Solving a recurrence by expansion

Example 1: $T(n) = T(n-3) + 2$ for $n \geq 0$; $T(0) = T(1) = T(2) = 5$.

Step 1: Calculate the first few values to get some insight and understanding

n	$T(n)$
0	5
1	5
2	5
3	7
4	7
5	7
6	9
7	9
8	9
9	11

Step 2: Expand the definition and look for a pattern

$$T(n) = T(n-3) + 2$$
$$= T(n-6) + 2 + 2$$
$$= T(n-9) + 2 + 2 + 2$$

Here we assumed n is a multiple of 3, for simplicity

$n/3$ copies of 2

$$= T(0) + 2 + 2 + 2 + \cdots + 2 + 2$$
$$= 5 + 2 \cdot \frac{n}{3}$$

So we conclude $T(n) = 5 + \frac{2n}{3}$, when n is a multiple of 3.

Step 3: check agreement with step 1. Also use an online tool like Wolfram

Alpha to check the answer.

(but it may not help!!)

Example 2: $T(n) = 5T(n-1) - 3, \quad T(0) = 2$

Step 1: Calculate the first few values to get some insight and understanding

n	$T(n)$
0	2
1	7
2	32
3	157
4	782
$\vdots$	$\vdots$

Step 2: Expand the definition and look for a pattern

$T(n) =$

expand $= 5T(n-1) - 3$

$= 5\left(5T(n-2)-3\right) - 3$

rewrite $= 5^2 T(n-2) - 3\times 5 - 3$

expand $= 5^2\left(5T(n-3)-3\right) - 3\times 5 - 3$

rewrite $= 5^3 T(n-3) - 3\times 5^2 - 3\times 5 - 3$

notice the pattern $= 5^n T(n-n) - 3\times 5^{n-1} - 3\times 5^{n-2} - \cdots - 3\times 5 - 3$

rewrite $= 2\times 5^n - 3\times\left(5^{n-1} + 5^{n-2} + \cdots + 5 + 1\right)$

Use formula for sum of geometric progression

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rewrite -2×5 $-5^{\sim}(5 \quad +5 \quad + \dots + 5 \quad +1)$

Use formula for sum of geometric progression (or just use Wolfram Alpha): this is $(5^n - 1)/4$

rewrite $= 2 \times 5^n - \dfrac{3}{4}(5^n - 1)$

$= \dfrac{5^{n+1} + 3}{4}$

Step 3: check agreement with step 1. Also use an online tool like Wolfram Alpha to check the answer.

WolframAlpha

solve T(n)=5T(n-1)-3, T(0)=2

NATURAL LANGUAGE   ∫ MATH INPUT   ⊞ EXTENDED KEYBOARD

Assuming "solve" is referring to equation solving | Use as referring to variable su|

Input interpretation

solve $\begin{array}{l} T(n) = 5\,T(n-1) - 3 \\ T(0) = 2 \end{array}$ for $T(n)$

Result   🔍 Enlarge

$T(n) = \dfrac{1}{4}\left(5^{n+1} + 3\right)$