Priors on exchangeable directed graphs

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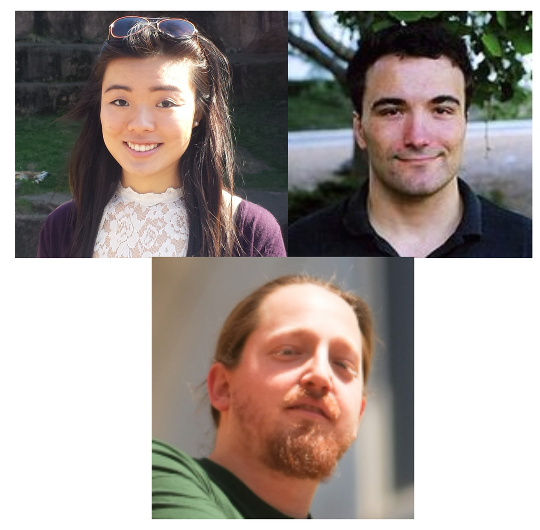


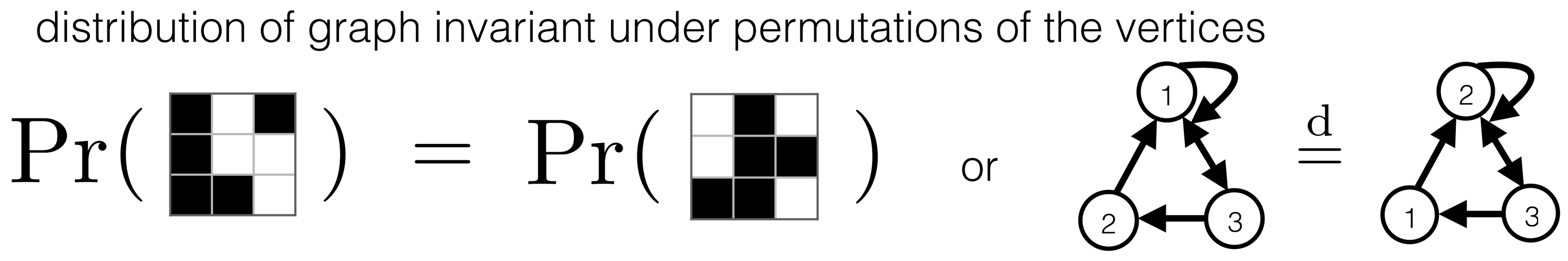
Motivation and overview

- Exchangeable *undirected* graphs well-studied:
  - Aldous-Hoover theorem (graphons)
  - Many models: e.g., stochastic block model
- *Directed* exchangeable graphs have a representation via Aldous-Hoover (digraphons)
- However, models have not been traditionally built using them; instead, many often use *asymmetric* measurable functions, which cannot capture the complete structure
- We show how to use digraphons to model graphs that can't be modeled using asymmetric functions
- We present a new Bayesian nonparametric model using digraphons

Background

Exchangeability

distribution of graph invariant under permutations of the vertices



Graphons

symmetric measurable function W: $[0,1]^2 \longrightarrow [0,1]$

sampling procedure:

$$U_i \overset{\text{iid}}{\sim} \text{Uniform}[0,1] \text{ for } i \in \mathbb{N},$$

$$G_{ij} \mid U_i, U_j \overset{\text{ind}}{\sim} \text{Bernoulli}(W(U_i,U_j)), \text{ for } i<j,$$

Asymmetric measurable functions

measurable function W_{asym}: $[0,1]^2 \longrightarrow [0,1]$, not symmetric

one way of sampling directed graph:

$$U_i \overset{\text{iid}}{\sim} \text{Uniform}[0,1] \text{ for } i \in \mathbb{N},$$

$$G_{ij} \mid U_i, U_j \overset{\text{ind}}{\sim} \text{Bernoulli}(W_{\text{asym}}(U_i,U_j)), \text{ for } i \neq j.$$

(treats edge directions as *independent*)

