Week 5

z-transform

↳ represent and analyse Discrete time signals | systems.

$x[n] \rightarrow$ signal.

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$$

→ Z-transform of $x(n)$

z = Complex number

$= re^{j\Omega}$

↑ Polar coordinate

$r = |z|$

$\Omega = \measuredangle z$.

$$\underbrace{x(n) \longleftrightarrow X(z)}_{\text{z-Transform pair.}}$$

Region of convergence (ROC)

↳ Range of values for which z-transform converges

$x(n) = a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}.$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

converges only if

$|a z^{-1}| < 1$

$\Rightarrow |a| < |z|$

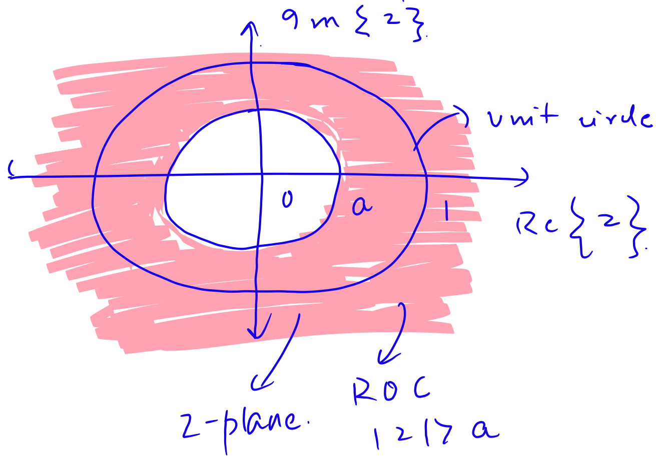
$$\text{or } \underbrace{|z| > |a|}_{\text{ROC}}.$$

$0 < a < 1$

$$X(z) = \frac{z}{z-a}$$

$\rightarrow$ zero at $z=0$

$\rightarrow$ pole at $z=a$



Consider now

$$x[n] = -a^n u(-n-1)$$

$$= \begin{cases} -a^n, & n \le -1 \\ 0, & \text{else.} \end{cases}$$

Left handed signal.

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$m = -n$

$$X(z) = -\sum_{m=1}^{\infty} a^{-m} z^{m}$$

$$= -\sum_{m=1}^{\infty} \left(a^{-1}z\right)^{m}$$

$$= -\frac{a^{-1}z}{1-a^{-1}z}$$

converges only for $|a^{-1}z| < 1$

$\Rightarrow \underbrace{|z| < |a|}_{\text{ROC}}$

$$X(z) = -\frac{a^{-1}z}{1-a^{-1}z}$$

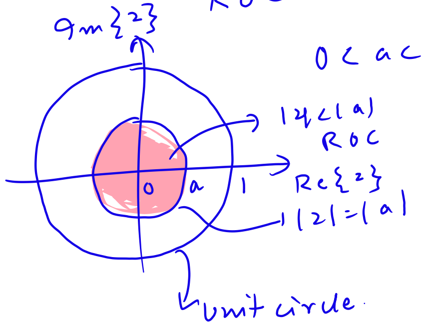
$$= -\frac{1}{az^{-1}-1}$$

$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$-a^n u(-n-1) \longleftrightarrow \frac{z}{z-a}$$

ROC : $|z| < |a|$

$0 < a < 1$



Therefore.

$$x(n) = a^n u(n) \xleftrightarrow{z} \frac{z}{z-a}, \quad \text{ROC } |z| > |a|$$

$$x(n) = -a^n u(-n-1) \xleftrightarrow{z} \frac{z}{z-a} \quad \text{ROC } \underbrace{|z| < |a|}$$

Same $X(z)$

Different ROCs.

$\Rightarrow$ To fully characterise the signal, one has to also specify ROC.

z <u>transform</u>.

unit <u>impulse</u>

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$$= z^{-0} = 1$$

$$\boxed{\delta(n) \longleftrightarrow 1} \rightarrow \text{ROC } 0 < |z| < \infty$$

<u>Unit step sequence</u>

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= \frac{1}{1 - z^{-1}}, \quad \underbrace{|z| > 1}_{\text{RO}}$$

$$\boxed{u(n) \longleftrightarrow \frac{1}{1 - z^{-1}}} \quad \text{ROC } |z| > 1$$

Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}.$$

$$x[n] = \cos(\Omega_0 n)\, u(n)$$

$$= \frac{1}{2}\left[e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right] u[n]$$

$$X(z) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right) u(n)\, z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} e^{j\Omega_0 n} z^{-n} + \frac{1}{2} \sum_{n=0}^{\infty} e^{-j\Omega_0 n} z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(e^{j\Omega_0} z^{-1}\right)^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(e^{-j\Omega_0} z^{-1}\right)^n$$

$$= \frac{1}{2} \frac{1}{1 - e^{j\Omega_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - e^{-j\Omega_0} z^{-1}}$$

$$= \frac{1}{2} \frac{2 - 2\cos\Omega_0 z^{-1}}{1 - 2\cos\Omega_0 z^{-1} + z^{-2}}$$

$$= \frac{1 - \cos\Omega_0 z^{-1}}{1 - 2\cos\Omega_0 z^{-1} + z^{-2}}$$

$$X(z) = \frac{z^2 - z\cos\Omega_0}{z^2 - 2z\cos\Omega_0 + 1}$$

$$P_1 = e^{j\Omega_0}, \quad P_2 = e^{-j\Omega_0}$$

$\cos(\Omega_0 n)$ ROC $|z| > |e^{j\Omega_0}|, |e^{-j\Omega_0}|$

$$\boxed{|z| > 1}$$

Properties of z-transform

Linearity

$x_1(n) \longleftrightarrow X_1(z)$, ROC $= R_1$

$x_2(n) \longleftrightarrow X_2(z)$;

ROC : R_2

$\Rightarrow a_1 x_1(n) + a_2 x_2(n)$

$\longleftrightarrow a_1 X_1(z) + a_2 X_2(z)$

ROC : $R_1 \cap R_2$

<u>TIME SHIFTING</u>

$x[n] \longleftrightarrow X(z)$

$x(n-n_0) \longleftrightarrow ?$

$\tilde{x}(n) \longleftrightarrow \tilde{X}(z)$

$$\tilde{X}(z) = \sum_{n=-\infty}^{\infty} \tilde{x}(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n}$$

$m = n - n_0$

$\Rightarrow n = m + n_0$

$$= \sum_{-\infty}^{\infty} x(m) z^{-(m+n_0)}$$

$$= z^{-n_0} \underbrace{\sum_{-\infty}^{\infty} x(m) z^{-m}}_{X(z)}$$

$= z^{-n_0} X(z)$

$\tilde{X}(z) = X(z) z^{-n_0}$

$$\boxed{x(n-n_0) \longleftrightarrow z^{-n_0} X(z)}$$

Delayed by n_0

z-transform.

For $n_0 = 1$.

unit delay.

$x(n-1) \longleftrightarrow z^{-1} X(z)$

Unit Delay operator.

$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow x[n-1]$

Unit Delay Block.

Time Reversal

$\tilde{x}(n) = x(-n)$

ROC $= R$

$$\tilde{X}(z) = \sum_{n=-\infty}^{\infty} \tilde{x}(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$m = -n$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{m}$$

$$= \sum_{m=-\infty}^{\infty} x(m) (z^{-1})^{-m}$$

$$= X(1/z)$$

$$\boxed{x(-n) \longleftrightarrow X(1/z)}$$

ROC

: $1/z \in R$

$\Rightarrow$ ROC $= 1/R$

$\tilde{R} = \{ t \mid 1/t \in R \}$

except zero

Multiplication By n

$x(n) \longleftrightarrow X(z)$

ROC $= R$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} (-n) x(n) z^{-n-1}$$

$$= -z^{-1} \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$\Rightarrow -z\frac{dX(z)}{dz} = \underbrace{\sum_{n=-\infty}^{\infty} n\,x(n)\,z^{-n}}_{\text{z-Transform of } n x(n)}$$

$$\boxed{n\,x(n) \longleftrightarrow -z\frac{dX(z)}{dz}}$$

Multiplication in time by n

Differentiation of z-transform

Accumulation:

$$y(n) = \sum_{k=-\infty}^{n} x(k) \qquad \int_{-\infty}^{t} x(\tau)\,d\tau$$

Accumulator

$\equiv$ Integrator for continuous time.

