Week 5 TA Sessions

Part I: Summary of Week V lecture

1) z transforms:

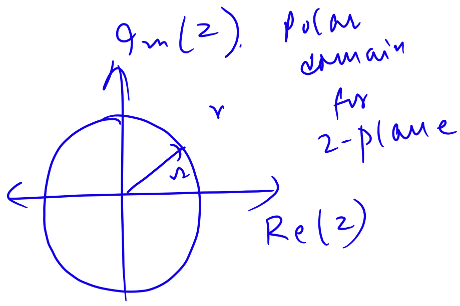
→ Represent and analyse discrete time signals and systems.

→ counterpart of Laplace transform.

$$X(z) \doteq \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

complex no.

$z = r e^{j\Omega}$

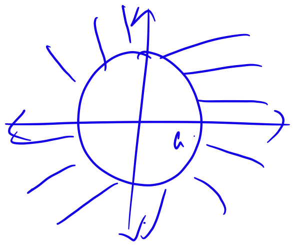


→ Region of convergence: Range of 'z' for which z transform converges.

eg $x(n) = a^n u(n) \overset{z}{\longleftrightarrow} X(z) = \dfrac{1}{1-az^{-1}}$

$|z| > |a|$

$= \dfrac{z}{z-a}$

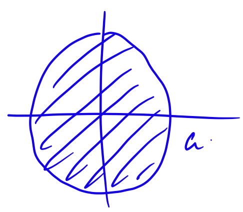


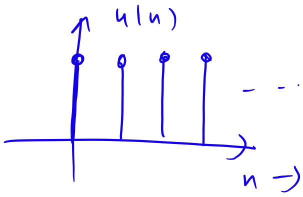
$$\sum_{i=1}^{\infty} x^i = \frac{1}{1-x}, \quad x<1$$

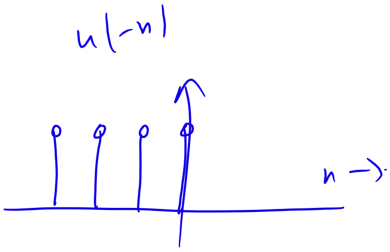
eg $x(n) = a^n u(-n-1)$.

$X(z) = \dfrac{a^{-1}z}{1-a^{-1}z}$; $|z| < |a|$

$= \dfrac{z}{z-a}$; $|z| < |a|$.







Diff time domain signals can have same z transforms.

→ To uniquely characterise the signal, one has to mention the ROC also.

② Properties of z transform.

(Ref oppenheim ch-10).

$$x(n) \longleftrightarrow X(z) \quad \text{ROC } R.$$
$$x_1(n) \longleftrightarrow X_1(z) \quad R_1$$
$$x_2(n) \longleftrightarrow X_2(z) \quad R_2$$

Linearity

$$a x_1(n) + b x_2(n) \rightarrow a X_1(z) + b X_2(z).$$

ROC: atleast intersection of R_1 & R_2

Time shifting

$$x(n-n_0) \rightarrow z^{-n_0} X(z)$$

ROC: R, except for the possible addition & deletion of origin.

Scaling in z domain

$$e^{j\omega_0 n} x(n) \longleftrightarrow X(e^{-j\omega_0} z) \quad R.$$

$z_0^n x(n) \rightarrow X(z/z_0)$ $\quad z_0 R$

$a^n x(n) \rightarrow X(a^{-1}z)$. scaled version of R.

Time Reversal

$x(-n) \rightarrow X(z^{-1})$ $\quad$ inverted R i.e. R^{-1}

Conjugation

$x^*(n) \longleftrightarrow X^*(z^*)$ $\quad R_x$

Convolution

$x_1(n) * x_2(n)$ $\quad X_1(z)\, X_2(z)$

at least intersection of R_1 & R_2

First difference

$x(n) - x(n-1)$

$\longleftrightarrow (1 - z^{-1}) X(z)$ $\quad$ at least intersection of R & $|z|>0$.

