TA session 2 (signals and systems).
(yash vanth. L).

Part I - Summary of week 2's lectures.

(1) linear systems

$$x(t) \longrightarrow \boxed{T(\cdot)} \longrightarrow y(t).$$

a) additivity :- if $x_1(t)$ and $x_2(t)$, are 2 signals, and $T(x_1(t)) = y_1(t)$ and $T(x_2(t)) = y_2(t)$ then,

$$T(x_1(t) + x_2(t)) = y_1(t) + y_2(t).$$

b) homogeneity

$$\text{if } T(x_1(t)) = y_1(t)$$

$$\text{Then } T(\underset{\substack{\uparrow \\ \text{any scalar \#.}}}{c}\, x_1(t)) = c\, y_1(t).$$

A linear system should satisfy both of these properties.

eg $y(t) = 2x(t) - 5$ ← Linear system.

$y(t) = 5x^2(t)$ ← Non-linear system.

(2) Time Invariant Systems

If $x(t) \xrightarrow{T(\cdot)} y(t)$.

then $x(t-\tau) \xrightarrow[\text{for any } \tau.]{T(\cdot)} y(t-\tau)$.

shifted (or) delayed input

shifted (or) delayed output.

LTI systems (Linear Time Invariant systems)

→ A system which is both linear and time invariant.

→ very important for 2 reasons:

(*) Most of the naturally occurring systems are LTI.

(*). These systems are mathematically characterizable very easily (∵ impulse response).

(3) Stable system (BIBO stability)

A bounded i/p should give a bounded o/p.

(ie) $\underset{\le K}{x(t)} \xrightarrow{T(\cdot)} \underset{\_\,K}{y(t)}$

(eg) $y(t) = \int_{t-T}^{t} x(t)\,dt$ — system

let $|x(t)| \le K$ — bounded input

$$\Rightarrow |y(t)| = \left| \int_{t-T}^{t} x(t)\,dt \right|$$

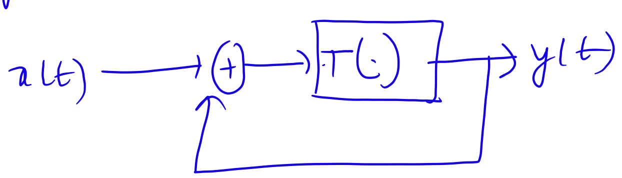
$$\le \int_{t-T}^{t} |x(t)|\,dt$$

$$\le \int_{t-T}^{t} K\,dt = \underline{\underline{KT}}$$

$\therefore |x(t)| \le K \Rightarrow |y(t)| \le KT$ ← another finite #

$y(t) = \int_{t-T}^{t} x(t)\,dt$ is a stable system.

→ feedback system may (or) may not be stable!



(4) Impulse response of LTI systems.

→ any LTI system can be completely specified by its impulse response!

→ It's the o/p of the system for a unit impulse fxn.

$$\underset{=\delta(t)}{x(t)} \longrightarrow \boxed{T(\cdot)} \longrightarrow \underset{= \text{impulse resp } h(t)}{y(t)}$$

(5) o/p of an LTI system

The o/p of any LTI system is given by convolution of i/p signal and the impulse response of the system.

$$y(t) = x(t) * h(t)$$

ie $$y(t) \triangleq \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau.$$

(6) Convolution and properties.

(a) Commutativity

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

(b) Associativity

$$y_1(t) = \underbrace{x_1(t) * x_2(t)}_{} * x_3(t)$$

$$= x_1'(t) * x_3(t)$$

III^ry. $$y_2(t) = x_1(t) * \underbrace{x_2(t) * x_3(t)}_{}$$

$$= x_1(t) * x_3'(t)$$

then $y_1(t) = y_2(t)$.

c) Distributivity

$$x_1(t) * (x_2(t) + x_3(t))$$

$$= x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

(7) Memoryless systems.

→ o/p depends only on current i/p.

→ an LTI system is NOT memoryless if

$$h(t) \neq 0 \text{ if } t \neq 0$$

⑧ Causal systems

→ o/p may depend only on current & past i/ps but not on future.

→ for LTI system to be causal

$h(t) = 0$ for $t < 0$.

⑨ Stable system

→ Bounded i/p gives a bounded o/p.

→ for LTI system to be BIBO stable.

h(t) should be absolutely integrable

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Suggested Textbooks-

1) Signals and Systems by Oppenheim.
2) Signals & system by Simon Haykins.

