Formal definitions of asymptotic notation

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> **Formal definition of big-O notation**. Let f and g be functions mapping nonnegative integers to nonnegative integers. We write $f \in O(g)$ if there is a positive constant C such that
>
> $$f(n) \leq Cg(n)$$
>
> for all sufficiently large n. "Sufficiently large" means that there exists N such that $f(n) \leq Cg(n)$ for all $n \geq N$. The notation "$f \in O(g)$" is read as "f is in big-O of g."

The next definition is exactly the same as the previous one, except that the inequality is reversed from $\leq$ to $\geq$ (highlighted in red).

> **Formal definition of big-Ω notation**. Let f and g be functions mapping nonnegative integers to nonnegative integers. We write $f \in \Omega(g)$ if there is a positive constant C such that
>
> $$f(n) \geq Cg(n)$$
>
> for all sufficiently large n. "Sufficiently large" means that there exists N such that $f(n) \geq Cg(n)$ for all $n \geq N$. The notation "$f \in \Omega(g)$" is read as "f is in big-Omega of g."

The next definition combines the previous two.

> **Formal definition of big-Θ notation**. Let f and g be functions mapping nonnegative integers to nonnegative integers. We write $f \in \Theta(g)$ if $f \in O(g)$ and $f \in \Omega(g)$.

Note: To apply this definition, you can use different values of C and N for the two parts of the definition. You could write this as C_1, N_1 for the big-O part and C_2, N_2 for the big-Omega part.

Examples of applying the formal definitions of asymptotic notation

Usually, we don't bother applying the formal definitions. We just convert expressions into asymptotic notation by finding the dominant term. For example, we know that $9n^3 + 20n + 2$ is in $O(n^3)$, because n^3 is the dominant term in this expression.

However, it is also important to have a mathematical understanding of the formal definitions. To demonstrate this understanding, you need to be able to apply the definitions explicitly. The following examples demonstrate how to do this. Note that we do not expect calculus-based proofs in this course. Therefore, it is permissible to use empirical evidence based on actual calculations, as in the examples below. However, an optional rigorous proof based on calculus is also given for those who are interested.

Example 1. Use the formal definition on $O(\cdot)$ to show that $9n^3 + 20n + 2 \in O(n^3)$.

Solution to Example 1. Write $f(n) = 9n^3 + 20n + 2$. We need to find C and N such that $f(n) \leq Cn^3$ whenever $n \geq N$. After some experimentation with different values of C, we find that a value of $C = 15$ looks promising, as demonstrated by the following table of values:

n	$f(n)$	$15n^3$
1	31	15
2	154	120
3	425	405
4	898	960
5	1627	1875
6	2666	3240
7	4069	5145
8	5890	7680

From this, it appears that $f(n) < 15n^3$ whenever $n \geq 4$. Therefore, we can take $C = 15$ and $N = 4$ in the definition of big-O, thus demonstrating that $f(n) \in O(n^3)$.

Further remarks about the solution to Example 1. There are many correct values of C, N that can be used to answer this problem. In fact, for any value of $C > 9$, there exist infinitely many suitable values of N. Here are a few possibilities: $C = 20, N = 2$; or $C = 20, N = 3$; or $C = 20, N = 50$; or $C = 12, N = 7$; or $C = 12, N = 8$; or $C = 12, N = 50$.

Optional calculus-based solution to Example 1. We are interested in comparing the values of $f(n)$ and $15n^3$. Let $g(n) = 15n^3 - f(n) = 6n^3 - 20n - 2$. The derivative of g is $g'(n) = 18n^2 - 20$, which is positive for $n \geq 2$. This guarantees that the difference between $15n^3$ and $f(n)$ will continue to increase after it first becomes positive at $n = 4$, as shown in the table above. Hence, we conclude that $f(n) < 15n^3$ whenever $n \geq 4$, as desired.

Example 2. Use the formal definition on $\Theta(\cdot)$ to show that $12n \log n + 20n \in \Theta(n \log n)$.

Solution to Example 2. Write $f(n) = 12n \log n + 20n$. We need to find C_1 and N_1 such that $f(n) \leq C_1 n \log n$ whenever $n \geq N_1$ (for the definition of big-O). And we need to find C_2 and N_2 such that $f(n) \geq C_2 n \log n$ whenever $n \geq N_2$ (for the definition of big-Omega). After some experimentation with different values, we find that $C_1 = 14$ and $C_2 = 3$ look promising, as demonstrated by the following table of values:

n	$f(n)$	$14n \log n$	$3n \log n$
1	32.00	0.00	0.00
2	65.00	28.00	6.00
3	97.58	66.57	14.26
4	130.00	112.00	24.00
5	162.32	162.53	34.83
6	194.58	217.14	46.53
7	226.81	275.12	58.95
8	259.00	336.00	72.00
9	291.17	399.41	85.59
10	323.32	465.07	99.66

From this, it appears that $f(n) < 14n \log n$ whenever $n \geq 5$. Therefore, we can take $C_1 = 14$ and $N_1 = 5$ in the definition of big-O, thus demonstrating that $f(n) \in O(n \log n)$. It also appears that $f(n) > 3n \log n$ whenever $n \geq 1$. Therefore, we can take $C_2 = 3$ and $N_2 = 1$ in the definition of big-Omega, thus demonstrating that $f(n) \in \Omega(n \log n)$. Finally, the fact that $f(n) \in O(n \log n)$ and $f(n) \in \Omega(n \log n)$ satisfies the definition for big-Theta, and we conclude that $f(n) \in \Theta(n \log n)$.

Further remarks about the solution to Example 1. There are many correct values of C_1, N_1, C_2, N_2 that can be used to answer this problem. In fact, for any values of $C_1 > 12$ and $C_2 < 12$, there exist infinitely many suitable values of N_1, N_2.