Week 8

Examples Problems Fourier Analysis:

① consider

$$x(t) = \cos(2t + \pi/4)$$

complex exponential Fourier series (CEFS).

$$w_0 = 2$$

$$\text{Period} = 2\pi/w_0 = T_0$$

$$= 2\pi/2 = \pi$$

$$w_0 = 2$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$x(t) = \cos(2t + \pi/4)$$

$$= \frac{e^{j(2t+\pi/4)} + e^{-j(2t+\pi/4)}}{2}$$

$$= \frac{1}{2}\left[e^{j\pi/4} e^{j2t}\right] + \frac{1}{2}\left[e^{-j2t} e^{-j\pi/4}\right]$$

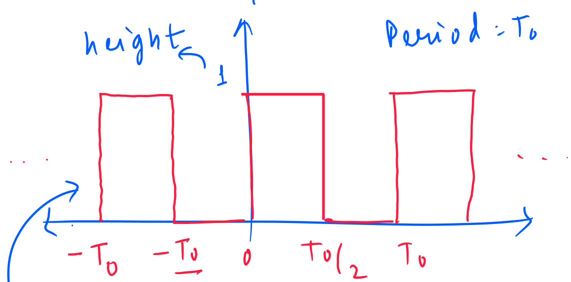
$$= c_1 e^{jw_0 t} + c_{-1} e^{-jw_0 t}$$

$$\boxed{c_1 = \frac{1}{2} e^{j\pi/4} = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + j/\sqrt{2}\right) = \frac{1+j}{2\sqrt{2}}}$$

$$\boxed{c_{-1} = \frac{1}{2} e^{-j\pi/4} = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right) = \frac{1-j}{2\sqrt{2}}}$$

$$c_k = 0 \quad \forall\ |k| \neq 1$$

② Periodic Square wave



Find complex exponential Fourier series, Trigonometric Fourier Series (TFS).

Fundamental Period $= T_0$

$\Rightarrow w_0 = 2\pi / T_0$

$$\Rightarrow x(t) = \underbrace{\sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}}_{\text{CEFS.}}$$

$$c_k = 1/T_0 \int_0^{T_0} x(t) e^{-jkw_0 t} dt$$

$$= \frac{1}{T_0} \int_0^{T_0/2} e^{-jkw_0 t} dt$$

$$= 1/T_0 \left[\frac{e^{-jkw_0 t}}{-jkw_0} \right]_0^{T_0/2}$$

$$= 1/T_0 \left[\frac{e^{-jkw_0 \frac{T_0}{2}} - 1}{-jkw_0} \right]$$

$(w_0 T_0 = 2\pi)$

$$= \frac{1}{T_0} \left[\frac{e^{-jk 2\pi/2} - 1}{-jkw_0} \right]$$

$$= \frac{1}{T_0} \left[\frac{1 - e^{-jk\pi}}{jkw_0} \right]$$

$\left[\because w_0 = 2\pi / T_0 \right]$

$$c_k = \frac{1}{jk 2\pi} \left[1 - e^{-jk\pi} \right]$$

$$(e^{-j\pi})^k = (-1)^k$$

$$= \frac{1}{jk 2\pi} \left[1 - (-1)^k \right]$$

$k =$ even ie $k = 2m$

$$c_k = \frac{1 - (-1)^{2m}}{j 2m \cdot 2\pi} = 0$$

$k =$ odd $\Rightarrow k = 2m+1$

$$c_k = \frac{1 - (-1)^{2m+1}}{j(2m+1) 2\pi}$$

$$c_{2m+1} = \frac{2}{2j(2m+1)\pi 2}$$

For any odd $k = 2m+1$

$$C_0 = 1/T_0 \int_0^{T_0} x(t)\,dt$$

$$= \frac{1}{T_0} \int_0^{T_0/2} 1\,dt$$

$$= \frac{1}{T_0} \frac{T_0}{2}$$

$$= 1/2$$

$$x(t) = \frac{1}{2} + \frac{1}{j\pi} \sum_{m=-\infty}^{\infty} \left(\frac{1}{2m+1}\right) e^{j(2m+1)w_0 t}$$

$$\boxed{x(t) = \frac{1}{2} + \frac{1}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{(2m+1)} e^{j(2m+1)w_0 t}}$$

→ CEFS

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Example Problems Fourier analysis.

$$\boxed{x(t) = \frac{1}{2} + \frac{1}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{(2m+1)} e^{j(2m+1)w_0 t}}$$

→ CEFS

TFS : $x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k w_0 t + \sum_{k=1}^{\infty} b_k \sin(k w_0 t)$

$$\frac{a_0}{2} = c_0$$

$$\Rightarrow a_0 = 2c_0 = 2 \times 1/2 = 1$$

$$a_k = C_k + C_{-k}$$

$C_k = 0$ for even k.

$a_k = 0$ if $k =$ even

$k =$ odd $= 2m+1$.

$$C_{2m+1} = \frac{1}{j\pi(2m+1)}.$$

$$\Rightarrow a_{2m+1} = C_{2m+1} + C_{-(2m+1)}$$

$$= \frac{1}{j\pi(2m+1)} + \frac{1}{j\pi(-2m-1)}$$

$$= 0$$

$$\therefore \boxed{a_k = 0 \ \forall k}$$

$b_k = j(c_k - c_{-k})$
$= 0$ if $k =$ even.

