Edge-exchangeable graphs and sparsity

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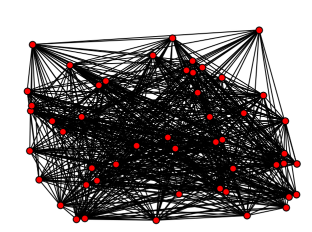
Motivation

- The traditional notion of exchangeability for random graphs is *vertex exchangeability*.
- Vertex-exchangeable graphs are dense or empty almost surely.
- But most real-world graphs are *sparse*.
- We introduce the alternative notion of **edge exchangeability**, and show that a wide class of edge-exchangeable models can produce sparse graphs.

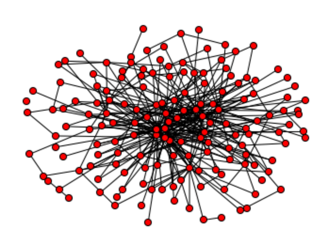
Projective graph sequences: We consider graph sequences $G_1, G_2, G_3, \ldots$, where at each step of the sequence, we only add vertices and edges to the graph (instead of deleting edges).

Dense graph sequence:
$[\#\text{edges}(G_n)] = \Omega([\#\text{vertices}(G_n)]^2)$
(quadratic growth)



generated from a **vertex-exchangeable** model

Sparse graph sequence
$[\#\text{edges}(G_n)] = o([\#\text{vertices}(G_n)]^2)$
(sub-quadratic growth)