Part II (Tutorials)

① Find the periodicity of

$x(t) = 10\cos(4\pi t) + 7\sin(8\pi t)$

Ans $x(t) = x_1(t) + x_2(t)$

$x_1(t) = 10\cos(4\pi t)$ → Periodic

$x_2(t) = 7\sin(8\pi t)$ → Periodic

Let T_1 be the period of $x_1(t)$.

$\Rightarrow x_1(t+T_1) = x_1(t)$.

$\Rightarrow 10\cos(4\pi(t+T_1)) = 10\cos(4\pi t)$

$\Rightarrow \cos(4\pi t + 4\pi T_1) = \cos(4\pi t)$

$\Rightarrow 4\pi T_1 = 2\pi n \; ; \; n \in \mathbb{Z}$

$\boxed{T_1 = n/2} \; ; \; n \in \mathbb{Z}$, @ $n=1$

Fundamental period: $T_1^f = 1/2$

Let T_2 be the period of $x_2(t) = 7\sin(8\pi t)$

$x_2(t+T_2) = x_2(t)$

$\Rightarrow 7\sin(4\pi(t+T_2)) = 7\sin(8\pi t)$

$\Rightarrow \sin(8\pi t + 8\pi T_2) = \sin(8\pi t)$

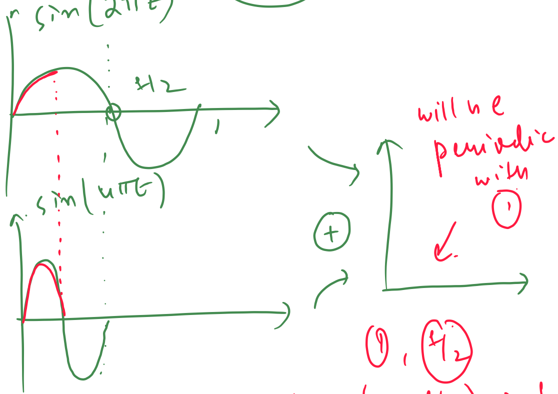
$8\pi T_2 = 2\pi n \; ; \; n \in \mathbb{Z}$

$T_2 = \dfrac{2\pi n}{8\pi} = n/4 \; ; \; n \in \mathbb{Z}$

$$\boxed{T_2^f = 1/4} \quad (n=1$$

eg $\sin(2\pi t)$ $T = 1$

$\sin(4\pi t)$ $T = 1/2$



(1), (1/2)

$LCM(1, 1/2) = 1$

$1 \times 2 = 2$, $LCM(2, 1) = 2$

$1/2 \times 2 = 1$, $LCM(1, 1/2) = 2/2 = 1$

least common multiple !

$\therefore T = LCM(1/2, 1/4)$ /

(i) step 1 → convert 1/2 & 1/4 to integer

eg let $w = 4$.

$1/2 \times w = 1/2 \times 4 = 2$

$1/4 \times w = 1/4 \times 4 = 1$

(ii) step 2 → take LCM of (2, 1) = 2

(iii) step 3 → $LCM(1/2, 1/4) = 2/w$

$= 2/4$

$= 1/2$

$$\boxed{T^f = 1/2}$$

(2) Check whether the following systems are time invariant or not.

a) $y(t) = 2t\,x(t)$.

Procedure

Operation 1:

$x(t) \longrightarrow \boxed{\text{Delay by } \tau} \longrightarrow x(t-\tau) \longrightarrow \boxed{T(\cdot)} \longrightarrow y_1(t)$

Operation 2

$x(t) \longrightarrow \boxed{T(\cdot)} \longrightarrow y(t) \longrightarrow \boxed{\text{Delay by } \tau} \longrightarrow y_2(t)$

$y(t-\tau)$

if $y_1(t) = y_2(t)$

$\Rightarrow T(\cdot)$ is TIV system.

Solution

Operation 1.

$x(t) \longrightarrow \boxed{\tau} \longrightarrow x(t-\tau) \longrightarrow \boxed{T(\cdot)} \longrightarrow 2t\,x(t-\tau)$.

Operation 2

$x(t) \longrightarrow \boxed{T(\cdot)} \longrightarrow 2t\,x(t) \longrightarrow \boxed{\tau} \longrightarrow 2(t-\tau)\,x(t-\tau)$.

$\because\ y_1(t) \neq y_2(t)$.

$\Rightarrow y(t) = 2t\,x(t)$ cannot be realized by TI system

b) $y(t) = x(t)\sin(20\pi t)$

Solution

Op-1 $x(t) \longrightarrow \boxed{\tau} \longrightarrow x(t-\tau) \longrightarrow \boxed{T(\cdot)} \longrightarrow x(t-\tau)\sin(20\pi t)$

$y_1(t)$.

