$\boxed{\text{W10 TA session}}$ Yashvanth. L

Part I - Summary of week X's lectures

(1) Discrete time fourier transforms.

→ Let $x[n]$ be discrete time signal

DTFT $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$

continuous freq signal

π

IDTFT : $x(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega k} d\Omega.$

$|X(\Omega)|$ → Magnitude spectrum.

$\angle X(\Omega)$ → Phase spectrum.

For real signals : $|X(\Omega)|$ → even symmetry

$\angle X(\Omega)$ → odd symmetry.

$|X(-\Omega)| = |X(\Omega)|$

Convergence of DTFT → if $x(n)$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

→ If ROC of $X(z)$ contains unit circle, $x(n)$ is absolutely summable.

→ Converse need not be true

$\sum_{n=-\infty}^{\infty} x[n] < \infty$ $\Rightarrow$ ROC of $X(z)$ should contain unit circle.

→ DTFT is periodic in 2π
ie $X(\Omega + 2\pi) = X(\Omega)$

→ frequency response of a discrete-time system.

$$x[n] \longrightarrow \boxed{h(n)} \longrightarrow y(n).$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$ → freq. response of system.

* Sampling is introducing periodicity in DFT

For any LTI system.

$$H(\Omega) = \sum_{k=0}^{N} b_k e^{-jk\Omega} \Big/ \sum_{k=0}^{N} a_k e^{-jk\Omega}.$$

Properties of DTFT : Ref Oppenheim

$x(n) \to$ Notes
$\updownarrow$
$X(e^{j\omega})$
Oppenheim

$x(n) \longrightarrow X(e^{j\omega})$ periodic with period 2π
$y(n) \longrightarrow Y(e^{j\omega})$

Linearity $\quad a x(n) + b y(n) \longrightarrow a X(e^{j\omega}) + b Y(e^{j\omega})$

Time shifting $\quad x(n-n_0) \longrightarrow e^{-j\omega n_0} X(e^{j\omega})$

Freq shift $\quad e^{j\omega_0 n} x(n) \longrightarrow X(e^{j(\omega-\omega_0)})$

Conjugation $\quad x^*(n) \longrightarrow X^*(e^{-j\omega})$

Time reversal $\quad x(-n) \longrightarrow X(e^{-j\omega})$

Time expansion	$x_{(k)}[n] = \begin{cases} x(n/k), & n = \text{mult of } k \\ 0, & n \neq \text{mult of } k. \end{cases}$	$X(e^{jkw})$
Convolution	$x(n) * y(n)$	$X(e^{jw})Y(e^{jw})$
Mult.	$x(n)y(n)$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\, Y(e^{j(w-\theta)})\, d\theta.$
Diff in time	$x(n) - x(n-1)$	$(1-e^{-jw})\, X(e^{jw})$
accumulation	$\sum_{k=-\infty}^{n} x(k)$	$\frac{1}{1-e^{-jw}} X(e^{jw})$
Diff in freq	$n\,x(n)$	$j\,\frac{dX(e^{jw})}{dw}.$
Conj symm for real signals.	$x(n)$ real	$X(e^{jw}) = X^*(e^{-jw})$ $\mathrm{Re}(X(e^{jw})) = \mathrm{Re}(X(e^{-jw}))$ $\mathrm{Im}(X(e^{jw})) = -\mathrm{Im}(X(e^{-jw}))$ $\lvert X(e^{jw})\rvert = \lvert X(e^{-jw})\rvert$ $\measuredangle X(e^{jw}) = -\measuredangle X(e^{-jw})$
Symmetry for real & even	$x(n)$ real even	$X(e^{jw})$ real & even.
" " " & odd	$x(n)$ real, odd	$X(e^{jw})$ is purely imag.

