priors on exchangeable directed graphs

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Cameron Freer (Gamalon and MIT)

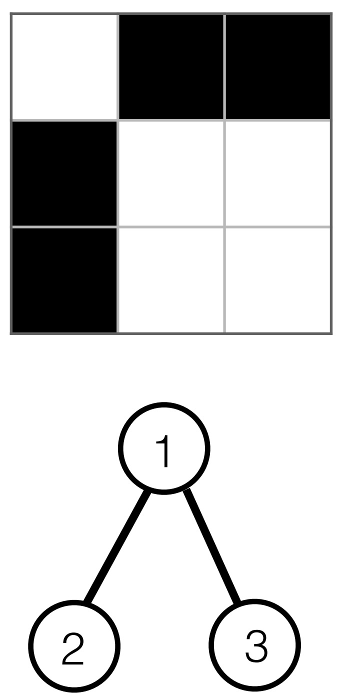
Overview

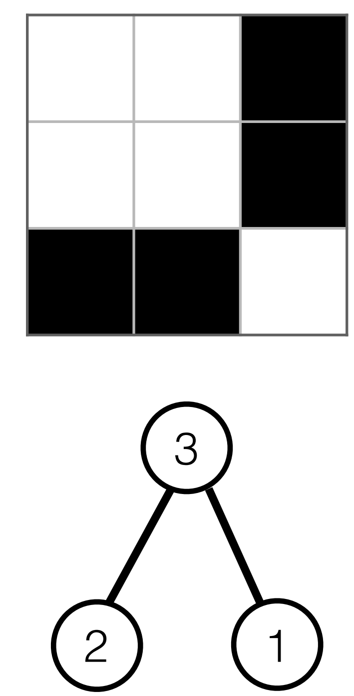
- We consider **exchangeable (dense) graphs**.
- Specifically, we are interested in **directed** graphs. However, most work has focused on undirected graphs.
- **Aldous–Hoover** has an analogous statement for directed graphs, which is more complicated than merely using an *asymmetric* function.
- Many natural nonparametric priors on exchangeable undirected graphs **extend to the directed case.**
- This perspective leads to natural priors on other exchangeable structures, such as **tournaments** and **directed acyclic graphs**.

Exchangeable graphs

exchangeability:

- order of vertices doesn’t affect distribution of graph

$\Pr($ 

$) = \Pr($ 

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Exchangeable graphs

Undirected	Directed
Any exchangeable random infinite graph is obtained as a mixture of $\mathbb{G}(\infty, W)$. $\mathbb{G}(\infty, W)$ is a sampling procedure from the **graphon** W. $W : [0,1]^2 \to [0,1]$ $W(x,y) = W(y,x)$	

Aldous–Hoover

Exchangeable graphs

Graphon:

$$W : [0,1]^2 \to [0,1]$$
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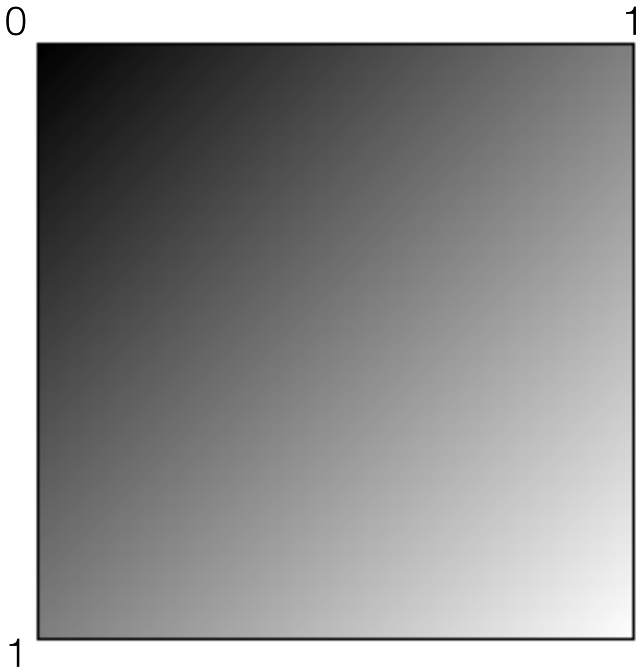

$$W(x,y) = \frac{(1-x)+(1-y)}{2}$$

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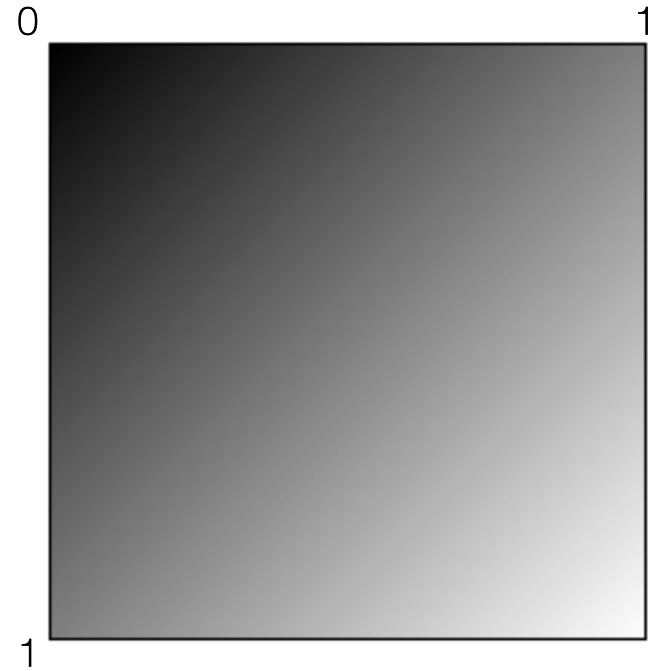

Sampling procedure:

$$U_i \sim \mathrm{Uniform}[0,1]$$
$$G_{ij} \sim \mathrm{Bern}(W(U_i, U_j)), i < j \quad (\text{Set } G_{ji} = G_{ij})$$

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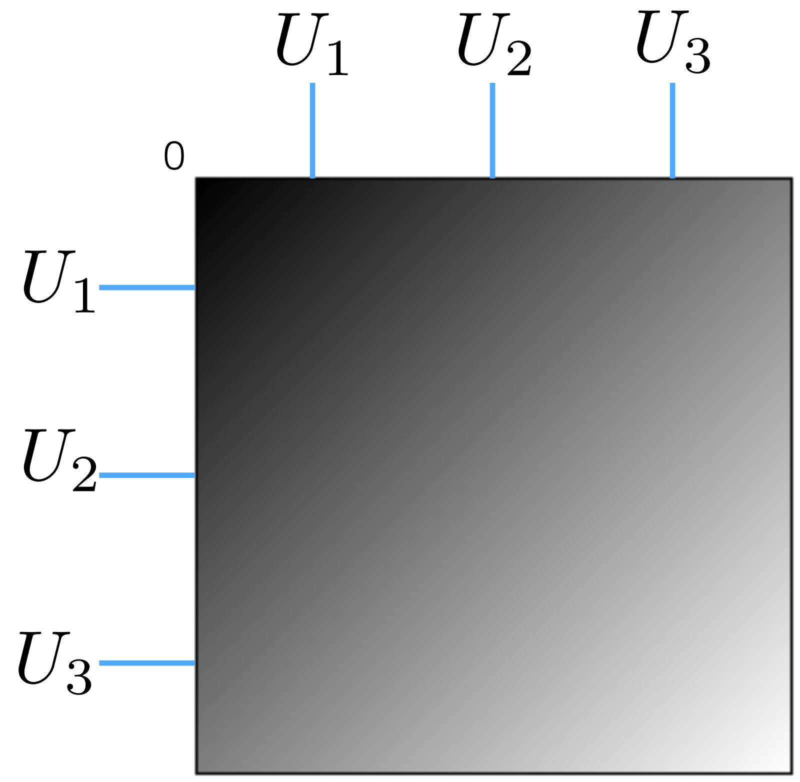

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 [Cai–Ackerman–Freer, 2015]

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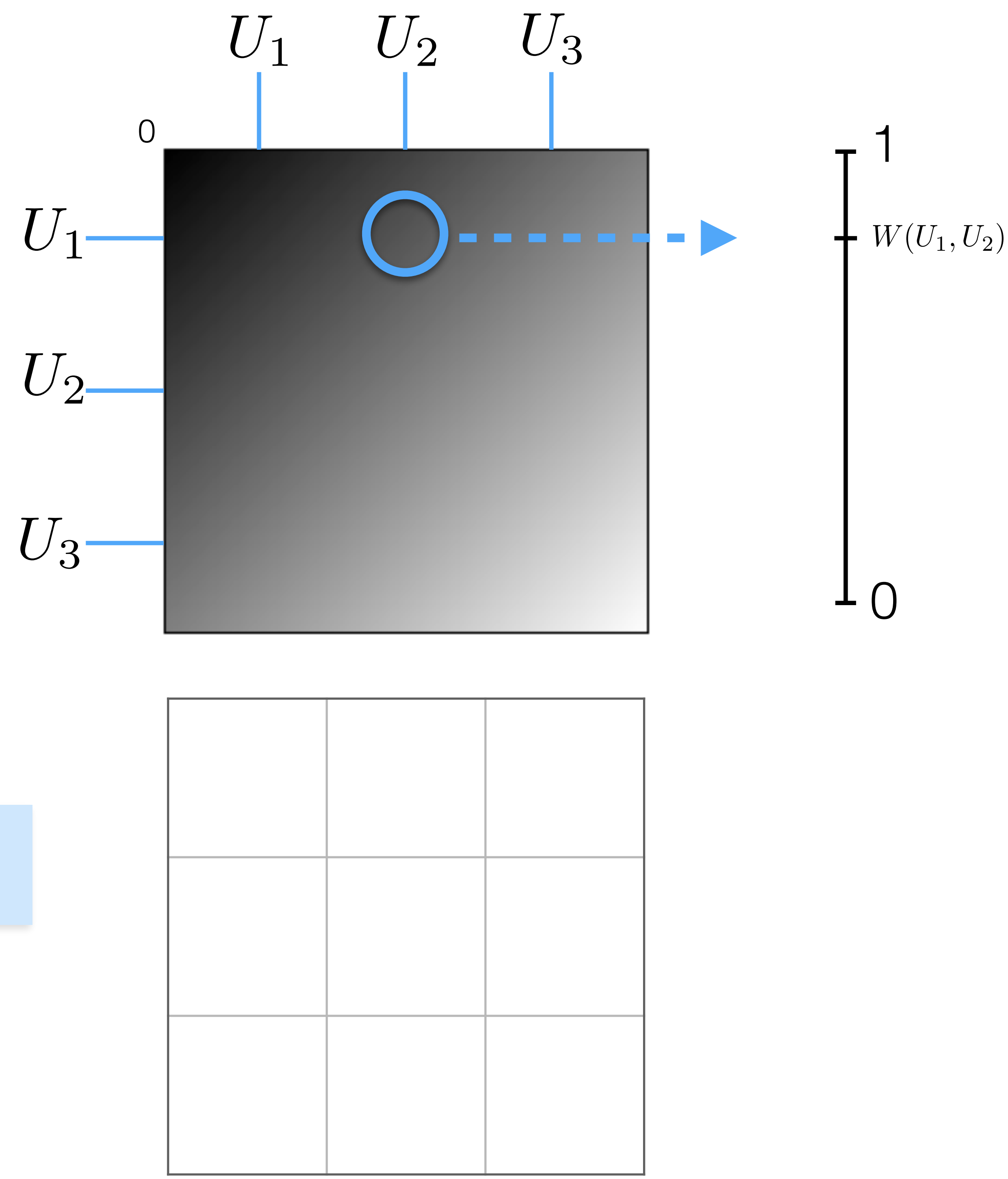

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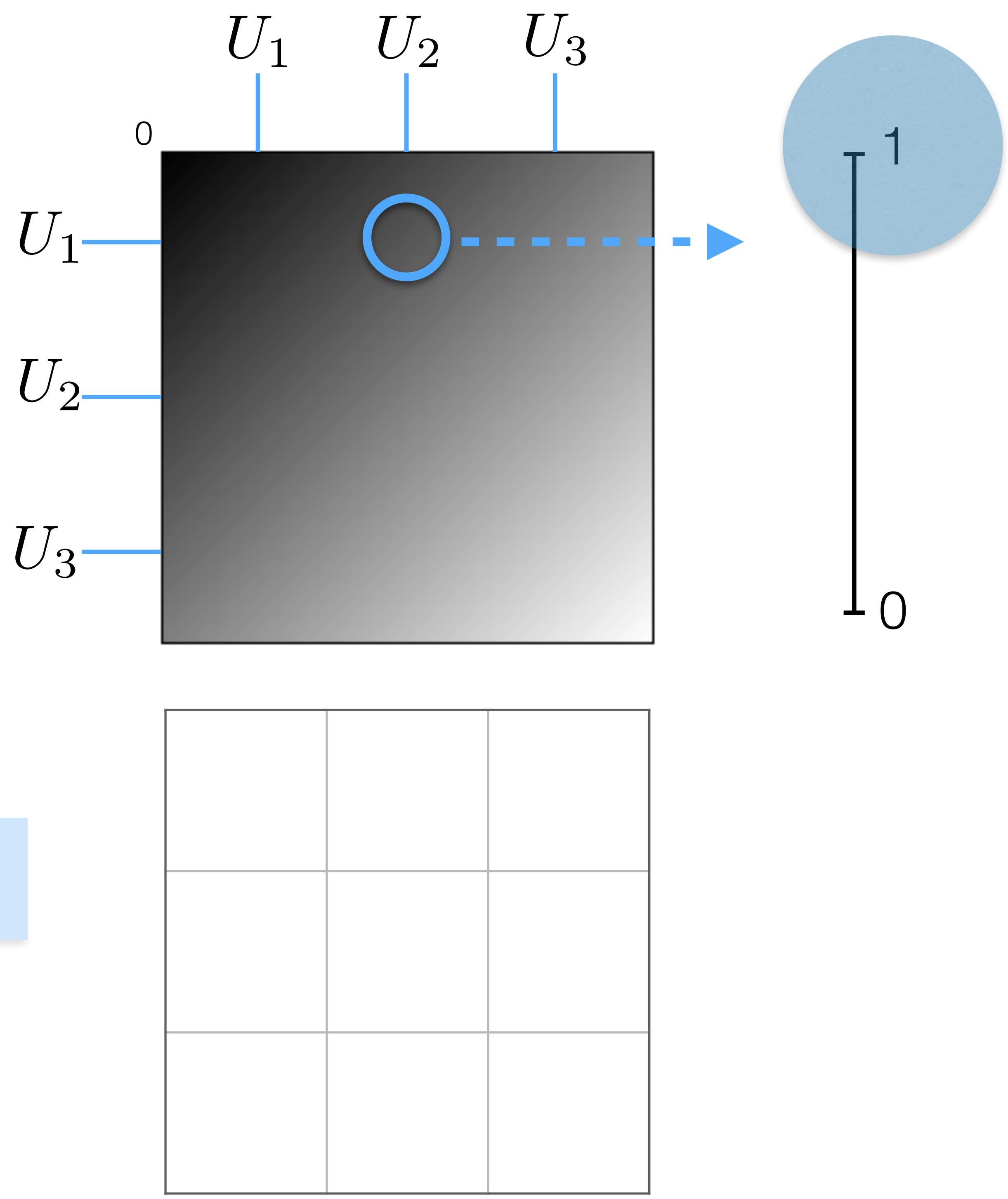

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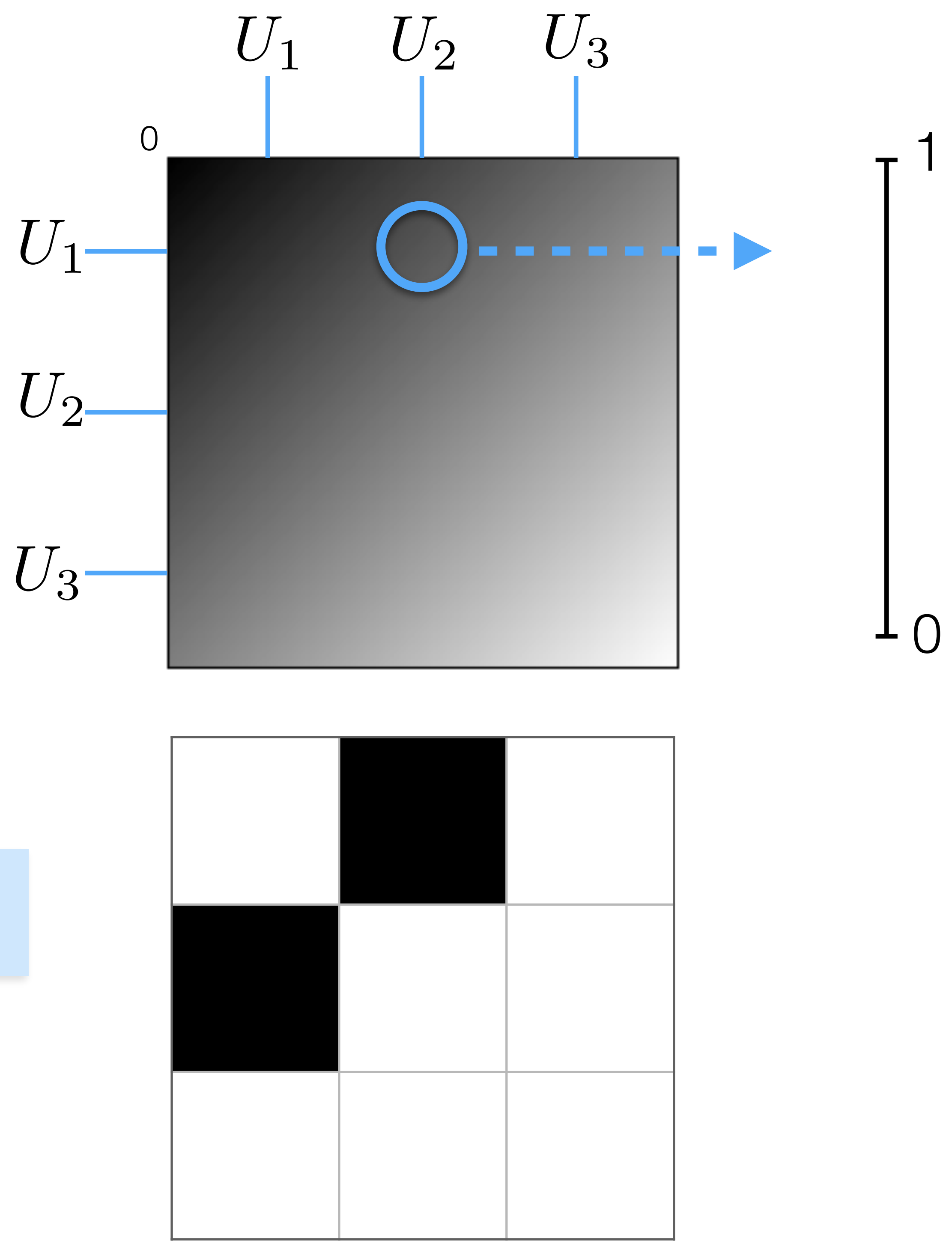

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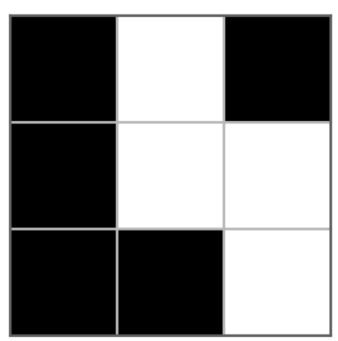
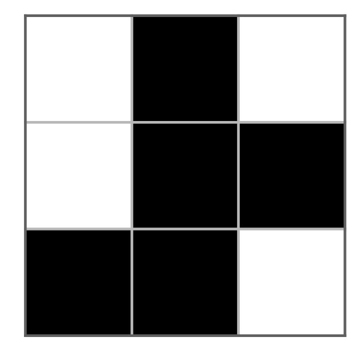
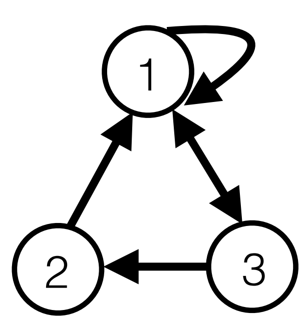
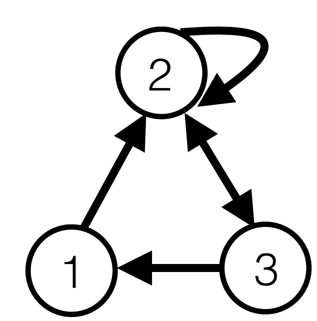
$\Pr($  $) = \Pr($  $)$





