$\boxed{\text{Week 6}}$

* Example Problems of Z transform

$$X(z) = \frac{z}{z-2} - \frac{2z}{z-3} + \frac{z}{(z-3)^2}$$

$$2 < |z| < 3$$

$$\frac{z}{z-2} \longleftrightarrow (2)^n u(n)$$

$$\frac{-2z}{z-3} \longleftrightarrow (-2)(-3^n u(-n-1))$$
$$= 2 \cdot 3^n u(-n-1)$$

Consider now the term

$$\frac{z}{(z-3)^2}$$

We use the property

$$-z \frac{d}{dz} \tilde{X}(z) \longleftrightarrow n\tilde{x}(n)$$

$$\tilde{x}(n) \longleftrightarrow \tilde{X}(z)$$

$$\tilde{X}(z) = \frac{z}{z-3} \longrightarrow \tilde{x}(n) = -3^n u(-n-1)$$

$$\frac{d}{dz}\tilde{X}(z) = \frac{d}{dz}\frac{z}{z-3}$$

$$= \frac{d}{dz}\frac{z-3+3}{z-3}$$

$$= \frac{d}{dz}\left(1 + \frac{3}{z-3}\right)$$

$$= \frac{-3}{(z-3)^2}$$

$$-z\frac{d}{dz}\tilde{X}(z) = -z \times \frac{-3}{(z-3)^2}$$

$$= \frac{3z}{(z-3)^2}$$

$$\frac{3z}{(z-3)^2} \longleftrightarrow n\tilde{x}(n)$$
$$= n(-3^n u(-n-1))$$

$$= -n3^n u(-n-1)$$

$+\frac{2}{(z-3)^2} \longleftrightarrow -n\,3^{n-1}u(-n-1)$

$$x(n) = 2^n u(n) + 2\,3^n u(-n-1) - n\cdot 3^{n-1}u(-n-1)$$

$\rightarrow$ Inverse z-transform.

(7) Consider the system with impulse response

$$h(n) = a^n u(n)$$

$$0 < a < 1$$

$$x(n) = u(n)$$

input

Find output using z-transform.

Solution: $h(n) = a^n u(n)$

$$H(z) = \frac{z}{z-a}, \quad |z| > a$$

ROC.

$x(n) = u(n)$

$$X(z) = \frac{z}{z-1}, \quad \text{ROC} \quad |z| > 1$$

$$y(n) = x(n) * h(n)$$

or $h(n) * x(n)$

$$Y(z) = H(z)\,X(z)$$

z transform of output

$$= \frac{z}{z-a}\,\frac{z}{z-1}.$$

$$= \frac{z^2}{(z-a)(z-1)}. \quad |z|>1$$

ROC $|z|>1 \cap |z|>a$

$= |z|>1 \quad \because a<1$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-a)}$$

$$= \frac{c_1}{z-1} + \frac{c_2}{z-a}$$

$$C_1 = (z-1)\frac{Y(z)}{z}\Big|_{z=1}$$

$$= \frac{z}{z-a}\Big|_{z=1}$$

$$= \frac{1}{1-a}.$$

$$C_2 = (z-a)\frac{Y(z)}{z}\Big|_{z=a}$$

$$= \frac{z}{z-1}\Big|_{z=a}$$

$$= \frac{a}{a-1} = \frac{-a}{1-a}$$

$$Y(z) = \frac{1}{1-a} \times \frac{z}{z-1} + \frac{a}{a-1}\frac{z}{z-a}$$

$|z|>1$

↳ RSS.

$$y(n) = \frac{1}{1-a}u(n) - \frac{a}{1-a}a^n u(n)$$

$$\Rightarrow \boxed{\frac{1-a^{n+1}}{1-a}u(n) = y(n)}$$

output signal $y(n)$ evaluated using z-Transform.

(8) Step response of LTI system.

$= a^n u(n)$

Impulse response? ie $h(n) = ?$

$x(n) = u(n)$ (∵ unit step response).