Parseval's relation for aperiodic signals

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{jw})|^2 dw.$$

Duality

$x(n) \longrightarrow X(w)$

$x(t) \overset{FS}{\longleftrightarrow} x(-n)$ (n^{th} F. coeff)

↳ Continuous time periodic

Basic DTFT pairs

signal	FT
$\sum_{k=\langle N\rangle} a_k e^{jk2\pi/N\, n}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(w - \frac{2\pi k}{N}\right)$
$e^{jw_0 n}$	$2\pi \sum_{l=-\infty}^{\infty} \delta(w - w_0 - 2\pi l)$
$\cos w_0 n.$	$\pi \sum_{l=-\infty}^{\infty} \left(\delta(w - w_0 - 2\pi l) + \delta(w + w_0 - 2\pi l)\right)$
$\sin w_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{\infty} \{\delta(w - w_0 - 2\pi l) - \delta(w + w_0 - 2\pi l)\}$
$x(n) = 1$	$2\pi \sum_{l=-\infty}^{\infty} \delta(w - 2\pi l)$
Periodic square wave $x(n) = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(w - \frac{2\pi k}{N}\right)$

$\sum_{k=-\infty}^{\infty} \delta(n-kN)$	$2\pi/N \sum_{k=-\infty}^{\infty} \delta\left(w-\frac{2\pi k}{N}\right)$
$a^n u(n)$, $\lvert a\rvert<1$	$\frac{1}{1-ae^{-jw}}$.
$a(n)=\begin{cases}1, & \lvert n\rvert\le N_1\\ 0, & \lvert n\rvert>N_1\end{cases}$	$\frac{\sin(w(N_1+1/2))}{\sin(w/2)}$
$\frac{\sin Wn}{\pi n}=\frac{W}{\pi}\,\mathrm{sinc}\left(\frac{W\pi}{\pi}\right)$ $0<W<\pi$	$X(w)=\begin{cases}1, & 0\le\lvert w\rvert\le W\\ 0, & W<\lvert w\rvert\le\pi\end{cases}$
$\delta(n)$	1
$u(n)$	$\frac{1}{1-e^{-jw}}+\sum_{k=-\infty}^{\infty}\pi\delta(w-2\pi k)$
$\delta(n-n_0)$	e^{-jwn_0}
$(n+1)\,a^n u(n)$, $\lvert a\rvert<1$	$\frac{1}{(1-ae^{-jw})^2}$
$\frac{(n+r-1)!}{n!\,(r-1)!}a^n u(n)$, $\lvert a\rvert<1$	$\frac{1}{(1-ae^{-jw})^r}$.

Part II Tutorials (DFT next week)

(1) Determine the discrete Fourier series representation for each of the following sequences (discrete time signals)

a) $x[n] = \cos(\pi/4\, n)$

b) $x(n) = \cos(\pi/3\, n) + \sin(\pi/4\, n)$

c) $x[n] = \cos^2(\pi/8\, n)$

Quick recollect:

Given $x[n]$ – discrete time periodic signal with period N_0

DFS
$$x(n) = \sum_{K=0}^{N_0-1} C_k\, e^{jk\Omega_0 n}$$

Property DFS coefficients ie C_k's are periodic with period : N_0

ie $\boxed{C_{K+N_0} = C_K \quad \forall K}$ $\quad \Omega_0 = 2\pi/N_0$

Solution

(i) $x[n] = \cos(\pi/4\, n) \rightarrow$ discrete time periodic signal

Period $= \dfrac{2\pi}{\pi/4} = \dfrac{2\pi}{\pi} \times 4 = 8$

$\boxed{N_0 = 8}$ $\quad \Omega_0 = 2\pi/N_0 = 2\pi/8 = \pi/4$

$\therefore$ DFS representation is:–

$$\cos(\pi/4\, n) = \sum_{K=0}^{7} C_K\, e^{jK\pi/4\, n} \quad \text{— (A)}$$

$\hookrightarrow$ find C_K!

By Euler's relation,

$$\cos(\pi/4\, n) = \frac{1}{2}\left\{e^{j\pi/4\, n} + e^{-j\pi/4\, n}\right\}$$

$$= \frac{1}{2}\left\{e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right\}$$

$$c_1 = 1/2, \quad c_{-1} = 1/2$$

$k = 0, 1, \ldots, 7$

Now:

$$c_{-1} = c_{-1+N_0} = c_{-1+8}$$

$$= c_7 = 1/2$$

$$c_1 = c_7 = 1/2$$

$$\cos(\pi/4\, n) = \frac{1}{2} e^{j\Omega_0 n} + \frac{1}{2} e^{j7\Omega_0 n} \quad \text{— (B)}$$

