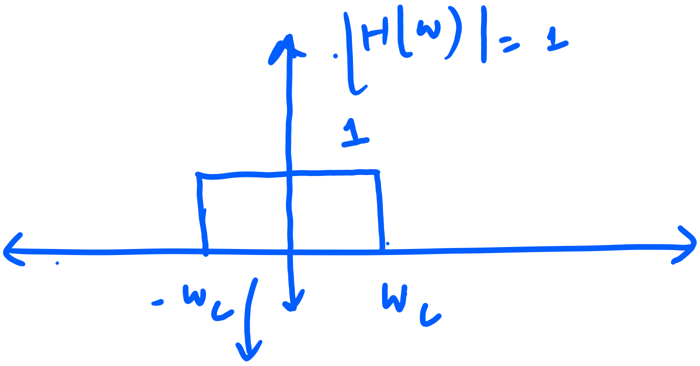
week 9.

1-50. F.T examples:

filtering:

#16. Consider ideal lowpass filter (LPF)



ideal LPF response

$$H(w) = \begin{cases} 1, & |w| \le w_c \\ 0, & \text{otherwise} \end{cases}$$

Input $\Rightarrow x(t) = e^{-2t} u(t)$

Find w_c such that, filter passes exactly $\frac{1}{2}$ of energy of input signal

$$x(t) = e^{-2t} u(t)$$

$$\Rightarrow X(w) = \frac{1}{jw + 2}$$

$$Y(w) = X(w) H(w)$$

$$= \begin{cases} \frac{1}{jw+2}, & |w| \le w_c \\ 0, & \text{otherwise} \end{cases}$$

Energy of signal

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_0^{\infty} e^{-4t}\, dt$$

$$= -\frac{1}{4} e^{-4t} \Big|_0^{\infty}$$

$$= \frac{1}{4} \rightarrow \text{Energy of input signal.}$$

ESD (Energy spectral density of output).

$$\underbrace{|Y(w)|^2}_{\text{ESD of output signal}} = \begin{cases} \frac{1}{4+w^2}, & |w| \leq w_c \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{Energy of output} = \int_{-\infty}^{\infty} |Y(w)|^2\, dw.$$

$$= \frac{1}{2\pi} \int_{-w_c}^{w_c} \frac{1}{4+w^2}\, dw$$

$$= \frac{1}{2\pi} \int_{-w_c}^{w_c} \frac{1}{4} \left(\frac{1}{1+\frac{w^2}{4}} \right) dw.$$

$$= \frac{1}{2\pi} \cdot \frac{1}{4} \tan^{-1}\left(w/2 \right) \Big|_{-w_c}^{w_c} \times 2$$

$$= \frac{1}{4\pi} \times 2 \tan^{-1} w/2$$

$$\boxed{\underset{\text{Energy of the output.}}{\mathcal{E}_y} = 1/2\pi \tan^{-1} \frac{w_c}{2}}$$

$$\mathcal{E}_y = \frac{1}{2} \mathcal{E}_x$$

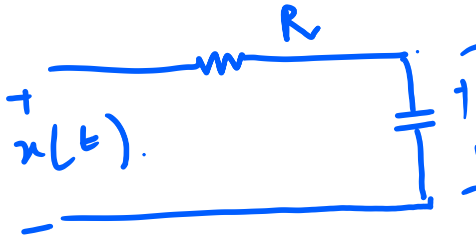
$$\Rightarrow \frac{1}{2\pi} \tan^{-1} \frac{w_c}{2} = \frac{1}{2} \cdot 1/4$$

$$\Rightarrow \tan^{-1}\frac{w_c}{2} = \pi/4$$

$$\Rightarrow \frac{w_c}{2} = 1$$

$$\Rightarrow \boxed{w_c = 2 \text{ rad/sec.}} \rightarrow \text{Filter.}$$

(17) RC filter (Rise time).



→ serial RC circuit

→ y(t).

a) Find unit step response.
b) Characterize Rise time in terms of 3dB frequency.
↳ Time taken to go from 10% to 90% of its final value

$$x(t) = y(t) + RC\frac{dy(t)}{dt}$$ → Differential equation for input/output relation of RC circuit

FT

$$\Rightarrow X(w) = Y(w) + RC\, jw\, Y(w)$$

$$\Rightarrow \frac{Y(w)}{X(w)} = \frac{1}{1 + jwRC} = H(w)$$
= Transfer function.