$k = 2m+1 =$ odd

$$b_k = j\left(\frac{1}{j\pi(2m+1)} - \frac{1}{j\pi(-2m-1)}\right)$$

$$\boxed{b_{2m+1} = \frac{2}{\pi(2m+1)}}$$

TFS

$$x(t) = 1/2 + \sum_{m=0}^{\infty} \frac{2}{\pi(2m+1)} \times \sin(2m+1)\omega_0 t$$

$$\boxed{x(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)\omega_0 t}{(2m+1)}}$$

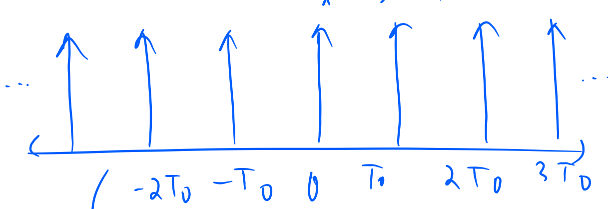
TFS of periodic square wave.

$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + 1/5 \sin 5\omega_0 t + \cdots\right)$$

TFS.

③ Periodic impulse train

spacing $= T_0$



Periodic impulse train.

Determine CEFS, TFS

$$\delta_{T_0} = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

impulses at integer multiples of T_0

$$x(t) = \delta(t) \quad -\frac{T_0}{2} \leq t < T_0/2$$

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-j2\pi k w_0 t}$$

$$= 1/T_0 \int_{-T_0/2}^{T_0/2} \delta(t)\, e^{-j2\pi k w_0 t}\, dt$$

$$= 1/T_0 \; e^{-j2\pi k w_0 t} \Big|_{t=0}$$

$$= 1/T_0$$

$$\therefore \boxed{c_k = 1/T_0}$$

CEFS:

$$\delta_{T_0}(t) = 1/T_0 \sum_{k=-\infty}^{\infty} e^{jk w_0 t}$$

TFS:

$$\delta_{T_0}(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k w_0 t + \sum_{k=1}^{\infty} b_k \sin k w_0 t$$

$$\frac{a_0}{2} = c_0 = 1/T_0$$

$$\boxed{a_0 = 2/T_0}$$

$$a_k = c_k + c_{-k} = \frac{2}{T_0}$$

$$b_k = j(c_k - c_{-k}) = j\left(1/T_0 - 1/T_0\right) = 0$$

$$\boxed{x(t) = 1/T_0 + \frac{2}{T_0} \sum_{k=1}^{\infty} \cos(k w_0 t)}$$

TFS of impulse train.

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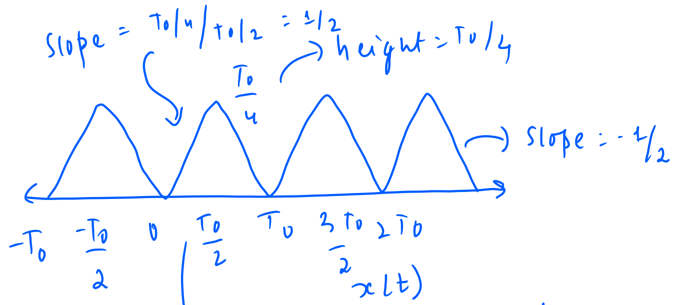
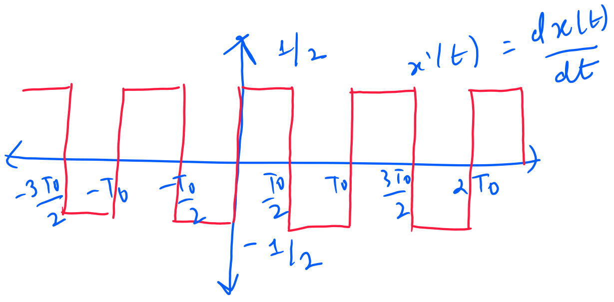


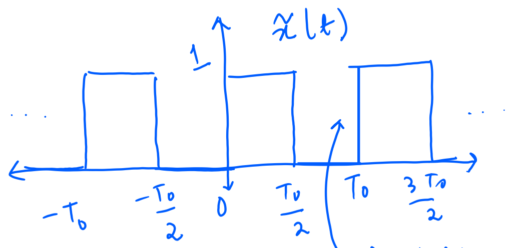
Periodic triangular wave

Determine CEFS, TFS

Consider $x'(t) = \frac{d\,x(t)}{dt}$



$x'(t) = \frac{1}{2} + x(t)$



Periodic square wave

$\tilde{x}(t) = x'(t) + 1/2$

$$\boxed{x'(t) = \tilde{x}(t) - 1/2}$$

lec-44- example problems Fourier analysis:

CEFS of $\tilde{x}(t)$

$$= \frac{1}{2} + \frac{1}{j\pi}\sum_{m=-\infty}^{\infty}\frac{e^{j(2m+1)\omega_0 t}}{(2m+1)} \quad (1)$$

CEFS of $x'(t) = \hat{x}(t) - 1/2$

$$= \frac{1}{j\pi}\sum_{m=-\infty}^{\infty}\frac{e^{j(2m+1)\omega_0 t}}{(2m+1)}$$

let CEFS of $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

$\Rightarrow$ CEFS of $x'(t) = \frac{d\,x(t)}{dt}$

$$= \sum_{k=-\infty}^{\infty} \underbrace{j k w_0 c_k}_{\tilde{c}_k} e^{jk w_0 t} \qquad (2)$$

$=$

From (1) and (2), compare the coefficients,

$$\tilde{c}_k = \frac{1}{j\pi}\,\frac{1}{(2m+1)} \qquad k = \text{odd} = 2m+1$$

If $k = \text{even}$.

