$\boxed{\text{Week 3}}$

DIFFERENTIAL EQUATION REPRESENTATION / DESCRIPTION OF AN LTI SYSTEM.

Input output relation of several systems can be represented by a differential equation

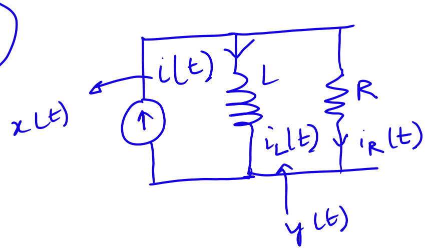
General form

$$\sum_{m=0}^{M} a_m \frac{d^m y(t)}{dt^m} = \sum_{m=0}^{N} b_m \frac{d^M x(t)}{dt^m}.$$

output of system.

input to system.

eg



$$\frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = i(t)$$

$$\underbrace{\frac{L}{R}\frac{dy(t)}{dt} + y(t) = x(t)}_{\text{Differential equation}}$$

representation of i/p o/p relation

Solution can be described

output $y(t)$ for given input signal $x(t)$.

$y(t) = y_p(t) + y_h(t)$

Particular solution

homogeneous solutions.

any general solution

x

$$\underbrace{\sum_{m=0}^{M} a_m \frac{d^m y_h(t)}{dt^m} = 0}_{\text{Homogeneous solution}}$$

+ M auxillary conditions.

are needed to determine the o/p signal

$$\underbrace{y(t), \frac{dy(t)}{dt}, \frac{d^2y(t)}{dt^2}, \ldots, \frac{d^{m-1}y(t)}{dt^{m-1}}}_{\text{at some pt to auxilliary conditions.}}$$

LINEAR → when is system described by above differential equ^n linear?

→ If all auxillary conditions :0

Auxillary conditions have to be zero.

TIME INVARIANT :

↳ when is the system time invariant?

→ for time invariant system has to be at initial rest

If $x(t) = 0$ for $t \le t_0$
Assume $y(t) = 0$ for $t \le t_0$

↓

initial condition become

$$y(t_0) = \left.\frac{dy}{dt}\right|_{t=t_0} = \cdots \cdots \frac{d^{m-1}y(t)}{dt^{m-1}} = 0$$

* Properties of Discrete time LTI systems

Impulse response

$$\delta(n) \longrightarrow \boxed{T(\cdot)} \longrightarrow h(n)$$

unit Impulse

Impulse response

$$\boxed{h(n) = T(\delta(n))}$$

Response to unit impulse.

Response to arbitrary input

signal: $x(n)$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, x[n-m].$$

Convolution sum $= x(m) * h(m)$

$$= h(m) * x(m).$$

MEMORYLESS

$$\boxed{h[n] = k\,\delta[n]}$$

impulse response of memoryless discrete LTI system.

CAUSAL

If $h[n] = 0$ for $n < 0$

$\Rightarrow$ o/p depends only on present and past values of input

BIBO stability

Discrete time LTI system = BIBO stable if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$\rightarrow$ finite

For continuous LTI system

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

EIGENFUNCTIONS OF DISCRETE TIME LTI SYSTEMS.

$$x[n] = z^n.$$

$$T(x(n)) = \sum_{m=-\infty}^{\infty} h(m)\, z^{n-m}.$$

$$= z^n \underbrace{\sum_{m=-\infty}^{\infty} h(m)\, z^{-m}}_{H(z)}$$

$$\boxed{T(z^n) = H(z)\, z^n}$$

eigen function of the

SYSTEMS DESCRIBED BY A DIFFERENCE EQUATION

Similar to continuous time LTI systems given by Differential equation.