$\rightarrow X(z) = \dfrac{z}{z-1}$.

input

$$y(n) = h(n) * x(n).$$
$$= a^n u(n).$$

$$Y(z) = \frac{z}{z-a}, \quad |z| > a.$$

output

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{z}{z-a}}{\frac{z}{z-1}} = \frac{z-1}{z-a}.$$

Transfer Function.

$$\frac{H(z)}{z} = \frac{(z-1)}{z(z-a)}$$ Poles $z = 0, a$.

$$= \frac{c_1}{z} + \frac{c_2}{z-a}.$$

$$c_1 = z \frac{H(z)}{z} \Big|_{z=0}$$

$$= \frac{z-1}{z-a} \Big|_{z=0} = 1/a$$

$$c_2 = \frac{z-a}{} \frac{H(z)}{z} \Big|_{z=a}$$

$$= \frac{z-1}{z} \Big|_{z=a}$$

$$= \frac{a-1}{a} = -\left(\frac{1-a}{a}\right)$$

$$\frac{H(z)}{z} = \frac{1}{a} \cdot \frac{1}{z} - \frac{1-a}{a} \frac{1}{z-a}$$

$$H(z) = \frac{1}{a} \cdot 1 - \frac{1-a}{a} \frac{z}{z-a}$$

$\delta(n)$ $\qquad$ $a^n u(n)$

$$h(n) = \frac{1}{a} \delta(n) - \frac{1-a}{a} a^n u(n)$$

$$h(0) = \frac{1}{a} - \frac{1-a}{a} = \frac{a}{a} = 1$$

$$h(n) = 0 - \left(\frac{1-a}{a}\right) a^n$$

$$= -(1-a)a^{n-1}$$

for $n > 0$ $n \geq 1$

$$\boxed{h(n) = \delta(n) - (1-a)a^{n-1}u(n-1).}$$

Impulse response of the system.

Example Problems Z transform:

(9) Consider the LTI system described by the difference equation

$$y(n) - \frac{5}{2}y(n-1) + y(n-2) = x(n)$$

find $H(z)$, $h(n)$ for above system.

Transfer function

impulse response

consider z transform

$$Y(z) - \frac{5}{2}z^{-1}Y(z) + z^{-2}Y(z) = X(z)$$

$$Y(z)\left(1 - \frac{5}{2}z^{-1} + z^{-2}\right) = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 5/2\, z^{-1} + z^{-2}}$$

$$\boxed{H(z) = \frac{z^2}{z^2 - 5/2\, z + 1}}$$

Transfer function.

Poles : $z = 2, 1/2$. $|z| > 2 = r_{max}$

$$H(z) = \frac{z^2}{(z-2)(z-1/2)}$$

$$\frac{H(z)}{z} = \frac{z}{(z-2)(z-1/2)}$$

PF expansion for inverse z transform.

$$= \frac{c_0}{z-2} + \frac{c_1}{z-1/2}$$

$$C_0 = \left.\cancel{(z-2)} \frac{H(z)}{z}\right|_{z=2}$$

$$= \left.\frac{z}{z-1/2}\right|_{z=2}$$

$$= \frac{2}{3/2} = 4/3.$$

$$c_1 = \left.(z-1/2)\frac{H(z)}{z}\right|_{z=1/2}$$

$$= \left.\frac{z}{z-2}\right|_{z=1/2}$$

$$= \frac{1/2}{1/2 - 2} = \frac{1/2}{-3/2}$$

$$= -1/3.$$

$$c_1 = -1/3.$$

$$\frac{H(z)}{z} = \frac{4}{3}\frac{1}{z-2} - \frac{1}{3(z-1/2)}$$

$$H(z) = \frac{4}{3}\frac{z}{z-2} - \frac{1}{3}\frac{z}{z-1/2}$$

$\Rightarrow$. $|z| > 2$.

Right handed signal.

$$\boxed{h(n) = 4/3\,(2)^n u(n) - 1/3\,(1/2)^n u(n)}$$

$\rightarrow$ Impulse response of causal LTI system described by difference equation given in the problem.

* Inverse Z transform

General inversion technique.

$X(z)$: z transform.

$$x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz.$$

(contour integral.

