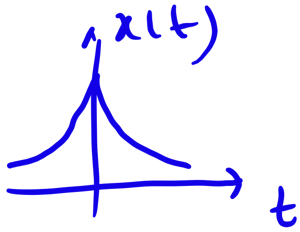
$\boxed{\text{Week 8}}$ TA session. Yashvanth L

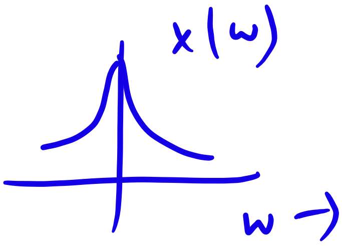
Part I - Summary of Week 7/8 lectures.

Some standard Fourier pairs solved in class.

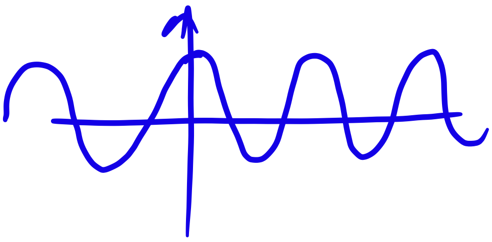
(i) $x(t) = e^{-a|t|}$; $a > 0$.



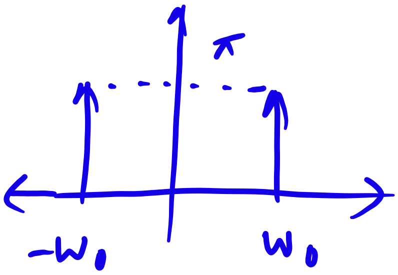
$$X(w) = \frac{2a}{a^2 + w^2}$$



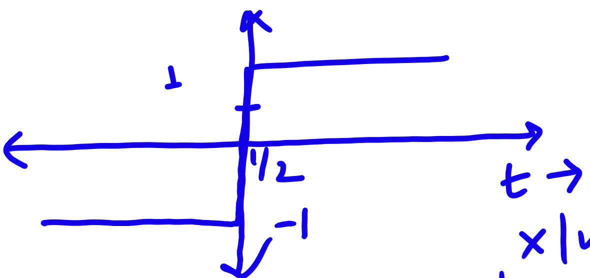
(ii) $x(t) = \cos(w_0 t)$



$$X(w) = \pi\,\delta(w - w_0) + \pi\,\delta(w + w_0)$$

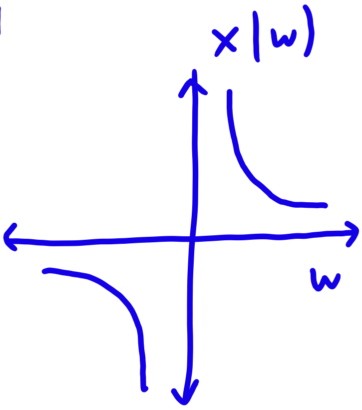


iii) $x(t) = \mathrm{sgn}(t)$ [signum function]



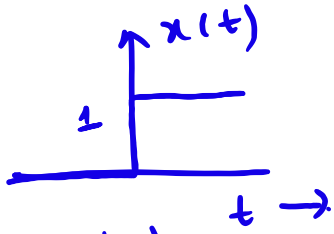
(it can be any # b/w -1 & 1 as integrating over time, value does not matter)

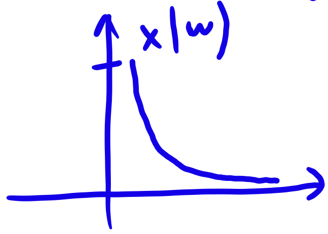
$$X(w) = \frac{2}{jw}$$



iv) $x(t) = u(t)$

$X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$





Part 2 Tutorials.

1) Let $x(t)$ have FT $X(\omega)$. Then what is the inverse F.T of $x(\omega)$?