Discrete time LTI systems can be described by a difference equation.

$$\underbrace{\sum_{m=0}^{M} a_m \overset{o/p.}{y(n-m)} = \sum_{m=0}^{N} b_m \overset{i/p.}{x(n-m)}}_{\text{Standard form of Difference equation}}$$

eg. Simplify $x(t) * u(t-t_0)$

convolute

$$= \int_{-\infty}^{\infty} x(\tau)\, u(t-t_0-\tau)\, d\tau$$

$\neq 0$ only for $t-t_0-\tau \geq 0$

$\Rightarrow \tau \le t - t_0$

$$= \boxed{\int_{-\infty}^{t-t_0} x(\tau)\, d\tau.}$$

eg $x(t) = u(t) \rightarrow$ input

$h(t) = e^{-t} u(t) \rightarrow$ impulse response

$y(t) = ? \rightarrow$ output

$$y(t) = x(t) * h(t)$$
$$= h(t) * x(t)$$
$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} e^{-\tau} \underbrace{u(\tau)}_{\substack{\neq 0 \\ \text{for } \tau \ge 0}} \underbrace{u(t-\tau)}_{\substack{\neq 0 \\ \text{for } t-\tau \ge 0 \\ \Rightarrow t \ge \tau \\ \text{or } \tau \le t}} d\tau.$$

$$= \begin{cases} \int_0^t e^{-\tau} d\tau & \text{if } t \ge 0 \\ 0, & \text{if } t < 0 \end{cases}$$

$$\int_0^t e^{-\tau} d\tau = -e^{-\tau}\Big|_0^t$$
$$= 1 - e^{-t} \quad \Big] \text{ when } t \ge 0$$

$$\boxed{y(t) = (1 - e^{-t})\, u(t)}$$

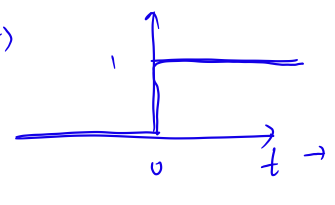
expression for final output signal.

* ~~Example~~ problems for the analysis of LTI systems.

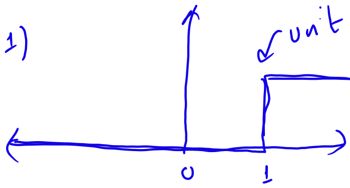
eg $x(t) = u(t) - u(t-1)$. unit step function.

$y(t) = x(t) * x(t) = ?$

w.r.t. $u(t)$



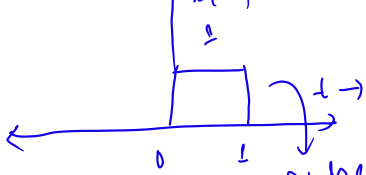
$u(t-1)$



unit step delayed by 1

$x(t) = u(t) - u(t-1)$ $= 1$ if $0 \le t \le 1$
$= 0$ otherwise.

$\therefore x(t)$

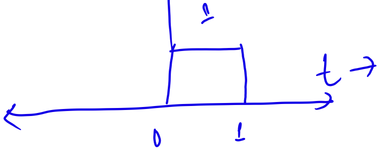


pulse of height $= 1$

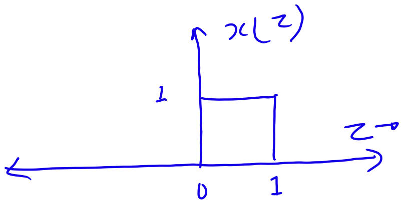
b/w 0, 1.

$*$

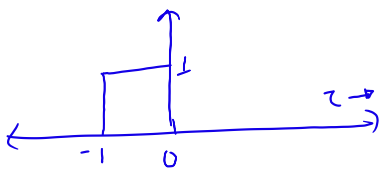
$\therefore x(t)$



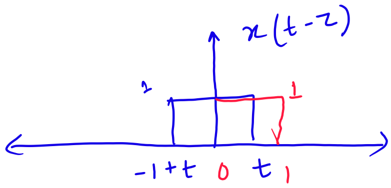
$x(\tau)$



$x(-\tau)$



$x(t-\tau)$



$$\int_{-\infty}^{\infty} x(\tau)\, x(t-\tau)\, d\tau$$

$= 1 \times t = t$

increases linearly with t for $0 \le t < 1$

For $t \geq 1$, decreases linearly until it reaches 0.