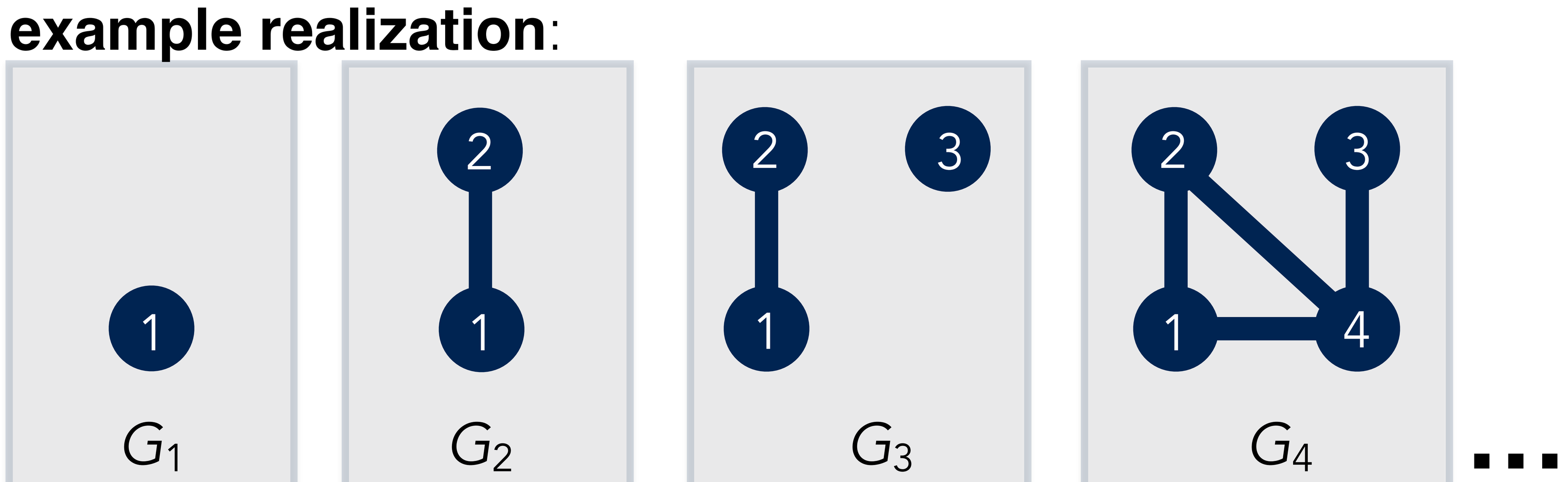


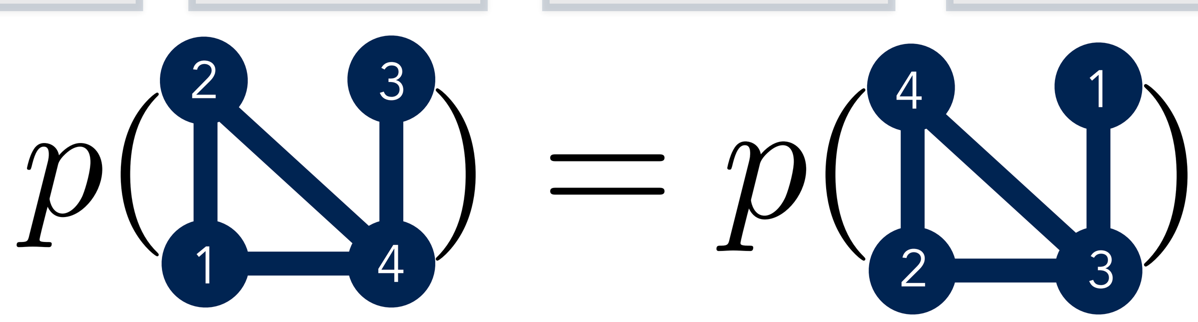
generated from an **edge-exchangeable** model

Vertex exchangeability

In vertex exchangeability, a new vertex joins the graph sequence at each step, is labeled with that step number, and instantiates all edges with *existing* vertices.

example realization:





Vertex exchangeability means that the distribution of any step in the random graph sequence is invariant to permutations of the vertex labels. *Examples*: Erdős–Rényi, stochastic block model

The Aldous-Hoover theorem implies that **vertex-exchangeable graphs are dense or empty with probability 1**, *graphon* $f : [0,1]^2 \to [0,1]$

$$\mathbb{E}([\#\text{edges}(G_n)]) = \mathbb{E}\left(\binom{n}{2}\frac{1}{2}\int_0^1\int_0^1 f(x,y)\,dx\,dy\right)$$

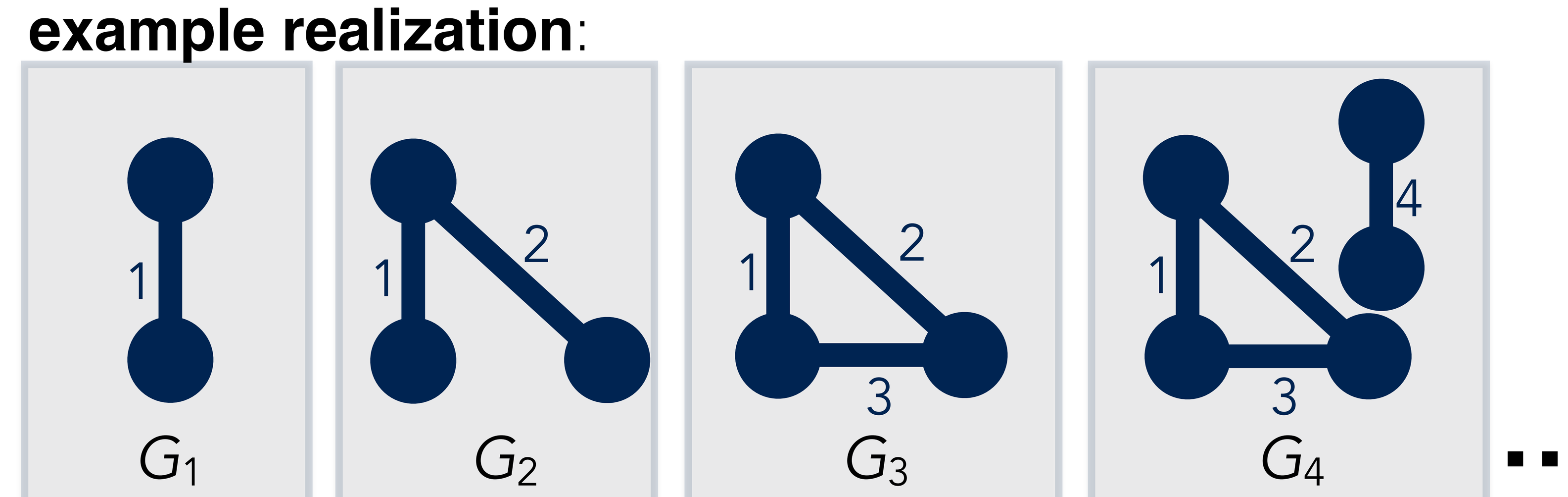
$$\sim c \cdot n^2 = c \cdot [\#\text{vertices}(G_n)]^2$$