$$\tilde{c}_k = 0$$

$$\Rightarrow j w_0 k c_k = 0$$

$$\Rightarrow c_k = 0 \text{ if } k \neq 0$$

$$\tilde{c}_{2m+1} = \frac{1}{j\pi}\,\frac{1}{(2m+1)}$$

$$\Rightarrow j w_0 (2m+1)\, c_{2m+1} = \frac{1}{j\pi(2m+1)}$$

$$\Rightarrow c_{2m+1} = \frac{1}{-\pi(2m+1)^2 w_0}$$

$$= \frac{1}{-\pi(2m+1)^2 \frac{2\pi}{T_0}}$$

$$\boxed{c_{2m+1} = \frac{-T_0}{-2\pi^2 (2m+1)^2}}$$

$$c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$

$$= \frac{1}{\cancel{T_0}}\,\frac{1}{2} \times \cancel{T_0} \times \frac{T_0}{4}$$

$$= \frac{T_0}{8}$$

$$\boxed{c_0 = \frac{T_0}{8}}$$

CEFS of $x(t)$

$$= \frac{T_0}{8} - \underbrace{\frac{T_0}{(2\pi)^2} \sum_{m=-\infty}^{\infty} \frac{e^{j(2m+1)\omega_0 t}}{(2m+1)^2}}_{\text{CEFS of Periodic Triangular wave}}$$

TFS : $\frac{a_0}{2} = c_0 = \frac{T_0}{8}$

$\Rightarrow a_0 = \frac{T_0}{4}$

$a_K = c_K + c_{-K}$

$K = \text{even} \Rightarrow c_K, c_{-K} = 0$

$\Rightarrow a_K = 0$

$K = \text{odd},$

$K = 2m+1$

$$a_{2m+1} = \frac{-T_0}{2\pi^2 (2m+1)^2} + \frac{-T_0}{2\pi^2(-2m-1)^2}$$

$$= \frac{-2T_0}{2\pi^2(2m+1)^2}$$

$$\boxed{a_{2m+1} = \frac{-T_0}{\pi^2 (2m+1)^2}}$$

$b_K = j(c_K - c_{-K})$

$= 0$ if $K = \text{even}$

$K = 2m+1$

$$= j\left(\frac{-T_0}{2\pi^2(2m+1)^2} - \frac{(-T_0)}{2\pi^2(-2m-1)^2}\right)$$

$b_{2m+1} = 0$

$\Rightarrow \boxed{b_K = 0 \quad \forall K}$

TFS of $x(t)$

$$\underbrace{= T_0/8 - \sum_{m=1}^{\infty} \frac{T_0 \, 2\cos(2m+1)\omega_0 t}{\pi^2 (2m+1)^2}}_{\text{TFS of periodic triangular wav}}$$

5. Periodic convolution

Consider 2 signals $x_1(t), x_2(t)$ that are periodic with common period T_0.

Periodic convolution:

$$\tilde{x}(t) = x_1(t) \circledast x_2(t)$$

↳ Periodic convolution

$$= \int_0^{T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

Derive CEFS of $\tilde{x}(t)$ given $x_1(t), x_2(t)$

$$x_1(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}$$

$$x_2(t) = \sum_{k=-\infty}^{\infty} e_k e^{jk\omega_0 t}$$

$$\tilde{x}(t) = \int_0^{T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \underbrace{\int_0^{T_0} x_1(\tau) \sum_{k=-\infty}^{\infty} e_k e^{jk\omega_0(t-\tau)}\, d\tau}_{\text{Interchange summation / integral}}$$

$$= \sum_{k=-\infty}^{\infty} e_k e^{jk\omega_0 t} \underbrace{\int_0^{T_0} x_1(\tau) e^{-j\omega_0 \tau k}\, d\tau}_{T_0 d_k}$$

$$d_k = \frac{1}{T_0}\int_0^{T_0} x_1(\tau) e^{-jk\omega_0 \tau}\, d\tau.$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} e_k e^{jk\omega_0 t}\, T_0 d_k$$

$$= \sum_{k=-\infty}^{\infty} \tilde{c}_k e^{jk\omega_0 t}$$

$$\boxed{\tilde{c}_k = T_0 d_k e_k}$$

k^{th} CEFS coefficient of periodic convolution of $x_1(t), x_2(t)$.

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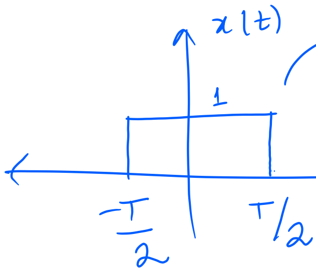
Example problems fourier analysis (CT signals) aperiodic

(6)



Rectangular pulse

$$x(t) = P_T(t)$$

$$= \begin{cases} 1, & |t| \le T/2 \\ 0, & \text{otherwise} \end{cases}$$

pulse of width = T
height = 1
centered at 0.