$y(t) = x(t) * x(t)$

Triangular pulse $= \begin{cases} 1 - |t-1|, & 0 \le t < 2 \\ 0, & \text{otherwise} \end{cases}$

$$y(t) = \begin{cases} 1-|t-1|, & 0 \le t < 2 \\ 0, & \text{otherwise} \end{cases}$$

eg. Periodic convolution

Let $x_1(t), x_2(t)$ = Periodic

T_0 ↳ common Period = t_0.

$$F(t) = \int_0^{T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= x_1(t) \circledast x_2(t).$$

Periodic convolution.

Show $F(t) = \displaystyle\int_{t_0}^{t_0+T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau$

$= F(t)$

Shifting integrally by t_0.

Solution: Let

$$\phi_t(\tau) = x_1(\tau)\, x_2(t-\tau)$$

For given t

$$\phi_t(\tau+T_0) = x_1(\tau+T_0)\, x_2(t-(\tau+T_0))$$

$$= x_1(\tau+T_0)\, x_2(t-\tau-T_0)$$

$$= x_1(\tau)\, x_2(t-\tau)$$

$$\phi_t(\tau+T_0) = \phi_t(\tau)$$

$\phi_t(\tau)$ = periodic

↳ T_0

$$\int_0^{T_0} \phi_t(\tau)\, d\tau = \int_{t_0}^{t_0+T_0} \phi_t(\tau)\, d\tau$$

$$= \int_{t_0}^{t_0+T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \tilde{F}(t)$$

For a periodic function integral over any period = same (eg) Consider system with impulse response

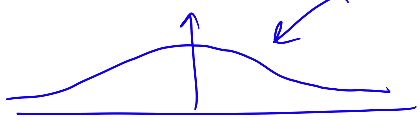
$$h(t) = e^{-t^2/2} \quad -\infty < t < \infty$$

Is this system BIBO stable?

<u>Solution</u> - For BIBO stability

$$\underbrace{\int_{-\infty}^{\infty} |h(t)|\, dt}_{\text{Finite}} < \infty$$

Gaussian pulse $e^{-t^2/2}$



"Bell" shaped function.

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} e^{-t^2/2}\, dt$$

$$= \sqrt{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt}_{=1}$$

Gaussian Density fxn

for $\sigma^2 = 1$

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma^2}$$

var $= \sigma^2$

$$= \sqrt{2\pi} < \infty$$

$\Rightarrow$ system is indeed BIBO stable

(eg) Consider LTI system

$$y(t) = \int_{-\infty}^{\infty} e^{-j\omega_0(t-\tau)} x(\tau)\, d\tau$$

output, input

Find $h(t)$, eigenvalue for function

impulse response

$$= \int_{-\infty}^{\infty} \underbrace{e^{-j\omega_0(t-\tau)}}_{h(t-\tau)} \underbrace{x(\tau)}_{x(\tau)}\, d\tau$$

$= h(t) * x(t)$.

Impulse response

$h(t) = e^{j\omega_0 t}$

* PROBLEMS FOR LTI SYSTEMS:

eg Consider LTI system

$$y(t) = \int_{-\infty}^{t} e^{-j\omega_0(t-\tau)} x(\tau)\, d\tau$$

Find $h(t)$

Impulse Response.

$$= \int_{-\infty}^{t} e^{-j\omega_0(t-\tau)} x(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \underbrace{e^{-j\omega_0(t-\tau)} u(t-\tau)}_{h(t-\tau)} \underbrace{x(\tau)}_{x(\tau)} d\tau$$

$$h(t) = e^{-j\omega_0 t} u(t)$$

$$h(t) * x(t) = \int_{-\infty}^{\infty} e^{-j\omega_0(t-\tau)} u(t-\tau)\, x(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j\omega\tau} u(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} e^{-j\omega_0\tau} x(t-\tau)\, d\tau$$