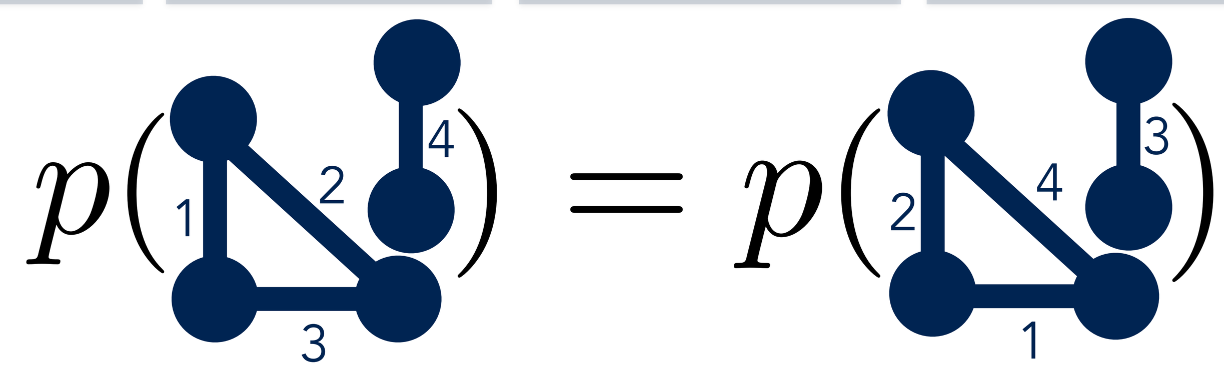
so we need something else that can give us sparse graphs.

Edge exchangeability

In edge exchangeability, a new *edge* joins the graph sequence at each step, is labeled with that step number, and instantiates any new vertices in that edge along the way.

example realization:





Edge exchangeability means that the distribution of any step in the random graph sequence is invariant to permutations of the edge labels.

We now show the *existence of sparse graphs* in a wide class of edge-exchangeable models.

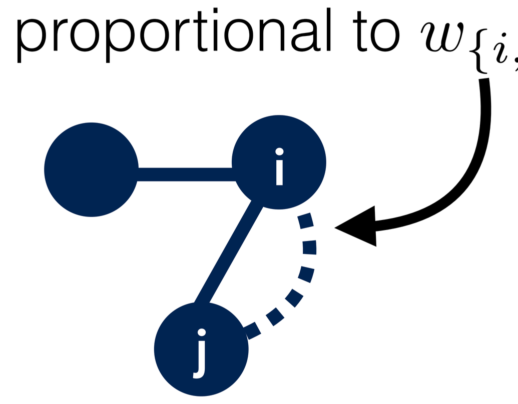
Graph frequency models

1. Draw edge weights $w_{\{i,j\}}$ from some distribution
2. Draw edge $\{i,j\}$ with probability proportional to $w_{\{i,j\}}$