$$P_T(t) \longleftrightarrow P_T(\omega)$$

$$P_T(\omega) = \int_{-\infty}^{\infty} P_T(t) e^{-j\omega t} dt$$

$$= \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$= \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T/2}^{T/2}$$

$$= \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$= \frac{-2j \sin\left(\frac{\omega T}{2}\right)}{-j\omega}.$$

$$= \frac{2 \sin \omega T/2}{\omega}.$$

$$= \frac{2\, T/2\, \sin(\omega T/2)}{\omega T/2}$$

$$= \frac{T \sin(\omega T/2)}{(\omega T/2)}$$

$$= \frac{T \sin \pi \left(\frac{\omega T}{2\pi}\right)}{\left(\pi \frac{\omega T}{2\pi}\right)}$$

$$\text{sinc } u = \frac{\sin \pi u}{\pi u}$$

'sinc' function

$$T \,\text{sinc}\left(\frac{\omega T}{2\pi}\right) = P_T(\omega)$$

$$\omega / 2\pi = f$$

$$\boxed{P_T(f) = T \,\text{sinc}(fT)}$$

$$T \,\text{sinc}\left(\frac{\omega T}{2\pi}\right) = \frac{T \sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)}$$

zero if $\frac{\omega T}{2} = k\pi$

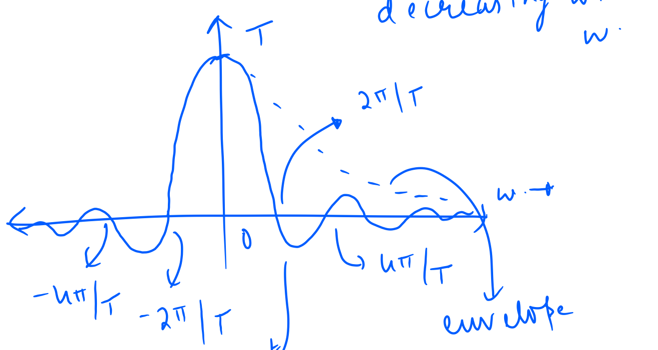
$$\omega = \frac{2k\pi}{T}$$

$$\omega = -\frac{2\pi}{T}, 0, 2\pi/T, \dots$$

$$\lim_{\omega \to 0} \frac{T \sin \frac{\omega T}{2}}{\frac{\omega T}{2}} = T$$

Envelope $= \frac{T}{\omega T/2} = 2/\omega$.

decreasing with ω.



$$T \,\text{sinc}(\omega T / 2\pi) = P_T(\omega)$$

8 FT of sinc pulse:

$$x(t) = \text{sinc}(\alpha t) = \frac{\sin(\pi \alpha t)}{\pi \alpha t}$$

$$X(\omega) = ?$$

Duality:

$$x(t) \longleftrightarrow X(\omega)$$

$$X(t) \longleftrightarrow 2\pi\, x(-\omega)$$

$$P_T(t) \longleftrightarrow T \operatorname{sinc}\left(\frac{\omega T}{2\pi}\right)$$

$$\Rightarrow T \operatorname{sinc}\left(\frac{tT}{2\pi}\right) \longleftrightarrow 2\pi P_T(-\omega)$$

$$= 2\pi P_T(\omega)$$

since $P_T(\omega)$ = even function

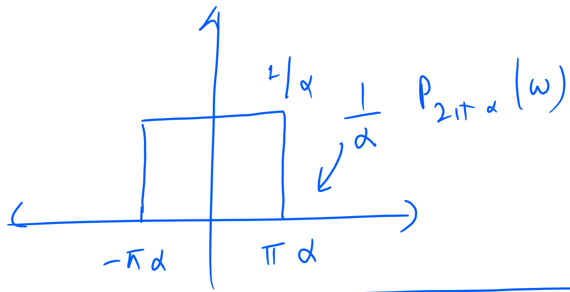
$$\Rightarrow \operatorname{sinc}\left(\frac{tT}{2\pi}\right) \longleftrightarrow \frac{2\pi}{T} P_T(\omega)$$

$$\frac{t}{2\pi} = \alpha \Rightarrow t = 2\pi\alpha$$

$$\operatorname{sinc}(\alpha t) \longleftrightarrow \frac{2\pi}{2\pi\alpha} P_{2\pi\alpha}(\omega)$$

$$= \frac{P_{2\pi\alpha}(\omega)}{\alpha}$$

Pulse of width $= 2\pi\alpha$
Height $= 1/\alpha$
centered at 0



$$\boxed{\operatorname{sinc}(\alpha t) \longleftrightarrow \frac{1}{\alpha} P_{2\pi\alpha}(\omega)}$$

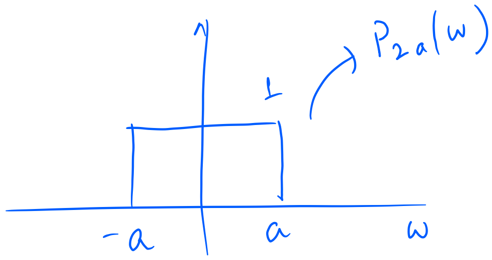
$$\alpha \operatorname{sinc}(\alpha t) \longleftrightarrow P_{2\pi\alpha}(\omega)$$

$$\alpha = a/\pi$$

$$\frac{a}{\pi} \operatorname{sinc}\left(\frac{a}{\pi} t\right) \longleftrightarrow P_{2a}(\omega)$$

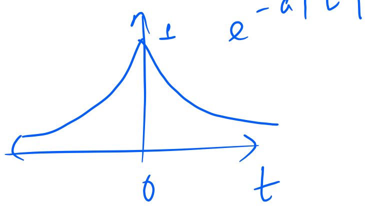
$$\Rightarrow \frac{a}{\pi} \frac{\sin at}{at} \longleftrightarrow P_{2a}(\omega)$$

$$\boxed{\frac{\sin at}{\pi t} \longleftrightarrow P_{2a}(\omega)}$$



lec-46 Fourier transform aperiodic signals.

8) $x(t) = e^{-a|t|}$ $\quad a > 0$.