$x(t) = e^{st}$

$$= \int_0^{\infty} e^{j\omega_0\tau} e^{s(t-\tau)}\, d\tau.$$

$$= e^{st} \int_0^{\infty} e^{-(s+j\omega_0)\tau}\, d\tau$$

$$= e^{st} \left. \frac{e^{-(s+j\omega_0)}}{-(s+j\omega_0)} \right|_0^{\infty}$$

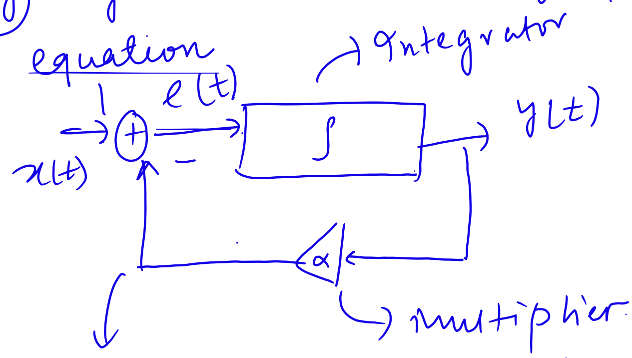
$$= e^{st} \left[0 - \frac{1}{-(s+j\omega_0)} \right]$$

$s > 0$

$$= e^{st} \times \underbrace{\left(\frac{1}{s+j\omega_0} \right)}_{\text{If } Re(s) > 0}$$

eigenvalue for eigen function e^{st}

eg: Systems Described by Differential equation



Determine DE representation of system above.

$$x(t) - \alpha\, y(t) = e(t).$$

$$\int_0^t e(\tau)\, d\tau = y(t)$$

Differentiating both sides of the above equation.

$$e(t) = \frac{dy(t)}{dt}.$$

$$x(t) - \alpha y(t) = \frac{dy(t)}{dt}.$$

$$\boxed{\frac{dy(t)}{dt} + \alpha y(t) = x(t).}$$

DE describing above system.

eg Consider s/s above

$$\frac{dy(t)}{dt} + \alpha y(t) = x(t).$$

auxiliary condition

$$y(0) = y_0$$

i/p, $\underbrace{x(t) = c e^{-\beta t} u(t)}_{\text{input signal.}}$

Soln $y(t) = y_h(t) + y_p(t).$

Homogeneous solution

Particular solution.

For $y_p(t)$:

Let $y_p(t) = K e^{-\beta t}$

$$\frac{dy_p(t)}{dt} + \alpha y_p(t) = x(t)$$

$$\Rightarrow -k\beta e^{-\beta t} + \alpha k e^{-\beta t} = x(t)$$

$$\Rightarrow k(\alpha - \beta) e^{-\beta t} = x(t) = c e^{-\beta t}$$

$$\Rightarrow (\alpha - \beta) = c/k$$

$$\Rightarrow \boxed{k = \frac{c}{(\alpha - \beta)}}$$

$$\Rightarrow \underbrace{y_p(t) = \frac{c}{\alpha - \beta} e^{-\beta t}}_{\text{particular solution.}}$$

