Formulas and Tables

Inferential Statistics

Contents

Descriptive statistics

Mean

For n observed values x of variable X, the mean equals

$$\overline{x} = \frac{\sum\limits_{i=1}^{n} x_i}{n}.$$

Mean of a frequency distribution

For n observed values x of variable X, with k different outcomes with frequency f, the mean equals

$$\overline{x} = \frac{\sum\limits_{i=1}^{k} f_i x_i}{n}.$$

For a dichotomous (binary) variable X with different outcomes $x = 0$ and $x = 1$, the mean equals the proportion of outcomes $x = 1$, referred to as p_x.

Median

The median is the middle observed value of all ordered observations. The median corresponds to the 50th percentile, P_{50} (see 'Percentiles' below).

Mode

The modus is the most frequent observed value.

Standard deviation

The standard deviation (as estimator for the population value σ) is

$$s_x = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}.$$

The standard deviation population value for a dichotomous (binary) variable is

$$\sigma_x = \sqrt{p_x(1-p_x)}.$$

Variance

The variance (as estimator for the population value σ^2) is

$$s_x^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}.$$

The variance population value for a dichotomous (binary) variable is

$$\sigma_x^2 = p_x(1 - p_x).$$

Percentiles

The pth percentile is the value for which p percent of observations is smaller or equal. For example, 50th percentile is the value for which holds that half of all observations are smaller or equal. This is referred to as P_{50} (which is equivalent to the median).

Interquartile distance

The interquartile distance is

$$IQR = Q_3 - Q_1,$$

where Q_3 corresponds to P_{75} and Q_1 corresponds to P_{25}.

Range

The range indicates within which distance from each other with all observed values are located. It is calculated by

$$range = maximum - minimum.$$

Z-score

The z-score, or standardized score

$$z_{x_i} = \frac{x_i - \overline{x}}{s_x}.$$

(This is a linear transformation with $a = -\overline{x}/s_x$ and $b = 1/s_x$, see 'Linear transformation' below).

Covariance

The covariance between x and y

$$s_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}).$$

The following rules apply with respect to the variance and covariance:

$$\begin{aligned} s_{xx} &= s_x^2 \\ s_{x+y}^2 &= s_x^2 + s_y^2 + 2s_{xy} \\ s_{x-y}^2 &= s_x^2 + s_y^2 - 2s_{xy}. \end{aligned}$$

For two dichotomous (binary) variables X and Y, where p_{xy} equals the probability of a score of 1 for both X and Y, the covariance population value equals

$$\sigma_{xy} = p_{xy} - p_x p_y.$$

Pearson's (product-moment) correlation coefficient

The correlation between x and y

$$\begin{aligned} r_{xy} &= \frac{s_{xy}}{s_x s_y} \\ &= \frac{1}{n-1}\sum_{i=1}^{n} z_{x_i} z_{y_i} \\ &= \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \overline{x}}{s_x}\right)\left(\frac{y_i - \overline{y}}{s_y}\right). \end{aligned}$$

Effect sizes correlation coefficient

- $r_{xy} = 0.1$ small effect
- $r_{xy} = 0.3$ medium effect
- $r_{xy} = 0.5$ large effect

Linear transformation

For a linear transformation $y_i = a + bx_i$ the following holds

$$\overline{y} = a + b \cdot \overline{x}$$

en

$$\begin{aligned} s_y^2 &= b^2 \cdot s_x^2 \\ s_y &= b \cdot s_x. \end{aligned}$$

Regression

Simple linear regression

Regression equation simple linear regression

$$\widehat{y}_i = a + bx_i,$$

where the regression coefficient is estimated by

$$b = r_{xy}\left(\frac{s_y}{s_x}\right)$$

and the intercept is estimated by

$$a = \overline{y} - b\overline{x}.$$

Residual

The residual (or prediction error) is

$$(y_i - \hat{y}_i),$$

where y_i is the observed value and $\hat{y}_i$ the predicted value for person i.

Sums of squares for y

$$\begin{aligned}
\sum_{i=1}^{n}(y_i - \overline{y})^2 &= \sum_{i=1}^{n}(\widehat{y}_i - \overline{y})^2 + \sum_{i=1}^{n}(y_i - \widehat{y}_i)^2, \\
\text{also referred to as:} & \\
SS_y &= SS_{\widehat{y}-\overline{y}} + SS_{y-\widehat{y}}, \\
\text{or as:} & \\
SS_{tot} &= SS_{reg} + SS_{res}
\end{aligned}$$

where SS_{tot} is the total sum of squares of y, SS_{reg} is the regression sum of squares 'explained' by the model and SS_{res} is the residual sum of squares.