Accumulation

$$\sum_{k=-\infty}^{n} x(k) \longleftrightarrow \frac{1}{1-z^{-1}} X(z)$$

at least intersection of R & $|z|>1$

Differentiation in z domain

$n\,x(n) \longleftrightarrow -z \dfrac{dX(z)}{dz}$ $\quad R$

③ Common examples

signal	Z.T	ROC
$\delta(n)$	1	All z
$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\|z\|>1$
$-u(-n-1)$	$\dfrac{1}{1-z^{-1}}$	$\|z\|<1$
$\delta(n-m)$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)

$\alpha^n u(n) \longleftrightarrow \dfrac{1}{1-\alpha z^{-1}} \qquad |z| > |\alpha|$

$-\alpha^n u(-n-1) \longleftrightarrow \dfrac{1}{1-\alpha z^{-1}} \qquad |z| < |\alpha|$

$n\alpha^n u(n) \qquad \dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} \qquad |z| > |\alpha|$

$-n\alpha^n u(-n-1) \qquad \dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} \qquad |z| < |\alpha|$

$\cos w_0 n \, u(n) \qquad \dfrac{1-\cos w_0 z^{-1}}{1-2\cos w_0 z^{-1} + z^{-2}} \qquad |z| > 1$

$\sin w_0 n \, u(n) \qquad \dfrac{\sin w_0 \, z^{-1}}{1-2\cos w_0 z^{-1} + z^{-2}} \qquad |z| > 1$

$r^n \cos w_0 n \, u(n) \qquad \dfrac{1-r\cos w_0 z^{-1}}{1-2r\cos w_0 z^{-1} + r^2 z^{-2}} \qquad |z| > r$

$r^n \sin w_0 n \, u(n) \qquad \dfrac{r\sin w_0 z^{-1}}{1-2r\cos w_0 z^{-1} + r^2 z^{-2}} \qquad |z| > r$

(ii) <u>Inverse z transform</u>

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

(i) <u>Power series expansion</u>

∴ Given $X(z)$, express $X(z)$ as a power series in z; then coefficient of z^{-n} is $x(n)$

(ii) <u>Partial fractions.</u>

$$X(z) = \frac{N(z)}{D(z)} = \frac{k(z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

$z_1, z_2, \ldots z_m \rightarrow$ zeros.

$p_1, p_2, \ldots p_n \rightarrow$ poles.

(*) All simple poles.

then $\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \cdots + \frac{c_n}{z-p_n}$.

$$c_0 = X(z)\Big|_{z=0}$$

$$c_k = (z-p_k)X(z)\Big|_{z=p_k}$$

(*) poles with Multiplicity

say p_i repeats 'r' times!

then for p_i

$$\frac{X(z)}{z} = \cdots + \frac{\lambda_1}{z-p_i} + \frac{\lambda_2}{(z-p_i)^2} + \cdots + \frac{\lambda_r}{(z-p_i)^r} + \cdots$$

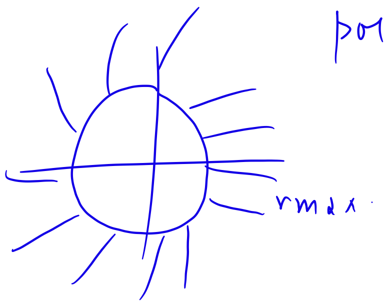
$$\therefore \lambda_{r-k} = \frac{1}{k!}\frac{d^k}{dz^k}(z-p_i)^r \frac{X(z)}{z}\Bigg|_{z=p_i}$$

(5) Properties of ROC

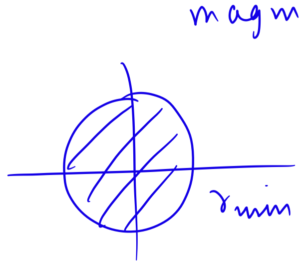
(1) ROC does not contain pole.

(2) For finite sequences, ROC is entire z-plane.

(3) For right sided signal, ROC is $|z| > r_{max}$ ← largest magnitude of poles of $X(z)$.

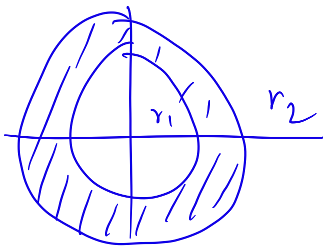


4) For left sided signal : ROC is $|z| < r_{min}$ → smallest magnitude of poles of $X(z)$



5) For 2 sided signal, ROC is annulus region

$$r_1 < |z| < r_2$$



(6) System function of LTI / LSI systems

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n).$$

$$X(z) \qquad H(z) \qquad Y(z)$$

$H(z) \triangleq$ System fxn / Transfer fxn.

$$\frac{Y(z)}{X(z)} = H(z) \longrightarrow \text{By convolution property}$$

(i) Causality

$h(n)$: right sided.