<u>For homogeneous solution</u>

Let $y_h(t) = \tilde{k} e^{st}$

$$\frac{dy_h(t)}{dt} + \alpha y(t) = 0$$

$$\tilde{k} e^{st} + \alpha \tilde{k} e^{st} = 0$$

$$\Rightarrow \tilde{k}(s + \alpha) e^{st} = 0$$

$$s = -\alpha.$$

$$y_h(t) = \tilde{k} e^{-\alpha t}.$$

Determine $\tilde{k}$

$$y(t) = y_h(t) + y_p(t)$$

$$= \tilde{k} e^{-\alpha t} + \frac{c}{\alpha - \beta}$$

use auxillary condition for $\tilde{k}$

$$y(0) = y_0 \Rightarrow \tilde{k} + \frac{c}{\alpha - \beta} = y_0$$

$$\Rightarrow \tilde{k} = -\frac{c}{(\alpha - \beta)} + y_0$$

$$y(t) = (y_0 - c/(\alpha - \beta)) e^{-\alpha t} + \frac{c}{(\alpha - \beta)} e^{-\beta t}$$

output signal for given system & auxillary condition.

for $t < 0$, $x(t) = 0$

$$\frac{dy(t)}{dt} + \alpha y(t) = 0$$

Let $y(t) = se^{-\alpha t}$

$\Rightarrow -\alpha s e^{-\alpha t}$

$\Rightarrow y(t) = se^{-\alpha t}$

s = constant

$y(0) = y_0$

$\Rightarrow s = y_0$

for $t < 0$, $\boxed{y(t) = y_0 e^{-\alpha t}}$

o/p signal for $t < 0$

zero input signal ie for $x(t) = 0$

$$y(t) = \left(y_0 - \frac{c}{\alpha - \beta}\right) e^{-\alpha t} + \left(\frac{c}{\alpha - \beta}\right) e^{-\beta t}$$

$$= \underbrace{y_0 e^{-\alpha t}}_{y_{zi}(t)} + \underbrace{\frac{c}{\alpha - \beta}\left(e^{-\beta t} - e^{-\alpha t}\right)}_{y_{zs}(t)}$$

$y_{zi}(t)$ → zero i/p signal

$y_{zs}(t)$ → zero state signal

Response with zero auxiliary conditions

o/p to zero input

Response to auxiliary conditions

10-1-16

:- DISCRETE TIME LTI SYSTEMS.

eg $x[n] = \alpha^n u[n]$ — input

$h[n] = \beta^n u[n]$.

impulse response

unit step function.

$y[n] = ?$

$$y[n] = x[n] * h[n]$$

$$= \sum_{m=-\infty}^{\infty} x(m)h(n-m)$$

$$= \sum_{m=-\infty}^{\infty} \alpha^m u(m) \times \beta^{n-m} u(n-m)$$

$u(m)$: 1, if $m \geq 0$; $= 0$, $m < 0$

$u(n-m)$: 1, $n-m \geq 0$; $m-n \leq 0$; 1, $m \leq n$; $= 0$, $m > n$.

$$= \begin{cases} \sum_{m=0}^{n} \alpha^m u(m) \beta^{n-m} u(n-m), & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$= 0$ only if $0 \leq m \leq n$.

$\Rightarrow$ For $n < 0$

$0/p = 0$

$$= \beta^n \sum_{m=0}^{n} (\alpha/\beta)^m$$

$$y[n] = \begin{cases} \dfrac{1-(\alpha/\beta)^{n+1}}{1-\alpha/\beta} \beta^n & \text{if } \alpha \neq \beta \\ \beta^n (n+1), & \text{if } \alpha = \beta. \end{cases}$$

For $n \geq 0$

$y[n] = 0$, for $n < 0$ ✓

eg $\quad y(n) = \sum_{k=-\infty}^{n} 2^{k-n} x(k+1)$

Is this causal?

output signal. input

Solution $\tilde{k} = k+1$

$\Rightarrow k = \tilde{k} - 1$

$$y[n] = \sum_{\tilde{k}=-\infty}^{n+1} 2^{\tilde{k}-n-1} x(\tilde{k})$$

$$= x(n+1) + \sum_{\tilde{k}=-\infty}^{n} 2^{\tilde{k}-n-1} x(\tilde{k})$$

System is not causal. since output $y(n)$ depends on $x(n+1)$ Future Input.