Unit step response:

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$U(w) = \pi\delta(w) + \frac{1}{jw}$$

$$Y(w) = U(w)\,H(w)$$

$$= \left(\pi\delta(w) + \frac{1}{jw}\right)\frac{1}{(1+jwRC)}$$

$$= \pi\delta(w)\,\frac{1}{1+jwRC} + \frac{1}{jw}\,\frac{1}{(1+jwRC)}$$

$$= \pi\delta(w)\,\frac{1}{1+jwRC}\Big|_{w=0} = \pi\delta(w)$$

$$= \pi\delta(w) + \frac{1}{jw}\,\frac{1}{1+jwRC}$$

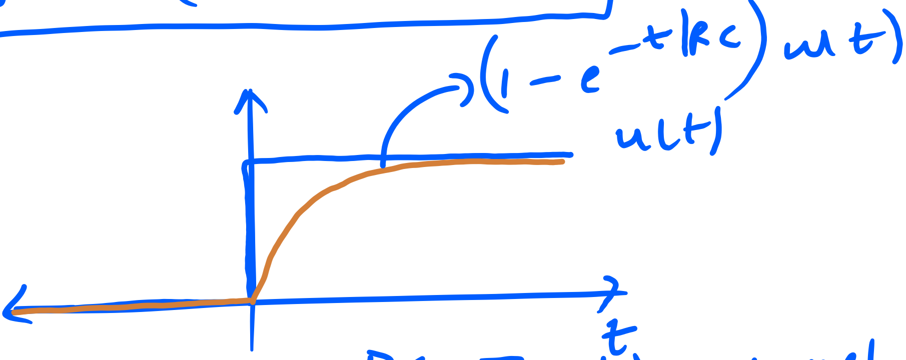
P.F expansion

$$= \pi\delta(w) + \frac{1}{jw} - \frac{RC}{jw+RC}$$

$$= \pi\delta(w) + \underbrace{\frac{1}{jw}}_{u(t)} - \underbrace{\frac{1}{\frac{1}{RC}+jw}}_{e^{-t/RC}\,u(t)}$$

$$y(t) \overset{IFT}{=} u(t) - e^{-t/RC}\,u(t)$$

$$\boxed{y(t) = \left(1 - e^{-t/RC}\right)u(t)}$$



$RC = \tau$ = time constant

small $\tau \Rightarrow$ fast rise large $\tau \Rightarrow$ slow rise.

1. Find Rise time
2. Relation to $f_{3dB} \rightarrow$ = 3 dB frequency.

lec-51: FT Problems:

Unit step response RC circuit.

$$y(t) = u(t)(1 - e^{-t/RC})$$

$t \to \infty$

$y(\infty) = 1 \leftarrow$ Final value = 1

$y(0) = 1 \times (1-1) = 0$

10% of Final value $= 0.1\, y(\infty)$
$= 0.1$

$$1 - e^{-t_{10}/RC} = 0.1$$

$$\Rightarrow e^{-t_{10}/RC} = 0.9$$

$$\Rightarrow \boxed{t_{10} = -RC \ln(0.9)}$$

$\hookrightarrow$ time taken for 10% of final value.

90% of final value $= 0.9 \times 1$
$= 0.9$

$$1 - e^{-t_{90}/RC} = 0.9$$

$$\Rightarrow e^{-t_{90}/RC} = 0.1$$

$$\Rightarrow \boxed{t_{90} = -RC \ln 0.1}$$

$\hookrightarrow$ time taken for 90% of final value

Rise time $= t_{90} - t_{10}$

$$= -RC\ln 0.1 + RC\ln 0.9$$

$$= RC\ln\left(\frac{0.9}{0.1}\right)$$

$$\boxed{\begin{aligned} T_{rise} &= RC\ln 9 \\ &= 2.1972\,RC \end{aligned}} \quad (1)$$

Rise time of RC

As RC increases Rise time increases.

Relate rise time to 3dB BW.

$$H(w) = \frac{1}{1 + jwRC}$$

$$= \frac{1}{1 + j\,w/w_0}$$

$$w_0 = 1/RC$$

3dB bandwidth.

$$H(w_0) = \frac{1}{1+j}$$

$$|H(w_0)| = \frac{1}{\sqrt{2}}$$

$$\Rightarrow |H(w_0)|^2 = 1/2$$

output power suppressed by 1/2.

= -3dB reduction

Therefore $w_0 = 1/RC$

= 3dB bandwidth.

$$T_{rise} = 2.1972\,RC$$

$$\omega_{3dB} = \frac{1}{RC}$$

$$f_{3dB} = \frac{\omega_{3dB}}{2\pi}$$

$$= \frac{1}{2\pi RC}$$

$$\boxed{RC = \frac{1}{2\pi f_{3dB}}} \quad \text{— (2)}$$

Substituting RC from (2) in (1)

$$\Rightarrow T_{rise} = 2.1972 \frac{1}{2\pi f_{3dB}}.$$

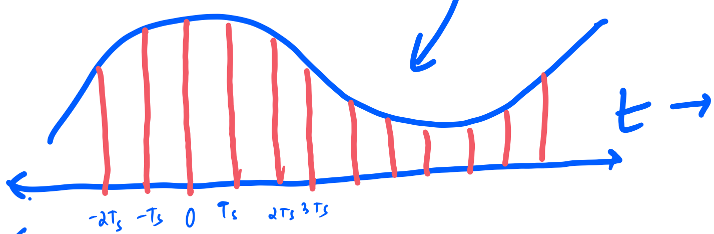
$$\boxed{T_{rise} = \frac{0.35}{f_{3dB}}.}$$

$$\rightarrow T_{rise} \propto \frac{1}{f_{3dB}}.$$

Rise time inversely proportional to 3dB frequency.

