Week 9 TA Session — Yashvanth L

Part I Summary of week IX's lectures.

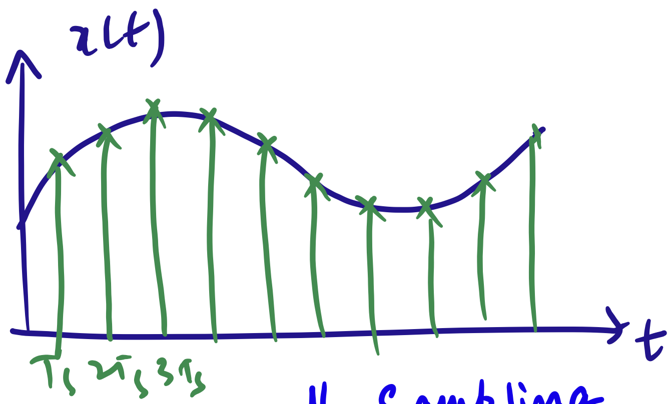
→ Problems on filter design.

→ Problems on RC circuits

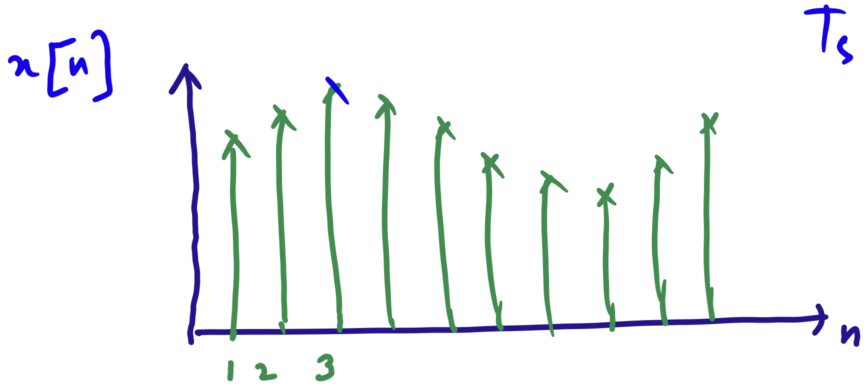
(*) Time constant — $\tau = RC$ (63% of final value)

(*) 3dB bandwidth — $\omega_0 = \frac{1}{RC}$

① Sampling of continuous time-signal to discrete-time signal.



⇓ Sampling



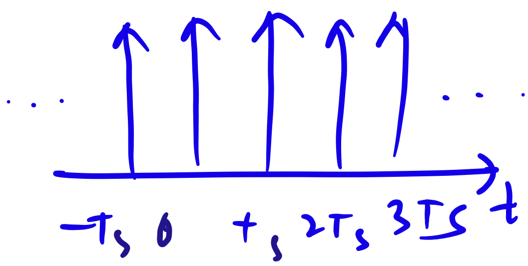
T_s - sampling Time

$x[n] \triangleq x(nT_s)$

$F_s = \frac{1}{T_s}$ ← Sampling Frequency

(*) Impulse train Sampling:

$$\delta_{T_s}(t) \triangleq \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$



Sampled signal

$x'(t) = x(t) \times \delta_{T_s}(t)$; $x(n) = x'(nT_s)$

$$\boxed{x_{T_s}(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\,\delta(t-kT_s)}$$

Fourier spectrum of sampled signal:

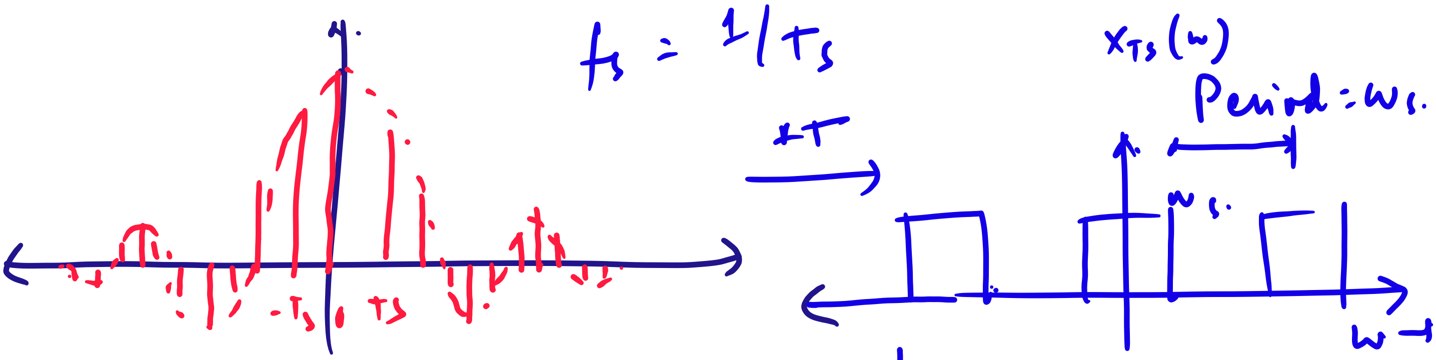
$$X_{T_s}(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) = \frac{\omega_s}{2\pi}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

→ Periodic version of $X(\omega)$.

eg

↓ sample at sampling freq

$f_s = 1/T_s$



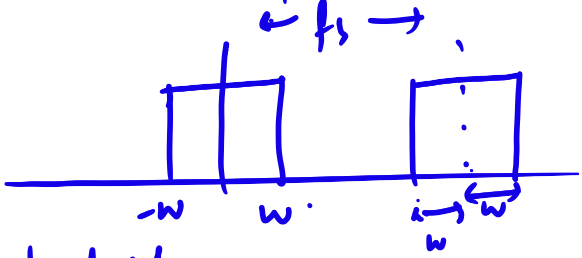
$\omega_s = 2\pi/T_s$.

