Week 12

L-69 Group / Phase Delays.

Consider an LTI system with impulse response $h(t)$

$$\underset{\text{input}}{x(t)} \longrightarrow \boxed{h(t)} \longrightarrow \underset{\text{output}}{y(t)}$$

$$h(t) \longleftrightarrow H(\omega).$$

→ Transfer function

→ Fourier Transform

$$Y(\omega) = X(\omega)\,H(\omega)$$

$$H(\omega) = |H(\omega)|\, e^{j\phi(\omega)}$$

$$\phi(\omega) = \measuredangle H(\omega)$$

Consider $x(t) = \underbrace{A\cos\omega_0 t}_{\text{Pure sinusoid}}$

given as input to LTI system

$$X(\omega) = A\left(\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\right)$$

$$= \pi A\,\delta(\omega-\omega_0) + \pi A\,\delta(\omega+\omega_0)$$

$$Y(\omega) = H(\omega)\,X(\omega)$$

$$= H(\omega)\,\pi A\,\delta(\omega-\omega_0) + \pi A\,\delta(\omega+\omega_0)\,H(\omega)$$

$$= \pi A\, H(\omega_0)\,\delta(\omega-\omega_0) + \pi A\, H(-\omega_0)\,\delta(\omega+\omega_0)$$

For real $h(t)$

$$\Rightarrow H(w_0) = H^*(-w_0)$$

$$H(w_0) = |H(w_0)| e^{j\phi(w_0)}$$

$$H(-w_0) = |H(w_0)| e^{-j\phi(w_0)}$$

$$Y(w) = \pi A |H(w_0)| e^{j\phi(w_0)} \delta(w - w_0)$$
$$+ \pi A |H(w_0)| e^{-j\phi(w_0)} \delta(w + w_0)$$

Take I.F.T.

$$\longleftrightarrow \quad A|H(w_0)| \frac{1}{2} e^{jw_0 t} e^{j\phi(w_0)}$$
$$+ A |H(w_0)| e^{-j\phi(w_0)} \frac{1}{2} e^{-jw_0 t}$$
$$= A |H(w_0)| \left\{ \frac{1}{2} e^{j(w_0 t + \phi(w_0))} + \frac{1}{2} e^{-j(w_0 t + \phi(w_0))} \right\}$$

$$= \underbrace{A|H(w_0)|}_{\text{gain}} \cos(w_0 t + \underbrace{\phi(w_0)}_{\text{phase offset.}})$$

$$= A|H(w_0)| \cos\left(w_0\left(t + \frac{\phi(w_0)}{w_0}\right)\right)$$

$$= A|H(w_0)| \cos\left(w_0\left(t - \underbrace{\left(-\frac{\phi(w_0)}{w_0}\right)}_{\tau_p}\right)\right)$$

$$= A|H(\omega_0)| \cos(\omega_0 (t - \tau_p))$$

Phase delay : τ_p.

Phase Delay

$$\boxed{\tau_p(\omega_0) := -\frac{\phi(\omega_0)}{\omega_0}}$$

Phase response of LTI system

Delay of carrier - input to LTI system.

<u>Group Delay</u> :-

Consider a modulated signal

$$x(t) = \underbrace{A\cos(\omega_m t)}_{\text{Message signal}} \times \underbrace{\cos(\omega_0 t)}_{\text{carrier signal.}}$$

$\omega_0 >> \omega_m$.

Carrier freq >> Message Bandwidth.

$$x(t) = \frac{A}{2}\left\{\cos((\omega_0 - \omega_m)t) + \cos((\omega_m + \omega_0)t)\right\}.$$

$\omega_0 - \omega_m = \omega_l$

$\omega_0 + \omega_m = \omega_h$

$$= \frac{A}{2}\left\{ \cos(w_\ell t) + \cos(w_h t) \right\}$$

Pass through LTI system.
LTI

$$A/2 \cos(w_\ell t) \xrightarrow{\text{output}} \frac{A}{2}|H(w_\ell)| \times \cos(w_\ell t + \phi(w_\ell))$$

$$A/2 \cos(w_h t) \longrightarrow \frac{A}{2}|H(w_h)| \cos(w_h t + \phi(w_h))$$

$\Rightarrow$ Net output

$$= A/2\, |H(w_\ell)| \cos(w_\ell t + \phi(w_\ell)) + A/2\, |H(w_h)| \cos(w_h t + \phi(w_h))$$

$$w_\ell = w_0 - w_m$$
$$w_h = w_0 + w_m$$

$$w_h - w_\ell = 2w_m << w_0$$

$$\Rightarrow \quad |H(w_\ell)| \approx |H(w_h)| \approx |H(w_0)|$$

(variation of mag response is negligible over $[w_\ell, w_h]$

L-70. <u>Group/Phase Delay</u>

Net output

$$= A/2\, |H(w_\ell)| \cos(w_\ell t + \phi(w_\ell))$$

$$+ \frac{A}{2} \left|H(w_u)\right| \cos\left(w_u t + \phi(w_u)\right)$$

$$\left|H(w_l)\right| \approx \left|H(w_u)\right| \approx H(w_0)$$

$$\approx \frac{A}{2} \left|H(w_0)\right| \cos\left(w_l t + \phi(w_l)\right)$$

$$+ \frac{A}{2} \left|H(w_u)\right| \cos\left(w_u t + \phi(w_u)\right)$$

$$= A\left|H(w_0)\right| \cos\left(\frac{w_u - w_l}{2} t + \frac{\phi(w_u) - \phi(w_l)}{2}\right)$$

$$\times \cos\left(\frac{w_u + w_l}{2} + \frac{\phi(w_u) + \phi(w_l)}{2}\right)$$

$$w_u = w_0 + w_m$$

$$w_l = w_0 - w_m$$

$$\Rightarrow \frac{w_u + w_l}{2} = w_0$$

$$\Rightarrow \frac{w_u - w_l}{2} = w_m$$

$$= A/2 \cos\left(w_m t + \frac{\phi(w_u) - \phi(w_l)}{2}\right)$$

$$\times \cos\left(w_0 t + \frac{\phi(w_u) + \phi(w_l)}{2}\right)$$

$$\phi(w + \Delta w) \approx \phi(w_0) + \left.\frac{d\phi}{dw}\right|_{w=w_0} \Delta w$$

Taylor series approximation

$$\phi(\omega_u) = \phi(\omega_0 + \omega_m)$$
$$\approx \phi(\omega_0) + \phi'(\omega_0)\,\omega_m$$

$$\phi(\omega_l) = \phi(\omega_0 - \omega_m)$$
$$\approx \phi(\omega_0) - \phi'(\omega_0)\,\omega_m$$

$$\Rightarrow \frac{\phi(\omega_u) + \phi(\omega_l)}{2} \approx \phi(\omega_0)$$

$$\frac{\phi(\omega_u) - \phi(\omega_l)}{2} \approx \phi'(\omega_0)\,\omega_m .$$
$$= \left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_0} \times \omega_m .$$

Net output

$$\approx A\,|H(\omega_0)| \cos\Big(\omega_m t + \underbrace{\left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_0}}_{\text{group delay}} \omega_m\Big) \times \cos\big(\omega_c t + \phi(\omega_0)\big)$$

$$= A\,|H(\omega_0)| \cos\big(\omega_m (t + \phi'(\omega_0))\big) \times \cos\Big(\omega_0\Big(t + \underbrace{\frac{\phi(\omega_0)}{\omega_0}}_{\tau_p \,=\, \text{phase delay}}\Big)\Big)$$

$$= A|H(w_0)| \cos(w_m(t-\tau_g)) \times \cos(w_0(t-\tau_p))$$

envelope message delayed by group delay τ_g

$$\boxed{\begin{aligned} \tau_g(w_0) &= -\frac{d\phi(w)}{dw}\Big|_{w=w_0} \\ \tau_p(w_0) &= -\frac{\phi(w_0)}{w_0} \end{aligned}}$$

τ_g → Group delay

τ_p → Phase delay.

Carrier is delayed by phase delay.