$$e^{j7\Omega_0 n} = e^{j7\pi/4\, n}$$

$$= e^{j(2\pi - \pi/4)\, n}$$

$$= \underbrace{e^{j2\pi n}}_{1} e^{-j\pi/4\, n}$$

$$= e^{-j\pi/4\, n}$$

$$= e^{-j\Omega_0 n}$$

Compare eqn (A) & (B) :-

$$c_1 = c_7 = 1/2$$

$$c_0 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$$

$$\& \cos(\pi/4\, n) = \sum_{k=0}^{7} c_k e^{jk\pi/4\, n}$$

→ DFS of $\cos(\pi/4\, n)$

b) $x(n) = \cos\left(\frac{\pi}{3}n\right) + \sin\left(\frac{\pi}{4}n\right)$

Solution : $x(n) = x_1(n) + x_2(n)$

$x_1(n) = \cos(\pi/_3 n)$

$x_2(n) = \sin(\pi/_4 n)$

$x_1(n) \rightarrow$ periodic . $N_{0_1} = \frac{2\pi}{\pi/3} = 6\pi/\pi = 6$

$x_2(n) \rightarrow$ periodic $N_{0_2} = \frac{2\pi}{\pi/4} = \frac{8\pi}{\pi} = 8$

$\therefore$ Period of $x(n) = LCM(6, 8)$

$= 24$

6	8
12	16
18	24
24	
30	
36	

$N_0 = 24$ (Period of $x(n)$)

$\Rightarrow \Omega_0 = 2\pi/N_0$

$= 2\pi/24$

$= \pi/12$

$\therefore$ DFS representation will be

$$x[n] = \sum_{k=0}^{23} c_k e^{jk\pi/_{12} n} \quad \text{—} \; (A)$$

$\hookrightarrow c_k$. To be found

$k = 0, 1, \ldots, 23$.

Recall

$x[n] = \cos(\pi/_3 n) + \sin(\pi/_4 n)$

By Euler's relation

$$= \frac{e^{j\pi/_3 n} + e^{-j\pi/_3 n}}{2} + \frac{e^{j\pi/_4 n} - e^{-j\pi/_4 n}}{2j}$$

$$= \frac{-1}{2j} e^{-j\pi/4\, n} + \frac{1}{2} e^{-j\pi/3\, n} + \frac{1}{2} e^{j\pi/3\, n} + \frac{1}{2j} e^{j\pi/4\, n}$$

$$\boxed{\begin{aligned} \pi/4 &= 3\pi/12 \\ &= 3\Omega_0 \\ \pi/3 &= 4\pi/12 \\ &= 4\Omega_0 \end{aligned}}$$

$$= \frac{-1}{2j} e^{-j3\Omega_0 n} + \frac{1}{2} e^{-j4\Omega_0 n} + \frac{1}{2} e^{j4\Omega_0 n} + \frac{1}{2j} e^{j3\Omega_0 n}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$c_{-3} \qquad + c_{-4} \qquad c_4 \qquad c_3$$

$$\Downarrow \qquad \Downarrow$$

$$c_{-3} = c_{-3+N_0} \qquad c_{-4} = c_{-4+N_0}$$

$$= c_{-3+24} \qquad = c_{-4+24}$$

$$= c_{21} = -1/2j \qquad = c_{20} = 1/2$$

$$\therefore x(n) = \frac{1}{2j} e^{j3\Omega_0 n} + \frac{1}{2} e^{j4\Omega_0 n} + \frac{1}{4} e^{j20\Omega_0 n} - \frac{1}{2j} e^{j21\Omega_0 n} \quad \text{— (B)}$$

Compare (A) & (B)

$$x(n) = \sum_{k=0}^{23} c_k e^{jk\pi/12\, n}$$

$$c_3 = 1/2j; \quad c_4 = c_{20} = 1/2; \quad c_{21} = -1/2j$$

$$c_k = 0 \quad \forall\ k \in \{0, \dots, 23\} \text{ except } 3, 4, 20, 21.$$

↳ DFS representation.

c) $x(n) = \cos^2(\pi/8\, n)$. $\cos 2\theta = 2\cos^2\theta - 1$

Sol^n $x(n) = \dfrac{1 + \cos(\pi/4\, n)}{2}$ $\Rightarrow \cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$

$N_0 = 2\pi / \pi/4 = 8$; $\Omega_0 = 2\pi/8 = \pi/4$.