Sol^n By Duality property of FT:

$x(t) \longleftrightarrow X(\omega)$

$x(-t) \longleftrightarrow X(-\omega)$.

$$x(t) \xleftrightarrow{FT} X(\omega)$$

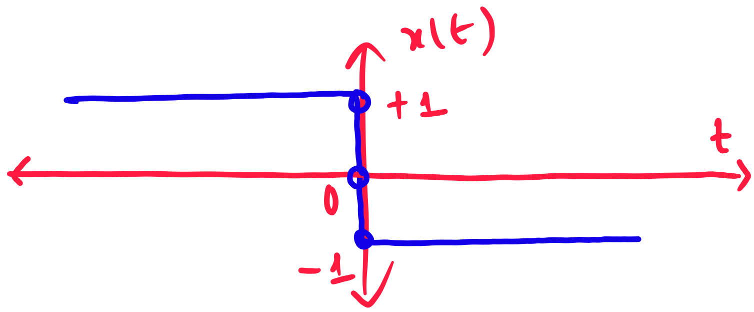
$$\boxed{X(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)}$$

$$\Rightarrow \frac{1}{2\pi} X(t) \xleftrightarrow{FT} x(-\omega)$$

$$\therefore \frac{1}{2\pi} X(-t) \longleftrightarrow x(\omega)$$

$\Rightarrow$ IFT of $x(\omega)$ is $\frac{1}{2\pi} X(-t)$. ☒

2) Consider



What is its Fourier Transform?

Sol^n $x(t) = -\text{sgn}(t)$

$X(\omega) = -X(\omega) = -2/j\omega$.

③ Given the frequency response :-

$$H(w) = \frac{100 + jw^2}{(1+jw)(10+jw)(1000+jw)}$$

what is the slope of Bode Magnitude plot b/w $100 << w << 1000$?

Solution If $w >> 100$:-

$1 + jw \simeq jw;$

$10 + jw \simeq jw;$

$1w + jw \simeq jw.$

If $w << 1000$:-

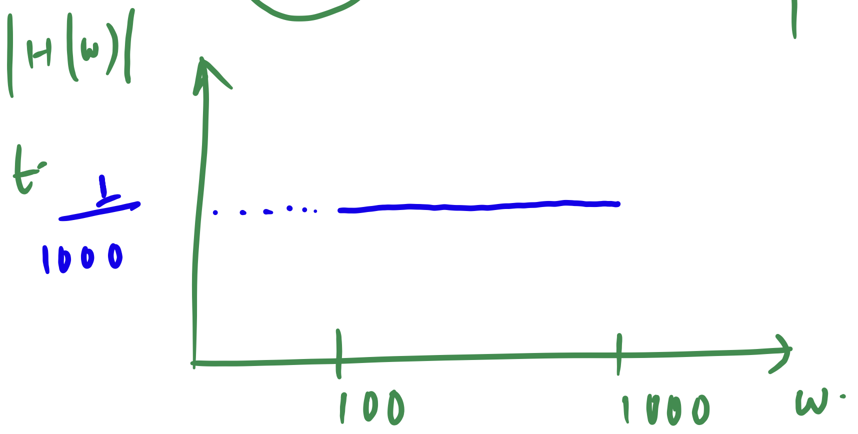
$1000 + jw \simeq 1000$

$\therefore$ using these expressions in $H(w)$ b/w $100 << w << 1000$

$$H(w) \simeq \cancel{(jw)^2} \Big/ \cancel{jw} \times \cancel{jw} \times 1000$$

$$= \frac{1}{1000}$$ → This is a constant and does not depend on w

Bode plot



$\therefore$ Slope $= 0$

④ Let $x(t)$ and $y(t)$ be two signals with F.T $X(w)$ & $Y(w)$. Then evaluate

$\int_{-\infty}^{\infty} X(-w)\, Y^*(w)\, dw$ in terms of $x(t)$ and $y(t)$.

Solution

$\int_{-\infty}^{\infty} X(-w)\, Y^*(w-\tilde{w})\, dw$ represents the convolution of $\underbrace{X(-w)\ \&\ Y^*(-w)}$

$$Z(\tilde{w}) \triangleq \int X(-w)\, Y^*(-(\tilde{w}-w))\, dw$$
$$= \int X(-w)\, Y^*(w-\tilde{w})\, dw$$

$$Z(\tilde{w}) = \int_{-\infty}^{\infty} X(-w)\, Y^*(w-\tilde{w})\, dw = X(-w) * Y^*(-w)$$

What is the IFT of $Z(\tilde{w}) = X(-w) * Y^*(-w)$?