Exchangeable graphs

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$\mathbb{G}(\infty, W)$ is a sampling procedure from the **graphon** W.	$\mathbb{G}(\infty, \boldsymbol{W})$ is a sampling procedure from the **digraphon** $\boldsymbol{W}$.
$W : [0,1]^2 \to [0,1]$ $W(x,y) = W(y,x)$	$\boldsymbol{W} = ?$
Aldous–Hoover; cf. Lovász–Szegedy	implicit in A–H; cf. Diaconis–Janson

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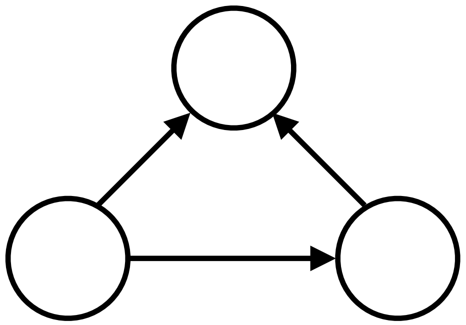

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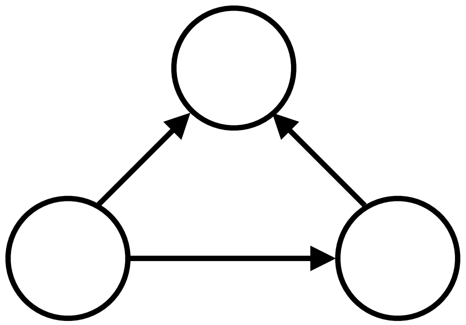
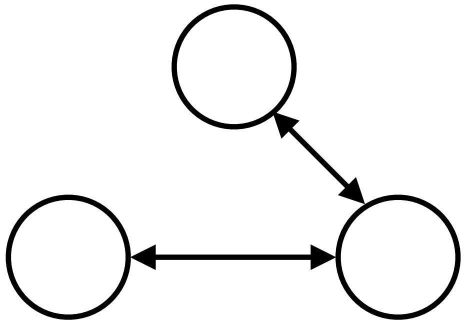
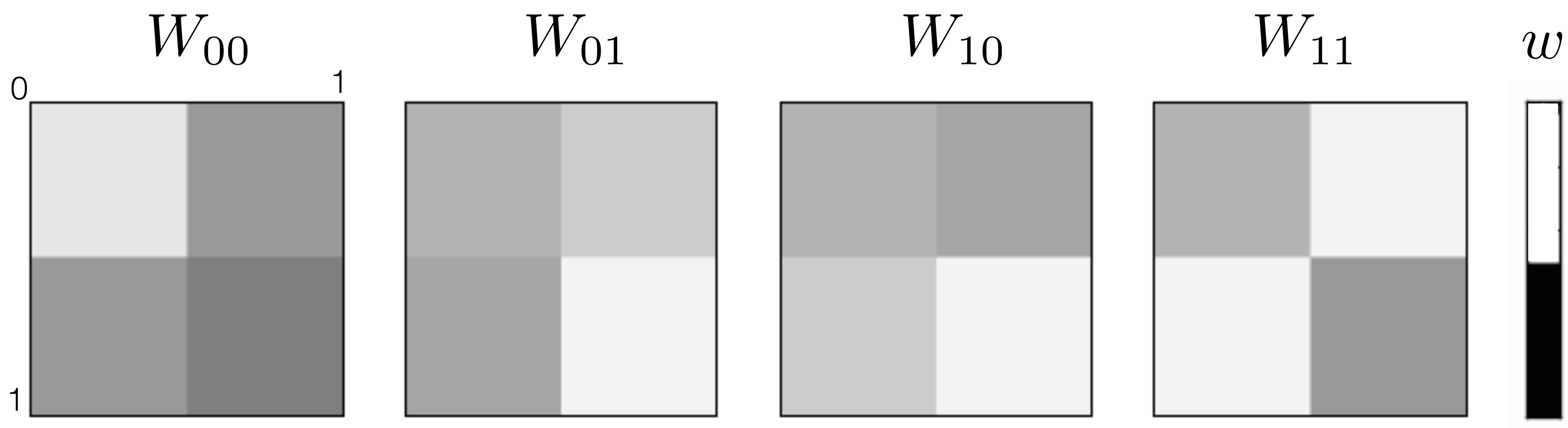


- **undirected** graphs (each pair has either both or no directed edges)



 [Cai–Ackerman–Freer, 2015]

Digraph representation



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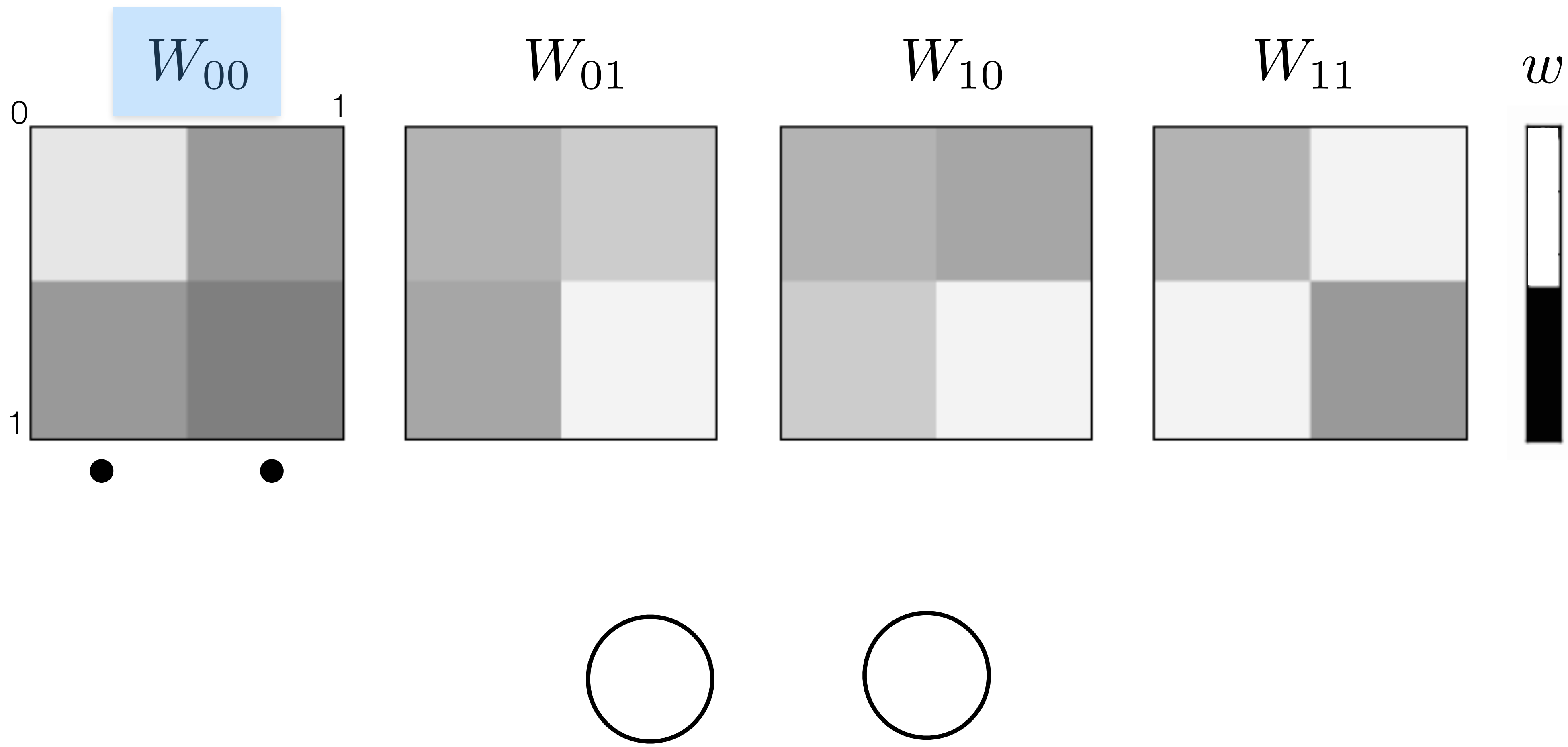

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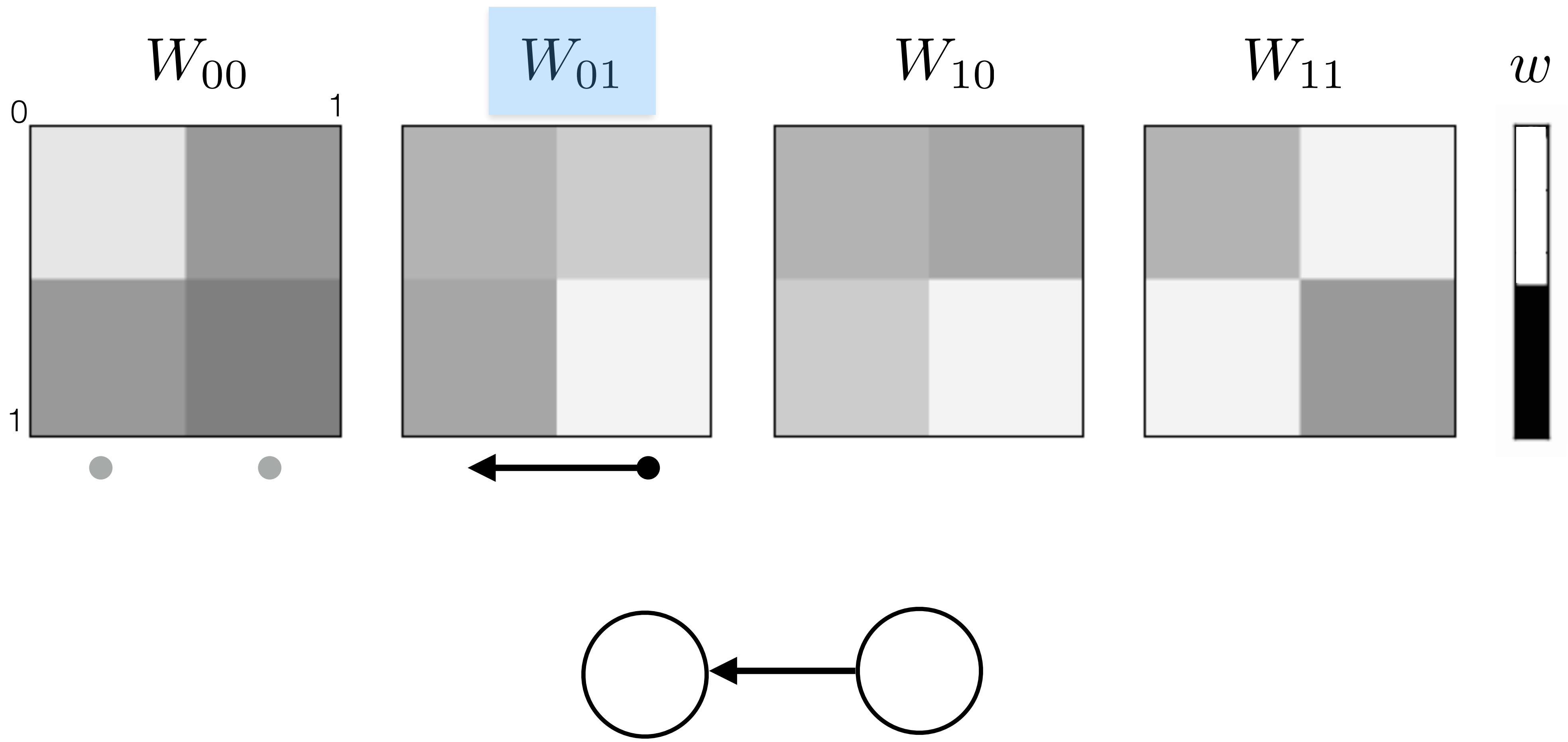

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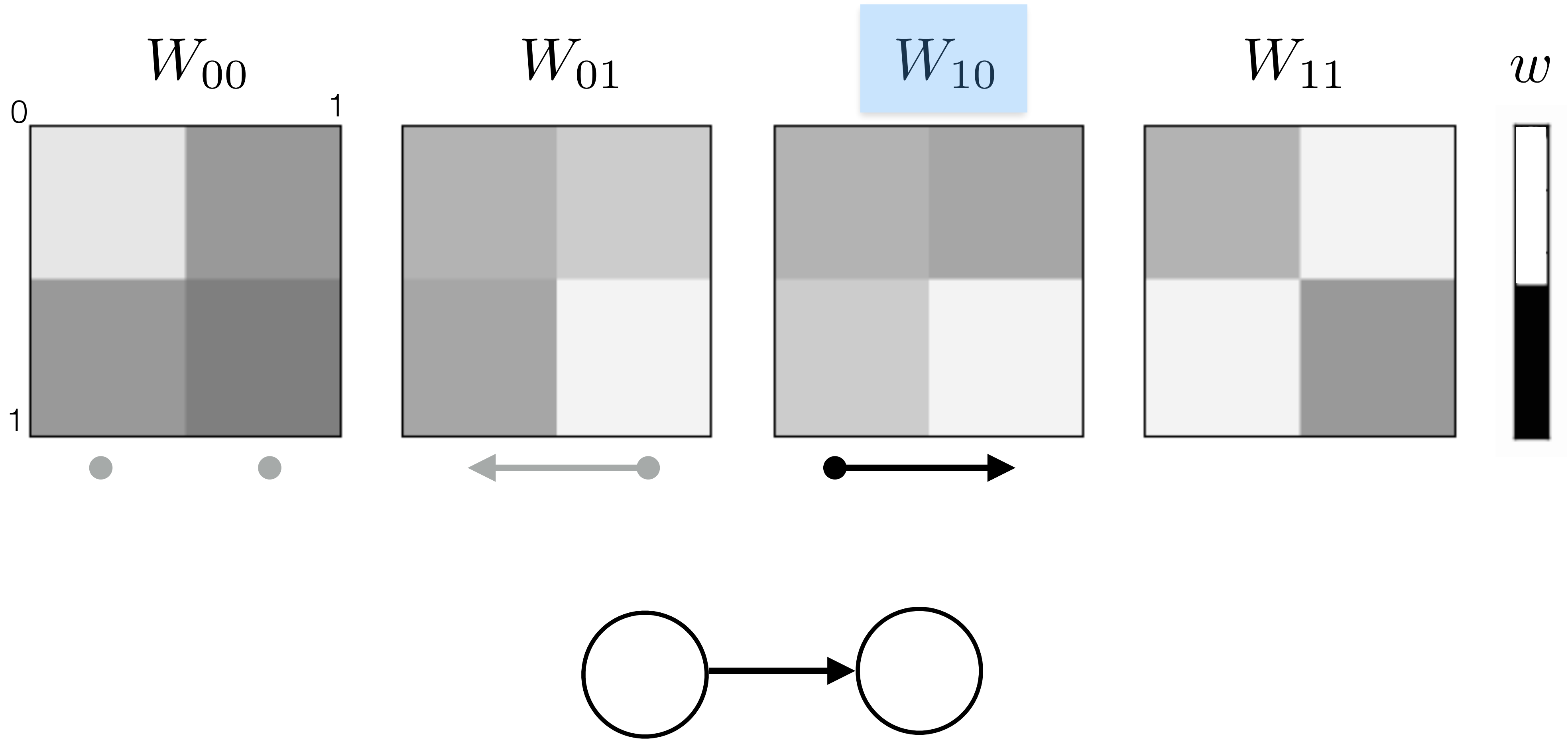

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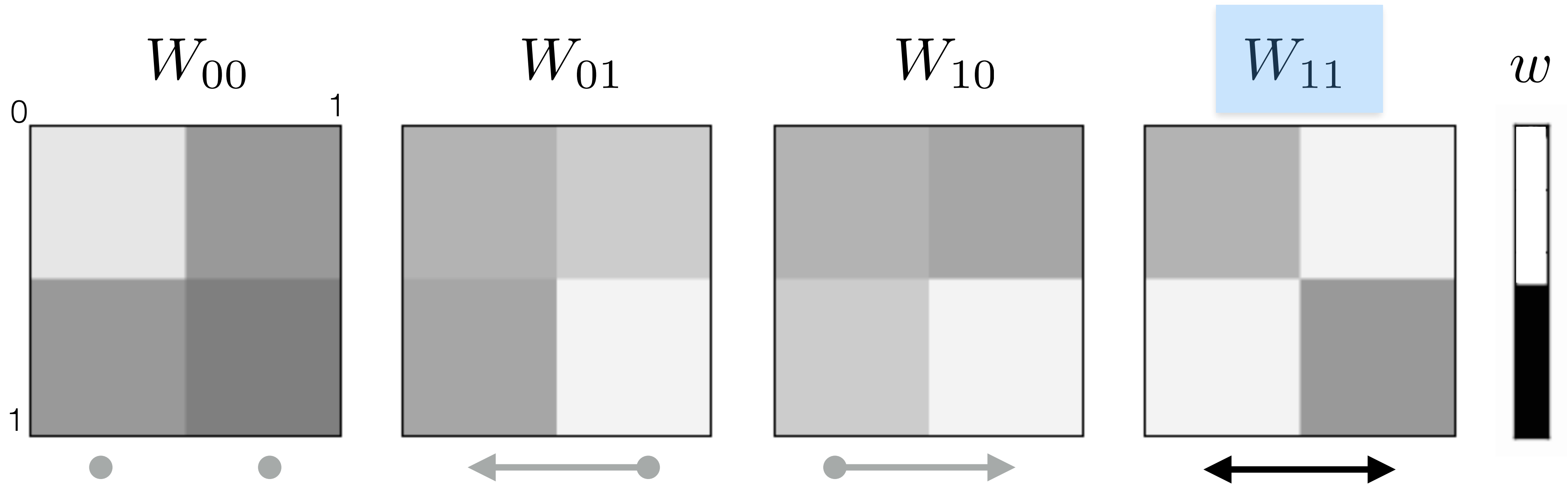