↓ periodic version of $X(\omega)$ with amplitude scaling

→ If the replicas overlap ⇒ "spectral aliasing"

→ to avoid spectral aliasing :-

$\leftarrow f_s \rightarrow$



$\boxed{f_s \geq 2W}$

↓

no overlaps !

∴ To avoid spectral aliasing, required sampling frequency :

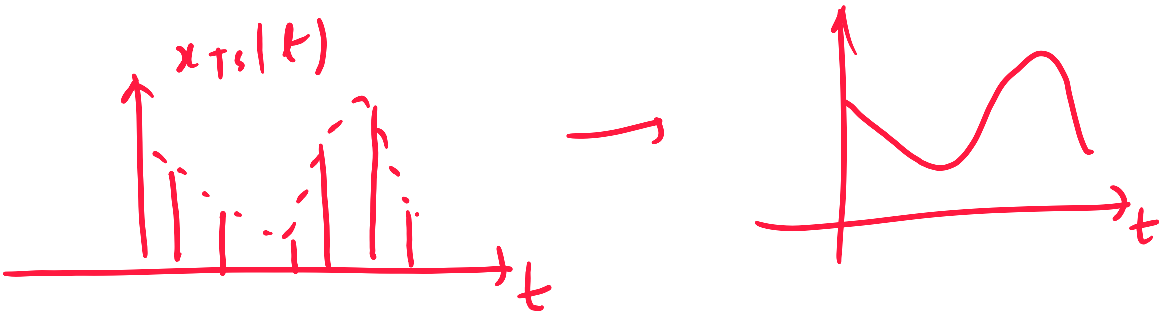
$$f_s \geq 2W.$$

where $W \rightarrow$ maximum frequency content in $x(t)$.

(*) why is "no spectral aliasing" important ?

→ Suppose from sampled signal $x_{T_s}(t)$, we want to recover $x(t)$ back!

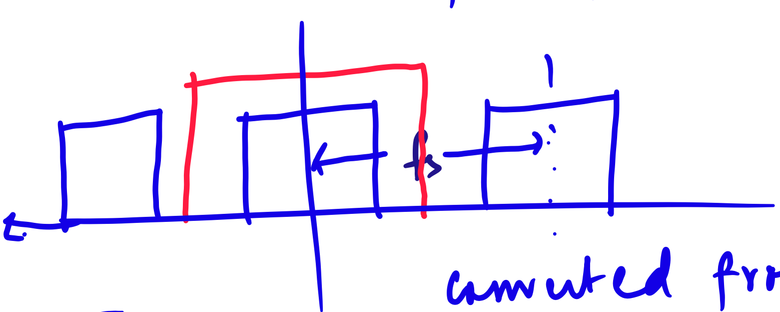
(ie)



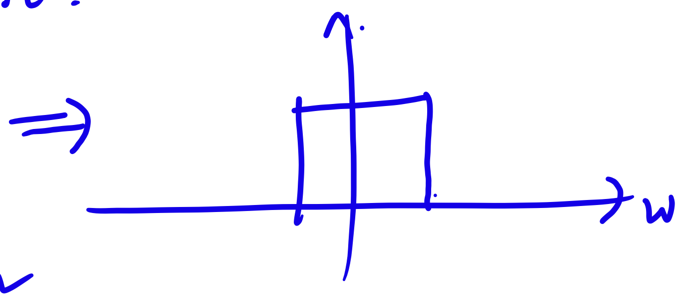
How to do ?

Sol^n Low pass filtering of sampled signal :-

This is because :-

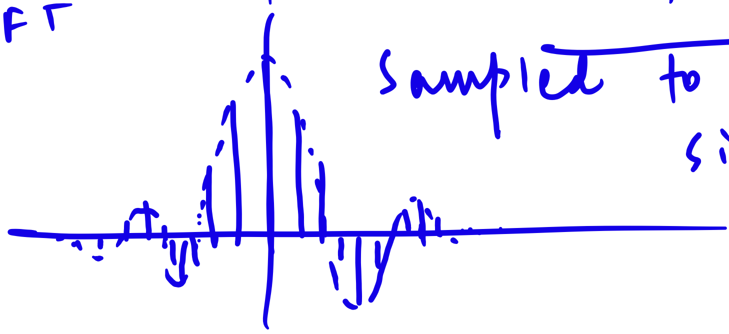


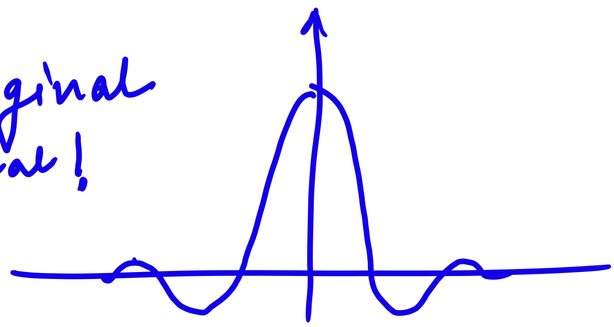
$\Rightarrow$



converted from sampled to original signal!

IFT





→ Now in order to have distortionless of the reconstruction, only one replica should be with LPF → Replicas should not overlap
⇒ No spectral aliasing is needed.

Nyquist Sampling Theorem

Any C.T signal $x(t)$ can be sampled and reconstructed from its sampled signal w/o any distortion if the sampling frequency

$$f_s \geq 2 f_m$$

$$\text{(i.e.)} \quad \omega_s \geq 2 \omega_m.$$

where f_m is the max^{m} freq component of $x(t)$.

→ A signal is said to be sampled at "Nyquist Rate" if $\omega_s = 2\omega_m$ $(f_s = 2f_m)$

Reconstructed Signal

→ Passing $X_{T_s}(\omega)$ through a LPF is similar to passing the sampled signal $x_{T_s}(t)$ through a system whose impulse response is "sinc function".