Proportion explained variation

The proportion explained variation (also called the proportional reduction in prediction error) is

$$
\begin{aligned}
r_{xy}^2 &= \frac{\sum_{i=1}^{n}(y_i - \overline{y})^2 - \sum_{i=1}^{n}(y_i - \widehat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \overline{y})^2}, \\
&= \frac{SS_{tot} - SS_{res}}{SS_{tot}}, \\
&= \frac{SS_{reg}}{SS_{tot}}
\end{aligned}
$$

t-test for regression coefficient b

The test statistic for regression coefficient b assuming H_0: $\beta = \beta_0 = 0$ is

$$t = \frac{(b - \beta_0)}{se_b}.$$

where se_b is calculated by software. The statistic follows a t distribution with $n-2$ degrees of freedom ($df = n-2$), when the assumptions hold.

Standardized residual

The standardized residual equals

$$\frac{y_i - \widehat{y_i}}{se_{y_i - \widehat{y_i}}},$$

where $se_{y_i - \widehat{y_i}}$, the standard error for the residual (also referred to as se_{res}) is calculated by software.

Residual standard deviation

The residual standard deviation based on n observations equals

$$s_{res} = \sqrt{\frac{\sum_{i=1}^{n} (y_i - \widehat{y}_i)^2}{n-k}},$$

in other words: $s_{res} = \sqrt{\frac{SS_{res}}{n-k}}$, where k equals the number of parameters in the regression equation ($k = 2$ for simple regression).

95% - prediction interval for y_i

$$\widehat{y}_i - 2s \leq y_i \leq \widehat{y}_i + 2s,$$

where s is the residual standard deviation and 2 is an approximation of $t_{\alpha/2}$.

95% - confidence interval for μ_y

$$\widehat{y} - 2(s/\sqrt{n}) \leq \mu_y \leq \widehat{y} + 2(s/\sqrt{n}),$$

where s is the residual standard deviation, 2 is an approximation of $t_{\alpha/2}$ and n is the number of observations.

Multiple linear regression

Regression equation simple multiple regression

For the independent variables (predictors) $x_1, x_2, x_3, \ldots, x_j, \ldots$

$$\widehat{y}_i = a + b_1 x_{i1} + b_2 x_{i2} + b_3 x_{i3} + \ldots + b_j x_{ij} + \ldots .$$

Proportion explained variation

The proportion explained variation (proportional reduction in prediction error), or squared multiple correlation coefficient is

$$R^2 = \frac{\sum_{i=1}^{n}(y_i - \overline{y})^2 - \sum_{i=1}^{n}(y_i - \widehat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \overline{y})^2},$$

oftewel: $R^2 = \frac{SS_{tot} - SS_{res}}{SS_{tot}} = \frac{SS_{reg}}{SS_{tot}}$.

Multiple correlation coefficient

$$R = \sqrt{R^2}$$

F-test statistic regression analysis

The null hypothesis that all regression coefficients equal zero is tested using

$$F = \frac{\dfrac{\sum_{i=1}^{n}(\widehat{y}_i - \overline{y})^2}{k-1}}{\dfrac{\sum_{i=1}^{n}(y_i - \widehat{y}_i)^2}{n-k}} = \frac{\dfrac{SS_{reg}}{df_{reg}}}{\dfrac{SS_{res}}{df_{res}}} = \frac{MS_{reg}}{MS_{res}},$$

where k equals the number of parameters in the regression equation and n the number of observations. The degrees of freedom are $df_{reg} = k - 1$ en $df_{res} = n - k$. df_{reg} and df_{res} are often referred to as df_1 and df_2.
MS denotes mean squares. MS_{res} denotes the residual variance.

Test statistic for b_j

The test statistic for regression coefficient b assuming H_0: $\beta = \beta_0 = 0$ is

$$t_{b_j} = \frac{(b_j - \beta_0)}{se_{b_j}},$$

where se_{b_j} is calculated by software. The statistic follows a t distribution with $n - k$ degrees of freedom, where k equals the number of parameters in the regression equation ($k = 2$ for simple regression), when the assumptions hold.

$100(1 - \alpha)\%$ - confidence interval for β_j

$$b_j - t_{\alpha/2} \cdot se_{b_j} \leq \beta_j \leq b_j + t_{\alpha/2} \cdot se_{b_j}.$$

Exponential regression

Regression equation simple exponential regression

$$\mu_y = \alpha\beta^x,$$

where $\beta > 0$ must hold. This populatie level equation provides the predicted value for the population mean of y for a given value of x.

Logistic regression

Log-odds (logit)

If y is a dichotomous (binary) variable taking on values 0 or 1 and we denote $p(y=1)$ as p, then:

$$\text{odds} = \frac{p}{1-p}$$
$$\text{log-odds} = \text{logit}(p) = \ln\left(\frac{p}{1-p}\right)$$
$$p = \frac{\text{odds}}{1+\text{odds}}$$

To calculate the probability p from a log-odds value requires the following rule: $e^{ln(x)} = x$.