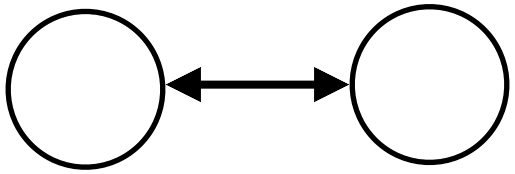

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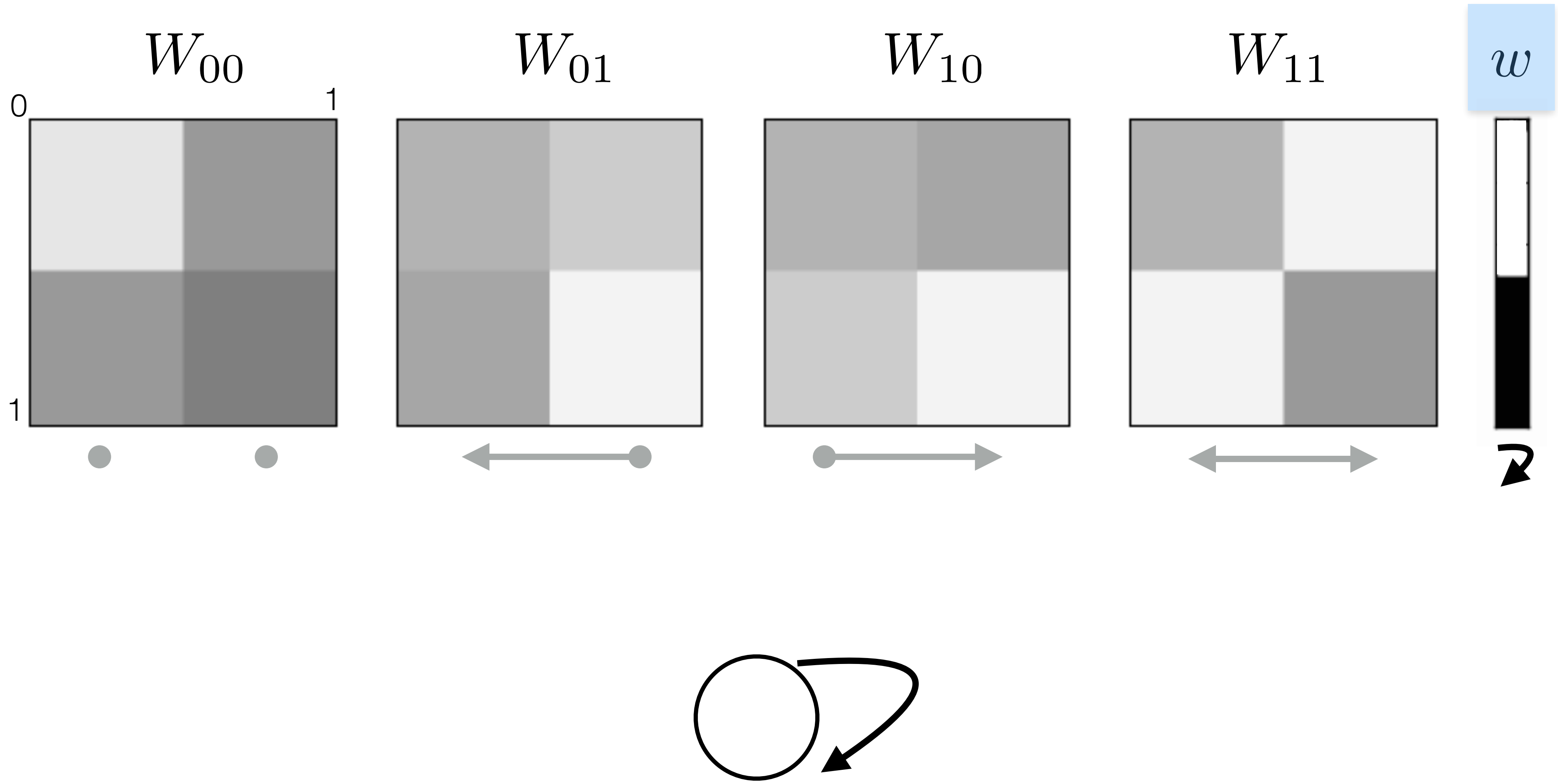

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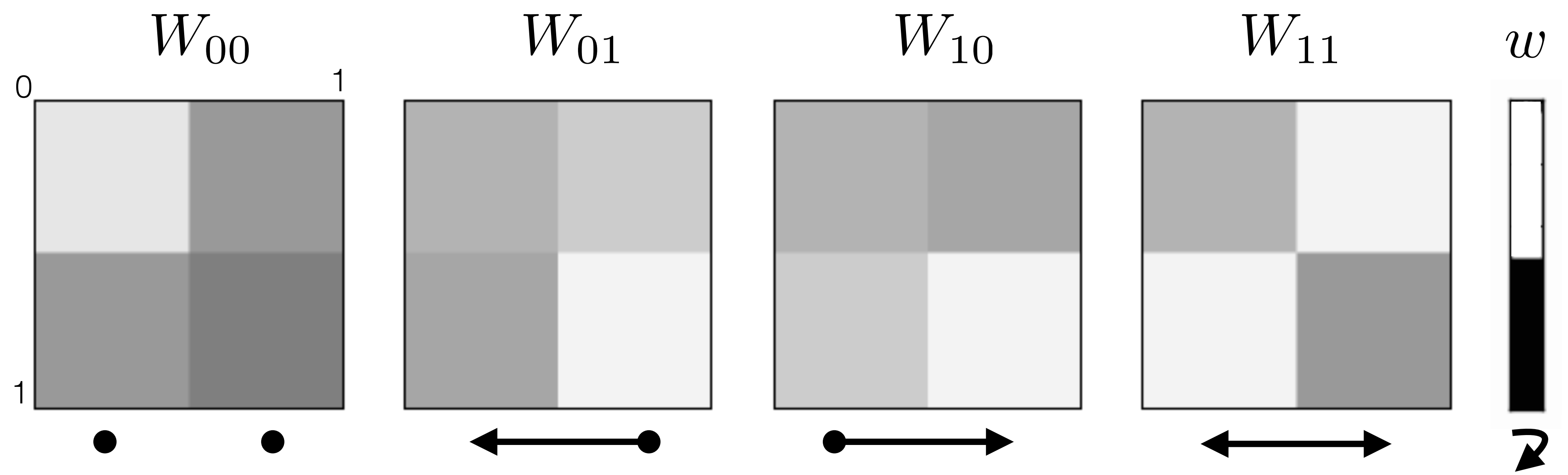

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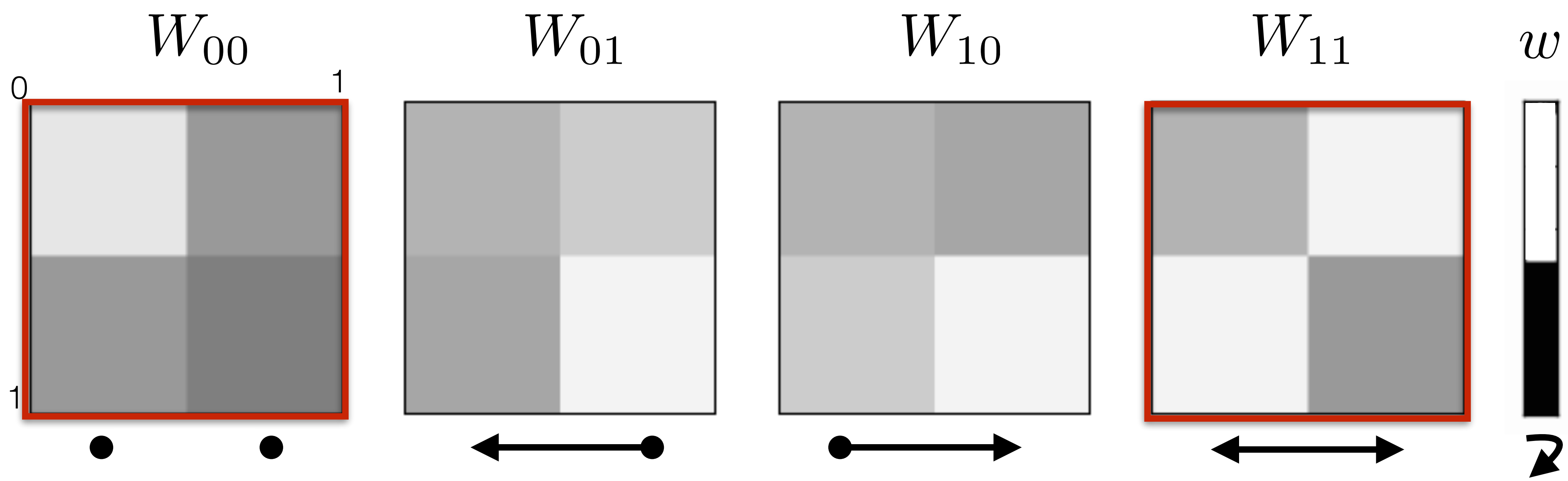

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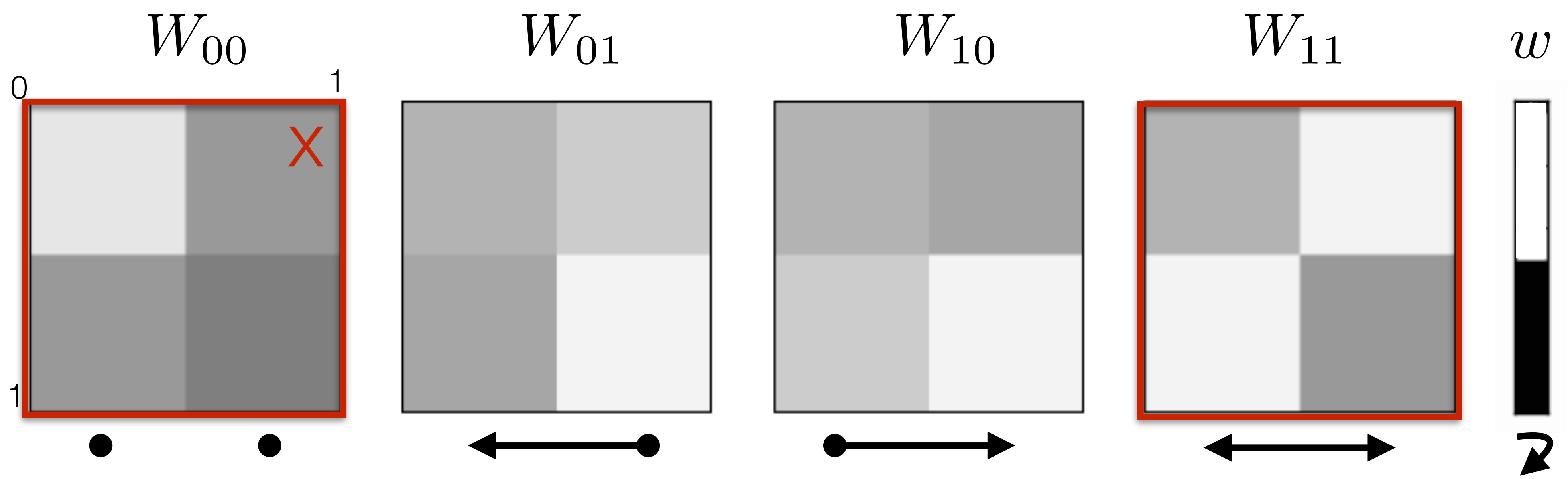

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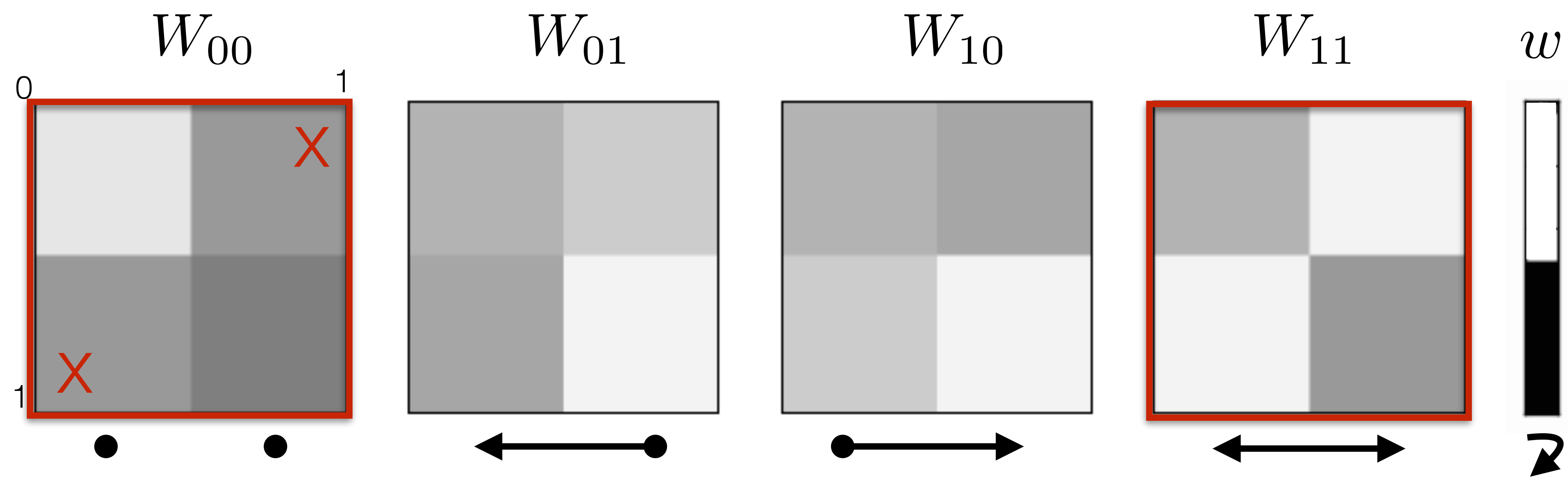

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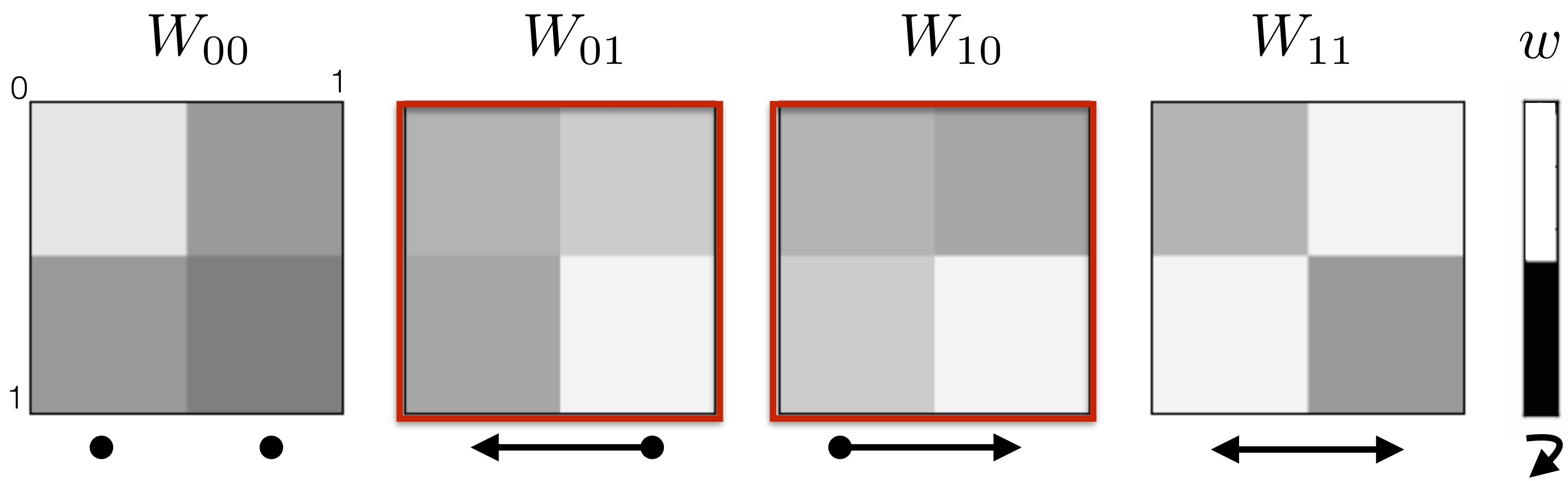

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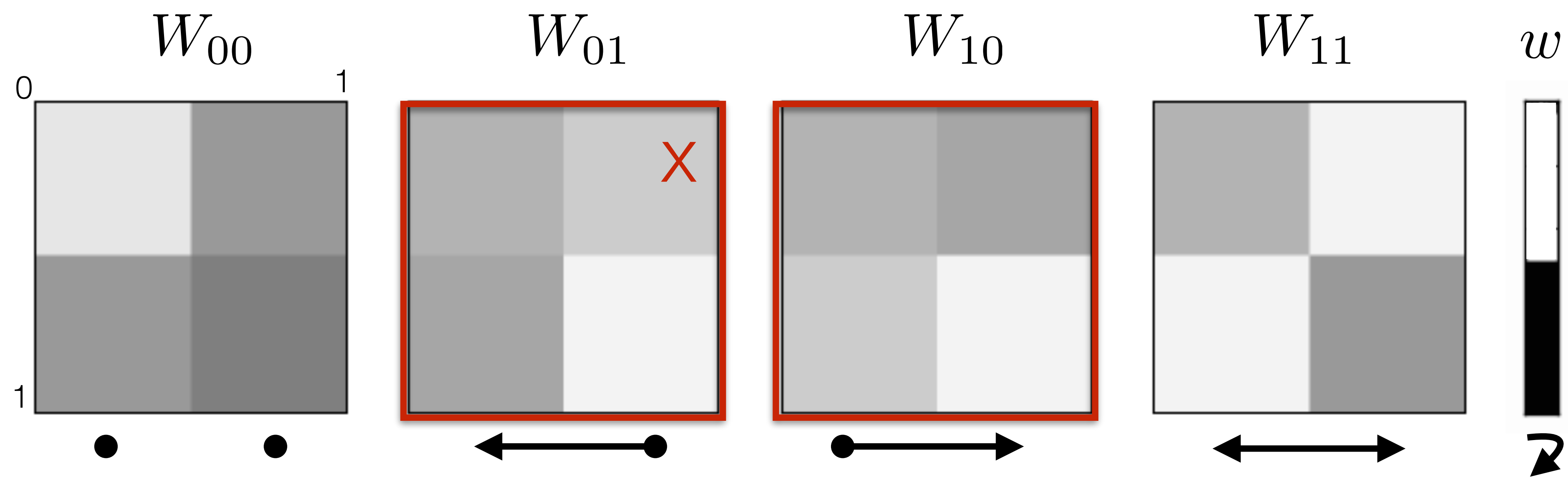

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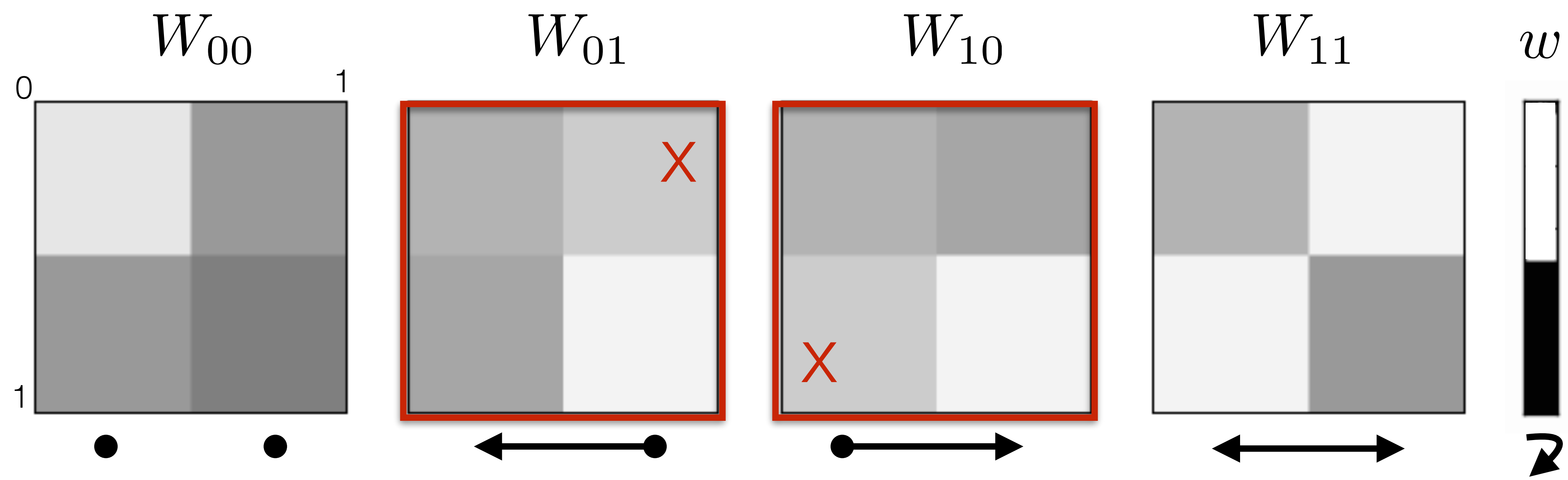