(*) $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) e^{jwt} dw.$

$\therefore$ For $X(-w)$

$$\text{IFT} = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(-w)\, e^{jwt} dw.$$

$$w' = -w. \quad = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(w') e^{jw'(-t)} dw' = x(-t).$$

(*) $y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(w) e^{jwt} dw$ —— (1)

Now for $Y^*(-w)$

$$\text{IFT} = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y^*(-w) e^{jwt} dw \quad \text{—— (2)}$$

eq (1) $\rightarrow$ Take conjugation on both sides.

$$\Rightarrow y^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y^*(w)\, e^{-jwt} dw.$$

Put $w' = -w$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(-w')e^{jw't}\,dw' \quad \text{—— (3)}$$

$\because$ (2) & (3) are same

$\Rightarrow$ IFT of $Y^*(-w) = y^*(t)$

$\therefore$ IFT of $X(-w) * Y^*(-w) = 2\pi x(-t)\, y^*(t)$

IFT of $\int_{-\infty}^{\infty} X(-w)\, Y^*(w-\tilde{w})\,dw = 2\pi\, x(-t)\, y^*(t)$

$\Rightarrow$ FT of $2\pi\, x(-t)\, y^*(t) = \int_{-\infty}^{\infty} X(-w)\, Y^*(w-\tilde{w})\,dw.$

$\hookrightarrow 2\pi \int_{-\infty}^{\infty} x(-t)\, y^*(t)\, e^{-j\tilde{w}t}\,dt = \int_{-\infty}^{\infty} X(-w)\, Y^*(w-\tilde{w})\,dw.$

$= Z(\tilde{w})$

$Z(\tilde{w}) \triangleq \int_{-\infty}^{\infty} X(-w)\, Y^*(w-\tilde{w})\,dw \Rightarrow$ convolution of $X(-w)$ & $Y^*(-w)$

$\therefore$ The Required answer is $Z(0)$ ie

$\tilde{w} = 0$ in above equation

$$Z(0) = 2\pi \int_{-\infty}^{\infty} x(-t)\, y^*(t)\,dt$$

$$\therefore \int_{-\infty}^{\infty} X(w)\, Y^*(w)\,dw = 2\pi \int_{-\infty}^{\infty} x(-t)\, y^*(t)\,dt$$

(5) Find the F.T of $x(t) = te^{-at}u(t)$; $a > 0$.

Soln: For $x(t) = e^{-at}u(t)$

$$X(w) = \frac{1}{a+jw}$$

$\therefore \quad t\,x(t) \longleftrightarrow +j \dfrac{d}{dw} X(w).$

$$t e^{-at} u(t) \longleftrightarrow +j \frac{d}{dw}\left(\frac{1}{jw+a}\right)$$

$$\frac{1}{x} \xrightarrow{d/dx} \frac{-1}{x^2}$$

$$= +j \frac{(a+jw)\cdot 0 - j}{(a+jw)^2}$$

$$= \frac{1}{(a+jw)^2}$$

⑥ If $x(t) = \dfrac{t^{n-1}}{(n-1)!} e^{-at} u(t)$, then show that

$$X(w) = \frac{1}{(a+jw)^n}.$$

Sol^n As per above,

$$e^{-at} u(t) \xleftrightarrow{F.T} \frac{1}{a+jw}.$$

differentiate $\dfrac{1}{a+jw}$ w.r.t 'f', $(f = w/2\pi)$ and apply the frequency domain differentiation property.

$$\frac{d}{df}\left[\underbrace{\frac{1}{a+jw}}_{w=2\pi f}\right] = \underbrace{\frac{-j2\pi}{(a+jw)^2}}_{Y(w).} \xleftrightarrow{IFT} -j2\pi t\, e^{-at} u(t)$$

$$\frac{\cancel{(a+jw)}^{0}\cdot 0 - j2\pi}{(a+jw)^2}.$$

$$= \frac{-j2\pi}{(a+jw)^2}. \quad \left[\begin{array}{l}\text{as } w = 2\pi f \\ \text{and diff} \\ \text{w.r.t } f.\end{array}\right]$$

Differentiate $Y(w)$ again w.r.t f and use properties of

FT :

$$-j2\pi \frac{d}{df}\left(\frac{1}{(a+jw)^2}\right) = -j2\pi \cdot \frac{-2 \times j2\pi}{(a+jw)^3}.$$

$$\Big\updownarrow \text{IFT}$$

$$(-j2\pi \times -j2\pi t)\, t e^{-at} u(t)$$

$$\therefore \frac{1}{2} t^2 e^{-at} u(t) \xleftrightarrow{F.T} \frac{1}{(a+jw)^3}$$

$\therefore$ Repeating 'n' times –

$$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t) \xleftrightarrow{F.T} \frac{1}{(a+jw)^n}.$$

7) Find the Fourier Transforms of the following signals :-

a) $x(t) = 1$

b) $x(t) = e^{jw_0 t}$

c) $x(t) = e^{-jw_0 t}$

d) $x(t) = \cos w_0 t$

e) $x(t) = \sin w_0 t$.