Regression equation simple logistic regression

$$p(Y=1) = \frac{e^{\alpha+\beta x}}{1+e^{\alpha+\beta x}},$$

for which the following holds:

$$\text{logit}\big(P(Y=1)\big) = \alpha + \beta x.$$

Binomial distribution

Binomial coefficient

For a binomial variable X, the number of possible combinations of x successes in n trials equals

$$\binom{n}{x} = \frac{n!}{x!(n-x)!},$$

where $0 \leq x \leq n$ and $n!$ (n-faculty) $= n(n-1)(n-2)\ldots 1$. By definition: $0! = 1$.

Formula binomial distribution

For a binomial variable X, the probability of x successes in n trials is

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x},$$

where p is the probability of success.

Expected value and standard deviation

The expected value of a discrete random variabele X with k possible outcomes, equals the mean of the distribution

$$\mu = \sum_{i=1}^{k} x_i P(x_i).$$

The standard deviation of this random variable is

$$\sigma = \sqrt{\sum_{i=1}^{k} (x_i - \mu)^2 P(x_i)}.$$

For a dichotomous (binary) binomial variable simpler formulas are available. The expected value of probability of success p with n trials is

$$\mu = np$$

and the standard deviation is

$$\sigma = \sqrt{np(1-p)}.$$

Normal distribution - z- and t-tests

Z-score

The z-score for an observation x of random variable X is

$$z_x = \frac{x - \mu}{\sigma}.$$

For z-scores it holds that $\mu_z = 0$ en $\sigma_z^2 = 1$. If X is normally distributed, then Z follows a standard normal distribution and for observations z_x of Z it holds that

$$P(X \geq x) = P(Z \geq z_x).$$

The p-values for the standard normal distribution can be found in Table 1.

Proportions

Standard error for a proportion with known population value p

$$se_p = \sqrt{\frac{p(1-p)}{n}}$$

Standard error for a proportion with unknown population value p

$$se_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$100(1-\alpha)\%$ - confidence interval for one proportion

$$\hat{p} - z_{\alpha/2} \cdot se_{\hat{p}} \leq p \leq \hat{p} + z_{\alpha/2} \cdot se_{\hat{p}},$$

where $P(Z \geq z_{\alpha/2}) = \alpha/2$ (z-score corresponding to the selected confidence level (for example $z = 1.96$ for a confidence level of 95%).

Minimal sample size to estimate a population proportion

$$n = \frac{\hat{p}(1-\hat{p})z^2}{m^2},$$

where $\hat{p}$ is the estimated proportion, m the margin of error and z the z-score corresponding to the selected confidence level (for example $z = 1.96$ for a confidence level of 95%).

z-test for one proportion

$$z = \frac{p - p_0}{se_0} = \frac{p - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

, where p_0 is the expected proportion under the null hypothesis.

Standard error for the difference between two proportions

$$se_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

, where $\hat{p}_1$ is the observed proportion based on n_1 observations in sample 1 and $\hat{p}_2$ the observed proportion based onn_2 observations in sample 2.

$100(1-\alpha)\%$ confidence interval for the difference between two proportions

$$(\hat{p}_1 - \hat{p}_2) - z_{\alpha/2} \cdot se_{\hat{p}_1 - \hat{p}_2} \leq (p_1 - p_2) \leq (\hat{p}_1 - \hat{p}_2) + z_{\alpha/2} \cdot se_{\hat{p}_1 - \hat{p}_2},$$

where $P(Z \geq z_{\alpha/2}) = \alpha/2$ (z-score corresponding to the selected confidence level (for example $z = 1.96$ for a confidence level of 95%).

z-test for the difference between two independent proportions

$$z = \frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2)}{se_0},$$

where se_0 is the standard error under the null hypothesis. If the null hypothesis assumes $p_1 = p_2$, then

$$se_0 = \sqrt{\frac{\hat{p}(1-\hat{p})}{n_1} + \frac{\hat{p}(1-\hat{p})}{n_2}} = \sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)},$$

where $\hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1+n_2}$ (referred to as the pooled proportion).

z-test for the difference between two dependent proportions - McNemar's test

With n_{01} denoting the number of observations with a score of 0 on variable A and a score of 1 on variable B, and n_{10} denoting the number of observations with a score of 1 on variable A and a score of 0 on variable B:

$$z = \frac{n_{01} - n_{10}}{\sqrt{n_{01} + n_{10}}},$$

see:

$$\begin{array}{c|cc} & \multicolumn{2}{c}{B} \\ \hline A & n_{00} & n_{01} \\ \hline & n_{10} & n_{11} \end{array}$$

Means

Expected value for a mean

$$E(\overline{X}) = \mu_{\overline{x}} = \mu$$

Standard error for a mean with known population variance σ

$$se_{\overline{X}} = \sigma/\sqrt{n}$$

Standard error for a mean with unknown population variance σ

$$se_{\overline{x}} = s/\sqrt{n}$$

$100(1-\alpha)\%$ confidence interval for one mean

$$\overline{x} - t_{\alpha/2} \cdot se_{\overline{x}} \leq \mu \leq \overline{x} + t_{\alpha/2} \cdot se_{\overline{x}},$$

where $P(T \geq t_{\alpha/2}) = \alpha/2$ for a t-distribution with $df = n-1$ degrees of freedom (t-score corresponding to the selected confidence level).