$\therefore$ DFS

$$x(n) = \sum_{k=0}^{7} c_k e^{jk\Omega_0 n}$$

$$= \sum_{k=0}^{7} c_k e^{jk\pi/4 n}$$

$\hookrightarrow$ Find c_k ; $k = 0, 1, \dots, 7$

— (A)

Now by Euler's relation.

$$x(n) = \cos^2(\pi/8\ n) = \left(\frac{e^{j\pi/8 n} + e^{-j\pi/8 n}}{2}\right)^2$$

$$= 1/4\, e^{j\pi/4 n} + 1/4\, e^{-j\pi/4 n} + 2 \cdot \frac{1}{2} \cdot 1/2$$

$$= 1/4\, e^{j\pi/4 n} + 1/4\, e^{-j\pi/4 n} + 1/2$$

$\downarrow$ c_1 $= c_1$; $\downarrow$ c_{-1} $= c_7$; $\downarrow$ c_0 $= c_0$; $\Omega_0 = \pi/4$

$c_{-k} = c_{-k+N_0}$

$c_{-1} = c_{-1+8}$
$= c_7$

$c_0 = 1/2$, $c_7 = 1/4$, $c_1 = 1/4$.

$$\therefore x(n) = 1/2 + 1/4\, e^{j\Omega_0 n} + 1/4\, e^{j7\Omega_0 n}.$$

(B)

Comparing (A) & (B)

$$\boxed{\begin{array}{l} \text{DFS} \quad x(n) = \sum_{n=0}^{7} c_k e^{jk\pi/4 n} \\ c_0 = 1/2,\ c_1 = 1/4,\ c_2 = c_3 = c_4 = c_5 = c_6 = 0, \\ c_7 = 1/4 \end{array}}$$

(2) Find the DTFT of $x(n) = -a^n u(-n-1)$; $a \in \mathbb{R}$, $a > 1$

Sol^n By z-transform.

$$X(z) = \frac{z}{z-a};$$

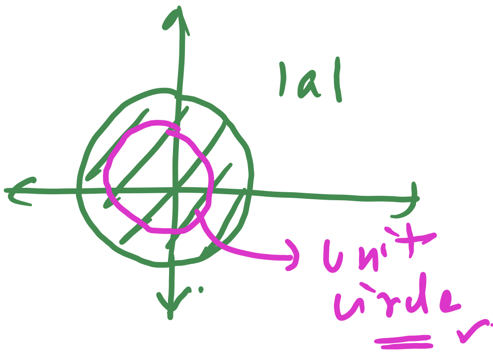
ROC

$|z| < |a|$.