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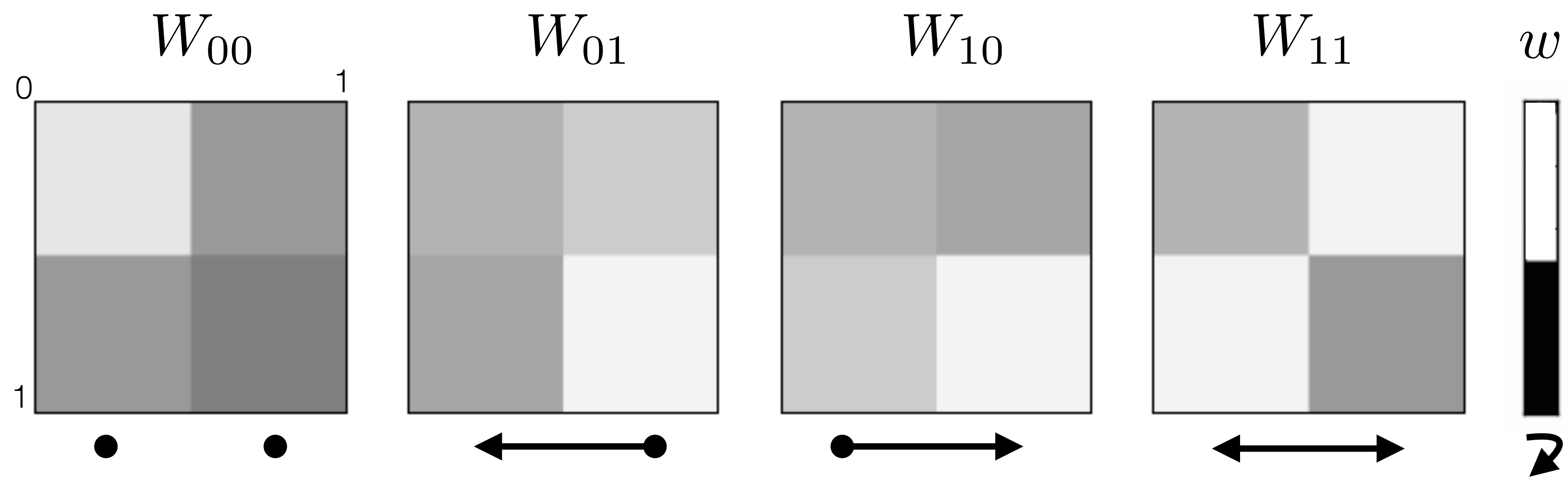

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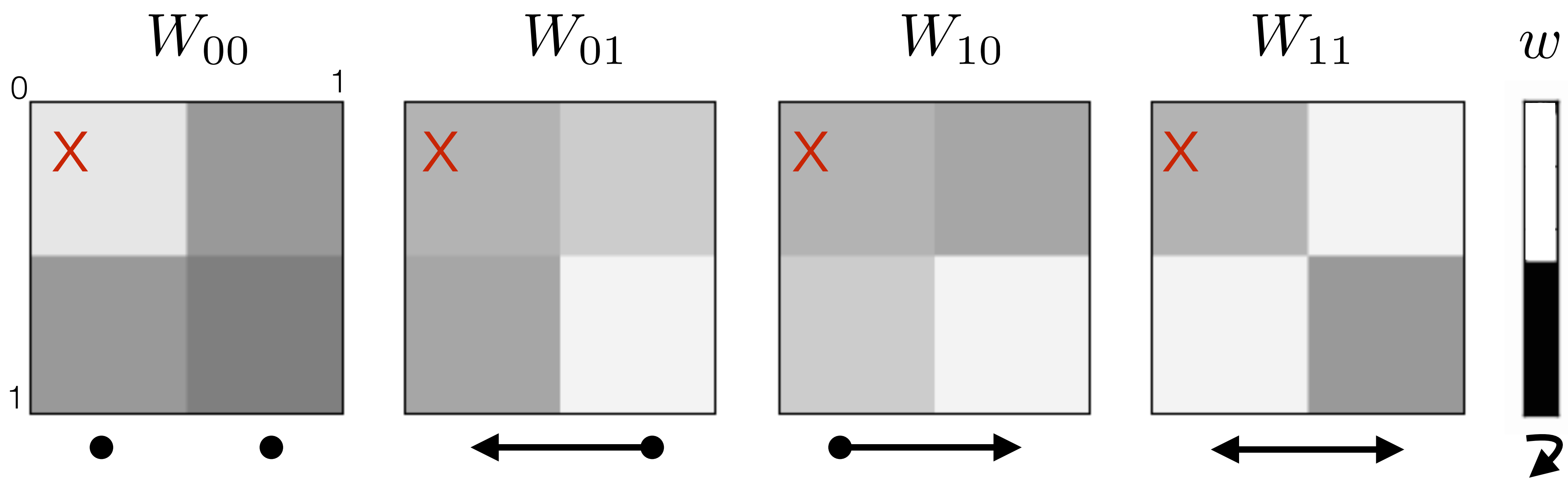

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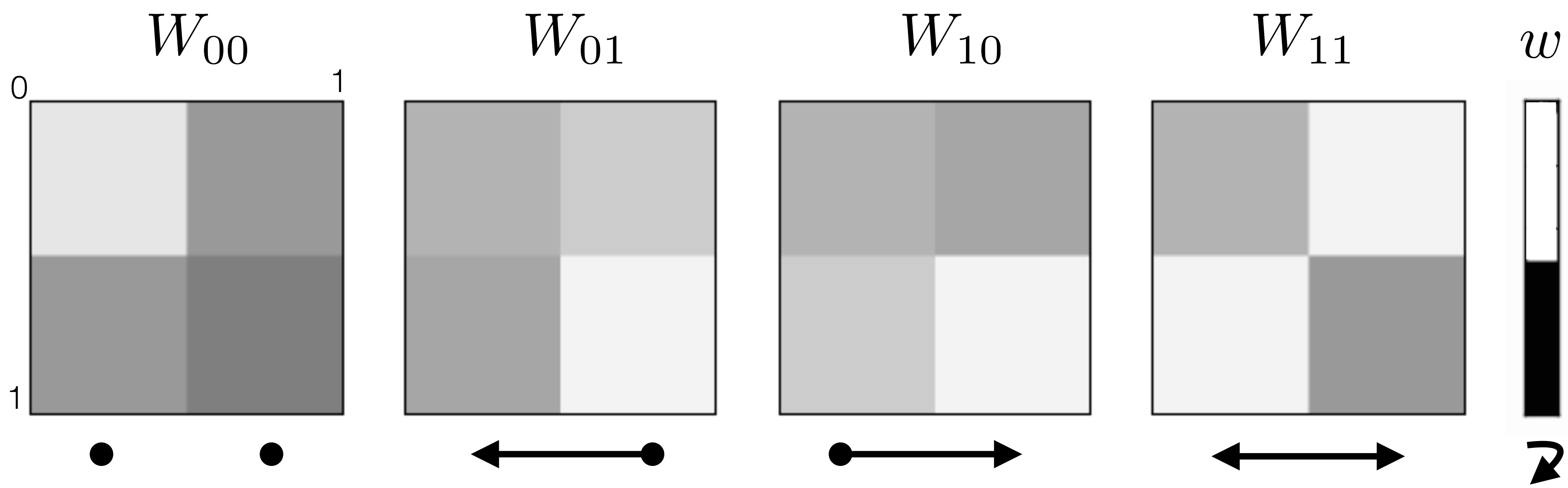
Sampling procedure



- Define $\boldsymbol{W}_4(x,y) := (W_{00}(x,y), W_{01}(x,y), W_{10}(x,y), W_{11}(x,y))$

$$U_i \sim \mathrm{Uniform}[0,1]$$
$$E_{ij} \sim \mathrm{Categorical}(\boldsymbol{W}_4(U_i,U_j)),\ \text{for } i<j$$
$$G_{ij}, G_{ji} = \text{directed edge(s) corresponding to } E_{ij}, \text{ for } i<j$$
$$G_{ii} = w(U_i)$$

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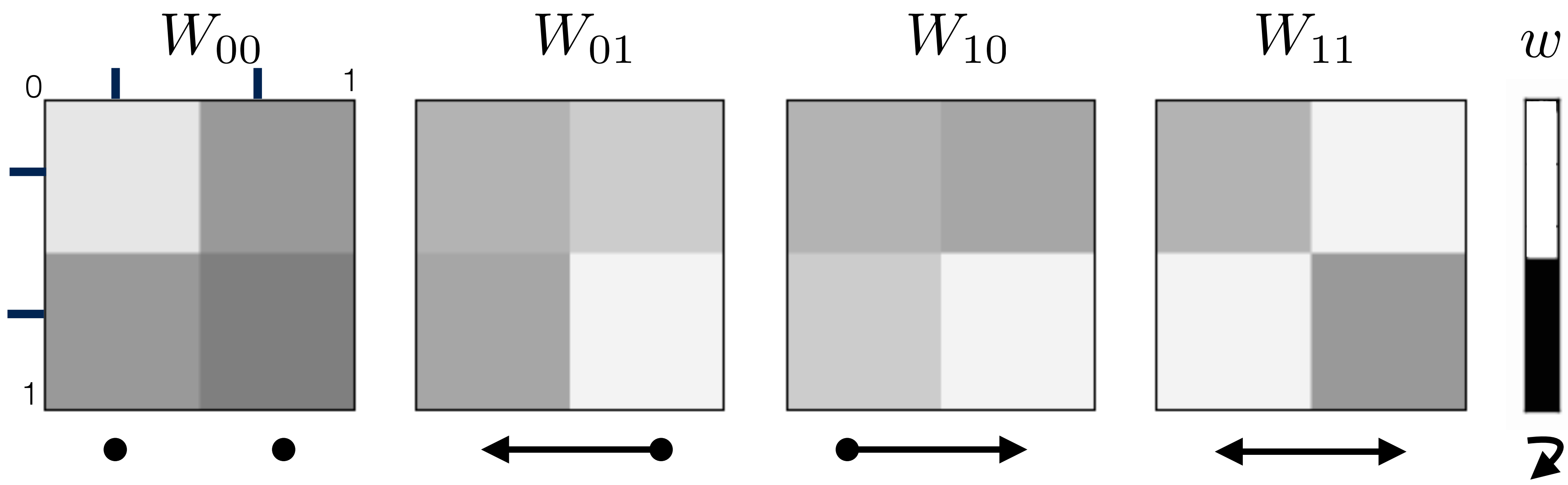

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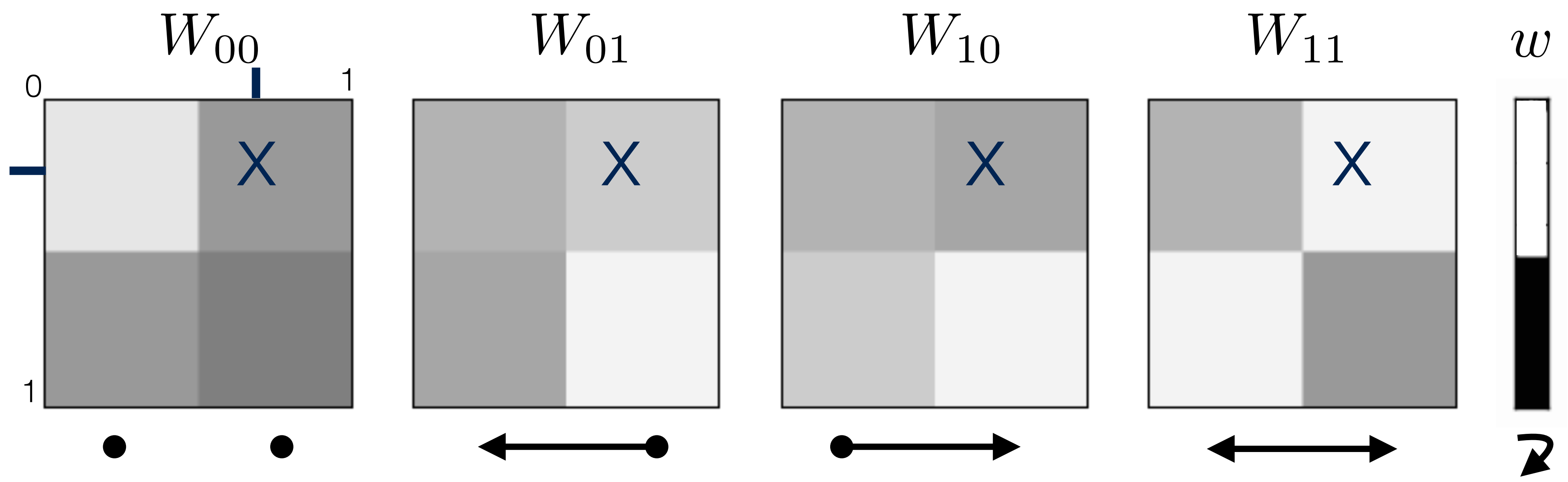

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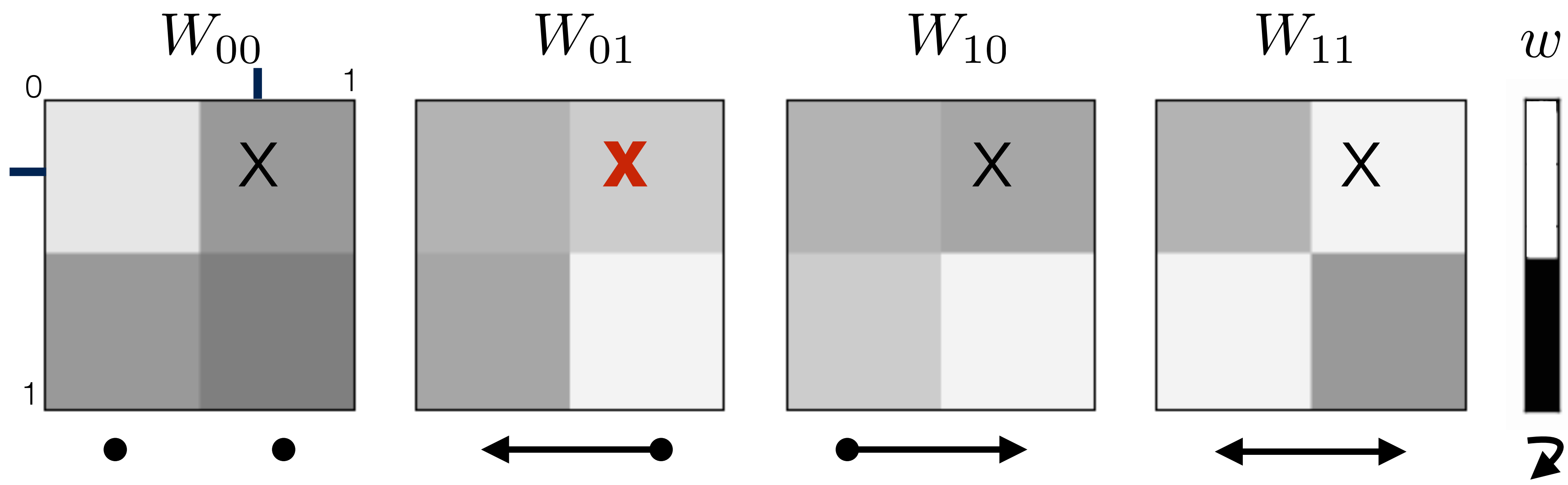

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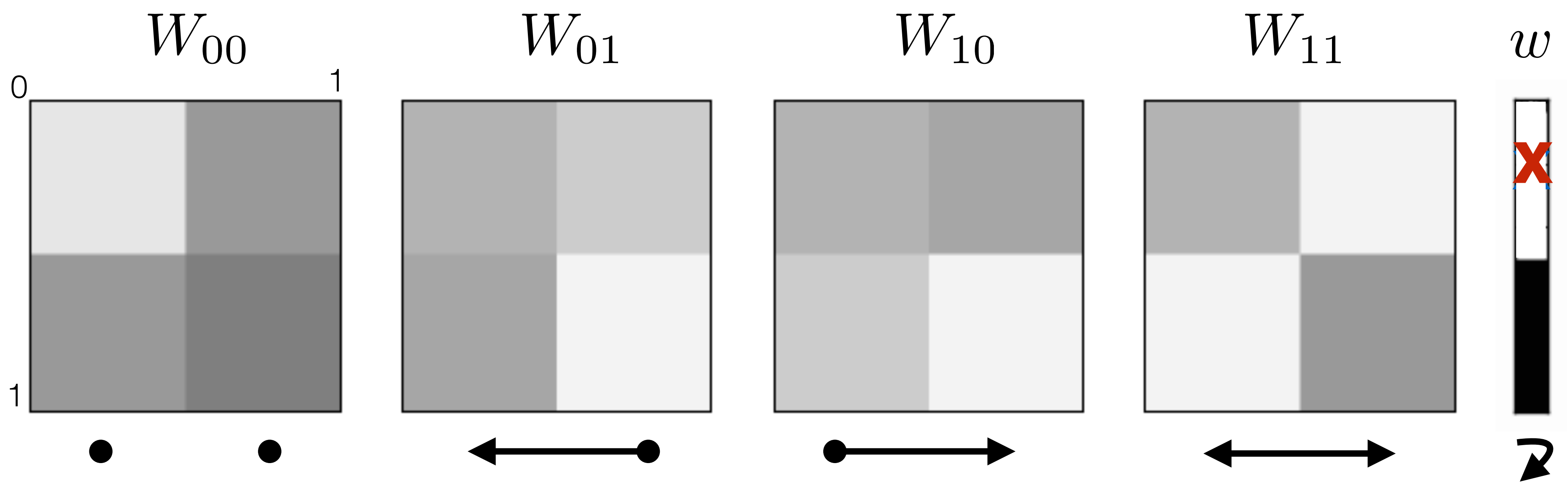

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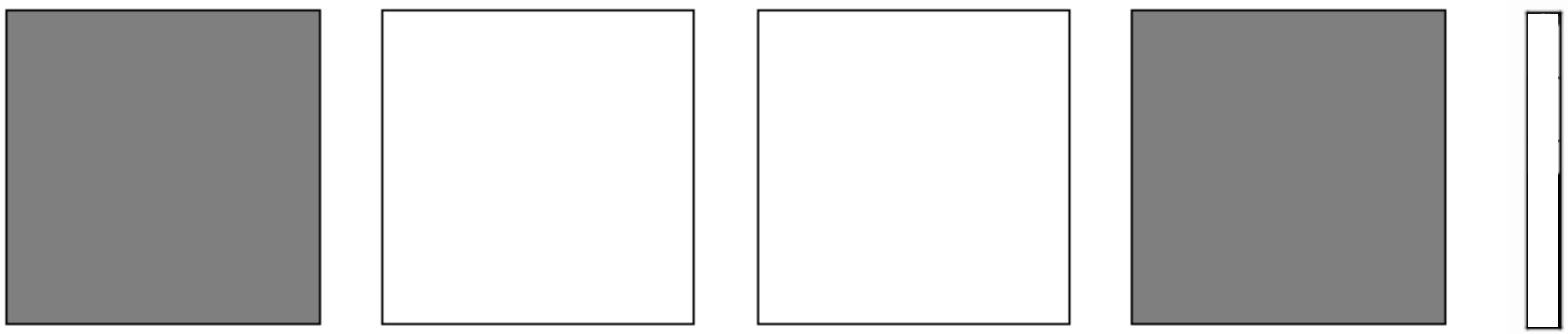
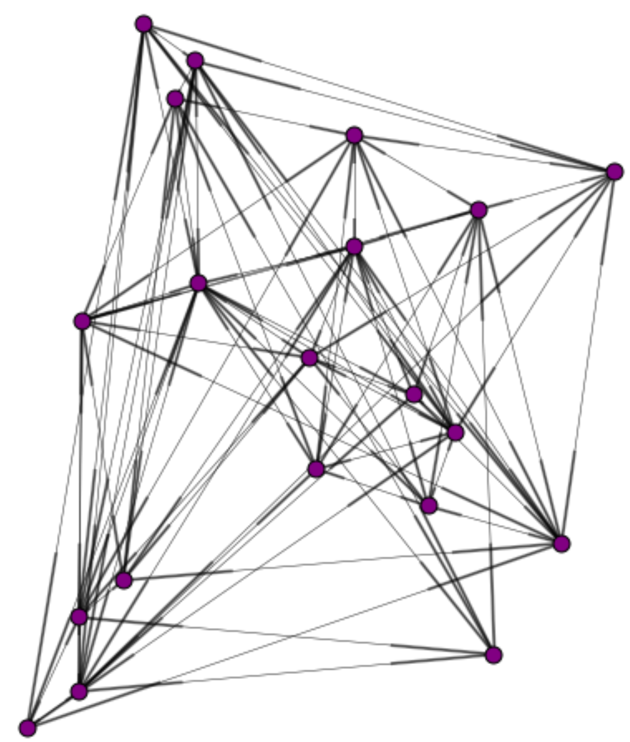
Special cases

- Undirected graphs (graphon) (e.g., ER(½)) $W : [0,1]^2 \to [0,1]$ $W(x,y) = W(y,x)$