For linear phase,

$$\phi(w) = Kw$$

$$\tau_p(w) = -\frac{\phi(w_0)}{w_0} = -K$$

$$\tau_g(w_0) = -\frac{d\phi(w)}{dw}\Big|_{w=w_0}$$

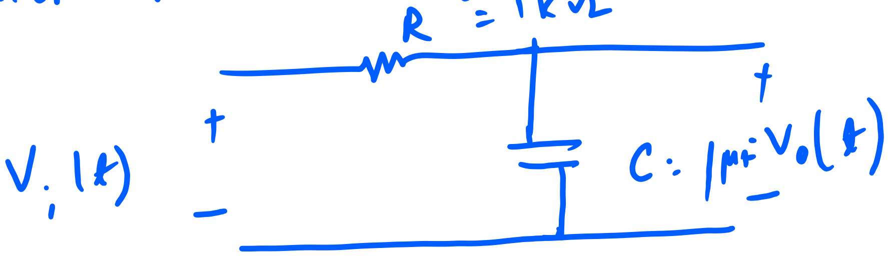
$$= -K$$

$$\Rightarrow \tau_p(w) = \tau_g(w)$$

$\Rightarrow$ $\boxed{\text{Phase Delay} = \text{group Delay.}}$

<u>Example:</u>

Consider RC circuit



$f = 100\,Hz.$

$\tau_g = ?$

$$H(w) = \frac{v_o(w)}{v_i(w)} = \frac{\frac{1}{jwC}}{R + 1/jwC}$$

$$= \frac{1}{1 + jwRC}$$

$$RC = 10^3 \times 10^{-6}$$
$$= 10^{-3}\ s$$

$$\phi(w) = -\tan^{-1}(wRC)$$

$$\tau_g = -\frac{d\phi(w)}{dw}$$

$$= \frac{RC}{1 + w^2R^2C^2}$$

$$= \frac{(10^{-3})}{1 + (2\pi\,100)^2 (10^{-6})}$$

$$\boxed{\tau_g = 0.717\,ms}$$

$f = 100\,Hz$

$w = 200\pi$ rad/s.

L-71 IIR Filter structures.

↳ Infinite Impulse response.

$$H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}$$

$$\frac{Y(z)}{X(z)} = \frac{p(z)}{d(z)}$$

This can be implemented as,

$$Y(z) = X(z)\,H(z)$$

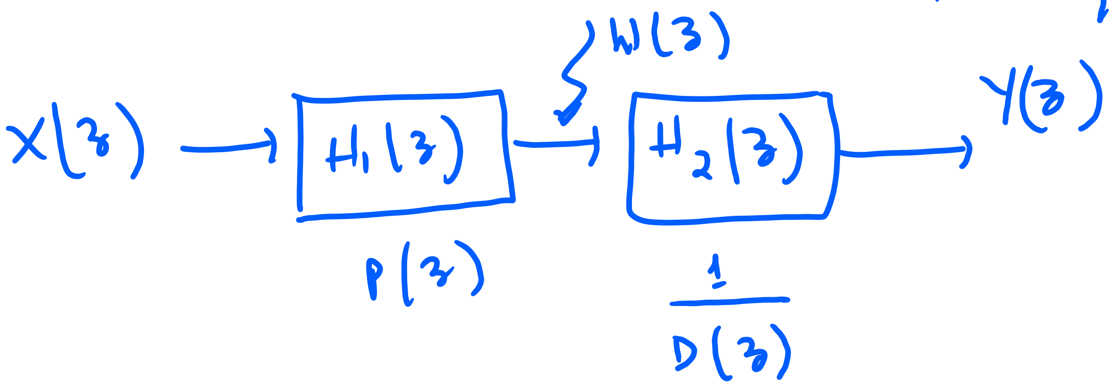
$$= X(z)\,\frac{P(z)}{D(z)}$$

$$= \underbrace{X(z)\,P(z)}_{W(z)}\,\frac{1}{D(z)}$$

$$= W(z)\cdot\frac{1}{D(z)}$$

$$= X(z)P(z)$$

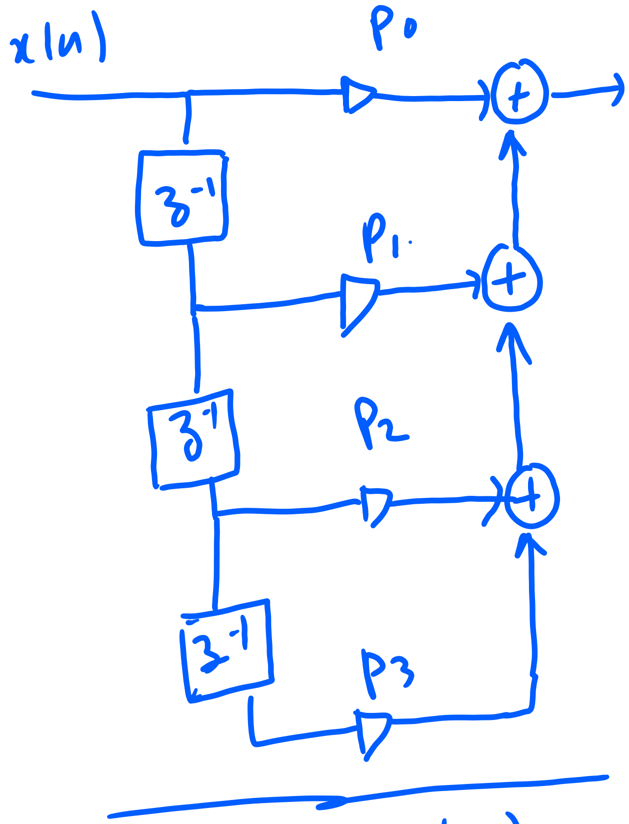
$$H(z) = P(z)\cdot\frac{1}{D(z)} \rightarrow \text{cascade of two systems.}$$



$$W(z) = X(z)\,P(z)$$

$$= X(z)\left(p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}\right)$$

$$\Rightarrow w(n) = p_0\,x(n) + p_1\,x(n-1) + p_2\,x(n-2) + p_3\,x(n-3).$$



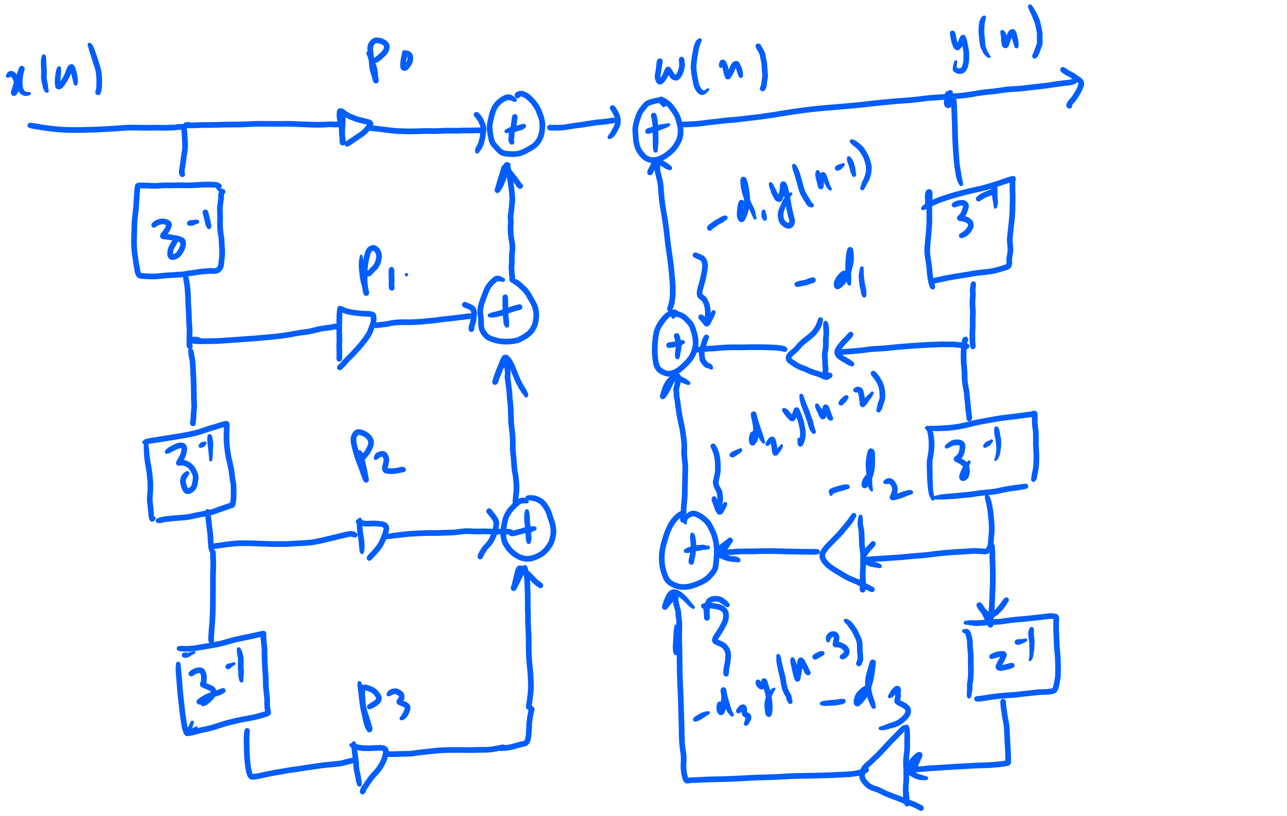
$w(n) = p_0 x(n) + p_1 x(n-1) + p_2 x(n-2) + p_3 x(n-3)$