$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \left.\frac{e^{(a-j\omega)t}}{a-j\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} + \frac{-1}{-(a+j\omega)}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{a+j\omega + a-j\omega}{a^2+\omega^2}$$

$$\boxed{X(\omega) = \frac{2a}{a^2+\omega^2}}$$

FT of $e^{-a|t|}$.

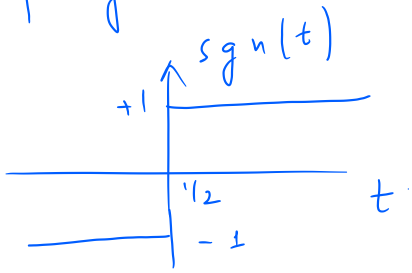
⑨ FT of $\cos w_0 t$

$$\cos w_0 t = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2}$$

$$X(w) = \frac{1}{2} 2\pi \delta(w - w_0) + \frac{1}{2} 2\pi \delta(w + w_0)$$

$$X(w) = \pi\delta(w - w_0) + \pi\delta(w + w_0)$$

⑩ FT of $\mathrm{sgn}(t)$



$$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \\ 1/2, & t = 0 \end{cases}$$

$$\tilde{x}(t) = \frac{d\,x(t)}{dt} = 2\delta(t)$$

$$\tilde{X}(w) = 2 \quad \left(\because \delta(t) \xleftrightarrow{FT} 1\right)$$

From properties of F.T,

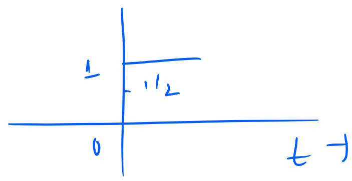
$$\tilde{X}(w) = jw\,X(w)$$

$$\Rightarrow jw\,X(w) = 2$$

$$\boxed{X(w) = \frac{2}{jw}}$$

⑪ FT of $u(t)$

↑ unit step fxn



$$u(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0 \end{cases}$$

$U(w) = ?$

$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$.

$\updownarrow$

$U(w) = \frac{1}{\cancel{2}} \cancel{2}\pi\delta(w) + \frac{1}{\cancel{2}} \frac{\cancel{2}}{jw}$.

$$\boxed{U(w) = \pi\delta(w) + \frac{1}{jw}}$$

Fourier Transform of unit step

(12) FT of even and odd components of $x(t)$

↓ Real
any signal $x(t)$

$X(w) = A(w) + jB(w)$

Real part of FT

Imaginary part of FT.

$x(t) = x_e(t) + x_o(t)$

even component of $x(t)$

odd component of $x(t)$

$x_e(t) = x_e(-t)$

$x_o(-t) = -x_o(t)$

$\Rightarrow x(t) = x_e(t) + x_o(t) \quad (1)$

$x(-t) = x_e(-t) + x_o(-t)$

$= x_e(t) - x_o(t) \quad (2)$

$$\boxed{\begin{aligned} x_e(t) &= \frac{x(t) + x(-t)}{2} \\ x_o(t) &= \frac{x(t) - x(-t)}{2} \end{aligned}}$$

even and odd components of $x(t)$

$$x_e(t) \longleftrightarrow X_e(\omega)$$

$$x_o(t) \longleftrightarrow X_o(\omega)$$

$$x(t) = x_e(t) + x_o(t)$$

$$\Rightarrow X(\omega) = X_e(\omega) + X_o(\omega) \quad \text{--- } 3$$

$$x(-t) = x_e(t) - x_o(t)$$

For real signal

$$x(-t) \longleftrightarrow X^*(\omega)$$

$$X^*(\omega) = X_e(\omega) - X_o(\omega). \quad \text{--- } 4$$

From 3, 4

$$X_e(\omega) = \frac{X(\omega) + X^*(\omega)}{2}$$

$$= \frac{2\,\mathrm{Re}\{X(\omega)\}}{2}$$

$$= A(\omega)$$

$$\boxed{X_e(\omega) = A(\omega)}$$

$$X_o(\omega) = \frac{X(\omega) - X^*(\omega)}{2}$$

$$= \frac{2j\,\mathrm{Im}\{X(\omega)\}}{2}$$

$$= jB\{\omega\}$$

$$\boxed{X(\omega) = jB(\omega)}$$

$$X(\omega) = A(\omega) + jB(\omega)$$

$$= X_e(\omega) + X_o(\omega)$$

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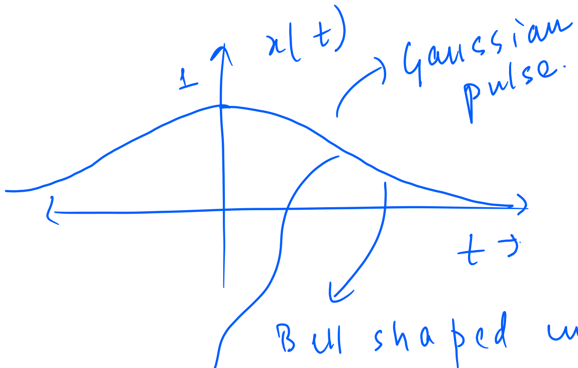
* Fourier transform (example problems).

