Week 7 TA Session

Part I Summary of Week VII's lecture

1) Fourier Transform

→ Defined for continuous aperiodic signals.

2) $X(\omega) \triangleq \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$

(or) $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt.$

$\omega \triangleq$ angular frequency.

-) Representation of time domain signal in frequency domain.

(3) ~~Inverse~~ Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

(or) $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df.$

(4) Spectrum

$$X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$$

Magnitude spectrum.

phase spectrum.

(5) For real signals

$$X^*(\omega) = X(-\omega)$$

$\Rightarrow |X(\omega)| = |X(-\omega)|$ (Even symmetry)

& $\angle X(-\omega) = -\angle X(\omega)$ (odd symmetry)

(6) Dirichlet conditions for convergence :—

1) $x(t)$ is absolutely integrable

2) $x(t)$ has finite maxima/minima.

3) $x(t)$ has finite discontinuities.

7) Laplace transform evaluated $s = 0 + j\omega$ gives the expression for Fourier Transform.

$s = \sigma + j\omega$. if $\sigma = 0 \Rightarrow s = j\omega$

8) Properties of FT:

Ref: Oppenheim ch-4.

$x(t) \longrightarrow X(j\omega)$ or $X(\omega)$

$y(t) \longleftrightarrow Y(j\omega)$ or $Y(\omega)$

Linearity $\quad a\,x(t) + b\,y(t) \longleftrightarrow a\,X(\omega) + b\,Y(\omega)$

Time shifting $\quad x(t - t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega)$

Freq Shifting $\quad e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega - \omega_0)$

Conjugation $\quad x^*(t) \longleftrightarrow X^*(-\omega)$.

Time reversal $\quad x(-t) \longleftrightarrow X(-\omega)$

Time & Frequency scaling $\quad x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

↳ compression when $a > 1$
expansion, for $a < 1$

Convolution $\quad x(t) * y(t) \longleftrightarrow X(\omega)\,Y(\omega)$

Multiplication $\quad x(t)\,y(t) \longleftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta)\,Y(\omega - \theta)\,d\theta$

Diff in time $\quad \frac{d\,x(t)}{dt} \longleftrightarrow j\omega\,X(\omega)$

Integration $\int_{-\infty}^{t} x(t)\,dt$ $\quad \frac{1}{j\omega} X(\omega) + \pi X(0)\,\delta(\omega)$

Diff in frequency $\quad t\,x(t)$ $\quad j\frac{d}{d\omega} X(\omega)$.

Conjugate symmetry for Real signals. $\quad x(t)$ real. $\quad X(\omega) = X^*(-\omega)$

$Re(X(\omega)) = Re(X(-\omega))$

$Im(X(\omega)) = -Im(X(-\omega))$

$|X(\omega)| = |X(-\omega)|$

$\measuredangle X(\omega) = -\measuredangle X(-\omega)$

symmetry for real & even signals $\quad x(t)$ R & E $\quad X(\omega)$ R + E

symmetry for real & odd signals $\quad x(t)$ R & O $\quad X(\omega)$ Purely Im & odd.

Even & odd Decomposition for real signals. $\quad x_e(t) = Ev(x(t))$ $\quad Re(X(\omega))$

$x_o(t) = Odd(x(t))$ $\quad j\,Im(X(\omega))$

$x(t)$ is real.

Duality $\quad x(t) \longleftrightarrow X(\omega)$

$X(t) \longleftrightarrow 2\pi\, x(-\omega)$

Parseval Relation for aperiodic signals.