Digraphons

5-tuple of measurable functions **W** = $(W_{00}, W_{01}, W_{10}, W_{11}, w)$, where W_{ab}: $[0,1]^2 \longrightarrow [0,1]$, w: $[0,1] \longrightarrow \{0,1\}$ and satisfy $W_{ab}(x,y) = W_{ba}(y,x)$, where $a,b \in \{0,1\}$, $x,y \in [0,1]$

sampling procedure: $\mathbb{G}(n, \mathbf{W})$

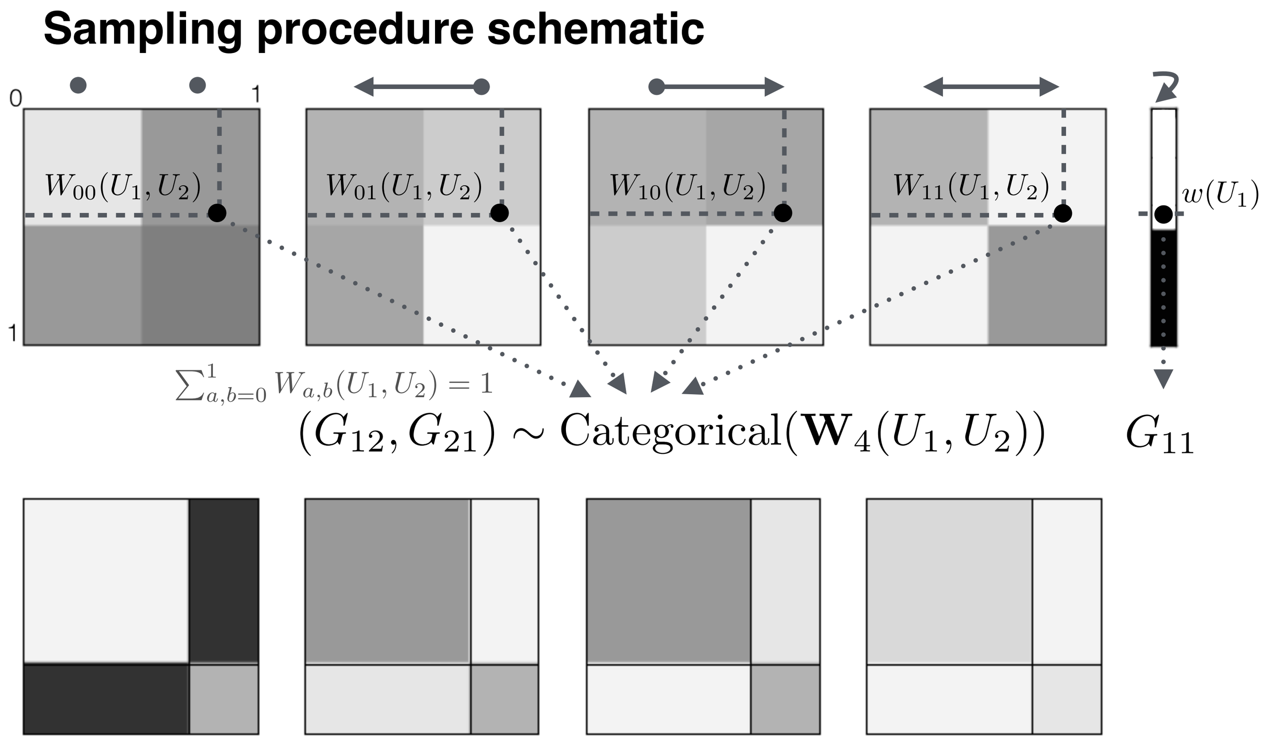
$$U_i \overset{\text{iid}}{\sim} \text{Uniform}[0,1] \text{ for } i \in \mathbb{N}$$

$$(G_{ij}, G_{ji}) \overset{\text{ind}}{\sim} \text{Categorical}(\mathbf{W}_4(U_i,U_j))$$

$$G_{ii} = w(U_i) \text{ for all } i$$

(considers edge directions *jointly*)