(18) Sampling of continuous signal

continuous time signal.



Equispaced Discrete time instants

1st step to for analog to Digital

Sampling → C.T ⟶ D.T

continuous time — Discrete time

T_s = sampling interval

$$w_s = \frac{2\pi}{T_s} = \text{sampling frequency}$$

$$F_s = 1/T_s$$

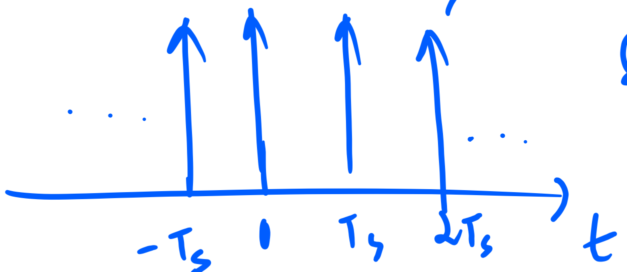
Impulse train sampling

→ popular technique for sampling

Impulse train

$$\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

Impulses at each multiple of T_s



Impulse train $\delta_{T_s}(t)$

CT signal

$$\underbrace{x(t) \times \delta_{T_s}(t)}_{\text{Impulse train sampling.}}$$

$$= x(t) \times \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

$$= \sum_{k=-\infty}^{\infty} x(t)\, \delta(t - kT_s)$$

spectrum of sampled signal

$$\underbrace{x_{T_s}(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\, \delta(t - kT_s)}_{\text{sampled signal}}$$

sampled value of signal at kT_s

$x(t) \longleftrightarrow X(w).$

$x_{T_s}(t) \longleftrightarrow ?$

$$\delta_{T_s} = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

Periodic signal

Period $= T_s$.

Fundamental freq $= 2\pi / T_s$

$= w_s$.

Therefore, we can derive the C.E.F.S.

$$\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_s t}$$

↳ CEFS of impulse train

L-52.

* Sampling:

$$c_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jkw_s t} dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jkw_s t} dt$$

$$= \frac{1}{T_s} e^{-jkw_s t} \Big|_{t=0}$$

$$\boxed{c_k = 1/T_s}$$

for all k.

$$\delta_{T_s}(t) = \underbrace{\frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jkw_s t}}_{\text{CEFS of impulse train}}$$

$$e^{jk\omega_s t} \longleftrightarrow 2\pi\,\delta(\omega - k\omega_s)$$

$$\delta_{T_s} = \frac{1}{T_s}\, 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$\underbrace{\delta_{T_s} = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)}_{\text{FT of impulse train}}$$

Spectrum of sampled signal

$$x_1(t) \cdot x_2(t) \longleftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

$$x_s(t) = x(t) \times \delta_{T_s}(t)$$

sampled signal $\longleftrightarrow \frac{1}{2\pi} X(\omega) * \delta_{T_s}(\omega)$

$$= \frac{1}{2\pi} X(\omega) * \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

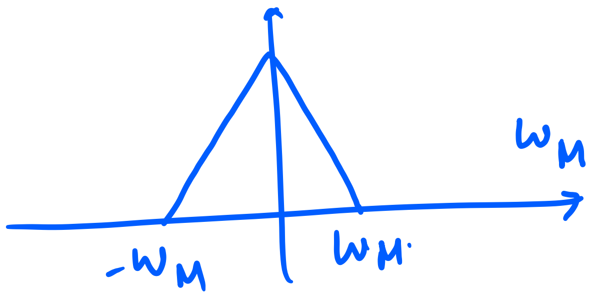
$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega) * \delta(\omega - k\omega_s)$$

$$\boxed{X_{T_s}(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)}$$

spectrum of sampled signal.