$$y(n) - y(n-1) = \sum_{k=-\infty}^{n} x(k) - \sum_{k=-\infty}^{n-1} x(k)$$

$\downarrow$ z-Transform. $= x(n)$

$$Y(z) - z^{-1}Y(z) = X(z)$$

$$\Rightarrow Y(z)\left(1-z^{-1}\right) = X(z)$$

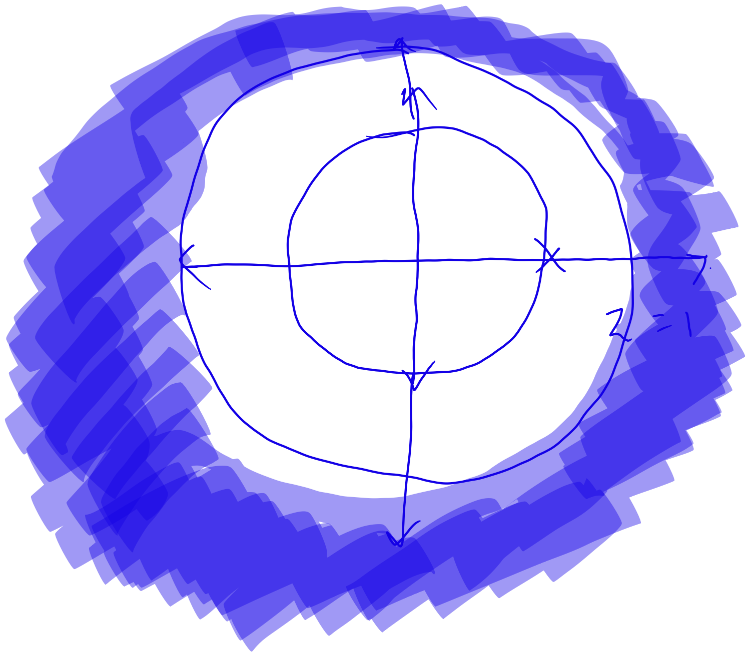
$$\Rightarrow \boxed{\frac{Y(z)}{X(z)} = \frac{1}{1-z^{-1}}}$$

$$Y(z) = \frac{z\,X(z)}{z-1}$$

adds poles at $z=1$.

$X(z)$: ROC $= R$

ROC : $R \cap |z|>1$



Convolution:

$x_1(n) \longrightarrow X_1(z)$

$x_2(n) \longrightarrow X_2(z)$

$y(n) = x_1(n) * x_2(n)$

$$y(n) = \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)$$

$\updownarrow$

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)}_{y(n)} z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x_1(m) \underline{\sum_{n=-\infty}^{\infty} x_2(n-m)\, z^{-n}}$$

interchanging order of sum.

z transform of $x_2(n)$ delayed by m.

$$= X_2(z) z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} x_1(m) X_2(z) z^{-m}$$

$$= X_2(z) \overbrace{\sum_{m=-\infty}^{\infty} x_1(m) z^{-m}}^{X_1(z)}$$

$= X_1(z)\, X_2(z)$

$$\boxed{Y(z) = X_1(z)\, X_2(z)}$$

convolution in time

= Multiplication in Z-transform Domain

Inverse Z-transform.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}.$$

express $X(z)$ as a power series.

$x(n)$ = coefficient of z^{-n}.

Partial fraction expansion

$$X(z) = \frac{N(z)}{D(z)}$$ Rational function of z.

$$= k \cdot \frac{(z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

$$\frac{z_1, z_2, \ldots, z_m.}{\text{zeros} = m}$$

$$\frac{p_1, p_2, \cdots p_n}{\text{poles} = n}$$

$n \geq m$.

Assume all poles are simple.

$\Rightarrow$ No repeated poles.

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \cdots + \frac{c_n}{z-p_n}$$

$$= \frac{c_0}{z} + \sum_{k=1}^{n} \frac{c_k}{(z-p_k)}$$

$$c_0 = X(z)\Big|_{z=0}$$

$$c_k = (z-p_k)\, X(z)\Big|_{z=p_k}$$

$$X(z) = c_0 + \frac{c_1 z}{z-p_1} + \frac{c_2 z}{z-p_2} + \cdots + \frac{c_n z}{(z-p_n)}$$

Compute z-transforms of individual terms.

Sum together

If $X(z)$ has multiple poles with multiplicity > 1

eg p_i of multiplicity $= r$.