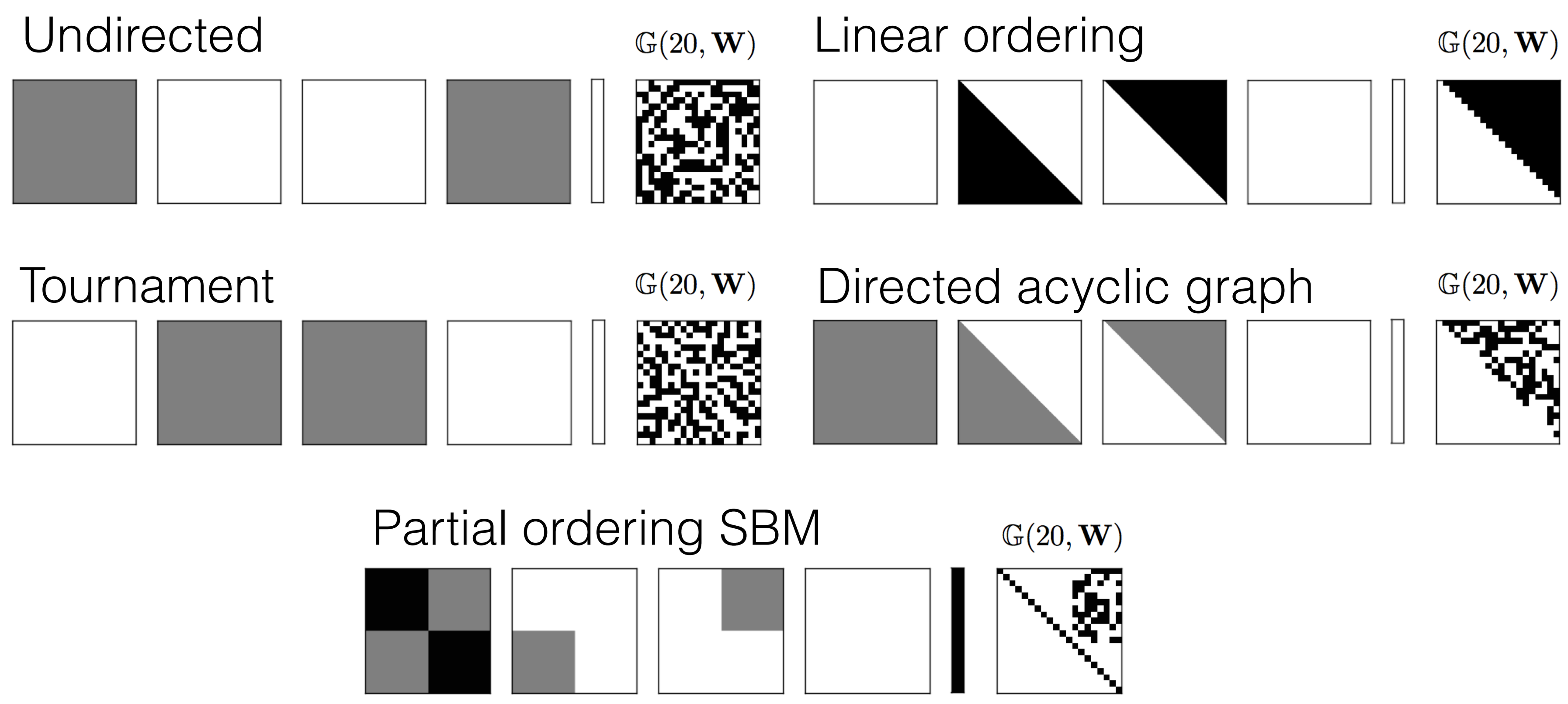
Sampling procedure schematic



Example of SBM digraphon, 0.7 division

Priors on digraphons

- W_{asym} cannot model many of types of directed graphs
- digraphons can capture this structure, e.g.,