shifted & added spectra of original signal $x(t)$

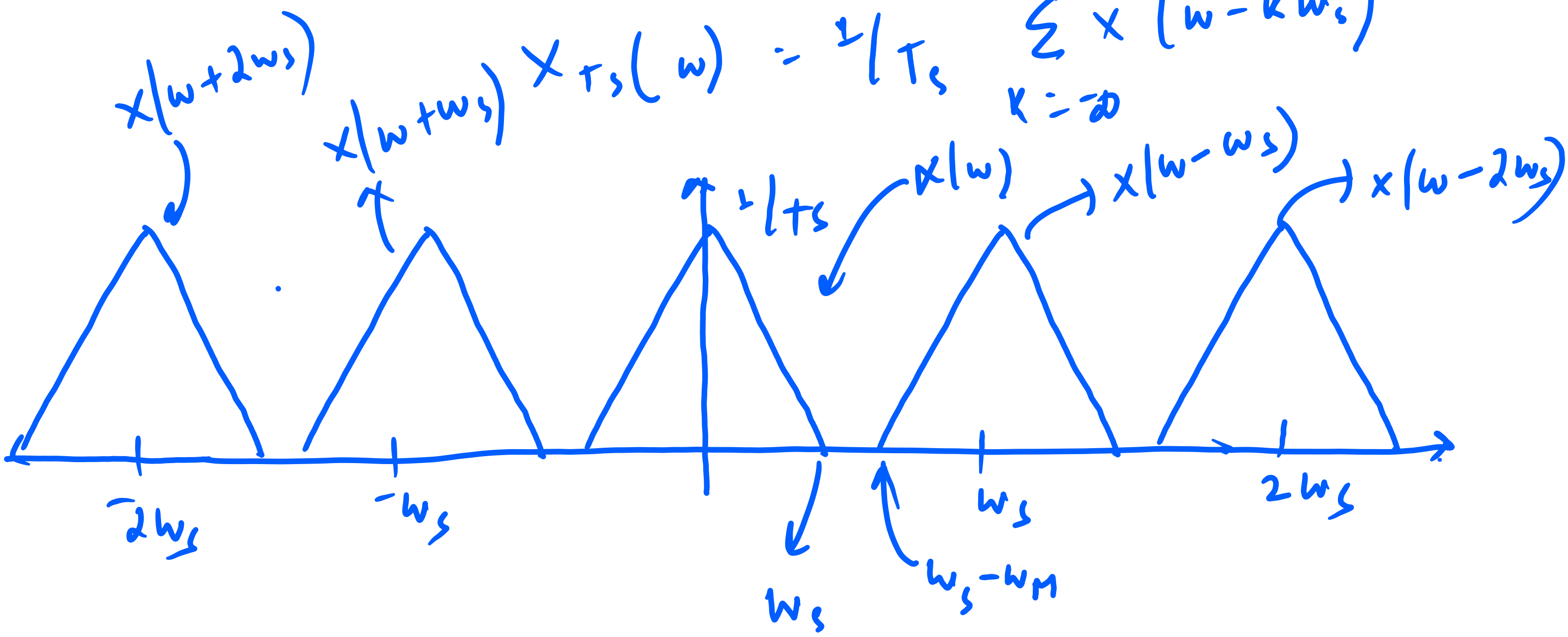
Consider $X(w)$



w_M: max freq

$X(w) = 0$ for $|w| \geq w_M$

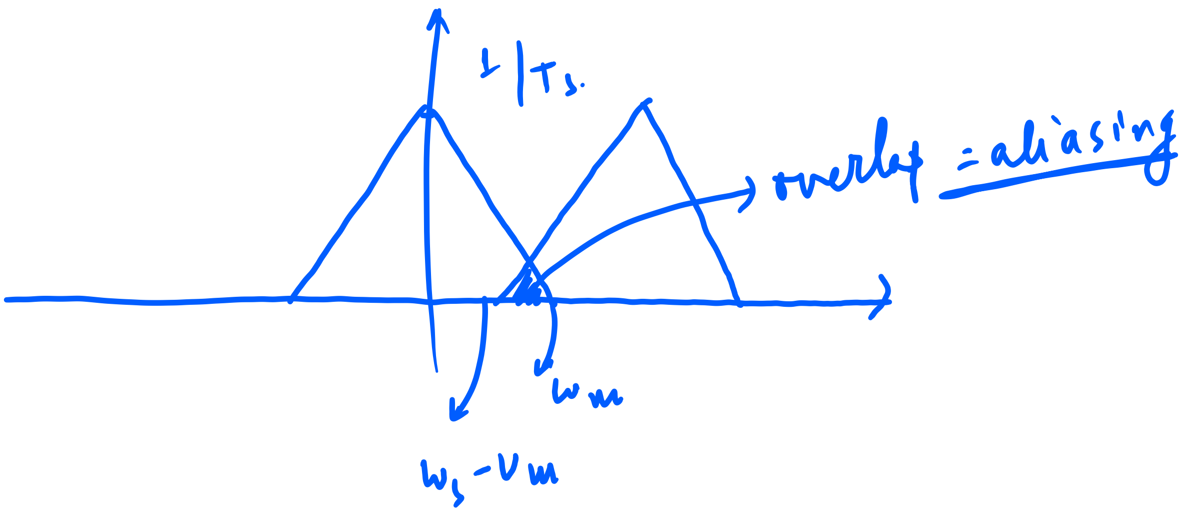
$$X_{T_s}(w) = 1/T_s \sum_{k=-\infty}^{\infty} X(w - k w_s)$$



$w_s - w_M \geq w_m$

$\boxed{w_s \geq 2w_m}$

No interference or distortion from spectrum overlap.



$w_s - w_m < w_m$
$\Rightarrow w_s < 2w_m$

$\Rightarrow$ overlap
= ALIASING
= Distortion.