$\therefore$ ROC of form $|z| > r_{max}$.

ii) Stability

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \Rightarrow z = e^{j\Omega} \text{ should be part of ROC.}$$

$\Rightarrow$ unit circle must be included in ROC

(7) Determining o/p of system using z transform.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n).$$

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$

Any LSI s/s can be represented using linear constant coefficient difference equation.

→ Take z transform on both sides

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Part II Tutorials

(i) ROC for $a^n u(n)$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n.$$

$$= \frac{1}{1-az^{-1}} \quad \text{if } |az^{-1}|<1$$

$$\Downarrow$$

$$|z|>|a|.$$

(1) let $x(n) = a^n u(n)$. find the z-transform of $n x(n)$

Sol^n^ we know that

$$n x(n) \xleftrightarrow{z} -z \frac{dX(z)}{dz}.$$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \qquad |z|>|a|$$

$$= \frac{z}{z-a}.$$

$$\therefore n x(n) \xleftrightarrow{z} -z \frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$n a^n u(n) \longleftrightarrow -z \left[\frac{(z-a)\cdot 1 - z\cdot 1}{(z-a)^2}\right]$$

$$= \frac{-z\left(\cancel{z}-a-\cancel{z}\right)}{(z-a)^2}$$

$$= \frac{+az}{(z-a)^2} \qquad \text{ROC } |z|>|a|.$$

(2) Show that if $x(n)$ is a real signal then the zeros of $X(z)$ occur in conjugate pairs.

Sol^n set of "z" for which $X(z) = 0$ is called "zeros" of $X(z)$.

To show: if $X(z) = 0$, then $X(z^*) = 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{—— (1)}$$

Take conjugation on both sides to (1)

$$X^*(z) = \sum_{n=-\infty}^{\infty} x^*(n) (z^*)^{-n}$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} x(n) (z^*)^{-n}}_{X(z^*)}$$

$x[n]$ is real

$\therefore$ If z_0 is zero of $X(z)$

$\Rightarrow z_0$ is zero of $X^*(z)$

$\Rightarrow z_0$ is zero of $X(z^*)$

or z_0^* is zero of $X(z)$

z_0 is zero of $X(z)$

$X(z_0) = 0$

$X^*(z_0) = 0$

$\Rightarrow X(z_0^*) = 0$

$\Rightarrow z_0^*$ is zero of $X(z)$

(3) Let $x(n) \overset{z}{\longleftrightarrow} X(z)$. Find the z-transform of $x(n_0 - n)$ for some fixed integer n_0

Sol^n $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$\sum_{n=-\infty}^{\infty} x(n_0-n)\, z^{-n} = \sum_{m=-\infty}^{\infty} x[m]\, z^{-(n_0-m)}$$

Put $m = n_0 - n$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x(m) z^{m}$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x(m) \underbrace{(z^{-1})}_{z'}{}^{-m}$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x(m) (z')^{-m}$$

$$= z^{-n_0} X(z')$$

$$= z^{-n_0} X(z')$$

$\therefore$ If $x(n) \xleftrightarrow{Z.T} X(z)$

$\Rightarrow x(n_0-n) \xleftrightarrow{Z.T} z^{-n_0} X(z^{-1})$

$x(n_0-n) = x(-(n-n_0))$

$x(-n) \longrightarrow X(z^{-1})$

$x(-(n-n_0)) \rightarrow z^{-n_0} X(z^{-1})$

(4) Find the ROC of $X(z)$ if

$x(n) = 5^{-n+1} u(n+1) - 4^n u(-n-1)$

Solution

(i) $5^{-n+1} u(n+1) \rightarrow$

$$= 5/5^n = 5 \cdot (1/5)^n u(n+1)$$

$\therefore$ ROC is of the form $|z| > 1/5$

$a^n u(n)$
$|z| > |a|$

(ii) $4^n u(-n-1)$ (left sided signal)

$\rightarrow |z| < 4$

$\therefore$ ROC of $X(z)$

$$= 1/5 < |z| < 4$$

⑤ Find the z-transform $X(z)$ and sketch the pole-zero plot with the ROC for the following signals.