Op 2 $x(t) \longrightarrow \boxed{T(\cdot)} \longrightarrow x(t)\sin(20\pi t) \longrightarrow \boxed{\tau} \longrightarrow x(t-\tau)\sin(20\pi(t-\tau))$

$y_1(t) \neq y_2(t) \Rightarrow$ NOT TI s/s.

c) $y(t) = 3x(t^2) \Rightarrow y(1) = 3x(1)$

$y(2) = 3x(4)$

$y(3) = 3x(9)$

Sol^n op 1

$x(t) \rightarrow \boxed{\tau} \rightarrow x(t-\tau) \rightarrow \boxed{T(\cdot)} \rightarrow 3x(t^2-\tau)$

op 2

$x(t) \rightarrow \boxed{T(\cdot)} \rightarrow 3x(t^2) \rightarrow \boxed{\tau} \rightarrow 3x((t-\tau)^2)$

$y_1(t) \neq y_2(t) \Rightarrow$ system is not TIV.

d) $y(t) = x(-t)$.

Sol^n op 1

$x(t) \rightarrow \boxed{\tau} \rightarrow x(t-\tau) \rightarrow \boxed{T(\cdot)} \rightarrow x(-t-\tau)$

op 2 $x(t) \rightarrow \boxed{T(\cdot)} \rightarrow x(-t) \rightarrow \boxed{\tau} \rightarrow = x(-t+\tau)$

$y_1(t) \neq y_2(t) \Rightarrow$ its a time varying s/s.

(e) $y(t) = x^2(t)$.

Sol^n op 1

$x(t) \rightarrow \boxed{\tau} \rightarrow x(t-\tau) \rightarrow \boxed{T(\cdot)} \rightarrow x^2(t-\tau)$.

op 2

$x(t) \rightarrow \boxed{T(\cdot)} \rightarrow x^2(t) \rightarrow \boxed{\tau} \rightarrow x^2(t-\tau)$.

$y_1(t) = y_2(t) \Rightarrow$ s/s is time invariant

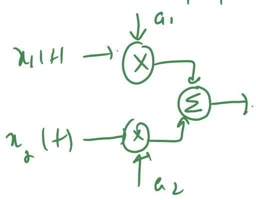
③ check whether the following system are Linear?

a) $y(t) = x(t^2)$.

Procedure to check linearity

→ Method of superposition.

op 1



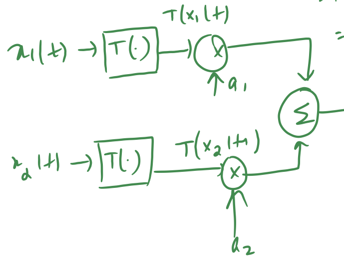
$a_1x_1(t) + a_2x_2(t) \rightarrow \boxed{T(\cdot)}$

$T(a_1x_1(t) + a_2x_2(t))$

$= T(a_1x_1(t)) + T(a_2x_2(t))$

$= a_1Tx_1(t) + a_2T(x_2(t))$

op 2



$y_2(t) = a_1T(x_1(t)) + a_2T(x_2(t))$

if $y_1(t) = y_2(t)$ → system is a linear system.

Sol^n $y_1(t) = T\{a_1 x_1(t) + a_2 x_2(t)\}$

$= a_1 x_1(t^2) + a_2 x_2(t^2)$ — (1)

$y_2(t) = a_1 T(x_1(t)) + a_2 T(x_2(t))$

$= a_1 x_1(t^2) + a_2 x_2(t^2)$ — (2)

$y_1(t) = y_2(t)$ → s/s is linear!

b) $y = x^2(t)$.

Sol^n Op 1

$y_1(t) = T\{\underbrace{a_1 x_1(t) + a_2 x_2(t)}_{x'(t)}\}$.

$= x'^2(t)$

$= \{a_1 x_1(t) + a_2 x_2(t)\}^2$ — (1)

Op 2 $y_2(t) = a_1 T\{x_1(t)\}$

$+ a_2 T\{x_2(t)\}$.

$= a_1 x_1^2(t) + a_2 x_2^2(t)$ — (2)

∴ $y_1(t) \neq y_2(t)$ ⇒ s/s is not linear.

c) $y(t) = e^{x(t)}$

Sol^n Op 1

$y_1(t) = T\{\underbrace{a_1 x_1(t) + a_2 x_2(t)}_{x'(t)}\}$

$= e^{x'(t)}$

$= e^{a_1 x_1(t) + a_2 x_2(t)}$.