$\boxed{w_s \geq 2w_m}$ = Nyquist rate

for no aliasing
NYQUIST SAMPLING Theorem

Nyquist Criterion.

Sampling freq $\geq$ 2 x Maximum frequency

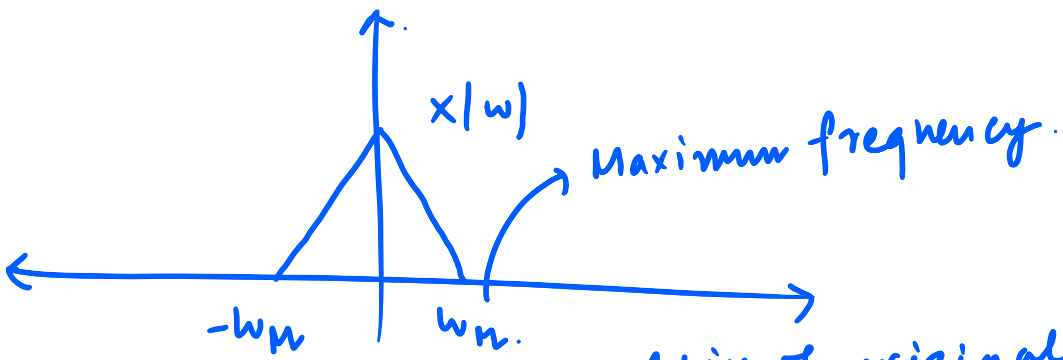
L:53. Sampling:

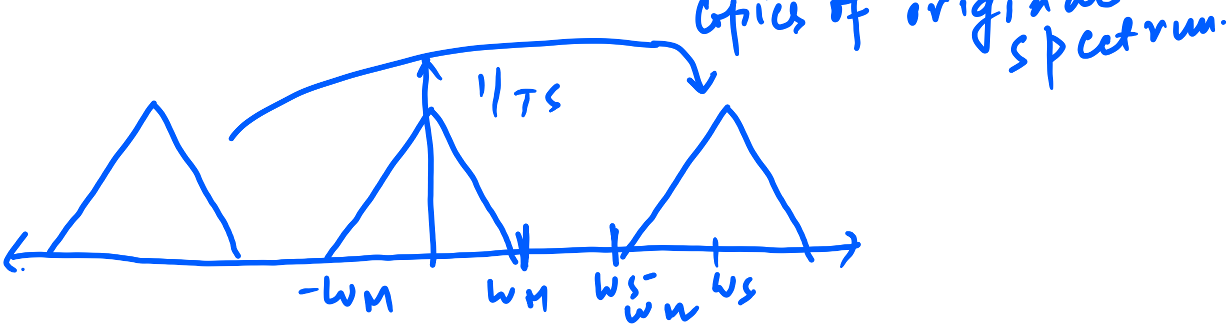
$x(t) \longleftrightarrow X(w)$

Spectrum of original signal

$$X_{T_s}(w) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(w - k w_s)$$

shifted by $k w_s$ and adding for all shifts. $k w_s$.





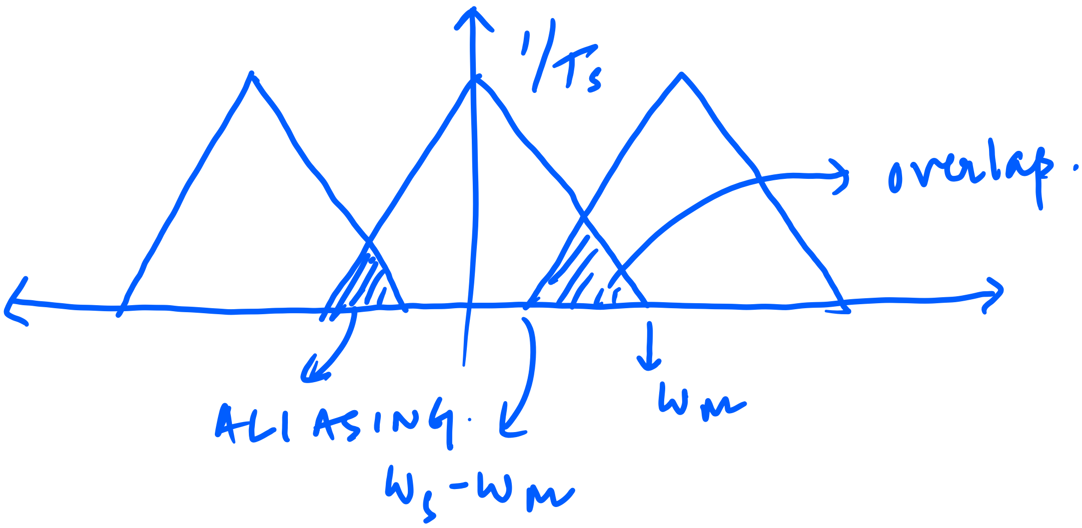
$\omega_s - \omega_M \geq \omega_M$

$\Rightarrow$ There is a guard band between copies

$\Rightarrow$ No distortion

ALIASING.