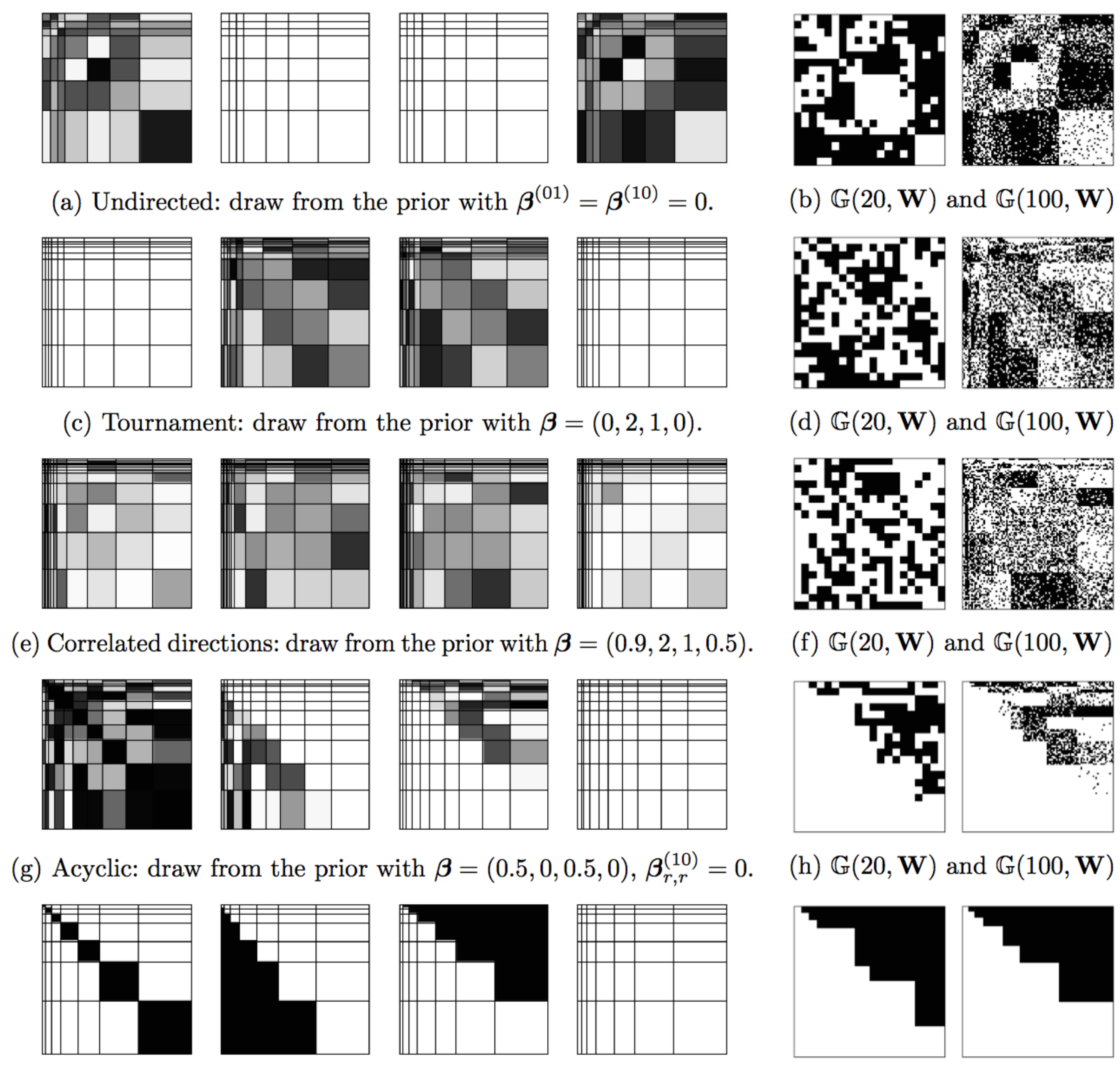
Example model: di-IRM

(Infinite relational model digraphon)

1. Draw partition ~ DP-Stick(α)
2. Draw weights ~ Dirichlet(β)

(such that symmetry requirements satisfied)

Examples of di-IRM with different parameter settings



(a) Undirected: draw from the prior with $\beta^{(01)} = \beta^{(10)} = 0$. (b) $\mathbb{G}(20,\mathbf{W})$ and $\mathbb{G}(100,\mathbf{W})$

(c) Tournament: draw from the prior with $\beta = (0,2,1,0)$. (d) $\mathbb{G}(20,\mathbf{W})$ and $\mathbb{G}(100,\mathbf{W})$

(e) Correlated directions: draw from the prior with $\beta = (0.9,2,1,0.5)$. (f) $\mathbb{G}(20,\mathbf{W})$ and $\mathbb{G}(100,\mathbf{W})$