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega.$$

(9) <u>FT of common signals.</u>

$$\sum_{n=-\infty}^{\infty} a_n e^{jk\omega_0 t} \longleftrightarrow 2\pi \sum_{n=-\infty}^{\infty} a_k\,\delta(\omega - k\omega_0)$$

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$$

$$\cos w_0 t \longrightarrow \pi\left[\delta(w-w_0) + \delta(w+w_0)\right]$$

$$\sin w_0 t \longleftrightarrow \frac{\pi}{j}\left[\delta(w-w_0) - \delta(w+w_0)\right]$$

$$1 \longleftrightarrow 2\pi\delta(w).$$

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases} \longleftrightarrow \sum_{n=-\infty}^{\infty} \frac{2\sin k w_0 T_1}{k}\,\delta(w - k w_0)$$

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) \longleftrightarrow \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(w - \frac{2\pi k}{T}\right)$$

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \longleftrightarrow \frac{2\sin w T_1}{w}.$$

$$\frac{\sin Wt}{\pi t} \longleftrightarrow X(w) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W. \end{cases}$$

$$\delta(t) \longleftrightarrow 1$$

$$u(t) \longleftrightarrow \frac{1}{jw} + \pi\delta(w)$$

$$\delta(t-t_0) \longleftrightarrow e^{-jwt_0}$$

$$e^{-at}u(t),\ Re\{a\} > 0 \qquad \frac{1}{a+jw}.$$

$$t\,e^{-at}u(t),\ Re\{a\} > 0 \qquad \frac{1}{(a+jw)^2}$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \qquad \frac{1}{(a+jw)^n}.$$

$$Re\{a\} > 0$$

(10)

) frequency response of the system.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t).$$

$X(w)\, H(w) = Y(w)$

$\Rightarrow \boxed{H(w) = \dfrac{Y(w)}{X(w)}}$ → Freq response.

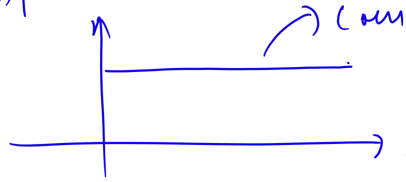
$x(t) = e^{jw_0 t}$ → eigen function!

11) Distortionless transmission via system

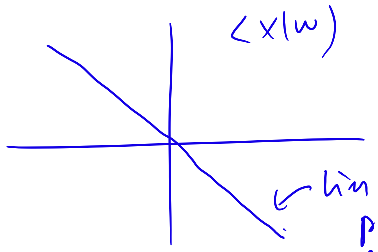
$y(t) = k\,x(t-t_0)$ ← only scaling & time delaying.

Such sys should have

$|H(w)|$.



→ constant mag response

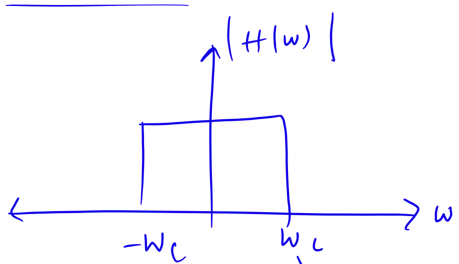


← linear phase response.

(12) Ideal filters:

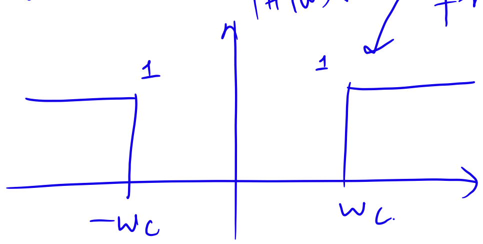
→ allows only certain frequencies to pass.

(i) Low pass.

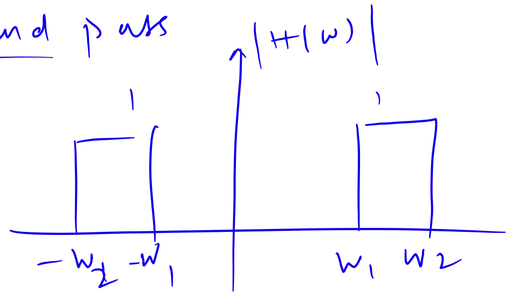


(ii) high pass:

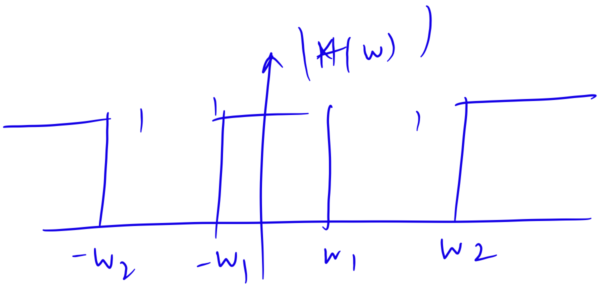
cut off frequency



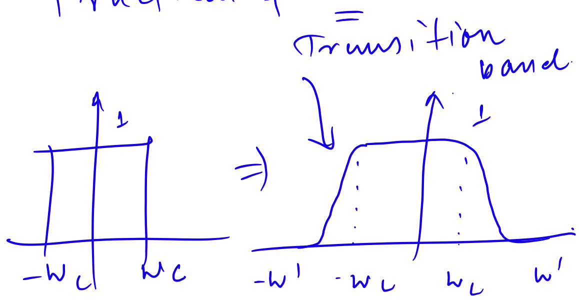
(iii) Band pass



(iv) Band stop (Notch filters)

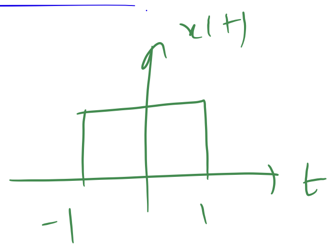


Practical filters.



Part II tutorials

eg $x(t)$:

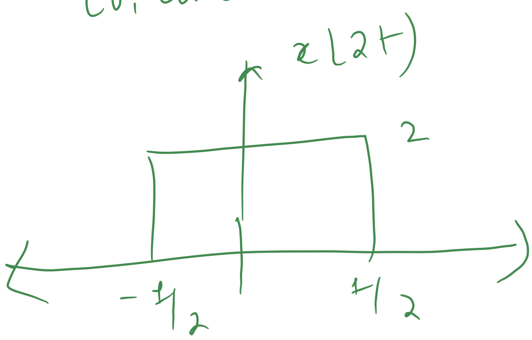


$a = 2 > 1$

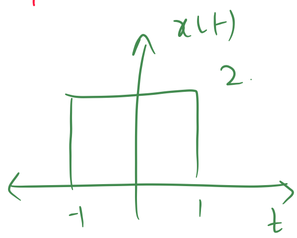
$x(2t)$:- ↳ $x(t) = \begin{cases} 2, & \text{if } -1 \le t \le 1 \\ 0, & \text{else} \end{cases}$

$$x(2t) = \begin{cases} 2, & \text{if } -1 \le 2t \le 1 \\ 0, & \text{else} \end{cases}$$

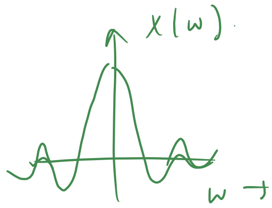
$$= \begin{cases} 2, & \text{if } -1/2 \le t \le 1/2 \\ 0, & \text{else.} \end{cases}$$



eg: Example of Freq. Duality.



⇓ $X(\omega)$

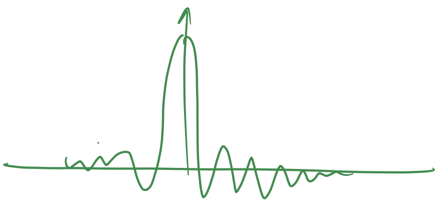


$X(\omega)$ = sinc fxn

$c_1 \operatorname{sinc}(c_2 \omega)$.

Suppose:-

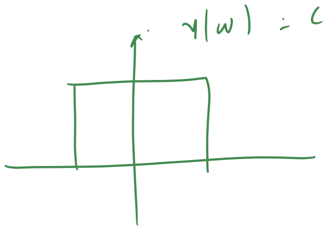
$y(t) = \operatorname{sinc}(t)$.