$\Rightarrow \boxed{\omega_s \geq 2\omega_m} \rightarrow$ Nyquist criteria



$\omega_s - \omega_M < \omega_m$

$\Rightarrow \boxed{\omega_s < 2\omega_m}$

$\Rightarrow$ aliasing.

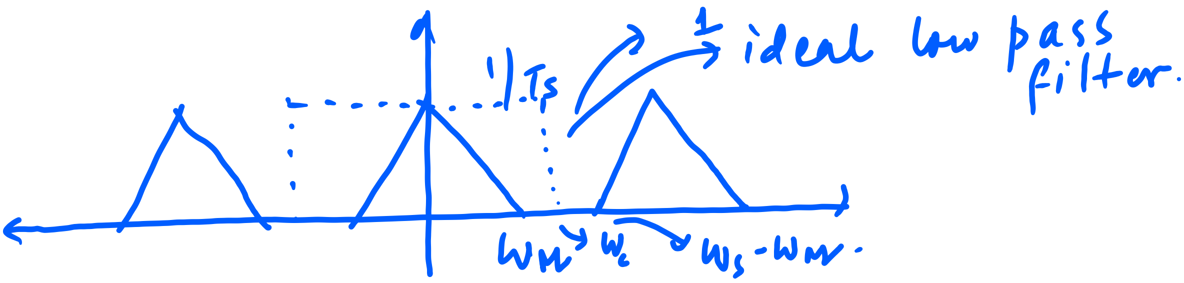
= DISTORTION.

Reconstruction:

Retrieve original signal from sampled signal.

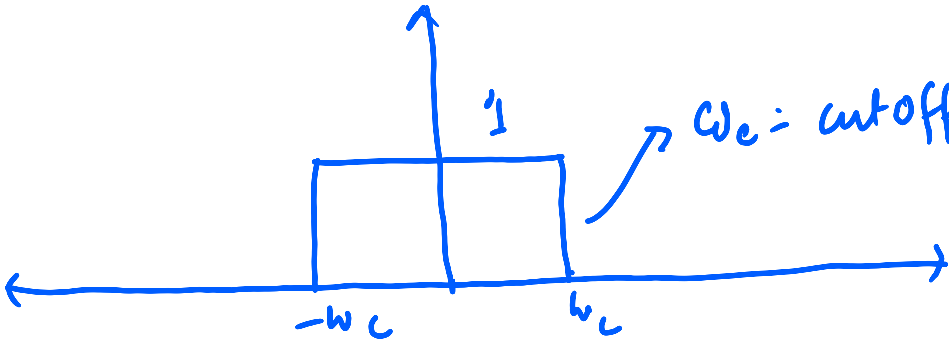
$\underbrace{\hspace{10cm}}$

Possible when there is no aliasing.



ω_c : cutoff

$$\underbrace{\omega_M \le \omega_c \le \omega_s - \omega_M}_{\text{Possible when } \omega_s \geq 2\omega_M}$$

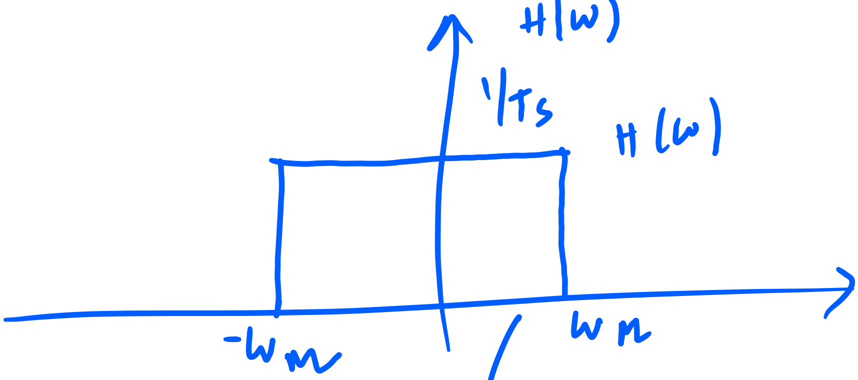


ω_c : cutoff freq.

Pass sampled signal $x_s(t)$ through ideal LPF

cutoff $= \omega_c$

$\omega_M \le \omega_c \le \omega_s - \omega_M$



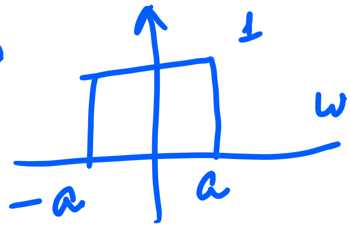
$$\underline{\text{choose } \omega_c = \omega_M.}$$

cut off freq = max freq of signal.

$$H(\omega) = \begin{cases} 1 & |\omega| \le \omega_M \\ 0, & \text{otherwise} \end{cases}$$