$X(z)\,P(z).$

$$Y(z) = \frac{W(z)}{D(z)} = \frac{W(z)}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}.$$

$\Rightarrow y(n) + d_1 y(n-1) + d_2 y(n-2) + d_3 y(n-3) = w(n)$

$\Rightarrow y(n) = w(n) - d_1 y(n-1) - d_2 y(n-2) - d_3 y(n-3)$



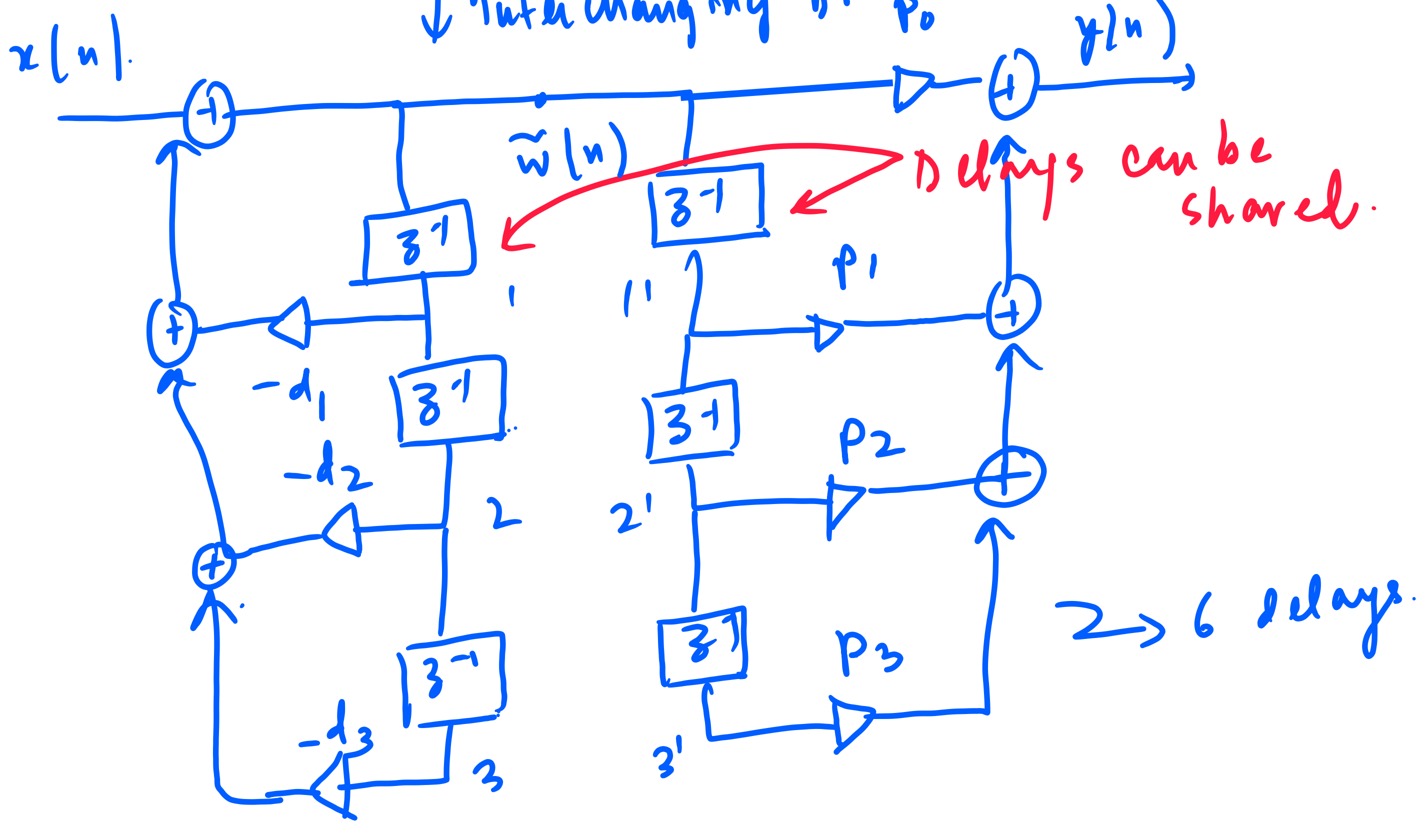
$\underbrace{\qquad\qquad}_{x(z)\,P(z)}$ $\qquad$ $\underbrace{\qquad\qquad}_{W(z)\frac{1}{D(z)} = Y(z)}$

$\underbrace{\qquad\qquad\qquad\qquad\qquad}$

Direct Form 1 (DF-I Realization).

Interchange 2 branches in DF1.

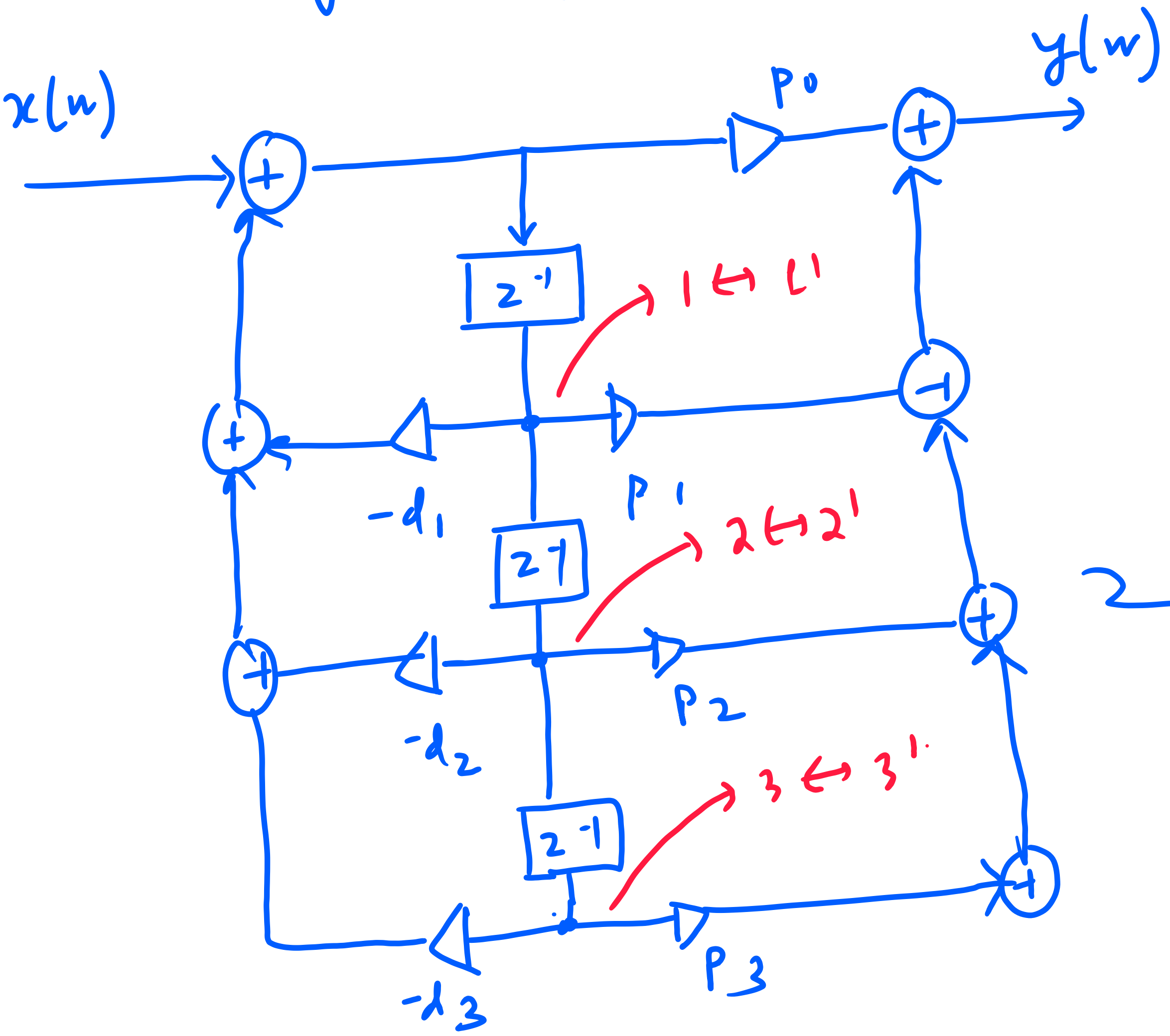
↓ Interchanging branches of DF1.