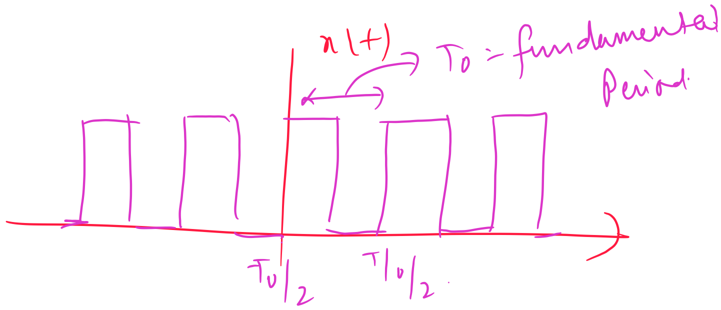
↓ $Y(\omega)$

$= x(-\omega)$

$Y(\omega) = c_1 x(-\omega)$



3) Consider the periodic square wave as shown below.



a) find complex exponential Fourier series of $x(t)$

b) find the trigonometric fourier series of $x(t)$

Sol^n

a) $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jk\omega_0 t}$; $\omega_0 = 2\pi/T_0$

$$c_k = 1/T_0 \int_0^{T_0} x(t) e^{-jk w_0 t} dt$$

$$= \frac{1}{T_0} \int_0^{T_0/2} x(t) e^{-j w_0 k t} dt + 1/T_0 \int_{T_0/2}^{T_0} x(t) e^{-jk w_0 t} dt$$

$$= \frac{1}{T_0} \int_0^{T_0/2} A e^{-jk w_0 t}$$

$$= \frac{A}{T_0} \int_0^{\frac{T_0}{2}} e^{-jk w_0 t} dt$$

$$= \frac{A}{T_0} \left[\frac{e^{-jk w_0 t}}{-jk w_0} \right]_0^{T_0/2}$$

$$= \frac{A}{jk w_0 T_0} \left\{ 1 - e^{-jk w_0 \frac{T_0}{2}} \right\}$$

$$w_0 = 2\pi / T_0 \Rightarrow w_0 T_0 = 2\pi$$

$$= \frac{A}{jk 2\pi} \left\{ 1 - e^{-jk \cancel{2}\pi/\cancel{2}} \right\}$$

$$= \frac{A}{jk 2\pi} \left\{ 1 - (-1)^k \right\},$$

$\underline{k \neq 0}$ (i) k is even , $k = 2m$; $m \in \mathbb{Z}$

$$\Rightarrow (-1)^k = 1 \Rightarrow c_k = 0$$

ii) k is odd $\Rightarrow k = 2m+1$, $m \in \mathbb{Z}$

$$(-1)^k = -1 \Rightarrow c_k = \frac{A}{jk\pi}$$

$\underline{\underline{k = 0}}$

$$c_k = \frac{A}{jk 2\pi} \left(1 - (-1)^k\right) \Big|_{k=0}$$

$(0/0)$ form $\rightarrow$ Not applicable

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt = 1/T_0 \int_0^{T_0/2} A\, dt$$

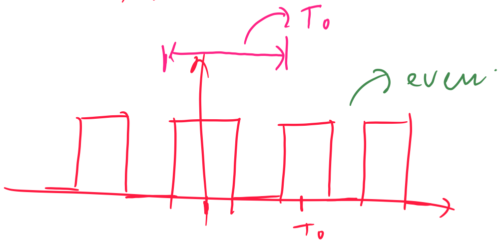
$$= \frac{A}{T_0} \cdot T_0/2 = A/2$$

Summarising

$$c_0 = A/2 \; ; \; c_{2m} = 0 \; ; \; c_{2m+1} = \frac{A}{j(2m+1)\pi}$$

$$x(t) = \frac{A}{2} + \frac{A}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{(2m+1)} e^{j(2m+1)\omega_0 t} \qquad \omega_1 = 2\pi/T_0$$