Then $\frac{X(z)}{z}$ will have terms of the form

$$\frac{\lambda_1}{z-p_i} + \frac{\lambda_2}{(z-p_i)^2} + \cdots + \frac{\lambda_r}{(z-p_i)^r}.$$

$$\boxed{\lambda_{r-k} = \frac{1}{k!} \frac{d^k}{dz^k} (z-p_i)^r \left.\frac{X(z)}{z}\right|_{z=p_i}}$$

Properties of ROC: $\rightarrow$ Region of convergence.

1. ROC does not contain any pole
2. If $x(n) =$ Finite sequence $\Rightarrow x(n) = 0$ for $n < N_1$ or $n > N_2$

and $X(z)$ converges for some value of z.

$\Rightarrow$ ROC $\rightarrow$ entire z plane

③ For a right handed signal

$\Rightarrow x(n) = 0$ for $n \le N_1$

ROC: $|z| > r_{max}$.

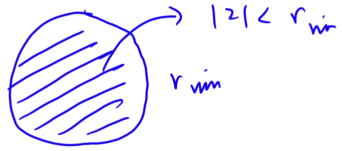
r_{max} = largest magnitude of poles of $x(z)$

④ For a left handed signal

$x(n) = 0$ for $n \geq N_2$.

ROC: $|z| < r_{min}$

r_{min} = minimum of magnitude of poles of $x(z)$



⑤ $x(n)$ = two sided signal

$\Rightarrow$ Infinite duration

$-\infty$ to ∞.

$\Rightarrow$ ROC : $r_1 < |z| < r_2$

r_1, r_2 = magnitudes of 2 poles of $x(z)$.

System function of Discrete time LTI systems.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

impulse response.

LTI system.

$y(n) = x(n) * h(n)$.

From properties of z-transform

$Y(z) = X(z) H(z)$

$H(z) = \dfrac{Y(z)}{X(z)}$

z-transform of $h(n)$

system function or Transfer function.

Properties of LTI systems.

Causality

$h(n) = 0 \quad \text{for } n < 0$

$\Rightarrow h(n) =$ Right handed signal.

$\Rightarrow$ ROC of form

$|z| > r_{max}$.

largest magnitude of poles of $H(z)$.

Stability

$\hookrightarrow$ BIBO stability

= Bounded input, bounded o/p stability

$$\underbrace{\sum_n |h(n)| < \infty}_{\text{finite}}$$

$\Rightarrow$ for system to be BIBO stable $\Rightarrow H(z)$ must include UNIT CIRCLE in ROC.

$z = e^{j\Omega}$, $|z| = 1$.

Proof: $H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\Omega n}$

$$|H(e^{j\Omega})| = \left| \sum_{n=-\infty}^{\infty} h(n) e^{-j\Omega n} \right|$$

$$\le \sum_{n=-\infty}^{\infty} |h(n)| \underbrace{|e^{-j\Omega n}|}_{=1}$$

$$= \sum_{n=\infty}^{\infty} |h(n)|$$

$$< \infty$$

Finite quantity

$|H(e^{j\Omega})| \le$ Finite

$|H(e^{j\Omega})| =$ Finite Qty

$\Rightarrow z = e^{j\Omega}$ in ROC

$\forall \Omega$.

$\Rightarrow e^{j\Omega} \in$ ROC $\forall \Omega$

unit circle $\in$ ROC

System function for LTI system described by a difference equation.

Input $x(n)$

output $y(n)$.

$$\sum_{k=0}^{N} a_k \, y(n-k) = \sum_{k=0}^{M} b_k \, x(n-k)$$

Difference equation.

$\rightarrow$ Taking Z-transform on Both sides.

$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\Rightarrow Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{M} a_k z^{-k}}$$

transfer

$\rightarrow$ Rational fxn.

Example problems - Z transform.

1. consider $x(n) = a^n \quad 0 \le n \le N-1$

0, otherwise.

find $X(z)$
poles/zeros.

$$X(n) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{N-1} a^n z^{-n}$$

$$= \sum_{n=0}^{N-1} (a z^{-1})^n.$$

$$\boxed{X(z) = \frac{1-(az^{-1})^N}{1-az^{-1}}}$$

$$X(z) = \frac{z^N - a^N}{z-a} \cdot \frac{1}{z^{N-1}}.$$

Poles $z = a$.