$1, 1'$
$2, 2'$
$3, 3'$ } signals at these node pairs are identical.

$\Rightarrow$ Delays can be shared

After sharing delays:



REDUCED TO 3 delays!

DF-II, DF-2
Direct form II realisation

L-72: IIR Filter Structures.

→ Transpose form.

$$H(z) = Y(z)/X(z)$$

$$= \frac{P(z)}{D(z)}$$

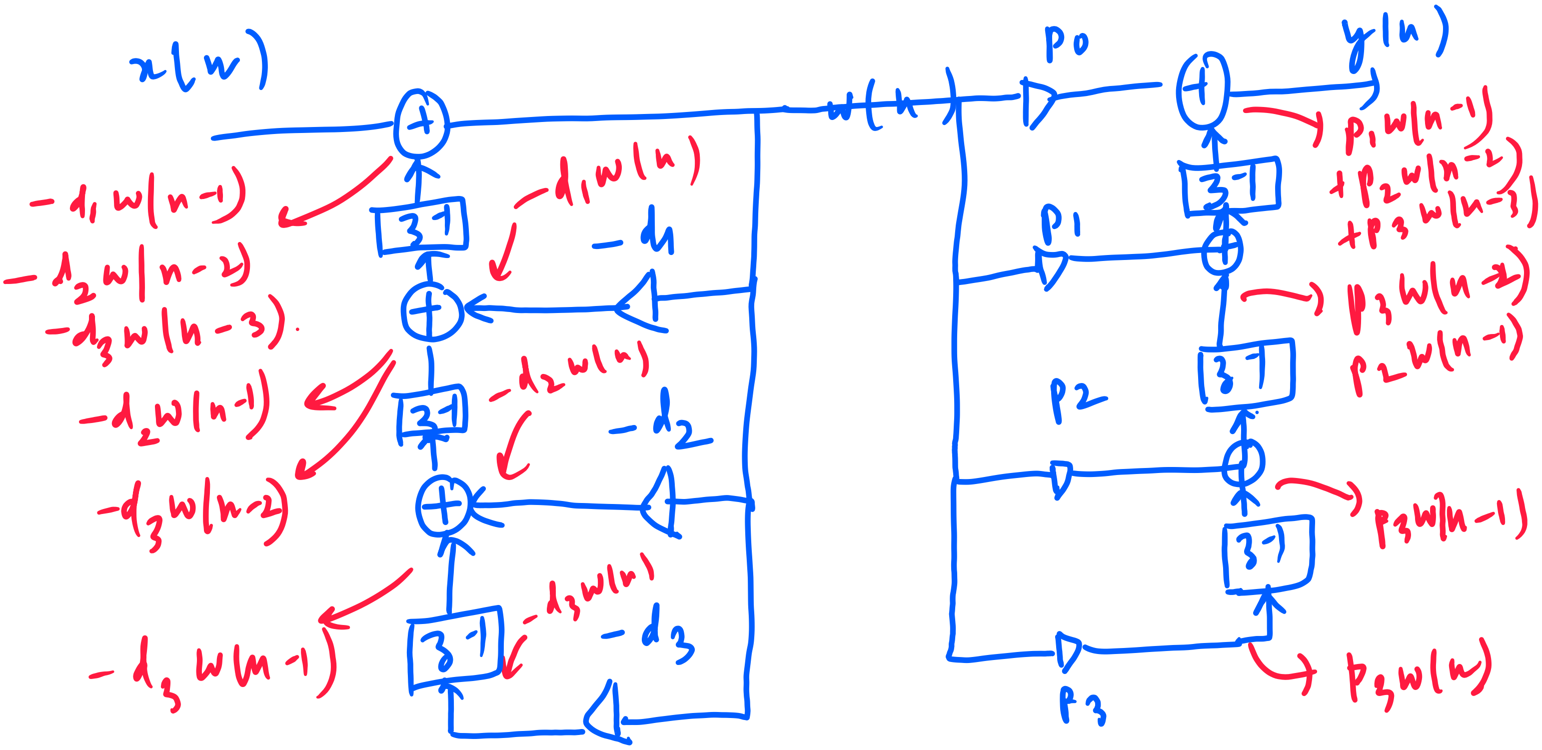
$$Y(z) = \underbrace{X(z) \frac{1}{D(z)}}_{W(z)} \cdot P(z)$$

$$= W(z)\, P(z)$$

$$X(z) = W(z)\, D(z)$$

$$= W(z)\,(1 + d_1 z + d_2 z^{-2} + d_3 z^{-3})$$

$$\Rightarrow x(n) - d_1 w(n-1) - d_2 w(n-2) - d_3 w(n-3) = w(n).$$



6 Delays
+ 3 adders.

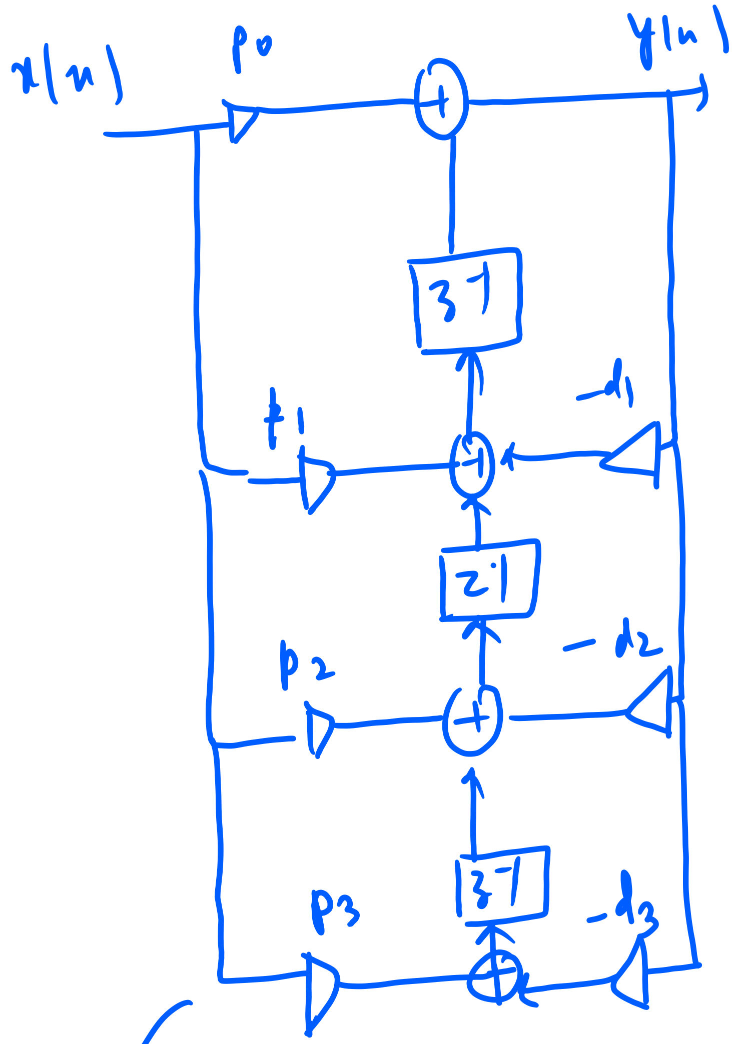
$$\underbrace{}_{W(z) = \frac{X(z)}{D(z)}}$$