$\therefore$ s/Is is non causal

Impulse response:

Set $x[n] = \delta[n]$.

$$y[n] = \sum_{k=-\infty}^{n} 2^{k-n} \delta(k+1)$$

$$= \begin{cases} 1 & \text{if } k = -1 \\ 0, & \text{if } k \neq -1 \end{cases}$$

non zero only if $n \geq -1$

$$h(n) = \begin{cases} 2^{-1-n} & \text{For } n \geq -1 \\ 0 & \text{otherwise.} \end{cases}$$

$$h[-1] = 2^{-1+1}, \quad n \geq -1$$

$$= 2^{0}, \quad n \geq -1$$

$$h[-1] = 1$$

$$\boxed{h(n) \neq 0 \text{ for } n < 0}$$

$\therefore$ System is NOT causal.

eg $h[n] = \alpha^{n} u[n]$ Is it BIBO stable?

BIBO = Bounded Input Bounded Output Stability

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |\alpha|^{n} u(n)$$

$$= \sum_{n=0}^{\infty} |\alpha|^{n}$$

$$= \frac{1}{1-|\alpha|} \quad \text{if } |\alpha| < 1$$

$= \begin{cases} \infty & \text{otherwise} \end{cases}$

$\Rightarrow$ system BIBO stable only if $|\alpha| < 1$ ✓

eg: $\underbrace{y[n] = \alpha y[n-1] + x[n]}_{\text{Difference equation}}$

what is impulse response of system?

(Assume initial Rest)

Set $x[n] = \delta[n]$

From initial rest

$\Rightarrow y[n] = 0$ for $n < 0$

$x(0) = \delta(0) = 1.$

$y(0) = \alpha y(-1) + x(0)$

$= 1.$

set $n=1$ since $y(-1) = 0$

$y(1) = \alpha y(0) + \cancel{x(0)}$

$= \alpha \times 1 = \alpha$ 0

$y(2) = \alpha y(1) + x(1)$

$= \alpha \alpha = \alpha^2$

$y(3) = \alpha^3$

$\vdots$

$\boxed{h(n) = \alpha^n}$

impulse response

Lec 17.

Laplace Transform.

→ Transform Domain Representation

→ Convenient representation for signals as well as LTI systems.

→ can be employed for analysis as well as to obtain valuable insights into behaviour of signals and systems.

LAPLACE TRANSFORM (LT).

Bilateral

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$s = \sigma + j\omega$ = complex valued.

$X(s)$ = Laplace Transform of $x(t)$

$$X(s) = \mathcal{L}(x(t))$$

$$\underbrace{x(t) \longleftrightarrow X(s)}_{\text{Signal / LT pair}}$$

ROC : Region of Convergence

Region of s-plane, or range of values of s, for which LT exists (converges).

eg: $x(t) = e^{-\alpha t} u(t)$. α = Real

$$X(s) = \int_{-\infty}^{\infty} e^{-\alpha t} u(t)\, e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-(\alpha+s)t}\, dt$$