(i) $x[n] = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{3}\right)^n u(n)$

(ii) $x(n) = \left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{2}\right)^n u(-n-1)$

(iii) $x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{3}\right)^n u(-n-1)$

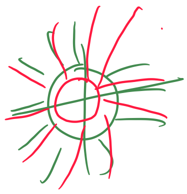
Solⁿ (i) $x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{3}\right)^n u(n)$

$\left(\frac{1}{2}\right)^n u(n) \overset{z}{\longleftrightarrow} \frac{z}{z - 1/2} \quad |z| > 1/2$

$\left(\frac{1}{3}\right)^n u(n) \overset{z.T}{\longleftrightarrow} \frac{z}{z - 1/3} \; ; |z| > 1/3.$

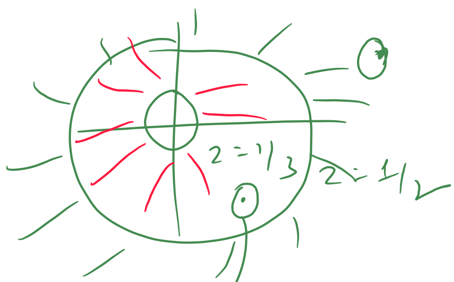
$\therefore X(z) = \frac{z}{z - 1/2} + \frac{z}{z - 1/3}$

$= \frac{2z(z - 5/12)}{(z - 1/2)(z - 1/3)} \; ; |z| >$



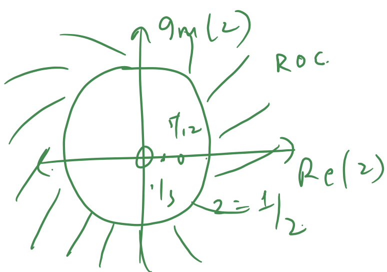
$\{|z| > 1/2\} \cap \{|z| > 1/3\}$

$= \{|z| > 1/2\}$



zeros : $5/12, 0$

poles : $1/2, 1/3$.



b) $x(n) = \underbrace{\left(\frac{1}{3}\right)^n u(n)}_{x_1(n)} + \underbrace{\left(\frac{1}{2}\right)^n u(-n-1)}_{x_2(n)}$

$x_1(n) = \left(\frac{1}{3}\right)^n u(n) \xleftrightarrow{z} \frac{z}{z - 1/3}$; $|z| > 1/3$

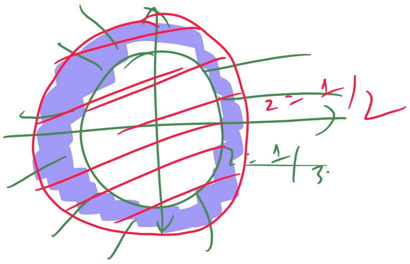
$x_2(n) = \left(\frac{1}{2}\right)^n u(-n-1) \xleftrightarrow{z} \frac{-z}{z - 1/2}$ $|z| < 1/2$

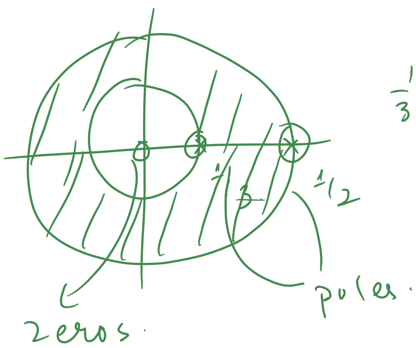
$X(z) = X_1(z) + X_2(z)$

$= \frac{z}{z - 1/2} - \frac{z}{z - 1/2}$

$= -1/6 \; \frac{z}{(z - 1/2)(z - 1/3)}$.

; ROC $\{|z| > \frac{1}{3}\} \cap \{|z| < 1/2$



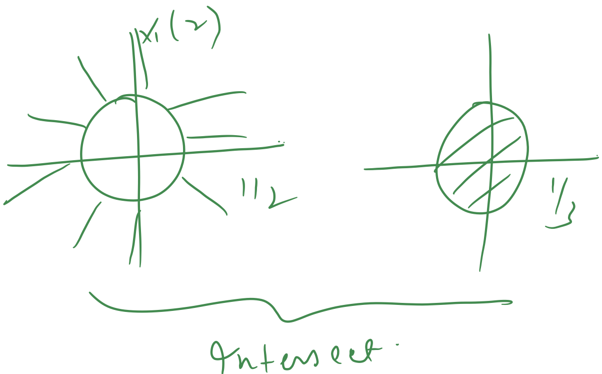


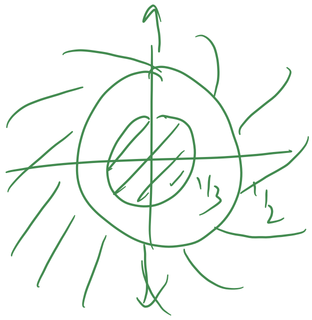
$\frac{1}{3} < |z| < 1/2$

c) $x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{3}\right)^n u(-n-1)$.