Direct form I_t (transpose)

$$Y(z) = W(z)\, P(z)$$
$$= W(z)\left(p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}\right)$$

$$\Rightarrow y(n) = p_0 w(n) + p_1 w(n-1) + p_2 w(n-2) + p_3 w(n-3).$$

<u>Interchange branches & merge common Delays &</u> <u>Adders</u>. This yields structure below



3 Delays
+ 4 adders.

Direct form II t → transpose.
→ Minimum # of delays and adders

<u>L-13.</u> <u>IIR filter structures.</u>
→ example

$$H(z) = \frac{1 - \frac{1}{2} z^{-1}}{1 - \frac{7}{8} z^{-1} + \frac{3}{32} z^{-2}}$$

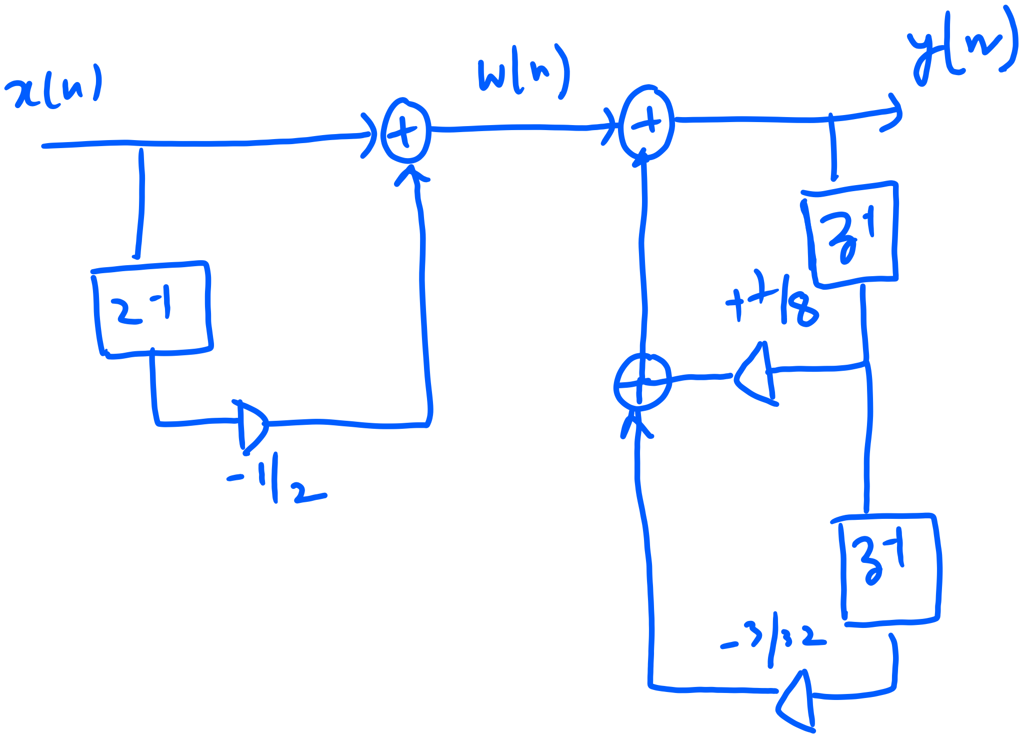
$$\Rightarrow P(z) = 1 - \frac{1}{2} z^{-1}$$

$$D(z) = 1 - \frac{7}{8} z^{-1} + \frac{3}{32} z^{-2}$$

$\Rightarrow \; \Rightarrow p_0 = 1, \; p_1 = -1/2$

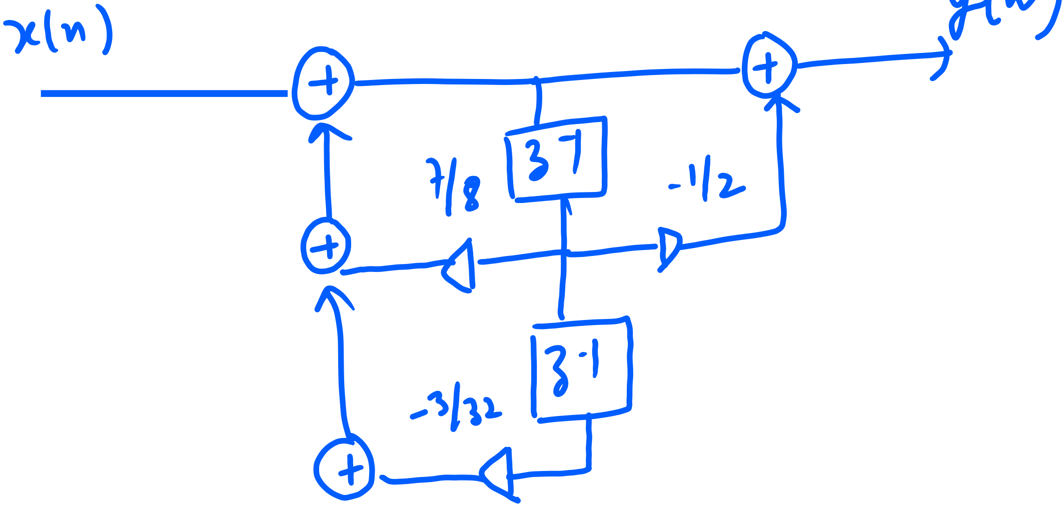
$\Rightarrow d_0 = 1, \; d_1 = -7/8, \; d_2 = 3/32.$

DF-I

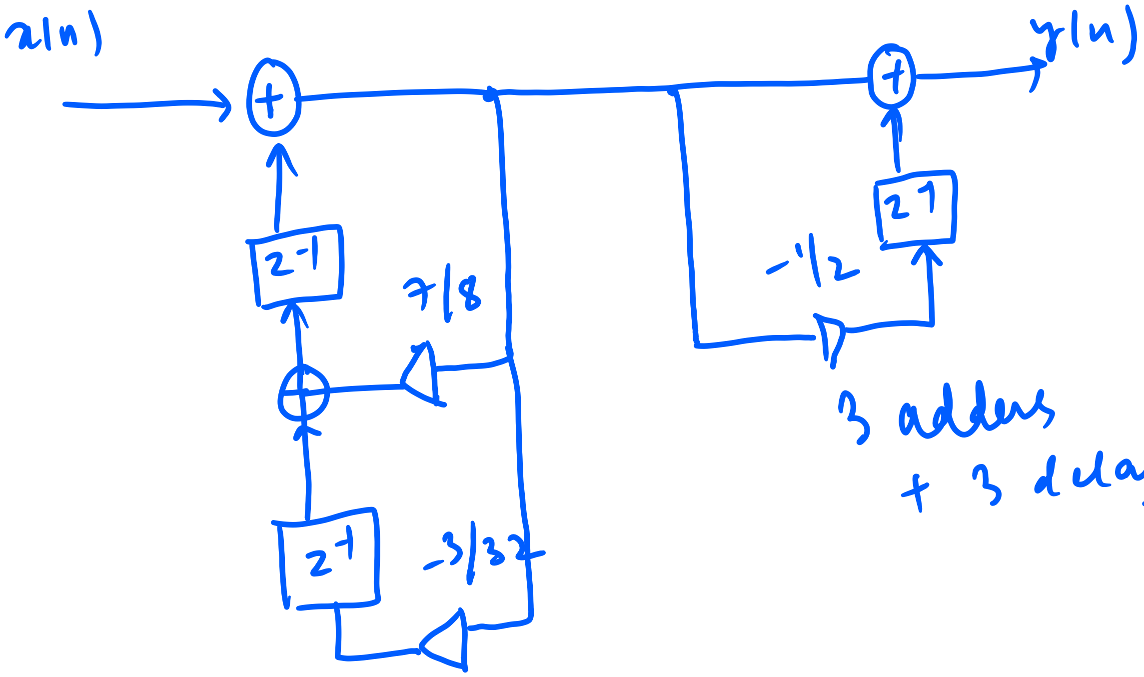


3 adders
3 delays.