Minimal sample size to estimate a population mean

$$n = \frac{\sigma^2 z^2}{m^2},$$

where σ is the (expected) standard deviation in the population, m the margin of error and z the z-score corresponding to the selected confidence level (for example $z = 1.96$ for a confidence level of 95%).

t-test for one independent mean

$$t_{\overline{x}} = \frac{\overline{x} - \mu_0}{se_{\overline{x}}} = \frac{\overline{x} - \mu_0}{s/\sqrt{n}}$$

μ_0 is the mean population value expected under the null hypothesis. If the null hypothesis holds and X is normally distributed, then $t_{\overline{x}}$ follows a t-distribution with $df = n-1$ degrees of freedom.

Standard error for the difference between two means

$$se_{\overline{x}_1-\overline{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

t-test for the difference between two independent means with unequal population variances

$$t_{\overline{x}_1-\overline{x}_2} = \frac{(\overline{x}_1 - \overline{x}_2) - (\mu_1 - \mu_2)}{se_{\overline{x}_1-\overline{x}_2}}.$$

If X_1 and X_2 are independent and normally distributed, $t_{\overline{x}_1-\overline{x}_2}$ approximately follows a t-distribution with degrees of freedom

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1-1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2-1}\left(\frac{s_2^2}{n_2}\right)^2}.$$

t-test for the difference between two independent means with equal population variances

$$t_{\overline{x}_1-\overline{x}_2} = \frac{(\overline{x}_1 - \overline{x}_2) - (\mu_1 - \mu_2)}{se_{\overline{x}_1-\overline{x}_2}},$$

met

$$se_{\overline{x}_1-\overline{x}_2} = s \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},$$

where

$$s = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{(n_1-1)+(n_2-1)}}$$

is referred to as the pooled standard deviation. If X_1 and X_2 are independent and normally distributed with equal variances, $t_{\overline{x}_1-\overline{x}_2}$ follows a t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

$100(1-\alpha)\%$ **confidence interval for the difference between two independent means**

$$(\overline{x}_1 - \overline{x}_2) - t_{\alpha/2} \cdot se_{\overline{x}_1 - \overline{x}_2} \leq (\mu_1 - \mu_2) \leq (\overline{x}_1 - \overline{x}_2) + t_{\alpha/2} \cdot se_{\overline{x}_1 - \overline{x}_2}$$

The degrees of freedom and $se_{\overline{x}_1 - \overline{x}_2}$ depend on whether the population variances are assumed to be equal or not, see the previous formulas for the appropriate method of calculation.

Standardized effect sizes for the difference between two means

$$d = \frac{\overline{x}_1 - \overline{x}_2}{s},$$

where s can refer to the pooled standard deviation or the standard deviation of either one of the samples.

- $d = 0.2$ small effect
- $d = 0.5$ medium effect
- $d = 0.8$ large effect

t-test for paired samples (two dependent means)

$$t_{\overline{x}_d} = \frac{\overline{x}_d - \mu_d}{se_{\overline{x}_d}},$$

where

$$se_{\overline{x}_d} = \sqrt{\frac{s_d^2}{n_d}} = \frac{s_d}{\sqrt{n_d}}$$

and $n_d - 1$ degrees of freedom.

$100(1-\alpha)\%$ **- confidence interval for paired samples**

$$\overline{x}_d - t_{\alpha/2} \cdot se_{\overline{x}_d} \leq \mu_d \leq \overline{x}_d + t_{\alpha/2} \cdot se_{\overline{x}_d},$$

with $n_d - 1$ degrees of freedom.

Analysis of variance (ANOVA)

One-way ANOVA

Sums of squares for y

With subscript i for participants/observations and j for groups, the total sum of squares of y can be expressed as

$$\begin{aligned}\sum_{j=1}^{g}\sum_{i=1}^{n_g}(y_{ij}-\overline{y})^2 &= \sum_{j=1}^{g} n_j(\overline{y}_j-\overline{y})^2 + \sum_{j=1}^{g}\sum_{i=1}^{n_g}(y_{ij}-\overline{y}_j)^2 \\ SS_{tot} &= SS_{between} + SS_{within},\end{aligned}$$

where $SS_{between}$ is the sum of squares of the differences between groups, SS_{within} the sum of squares of the differences within the groups, n_j the number of observations in group j and g the number of groups.