$x_1(n) = \left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{z} \frac{z}{z - 1/2}$ $|z| > 1/2$

$x_2(n) = \left(\frac{1}{3}\right)^n u(-n-1) \xleftrightarrow{z} \frac{-z}{z - 1/3}$; $|z| < 1/3$





No common ROC

$\Rightarrow$ at no point on the z-plane $x(z) = X_1(z) + X_2(z)$ converges.

$\therefore$ $x(n)$ does not have $X(z)$

⑥ Let $x(n) = a^{|n|}$; $a > 0$.

a) sketch $x(n)$ for $a < 1$ and $a > 1$.

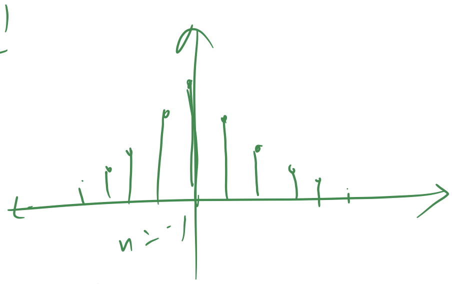
b) Find $X(z)$ and sketch the pole-zero plot and ROC for $a < 1$ and $a > 1$.

Soln Recall

$$|n| = \begin{cases} n, & n \geq 0 \\ -n, & \text{if } n < 0 \end{cases}$$

$$x(n) = \begin{cases} a^n, & n \geq 0 \ ✓ \\ a^{-n}, & n < 0 \ ✓ \end{cases}$$

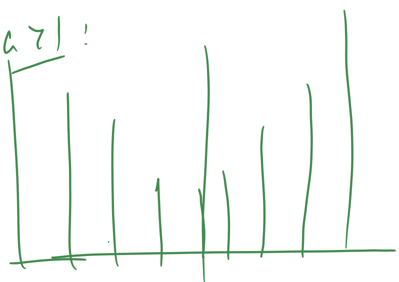
(i) $a < 1$



$a^{-n} = a^{-(-1)} = a$

$a^{-(-2)} = a^2 = x(2)$.

(ii) $a > 1$:



h) $x(n) = \begin{cases} a^n ; & n \geq 0 \\ a^{-n} ; & n < 0 \end{cases}$

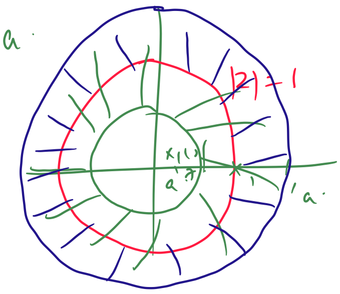
$\therefore x(n) = \underbrace{a^n u(n)}_{x_1(n)} + \underbrace{a^{-n} u(-n-1)}_{x_2(n)}$

$x_1(n) \xleftrightarrow{z} \dfrac{z}{z-a}$; $|z| > a$

$x_2(n) \xleftrightarrow{z} \dfrac{z}{z-1/a}$; $|z| < a$

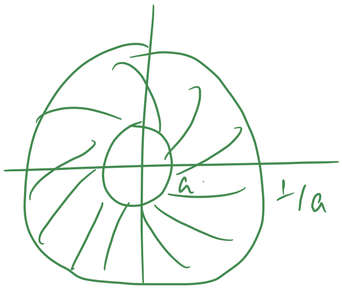
a < 1

$|z| > a$



$\Downarrow$

Common ROC



$X(z) = X_1(z) + X_2(z)$

$= \dfrac{z}{z-a} - \dfrac{z}{z-1/a}$

$= \dfrac{a^2-1}{a} \dfrac{z}{(z-a)(z-1/a)}$

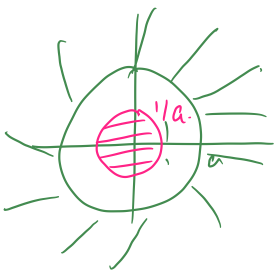
$a < |z| < 1/a$

zeros : 0

poles : a, 1/a

a > 1 :

$X_1(z)$



$X_2(z)$ $|z| < 1/a$

No common intersection

$\therefore X(z)$ DNE.