$= e^{a_1 x_1(t)} \cdot e^{a_2 x_2(t)}$ — (1)

Op 2 $y_2(t) = a_1 T\{x_1(t)\}$

$+ a_2 T\{x_2(t)\}$

$= a_1 e^{x_1(t)} + a_2 e^{x_2(t)}$ — (2)

$y_1(t) \neq y_2(t)$ ⇒ system is non-linear.

d). $y(t) = 6 \frac{dx(t)}{dt} - 7x(t)$.

Solⁿ op 1 $y_1(t) = T\{\underbrace{a_1 x_1(t) + a_2 x_2(t)}_{x'(t)}\}$

$= 6 \frac{dx'(t)}{dt} - 7x'(t)$ — (1)

$\frac{dx'(t)}{dx} = \frac{d}{dx}\{a_1 x_1(t) + a_2 x_2(t)\}$

$= a_1 \, d\, x_1(t)/dt + a_2 \frac{d\, x_2(t)}{dx} \rightarrow (2)$

Put (2) in (1)

$y_1(t) = 6a_1 \, d/_{dx}\, x_1(t) + 6a_2 \frac{d}{dx} x_2(t)$
$- 7a_1 x_1(t)$
$- 7a_2 x_2(t) \rightarrow (3)$.

op 2

$y_2(t) = a_1 T\{x_1(t)\} + a_2 T\{x_2(t)\}$.

$= a_1\{6 \frac{d}{dx} x_1(t) - 7x_1(t)\}$

$+ a_2\{6 \frac{d}{dx} x_2(t) - 7x_2(t)\}$

$= 6a_1 \frac{dx_1(t)}{dx} + 6a_2 \frac{dx_2(t)}{dx}$

$- 7a_1 x_1(t) - 7a_2 x_2(t)$ — (4)

$y_1(t) = y_2(t) \Rightarrow T(\cdot)$ is linear!

(4) Check whether the following system is BIBO stable.

$h(t) = te^{-3t} u(t) \rightarrow$ unit step function.

Solⁿ $h(t)$ should be absolutely integrable for BIBO stability.

$\therefore \int_{-\infty}^{\infty} |h(t)|\, dt$

$= \int_{-\infty}^{\infty} t\, e^{-3t} u(t)\, dt$

$= \int_{0}^{\infty} te^{-3t}\, dt$

integration by parts

$\int uv = u\int v - \int [du \int v]$

$= t \frac{e^{-3t}}{-3}\Big|_0^{\infty} - \int_0^{\infty} 1. \frac{e^{-3t}}{-3}$

$$= \frac{\infty e^{-\infty} - 0}{-3} + \frac{e^0}{9} - \frac{e^{-\infty}}{9}$$

$$= 1/9$$

$$\Rightarrow \int_{-\infty}^{\infty} |h(t)|\, dt = 1/9 < \infty$$

$\Rightarrow$ system is BIBO stable.

⑤ Determine the range of values of "a" and "b" for the stability of an LTI system, whose impulse response is

$$h(t) = e^{at} u(t) + e^{-bt} u(t)$$

Sol^n for BIBO stability

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} |e^{at} u(t) + e^{-bt} u(t)|\, dt$$

$$= \left| \int_{-\infty}^{\infty} e^{at} u(t) + e^{-bt} u(t)\, dt \right|$$

This is because the integrand is real and positive.

$|x_1 + x_2| = |x_1| + |x_2|$

real and positive.

$|\Sigma x_i| = \Sigma |x_i|$ → if x_i is real and positive

$|1+2+3+4| = |1| + |2| + |3| + |4|$.

$$\Rightarrow \left| \int f(x)\, dx \right| = \int f(x)\, dx$$

$$= \left| \int_0^{\infty} e^{at} dt + \int_0^{\infty} e^{-bt} dt \right|$$

$$= \left| \frac{e^{a\infty}}{a} - \frac{e^{a \cdot 0}}{a} + \frac{e^{-b \times \infty}}{-b} - \frac{e^{-b \times 0}}{-b} \right|$$

$$= \left| \frac{e^{a\infty}}{a} - 1/a + \frac{e^{-b \cdot \infty}}{-b} + 1/b \right|$$

$a < 0$, $b > 0$ ✓ $\} \; e^{a \times \infty} \Rightarrow e^{-\infty} \to 0$ ✓

$$= \left| \frac{0}{a} - 1/a + 0/b + 1/b \right|$$

$$= |1/b - 1/a|$$

$$= \left| \frac{a-b}{ab} \right| \quad \begin{matrix} a > -\infty \\ b < \infty \end{matrix}$$

$\hookrightarrow < \infty$ ✓

$\therefore$ Range of a & b are:

$-\infty < a < 0$

$0 < b < \infty$