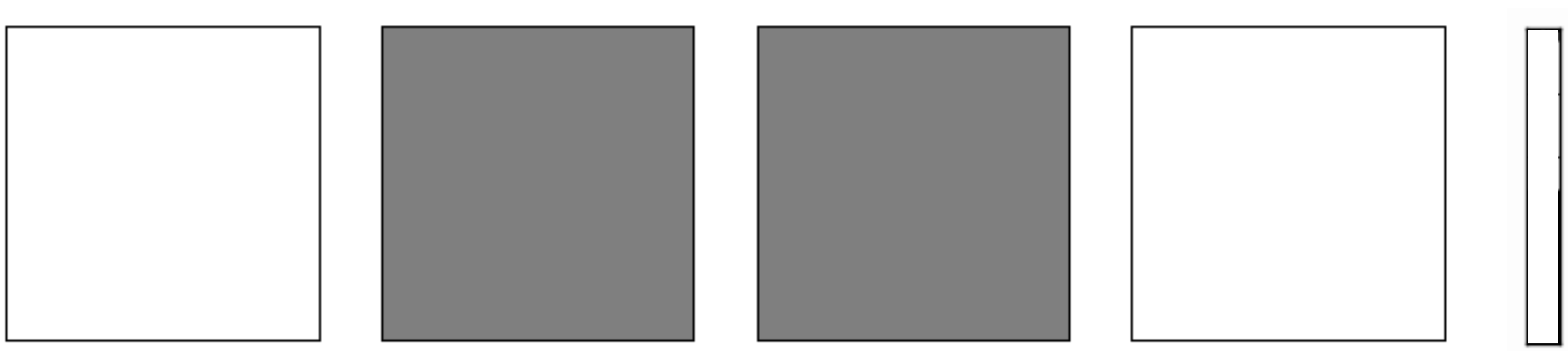


W_{00} W_{01} W_{10} W_{11} w

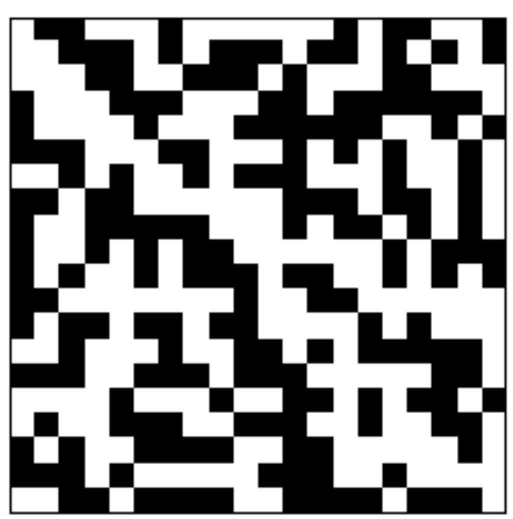


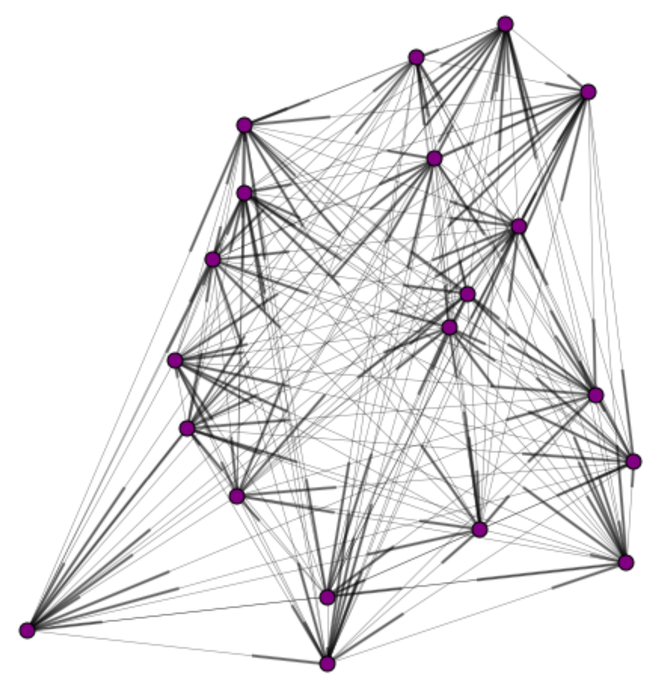
Special cases

- Tournaments (e.g., generic tournament)



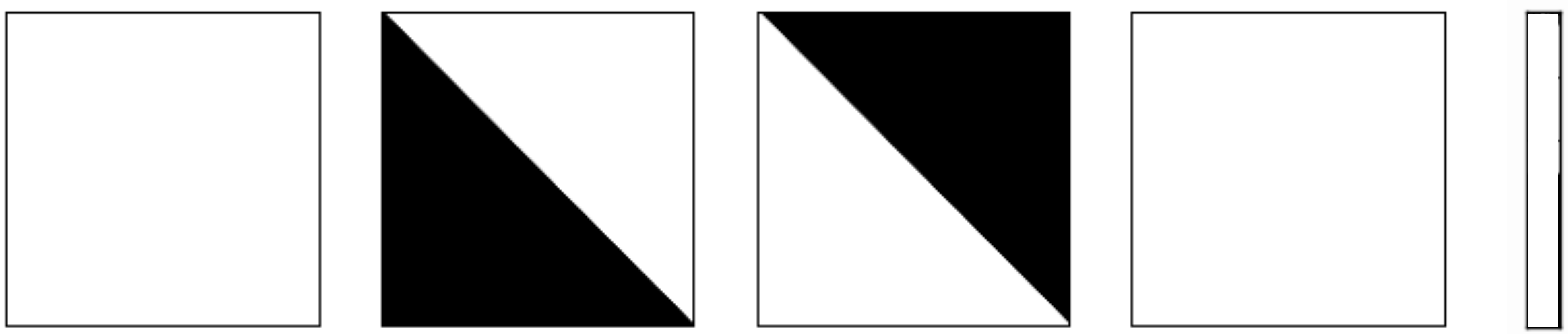
W_{00} W_{01} W_{10} W_{11} w



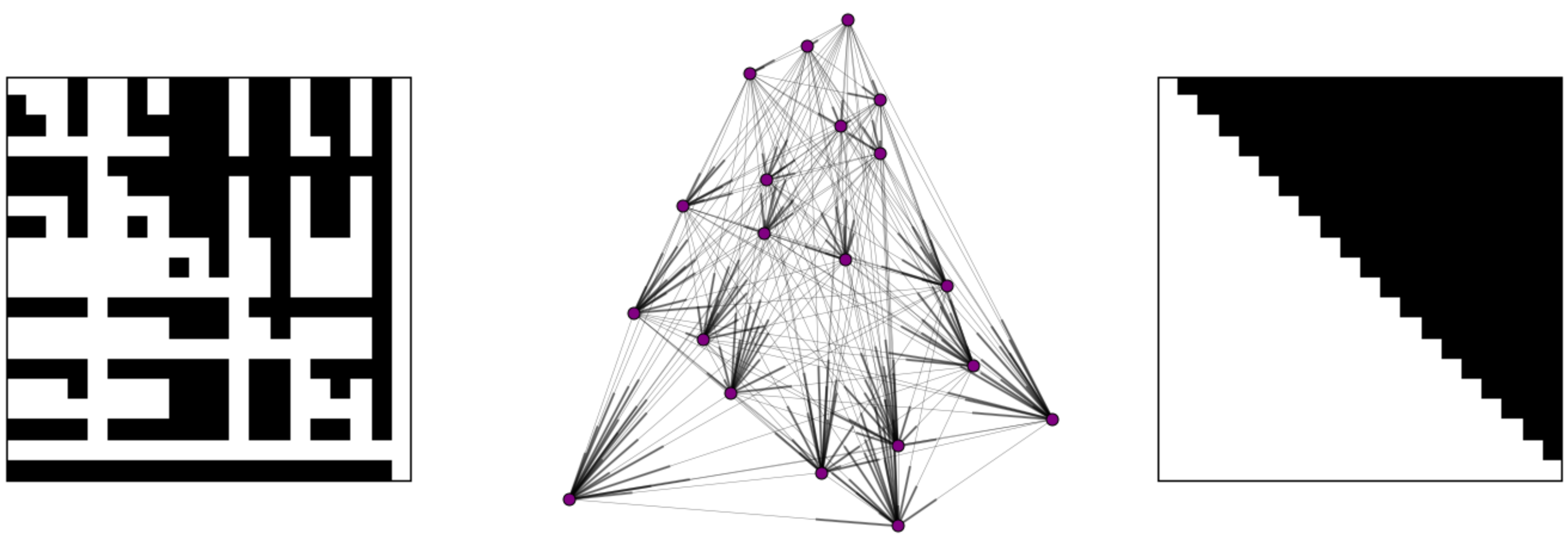


Special cases

- Linear ordering (the only one, by Glasner–Weiss)

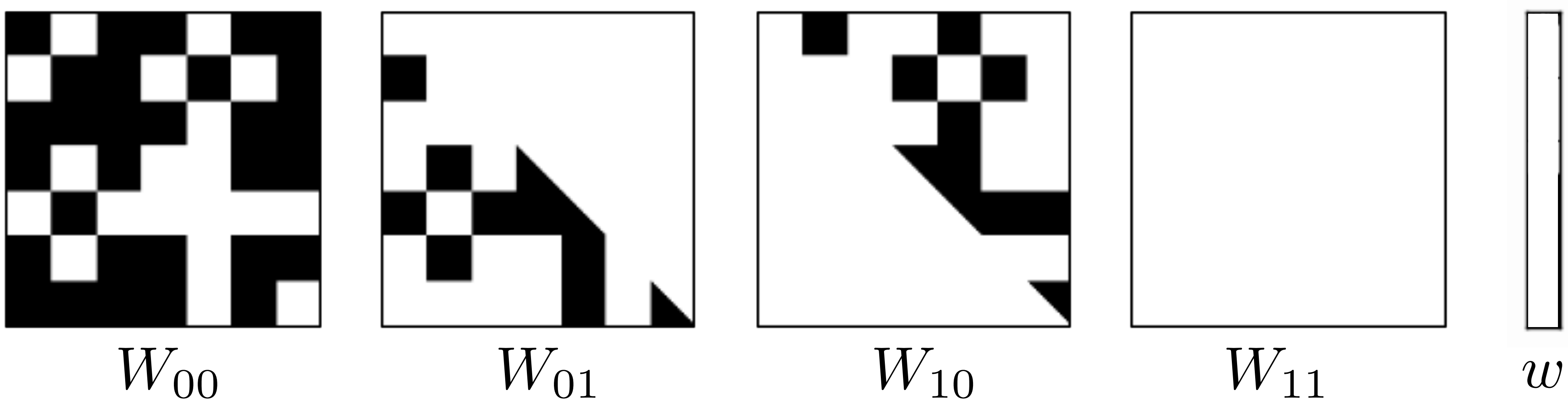


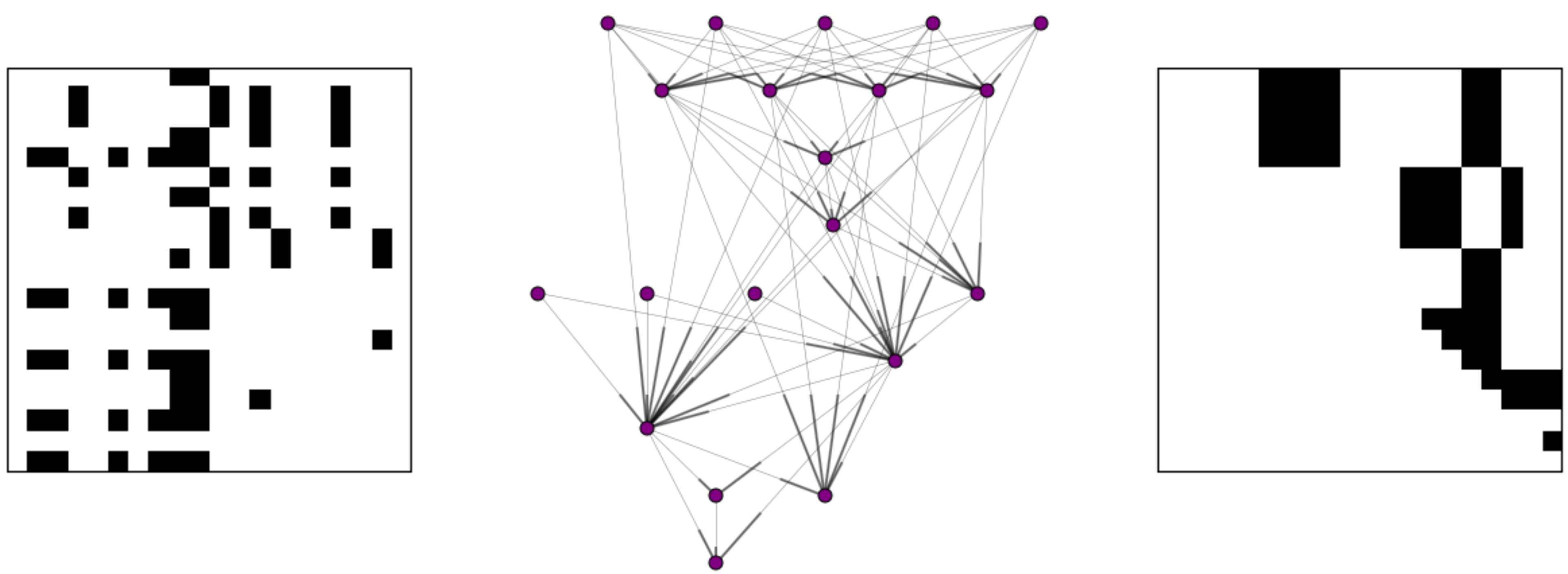
W_{00} W_{01} W_{10} W_{11} w



Special cases

- Directed acyclic graphs (DAGs)

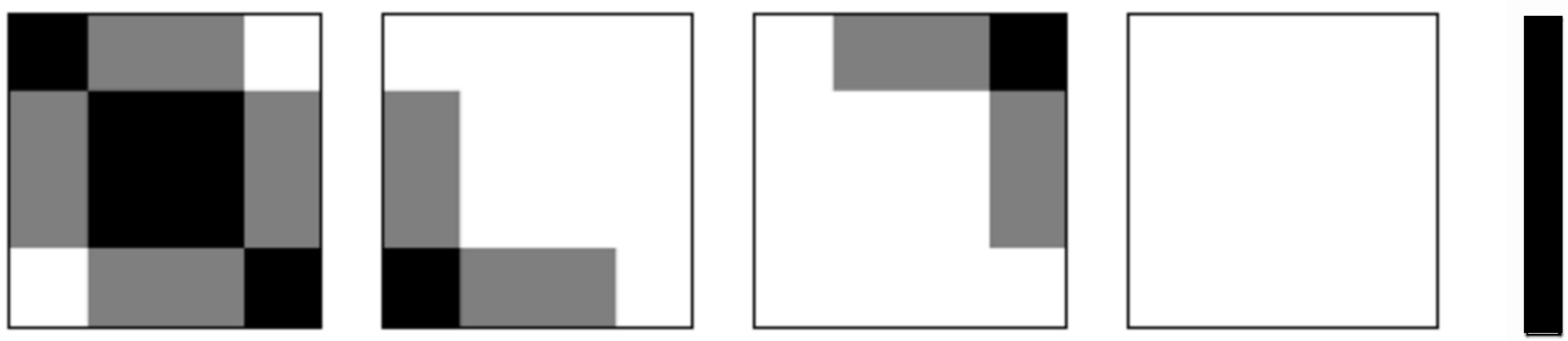




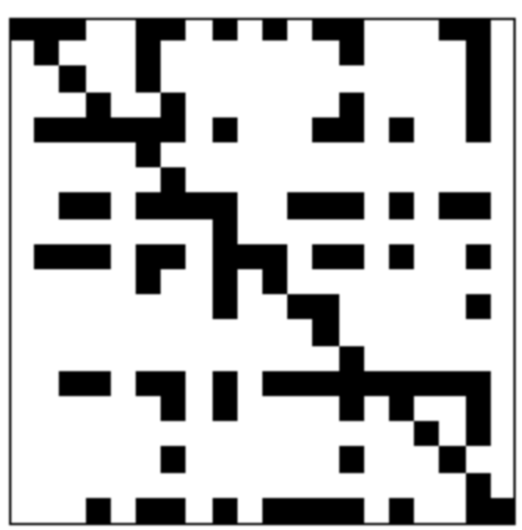
[Cai–Ackerman–Freer, 2015]

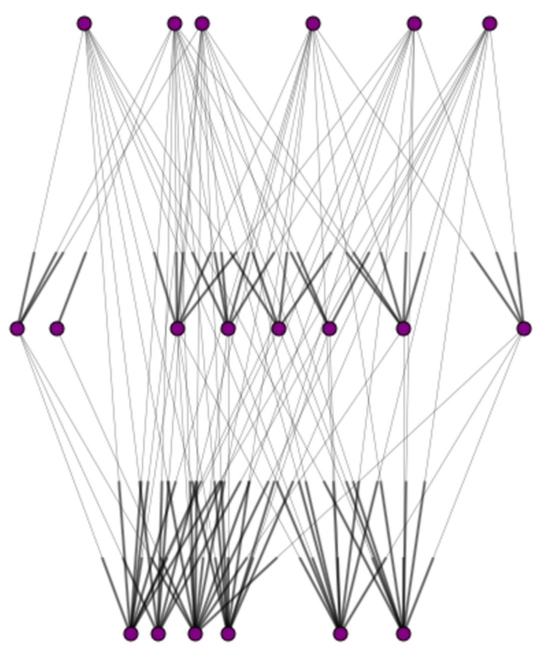
Special cases

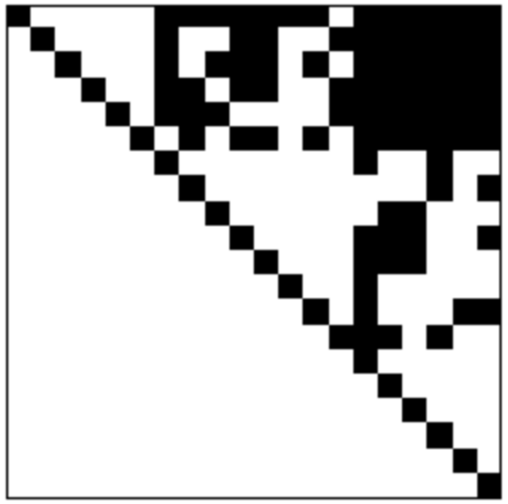
- Partial ordering (poset)



W_{00} W_{01} W_{10} W_{11} w

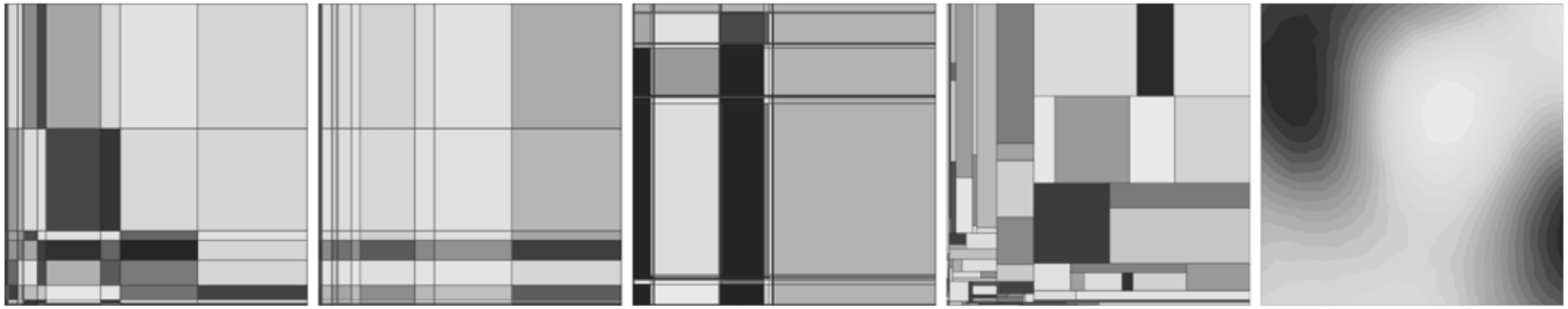






Priors on digraphons

- Can extend literature for graphon priors (cf. Orbanz–Roy) to directed graphs, e.g., infinite relational model



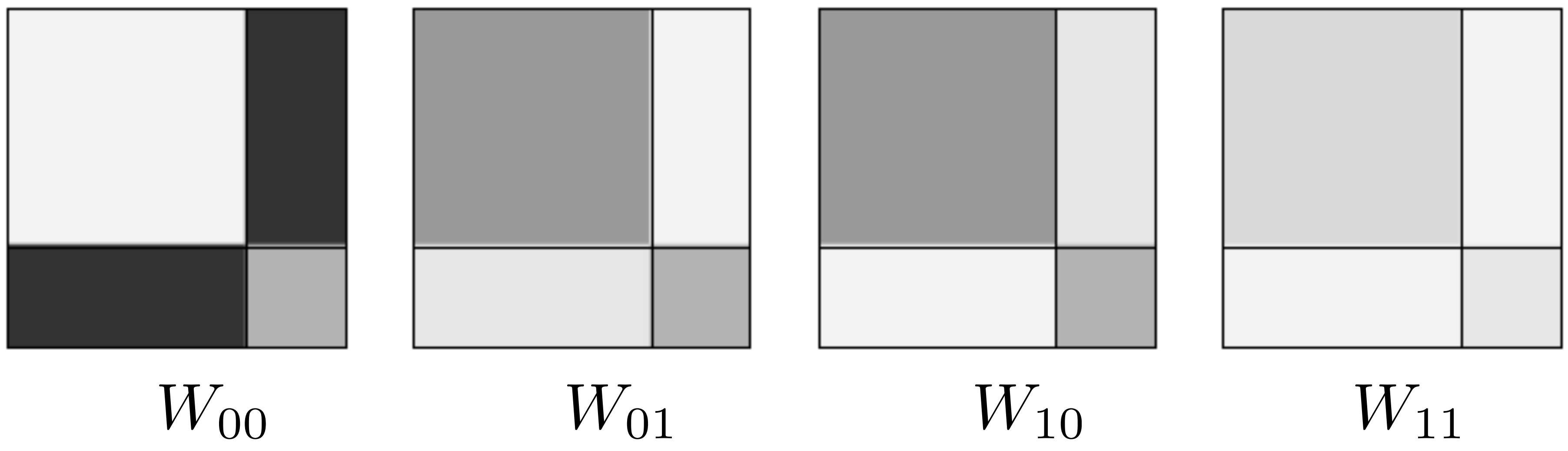
[Orbanz–Roy, 2015]

- Some of these models are already intended for directed graphs, via an **asymmetric** measurable function (to describe independent edge directions).