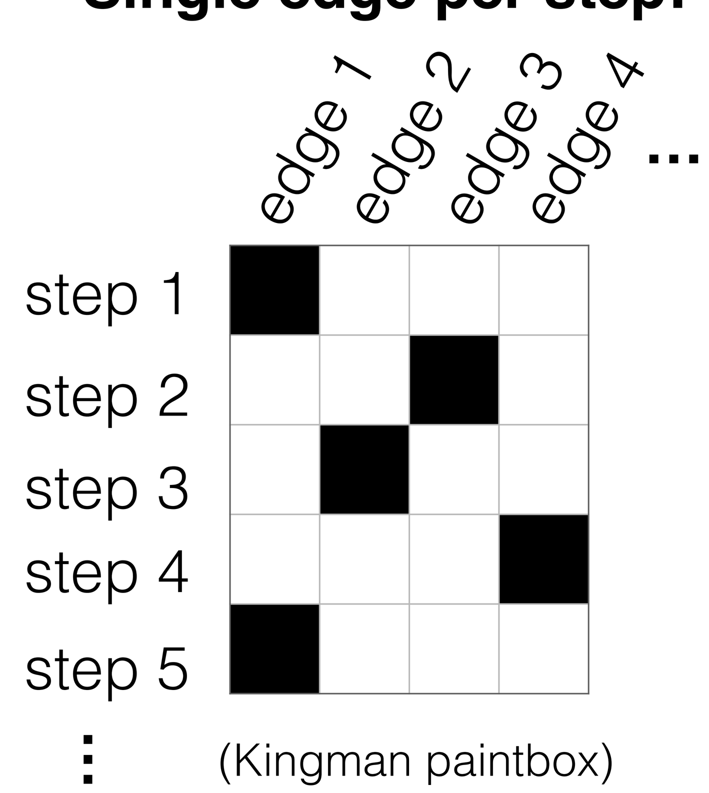
Discrete random measure:

$$W = \sum_{\{i,j\}} w_{\{i,j\}} \delta_{\theta_{\{i,j\}}}$$

edges drawn conditionally iid given W
implies edge exchangeability

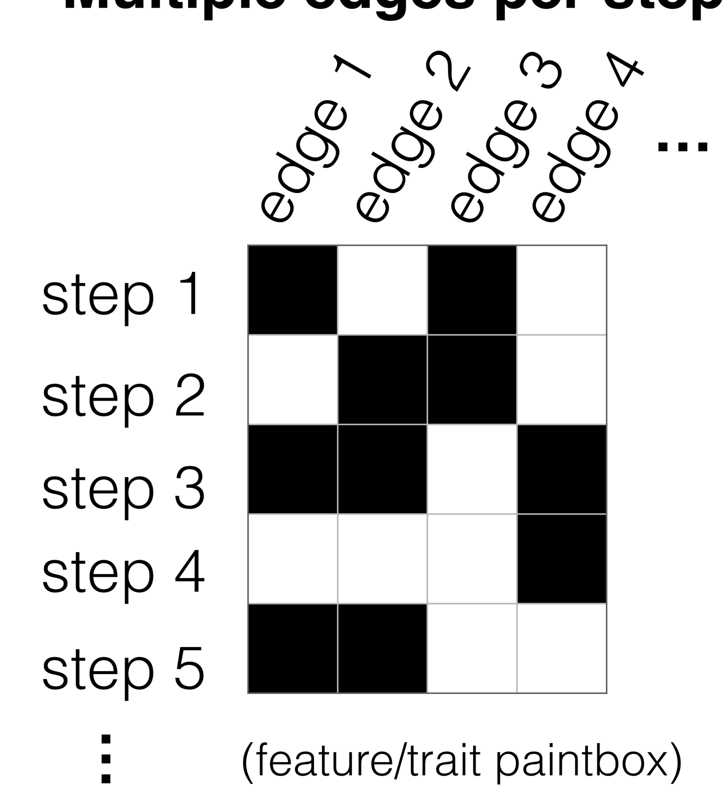


Single edge per step:



(Kingman paintbox)

Multiple edges per step:



(feature/trait paintbox)