↳ (*) due to convolution theorem of F.T

$$\therefore x(t) = \sum_{k=-\infty}^{\infty} x(kT_s) \, \text{sinc}(\omega_m t - k\pi)$$

2 Discrete Fourier Series:-

Let $x[n]$ be a discrete-time signal with period N_0

$$x[n] = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n}$$

at frequencies $k\Omega_0$; $k = 0, 1, \ldots, N_0 - 1$

(ie) $= 0, \frac{2\pi}{N_0}, \ldots, \frac{2\pi}{N_0}(N_0 - 1)$

$$c_l = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] \, e^{-jl\Omega_0 n}$$

$$c_0 = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x(n)$$

Mean of the signal over one period.

DC component

→ DFS is guaranteed to converge because of its finite summation.

→ DFS coefficients are periodic :

$$\boxed{c_{k+N_0} = c_k \ \forall k}$$

eg

$\boxed{N_0 = 4}$

$c_4 = c_0 = c_8$.

$c_1 = c_5 = c_9$

$c_2 = c_6 = c_{10}$

$c_3 = c_7 = c_{11}$

→ Duality

$$x(n) \xleftrightarrow{DFS} c(k)$$

$$c(n) \xleftrightarrow{DFS} N_0 \, x(-k)$$

→ For real signals.

$c_{-k} = c_k^*$ [conjugate property]

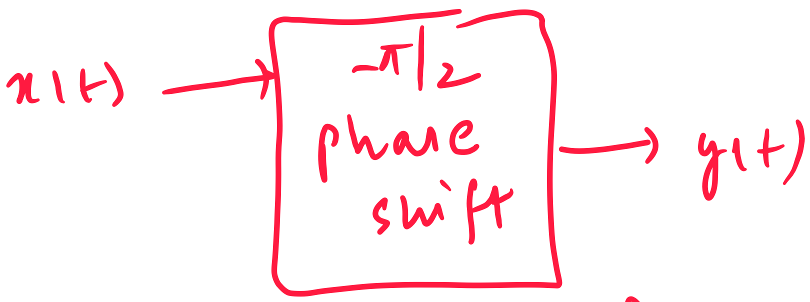
→ Parseval's theorem

$$\sum_{k=0}^{N_0-1} |c_k|^2 = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x(n)|^2 \rightarrow \text{Energy remains const in time \& freq domains}$$

Part II Tutorials

① An ideal $(-\pi/2)$ phase shifter is defined by the freq response

$$H(\omega) = \begin{cases} e^{-j\pi/2} & ; \omega > 0 \\ e^{j\pi/2} & ; \omega < 0 \end{cases}$$



$h(t) \longrightarrow$ freq response

(i) Find the impulse response h(t) of this phase shifter

(ii) Find the o/p of y(t) when $x(t) = \cos(\omega_0 t)$

Sol

$$e^{-j\pi/2} = \cos(-\pi/2) + j\sin(-\pi/2)$$

$$= 0 - j = -j$$

$|||^{r}$ $$e^{j\pi/2} = \cos(\pi/2) + j\sin(\pi/2)$$

$$= 0 + j = \underline{j}$$

$$\Rightarrow H(\omega) = \begin{cases} -j ; & \omega > 0 \\ j ; & \omega < 0 \end{cases}$$

$$= - \begin{cases} j , & \omega > 0 \\ -j ; & \omega < 0 \end{cases}$$

$= -j \, sgn(w)$.

$\therefore H(w) = -j \, sgn(w)$

<u>Recall</u> $\quad sgn(t) \xleftrightarrow{FT} \dfrac{2}{jw}$ → known!

$$\frac{2}{jt} \rightarrow 2\pi \, sgn(-w)$$
$$\cdot \; -2\pi \, sgn(w)$$

$$\Rightarrow \frac{1}{jt} \xleftrightarrow{F.T} -\pi \, sgn(w)$$

$$\frac{1}{j\pi t} \xleftrightarrow{FT} -sgn(w)$$

$$\frac{1}{\pi t} \xleftrightarrow{FT} -j \, sgn(w)$$
$$= H(w)$$

⇒ Impulse response of above system is

$$h(t) = \frac{1}{\pi t}$$

— Hilbert transformer

(ii) $y(t) = ?$ if $x(t) = \cos w_0 t$

$$Y(w) = X(w) H(w)$$

$$X(w) = \pi [\delta(w - w_0) + \delta(w + w_0)]$$

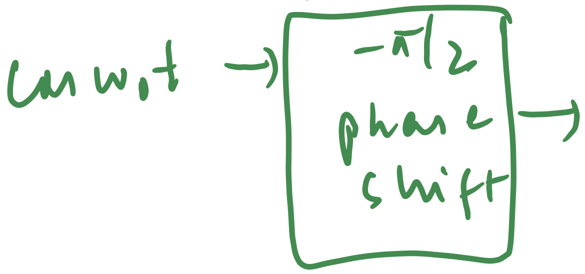
$$Y(w) = -j \pi \, sgn(w) \, \delta(w - w_0) - j\pi \, sgn(w) \, \delta(w + w_0)$$

$$= -j\pi \, \underbrace{sgn(w_0)}_{=1} \, \delta(w - w_0)$$
$$-j\pi \, \underbrace{sgn(-w_0)}_{=-1} \, \delta(w + w_0)$$