Γ = closed contour in counter clockwise sense enclosing all the poles of $X(z) z^{n-1}$ inside ROC.

lies inside ROC

$$x(n) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1}.$$

$$= \sum_{i=1}^{P} \underbrace{\operatorname{Res}_{z \to p_i} \left[X(z) z^{n-1} \right]}_{\text{Residue at pole } p_i}$$

P = # poles of $X(z) z^{n-1}$

Sum of all residues at poles.

Residue → let p_i = simple pole

⇒ Multiplicity = 1

Residue

$$= \lim_{z \to p_i} (z - p_i) X(z) z^{n-1}$$

$$= (z - p_i) X(z) z^{n-1} \Big|_{z = p_i}$$

p_i : pole with multiplicity r.

$$\text{Residue} = \frac{1}{(r-1)!} \lim_{z \to p_i} \frac{d^{r-1}}{dz^{r-1}} (z - p_i)^r X(z) z^{n-1}$$

Residue of pole p_i with Multiplicity $r \geq 1$.

① Inverse z-transform of

$$X(z) = \frac{1}{2(z-1)(z+1/2)}$$

For $n > 0$, we have

$$X(z)\, z^{n-1} = \frac{z^{n-1}}{2(z-1)(z+1/2)}$$

Poles $z = 1, -1/2$

$$x(n) = \operatorname{Res}_{z=1}\left(X(z)\, z^{n-1}\right) + \operatorname{Res}_{z=-1/2}\left(X(z)\, z^{n-1}\right)$$

$$\operatorname{Res}_{z=1}\left(X(z)\, z^{n-1}\right) = (z-1)\, X(z)\, z^{n-1}\Big|_{z=1}$$

$$= \frac{z^{n-1}}{2(z+1/2)}\Big|_{z=1}$$

$$= \frac{1}{\cancel{2}\, 3/\cancel{2}}$$

$$= 1/3.$$

↳ residue at $z = 1$

$$\operatorname{Res}_{z=-1/2}\left(X(z)\, z^{n-1}\right) = \left(z + 1/2\right) X(z)\, z^{n-1}\Big|_{z=-1/2}$$

$$= \frac{z^{n-1}}{2(z-1)}\Big|_{z=-1/2}$$

$$= \left(-\frac{1}{2}\right)^{n-1} \frac{1}{\cancel{2} \times \left(-3/\cancel{2}\right)}$$

Residue at $z = -1/2$. $= \left(\frac{-1}{3}\right)\left(\frac{-1}{2}\right)^{n-1}$

$$x(n) = \frac{1}{3} - \frac{1}{3}\left(\frac{-1}{2}\right)^{n-1}$$

↳ For $n > 0$

For $n=0$

$$X(z)\, z^{n-1} = \frac{X(z)}{z}$$

$$X(z)\,\frac{1}{z} = \frac{1}{2z(z-1)(z+1/2)}$$

additional pole
$z=0$

Poles at $z = 0, 1, -1/2$

Residue at $z=0$

$$= z \left.\frac{X(z)}{z}\right|_{z=0}$$

$$= \left.\frac{1}{2(z-1)(z+1/2)}\right|_{z=0}$$

$$= \frac{1}{2\times(-1)\times 1/2}$$

$$= -1$$

Residue at $z=1$.

$$= (z-1)\left.\frac{X(z)}{z}\right|_{z=1}$$

$$= \left.\frac{1}{2z(z+1/2)}\right|_{z=1}.$$

$$= \frac{1}{2\times\frac{3}{2}}.$$

Residue at $z=1$ $= 1/3$.

Residue at $z=-1/2$

$$= (z+1/2)\left.\frac{X(z)}{z}\right|_{z=-\frac{1}{2}}$$

$$= \left.\frac{1}{2(z-1)(z)}\right|_{z=-1/2}$$

$$= \frac{1}{\cancel{2}\cdot(-3/\cancel{2})(-1/2)}$$

$$= 2/3.$$

$x(0) = -1 + 1/3 + 2/3 = 0$

$x(n) = \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^{n-1} \quad n \geq 1$

$x(n) = \left\{\frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^{n-1}\right\} u(n-1)$

inverse z transform.