(a) $y(t) = \delta(t) \xleftrightarrow{FT} 1$ $\quad Y(w) = 1$

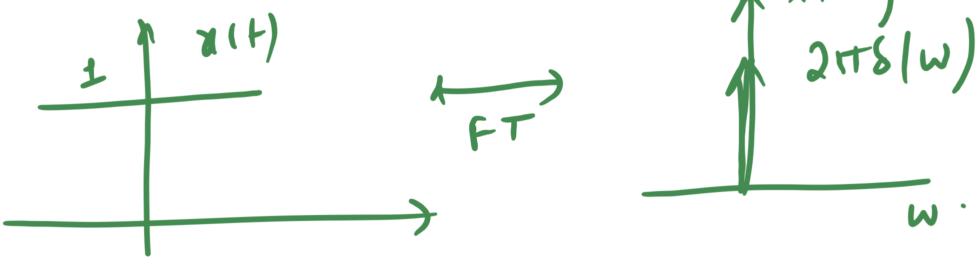
$x(t) = Y(t) \xrightarrow{FT} 2\pi\, y(-w)$.

↓

Duality property

$= 2\pi\, \delta(-w) = 2\pi\, \delta(w)$.

$\therefore 1 \xleftrightarrow{FT} 2\pi\, \delta(w)$



b) $x(t) = e^{jw_0 t}$

$$1 \longrightarrow 2\pi\delta(w)$$

$$1 \cdot e^{jw_0 t} \longleftrightarrow 2\pi\delta(w - w_0).$$

c) $x(t) = e^{-jw_0 t}$

$$1 \longrightarrow 2\pi\delta(w)$$

$$1\, e^{-jw_0 t} \longleftrightarrow 2\pi\,\delta(w + w_0)$$

d) $x(t) = \cos(w_0 t)$

$$\cos(w_0 t) = \frac{1}{2} e^{jw_0 t} + \frac{1}{2} e^{-jw_0 t}$$

using (b) and (c)

$$\cos w_0 t \xleftrightarrow{FT} \frac{2\pi}{2}\,\delta(w - w_0) + \frac{2\pi}{2}\delta(w + w_0)$$

$$= \pi\delta(w - w_0) + \pi\delta(w + w_0)$$

e) $x(t) = \sin w_0 t$

$$\sin w_0 t = \frac{1}{2j}\left\{e^{jw_0 t} - e^{-jw_0 t}\right\}$$

Using (b) and (c)

$$X(w) = \frac{1}{2j}\left\{2\pi\delta(w - w_0) - 2\pi\delta(w + w_0)\right\}$$

$$= -j\pi\left\{\delta(w - w_0) - \delta(w + w_0)\right\}$$

8) Find the FT of a periodic signal $x(t)$ with period T_0

Sol^n $\because x(t)$ is periodic

$\Rightarrow$ we can write Fourier series of $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+jkw_0 t} \quad ; \; w_0 = 2\pi/T_0$$

$$\therefore X(w) = \sum_{k=-\infty}^{\infty} C_k F\{e^{jkw_0 t}\}$$

Linearity of F.T

$$x_1(t) + x_2(t) + x_3(t) \longrightarrow X_1(w) + X_2(w) + \dots$$

From Q-7, recall

$$e^{jw_0 t} \xleftrightarrow{FT} 2\pi\delta(w-w_0)$$

$$\therefore e^{jkw_0 t} \xleftrightarrow{FT} 2\pi\delta(w-kw_0) \quad \text{— (2)}$$

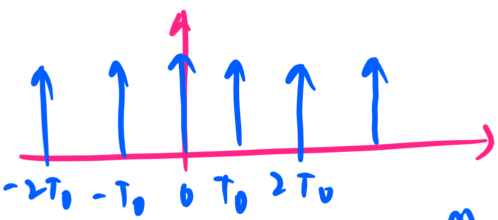
Put (2) in (1)

$$X(w) = 2\pi \sum_{k=-\infty}^{\infty} C_k\, \delta(w-kw_0)$$

eg

$$x_1(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT_0)$$

"Impulse train"!