Directed block models

- In a block model, pairs of regions can vary in how tournament-like, as well as how dense they are:
  - e.g., Directed stochastic block model [Wang–Wong, 1987]

Example of SBM digraphon, 0.7 division

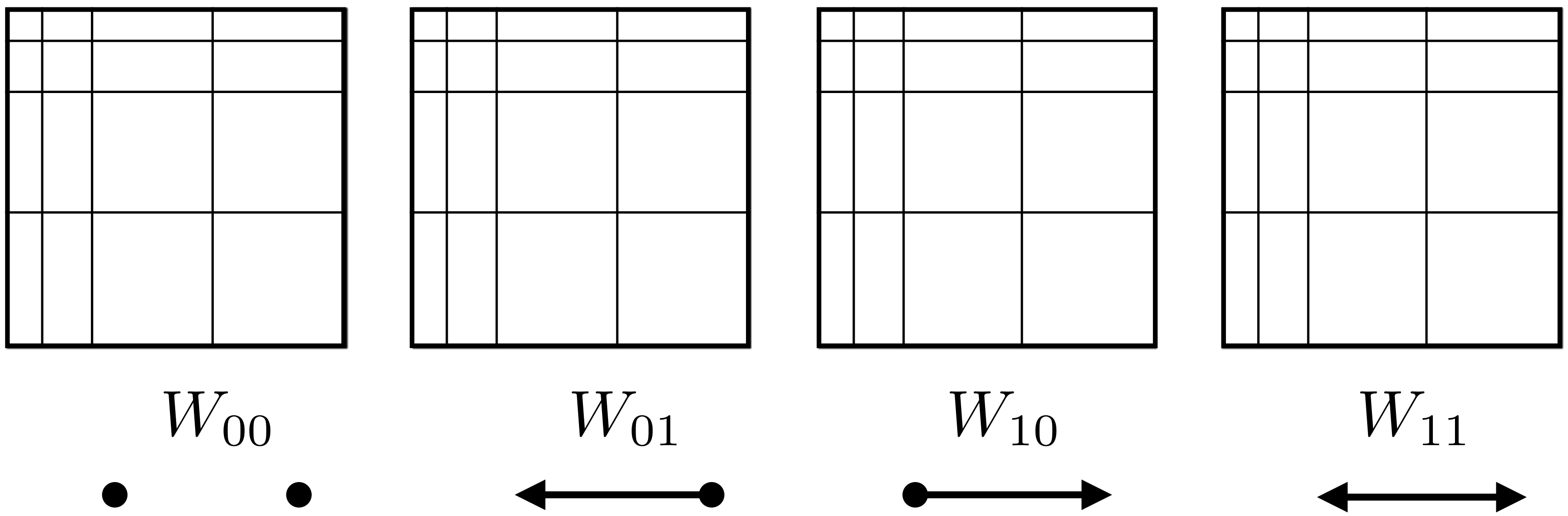


[Cai–Ackerman–Freer, 2015]

Infinite digraph block model

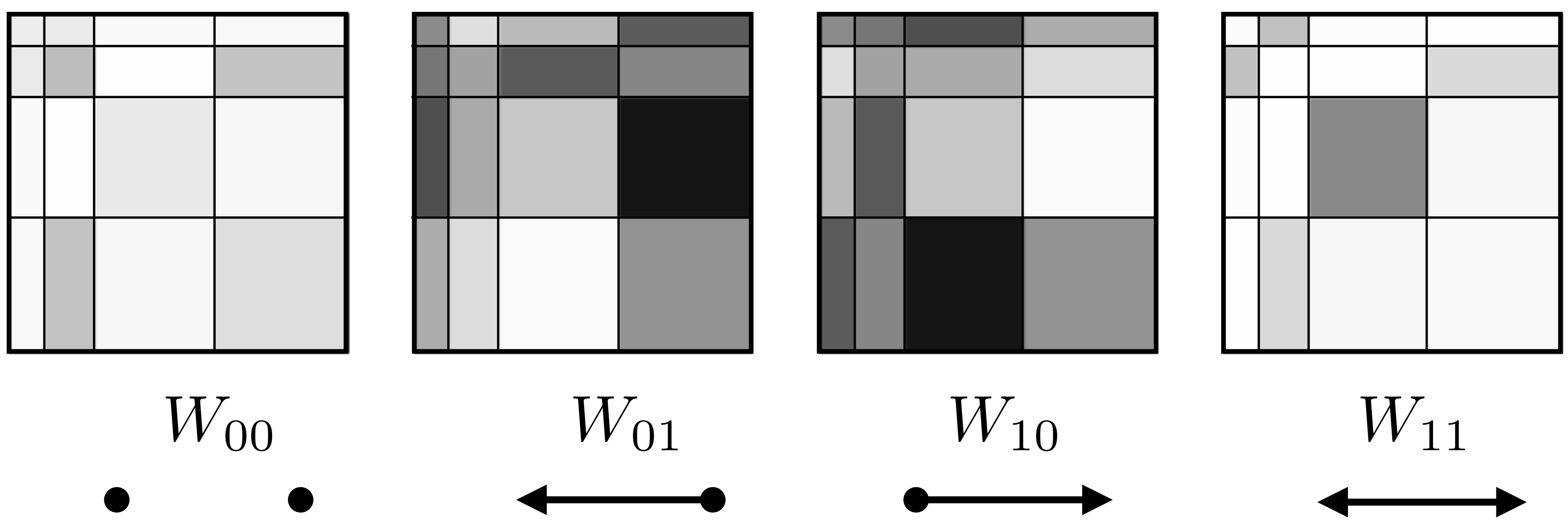
W_{00} $\quad$ W_{01} $\quad$ W_{10} $\quad$ W_{11}

Infinite digraph block model



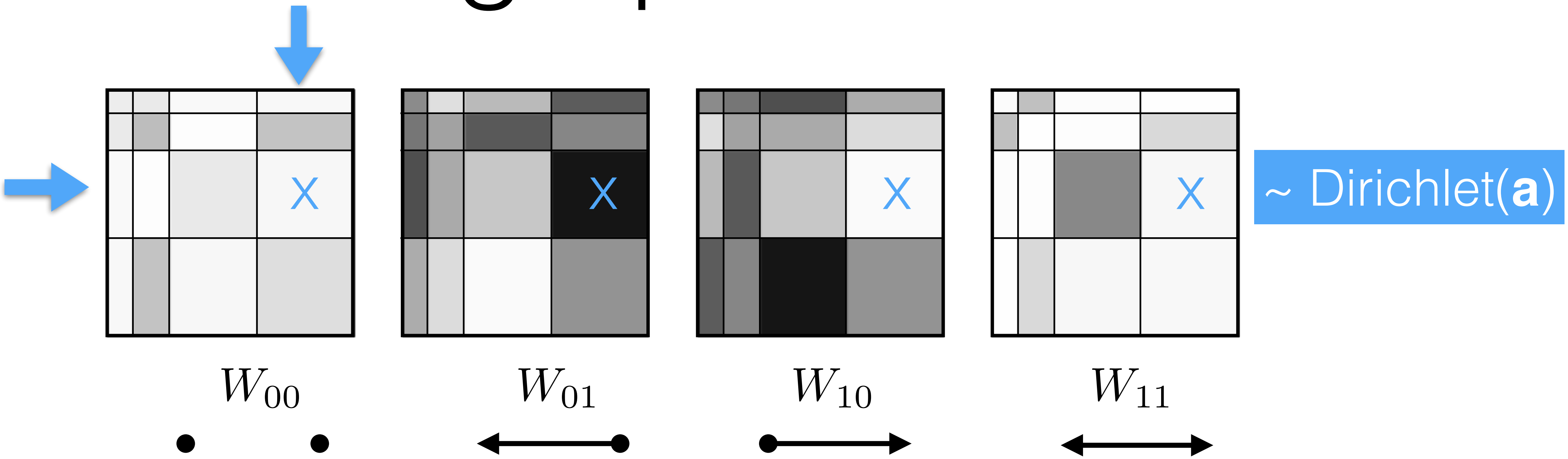
1. Draw partition ~ DP-Stick(alpha)

Infinite digraph block model



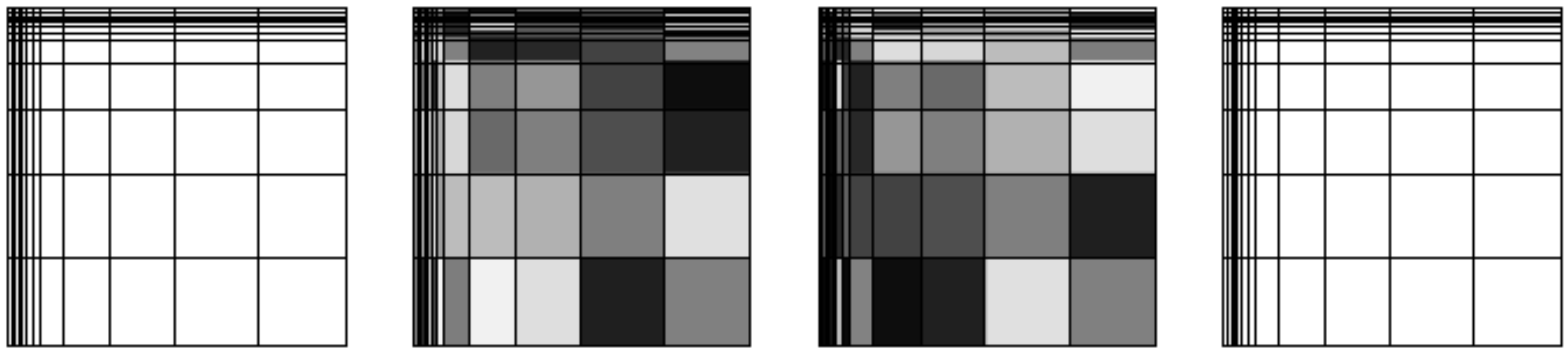
1. Draw partition ~ DP-Stick(alpha)
2. Draw weights ~ Dirichlet(**a**)

Infinite digraph block model



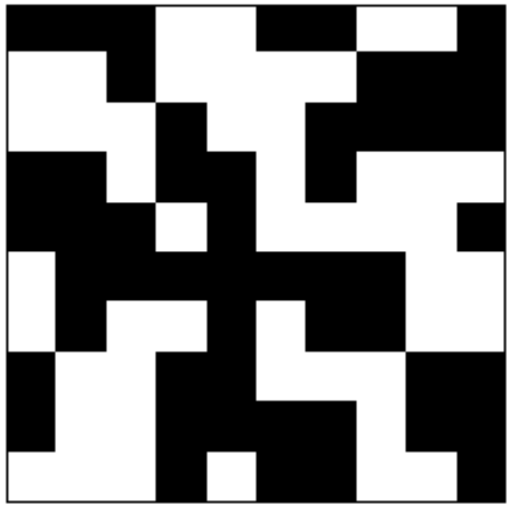
1. Draw partition ~ DP-Stick(alpha)
2. Draw weights ~ Dirichlet(**a**)

Infinite digraph block model

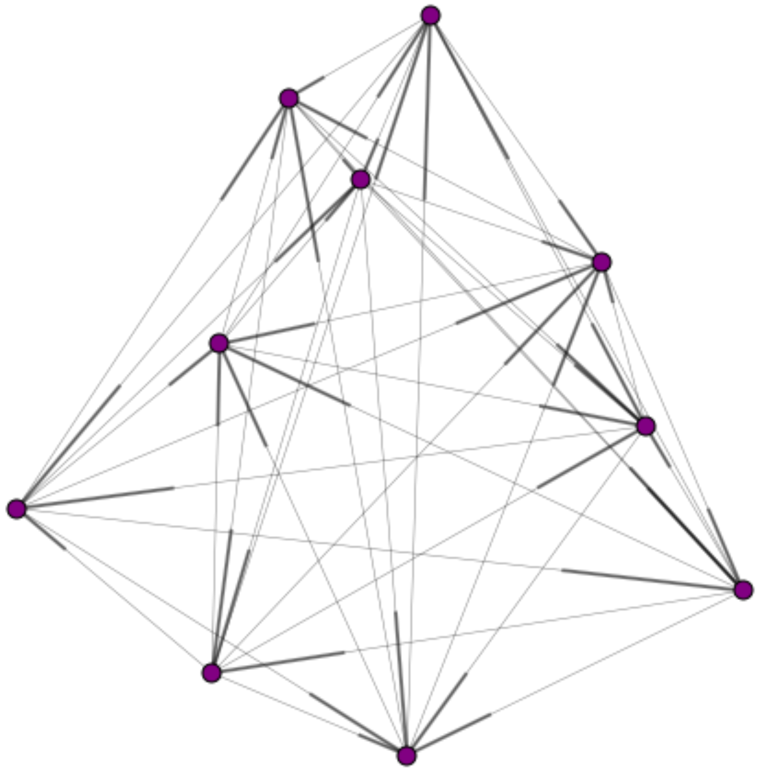


Tournament

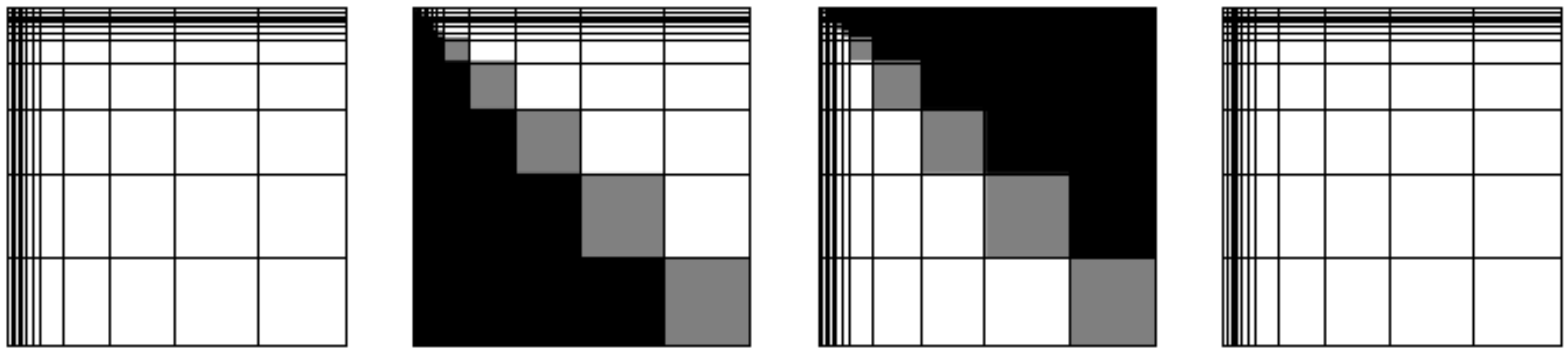
Dirichlet parameter a = [0, 2, 1, 0]



sample

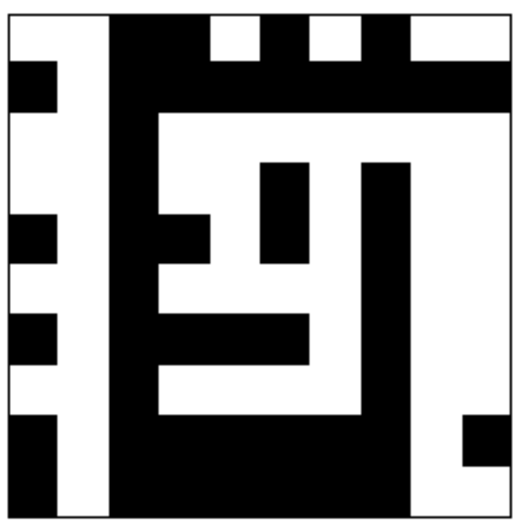


Infinite digraph block model

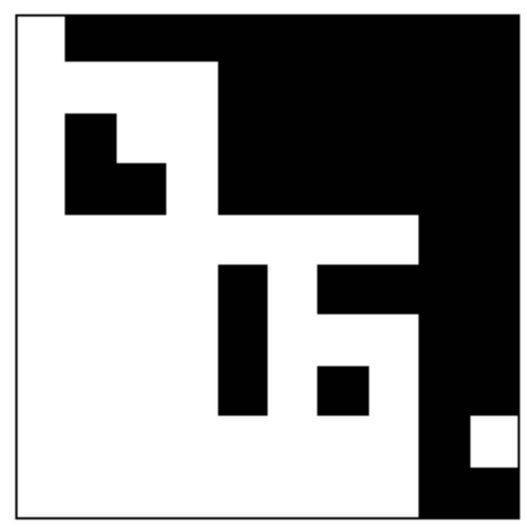


Almost totally ordered

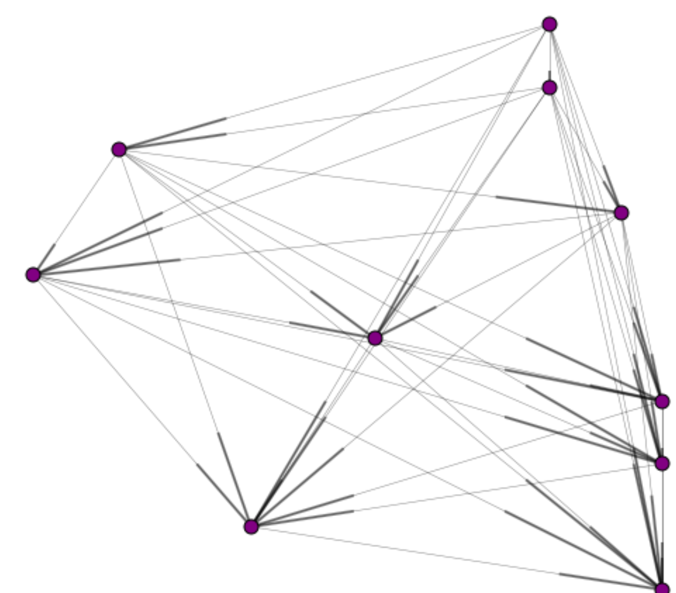
Dirichlet parameter a = [0, 0, 1, 0]