(2) Find inverse z transform of $X(z) = \dfrac{z^2}{(z-1)^2 (z-e^{-aT})}$

Consider $X(z) z^{k-1}$.

$$= \frac{z^{k+1}}{(z-1)^2 (z-e^{-aT})}$$

Pole of Multiplicity $= 2$. @ $z = 1$.

Simple pole at $z = e^{-aT}$

Residue at $z = e^{-aT}$

$$= (z - e^{-aT}) \, X(z) z^{k-1}$$

$$= \left.\frac{z^{k+1}}{(z-1)^2}\right|_{z = e^{-aT}}$$

$$= \frac{e^{-a(k+1)T}}{(e^{-aT} - 1)^2}$$

$$= \frac{e^{-a(k+1)T}}{(1 - e^{-aT})^2}$$

Residue at $z = e^{-aT}$

Residue at $z = 1$.

✓ Pole of Multiplicity $= 2$

$r=2$

$$\frac{1}{(2-1)!}\frac{d}{dz}(z-1)^2 X(z) z^{k-1}\Big|_{z=1}$$

$$= \frac{d}{dz}\frac{z^{k+1}}{z-e^{-aT}}\Big|_{z=1}$$

$$= \frac{(k+1)z^k}{(z-e^{-aT})} - \frac{z^{k+1}}{(z-e^{-aT})^2}\Big|_{z=1}$$

$$= \left(\frac{k+1}{1-e^{-aT}}\right) - \frac{1}{(1-e^{-aT})^2}$$

$$= \underbrace{\frac{k}{1-e^{-aT}} - \frac{e^{-aT}}{(1-e^{-aT})^2}}_{\text{Residue at } z=1.}$$

$$x(n) = \frac{e^{-a(k+1)T}}{(1-e^{-aT})^2} + \frac{k}{1-e^{-aT}} - \frac{e^{-aT}}{(1-e^{-aT})^2}$$

$$= \underbrace{\frac{k}{1-e^{-aT}} - \frac{e^{-aT}(1-e^{-aT})}{(1-e^{-aT})^2}}_{\text{for } k \geq 0.}$$

* Fourier analysis continuous time signal and systems.

→ Transforms signals or systems into spectral / frequency domain.

→ One of the most commonly employed and insightful transforms.

Consider the complex exponential

$$e^{j\omega_0 t}$$

$$= \cos(\omega_0 t) + j\sin(\omega_0 t)$$

periodic with period

$$T_0 = 2\pi / \omega_0$$

$$e^{j\omega_0 (t + kT_0)}$$

$$= e^{j\omega_0 t + j\omega_0 k\, 2\pi/\omega_0}$$

$$= e^{j\omega_0 t + 2\pi j k}$$

$$= e^{j\omega_0 t} \underbrace{e^{j2\pi k}}_{1}$$

$$= e^{j\omega_0 t}$$

$$T_0 = 2\pi / \omega_0$$

Fundamental period.

ω_0 = Fundamental Angular frequency

$$F_0 = 2\pi / \omega_0 \Rightarrow \omega_0 = 2\pi F_0$$

Fundamental Frequency.

COMPLEX EXPONENTIAL OR FOURIER SERIES REPRESENTATION

Continuous periodic signal $x(t)$

Fundamental Period $= T_0$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$\omega_0 = 2\pi / T_0$$

sum of complex exponentials frequency ω_0 and various harmonics $k\omega_0$.

c_k — Fourier series coefficients

$$c_k = 1/T_0 \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= 1/T_0 \int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

This can be verified as follows.

$$\frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= 1/T_0 \int_{T_0} \underbrace{\left(\sum_{l=-\infty}^{\infty} c_l\, e^{jl\omega_0 t} \right)}_{x(t)} e^{-jk\omega_0 t}\, dt$$

$$= \sum_{l=-\infty}^{\infty} c_l \underbrace{\frac{1}{T_0} \int_{T_0} e^{j(l-k)\omega_0 t}\, dt}$$

$$= \sum_{l=-\infty}^{\infty} c_l\, \delta(l-k)$$

If $l = k$

$$1/T_0 \int_{T_0} 1 \cdot dt = 1/T_0 \times T_0$$

$$= 1$$

If $l \neq k$, let $l - k = m$

$$\underbrace{1/T_0 \int_{T_0} e^{jm\omega_0 t}\, dt}$$

Integral of the Harmonic over fundamental period.