(7) Consider a discrete time LTI system with impulse response

$$h(n) = \alpha^n u(n)$$

examine the conditions under which

(i) system is causal

(ii) the system is BIBO stable

using z transforms.

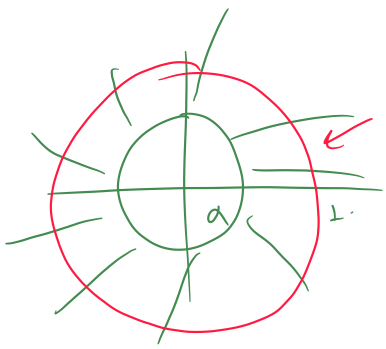
Sol^n $h(n) = \alpha^n u(n) \xleftrightarrow{z} \frac{z}{z-\alpha}$; $|z| > |\alpha|$

$\therefore$ ROC is $|z| > |\alpha|$ $\Rightarrow$ ROC is exterior part of the circle!

$\therefore$ s/s is stable for any value of α

(ii) When is the s/s stable?

$\Rightarrow$ $|\alpha| < 1$

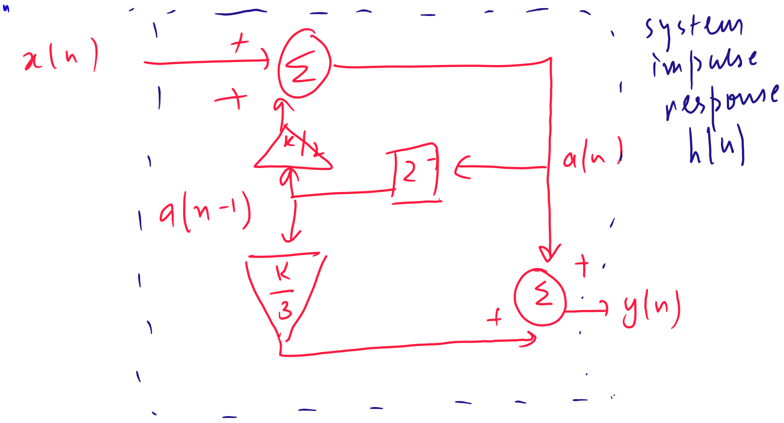


$\therefore$ ROC contains unit circle if $|\alpha| < 1$ and otherwise no!

$\therefore$ s/s is BIBO stable if $|\alpha| \leq 1$ & not stable if $|\alpha| > 1$.

(8) Consider the discrete time causal system as shown below.

For what values of K, is the system BIBO stable?



Solⁿ i) $q(n) = x(n) + \frac{K}{2} q(n-1)$

$y(n) = q(n) + \frac{K}{3} q(n-1)$

Take z transforms on both sides to both equations.

$\Rightarrow Q(z) = X(z) + \frac{K}{2} z^{-1} Q(z)$

$Y(z) = Q(z) + \frac{K}{3} z^{-1} Q(z)$

Rearranging both equations.

$$\left(1 - K/2\, z^{-1}\right) Q(z) = X(z) \quad \text{— (1)}$$

$$\left(1 + \frac{K}{3} z^{-1}\right) Q(z) = Y(z)$$

Taking ratio of (2) & (1)

$$\frac{Y(z)}{X(z)} = \frac{Q(z)\left(1 + K/3\, z^{-1}\right)}{Q(z)\left(1 - \frac{K}{2} z^{-1}\right)}$$

H(z)

$$= \frac{1 + K/3\, z^{-1}}{1 - K/2\, z^{-1}} = \frac{z + K/3}{z - K/2}$$

ROC for H(z)

Poles of $H(z) = K/2$

$\therefore$ Your ROC must be of the form $|z| > |K/2|$ or $|z| < |K/2|$

But system is causal, ROC is $|z| > |K/2|$.