$h(t) = ?$

$P_{2a}(\omega) \longleftrightarrow \dfrac{\sin at}{\pi t}$



$$\Rightarrow P_{2w_M}(w) \longleftrightarrow \frac{\sin w_M t}{\pi t}$$

$$\Rightarrow \underbrace{T_s \, P_{2w_M}(w)}_{H(w)} \longleftrightarrow T_s \frac{\sin(w_m t)}{\pi t}$$

Impulse response of ideal LPF

pass $x_s(t)$ through ideal LPF

$$x_R(t) = x_s(t) * h(t)$$

Reconstructed signal

$$= \sum_{k=-\infty}^{\infty} x(kT_s)\,\delta(t-kT_s) * \frac{\sin w_M t}{\pi t} T_s$$

$$w_s = 2w_M$$

$$\Rightarrow 2\pi / T_s = 2w_M$$

$$\Rightarrow w_M = \pi / T_s$$

$$x_R(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\,\delta(t-kT_s) * \frac{\sin(w_M t)}{w_M t}$$

Interpolation using sinc in time

$$\underbrace{x_R(t) = \sum_{k=-\infty}^{\infty} x(kT_s) \frac{\sin(w_M(t-kT_s))}{w_M(t-kT_s)}}_{\text{Reconstructed signal}}$$

$x_R(t) = x(t)$ only if NO ALIASING

$\Rightarrow \omega_s \geq 2\omega_m$

$$\underbrace{= \sum_{k=-\infty}^{\infty} x(kT_s) \frac{\sin(\omega_M t - k\pi)}{\omega_M t - k\pi}}_{\text{Reconstructed signal from sampled signal}}$$

L-54. Fourier analysis discrete time signals and systems.

Discrete Fourier Series.

Discrete time periodic signals:

$$\underbrace{x(n+N_0) = x(n)}, \text{ for all } n.$$

↑ Discrete time periodic signal = Periodic with period $= N_0$

eg $x(n) = e^{j\Omega_0 n} \rightarrow$ Periodic signal

$\Omega_0 = 2\pi/N_0$ period $= N_0$.

$= e^{j2\pi/N_0 \, n}$

$$x(n + kN_0) = e^{j2\pi/N_0 (n + kN_0)}$$

$$= e^{j2\pi n/N_0} \underbrace{e^{j2\pi k}}_{1}$$

$$= e^{j2\pi/N_0 \, n}$$

$$= e^{j\Omega_0 n}$$

$$= x(n)$$

$$\boxed{x(n + kN_0) = x(n)}$$

Discrete Fourier Series Representation :

$x(n) \rightarrow$ Fundamental Period $= N_0$

$$x(n) = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n}$$

$c_k \rightarrow$ coefficients of discrete Fourier series.

Frequencies $= k\Omega_0$, $k = 0, 1, \ldots, N_0-1$

$= 0, 2\pi/N_0, 4\pi/N_0, \ldots, \frac{2\pi}{N_0}(N_0-1)$

$$\underbrace{0, 2\pi/N_0, 4\pi/N_0, \ldots, \frac{2\pi}{N_0}(N_0-1)}_{\text{Finite set of Frequencies.}}$$

$C_l = ?$

Consider

$$\sum_{n=0}^{N_0-1} x(n) e^{-jl\Omega_0 n}$$

$$= \sum_{n=0}^{N_0-1} \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n} e^{-jl\Omega_0 n}$$

$$= \sum_{k=0}^{N_0-1} c_k \underbrace{\sum_{n=0}^{N_0-1} e^{j\Omega_0(k-l)n}}_{}$$

$= 0$, for $k \neq l$

$$\frac{1 - e^{-j\Omega_0(k-l)N_0}}{1 - e^{-j\Omega_0(k-l)}}$$

$\left(1 - e^{-j(k-l)2\pi}\right) / 1 - e^{-j\Omega_0(k-l)}$

$= N_0 \quad , \text{if } k = l$

$= c_l N_0 .$

$$N_0 c_l = \sum_{n=0}^{N_0-1} x(n) e^{-jl\Omega_0 n}$$

$$\boxed{\Rightarrow c_l = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x(n) e^{-jl\Omega_0 n}}$$

$$= \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x(n) e^{-jl\Omega_0 n}$$

Finite sum

Summation can be over any contiguous N samples.

$l = 0$

$$c_0 = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x(n) = \text{mean of samples over period } N_0$$

DC coeff

Convergence of DFS:

Guaranteed convergence because finite sum.