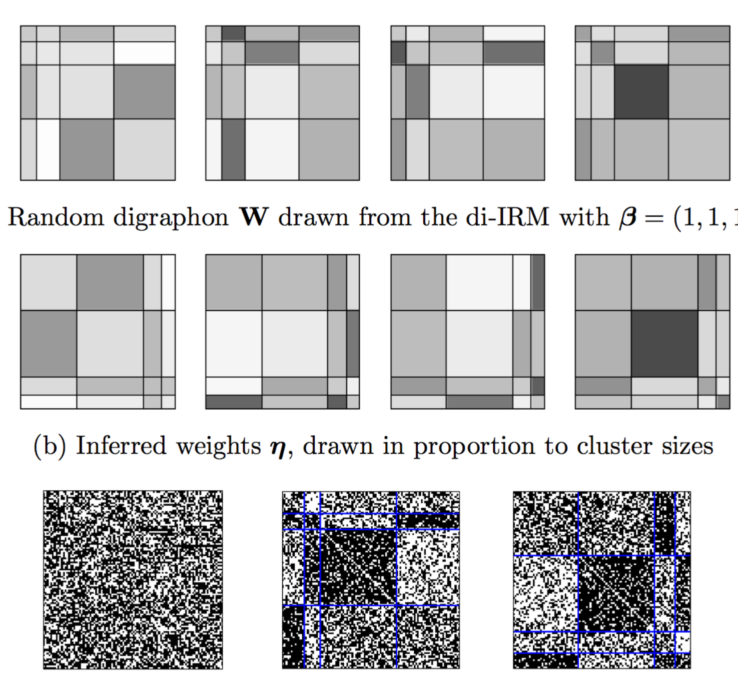
(g) Acyclic: draw from the prior with $\beta = (0.5,0,0.5,0)$, $\beta^{(10)}_{r,r} = 0$. (h) $\mathbb{G}(20,\mathbf{W})$ and $\mathbb{G}(100,\mathbf{W})$

(i) Near-ordering: draw from the prior with $\beta = (0,0,1,0)$, $\beta^{(10)}_{r,r} = 0$. (j) $\mathbb{G}(20,\mathbf{W})$ and $\mathbb{G}(100,\mathbf{W})$

Experiments

Synthetic data, collapsed Gibbs sampling

(1) Uniform di-IRM

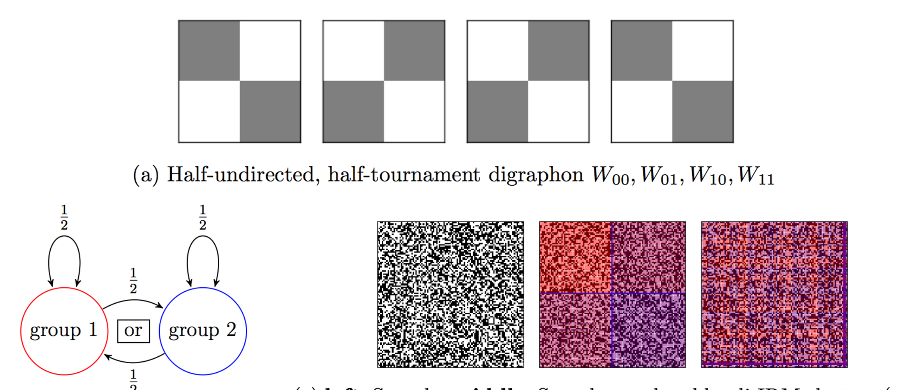


(a) Random digraphon **W** drawn from the di-IRM with $\beta = (1,1,1,1)$

(b) Inferred weights η, drawn in proportion to cluster sizes

(c) Sampled order (d) True clusters (e) Inferred clusters

(2) Half-undirected, half-tournament di-IRM



(a) Half-undirected, half-tournament digraphon $W_{00}, W_{01}, W_{10}, W_{11}$

(b) Schematic of example

(c) **left**: Sample; **middle**: Sample reordered by di-IRM clusters (true clusters are colored), **right**: Sample reordered by IRM clusters