13 FT of Gaussian Pulse

$x(t) = e^{-at^2}$



Gaussian Minimum Shift Keying (GMSK)

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$$

$$\frac{d\,X(\omega)}{d\omega} = \int_{-\infty}^{\infty} e^{-at^2} (-jt) e^{-j\omega t}\, dt$$

$$= \underbrace{-\int_{-\infty}^{\infty} jt\, e^{-at^2} e^{-j\omega t}\, dt}_{\text{Integration by parts}}$$

$$= \frac{j}{2a} e^{-at^2} e^{-j\omega t} \Big|_{-\infty}^{\infty}$$

$$\frac{-j}{2a} \int_{-\infty}^{\infty} e^{-at^2} (-j\omega) e^{-j\omega t}\, dt$$

$$= \frac{-\omega}{2a} \underbrace{\int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt}_{X(\omega)}$$

$$\boxed{\frac{dX(\omega)}{d\omega} = \frac{-\omega}{2a} X(\omega)}$$

$$\Rightarrow \underbrace{\frac{d\,X(\omega)}{X(\omega)} = -\frac{\omega}{2a}\, d\omega}_{\text{Differential equation.}}$$

Integrating on both sides

$$\Rightarrow \int_0^{w} \frac{dX(w)}{X(w)} = \int_0^{w} -\frac{w}{2a}\, dw.$$

$$\Rightarrow \ln X(w) \Big|_0^{w} = -\frac{w^2}{4a} \Big|_0^{w}.$$

$$\Rightarrow \ln X(w) - \ln X(0) = -\frac{w^2}{4a} = -\frac{w^2}{4a}.$$

$$\Rightarrow \ln\left(\frac{X(w)}{X(0)}\right) = -\frac{w^2}{4a}.$$

$$\boxed{X(w) = X(0)\, e^{-w^2/4a}}$$

evaluate $X(0)$ — (1)

$$X(0) = \int_{-\infty}^{\infty} e^{-at^2}\, dt$$

$$\frac{1}{\sqrt{2\pi \cdot \frac{1}{2a}}}\, e^{-\frac{t^2}{2 \cdot 1/2a}}.$$

Gaussian PDF
mean $= 0$
var $= 1/2a$.

$$\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-t^2/2\sigma^2}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \cdot 1/2a}}\, e^{-\frac{t^2}{2 \cdot 1/2a}}\, dt = 1$$

$$\int_{-\infty}^{\infty} e^{-at^2}\, dt = \sqrt{2\pi \cdot 1/2a}.$$

$$= \sqrt{\pi/a}.$$

$$\boxed{X(0) = \sqrt{\pi/a}}$$

$$\boxed{X(w) = \sqrt{\pi/a}\; e^{-w^2/4a}}$$

FT of Gaussian pulse

substituting $X(0)$ in (1)

$$e^{-at^2} \longleftrightarrow \sqrt{\pi/a}\; e^{-\omega^2/4a}.$$

(14) LTI systems.

Consider LTI system

$$\frac{dy(t)}{dt} + 2\underbrace{y(t)}_{\text{output}} = \underbrace{x(t)}_{\text{input}}$$

DE describes LTI systems.

Find output to $x(t) = e^{-t}u(t)$

Taking FT on both sides

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega)$$

$$Y(\omega)(j\omega + 2) = X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{2+j\omega} = H(\omega).$$

Frequency response of LTI system.

$$x(t) = e^{-t}u(t)$$

$$X(\omega) = \frac{1}{j\omega+1}$$

$$Y(\omega) = H(\omega)X(\omega)$$

$$= \frac{1}{j\omega+2} \cdot \frac{1}{j\omega+1}$$

$$= \underbrace{\frac{1}{j\omega+1} - \frac{1}{j\omega+2}}_{\text{PF expansion}}$$

Taking Inverse FT

$$\Rightarrow y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

$$\boxed{y(t) = (e^{-t} - e^{-2t})u(t)}$$

output of LTI system to given input $e^{-t}u(t)$

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FT Problems: Bode plot

(15) Bode plot

↳ Graphical representation of $H(w)$ for an LTI system.