$= 0$

$$\underbrace{1/T_0 \int_{T_0} e^{j(l-k)\omega_0 t}\, dt}_{} \rightarrow \delta(l-k)$$

$$= \begin{cases} 1, & l = k \\ 0, & l \neq k \end{cases}$$

$$\frac{1}{T_0} \int_{T_0} x(t)\, e^{-j\omega_0 k t}\, dt$$

$$= \sum_{l=-\infty}^{\infty} c_l\, \delta(l-k)$$

$$= c_k$$

$$c_k = 1/T_0 \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

kth fourier coefficients

$k = 0$

$$c_0 = 1/T_0 \cdot \underbrace{\int_{T_0} x(t)\, dt}$$

Mean or average of $x(t)$ over period.

DC coefficient

$k = 0$

$\Rightarrow$ 0 frequency.

FOURIER ANALYSIS

CONTINUOUS TIME SIGNALS.

Fourier Series.

$x(t) \rightsquigarrow$ Periodic Signal

$\hookrightarrow$ Fundamental period $= T_0$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$\hookrightarrow$ Sum of complex exponentials at ω_0 and $k\omega_0$.

Harmonics.

$$\omega_0 = 2\pi / T_0$$

$\hookrightarrow$ Fundamental angular frequency.

$$c_k = 1/T_0 \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

kth Fourier series coefficient.

If $x(t)$ = Real,

$\hookrightarrow$ Real Signal

$$c_K = 1/T_0 \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$c_K^* = \left(1/T_0 \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt\right)^*$$

$$= \frac{1}{T_0} \int_{T_0} \underbrace{x^*(t)}\, e^{jk\omega_0 t}\, dt$$

↳ as $x(t)$ is real
$x^*(t) = x(t)$

$$c_K^* = \underbrace{\frac{1}{T_0} \int x(t)\, e^{jk\omega_0 t}\, dt}_{c_{-K}}$$

For a real signal $x(t)$,

we have,

$$\boxed{c_K^* = c_{-K}}$$ → conjugate symmetry

↳ $|c_K| = |c_K^*| = |c_{-K}|$

↳ Magnitude of Fourier coefficients exhibits even symmetry.

$$\measuredangle c_K = -\measuredangle c_K^*$$
$$= -\measuredangle c_{-K}$$

$$\measuredangle c_K = -\measuredangle c_{-K}$$

phase spectrum exhibits odd symmetry

TRIGONOMETRIC FOURIER SERIES:

$$x(t) = \frac{a_0}{2} + \sum_{K=1}^{\infty} a_K \cos(k\omega_0 t) + b_K \sin(k\omega_0 t)$$

a_0, a_K, b_K can be complex.

← Factor of 2

$$a_K = \frac{2}{T_0} \int_{T_0} x(t) \cos(k\omega_0 t)\, dt$$

$$b_K = \frac{2}{T_0} \int_{T_0} x(t) \sin(k\omega_0 t)\, dt$$

Fourier Series.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k . e^{jk\omega_0 t}$$

$$= c_0 + \sum_{k=1}^{\infty} c_n e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}$$

$$= c_0 + \sum_{k=1}^{\infty} \left((c_k + c_{-k}) \cos(k\omega_0 t) + j(c_k - c_{-k}) \sin(k\omega_0 t) \right)$$

By comparison of coeffs b/w Fourier series & Trigonometric

Fourier series.

$$c_0 = \frac{a_0}{2}$$

$$a_k = c_k + c_{-k}.$$

$$b_k = j(c_k - c_{-k})$$

$$j b_k = -c_k + c_{-k}$$

$$\Rightarrow \frac{a_k - j b_k}{2} = c_k$$

$$\Rightarrow \frac{a_k + j b_k}{2} = c_{-k}.$$

If $x(t)$ is Real, Then, a_k, b_k are Real

$$a_k = c_k + c_{-k}.$$

$$= c_k + c_k^*$$

$$\boxed{a_k = 2\,\mathrm{Re}\{c_k\}}$$

$x(t)$ is Real.