$z = 0$ Pole of multiplicity $N-1$.

zeros : $z^N = a^N$

$= a^N \cdot 1$

$$z = a\, e^{j\frac{2\pi k}{N}}.$$

$e^{j2\pi k/N}$ = one of roots of unity

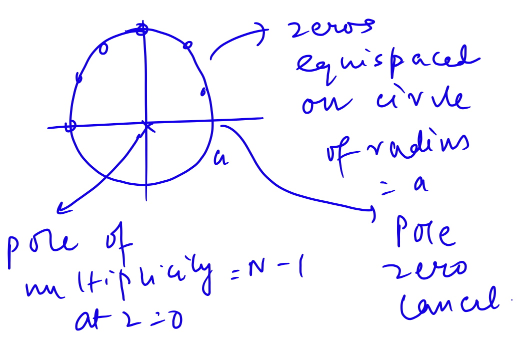
solution of $x^N = 1$.

zeros. $a\, e^{j\frac{2\pi k}{N}}$

$k = 0, 1, \dots, N-1$

For $k=0$; zero at $z=a$.

pole, zero at $z=a$ cancel.



zeros: $e^{j2\pi k/N}$

$k = 1, 2, \dots, N-1$

Poles: at 0.

multiplicity $N-1$

(2) $x(n) = \left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{2}\right)^n u(-n-1)$

z-transform ROC.

Pole zero plot

Results

$a^n u(n) \longleftrightarrow \frac{z}{z-a}$, ROC $|z| > |a|$

$-a^n u(-n-1) \longleftrightarrow \frac{z}{z-a}$ ROC $|z| < |a|$

$\left(\frac{1}{3}\right)^n u(n) \longleftrightarrow \frac{z}{z-1/3}$ ROC $|z| > 1/3$.

$\left(\frac{1}{2}\right)^n u(-n-1) \longleftrightarrow \frac{-z}{z-1/2}$.

ROC $|z| < 1/2$

$X(z) = \frac{z}{z-1/3} - \frac{z}{z-1/2}$

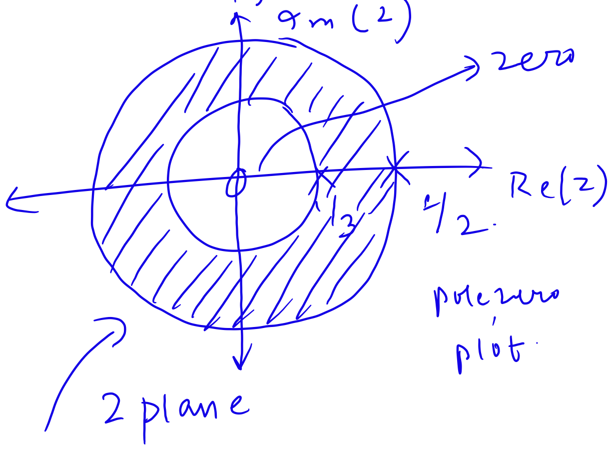
ROC $|z| > 1/3 \cap |z| < 1/2$

$= \frac{1}{3} < |z| < 1/2$

$= \frac{z \cdot (-1/6)}{(z-1/3)(z-1/2)}$

$X(z) = -1/6 \frac{z}{(z-1/3)(z-1/2)}$

ROC $1/3 < |z| < 1/2$



poles $= 1/3, 1/2$, zero

(3) z transform of

$$x(n) = a^{n+2} u(n+2)$$

$$\tilde{x}(n) = a^n u(n)$$

$$x(n) = \tilde{x}(n+2)$$

↳ time advanced version of $\tilde{x}(n)$

$$\tilde{X}(z) = \frac{z}{z-a}$$

$$X(z) = z^2 \tilde{X}(z) = \frac{z^3}{z-a}$$

ROC $|z| > |a|$

<u>Example problems z-transform:</u>

(4) Power series to find inverse z transform of $\frac{1}{1-az^{-1}}$, $\underbrace{|z| > |a|}_{\text{ROC}}$

Right handed signal

$$\begin{array}{r|l} & 1 + az^{-1} + a^2 z^{-2} \\ \hline 1 - az^{-1} & 1 \\ & 1 - az^{-1} \\ \hline & az^{-1} \\ & az^{-1} - a^2 z^{-2} \\ \hline & a^2 z^{-2} \\ & a^2 z^{-2} - a^3 z^{-3} \\ \hline & a^3 z^{-3} \end{array}$$

$$X(z) = \frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$$

Power series in z.