a) $H(w) = 1 + \frac{jw}{100}$

Consider dB Power

$$G(w) = 10\log_{10}|H(w)|^2$$

$$\boxed{G(w) = 20\log_{10} H(w)}$$

↳ dB power definition

$$G(w) = 20\log_{10}\left(\left|1 + \frac{jw}{100}\right|\right)$$

$\underline{w_c} = 100$ Rad/s

corner frequency

$w << w_c$

$\Rightarrow w/w_c << 1$

$\Rightarrow jw/w_c + 1 \simeq 1$

$$G(w) = 20\log\left|\left(1 + \frac{jw}{w_c}\right)\right|$$

$$\approx 20\log 1 = 0$$

$\frac{w}{w_c} >> 1$

$\Rightarrow 1 + j\frac{w}{w_c} \approx jw/w_c$

$$G(w) \approx 20\log_{10}\left|\frac{jw}{100}\right|$$

$$= 20\log_{10} w - 20\log_{10} 10^2$$

$$\underbrace{G(w) = 20\log_{10} w - 40}_{w >> w_c}$$

$$G(10\omega) = 20\log_{10} 10\omega - 40$$

$$= 20\log_{10}\omega + 20\log_{10}10 - 40$$

$$= 20\log_{10}\omega + 20 - 40$$

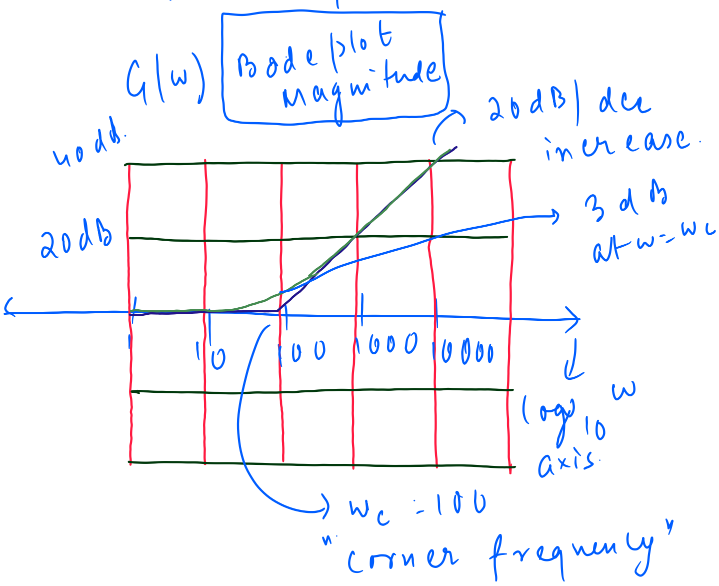
$$= 20 + \underbrace{20\log_{10}\omega - 40}_{G(\omega)}$$

$$= 20 + G(\omega)$$

$$\boxed{G(10\omega) = 20 + G(\omega)}$$

For a factor of 10 increase in ω
$G(\omega)$ increases by 20 dB.

$\Rightarrow$ 20 dB / dec



at $\omega = \omega_c = 100$

$$G(\omega) = 20\log_{10}\left|1 + j\frac{100}{100}\right|$$

$$= 20\log_{10}(|1+j|)$$

$$= 20\log_{10}(\sqrt{2}) = 10\log_{10}(2)$$

$$\simeq 3\text{ dB}.$$

Consider now Phase:

$$\theta_H = \measuredangle\left(1 + \frac{j\omega}{100}\right)$$

$\omega << \omega_c$ ie $\omega << 100$

$$\theta_H \simeq \measuredangle 1 = 0^\circ$$

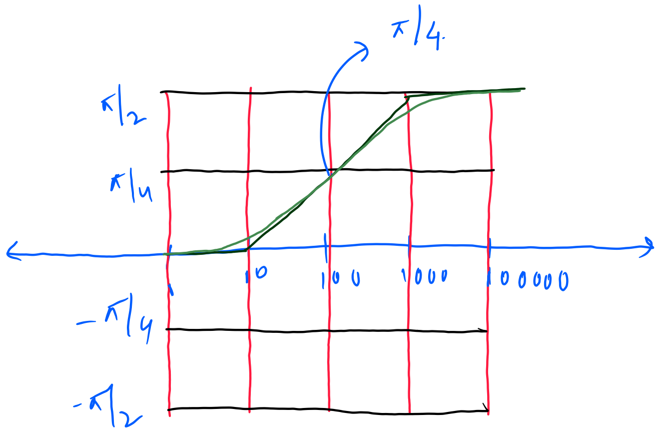
$\omega >> 100$

$\Rightarrow \theta_H \approx \angle j(\omega/100)$

$= \pi/2.$

$\omega = 100$

$\theta_H = \angle(1+j) = \pi/4.$



b) Bode plot:

$\hookrightarrow H(\omega) = \dfrac{10 + j\omega}{(1+j\omega)(100+j\omega)}$

$H(\omega) = \dfrac{1/10\left(1 + j\frac{\omega}{10}\right)}{(1+j\omega)\left(1 + j\frac{\omega}{100}\right)}$

3 corner frequencies.

$\omega = 1, 10, 100.$

$\omega << 1.$

$G(\omega) = 20\log_{10}|H(\omega)|$

$\approx 20\log_{10}\dfrac{1/10 \times 1}{1 \times 1}$

$= -20\,dB.$

$1 << \omega << 10$

$G(\omega) \approx 20\log_{10}\dfrac{1/10 \times 1}{j\omega \times 1}$

$= \underline{-20 - 20\log_{10}\omega}.$

20 dB/ dec decreases

Signals and Systems

L-49.

Bode plots

$10 << w << 100.$

$$G(w) \approx 20 \log_{10} \left| \frac{\frac{1}{10} j\frac{w}{10}}{jw \times 1} \right|$$

$$= 20 \log_{10} \frac{1}{100}$$

$$= -40 \, dB.$$

$w >> 100$ rad/s.

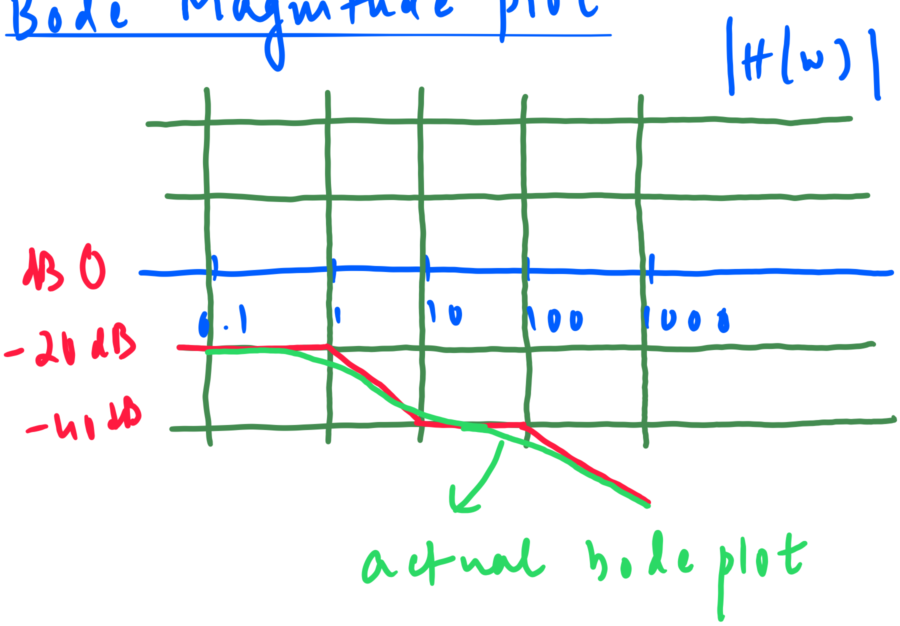
$$G(w) \approx 20 \log_{10} \left| \frac{\frac{1}{10} \times \frac{jw}{10}}{jw \times \frac{jw}{100}} \right|$$