Now for the system to be BIBO stable ROC should include unit circle (ie) $|z| = 1$

$$\therefore \left|\frac{K}{2}\right| < 1 \Rightarrow |K| < 2$$

$\boxed{/}$

(9) Find the inverse z-transform of $X(z) = \frac{2z^3 - 5z^2 + z + 3}{(z-1)(z-2)}$;

$|z| < 1$.

Soln 1st step Make $X(z)$ into a proper rational function

$$
\begin{array}{r l}
 & 2z + 1 \\
z^2 - 3z + 2 & \big) \; 2z^3 - 5z^2 + z + 3 \\
 & -\;(2z^3 - 6z^2 + 4z) \\
 & \quad z^2 - 3z + 3 \\
 & -\;(z^2 - 3z + 2) \\
 & \quad 1
\end{array}
$$

$$= 2z + 1 + \frac{1}{z^2 - 3z + 2}$$

Proper Rational Fraction $\qquad X_1(z)$

Compute the partial fraction expansion of $\frac{X_1(z)}{z}$.

$$\frac{X_1(z)}{z} = \frac{1}{z(z-1)(z-2)} = \frac{c_1}{z} + \frac{c_2}{z-1} + \frac{c_3}{z-2}$$

$$c_1 = \frac{1}{(z-1)(z-2)}\Big|_{z=0} = 1/2$$

$$c_2 = (z-1)\,X(z)\Big|_{z=1} = -1$$

$$c_3 = (z-2)\,X(z)\Big|_{z=2} = 1/2$$

$$\frac{X_1(z)}{z} = \frac{1/2}{z} - \frac{1}{z-1} + \frac{1}{2}\frac{1}{z-2}$$

$$X_1(z) = 1/2 - \frac{z}{z-1} + \frac{1}{2}\frac{z}{z-2}$$

$$X(z) = 2z + \frac{3}{2} - \frac{z}{z-1} + \frac{1}{2}\frac{z}{z-2} \quad ; |z|<1$$

$$= 2\delta(n+1) + \frac{3}{2}\delta(n) + u(-n-1) - \frac{1}{2}2^{n}u(-n-1)$$

left sided signal because ROC is interior part of circle.

$$\boxed{x(n) = 2\delta(n+1) + \frac{3}{2}\delta(n) + (1-2^{n-1})u(-n-1)}$$

(10) Find the inverse Z-T of

$$X(z) = \frac{2+z^{-2}+3z^{-4}}{z^2+4z+3} \quad ; |z|>0$$

Soln

$$X(z) = \frac{2+z^{-2}+3z^{-4}}{z^2+4z+3}$$

$$= \frac{2+z^{-2}+3z^{-4}}{z} \cdot \frac{z}{z^2+4z+3}$$

$$= (2z^{-1}+z^{-3}+3z^{-5})\underbrace{\left(\frac{z}{z^2+4z+3}\right)}_{X_1(z)}$$

Compute partial fraction expansion of $X_1(z)$

$$X_1(z) = \frac{z}{z^2+4z+3}$$

$$\Rightarrow \frac{X_1(z)}{z} = \frac{1}{z^2+4z+3} = \frac{1}{(z+1)(z+3)}$$

$$= \frac{c_1}{z+1} + \frac{c_2}{z+3}$$

$$c_1 = \left.\frac{1}{z+3}\right|_{z=-1} = 1/2$$

$$c_2 = \frac{1}{z+1}\Big|_{z=-3} = -1/2$$

$$\therefore X_1 = \frac{1}{2}\frac{z}{z+1} - \frac{1}{2}\frac{z}{z+3} \; ; |z|>0$$

$$X(z) = \left(2z^{-1} + z^{-3} + 3z^{-5}\right) X_1(z)$$

$$\Rightarrow x(n) = 2x_1(n-1) + x_1(n-3) + 3x_1(n-5)$$

What remain is to compute $x_1(n)$: —

$$\therefore x_1(n) = \frac{1}{2}(-1)^n u(n) - \frac{1}{2}(-3)^n u(n)$$

$$= \frac{1}{2}\left\{(-1)^n - (-3)^n\right\} u(n) \quad \text{— (2)}$$

Now put (2) in (1):

$$x(n) = \left\{(-1)^{n-1} - (-3)^{n-1}\right\} u(n-1) + \frac{1}{2}\left\{(-1)^{n-3} - (-3)^{n-3}\right\} u(n-3) + 3/2\left\{(-1)^{n-5} - (-3)^{n-5}\right\} u(n-5)$$