$$= -j\pi \, \delta(w - w_0) + j\pi \, \delta(w + w_0)$$

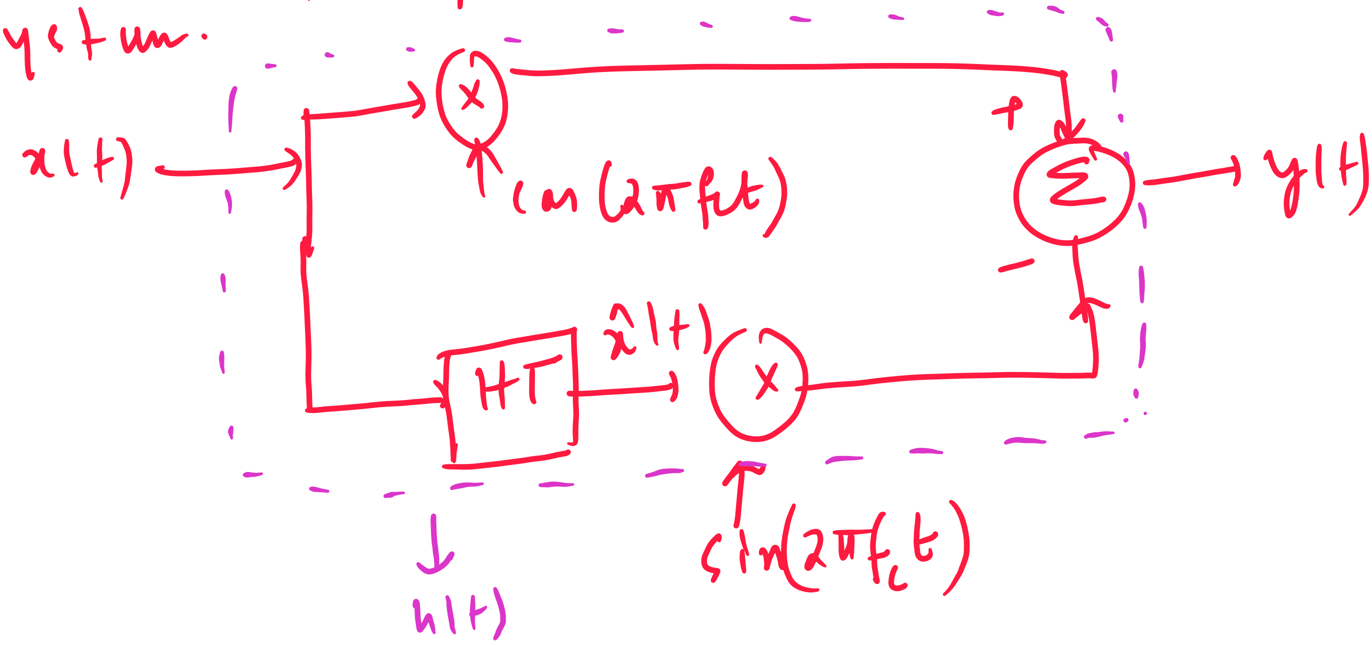
$$= -j\pi\left[\delta(w-w_0) - \delta(w+w_0)\right]$$

$$y(t) = \sin(w_0 t)$$

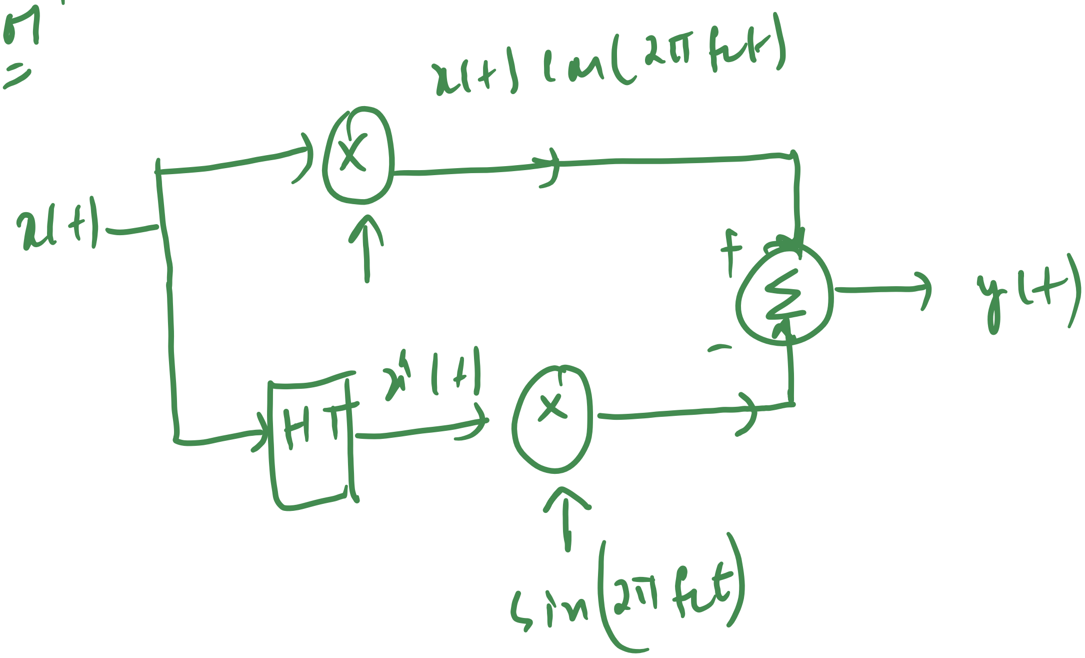


$$\cos(w_0 t - \pi/2)$$
$$= \cos(\pi/2 - w_0 t)$$
$$= \sin w_0 t$$

(2) let $x(t) = \cos(2\pi f_0 t)$. let $\hat{x}(t)$ be the hilbert transform. Find the o/p of the system.