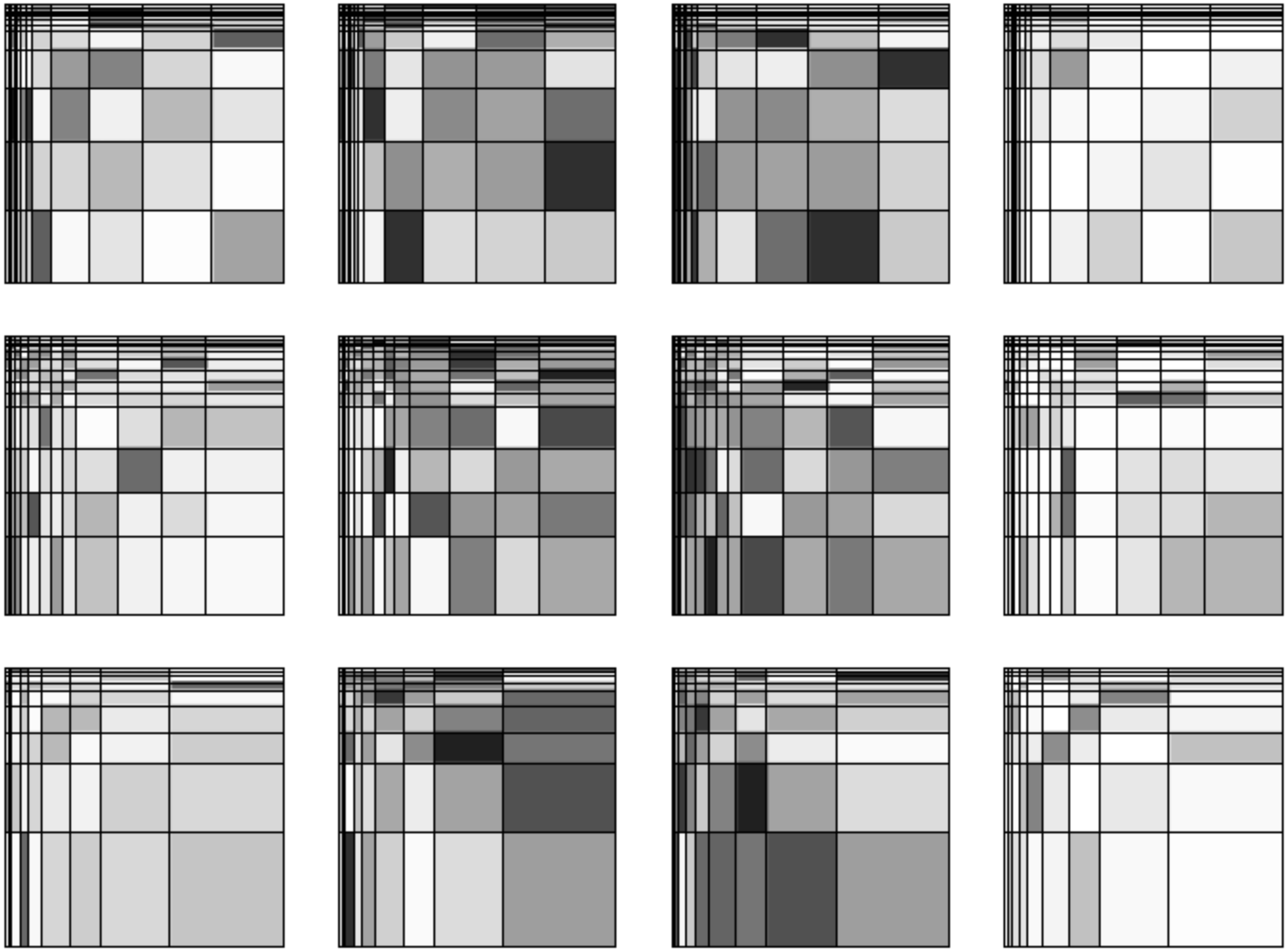
sample



reordered



Infinite digraph block model



Dirichlet parameter a = [0.9, 2.0, 1.0, 0.5]

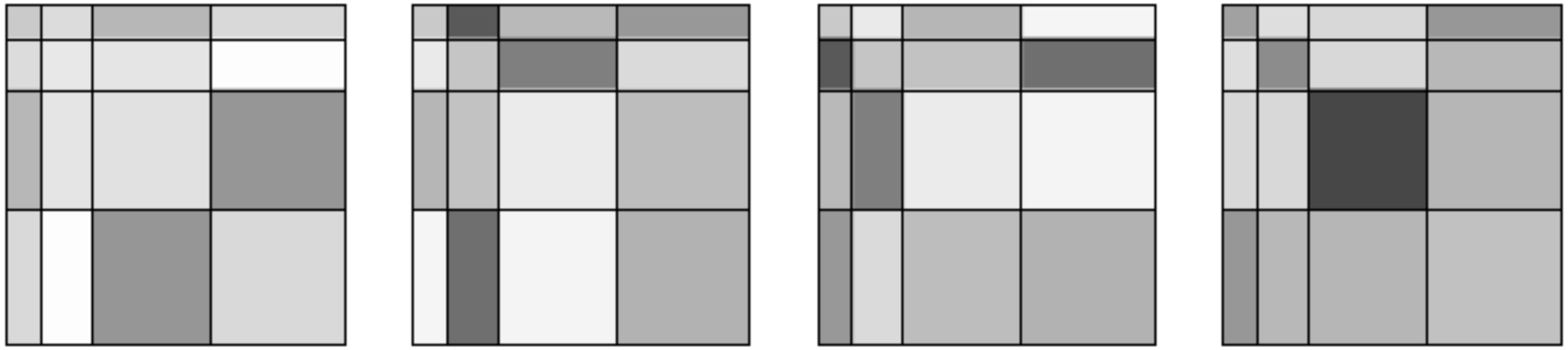
Infinite digraph block model

- Inference via collapsed Gibbs sampling of cluster assignments

$$p(z_i|z_{-i}, G) \propto p(z_i|z_{-i})p(G|z_i, z_{-i})$$

cluster assignment of vertex i   cluster assignments of all vertices except i

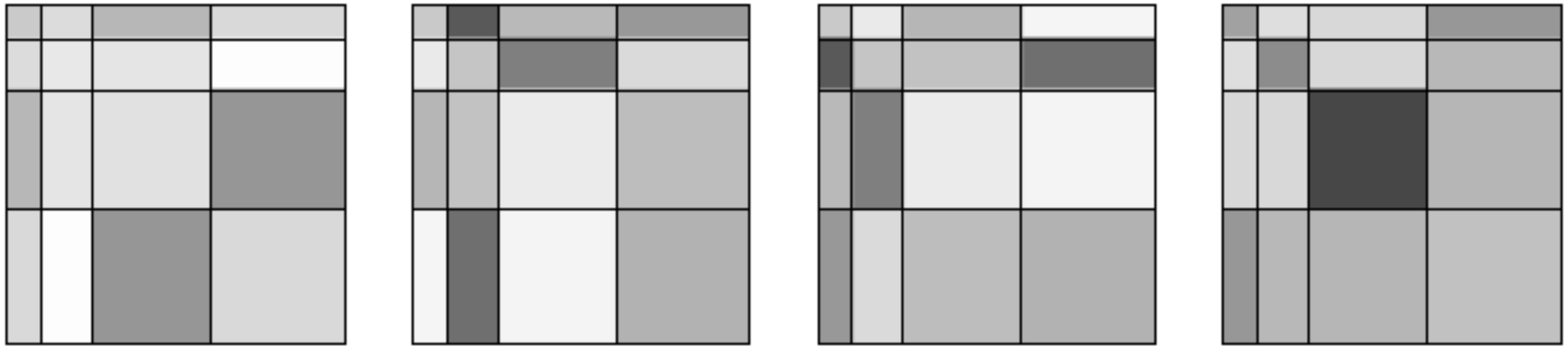
- Experiments using synthetic data



Dirichlet parameter a = [1, 1, 1, 1]

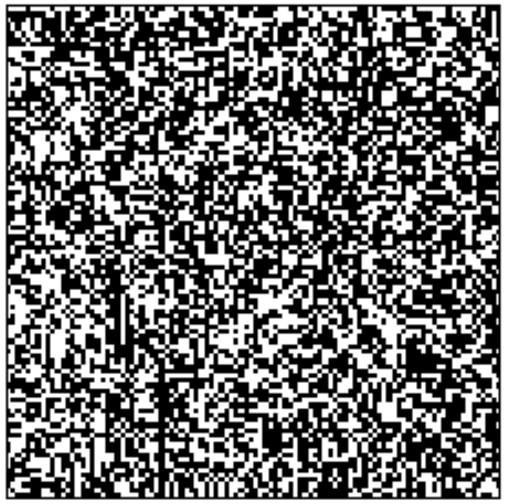
Infinite digraph block model

- Random digraphon

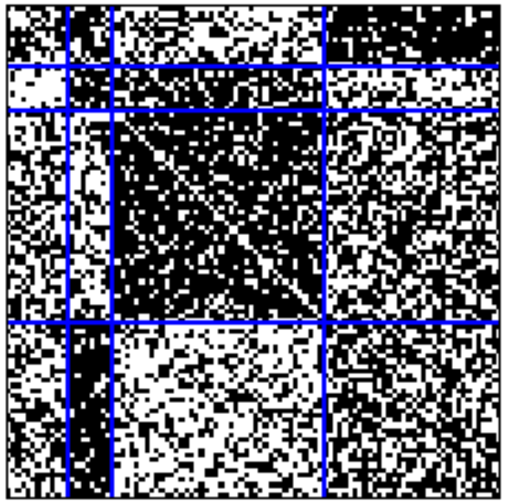


Dirichlet parameter a = [1, 1, 1, 1]

- Random sample, 100 vertices



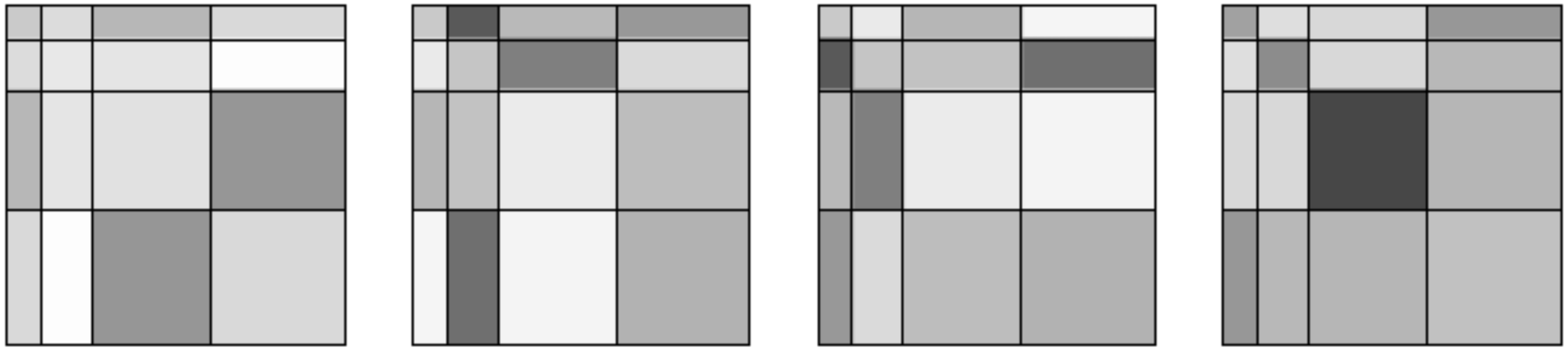
original ordering



resorted by increasing
uniform random variables

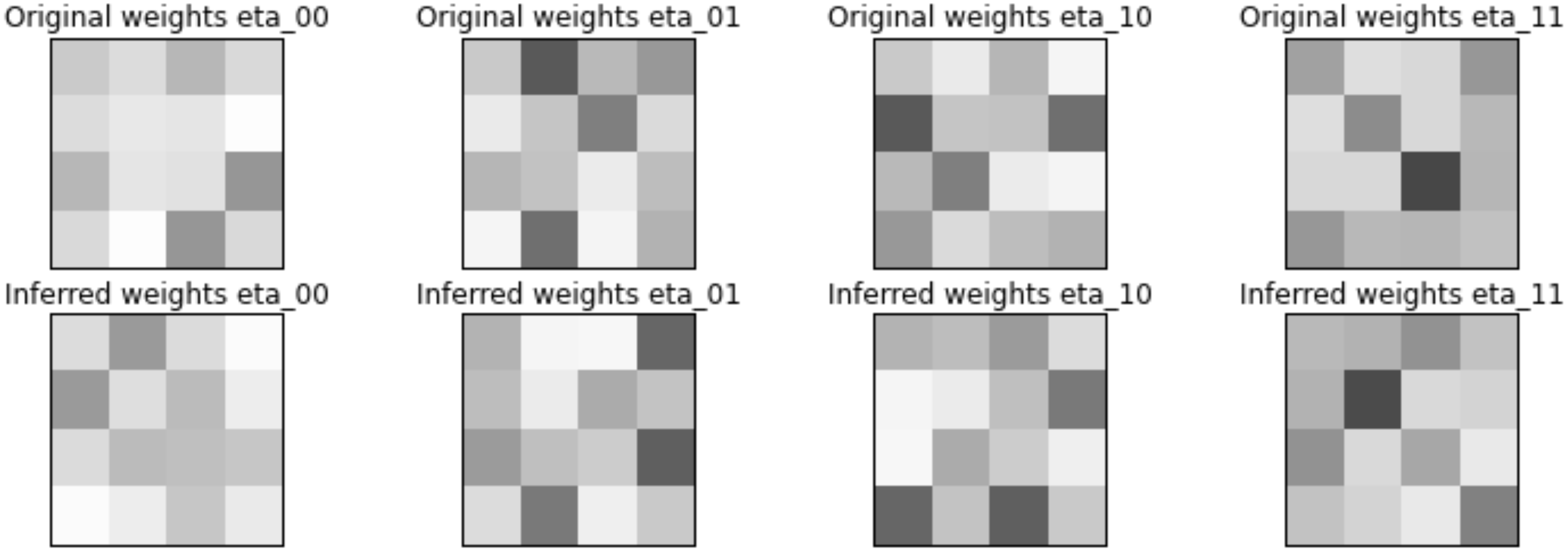
Infinite digraph block model

- Random digraphon



Dirichlet parameter a = [1, 1, 1, 1]

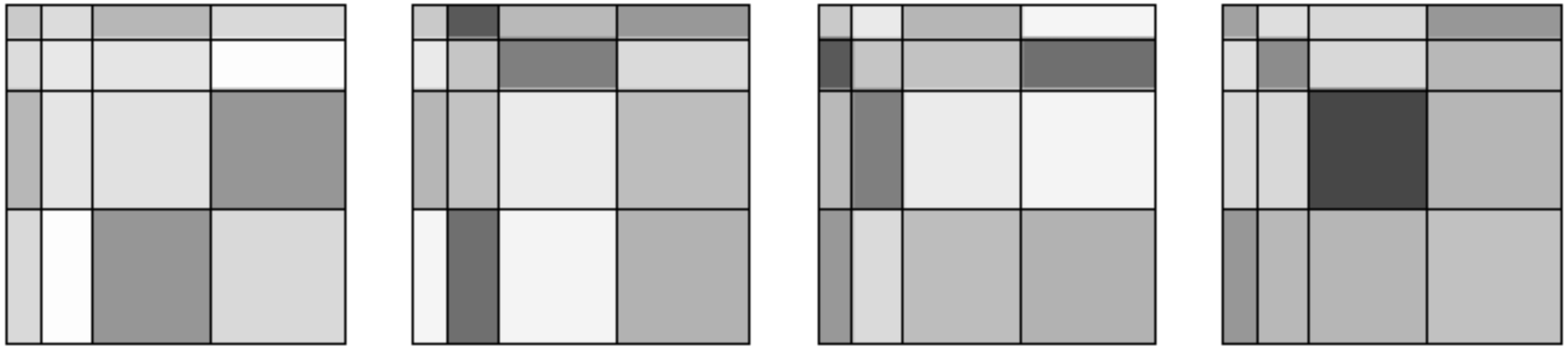
- Collapsed Gibbs sampling



[Cai–Ackerman–Freer, 2015]

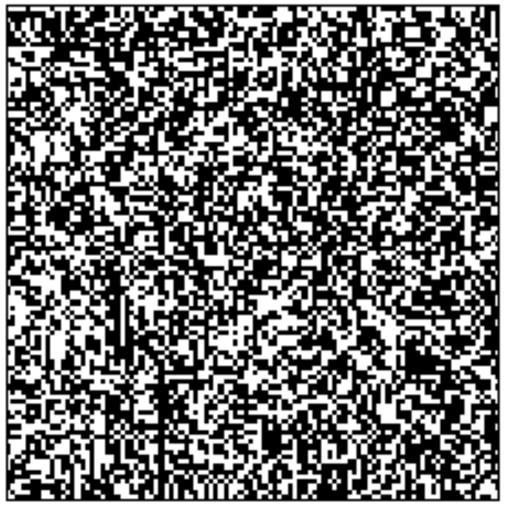
Infinite digraph block model

- Random digraphon

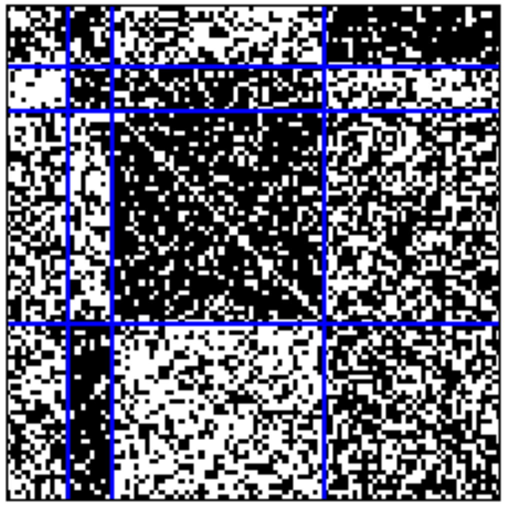


Dirichlet parameter a = [1, 1, 1, 1]