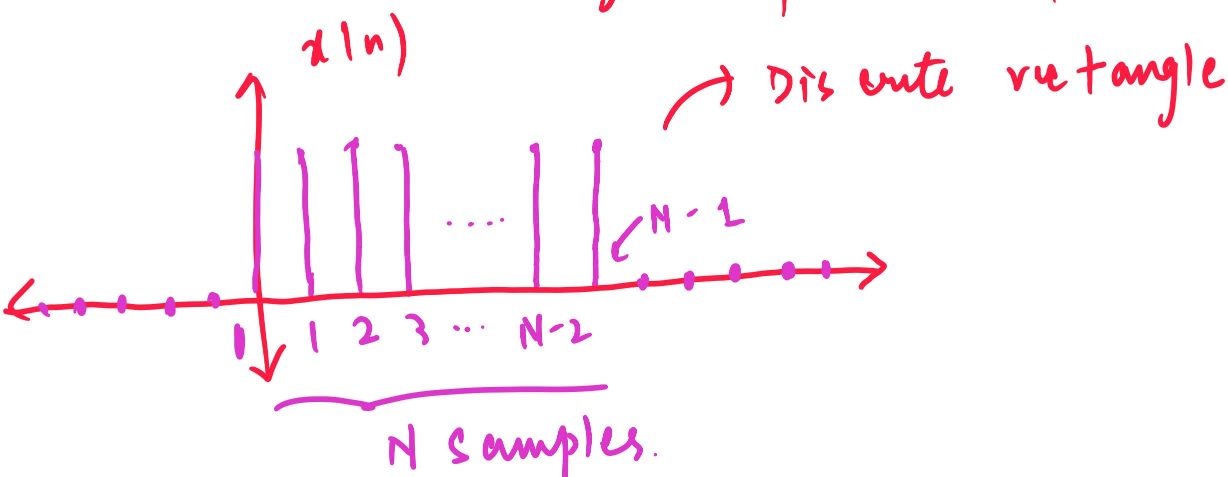
$z = 1$ is part of ROC

∴ Put $z = e^{jw}$ For DTFT

$$X(w) = \frac{1}{1 - a e^{-jw}} \text{ or } \frac{e^{jw}}{e^{jw} - a}.$$



(3) Find DTFT of the rectangular pulse sequence:



$= u(n) - u(n-N)$

$= u(n) - u(n-N)$

Sol^n

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$= \sum_{n=0}^{N-1} e^{-jnw}$$

$$= 1 + e^{-j\Omega} + e^{-j2\Omega} + \dots + e^{-j\Omega(N-1)}$$

$$= \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}}$$

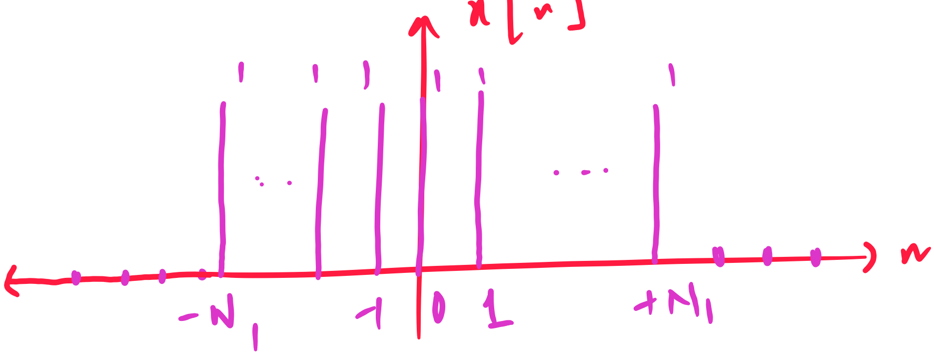
$$= \frac{e^{-j\Omega N/2}\left(e^{j\Omega N/2} - e^{-j\Omega N/2}\right)}{e^{-j\Omega/2}\left(e^{j\Omega/2} - e^{-j\Omega/2}\right)}$$

$$= e^{-j\Omega\left(\frac{N-1}{2}\right)} \frac{\cancel{2j}\left(\sin \Omega N/2\right)}{\cancel{2j} \sin\left(\Omega/2\right)}$$

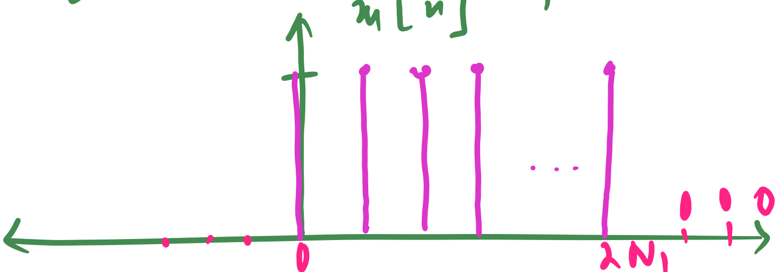
$$= e^{-j\Omega\frac{(N-1)}{2}} \frac{\sin\left(\frac{\Omega N}{2}\right)}{\sin\left(\Omega/2\right)}$$

$$\therefore \boxed{X(\Omega) = \sin\left(\frac{\Omega N}{2}\right) \Big/ \sin(\Omega/2)\, e^{-j\Omega\frac{(N-1)}{2}}}$$

(ii) Find the DTFT $X(\Omega)$ of the following signal:



Solution :- Let $x_1(n)$ be defined as:



Now $x(n) = x_1(n+N_1)$

$$\therefore X(\Omega) = e^{+j\Omega N_1} X_1(\Omega) \quad \text{—} \; (1)$$

To find $X_1(\Omega)$, recall that $x_1(n)$ is same as $x(n)$ in prev. question with

$$N = 2N_1+1$$

From prev Q:-

$$X_1(\Omega) = e^{-j\Omega\left(\frac{2N_1+1-1}{2}\right)} \frac{\sin\left((2N_1+1)\Omega/2\right)}{\sin(\Omega/2)}$$

$$= e^{-j\Omega N_1} \frac{\sin\left(\frac{2N_1-\Omega}{2} + \Omega/2\right)}{\sin\left(\frac{\Omega}{2}\right)} \quad \text{—} \; (2)$$

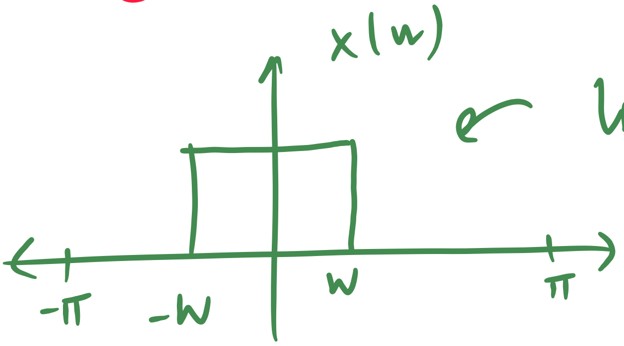
Putting (2) in (1)

$$\boxed{X(\Omega) = \frac{\sin\left[\Omega\left(N_1+1/2\right)\right]}{\sin(\Omega/2)}}$$

(5) Find the IDTFT of the following freq domain response

$$X(\Omega) = \begin{cases} 1, & \text{if } |\Omega| \le W \\ 0, & \text{if } W < |\Omega| \le \pi \end{cases}$$

Sol^n



low pass filter response

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} 1 \; e^{j\Omega n}\, d\Omega.$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi} \left. \frac{e^{j\Omega n}}{jn} \right|_{-W}^{W}.$$

$$= \frac{1}{2\pi}\left[\frac{e^{jWn} - e^{-jWn}}{jn}\right]$$

$$= \frac{1}{\pi n} \sin(nW)$$

$$\therefore \; x(n) = \frac{\sin(Wn)}{\pi n}$$

⑥ Find the IDTFT of

$X(\Omega) = 2\pi\delta(\Omega - \Omega_0)$; $(\Omega), |\Omega_0| < \pi$

Sol^n:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi} \cancel{2\pi}\, \delta(\Omega - \Omega_0)\, e^{j\Omega n}\, d\Omega.$$

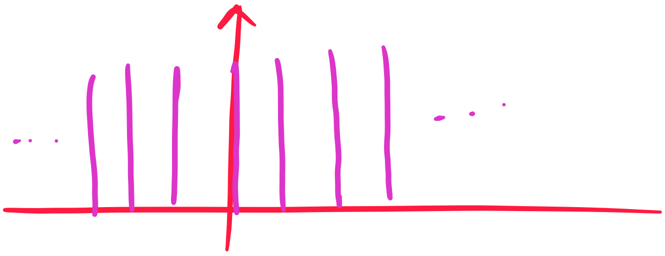
$$= e^{j\Omega_0 n}$$

By sifting property

$$\therefore \; e^{j\Omega_0 n} \xleftrightarrow{DTFT} 2\pi\,\delta(\Omega - \Omega_0)$$

$$\Leftrightarrow \; \frac{1}{2\pi} e^{j\Omega_0 n} \xleftrightarrow{DTFT} \delta(\Omega - \Omega_0)$$

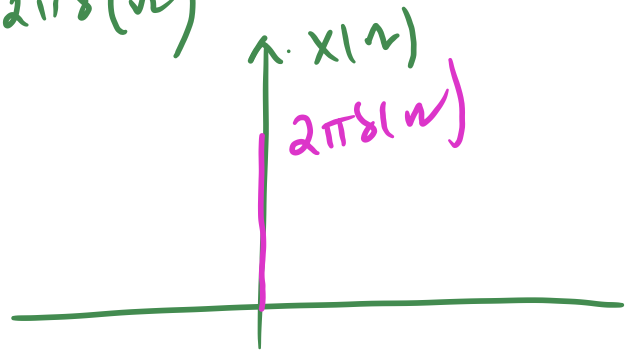
(7) Find the DTFT of $x(n) = 1 \; \forall n$.