Sparsity results

Consider a graph frequency model with edge frequencies

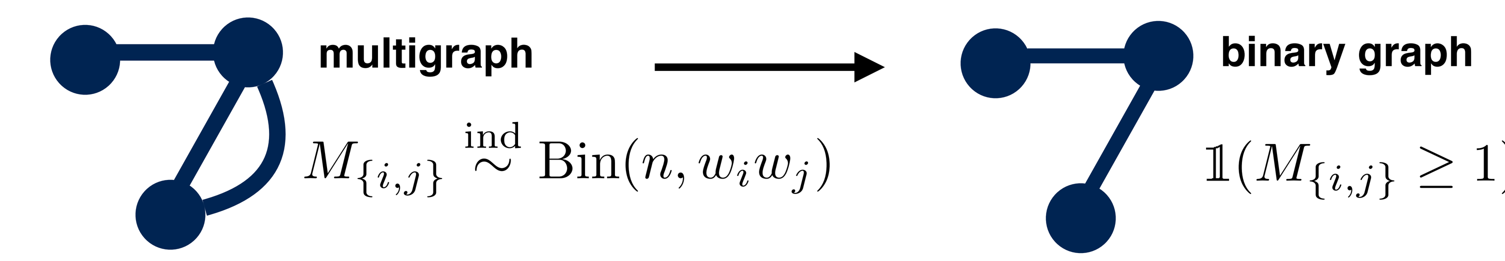
$$w_{\{i,j\}} = w_i w_j \quad \text{and} \quad (w_i) \sim \text{Poisson Point Process}(\nu),$$

and the rate measure ν has ***power law tails:***

$$\int_x^1 \nu(dw) \overset{x\to 0}{\sim} x^{-a}\ell(x^{-1}) \qquad \forall c>0, \ell(cx)/\ell(x) \overset{x\to\infty}{\to} 1$$

ν is ***regularly varying*** with exponent a $\quad$ $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ is ***slowly varying***

Generate a multigraph $M_{\{i,j\}}$ and also construct binary graph:



Theorem. Suppose the rate measure ν is regularly varying with exponent a. Let $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ be a slowly varying function.

# of instantiated vertices: $\Theta(n^a \ell(n))$ a.s.

Multigraph # of edges: $\Theta(n)$ a.s.

Binary graph # of edges: $O(\min\{n^{(1+a)/2}, n^{3a/2}\ell(n)\}\,\ell(n^{1/2}))$ a.s.

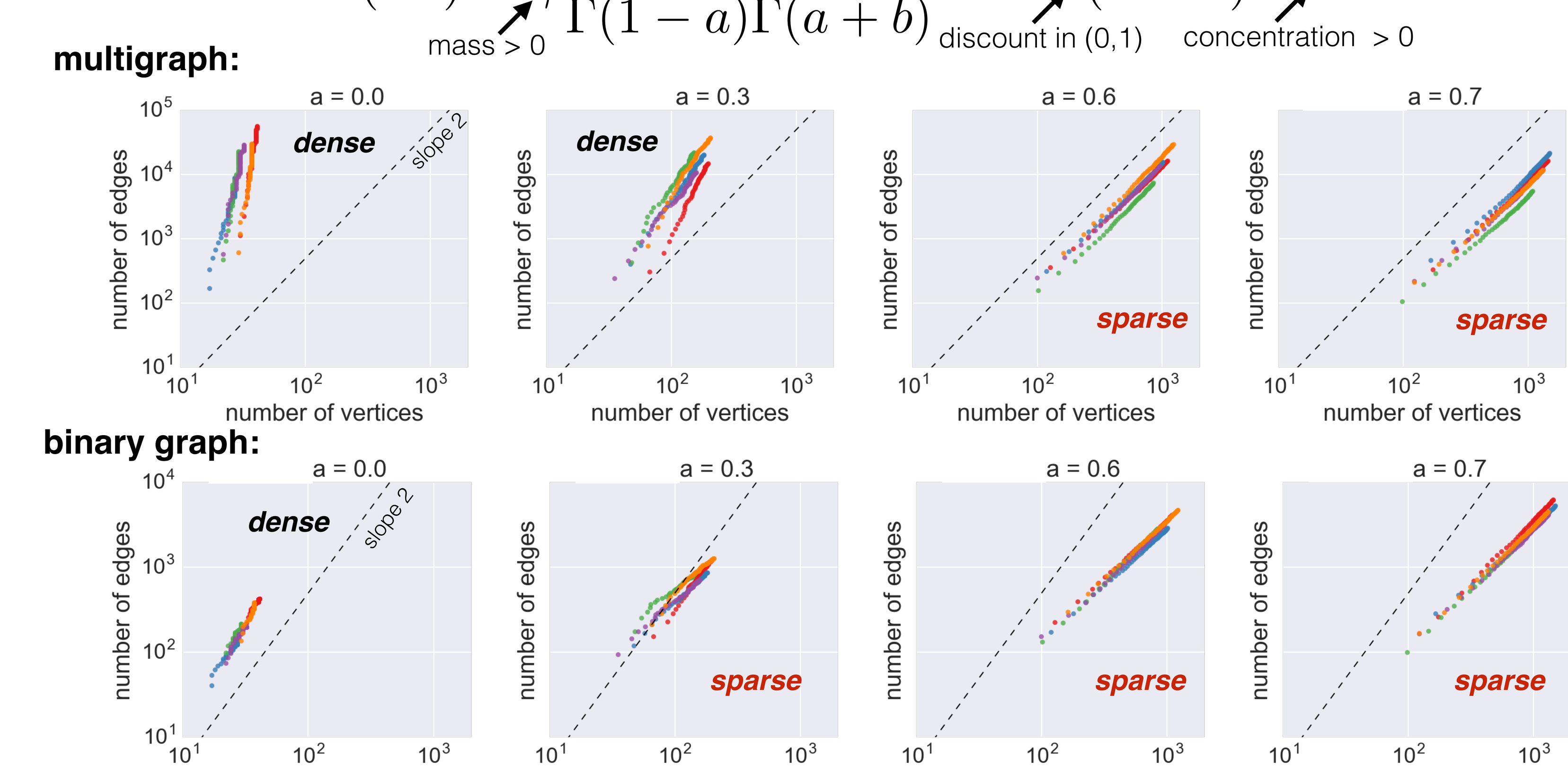
Thus, the multigraph is sparse when $1/2 < a < 1$;
the binary graph is sparse when $0 < a < 1$, and dense otherwise.