② Consider the periodic square wave $x(t)$ shown below:-



a) Find complex and trigonometric Fourier series of $x(t)$

Soln: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$; $\omega_0 = 2\pi/T_0$

$$c_k = 1/T_0 \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} 0 + \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A e^{-jk\omega_0 t}\, dt$$

$$+ 1/T_0 \int_{T_0/4}^{T_0/2} 0$$

$$= A/T_0 \int_{-T_0/4}^{T_0/4} e^{-jk\omega_0 t}$$

$$= \frac{A}{T_0} \left. \frac{e^{-jk\omega_0 t}}{-jk\omega_0} \right|_{-T_0/4}^{T_0/4}$$

$$= A/T_0 \frac{\left(e^{-jk\omega_0 T_0/4} - e^{jk\omega_0 T_0/4}\right)}{-jk\omega_0}$$

$$= \frac{A}{jk\omega_0 T_0}\left[e^{jk\omega_0 T_0/4} - e^{-jk\omega_0 T_0/4}\right] \quad -(1)$$

$\omega_0 = 2\pi/T_0,$

$\omega_0 T_0 = 2\pi \quad -(2)$

Put (2) in (1)

$$= \frac{A}{jk2\pi}\left[e^{jk2\pi/4} - e^{-jk2\pi/4}\right]$$

$$= \frac{A}{k\pi}\left[\frac{e^{jk\pi/2} - e^{-jk\pi/2}}{2j}\right]$$

$$= \frac{A}{k\pi}\sin(k\pi/2).$$

Case $k \neq 0$

If k is even : $k = 2m$

$$\Rightarrow c_k = c_{2m} = \frac{A}{2m\pi}\sin\left(\frac{\cancel{2}m\pi}{\cancel{2}}\right) = 0$$

If k is odd : $\Rightarrow k = 2m+1$.

$$c_k = c_{2m+1} = \frac{A}{(2m+1)\pi}\sin\left(\frac{(2m+1)\pi}{2}\right)$$

$$= \frac{A}{(2m+1)\pi}\sin\left(\frac{2m\pi}{2} + \pi/2\right)$$

$$= \frac{A}{(2m+1)\pi}\cos\left(\frac{\cancel{2}m\pi}{\cancel{2}}\right)$$

$$= \frac{A}{(2m+1)\pi}(-1)^m.$$

$k=0$ $\quad C_0 = 1/T_0\int_0^{T_0} x(t)\,dt = A/2$

Substitute for all these values

complex exponential Fourier Series.

$$\boxed{x(t) = A/2 + \frac{A}{\pi}\left[\sum_{m=-\infty}^{\infty} (-1)^m/(2m+1)\, e^{j(2m+1)\omega_0 t}\right] \quad \omega_0 = 2\pi/T_0}$$

(ii) $\frac{a_0}{2} = c_0 = A/2 \Rightarrow \boxed{a_0 = A}$

$a_{2m} = 2\,\mathrm{Re}\{c_{2m}\} = 0 \; ; m \neq 0$

$x(t)$ is Real

$a_{2m+1} = 2\,\mathrm{Re}\{c_{2m+1}\}$

$= \frac{(-1)^m 2A}{(2m+1)\pi}$

$b_{2m} = -2\,\mathrm{Im}\{c_{2m}\} = 0$
$b_{2m+1} = -2\,\mathrm{Im}\{c_{2m+1}\} = 0$ $\Big\}$ $b_k = 0 \; \forall k$

Recall:
$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k w_0 t) + \cancel{\sum_{k=1}^{\infty} b_k \sin(k w_0 t)}$ 0 as $b_k = 0 \; \forall k$

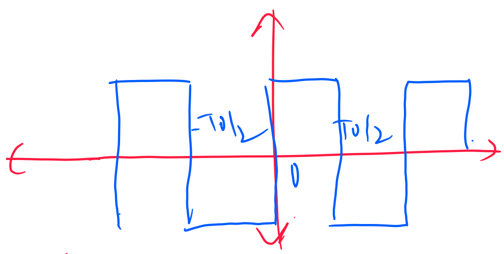
$A/2 + \sum_{m=1}^{\infty} a_{2m+1} \cos((2m+1) w_0 t)$.