Sol^n Put $\Omega_0 = 0$:

from eqn (4) of prev. answer.

$$1 \xleftrightarrow{DTFT} 2\pi\delta(\Omega)$$



(8) Find DTFT of

$$x(n) = -\sin(\Omega_0 n) \quad ; \quad |\Omega_0| \le \pi$$

Sol^n

$$x(n) = -\left(\frac{e^{j\Omega_0 n} - e^{-j\Omega_0 n}}{2j}\right)$$

$$= \frac{e^{-j\Omega_0 n}}{2j} - \frac{e^{j\Omega_0 n}}{2j}$$

$$\xleftrightarrow{DFT} \frac{2\pi\delta(\Omega+\Omega_0)}{2j} - \frac{2\pi\delta(\Omega-\Omega_0)}{2j}$$

$$= -\pi j\,\delta(\Omega+\Omega_0) + \pi j\,\delta(\Omega-\Omega_0)$$

$$\boxed{X(\Omega) = \pi j\left(\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)\right)}$$

(9) Using convolution property of DTFT, find IDTFT of

$$X(\Omega) = \frac{1}{(1-ae^{-j\Omega})^2} \; ; |a|<1.$$

Sol From class lecture

$$a^n u(n) \xleftrightarrow{FT} \frac{1}{1-ae^{-j\Omega}}.$$

$$\therefore x(n) = \frac{1}{(1-ae^{-j\Omega})} \frac{1}{(1-ae^{-j\Omega})}$$

$\therefore$ By convolution theorem.

$$x(n) = a^n u(n) \circledast a^n u(n)$$

$$= \sum_{k=-\infty}^{\infty} a^k u(k)\, a^{n-k} u(n-k)$$

$u(k) = 0$, only for $k<0$

$u(n-k) = 0$ if $n-k<0 \Rightarrow \boxed{k>n}$

$$= \sum_{k=0}^{n} a^k a^{n-k} = \sum_{k=0}^{n} a^n$$

$$= a^n \sum_{k=0}^{n} 1$$

$$= a^n (n+1)$$

valid only for $n>0$

for $n<0$ $\quad x(n) = 0$

$$\therefore \boxed{x(n) = (n+1) a^n u(n)}$$

(10) Find IDFT of $x(\Omega) = \dfrac{e^{-j\Omega}}{(1-ae^{-j\Omega})^2}$

Sol^n By time shifting property

$$x(n) \longrightarrow X(\Omega)$$
$$x(n-n_0) \longrightarrow X(\Omega)\, e^{-j\Omega n_0}$$

In the Question

let $x(n)$ be same as prev. Q & $n_0 = 1$

$\therefore$ IDTFT $= x(n-1)$

$= n a^{n-1} u(n-1)$

W-10 Assignment

① for gaussian pulse

$$e^{-at^2} \xleftrightarrow{FT} \sqrt{\pi/a}\; e^{-w^2/4a}.$$

$a = 2$ $\quad \sqrt{2}\sqrt{\pi/2\times 2}\; e^{-w^2/8} \xleftrightarrow{IFT} e^{-2t^2}$

$$\Rightarrow \sqrt{\pi/4}\; e^{-w^2/8} \xleftrightarrow{IFT} \frac{e^{-2t^2}}{\sqrt{2}}.$$

② $x(t) = \cos(2\pi f_0 t)$

$$x(t) \longrightarrow \boxed{\text{shift by } \pi/2} \longrightarrow \hat{x}(t)$$

$$x(t - \pi/2) = \cos(2\pi f_0 t - \pi/2)$$
$$= \sin 2\pi f_0 t$$

$$x(t)\cos(2\pi f_c t) - \hat{x}(t)\sin(2\pi f_c t)$$
$$= \cos(2\pi f_0 t)\cos(2\pi f_c t) - \sin(2\pi f_0 t)\sin(2\pi f_c t)$$