F-test ANOVA

$$F = \frac{MS_{between}}{MS_{within}} = \frac{SS_{between}/(g-1)}{SS_{within}/(n-g)} = \frac{\left(\sum_{j=1}^{g} n_j(\overline{y}_j-\overline{y})^2\right)/(g-1)}{\left(\sum_{j=1}^{g}\sum_{i=1}^{n_g}(y_{ij}-\overline{y}_j)^2\right)/(n-g)},$$

met $df_1 = g-1$ en $df_2 = n-g$, waarbij n het totaal aantal waarnemingen is, g het aantal groepen, en n_j het aantal waarnemingen in groep j.

F-test ANOVA - equal group sizes

$$F = \frac{MS_{between}}{MS_{within}} = \frac{\left(n_c \sum_{j=1}^{g}(\overline{y}_j-\overline{y})^2\right)/(g-1)}{\left(\sum_{j=1}^{g} s_j^2\right)/g}$$

with $df_1 = g-1$ (between) and $df_2 = n-g$ (within) degrees of freedom, where n_c denotes the number of observations in each group ($n_1 = n_2 = \cdots = n_g = n_c$).

Proportion explained variation (eta^2)

$$\eta^2 = \frac{SS_{between}}{SS_{tot}} = 1 - \frac{SS_{within}}{SS_{tot}}$$

Effect sizes ANOVA

$$f = \sqrt{\frac{SS_{between}}{SS_{within}}} = \sqrt{\frac{\eta^2}{1-\eta^2}}$$

- $f = 0.10$ small effect
- $f = 0.25$ medium effect
- $f = 0.40$ large effect

$100(1-\alpha)\%$ - confidence interval post hoc comparisons groups j and k

$$(\overline{y}_j - \overline{y}_k) - t_{\alpha/2} \cdot se_{\overline{y}_j - \overline{y}_k} \leq (\mu_j - \mu_k) \leq (\overline{y}_j - \overline{y}_k) + t_{\alpha/2} \cdot se_{\overline{y}_j - \overline{y}_k},$$

with

$$se_{\overline{y}_j - \overline{y}_k} = \sqrt{\frac{SS_{within}}{n-g}\left(\frac{1}{n_j} + \frac{1}{n_k}\right)},$$

where n is the **total** sample size, g is the **total** number of groups and $df = (n-g)$ are the degrees of freedom associated with MS_{within}.

Two-way ANOVA

Sums of squares for y

With factors A and B the total sum of squares of y can be expressed as

$$SS_T = SS_A + SS_B + SS_{AB} + SS_{within}.$$

Mean sums of squares for y (mean squares)

With a levels for factor A and b levels for factor B the mean squares are

$$\begin{aligned} MS_A &= SS_A/(a-1) \\ MS_B &= SS_B/(b-1) \\ MS_{AB} &= SS_{AB}/((a-1)(b-1)) \\ MS_{within} &= SS_{within}/(n-ab) \end{aligned}$$

NB MS_{within} is also referred to as MS_{error}.

F-test interaction $A \times B$

$$F = MS_{AB}/MS_{within}$$

with $df_1 = (a-1)(b-1)$ and $df_2 = n - ab$ (also: df_{AB} and df_{error}).

F-test main effect A

$$F = MS_A/MS_{within}$$

with $df_1 = a - 1$ and $df_2 = n - ab$ (also: df_A and df_{error}).

F-test main effect B

$$F = MS_B/MS_{within}$$

with $df_1 = b - 1$ and $df_2 = n - ab$ (also: df_B and df_{error}).

Cross tables - χ^2-test

Pearson's χ^2-test for cross tables

O_{ab} is the number of observations in row a and column b in a $A \times B$ cross table. E_{ab} is the expected number of observations - given the marginals (row and column totals) - if A and B are independent. Pearson's χ^2 is

$$\chi^2 = \sum_{a=1}^{A} \sum_{b=1}^{B} \frac{(O_{ab} - E_{ab})^2}{E_{ab}},$$

where E_{ab} is the product of the appropriate row and column total divided by n (row total times column total divided by the grand total). χ^2 has $df = (A-1)(B-1)$ degrees of freedom.

Effect sizes χ^2

$$w = \sqrt{\frac{\chi^2}{n}}$$

- $w = 0.1$ small effect
- $w = 0.3$ medium effect
- $w = 0.5$ large effect

Cohen's κ (kappa)

For a A by A table

$$\kappa = \frac{F_a - E_a}{1 - E_a},$$

where

- F_a is the sum of the observations on the diagonal and
- E_a is the sum of the observations on the diagonal if the rows and columns are independent (the product of the appropriate row and column total divided by n).