Follows from power series X(z)

Inverse z transform follows from

Power series $X(z)$

$$\boxed{x(n) = a^n u(n)}$$

(5) evaluate Inverse z-transform of

$$\frac{3z^2 - 5z}{z^2 - 3z + 2} = X(z)$$

ROC $\underbrace{|z| < 1}$

Left handed sequence

Rational function of z.

$z^2 - 3z + 2 = (z-1)(z-2)$

poles $z = 1, 2$.

$$X(z) = \frac{3z^2 - 5z}{(z-1)(z-2)}$$

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-1} + \frac{c_2}{z-2}$$

Partial fraction expansion

$3z^2 - 5z = 0$

$z(3z - 5) = 0$, $\underline{z = 0, z = 5/3}$.

zeros.

$$c_0 = X(z)\Big|_{z=0}$$

$$= \frac{3z^2 - 5z}{(z-1)(z-2)}\Big|_{z=0}$$

$c_0 = 0$

$$c_1 = (z-1)\frac{X(z)}{z}\Big|_{z=1}$$

$$= \frac{(3z-5)}{z-2}\Big|_{z=1}$$

$$= \frac{-2}{-1} = 2$$

$$c_2 = \frac{(z-2)X(z)}{z}\Big|_{z=2}$$

$$= \frac{3z-5}{z-1}\Big|_{z=2}$$

$$= 1/1 = 1$$

$$\boxed{c_0 = 0,\ c_1 = 2,\ c_2 = 1}$$

↳ coefficients in PF expansion of $X(z)/z$

$$\therefore \frac{X(z)}{z} = \frac{2}{z-1} + \frac{1}{z-2}$$

$$X(z) = \frac{2z}{z-1} + \frac{z}{z-2} \qquad \text{ROC } |z| < 1$$

$$\frac{z}{z-a} \longleftrightarrow -a^n u(-n-1) \qquad |z| < a.$$

$$\boxed{x(n) = -2u(-n-1) - 2^n u(-n-1)}$$

↳ Inverse z transform.

(6) Inverse z transform of

$$X(z) = \frac{z(-z^2+5z-5)}{(z-2)(z-3)^2}$$

$$\text{ROC } \underline{2 < |z| < 3}.$$

infinite signal. from $-\infty$ to ∞

Poles

$z = 2, 3$ ↳ multiplicity $= 2$

PF expansion of $\frac{X(z)}{z}$

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-2} + \frac{d_1}{z-3} + \frac{d_2}{(z-3)^2}$$

$$c_0 = X(z)\Big|_{z=0}$$

$$\boxed{c_0 = 0}$$

$$c_1 = (z-2)\frac{X(z)}{z}\Big|_{z=2}$$

$$= \frac{-z^2+5z-5}{(z-3)^2}\Big|_{z=2}$$

$$= \frac{-4+10-5}{1} = \frac{1}{1} = 1$$

$$\boxed{c_1 = 1}$$

$$d_2 = (z-3)^2 \frac{X(z)}{z}\Big|_{z=3}$$

$$= \left. \frac{-z^2+5z-5}{z-2} \right|_{z=3}$$

$$= \frac{-9+15-5}{1}$$

$$= 1$$

$$\boxed{A_2 = 1}$$

$r=2$

$$A_1 = A_{2-1} = \frac{1}{1!} \frac{d}{dz} \left. (z-3)^2 \frac{X(z)}{z} \right|_{z=3}$$

$$(z-3)^2 \frac{X(z)}{z}$$

$$= \frac{-z^2+5z-5}{z-2}$$

$$= \frac{-(z-2)^2+z-1}{z-2}$$

$$= -(z-2) + \frac{z-2+1}{z-2}$$

$$= -(z-2) + 1 + \frac{1}{z-2}$$

$$= -z+3+\frac{1}{z-2}$$

$$(z-3)^2 \frac{X(z)}{z}$$

$$\frac{d}{dz} (z-3)^2 \frac{X(z)}{z} = -1 - \frac{1}{(z-2)^2}$$

$$\left. \frac{d}{dz} (z-3)^2 \frac{X(z)}{z} \right|_{z=3}$$

$$= -1 - \frac{1}{1}$$

$$= -2$$

$$\frac{X(z)}{z} = \frac{1}{z-2} - \frac{2}{z-3} + \frac{1}{(z-3)^2}$$

$$X(z) = \frac{z}{z-2} - \frac{2z}{z-3} + \frac{z}{(z-3)^2}$$

ROC: $2 < |z| < 3$.