{x} = \tilde{W}_N \bar{X} = \tilde{W}_N W_N \bar{x}$$

$$\Rightarrow \bar{x} = \tilde{W}_N W_N \bar{x}$$

$$\boxed{\tilde{W}_N W_N = I}$$

$$\tilde{W}_N = \tilde{W}_N^{-1}$$

$$\tilde{W}_N = W_N^{-1}.$$

$$W_N(2,3) = W_N^2 = W_N(3,2)$$

$$W_N(i,j) = W_N(j,i).$$

$$\boxed{W_N = W_N^T}$$

Prop 1 $\quad \tilde{W}_N = W_N^{-1}$

Prob 2 $\quad \Rightarrow W_N = W_N^T$

$$\tilde{W}_N(i,j) = \tilde{W}_N(j,i)$$

$$\Rightarrow \tilde{W}_N = \tilde{W}_N^T$$

Prop 3 $\quad$ Ex $\quad W_N(2,3) = W_N^2$

$$= e^{j4\pi/N}$$

$$\tilde{W}_N(2,3) = 1/N \; W_N^{-2}$$

$$= \frac{1}{N} e^{j4\pi/N}$$

$$\Rightarrow \tilde{W}_N(2,3) = 1/N \left(W_N(2,3) \right)^*$$

$$\tilde{W}_N = \frac{1}{N} W_N^*$$

$$= \frac{1}{N} \left(W_N^T \right)^*$$

$$\boxed{\tilde{W}_N = \frac{1}{N} W_N^H}$$

Prop 4

$$\tilde{W}_N W_N = I_{N \times N}$$

$$\Rightarrow \frac{1}{N} W_N^H W_N = I$$

$$\Rightarrow \boxed{W_N^H W_N = N\, I}$$

$N \times N$ identity matrix

$$W_N^{-1} = \frac{1}{N} W_N^H$$

$$= \frac{1}{N} W_N^*$$

Ex: $N=4$, DFT, IDFT matrices.

$$W_4 = e^{-j2\pi/4} = e^{-j\pi/2}$$

$$= -j$$

$$\underset{4\times 4}{\overline{W}_4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 \\ 1 & W_N^2 & W_N^4 & W_4^6 \\ 1 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \rightarrow N=4$$

$$\underbrace{= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}}_{4\times 4 \text{ DFT matrix}}$$

$$\tilde{W}_N = \frac{1}{4} W_4^*$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$\hookrightarrow$ 4x4 IDFT matrix

$$\tilde{W}_N = 1/N \, W_N^*$$

$$\tilde{W}_N \overline{W}_N = I$$

$$\overline{W}_N^H \, \overline{W}_N = N \, I_{N\times N}.$$

HW $\Rightarrow$ verify these properties in <u>MATLAB</u>