DF-II - 3 adders
2 delays.

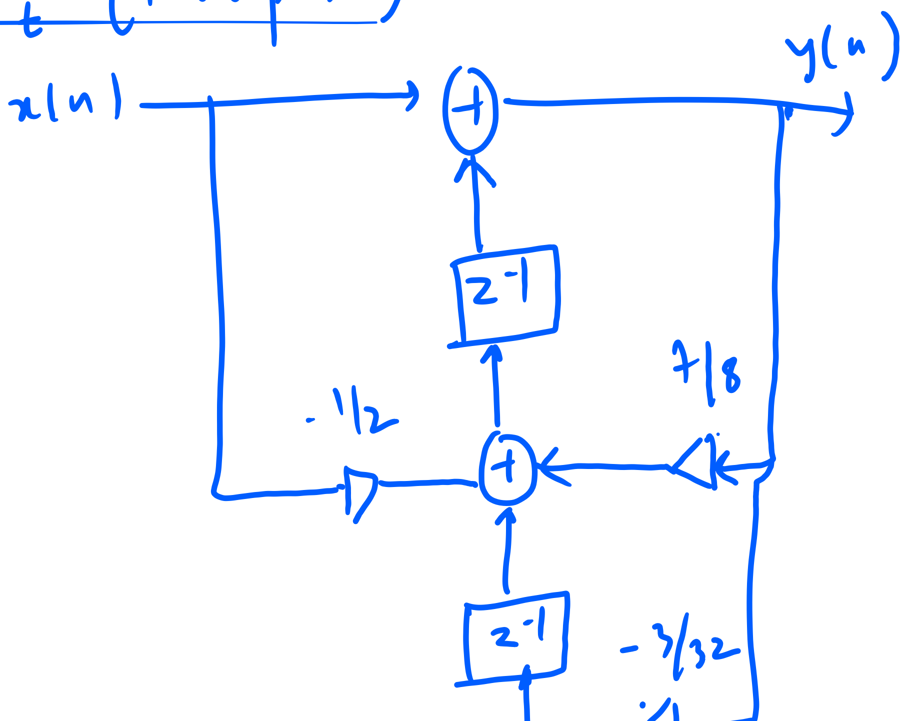


DF - I$_t$ (transpose)



3 adders
+ 3 delay elements

DF - II$_t$ (transpose)



2 adders
2 delays.

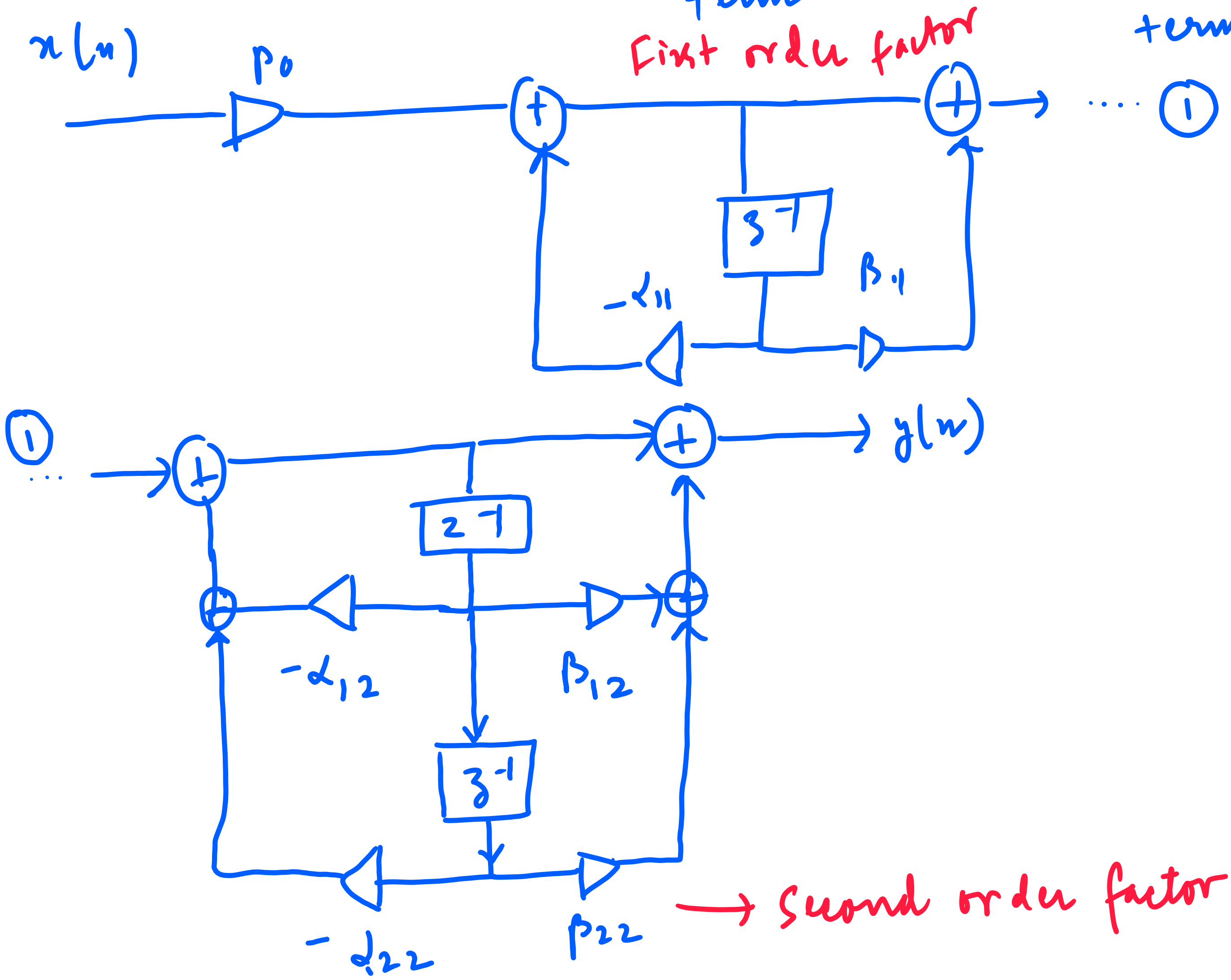
L-74. Cascade form : IIR filters.

$$H(z) = \frac{P(z)}{D(z)}$$

Factored into first and 2nd order factors.

$$H(z) = p_0 \underbrace{\left(\frac{1+\beta_{11} z^{-1}}{1+\alpha_{11} z^{-1}}\right)}_{\text{1st order term}} \underbrace{\left(\frac{1+\beta_{12} z^{-1}+\beta_{22} z^{-2}}{1+\alpha_{12} z^{-1}+\alpha_{22} z^{-2}}\right)}_{\text{2nd order term}}$$

<u>Structure</u>:

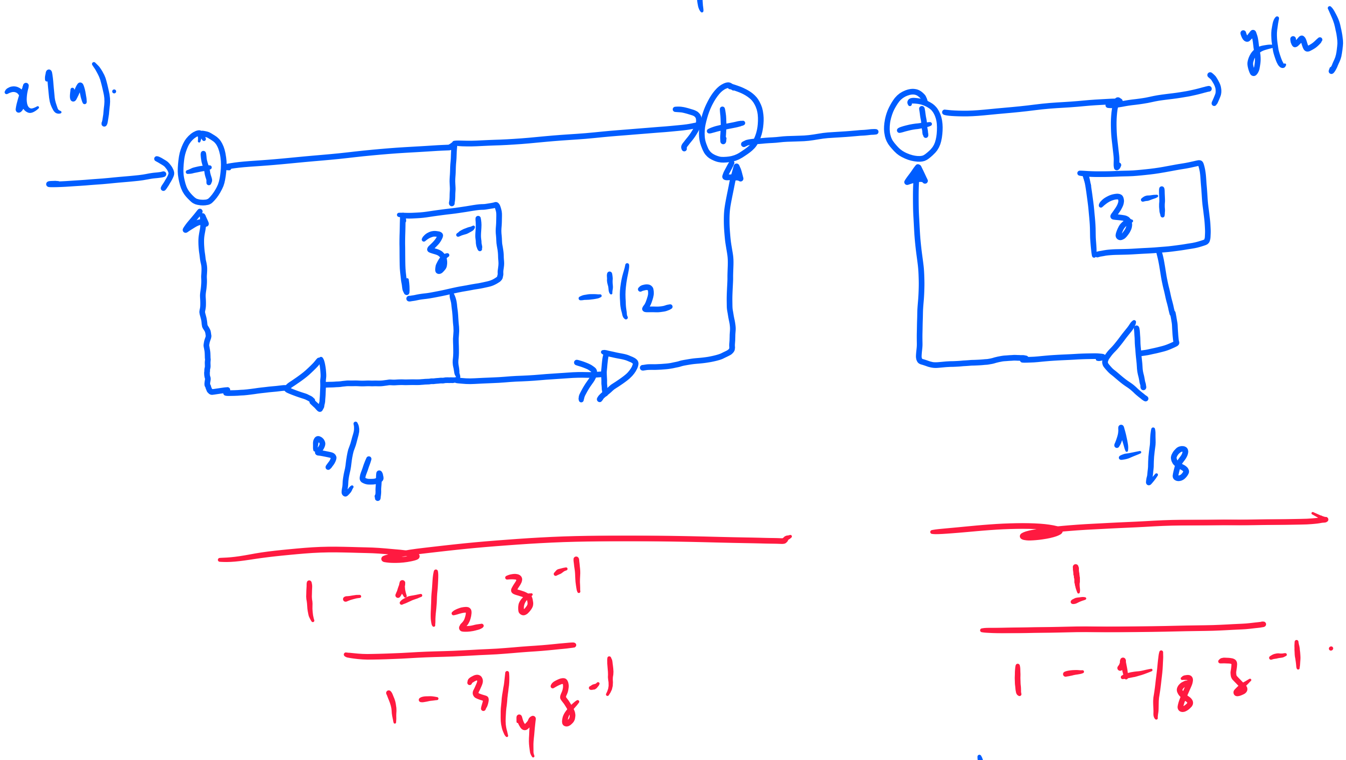


<u>Cascade form example</u>

$$H(z) = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}}$$

$$= \frac{1-\frac{1}{2}z^{-1}}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)}$$

$$= \frac{1 - \frac{1}{2} z^{-1}}{1 - \frac{3}{4} z^{-1}} \cdot \frac{1}{1 - \frac{1}{8} z^{-1}}$$



CASCADE FORM of $\frac{1 - \frac{1}{2} z^{-1}}{\left(1 - \frac{3}{4} z^{-1}\right)\left(1 - \frac{1}{8} z^{-1}\right)}$

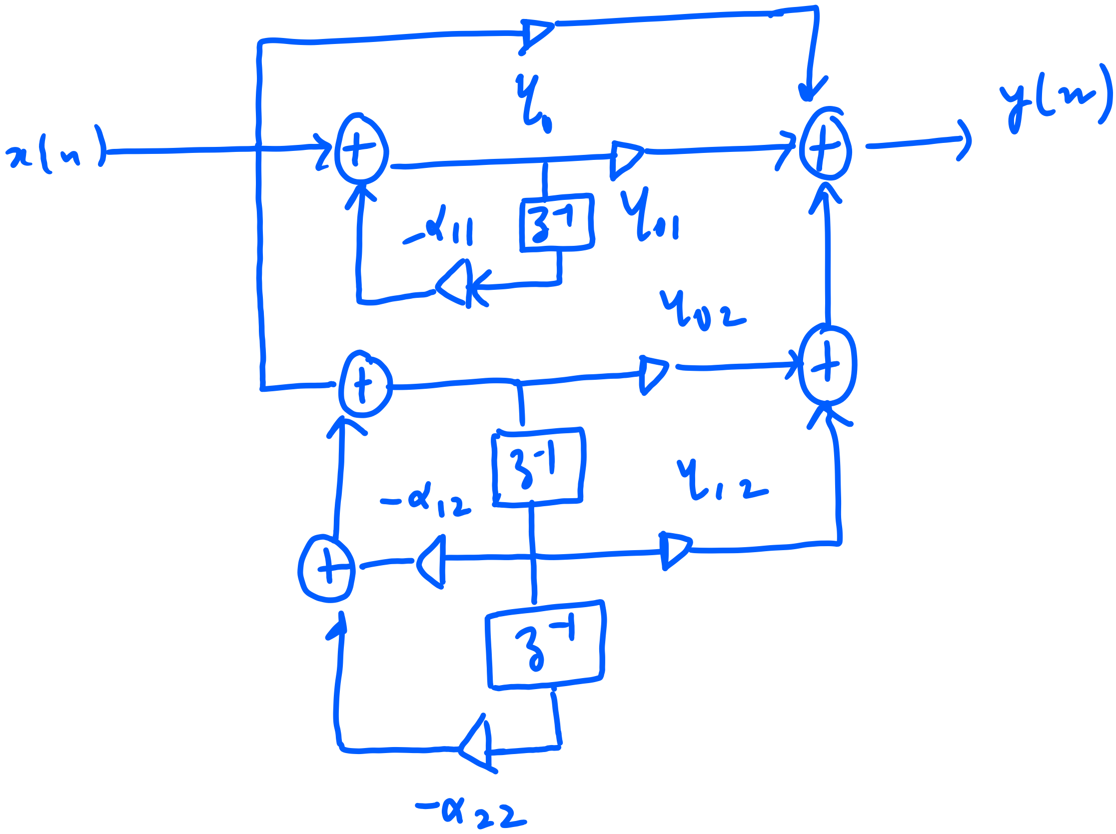
<u>Parallel form I:</u>

$$H(z) = \frac{P(z)}{D(z)}$$

Perform partial fraction expansion in z^{-1}.

$$H(z) = \gamma_0 + \frac{\gamma_{01}}{1 + \alpha_{11} z^{-1}} + \frac{\gamma_{02} + \gamma_{12} z^{-1}}{1 + \alpha_{12} z^{-1} + \alpha_{22} z^{-2}}.$$