$= A/2 + \frac{2A}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} \cos((2m+1) w_0 t)$

$= A/2 + \frac{2A}{\pi} \left\{ \cos w_0 t - \frac{1}{3} \cos 3 w_0 t + 1/5 \cos 5 w_0 t - \dots \right\}$

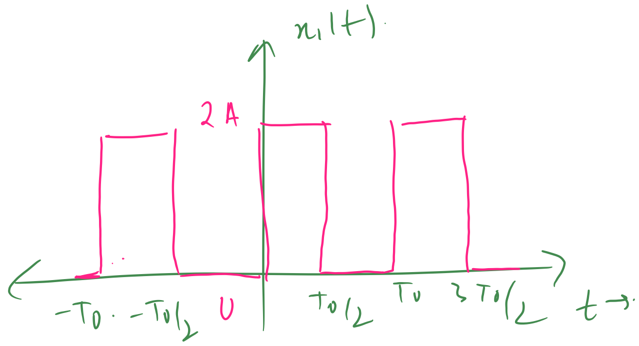
$\therefore x(t)$ contains odd harmonics!

(3) Consider the periodic square wave shown below:-

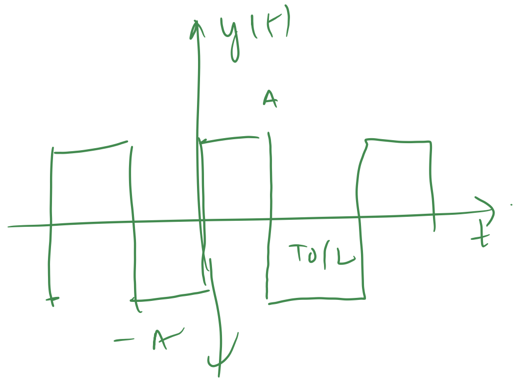


a) find complex exponential and trigonometric FS expansion of $x(t)$.

Sol^n Consider the signal $x_1(t)$ as



let $y(t) = x_1(t) - A$



we observe $y(t) = x(t)$.

$$x(t) = x_1(t) - A$$

$\therefore$ From 1st question, Replace A by 2A.

$\therefore$ FS expansion of $x(t)$

$$A + \frac{2A}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}.$$

$\therefore$ Put (2) in (1)

$$\boxed{\Rightarrow x(t) = \frac{2A}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t} \; ; \; \omega_0 = \frac{2\pi}{T_0}}$$

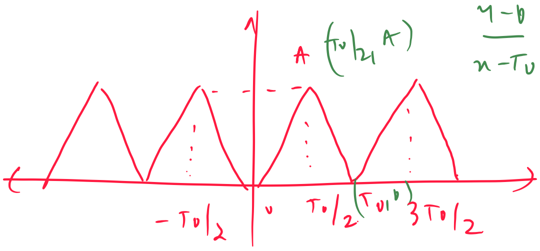
$\hookrightarrow$ complex FS expansion of $x(t)$.

III^{rly}, the Trigonometric FS of $x(t)$

$$x(t) = \frac{4A}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)} \sin((2m+1)\omega_0 t)$$

$$\omega_0 = 2\pi / T_0$$

(ii) Consider the triangular wave $x(t)$ as shown below :



$\frac{y-0}{x-T_0} = \frac{A-0}{\frac{T_0}{2}-T_0}$

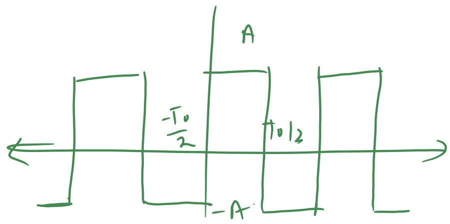
a) Find complex exponential F.S.

b) Find trigonometric F-S of $x(t)$.