Solⁿ



$$y(t) = x(t)\cos(2\pi f_c t) - \hat{x}(t)\sin(2\pi f_c t)$$

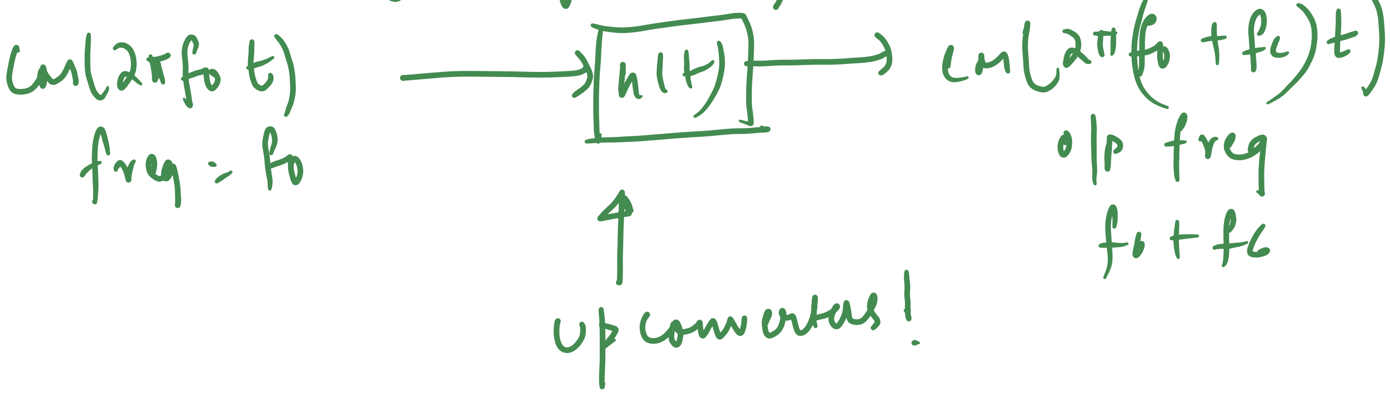
$\hat{x}(t)$ = HT of $x(t)$

If $x(t) = \cos(2\pi f_0 t)$

$$\Rightarrow \hat{x}(t) = \cos(2\pi f_0 t - \pi/2)$$
$$= \sin(2\pi f_0 t)$$

$$y(t) = \cos(2\pi f_0 t)\cos(2\pi f_c t) - \sin(2\pi f_0 t) \times \sin(2\pi f_c t)$$

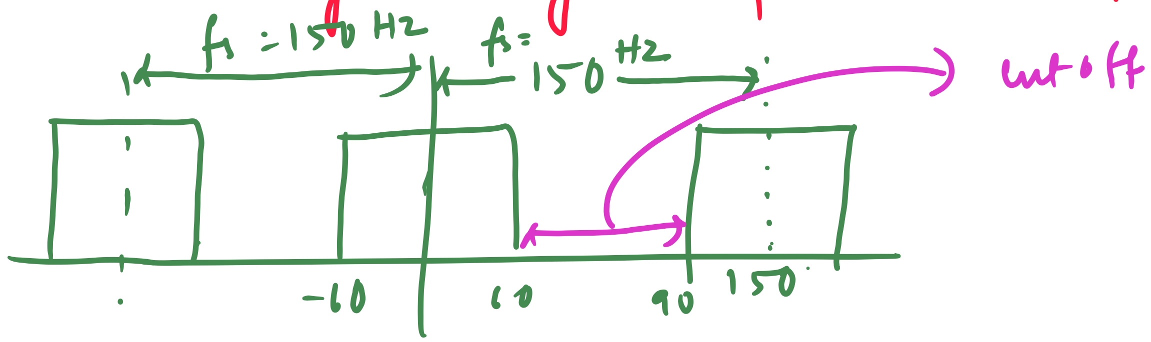
$$= \cos(2\pi f_0 t + 2\pi f_c t)$$
$$= \cos(2\pi(f_0 + f_c)t)$$



→ application in communication systems.

(3) Consider a signal with maximum frequency of 60 Hz, sampled at $f_s = 150$ Hz using a unit impulse train. What should be the cutoff frequency range of the low pass filter that can be used to recover the original signal from its sampled version?

Solution



$$\boxed{60 \le f_c \le 90 \text{ Hz}}$$

4) What is the minimum sampling frequency required to sample the signal

$$x(t) = \frac{\sin(2\pi t)}{2\pi t} \times \cos(2\pi t)$$

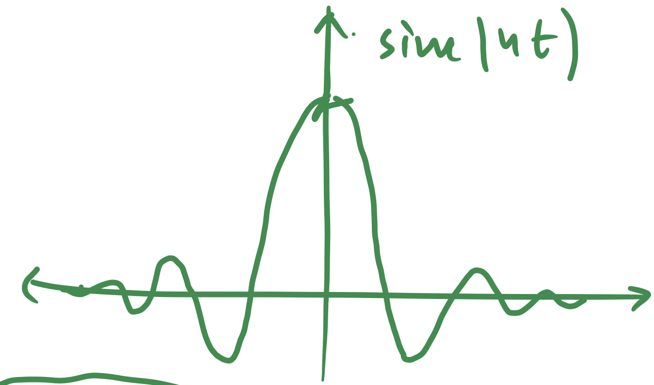
Solution

$$x(t) = \frac{1}{2}\,\frac{\sin(4\pi t)}{2\pi t}$$

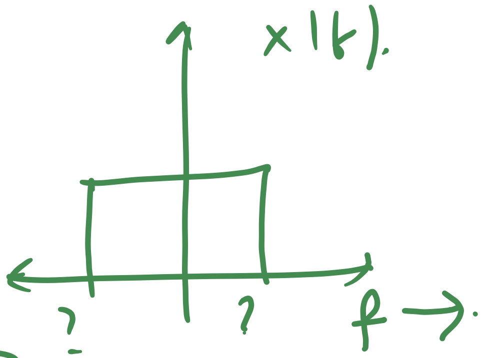
$$= \frac{\sin(4\pi t)}{4\pi t}$$

$$= \text{sinc}(4t)$$

$$\text{sinc}(at) = \frac{\sin(\pi a t)}{\pi a t}.$$

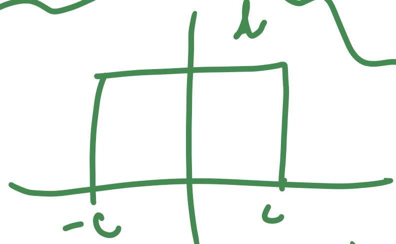


$\xrightarrow{FT}$



Short cut

$a\,\text{sinc}(bt) \longrightarrow$



$b = 2c$ (width of rectangle)