$$x(t) = \sum_{K=-\infty}^{\infty} 1/T_0\, e^{jkw_0 t} \; ; \; w_0 = 2\pi/T_0.$$

$\rightarrow c_K.$

Complex exponential F.S

$$\text{then } X(w) = \frac{2\pi}{T_0} \sum_{K=-\infty}^{\infty} \delta(w-kw_0)$$

$$\therefore X(w) = w_0 \sum_{k=-\infty}^{\infty} \delta(w-kw_0).$$

Summary

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

$$\Big\updownarrow \text{FT}$$

$$X(\omega) = \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

Impulse train in time domain $\xleftrightarrow{FT}$ Impulse train in freq domain with a change in amplitude!

Q) Show that $x(t)\cos(\omega_0 t) \xleftrightarrow{FT} \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0)$

III^ry $x(t)\sin(\omega_0 t) \xleftrightarrow{FT} -j\left[\frac{1}{2} X(\omega - \omega_0) - \frac{1}{2} X(\omega + \omega_0)\right]$

Sol^n Recall $\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$

$x(t)\cos\omega_0 t \rightarrow \frac{x(t)}{2} e^{j\omega_0 t} + \frac{x(t)}{2} e^{-j\omega_0 t}$

Take FT on both sides:

$\Rightarrow \mathcal{F}\left[x(t)\cos\omega_0 t\right] = \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0)$

use frequency shifting property.

III^ry try 2nd one

→ These 2 statements are called "Modulation Theorem" of FTs

→ applications :- Communication systems.

10 Find F.T of

a) $x(t) = 1/t$

b) $x(t) = \dfrac{1}{1+t^2}$

Solution (i) $\mathrm{sgn}(t) \xleftrightarrow{FT} \dfrac{2}{jw}$.

$$\underbrace{\frac{j}{2}\,\mathrm{sgn}(t)}_{y(t)} \xleftrightarrow{FT} \underbrace{\frac{1}{w}}_{Y(w)}$$

what is FT of $Y(t) = 1/t$

$= 2\pi\, y(-w)$

$= 2\pi \dfrac{j}{2}\,\mathrm{sgn}(-t)$

$= j\pi\,\mathrm{sgn}(-w)$

$y(t) \leftrightarrow Y(w)$

$Y(t) \leftrightarrow 2\pi\, y(-w)$.

$$F\{1/t\} \leftrightarrow -j\pi\,\mathrm{sgn}(w)$$

(ii) $x(t) = \dfrac{1}{1+t^2}$

Recall $\quad e^{-a|t|} \longleftrightarrow \dfrac{2a}{a^2+w^2}$

$$e^{-|t|} \longleftrightarrow \frac{2}{1+w^2}$$

$$\frac{1}{2} e^{-|t|} \xleftrightarrow{FT} \frac{1}{1+w^2}$$

$$\underbrace{\frac{1}{2} e^{-|t|}}_{y(t)} \qquad \underbrace{\frac{1}{1+w^2}}_{Y(w)}$$

Q. What is the FT of $Y(t) = \frac{1}{1+t^2}$?

By Duality theorem

$$\mathcal{F}\left\{\frac{1}{1+t^2}\right\} = 2\pi\, y(-w)$$

$$= 2\pi \cdot \frac{1}{2} e^{-|-w|}$$

$$= \pi e^{-|-w|}$$

$$= \pi e^{-|w|}$$

W8 Assignment 8.

(1). $x(t)$ cannot be unbounded at a finite # of discontinuities in any interval.

(2) $u(t) \longrightarrow \pi\delta(w) + \frac{1}{jw}$. } (see theory by Prof. S+A)

(3) Not done

(4) $x(t) \longleftrightarrow X(w)$.

$$X(t) \longleftrightarrow \frac{2\pi\, x(-w)}{2\pi} \times 2\pi$$

$$\frac{X(t)}{4\pi^2} \longleftrightarrow \frac{x(-w)}{2\pi} \checkmark$$

↳ IFT

(5) $x(t) \longrightarrow X(w)$

$$\frac{dx(t)}{dt} \longleftrightarrow jw\, X(w).$$