Non-parametric tests

Wilcoxon-Mann-Whitney-test for two independent groups

$$z_{r_1} = \frac{r_1 - n_1(n+1)/2 \pm \mathbf{0.5}}{\sqrt{n_1 n_2 (n+1)/12}},$$

where r_1 is the sum of the rank numbers in the smallest group ($n_1 \leq n_2$) and $n = n_1 + n_2$ and the continuity correction $\pm\mathbf{0.5}$ equals

- $+0.5$ for a left-sided test
- -0.5 for a right-sided test
- $+0.5$ for a two-sided test if $r_1 \leq n_1(n+1)/2$, and
- -0.5 for a two-sided test if $r_1 > n_1(n+1)/2$.

Kruskal-Wallis-test for more than two independent groups

$$\chi^2 = \left(\frac{12}{n(n+1)}\right) \sum_{j=1}^{g} n_j (\overline{r}_j - \overline{r})^2,$$

where $\overline{r}_j$ is the mean of rank numbers in group j of size n_j, with g groups in total and $df = g - 1$ degrees of freedom.

Z-score sign-test for two dependent groups

$$z_p = \frac{p - 0.5}{\sqrt{0.25/n}},$$

where n is the number of pairs of observations.

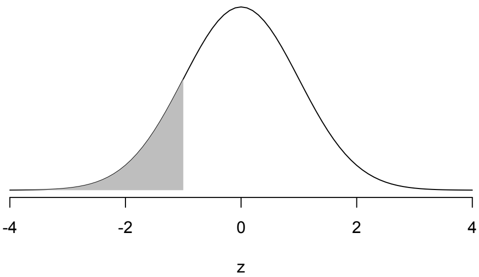
Wilcoxon's signed rank-test for two dependent groups

$$W_+ = \sum r_{d_i},$$

where r_{d_i} denotes the rank score of the positive difference score i.

Table 1a: Standard normal distribution

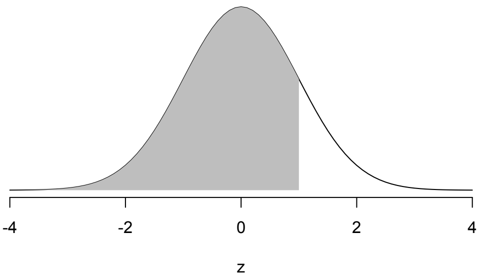
left-sided p-values for negative values of z



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

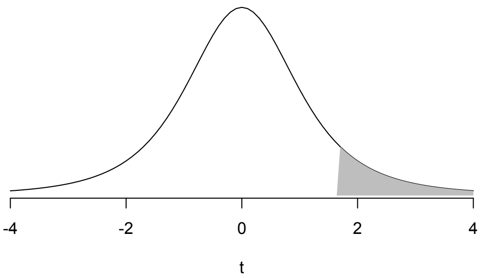
Tabel 1b: Standard normal distribution

left-sided p-values for positive values of z



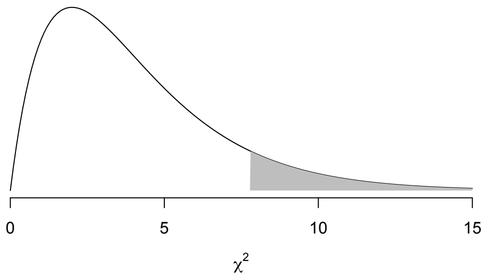
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Table 2: Critical values Student t-distribution