$a = 2cd$ (area of rectangle)

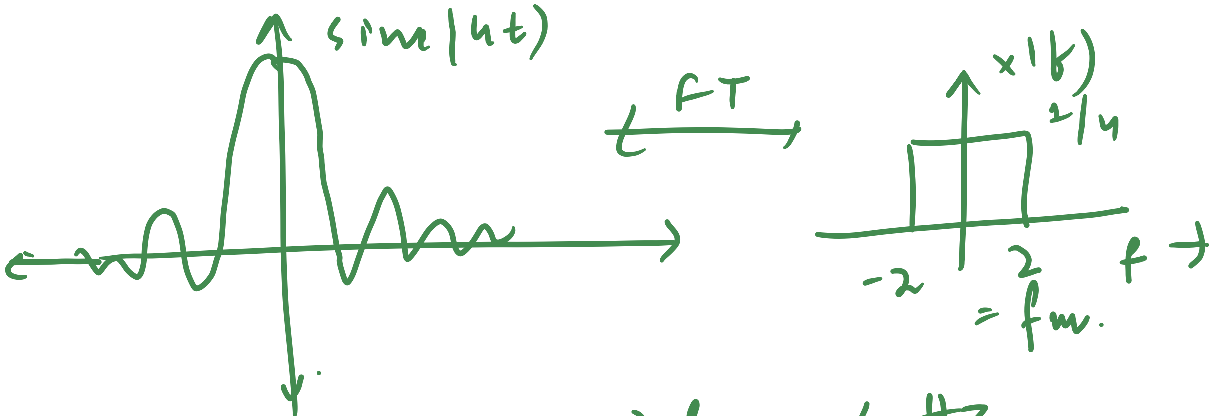
In our problem

$$x(t) = \text{sinc}(4t)$$

$$a = 1,\quad b = 4, \; \Big\} \Rightarrow \begin{array}{l} 4 = 2c \\ 1 = 2cd \end{array}$$

$c = 2,$

$d = 1/4.$



$\therefore$ Nyquist rate $= 2 f_m = \underline{4\ Hz}$.

(5) Consider an RC circuit with $R = 12\,k\Omega$ and $C = 50\,nF$. A unit step voltage is applied across RC circuit. What is the rise time for this RC circuit.

(Time taken for voltage across capacitor to rise from 10% to 90% of final values?)

Ans: $2.1972\,RC$

$= 2.1972 \times 12 \times 10^{3} \times 50 \times 10^{-9}$

$= 2.2 \times 600 \times 10^{-6}$

$= 13.2 \times 10^{-4}$

$= 1.32 \times 10^{-3}$

$= \underline{1.32\ ms}$

(6) Consider a sinusoid frequency of 500 Hz sampled at $f_s = 1000$ Hz. Let the sampled signal be filtered with ideal low pass filter with cutoff frequency $f_c = 3000$ Hz. What are the output frequencies?

Solution: $x(t) = \cos(2\pi 500 t)$

$x_{T_s}(t) = \cos(2\pi 500 t) \Big|_{t = n/1000}$

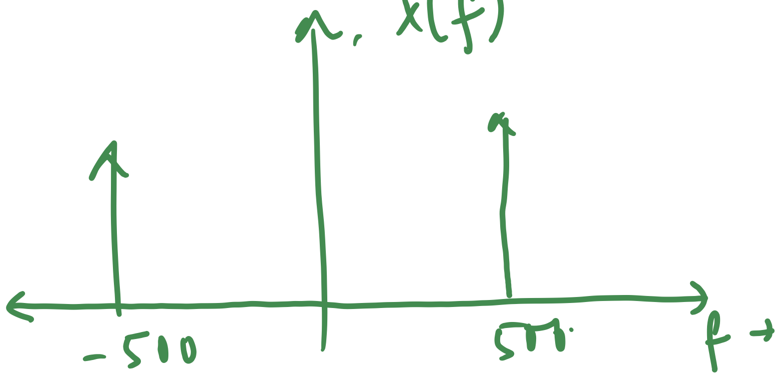
$x_R(t) = x_{Ts}(t) * h(t)$

↳ LPF of

$f_c = 3000$ Hz.

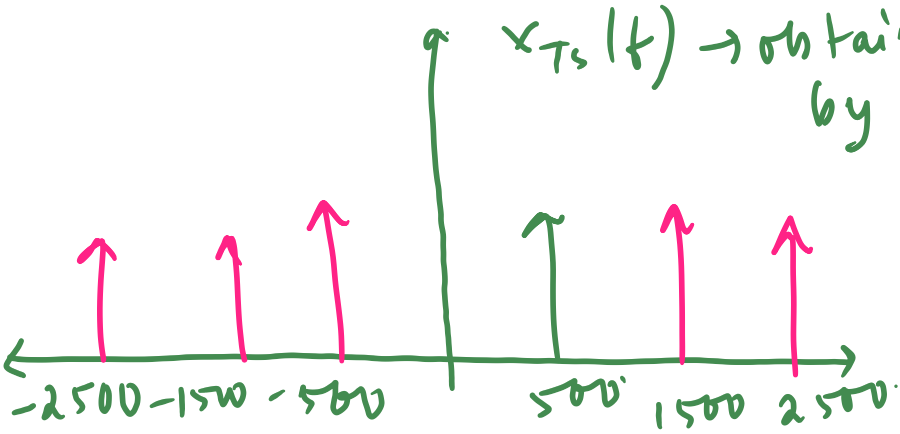
↑

freq components = ?



$x_{Ts}(f)$ → obtained by shifting right impulse

$f_s = 1000$.