Periodicity of DFS coefficients

$$c_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x(n) e^{-jk\Omega_0 n}$$

$$c_{k+N_0} = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x(n) e^{-j(k+N_0)\Omega_0 n}$$

$$= \frac{1}{N_0} \sum_{n=0}^{N_0-1} x(n) e^{-jk\Omega_0 n} \times \underbrace{e^{-j2\pi n}}_{1}$$

$$C_{k+N_0} = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x(n)\, e^{-jk\Omega_0 n}.$$

$$\therefore \boxed{C_{k+N_0} = C_k} \quad \forall k.$$

$\Rightarrow$ DFS coefficients are periodic

> Spectral coefficients of $x(n)$

L-55 Fourier Analysis :

D.T. signals :

Discrete Fourier Series

$$x(n) = \sum_{n=0}^{N_0-1} C_k e^{jk\Omega_0 n} \rightarrow \text{DFS}$$

$$C_k = 1/N_0 \sum_{n=0}^{N-1} x(n) e^{-jk\Omega_0 n} \longrightarrow \text{Inverse DFS}$$

Duality

$$x(n) = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n}$$

$C_k = c(n) = ?$ = Periodic in time with period N_0

$\hookrightarrow$ Periodic with period N_0

$$x(N_0 - n) = \sum_{k=0}^{N_0-1} C_k e^{jk\Omega_0 (N_0 - n)}$$

$$\Omega_0 = 2\pi / N_0$$

$$\Rightarrow \Omega_0 N_0 = 2\pi$$

$$= \sum_{k=0}^{N_0-1} C_k \underbrace{e^{jk2\pi}}_{=1} e^{-jk\Omega_0 n}.$$

$$x(N_0 - n) = \sum_{k=0}^{N_0-1} c_k e^{-jk\Omega_0 n}$$

$$c_k = c(n)$$

$$x(n) = x_k.$$

$$x_{N_0-k} = \sum_{n=0}^{N_0-1} c(n) e^{-jk\Omega_0 n}.$$

$$= \frac{1}{N_0} \sum_{n=0}^{N_0-1} \underbrace{N_0 c(n)}_{\tilde{c}(n)} e^{-jk\Omega_0 n}$$

Inverse DFS.

$$\tilde{c}(n) \longleftrightarrow x_{N_0-k}$$

$$= x_{-k}.$$

$$= \frac{1}{N_0} c(n)$$

$$\frac{1}{N_0} c(n) \longleftrightarrow x_{-k}.$$

$$\Rightarrow \boxed{c_k \overset{DFS}{\longleftrightarrow} N_0\, x(-n)}$$

= Duality property

$$\boxed{\underset{\underset{c_k}{\parallel}}{c(k)} \longleftrightarrow N_0 \underset{\underset{x(-n)}{\parallel}}{x_{-n}}}$$

<u>DFS of real signal</u>

$$x(n) = \text{real}$$

$$x^*(n) = x(n)$$

$$c_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x(n)e^{-jk\Omega_0 n}$$

$$c_{N_0-k} = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x(n)e^{-j(N_0-k)\Omega_0 n}$$

$$= \frac{1}{N_0}\sum_{n=0}^{N_0-1} x(n)\left(\underbrace{e^{-j2\pi n}}_{\to 1} \times e^{-j\Omega_0 n}\right)$$

$$c_{N_0-k} = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x(n)e^{-j\Omega_0 n}$$

$$\Rightarrow c^*_{N_0-k} = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x^*(n)e^{j\Omega_0 n}$$

$$= \underbrace{\frac{1}{N_0}\sum_{n=0}^{N_0-1} x(n)e^{j\Omega_0 n}}_{c_k}$$

$$\Rightarrow \boxed{c^*_{N_0-k} = c_k}$$

For real signal

c_{-k}

$$\Rightarrow \boxed{c^*_{-k} = c_k}$$

PARSEVAL'S THEOREM

$$\sum_{k=0}^{N_0-1} |c_k|^2 = \sum_{k=0}^{N_0-1}\left|\frac{1}{N_0}\sum_{n=0}^{N_0-1} x(n)e^{-jk\Omega_0 n}\right|^2$$

$$= \sum_{k=0}^{N_0-1}\frac{1}{N_0^2}\left(\sum_{n=0}^{N_0-1} x(n)e^{-jk\Omega_0 n}\right)$$

$$\times \left(\sum_{m=0}^{N_0-1} x(m) e^{-jk\Omega_0 m} \right)^*$$

$$= \frac{1}{N_0^2} \sum_{k=0}^{N_0-1} \sum_{n=0}^{N_0-1} \sum_{m=0}^{N_0-1} x(n) x^*(m) e^{jk(n-m)\Omega_0}$$

$$= \frac{1}{N_0} \sum_{n=0}^{N_0-1} \sum_{m=0}^{N_0-1} x(n) x^*(m) \underline{\sum_{k=0}^{N_0-1} e^{jk(n-m)\Omega_0}}$$

$$\begin{cases} = 0, & \text{if } n \neq m \\ = N_0, & \text{if } n = m \end{cases}$$

$$= N_0 \delta(n-m)$$

$$= \frac{1}{N_0^2} \sum_{n=0}^{N_0-1} \sum_{m=0}^{N_0-1} x(n) x^*(m) \times N_0 \delta(n-m)$$

$$= \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x(n)|^2$$

$$\sum_{k=0}^{N_0-1} |C_k|^2 = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x(n)|^2.$$

Parseval's relation for DFS.