Parallel form - I

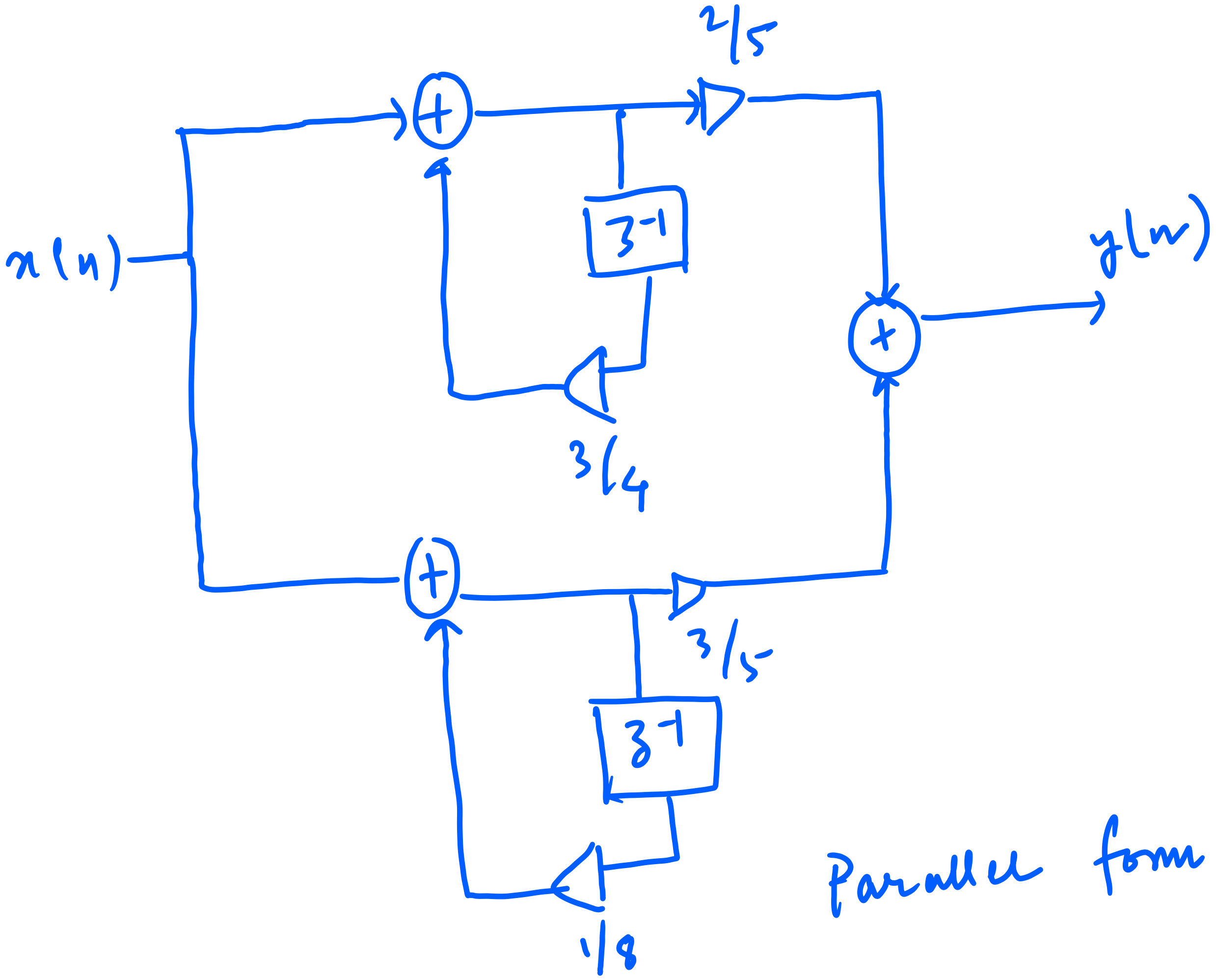


1-75 parallel form - I

$$H(z) = \frac{1 - \frac{1}{2} z^{-1}}{1 - \frac{7}{8} z^{-1} + \frac{3}{32} z^{-2}}$$

$$= \frac{1 - \frac{1}{2} z^{-1}}{\left(1 - \frac{3}{4} z^{-1}\right)\left(1 - \frac{1}{8} z^{-1}\right)}$$

$$= \frac{2/5}{\left(1 - 3/4\, z^{-1}\right)} + \frac{3/5}{\left(1 - 1/8\, z^{-1}\right)}$$



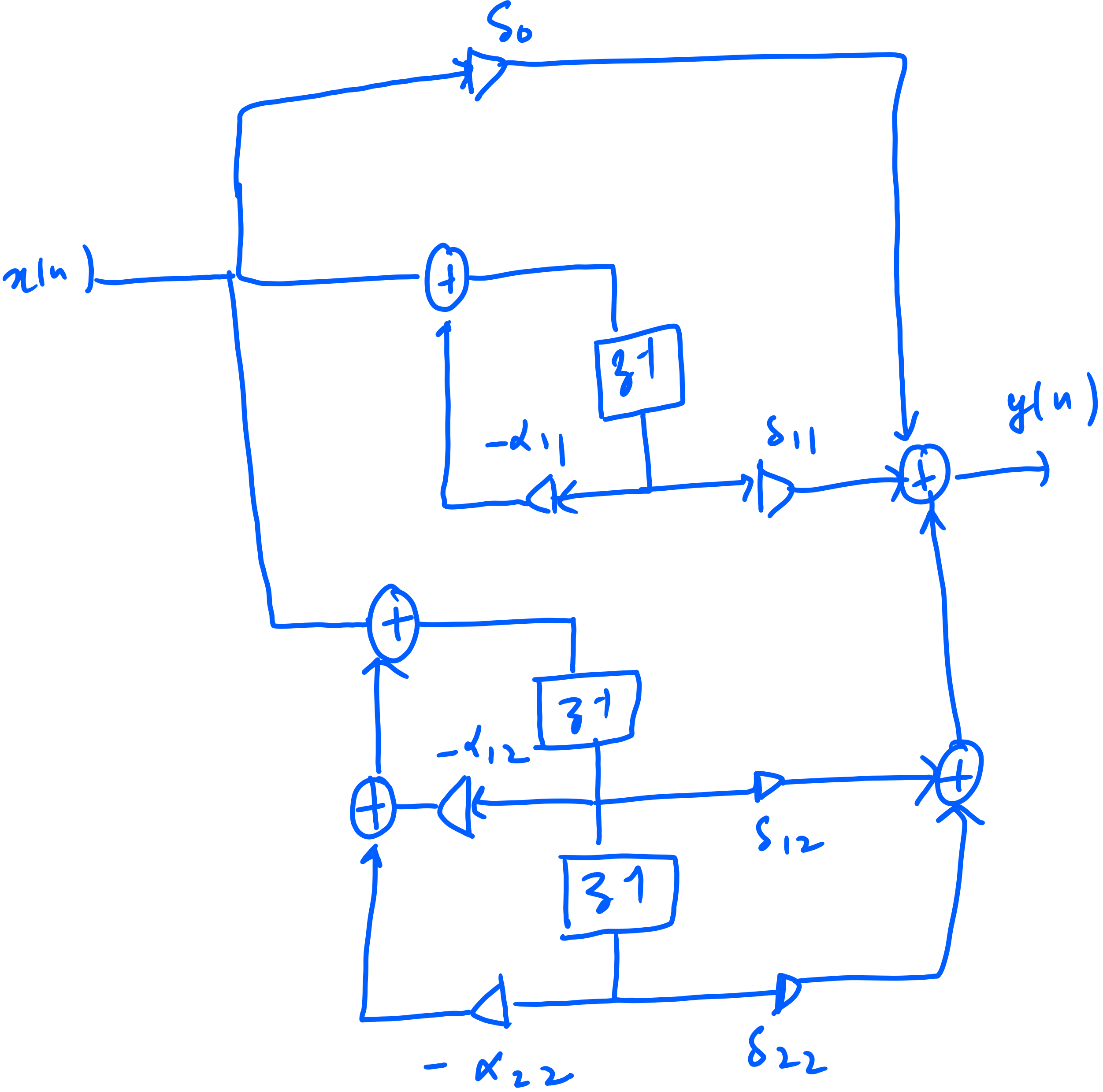
Parallel form - I

Parallel form - II

$$H(z) = \frac{P(z)}{D(z)}$$

Partial fraction expansion in z for parallel form II

$$H(z) = \delta_0 + \frac{\delta_{11} z^{-1}}{1 - \alpha_{11} z^{-1}} + \frac{\delta_{12} z^{-1} + \delta_{22} z^{-2}}{1 + \alpha_{12} z^{-1} + \alpha_{22} z^{-2}}.$$



PF - II example:

$$H(z) = \frac{1 - \frac{1}{2} z^{-1}}{1 - \frac{7}{8} z^{-1} + \frac{3}{32} z^{-2}}$$

$$= \frac{z^2 - \frac{1}{2} z}{z^2 - \frac{7}{8} z + \frac{3}{32}}$$

$$= \frac{z^2 - \frac{7}{8} z + \frac{3}{32} + \frac{3}{8} z - \frac{3}{32}}{z^2 - \frac{7}{8} z + \frac{3}{32}}$$

$$= 1 + \frac{3/8\, z - 3/32}{(z - 3/4)(z - 1/8)}$$

$$= 1 + \frac{a}{z - 3/4} + \frac{b}{z - 1/8}$$

$a + b = 1$

$\frac{1}{8}a + \frac{3}{4}b = 3/32$

$\Rightarrow a = 3/10, \quad b = 3/40$

$$\Rightarrow H(z) = 1 + \frac{3/10}{(z - 3/4)} + \frac{3/40}{(z - 1/8)}$$

$$= 1 + \frac{3/10\, z^{-1}}{1 - 3/4\, z^{-1}} + \frac{3/40\, z^{-1}}{1 - 1/8\, z^{-1}}$$

Parallel form-II