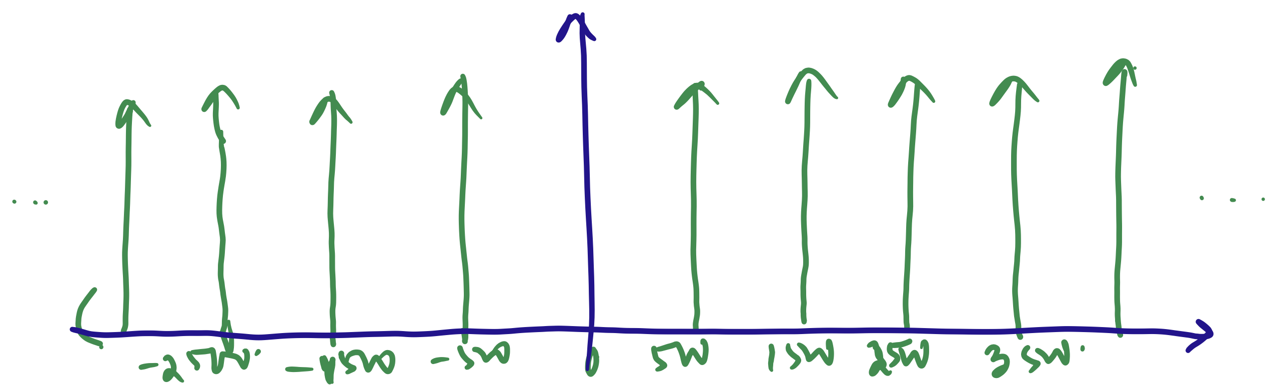


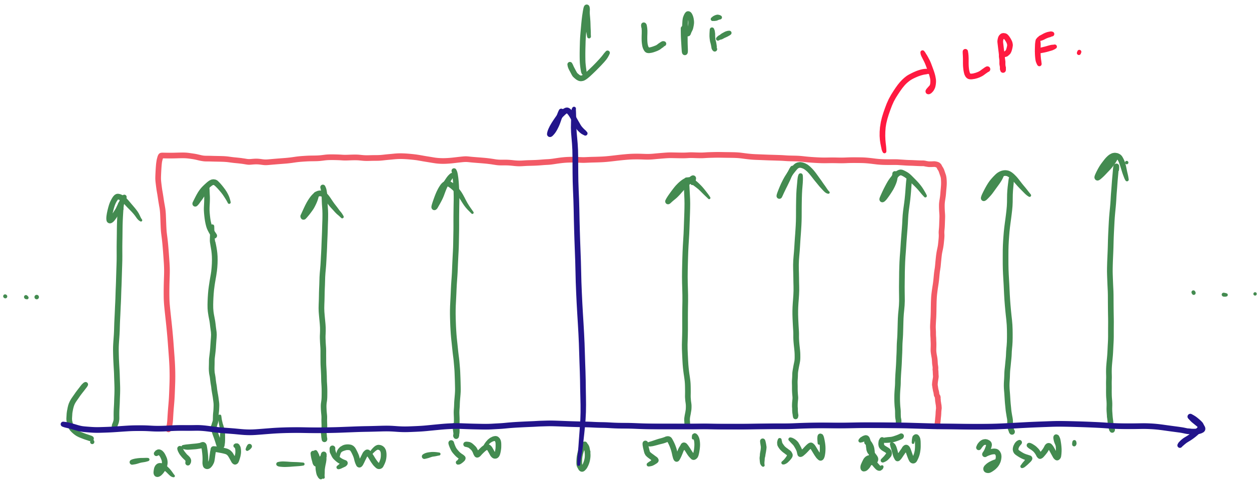
(1)

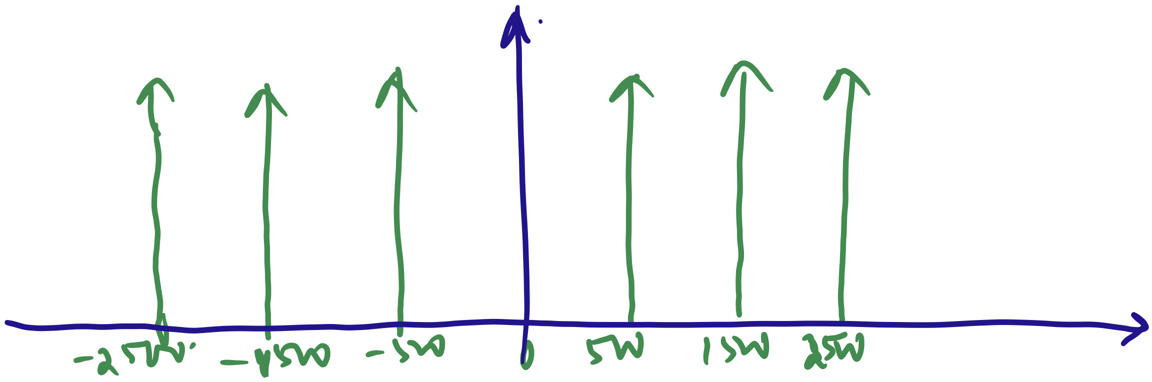
$X_{Ts}(f)$ - obtained by shifting left impulse

(2)

$X_{Ts}(f) = x_{Ts}(f)$ in (1) + $x_{Ts}(f)$ in (2)







$\therefore$ o/p frequencies = 500 Hz, 1500 Hz, 2500 Hz.