	right-sided p-value						
df	0.250	0.100	0.050	0.025	0.010	0.005	0.001
1	1.0000	3.0777	6.3138	12.7062	31.8205	63.6567	318.3088
2	0.8165	1.8856	2.9200	4.3027	6.9646	9.9248	22.3271
3	0.7649	1.6377	2.3534	3.1824	4.5407	5.8409	10.2145
4	0.7407	1.5332	2.1318	2.7764	3.7469	4.6041	7.1732
5	0.7267	1.4759	2.0150	2.5706	3.3649	4.0321	5.8934
6	0.7176	1.4398	1.9432	2.4469	3.1427	3.7074	5.2076
7	0.7111	1.4149	1.8946	2.3646	2.9980	3.4995	4.7853
8	0.7064	1.3968	1.8595	2.3060	2.8965	3.3554	4.5008
9	0.7027	1.3830	1.8331	2.2622	2.8214	3.2498	4.2968
10	0.6998	1.3722	1.8125	2.2281	2.7638	3.1693	4.1437
11	0.6974	1.3634	1.7959	2.2010	2.7181	3.1058	4.0247
12	0.6955	1.3562	1.7823	2.1788	2.6810	3.0545	3.9296
13	0.6938	1.3502	1.7709	2.1604	2.6503	3.0123	3.8520
14	0.6924	1.3450	1.7613	2.1448	2.6245	2.9768	3.7874
15	0.6912	1.3406	1.7531	2.1314	2.6025	2.9467	3.7328
16	0.6901	1.3368	1.7459	2.1199	2.5835	2.9208	3.6862
17	0.6892	1.3334	1.7396	2.1098	2.5669	2.8982	3.6458
18	0.6884	1.3304	1.7341	2.1009	2.5524	2.8784	3.6105
19	0.6876	1.3277	1.7291	2.0930	2.5395	2.8609	3.5794
20	0.6870	1.3253	1.7247	2.0860	2.5280	2.8453	3.5518
21	0.6864	1.3232	1.7207	2.0796	2.5176	2.8314	3.5272
22	0.6858	1.3212	1.7171	2.0739	2.5083	2.8188	3.5050
23	0.6853	1.3195	1.7139	2.0687	2.4999	2.8073	3.4850
24	0.6848	1.3178	1.7109	2.0639	2.4922	2.7969	3.4668
25	0.6844	1.3163	1.7081	2.0595	2.4851	2.7874	3.4502
26	0.6840	1.3150	1.7056	2.0555	2.4786	2.7787	3.4350
27	0.6837	1.3137	1.7033	2.0518	2.4727	2.7707	3.4210
28	0.6834	1.3125	1.7011	2.0484	2.4671	2.7633	3.4082
29	0.6830	1.3114	1.6991	2.0452	2.4620	2.7564	3.3962
30	0.6828	1.3104	1.6973	2.0423	2.4573	2.7500	3.3852
40	0.6807	1.3031	1.6839	2.0211	2.4233	2.7045	3.3069
50	0.6794	1.2987	1.6759	2.0086	2.4033	2.6778	3.2614
60	0.6786	1.2958	1.6706	2.0003	2.3901	2.6603	3.2317
80	0.6776	1.2922	1.6641	1.9901	2.3739	2.6387	3.1953
100	0.6770	1.2901	1.6602	1.9840	2.3642	2.6259	3.1737
∞	0.6745	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902

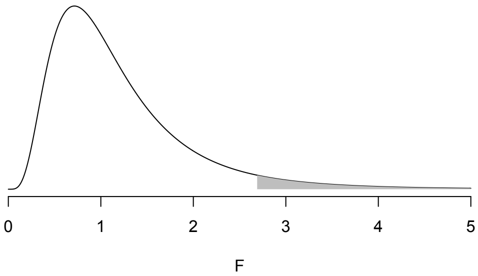
Table 3: Critical values χ^2-distribution



df	right-sided p-value 0.250	0.100	0.050	0.025	0.010	0.005	0.001
1	1.3233	2.7055	3.8415	5.0239	6.6349	7.8794	10.8276
2	2.7726	4.6052	5.9915	7.3778	9.2103	10.5966	13.8155
3	4.1083	6.2514	7.8147	9.3484	11.3449	12.8382	16.2662
4	5.3853	7.7794	9.4877	11.1433	13.2767	14.8603	18.4668
5	6.6257	9.2364	11.0705	12.8325	15.0863	16.7496	20.5150
6	7.8408	10.6446	12.5916	14.4494	16.8119	18.5476	22.4577
7	9.0371	12.0170	14.0671	16.0128	18.4753	20.2777	24.3219
8	10.2189	13.3616	15.5073	17.5345	20.0902	21.9550	26.1245
9	11.3888	14.6837	16.9190	19.0228	21.6660	23.5894	27.8772
10	12.5489	15.9872	18.3070	20.4832	23.2093	25.1882	29.5883
11	13.7007	17.2750	19.6751	21.9200	24.7250	26.7568	31.2641
12	14.8454	18.5493	21.0261	23.3367	26.2170	28.2995	32.9095
13	15.9839	19.8119	22.3620	24.7356	27.6882	29.8195	34.5282
14	17.1169	21.0641	23.6848	26.1189	29.1412	31.3193	36.1233
15	18.2451	22.3071	24.9958	27.4884	30.5779	32.8013	37.6973
16	19.3689	23.5418	26.2962	28.8454	31.9999	34.2672	39.2524
17	20.4887	24.7690	27.5871	30.1910	33.4087	35.7185	40.7902
18	21.6049	25.9894	28.8693	31.5264	34.8053	37.1565	42.3124
19	22.7178	27.2036	30.1435	32.8523	36.1909	38.5823	43.8202
20	23.8277	28.4120	31.4104	34.1696	37.5662	39.9968	45.3147
25	29.3389	34.3816	37.6525	40.6465	44.3141	46.9279	52.6197
30	34.7997	40.2560	43.7730	46.9792	50.8922	53.6720	59.7031
40	45.6160	51.8051	55.7585	59.3417	63.6907	66.7660	73.4020
50	56.3336	63.1671	67.5048	71.4202	76.1539	79.4900	86.6608
60	66.9815	74.3970	79.0819	83.2977	88.3794	91.9517	99.6072
70	77.5767	85.5270	90.5312	95.0232	100.4252	104.2149	112.3169
80	88.1303	96.5782	101.8795	106.6286	112.3288	116.3211	124.8392
90	98.6499	107.5650	113.1453	118.1359	124.1163	128.2989	137.2084
100	109.1412	118.4980	124.3421	129.5612	135.8067	140.1695	149.4493