Solⁿ Consider $x'(t) = \frac{d}{dt} x(t)$.

$$x(t) = \begin{cases} \frac{2}{T_0} At & ; 0 \le t \\ \frac{2A}{T_0}(T_0 - t) & ; \frac{T_0}{2} \le t \le T_0 \end{cases}$$

$$x'(t) = \begin{cases} 2A/T_0 & , 0 \le t \le T_0/2 \\ -2A/T_0 & , \frac{T_0}{2} \le t \le T_0 \end{cases}$$



FS of $x'(t)$ from Q. 3. when

$A \longrightarrow 2A/T_0$ is $\rightarrow$

$$x'(t) = \frac{4A}{j\pi T_0} \sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t} \; ; \omega_0 = 2\pi/T_0 \quad \text{--- (1)}$$

Now actual F.S of $x(t)$.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jk\omega_0 t} \quad ; \omega_0 = 2\pi/T_0$$

$$\therefore \frac{d\,x(t)}{dt} = \sum_{n=-\infty}^{\infty} C_k jk\omega_0 e^{jk\omega_0 t} \quad \text{--- (2)}$$

But $x'(t) = \frac{d}{dt} x(t)$.

$\therefore$ compare (1) & (2)

when $k = 2m, \ne 0 \Rightarrow c_k jk\omega_0 = 0$

$\Rightarrow c_k = 0$.

when $k = 2m+1$:-

$$C_k jk\omega_0 = \frac{4A}{jkT_0\pi} \cdot \frac{1}{k}$$

$$\Rightarrow c_k = \frac{-4A}{k^2 \pi T_0 w_0} = \frac{-4A}{k^2 \cdot 2\pi^2} = \frac{-2A}{k^2\pi^2}$$

$w_0 = 2\pi / T_0$

$\Rightarrow w_0 T_0 = 2\pi$

$$c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt = A/2$$

$\underline{k=0}$

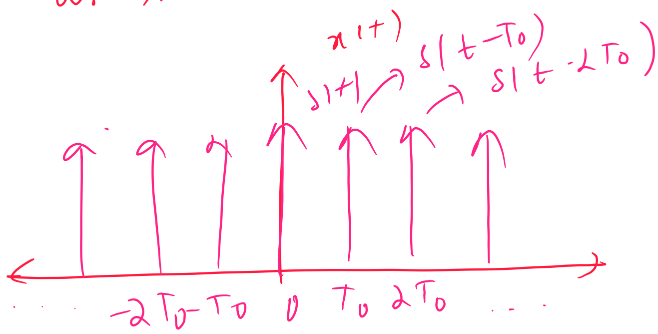
Collect the above values and use in (A) :

$$x(t) = \frac{A}{2} - \frac{2A}{\pi^2}\sum_{m=-\infty}^{\infty} \frac{1}{(2m+1)^2} e^{j(2m+1)w_0 t}$$

m'ly :- for trigonometric F-S:

$$x(t) = \frac{A}{2} - \frac{4A}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} \times \cos((2m+1) w_0 t)$$

Q5 Consider periodic impulse train as shown below :-



a) Find complex exponential FS

b) Find trigonometric exponential F S.

$\underline{\underline{Sol^n}}$ a) $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}$; $w_0 = 2\pi / T_0$

$$c_k = 1/T_0 \int_{-T_0/2}^{T_0/2} x(t) e^{-jkw_0 t}\, dt$$

$$= 1/T_0 \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jk\omega_0 t} dt$$

$$= 1/T_0 \; e^{j0} = 1/T_0$$

$$\therefore \boxed{x(t) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} \quad ; \omega_0 = 2\pi/T_0}$$

(ii) $$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + b_k \sin k\omega_0 t$$

$$a_k = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} (x(t) \cos k\omega_0 t) dt$$

$$= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) \cos(k\omega_0 t) dt$$

$$= 1$$

$$= 2/T_0 \times \cos(0) = 2/T_0.$$

$$a_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) dt = 1/T_0.$$

$$b_k = 2/T_0 \int_{-T_0/2}^{T_0/2} x(t) \sin(k\omega_0 t) dt$$

$$= 2/T_0 \int_{-T_0/2}^{T_0/2} \delta(t) \sin(k\omega_0 t) dt$$

$$= 2/T_0 \sin(0) = 0.$$

Collecting values for a_0, a_n, b_n in (A)

$$x(t) = 1/T_0 + \frac{2}{T_0} \sum_{k=1}^{\infty} \cos(k\omega_0 t);$$

$$\omega_0 = 2\pi/T_0$$

Trigonometric F.S.