7) Find the Nyquist rate of :

$$x(t) = \left(\frac{\sin(400\pi t)}{\pi t}\right)^2$$

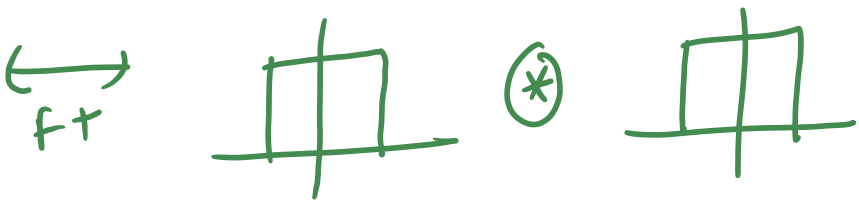
Solution

$$x(t) = \left(\frac{\sin(400\pi t)}{400\pi t} \times 400\right)^2$$

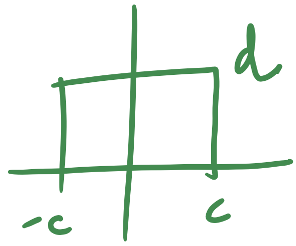
$$= 16 \times 10^4 \times \left(\frac{\sin(400\pi t)}{400\pi t}\right)^2$$

$$= 16 \times 10^4 \times \operatorname{sinc}^2(400t)$$

$$\operatorname{sinc}^2(400t) \longleftrightarrow \operatorname{sinc}(400t) \times \operatorname{sinc}(400t)$$



$$\sin(400t) \xrightarrow{FT}$$



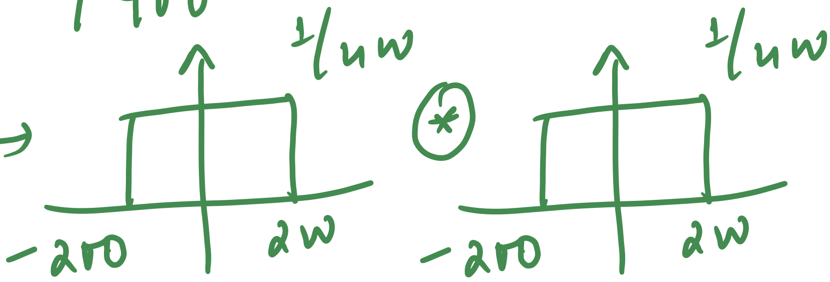
$a = 1$

$b = 400$

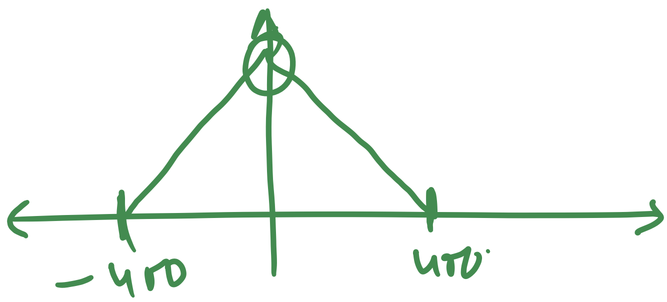
$$2c = b \Rightarrow c = 200$$

$$2cd = a \Rightarrow d = 1/400$$

$$\sin^2(400t) \xleftrightarrow{FT}$$



$=$

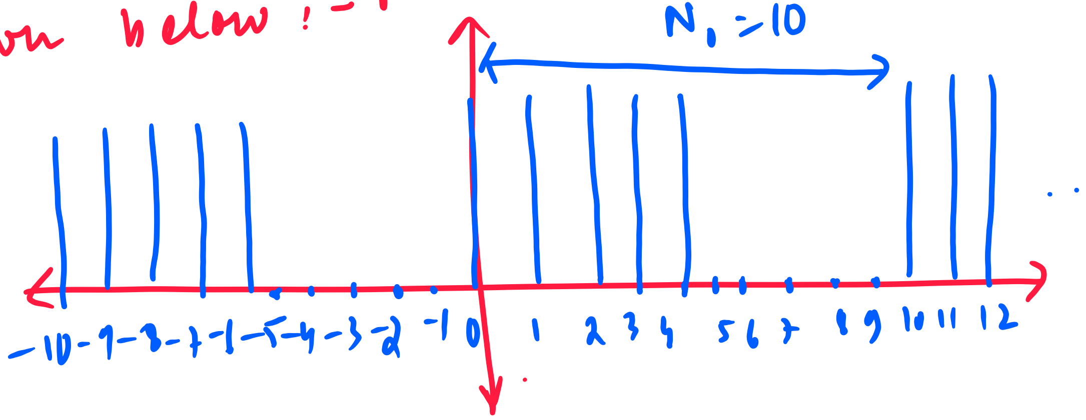


$\therefore$ Max freq content in $x(t)$ is

$f_m = 400$ Hz.

$\therefore$ Nyquist rate $= 2f_m = \underline{\underline{800 \text{ Hz}}}$.

(8) Consider the periodic sequence $x[n]$ as shown below :-



Determine the discrete Fourier coefficients c_k & sketch the magnitude spectrum!

Solution By DFS:-

$$x(n) = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n}$$

$$\Omega_0 = \frac{2\pi}{N_0} = 2\pi/10 = \pi/5$$

$$x[n] = \sum_{k=0}^{N_0-1} c_k e^{jk(\pi/5)}$$

to find c_k:

$$c_k = 1/N_0 \sum_{n=0}^{N_0-1} x[n] e^{-jk\Omega_0 n}$$

$$= \frac{1}{N_0} \sum_{n=0}^{9} x[n] e^{-jk(\pi/5)n}.$$

10

$$= \frac{1}{10}\left[\frac{1-e^{-jk\pi/5\times 5}}{1-e^{-jk\pi/5}}\right]$$

Geometric series expansion

$$= \frac{1}{10} \frac{1-e^{-jk\pi}}{1-e^{-jk\pi/5}}$$

$$c_k = 1/10\left(\frac{e^{-jk\pi/2}\left(e^{jk\pi/2} - e^{-jk\pi/2}\right)}{e^{-jk\pi/10}\left(e^{jk\pi/10} - e^{-jk\pi/10}\right)}\right)$$

$$|c_k| = \frac{1}{10}\left|\frac{e^{jk\pi/2} - e^{-jk\pi/2}}{e^{jk\pi/10} - e^{-jk\pi/10}}\right| \qquad |e^{j\theta}| = 1.$$

$$= \frac{1}{10}\,\frac{\sin k\pi/2}{\sin(k\pi/10)}$$

Euler's relation

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$