$$b_k = j(c_k - c_{-k})$$

$$= j(c_k - c_k^*)$$

$$= j\,2j\,\mathrm{Im}(c_k)$$

$$= -2\,\mathrm{Im}(c_k)$$

$\boxed{b_K = -2\,Im(C_K)}$

→ for real signal $x(t)$

<u>Even and ODD SIGNALS.</u>

If $x(t)$ is even

Then $b_K = 0$

coefficients of $\sin(k\omega_0 t)$

odd function.

$$\Rightarrow x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_K \cos k\omega_0 t$$

→ even signal.

→ only cos terms remain.

If $x(t)$ = odd.

$a_K = 0$

coefficients of $\cos(k\omega_0 t)$

even function.

$$x(t) = \sum_{K=1}^{\infty} b_K \sin(k\omega_0 t)$$

$\omega_0 = 2\pi/T_0$

fundamental angular frequency.

* <u>Conditions for existence of fourier series.</u>

condition for Fourier series to <u>converge</u>.

<u>Dirichlet</u> conditions.

Periodic signal $x(t)$ has a Fourier series representation if it satisfies the Dirichlet conditions

1. $x(t)$ is absolutely integrable over a period.

$$\Rightarrow \underbrace{\int_{T_0} |x(t)|\,dt}_{\text{Finite quantity}} < \infty$$

2. $x(t)$ has a finite # of maxima or minima in any finite interval over t.

3. $x(t)$ has a finite # of discontinuity within any finite interval on t and each of these discontinuities is finite

NOTE: The above Dirichlet conditions are sufficient but NOT necessary

$\Rightarrow$ Fourier series exist if Dirichlet conditions are satisfied.

AMPLITUDE and Phase spectrum: $x(t)$

Continuous periodic signal

Complex Fourier series coefficients of $x(t)$ can be expressed as

$$\underbrace{C_k}_{} = |c_k|\, e^{j\phi_k}$$

phase of C_k

Magnitude of C_k.

$$\phi_k = \measuredangle C_k.$$

Plot of $|C_k|$ vs. angular frequency ω — Termed as amplitude spectrum.

$$\underbrace{\phi_k = \measuredangle C_k \text{ vs } \omega}_{\text{phase spectrum.}}$$

Note: $|c_k|$, ϕ_k — Not continuous functions of ω.

Exist only at discrete Frequency $k\omega_0$

k = integer

Fundamental angular frequency

Hence Termed as discrete frequency spectrum or line spectrum.

For a real periodic signals $x(t)$ we have

$$c_{-k} = c_k^*$$

$$\Rightarrow |c_{-k}| = |c_k|$$

$\Rightarrow$ Magnitude spectrum = even fxn of ω.

$$\Rightarrow \phi_{-k} = -\phi_k$$

$\Rightarrow$ Phase spectrum = odd function.

Power of periodic signal

Power of periodic signal

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt$$

Consider Fourier series of $x(t)$ given as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$|x(t)|^2 = x(t)\, x^*(t)$$

$$= \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right) \times \left(\sum_{m=-\infty}^{\infty} c_m e^{jm\omega_0 t} \right)^*$$

$$= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_k c_m^* e^{j(k-m)\omega_0 t}$$

Therefore, for power, we have

$$\frac{1}{T_0} \int |x(t)|^2 dt$$

$$= \frac{1}{T_0} \int_{T_0} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_k c_m^* \times e^{j(k-m)\omega_0 t} dt$$

Interchange sum / integral

$$= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_k c_m^* \underbrace{\frac{1}{T_0} \int_{T_0} e^{j(k-m)\omega_0 t} dt}_{\begin{cases} = 1 \text{ if } k = m \\ = 0 \text{ if } k \neq m \end{cases} = \delta(k-m)}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_k c_m^* \delta(k-m)$$

$$= \sum_{k=-\infty}^{\infty} |c_k|^2.$$

$$\boxed{P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2}$$

PARSEVAL'S THEOREM /

Parseval's Identity