Table 4: Critical values F-distribution

for $\alpha = 0.05$



df_2 \ df_1	1	2	3	4	5	6	8	12	24	∞
1	161.45	199.50	215.71	224.58	230.16	233.99	238.88	243.91	249.05	254.31
2	18.513	19.000	19.164	19.247	19.296	19.330	19.371	19.413	19.454	19.496
3	10.128	9.5521	9.2766	9.1172	9.0135	8.9406	8.8452	8.7446	8.6385	8.5264
4	7.7086	6.9443	6.5914	6.3882	6.2561	6.1631	6.0410	5.9117	5.7744	5.6281
5	6.6079	5.7861	5.4095	5.1922	5.0503	4.9503	4.8183	4.6777	4.5272	4.3650
6	5.9874	5.1433	4.7571	4.5337	4.3874	4.2839	4.1468	3.9999	3.8415	3.6689
7	5.5914	4.7374	4.3468	4.1203	3.9715	3.8660	3.7257	3.5747	3.4105	3.2298
8	5.3177	4.4590	4.0662	3.8379	3.6875	3.5806	3.4381	3.2839	3.1152	2.9276
9	5.1174	4.2565	3.8625	3.6331	3.4817	3.3738	3.2296	3.0729	2.9005	2.7067
10	4.9646	4.1028	3.7083	3.4780	3.3258	3.2172	3.0717	2.9130	2.7372	2.5379
11	4.8443	3.9823	3.5874	3.3567	3.2039	3.0946	2.9480	2.7876	2.6090	2.4045
12	4.7472	3.8853	3.4903	3.2592	3.1059	2.9961	2.8486	2.6866	2.5055	2.2962
13	4.6672	3.8056	3.4105	3.1791	3.0254	2.9153	2.7669	2.6037	2.4202	2.2064
14	4.6001	3.7389	3.3439	3.1122	2.9582	2.8477	2.6987	2.5342	2.3487	2.1307
15	4.5431	3.6823	3.2874	3.0556	2.9013	2.7905	2.6408	2.4753	2.2878	2.0658
16	4.4940	3.6337	3.2389	3.0069	2.8524	2.7413	2.5911	2.4247	2.2354	2.0096
17	4.4513	3.5915	3.1968	2.9647	2.8100	2.6987	2.5480	2.3807	2.1898	1.9604
18	4.4139	3.5546	3.1599	2.9277	2.7729	2.6613	2.5102	2.3421	2.1497	1.9168
19	4.3807	3.5219	3.1274	2.8951	2.7401	2.6283	2.4768	2.3080	2.1141	1.8780
20	4.3512	3.4928	3.0984	2.8661	2.7109	2.5990	2.4471	2.2776	2.0825	1.8432
21	4.3248	3.4668	3.0725	2.8401	2.6848	2.5727	2.4205	2.2504	2.0540	1.8117
22	4.3009	3.4434	3.0491	2.8167	2.6613	2.5491	2.3965	2.2258	2.0283	1.7831
23	4.2793	3.4221	3.0280	2.7955	2.6400	2.5277	2.3748	2.2036	2.0050	1.7570
24	4.2597	3.4028	3.0088	2.7763	2.6207	2.5082	2.3551	2.1834	1.9838	1.7330
25	4.2417	3.3852	2.9912	2.7587	2.6030	2.4904	2.3371	2.1649	1.9643	1.7110
26	4.2252	3.3690	2.9752	2.7426	2.5868	2.4741	2.3205	2.1479	1.9464	1.6906
27	4.2100	3.3541	2.9604	2.7278	2.5719	2.4591	2.3053	2.1323	1.9299	1.6717
28	4.1960	3.3404	2.9467	2.7141	2.5581	2.4453	2.2913	2.1179	1.9147	1.6541
29	4.1830	3.3277	2.9340	2.7014	2.5454	2.4324	2.2783	2.1045	1.9005	1.6376
30	4.1709	3.3158	2.9223	2.6896	2.5336	2.4205	2.2662	2.0921	1.8874	1.6223
40	4.0847	3.2317	2.8387	2.6060	2.4495	2.3359	2.1802	2.0035	1.7929	1.5089
60	4.0012	3.1504	2.7581	2.5252	2.3683	2.2541	2.0970	1.9174	1.7001	1.3893
120	3.9201	3.0718	2.6802	2.4472	2.2899	2.1750	2.0164	1.8337	1.6084	1.2539
∞	3.8415	2.9957	2.6049	2.3719	2.2141	2.0986	1.9384	1.7522	1.5173	1.0000