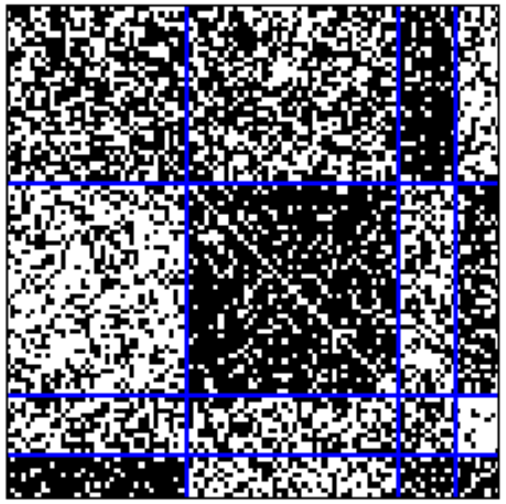
- Collapsed Gibbs sampling



original ordering



resorted by increasing uniform random variables

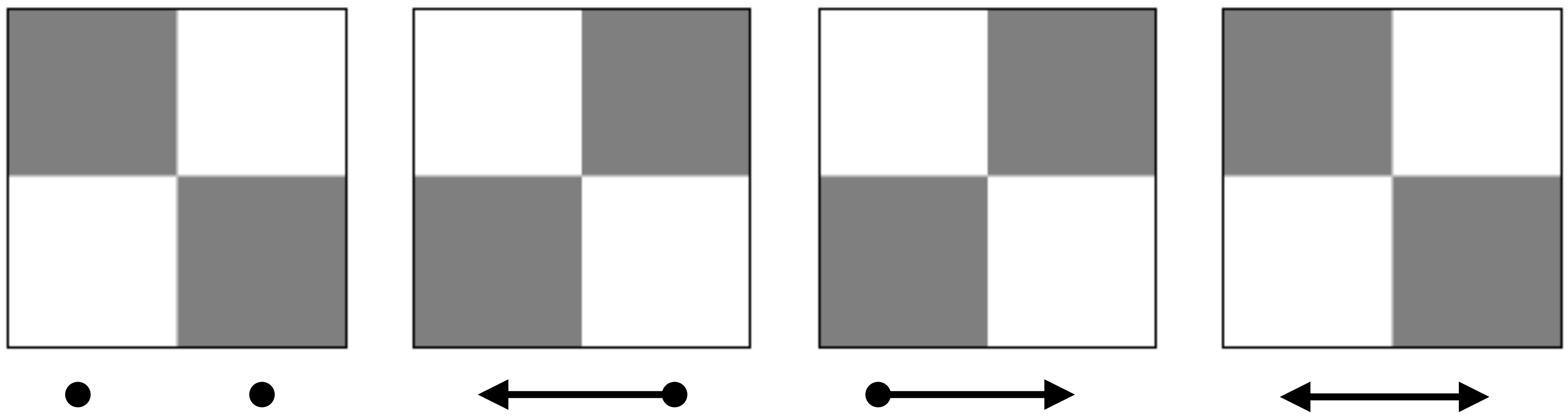


resorted by inferred cluster

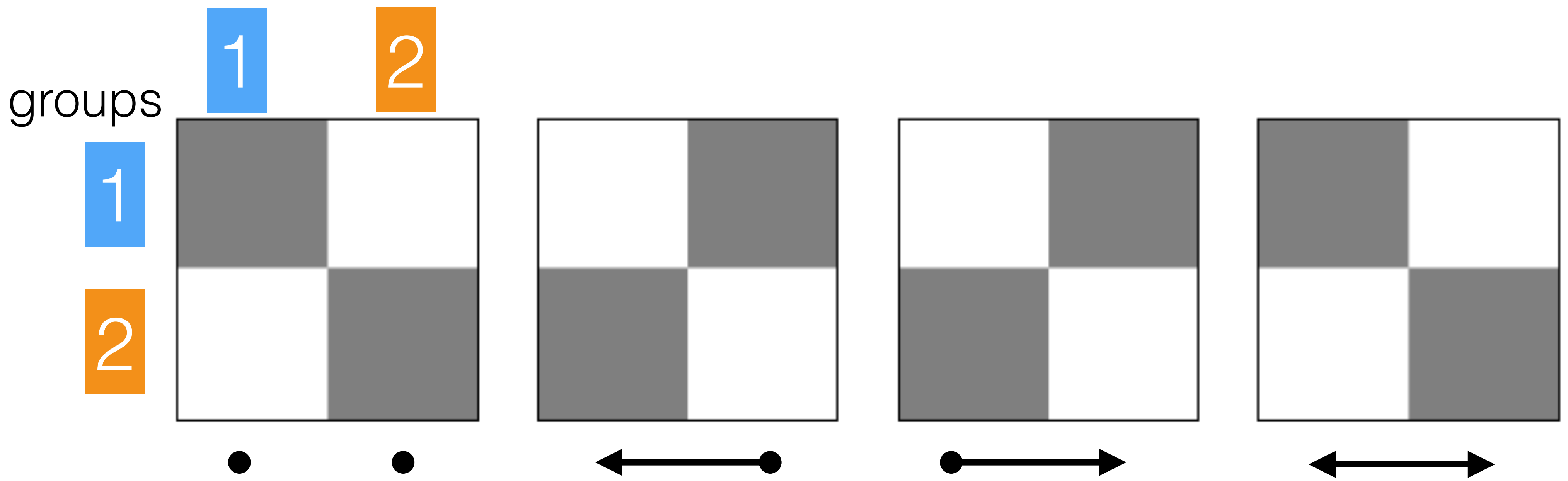
 [Cai–Ackerman–Freer, 2015]

Infinite digraph block model

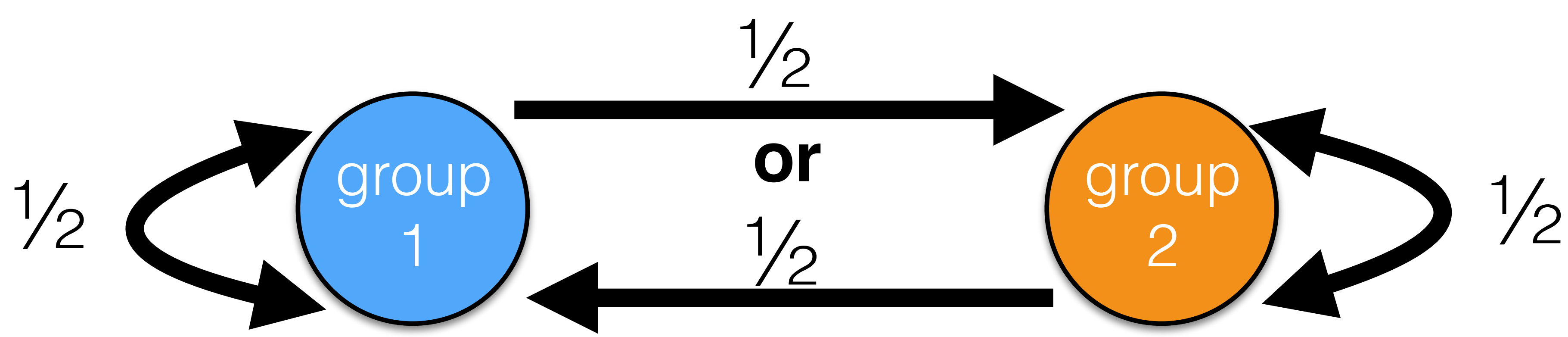
- Random digraphon: ER + tournament



Infinite digraph block model

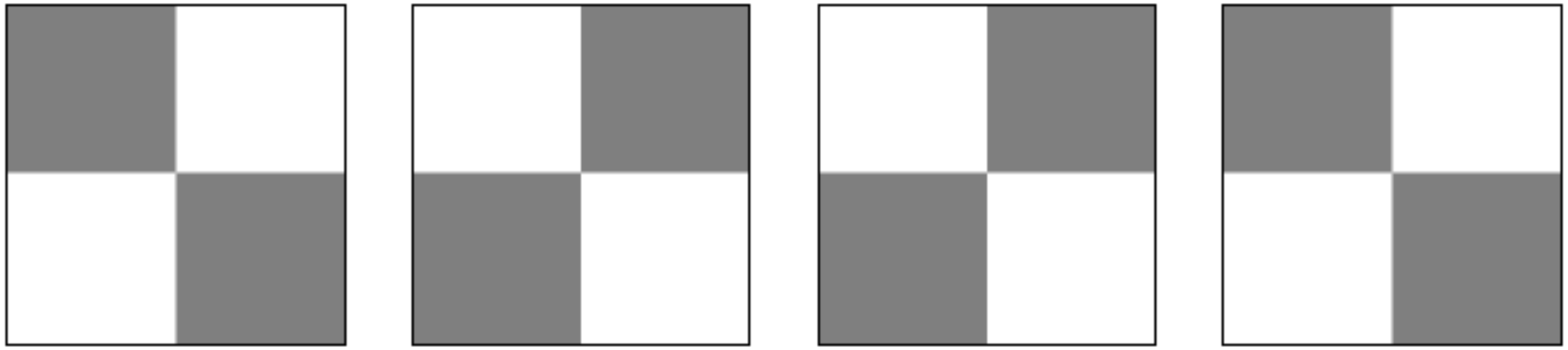


- Schematic of the sampled digraph:

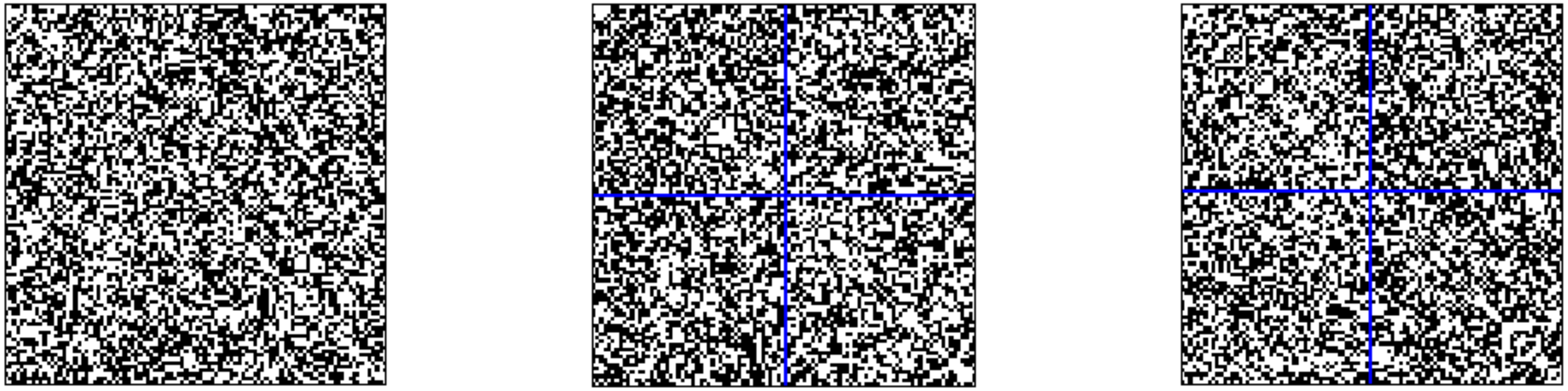


Infinite digraph block model

- Random digraphon: ER + tournament



- Collapsed Gibbs sampling for **this model**



original ordering

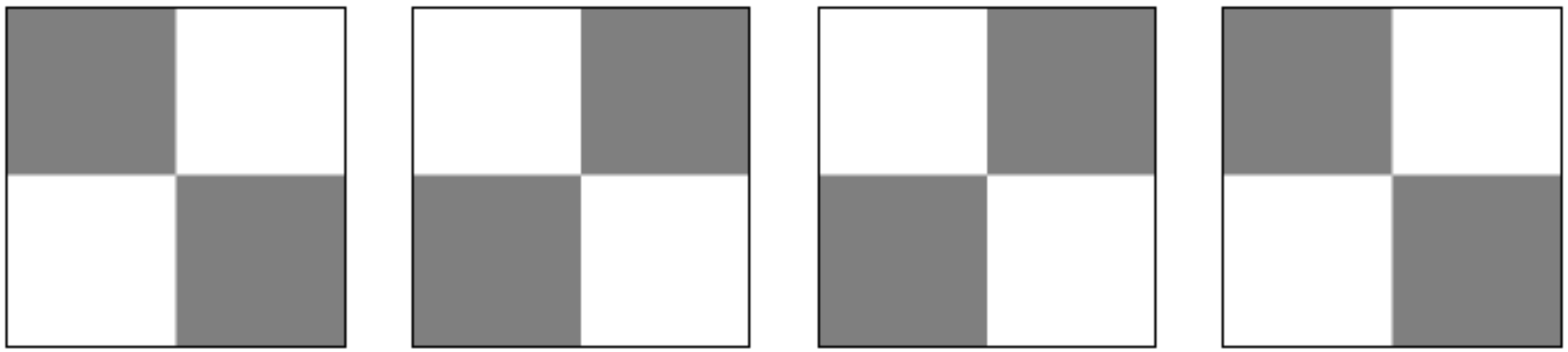
resorted by increasing uniform random variables

resorted by inferred cluster

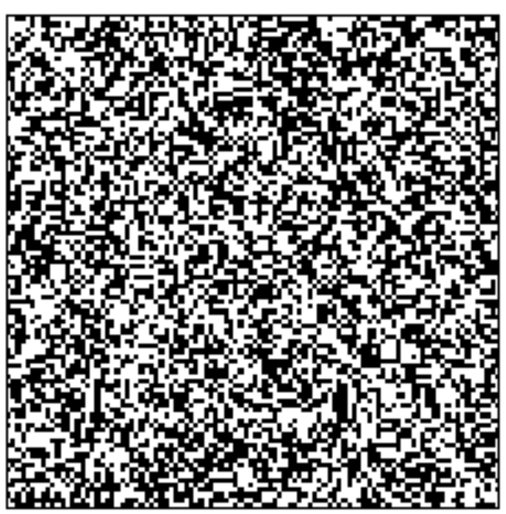
[Cai–Ackerman–Freer, 2015]

Infinite digraph block model

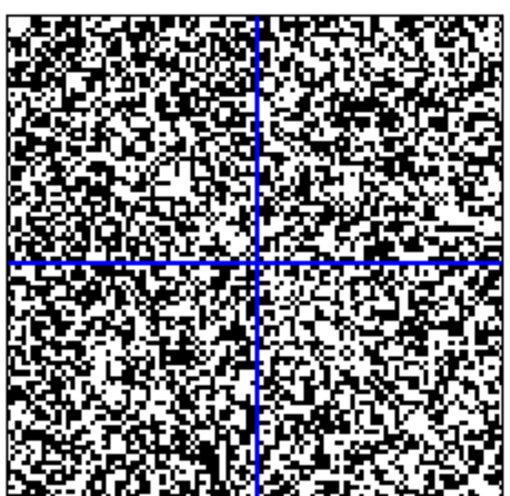
- Random digraphon: ER + tournament



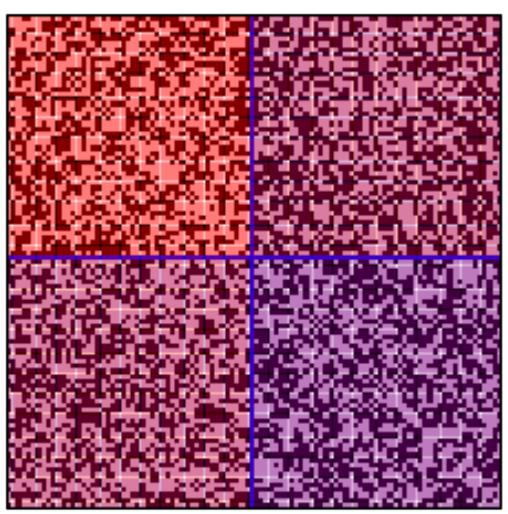
- Collapsed Gibbs sampling for **this model**



original ordering



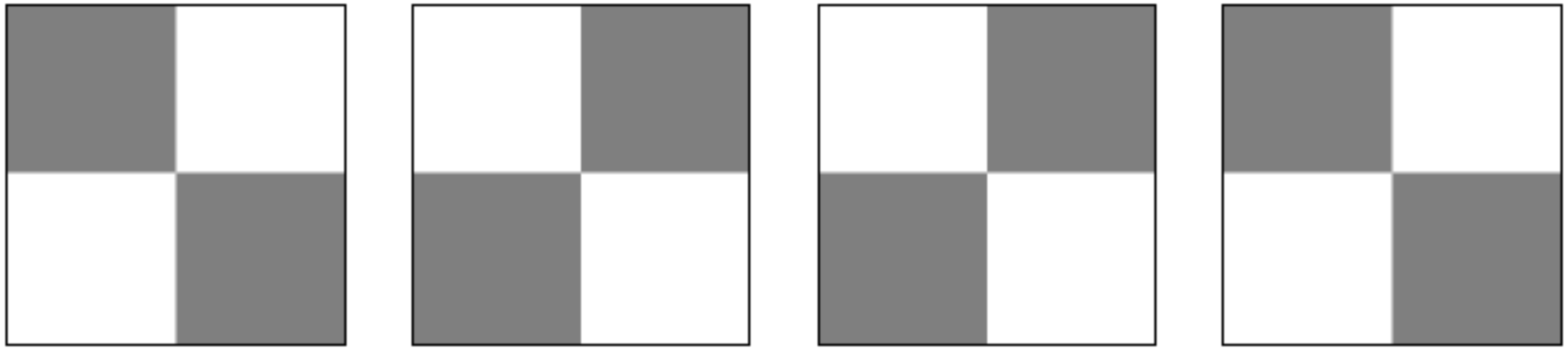
resorted by increasing uniform random variables



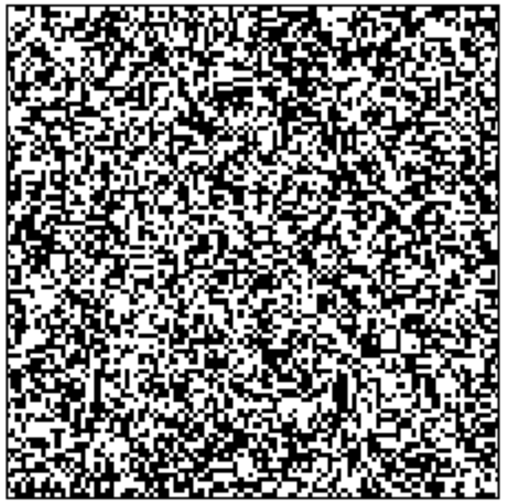
resorted by inferred cluster

Infinite digraph block model

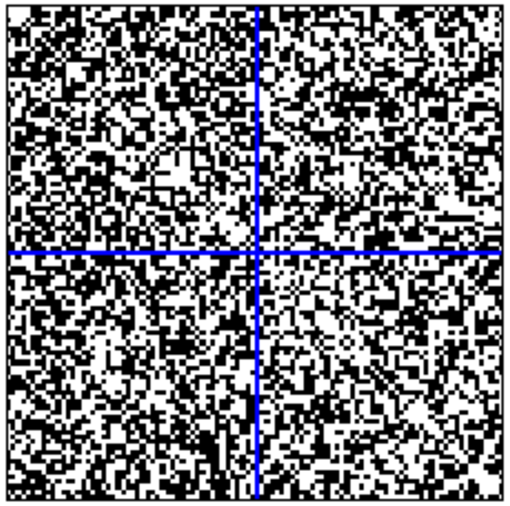
- Random digraphon: ER + tournament



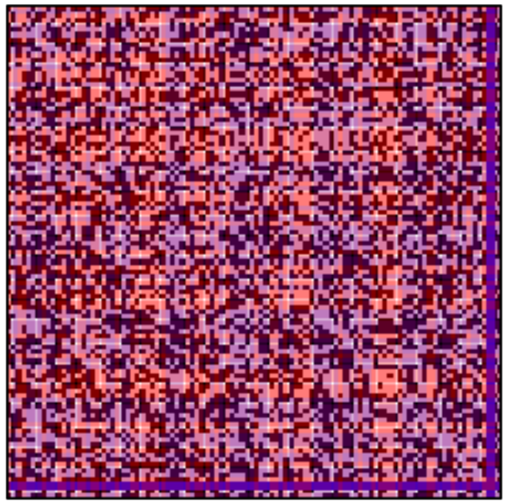
- Collapsed Gibbs sampling for the **infinite relational model**



original ordering



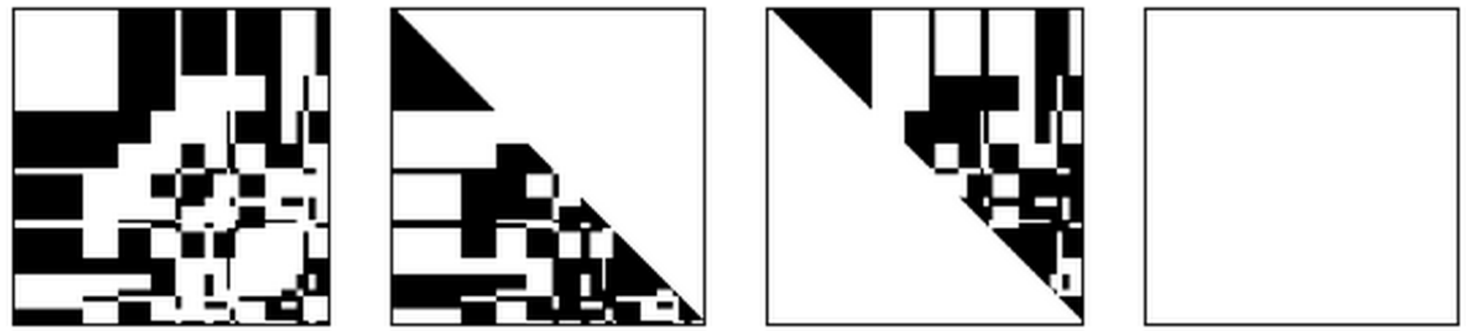
resorted by increasing uniform random variables



resorted by inferred cluster

Conclusions

- Summary
- Discussion:



  - other types of block models:
    - could consider other partitions, e.g., **Pitman–Yor**
  - other priors, e.g., **Gaussian process** (as in Lloyd et al.)
  - Aldous–Hoover already describes exchangeable **hypergraphs**
  - **sparsity**

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