$$= 20 \log_{10} \frac{1}{w}$$

$$= -20 \log w$$

decreases 20 dB/dec

Bode Magnitude plot



Bode plot phase

$$H(\omega) = \frac{\frac{1}{10}\left(1+j\frac{\omega}{10}\right)}{(1+j\omega)\left(1+j\frac{\omega}{100}\right)}$$

$\omega \ll 1$

$$\theta_H = \measuredangle \tfrac{1}{10} + \measuredangle 1 - \measuredangle 1 - \measuredangle 1$$

$$\theta_H = 0^\circ$$

$\omega = 1 \qquad \omega \ll 10$

$$\theta_H = \measuredangle \tfrac{1}{10} + \measuredangle 1 - \pi/4 - \measuredangle 1$$

$$= -\pi/4.$$

$1 \ll \omega \ll 10$

$$\theta_H = 0^\circ + 0^\circ - \pi/2 - 0^\circ$$

$$= -\pi/2$$

$\omega = 10, \quad \omega \ll 100$

$$\theta_H = 0^\circ + \pi/4 - \pi/2 - 0^\circ$$

$$= -\pi/4.$$

$10 \ll \omega \ll 100.$

$$\theta_H = 0^\circ + \pi/2 - \pi/2 - 0^\circ$$

$$= 0^\circ$$

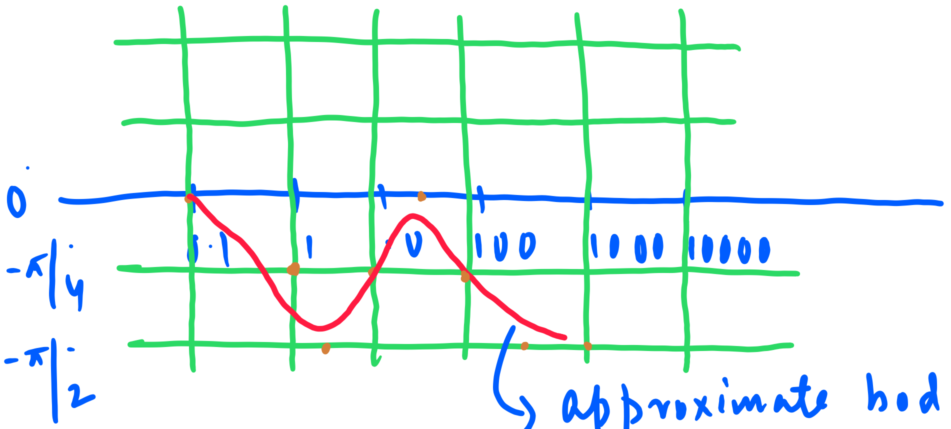
$\omega = 100$

$w = 100$

$$\theta_H = 0^\circ + \pi/2 - \pi/2 - \pi/4$$
$$= -\pi/4$$

$w \gg 100$

$$\theta_H = 0^\circ + \pi/2 - \pi/2 - \pi/2$$
$$= -\pi/2.$$



approximate bode plot of phase

(15) <u>Phase shifter</u> :

$$H(w) = \begin{cases} e^{-j\pi/2} & w > 0 \\ e^{j\pi/2} & w < 0 \end{cases}$$

impulse response?

Hilbert transform.
(Amplitude Modulation)
SSB - single sideband Modulated signal.

$$H(w) = \begin{cases} -j, & w > 0 \\ j, & w < 0 \\ 0, & w = 0 \end{cases}$$

$$H(w) = -j\,\mathrm{sgn}(w)$$

Duality

$$x(t) \longleftrightarrow X(\omega)$$

$$X(t) \longleftrightarrow 2\pi\, x(-\omega)$$

$$\operatorname{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$

$\downarrow$ using duality principle

$$\frac{2}{jt} \longleftrightarrow \underbrace{2\pi \operatorname{sgn}(-\omega)}_{\text{odd function}}$$

$$= -2\pi \operatorname{sgn}(\omega)$$

$$\Rightarrow \frac{1}{t} \longleftrightarrow -j\pi \operatorname{sgn}(\omega)$$

$$\Rightarrow \boxed{\frac{1}{\pi t} \longleftrightarrow -j \operatorname{sgn}(\omega)}$$

$\frac{1}{\pi t}$ → impulse response of Hilbert Transform

$$\underset{\text{input}}{x(t)} \longrightarrow \boxed{h(t) = 1/\pi t} \longrightarrow \underset{\text{output}}{y(t)}$$

$$\Rightarrow y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) \frac{1}{\pi(t-\tau)}\, d\tau$$

$$y(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} x(\tau) \frac{1}{(t-\tau)}\, d\tau$$

$y(t)$ → output of Hilbert Transformer

= Hilbert transform of $x(t)$.