Simulations

We generate the weights (w_i) from a Poisson point process with the **3-parameter beta process** rate measure:

$$\nu(dw) = \gamma \frac{\Gamma(1+b)}{\Gamma(1-a)\Gamma(a+b)} w^{-1-a}(1-w)^{a+b-1}dw$$

mass > 0 $\quad$ discount in (0,1) $\quad$ concentration > 0



Empirically, we get a range of dense and sparse behavior that agrees with the above theorem.

References

1. D. J. Aldous. Exchangeability and related topics. Springer Lecture Notes in Math, 1985.
2. T. Broderick, J. Pitman, and M. I. Jordan. Feature allocations, probability functions, and paintboxes. Bayesian Analysis, 8(4):801–836, 2013.
3. T. Broderick, M. I. Jordan, and J. Pitman. Beta processes, stick-breaking, and power laws. Bayesian Analysis, 7(2):439–475, 2012.
4. T. Broderick and D. Cai. Edge-exchangeable graphs, sparsity, and power laws. NIPS Workshop on Bayesian Nonparametrics, 2015 & NIPS workshop on Networks, 2015.
5. D. Cai and T. Broderick. Completely random measures for modeling power laws in sparse graphs. NIPS Workshop on Networks, 2015.
6. T. Campbell, D. Cai, T. Broderick. Exchangeable trait allocations. arXiv:1609.09147, 2016.
7. F. Caron and E. B. Fox. Bayesian nonparametric models of sparse and exchangeable random graphs. arXiv:1401.1137v3, 2015.
8. H. Crane and W. Dempsey. Edge exchangeable models for network data. arXiv:1603.04571, 2016.
9. D. N. Hoover. Relations on probability spaces and arrays of random variables. Preprint, Institute for Advanced Study, Princeton, NJ, 1979.
10. J. F. C. Kingman. The representation of partition structures. Journal of the London Mathematical Society, 2(2):374–380, 1978.
11. P. Orbanz and D. M. Roy. Bayesian models of graphs, arrays and other exchangeable random structures. PAMI, IEEE Transactions on, 37(2):437–461, 2015.