$$= \left. \frac{e^{-(s+\alpha)t}}{-(s+\alpha)} \right|_0^{\infty}$$

$$= \frac{-1}{-(s+\alpha)}$$

$e^{-(s+\alpha)t} \rightarrow 0 \quad \forall \; s+\alpha > 0$

$$= \frac{1}{(s+\alpha)} \quad \text{only if}$$

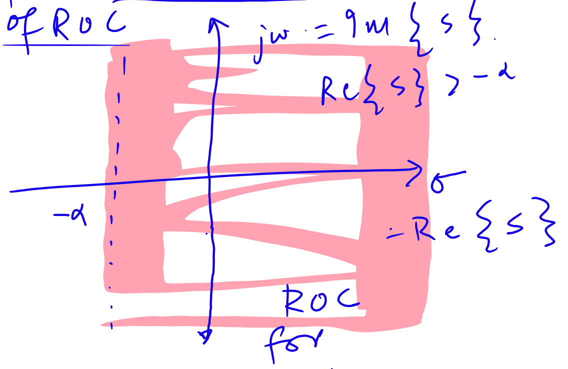
$\Rightarrow Re\{s\} + \alpha > 0$

$\Rightarrow \underbrace{Re\{s\} > -\alpha}_{ROC}$

$$e^{-st}u(t) \longleftrightarrow \frac{1}{s+\alpha}$$

$\underbrace{\text{if } Re\{s\} > -\alpha}_{\text{ROC / Region of Convergence}}$

s plane Representation of ROC



ROC for $e^{-\alpha t}u(t)$.

consider now.

$x(t) = -e^{-\alpha t}u(-t)$

nonzero only for $t \le 0$

$$X(s) = \int_{-\infty}^{\infty} -e^{-\alpha t}u(-t)e^{-st}dt$$

$$= -\left.\frac{e^{-(s+\alpha)t}}{-(s+\alpha)}\right|_{-\infty}^{0}$$

$$= \frac{1}{s+\alpha} - 0$$

$e^{-(s+\alpha)t} \longrightarrow 0$ for $t \to -\infty$

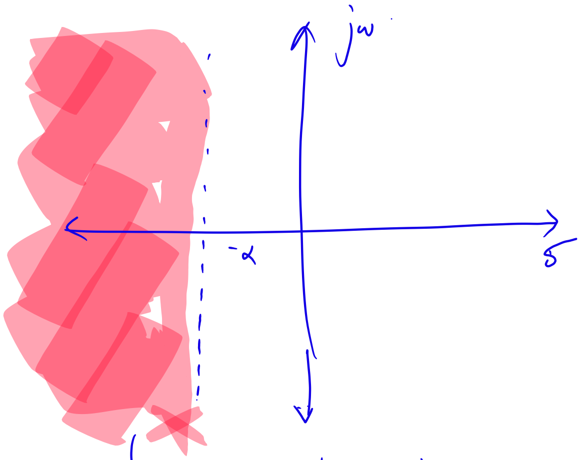
only if $Re\{s\} + \alpha < 0$

$\Rightarrow \underbrace{Re\{s\} < -\alpha}_{ROC}$

$$-e^{-\alpha t}u(-t) \longleftrightarrow \frac{1}{s+\alpha}$$

$\underbrace{\text{if } Re\{s\} < -\alpha}_{ROC}$

Region of convergence



$\hookrightarrow Re(s) < -\alpha$

ROC of $-e^{-\alpha t}u(-t)$

Both $e^{-\alpha t}u(t)$, $-e^{-\alpha t}u(-t)$ } have same LT

Two Distinct signals.

$e^{-\alpha t}u(t) \longleftrightarrow \frac{1}{s+\alpha}$, ROC $Re\{s\} > -\alpha$

$-e^{-\alpha t}u(-t) \longleftrightarrow \frac{1}{s+\alpha}$ ROC $Re\{s\} < -\alpha$

$\hookrightarrow$ same LT
But diff ROCs.

For LT to be unique, ROC must also be satisfied as part of LT

LAPLACE TRANSFORMS

Unit Impulse function.

$$x(t) = \delta(t)$$

$$X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = \mathcal{L}[\delta(t)]$$

$$= e^{-st}\Big|_{t=0}$$

$$= 1 \quad \forall s$$

$\hookrightarrow$ ROC : all s.
entire splane.

UNIT STEP FUNCTION

$$x(t) = u(t)$$

$$\mathcal{L}(u(t)) = \int_{-\infty}^{\infty} u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-st}\, dt$$

$$= \left. \frac{e^{-st}}{-s} \right|_{0}^{\infty}$$

$$= 0 - \frac{1}{-s}$$

where
$Re\{s\} > 0$

$$= 1/s \quad \text{for } \underline{Re\{s\} > 0}$$

ROC.
Region of Convergence